Reflection on Methods for Finding a Basis for Null Subspace

★ idea from: HW5 Problem 1.

- Problem Statement

  Given a matrix $A \in \mathbb{R}^{m\times n}$, the null space of A, denoted as null(A), is defined as:

$$\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

  A basis for null(A) consists of a set of linearly independent vectors that span this subspace.

- Problem encountered & Smart Way to solve:
  - develop two approaches to find such basis.

  1) Row Reduction to RREF (row-reduced echelon form)

  2) Direct Substitution

Method 1 : RREF

- Perform row reduction on A to compute its row-reduced echelon form
  - Given $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 2 & 4 & -2 & 6 \end{bmatrix}$

  - The rank of A is 2 (two pivot columns)

    ⇒ by Rank-nullity theorem:

$$\begin{aligned} \dim(\text{null}(A)) &= n - \text{rank}(A) \\ &= 4 - 2 \\ &= 2. \end{aligned}$$

- By RREF:

$$\text{Matrix } A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 - 2R_1$$

$$\sim \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad R_1 = R_1 - 2R_2$$

- Identify pivot & free variables:
  - Pivot columns: x_1, x_2
  - Free variables: x_3, x_4
- Solve from the Pivot variables x_1, x_2 in terms of free variables:
  - From row 1: $x_1 = -x_3 + x_4$
  - From row 2: $x_2 = x_3 - 2x_4$
- Parametrize the solution using $x_3 = t$ & $x_4 = s$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t+s \\ t-2s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Basis (null(A)) = $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Method 2: Direct Substitution

- Write out the system of equations implied by $Ax=0$:

$$x_1 + 2x_2 - x_3 + 3x_4 = 0$$
$$x_2 - x_3 + 2x_4 = 0$$
$$2x_1 + 4x_2 - 2x_3 + 6x_4 = 0$$

- Recompute equations: solve for x_1 & x_2 in terms of x_3 & x_4:

$$x_1 = -2x_2 + x_3 - 3x_4 \quad ①$$
$$x_2 = x_3 - 2x_4 \quad ②$$

- substitute ② into ①:

$$x_1 = -2(x_3 - 2x_4) + x_3 - 3x_4$$
$$= -2x_3 + x_3 + 4x_4 - 3x_4$$
$$= -x_3 + x_4$$

- Assign free variables ($x_3 = t$, $x_4 = s$):

$$x_1 = -x_3 + x_4 = -t + s$$
$$x_2 = x_3 - 2x_4 = t - 2s$$
$$x_3 = t$$
$$x_4 = s$$

$$\Rightarrow x = t\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{Basis}(\text{null}(A)) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Comparison Between Two Methods

Aspect	RREF	Direct Substitution
Systematic Approach	Eliminate dependencies step-by-step, less prone to errors	relies on identifying dependencies manually
Ease of use	Straightforward with clear rules (Gaussian elimination)	Simpler for small systems but requires attention
Scalability	efficient for large matrix ($n > 4$) and sparse matrices (many 0's)	better for small matrix ($n \leq 4$)
Basis Construction	Produces an independent and minimal basis directly	needs manual validation.
Computational steps	fewer step for structured reduction.	relies on solving and substituting equation.
Flexibility	require standard algorithmic structure	more intuitive for special cases (e.g. diagonal / triangular)

Most Serious Problem	Computational complexity and human error due to numerous steps increasing the chance of mistakes	misidentification of independence due to reliance on manual tracking of row relationships