A Novel Method for Ranking Group Performance

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Abstract

This paper introduces a novel non-parametric method for calculating group-level performance scores, incorporating rank sums, weight bias adjustments for unequal group sizes, and cross-collection interpretability. Theoretical derivations, Empirical validations and detailed example are discussed.

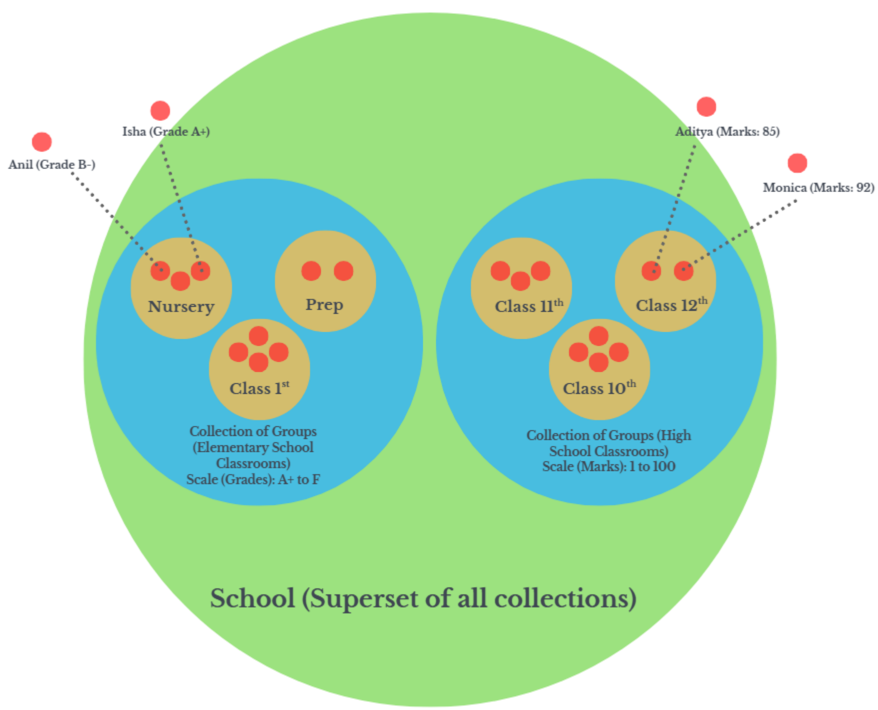
Keywords: Normalized Rank Comparison, Performance Analysis, Weighted Bias Adjustment, Group Performance Metrics, Statistical Methods

A Novel Method for Ranking Group Performance

This formula can numerically identify the performance of a group P_i within a collection of groups containing uneven number of elements. This novel approach takes into account weight bias inherent in the formula itself. The performance value, P_i numerically falls in the normalized range of [0,1] from non-dominant to dominant performance within their collection, which gives a more interpretable and practical scale, which could be visualized as 0 to 100% dominance in performance via $P_i * 100$. This standardized scale can be used to compare within cross-collection which have different scales, see the Figure 1 to which , this formula can be applied towards.

Figure 1

Superset Collection of Elementary Grade Groups and High School Groups



The inspiration for the formula came from the idea from Mann and Whitney, 1947, which uses median to calculate p-value significance. Though, the idea is inspired from their, but applied in terms of assessing performance comparison for groups having uneven elements, and expanded to create a standardize scale with cross-collection interpretability.

Normalized Group Performance Assessment Formulae

$$\boxed{\mathbf{P_i} = \frac{-\mathbf{n_i} + \sum_{\mathbf{j=1}}^{\mathbf{n_i}} \mathbf{r_j}}{\mathbf{n_i} \cdot (\mathbf{N-1})}} \tag{1}$$

- **Ranks**: Sequential ranks assigned to items within and across groups. Tied ranks are averaged out. Note ranks are in ascending order of dominance i.e. rank 1 is the lowest, with rank N^{th} as the highest. This is done to keep symmetry with usage of ranks in other statistics methods such as Mann and Whitney, 1947 etc.

- P_i: Normalized Performance value in the range of [0, 1]

- $\sum r_i$: Sum of ranks for group i.

- n_i: number of elements within the i^{th} group

- N: Total number of elements across all group combined.
  There are

$$k = \text{number of groups}, \quad n_i = \text{number of items in each group}$$

$$\mathrm{N} = \sum_{i}^{k} n_i, \quad \text{where N = total number of items} \tag{2}$$

Applications

This is a list of Applications scenario, the formula was designed for:

- **Educational**: Awarding Best Class Performance in a school, when classes have uneven students, and even different grading scales.

- **Organizational**: To identify best-performing team in the organization, and least-performing, for restructuring and guidance purposes, when number of person per team are uneven, and performance scale at various sector of an organization is different.

- **Business**: Often businesses have economy segment and luxury segment for services and products. Often, just price is not an enough indicator what segment of products fared better. If, a ranking is created, on customer feedback, return on investment, less-after-cost-maintenance, expenses etc. Then, this formula can be used to find the dominant product/services of their business, when there is uneven sale of each product/services sold.

There could be applications outside the scope of this paper. But, when designing the formula these above listed applications were being thought of.

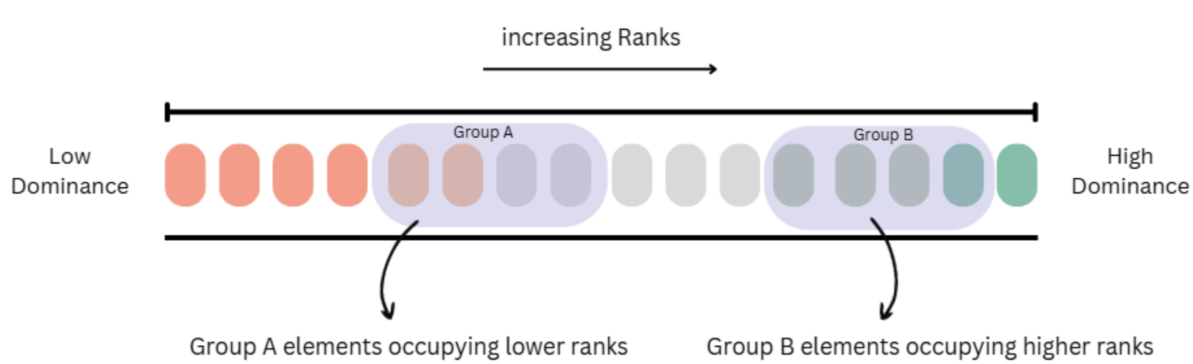
Mathematical derivation, along with its theoretical justification, would be discussed, with its analogous visual intuition where possible. Maxima and Minima, range of dominance [0, 1], would be proven as well.

Non-normalized Performance

If you see, the Figure 2, is intuitively easy to understand that Group B performance is more dominant, that Group A, simply because Group B occupy the higher ranks, and Group A lower ranks.

Figure 2

Visual intuition of the Dominance Formula, where Dominance: Group B > Group A



This non-normalized performance value, with uneven number of groups is given by:

$$p_i^{(non-normalized)} = w_i \cdot \frac{\sum r_i}{S_{UDH}} \tag{3}$$

This later non-normalized dominance would be used to derive the normalized version of the formula. Therefore, non-normalized performance derivation would be discussed first.

- Sum of Upper Dominant Half (S_{UDH}):

$$S_{UDH} = \frac{N(N+1)}{2} - \frac{a(a+1)}{2}, \quad a = \lfloor N/2 \rfloor \tag{4}$$

- Weight Bias (w_i):

$$w_i = \frac{N}{kn_i} \tag{5}$$

Derivation of Non-normalized Performance

From, the understanding we gain from Figure 2, we can easily, ascertain the local dominance, if we divide the scale into two halves of lower ranks half, and upper rank half. So, the group with ranks in the upper half would have a more prominent or dominant performance. So, we need to figure out what's the ranks of the upper dominant half is.

Sum of Upper Dominant Half

1. Purpose of UDH The **Upper Dominant Half (UDH)** represents the **rank sum of the dominant half** of a dataset, providing a benchmark for performance potential in rank-based comparisons.

2. Total Rank Sum (Full Dataset) For a dataset with **N items**, the **total rank sum** is the sum of integers from **1 to N**:

$$S = 1 + 2 + 3 + \ldots + N \tag{6}$$

Using the formula for the sum of the first N integers:

$$S = \frac{N(N+1)}{2} \tag{7}$$

3. Split the Dataset into Halves

   - $a = \lfloor \frac{N}{2} \rfloor$ represents the size of the **non-dominant half** (lower half).
   - The **dominant upper half** includes the rank from $a+1$ to N^{th} rank items.

4. Rank Sum of Non-Dominant Half (Lower Half) The **non-dominant half** (lower half) consists of the **smallest** a **ranks**. Its rank sum is:

$$S_{lower} = 1 + 2 + \ldots + a \tag{8}$$

   Using the formula for sum of integers:

$$S_{lower} = \frac{a(a+1)}{2} \tag{9}$$

5. Rank Sum of Dominant Half (Upper Half) The **Upper Dominant Half** is calculated by subtracting the rank sum of the **lower half** from the **total rank sum**:

$$S_{UDH} = S - S_{lower} \tag{10}$$

   Substituting the formulas:

$$S_{UDH} = \frac{N(N+1)}{2} - \frac{a(a+1)}{2} \tag{11}$$

6. Final Formula

$$S_{UDH} = \frac{N(N+1)}{2} - \frac{a(a+1)}{2} \tag{12}$$

   - N: Total number of items.
   - a: Size of the lower **dominant half**, calculated as the floor of half of the net total number of items across all groups, so as to be inclusive of both odd and even count of N items:

$$a = \lfloor \frac{N}{2} \rfloor \tag{13}$$

Non-normalized Performance Formulae (Biased)

So, to if we compare the ranked sum of the group with the upper dominant half, we can get a numerical understanding of performance value

$$p_i^{(biased)} = \frac{\sum r_i}{S_{UDH}} \tag{14}$$

- $\sum r_i$: Sum of ranks for group i.
- S_{UDH}: Sum of Upper Dominant Half

As, the formula doesn't take into account weight bias. A high number of items can still occupy lower ranks, but in account of total items in group being large, its ranked sum also gives a higher value. That's why weigh bias was added to adjust this non-normalized performance value.

Weight Bias Derivation

To derive the weight bias (w_i), we first understand, that if all the groups have equal number of items, then $w_i = 1$ i.e. no adjustment needed. That means all the groups have the same number of elements in them. The items expected in each group, can be written as:

$$X_e = \frac{N}{k} \tag{15}$$

where:

- X_e: Expected item count per group.
- k: Number of groups in collection.
- N: Total number of items across all groups.

So, now we calculate the fractional change needed to reach expected group item

$$fractional\ change = \frac{X_e - n_i}{n_i} \tag{16}$$

If we want to adjust our biased performance, $p_i^{(biased)}$, according to the fractional change, then,

$$p_i^{(unbiased)} = p_i^{(biased)} + p_i^{(biased)} * fractional\ change \tag{17}$$

in other words, taking $p_i^{(biased)}$ as common, we can rewrite it as

$$p_i^{(unbiased)} = p_i^{(biased)} * (1 + fractional\ change) \tag{18}$$

So, the term needed to adjust from $p_i^{(biased)}$ to $p_i^{(unbiased)}$, can be written as:

$$w_i = 1 + fractional\ change \tag{19}$$

substituting everything we get, the derivation of weight bias:

$$w_i = \frac{N}{kn_i} \tag{20}$$

where:

- w_i: Weight bias for group i.
- n_i: Number of items in group i.
- N: Total number of items.
- k: Number of groups.

Final Formula for unbiased (but non-normalized) Performance Value

Therefore, from Equation 12, Equation 14, Equation 18, and Equation 20

we derived unbiased(but non-normalized) formulae, as stated in the beginning of the section, in Equation 3:

$$p_i^{(non-normalized)} = w_i \cdot \frac{\sum r_i}{S_{UDH}} \tag{21}$$

Note in this research paper, $p_i^{(non-normalized)} = p_i^{(unbiased)}$ are same, but for standardization sake, we will use $p_i^{(non-normalized)}$ moving forward.

Theoretical Justification

We explicitly derive this adjustment to satisfy several criteria:

- The adjustment explicitly becomes smaller as the number of groups k or total observations N increases, ensuring minimal interference in larger datasets. Mathematically, we justify the chosen form by observing its desirable properties in handling various scenarios:

$$\lim_{N\to\infty} \frac{8}{k(3N+2)} = 0,$$
$$\lim_{k\to\infty} \frac{8}{k(3N+2)} = 0.$$

- Explicitly provides a meaningful zero baseline for dominance measures.

- dominance scores become standardized, allowing direct comparisons across multiple studies or scenario

Derivation of Adjustment Term

We aim to minimize the dominance measure $U_i^{(local)}$ defined as:

$$U_i^{(local)} = w_i \cdot \frac{\sum r_i}{S_{UDH}} \tag{22}$$

This is locally valid dominance measure for a single collection.

Now, if we always assume that the net total number of items across all groups is even, it is safe to assume explicitly that:

$$a = \frac{N}{2} \tag{23}$$

Thus, the lower and upper halves would be explicitly:

$$\left[1, \frac{N}{2}\right] \quad \text{and} \quad \left[\frac{N}{2}+1, N\right]$$

Simplifying explicitly, we get:

$$\begin{aligned} S_{UDH} &= \frac{N}{2} \times \frac{\left(\frac{N}{2} + 1 + N\right)}{2} \\ &= \frac{N}{4} \times \frac{3N+2}{2} \\ &= \frac{N(3N+2)}{8} \end{aligned}$$

Now explicitly substituting everything into our dominance equation:

$$\begin{aligned} U_i &= \frac{N}{kn_i} \times \frac{\sum r_i \times 8}{N(3N+2)} \\ &= \frac{8}{kn_i} \times \frac{\sum r_i}{3N+2} \end{aligned}$$

This explicitly represents the simplified and explicitly minimized form of the dominance measure.

Minimizing Expression

For the given expression to be minimized, $\sum r_i$ should be low. Consider exdplicitly:

$$\frac{\sum r_i}{n_i} = \frac{\frac{n_i}{2}(1+n_i)}{n_i} = \frac{1}{2}(n_i + n_i^2) \tag{24}$$

Differentiating explicitly and setting to zero for minimization:

$$\frac{d}{dn_i}\left[\frac{1}{2}(n_i + n_i^2)\right] = \frac{1}{2}(1 + 2n_i) = 0 \tag{25}$$

Solving explicitly, we find:

$$n_i = -\frac{1}{2} \tag{26}$$

Explicitly confirming via second derivative:

$$\frac{d^2}{dn_i^2}\left[\frac{1}{2}(n_i + n_i^2)\right] = 1 \tag{27}$$

The second derivative explicitly indicates a minimum. However, explicitly, since n_i represents a countable quantity and cannot be negative or fractional, we evaluate explicitly at the nearest valid positive integer, i.e., $n_i = 1$:

$$f(1) = \frac{1}{2}(1 + 1^2) = 1 \tag{28}$$

Thus explicitly, the minimum feasible value of the expression $\frac{\sum r_i}{n_i}$ is: 1

Minimum Dominance Value Analysis

This explicitly indicates that the lowest dominance value occurs when there is a singular group within the collection having one element with the lowest rank. Additionally, explicitly, the number of groups (k) and total number of items (N) across all groups should be high.

Therefore, explicitly, the minimum dominance value depends on the term:

$$\frac{8}{k(3N + 2)} \tag{29}$$

Final Adjusted Dominance Score

The dominance score, explicitly adjusted, is presented as:

$$\mathbf{U_i = U_i^{(local)} - \frac{8}{k(3N + 2)}} \tag{30}$$

Thus explicitly, the adjusted dominance measure is both fairer and more interpretable across diverse datasets.

Validation of Dominance Formula

- Mathematical proof of sympy
- Numerical proof annd implementation in Python
- cross-validation from a different method

Detailed Example of Dominance Method

To explicitly demonstrate the utility and robustness of the normalized dominance scoring method, we consider a complex example meeting the following conditions:

- Multiple groups (in example four groups)
- Unequal number of items per group (minimum three items per group)
- Tied ranks
- Odd total number of participants ($N = 15$)

Data and Initial Rankings

Participants from four intervention groups (A, B, C, D) were ranked based on their effectiveness scores. Ties were assigned average ranks explicitly in Table 1:

Calculations

Step 1: Rank Sums

- $r_A = 13.5 + 9 + 6 + 2.5 = 31$
- $r_B = 12 + 10 + 15 = 37$
- $r_C = 11 + 5 + 8 = 24$
- $r_D = 13.5 + 2.5 + 4 + 7 + 1 = 28$

Step 2: Define Parameters

Total participants: $N = 15$, Number of groups: $k = 4$.

Group sizes:

- $n_A = 4$, $n_B = 3$, $n_C = 3$, $n_D = 5$

Compute explicitly $a = \lfloor N/2 \rfloor = \lfloor 15/2 \rfloor = 7$.

Step 3: Calculate Upper Dominant Half

Table 1

Effectiveness Scores and Verified Assigned Ranks

Participant	Group	Score	Rank
1	A	95	13.5 (tie)
2	A	48	9
3	A	25	6
4	A	15	2.5 (tie)
5	B	83	12
6	B	57	10
7	B	100	15
8	C	70	11
9	C	20	5
10	C	40	8
11	D	95	13.5 (tie)
12	D	15	2.5 (tie)
13	D	18	4
14	D	30	7
15	D	10	1

Lower Dominant Half (LDH):

$$S_{lower} = \frac{a(a+1)}{2} = \frac{7 \times 8}{2} = 28 \tag{31}$$

Total sum of ranks:

$$S = \frac{N(N+1)}{2} = \frac{15 \times 16}{2} = 120 \tag{32}$$

Upper Dominant Half (UDH):

$$S_{UDH} = S - S_{lower} = 120 - 28 = 92 \tag{33}$$

Step 4: Weight Bias Calculation

$$w_i = \frac{N}{k \cdot n_i} \tag{34}$$

Calculate explicitly:

- $w_A = \frac{15}{4 \times 4} = 0.9375$
- $w_B = \frac{15}{4 \times 3} = 1.25$
- $w_C = \frac{15}{4 \times 3} = 1.25$
- $w_D = \frac{15}{4 \times 5} = 0.75$

Step 5: Dominance Scores

Dominance scores explicitly computed as:

$$U_i = w_i \cdot \frac{r_i}{S_{UDH}} \tag{35}$$

Calculate:

- $U_A = 0.9375 \times \frac{31}{92} = 0.316$
- $U_B = 1.25 \times \frac{37}{92} = 0.503$
- $U_C = 1.25 \times \frac{24}{92} = 0.326$
- $U_D = 0.75 \times \frac{28}{92} = 0.228$

Summary of Dominance Results

Results have been summarized in Table 2

Interpretation of Results

Group B achieves the highest dominance score, demonstrating superior effectiveness. Despite the variations in group size and tied ranks, the method accurately balances these complexities. Group D has the lowest dominance score, emphasizing the method's robustness and fairness in clearly assessing relative performance.

Validation of Results

Table 2

Dominance Calculation Results

Group	Rank Sum (r_i)	Weight Bias (w_i)	Dominance Score (U_i)
A	31	0.9375	0.316
B	37	1.2500	0.503 (Most Dominant)
C	24	1.2500	0.326
D	28	0.7500	0.228 (Least Dominant)

References

Mann, H. B., & Whitney, D. R. (1947). On a test of whether one of two random variables is stochastically larger than the other. *The Annals of Mathematical Statistics*, *18*(1), 50–60. https://doi.org/10.1214/aoms/1177730491