Ring: A ring $(R,+,\cdot)$ is a set R together with two binary operations $+$ (addition) and $\cdot$ (multiplication) defined on R such that the following axioms are satisfied:-

(R_1) $(a+b)+c = a+(b+c)$ $\forall\, a,b,c \in R$

(R_2) $a+b = b+a$ $\forall\, a,b \in R$

(R_3) There exist an element $0 \in R$ such that $0+a = a$ $\forall\, a \in R$

(R_4) $\forall\, a \in R$ there exist an element $-a \in R$ such that $a+(-a) = 0$.

(R_5) $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\forall\, a,b,c \in R$

(R_6) $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$ $\forall\, a,b,c \in R$ (Left distributive law)

(R_7) $(b+c)\cdot a = (b\cdot a)+(c\cdot a)$ $\forall\, a,b,c \in R$ (Right distributive law)

We call 0, the zero element of the Ring $(R,+,\cdot)$.

That is, an algebraic system $(R,+,\cdot)$ is called a ring if

(i) $(R,+)$ is an abelian group

(ii) $(R,\cdot)$ is a semigroup i.e $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\forall\, a,b,c \in R$

(iii) The operation $\cdot$ is distributive over the operation $+$.

* A ring R is called commutative if $a\cdot b = b\cdot a$ $\forall\, a,b \in R$
* In a ring, an element $e \in R$ is called a unity (identity) element if $ea = ae = a$ $\forall\, a \in R$. An unity element of a ring R (if it exists) is an element of the semigroup $(R,\cdot)$. The unity of ring (if it exists) is generally denoted by 1. A ring R is called a ring with unity if it has an unity element.

Examples :- (1) $(\mathbb{Z}, +, \cdot)$
Unity 1

(2) $(2\mathbb{Z}, +, \cdot)$
Unity not belongs.

(3) The set $Z_n = \{0, 1, 2, 3, \ldots n-1\}$ under addition and multiplication modulo n is a commutative ring with unity 1.

(4) The set $M_2(\mathbb{Z})$ of 2×2 matrices with integers elements is a non-commutative ring with unity.

Some elementary properties of Ring:-

Let $a, b, c \in R$ then

(1) $a \cdot 0 = 0 \cdot a = 0$

(2) $a \cdot (-b) = (-a)b = -(a \cdot b)$

(3) $(-a) \cdot (-b) = a \cdot b$

(4) $a \cdot (b-c) = a \cdot b - a \cdot c$ and $(b-c) \cdot a = b \cdot a - c \cdot a$.

Ring with zero divisor:- If a and b are two non-zero elements of a ring R such that $ab = 0$, then a and b are divisors of 0 (or 0 devisors). In particular, a is left divisor of 0 and b is right divisor of 0.

In a commutative ring, every left divisor of 0 is also a right divisor of 0 and conversely.

* The ring of integers do not have zero divisors.

* Suppose M is ring of all 2×2 matrices with their elements as integers, the addition and multiplication of matrices being the two ring composition. then M is a ring with zero divisors.

Null matrix $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ is the zero elements of this ring.

Now $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ are two non-zero elements of this ring.

We have $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = BA$

thus the product of two non-zero elements of the ring is equal to the zero element of the ring. therefore, M is a ring with zero divisors.

& Ex: $Z_4 = \{0, 1, 2, 3, +_4, \times_4\}$

$2 \cdot 2 = 0$, Ring with zero divisor.

Integral domain: A ring containing at least two elements is called an integral domain if it

(i) is commutative (ii) has unity element (iii) is without zero divisors.

Thus, in an integral domain a product is 0 only when one of the factor is 0, that is $ab=0$ only, when $a=0$ or $b=0$.

Ex:
$(\mathbb{Z}, +, \cdot)$ — I.D
$(\mathbb{R}, +, \cdot)$ — I.D
$(2\mathbb{Z}, +, \cdot)$ — × (unity ×)
$M_2(\mathbb{Z})$ — × [not commutative, also has zero divisors]

Field:

A ring containing at least two elements is called a field if it

(1) is commutative
(2) has unity
(3) is such that every non-zero element has multiplicative inverse in R

That is to say, a system $(R, +, \cdot)$ is a field if

(i) $(R, +)$ an abelian group
(2) $(R', \cdot)$ is a abelian group where $R' = R - \{0\}$
(3) the distributive laws

i.e $a \cdot (b+c) = ab + ac$
$(b+c) \cdot a = b \cdot a + c \cdot a$ hold $\forall a, b, c \in R$