In this reflection we will give an alternative proof of the Riesz's Representation Theorem for V a Real inner product vector space. As a consequence of this proof we are going to show that there exist a natural isomorphism (it doesn't not depend on the choices of the basis) between V and $V' = \mathcal{L}(\mathcal{V}, \mathbb{R})$, the dual space. Notice that these vector spaces are always isomorphic regardless if V has an inner product or not but the standard isomorphism does depends on the choice of the basis. This is Problem 17 on section 6B. We define a map from V to its dual space, i.e.

$$\begin{aligned} \Phi : V &\to V' \\ u &\mapsto \Phi u \end{aligned}$$

Given $u \in V$ we define Φu as the following maps:

$$\begin{aligned} \Phi u : V &\to \mathbb{F} \\ v &\mapsto \Phi u(v) = \langle v, u \rangle \end{aligned}$$

The plan is the following:

- **Step 1:** We prove that if $\mathbb{F} = \mathbb{R}$, then Φ is well defined and it is a linear map from V to V'
- **Step 1.5:** This is a remark that this argument only works for real vector spaces. If $\mathbb{F} = \mathbb{C}$ then we will show that Φ is not linear.
- **Step 2:** Φ is injective, we are going to do this by hand and using the properties of inner products.
- **Step 3:** Φ is also surjective, and therefor an isomorphism. To do so we will do some dimension counting and the Rank-Nullity formula. (without using Riesz Representation)
- **Step 4:** We can use this isomorphism to give an alternative proof of the Riesz Representation theorem by proving that V and V' are isomorphic. What is really surprising is that this isomorphism does not depend on the choice of the basis of V.

Step 1: Assume now that $\mathbb{F} = \mathbb{R}$, then we will show that Φ is well defined and it is a linear map from V to V'. Notice first that for each $u \in V$ then Φu is defined with the inner product: $\langle v, u \rangle \in \mathbb{R}$, this tell us that Φu is well defined as as a map from V to $\mathbb{F}$. We now need to show that it is also linear.

$$\begin{aligned} \Phi : V &\to V' \\ u &\mapsto \Phi u(v) = \langle v, u \rangle \end{aligned}$$

To do so, fix any $u, w \in V$ we compute:

$$\Phi(u + w)(v) = \langle v, u + w \rangle = \langle v, u \rangle + \langle v, w \rangle = \Phi(u)(v) + \Phi(w)(v)$$

Similarly fir any $\lambda \in \mathbb{R}$ then we have:

$$\Phi(\lambda u)(v) = \langle v, \lambda u \rangle = \bar{\lambda}\langle v, u \rangle = \lambda \langle v, u \rangle = \lambda \Phi(u)(v)$$

Notice that in this last step we used the fact that V is a real vector space ($\lambda = \bar{\lambda}$). We can now conclude that $\Phi \in V'$

Step 1.5 - Remark. Notice that if $\mathbb{F} = \mathbb{C}$ then we would have:

$$\Phi(\lambda u)(v) = \bar{\lambda}\Phi(u)(v)$$

Since in general for a complex number $\lambda \neq \bar{\lambda}$, our map Φ would not be linear. Highlighting the fact that this reflection only works for real vector spaces.

Step 2: Now we will show that Φ is injective. To do so fix any $u, w \in V$ and assume $\Phi u = \Phi w$ which means that

$$\text{For all } v \in V : \Phi u(v) = \Phi w(v)$$

Which is equivalent to have, for any $v \in V$:

$$\langle v, u \rangle = \langle v, w \rangle \iff \langle v, u - w \rangle = 0.$$

Since this is true for any $v \in V$, I can pick $v = u - w$ and I obtain:

$$\langle u - w, u - w \rangle = 0 \iff ||u - w||^2 = 0 \iff u - w = 0 \iff u = w$$

Which proves that Φ is injective.

Step 3: Now we are going to show that the map is also surjective and therefor an isomorphism. In a remark later in this reflection, we will show how this is immediate if we use Riesz Representation Theorem but since we want to give an alternative proof we are going to use a different argument of counting dimension, again we have, $\Phi : V \to V'$, we apply the Rank nullity formula we can write:

$$\dim V = \dim \operatorname{Null} \Phi + \dim \operatorname{Range} \Phi = \dim \operatorname{Range} \Phi$$

Since we have showed at Step 2 that this map is injective, so it remains to show that $\dim V' = \dim V$, we are using a result from Chapter 3:

$$\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$$

Since $V' = \mathcal{L}(V, \mathbb{R})$ we can conclude that:

$$\dim V' = \dim V \cdot \dim \mathbb{R} = n \cdot 1 = n$$

The conclusion follows from the fact that finite dimensional vector spaces sharing the same dimension are always isomorphic.

Remark. If we were to use Riesz Representation Theorem the surjectivity of $\Phi : V \to V'$ follows immediately. To see this, assume by by contradicion there exist at least one $f \in V'$ such that:

$$\Phi(v) \neq f \text{ for all } v \in V.$$

But thanks to the Riesz Representation theorem there exist a unique $u \in V$ such that $f(v) = \langle v, u\rangle$, if we pick $v = u$ we reach a contradiction. We don't want to use the Riesz's Representation theorem because we are actually trying it to prove it, and we are now ready to do so in the next step.

Step 4 Recall the Riesz Representation theorem, let V be a finite dimensional vector space and pick $\phi \in V'$ then there exist a unique u in V such that:

$$\phi(v) = \langle v, u\rangle \text{ for all } v \in V.$$

Notice that this map is precisely what we call Φu, to prove the theorem fix any $\phi \in V'$ then since $\Phi : V \to V'$ is surjective we can find an $u \in V$ such that $\Phi u = \phi$ notice that this u is also unique since Φ is injective, and we obtain:

$$\phi(v) = \Phi u(v) = \langle v, u\rangle \text{ for all } v \in V$$

As required and this concludes the proof.