MAT201A Homework 7
Fall 2019
Professor Qinglan Xia
Due Date: Wednesday, November 13th at 9:00am

1. For any $f \in C([0,1])$, define

$$||f||_1 := \left(\int_0^1 |f(x)|^2 \right)^{1/2}$$

and

$$||f||_2 := \left(\int_0^1 (1+x)|f(x)|^2 \right)^{1/2}.$$

Show that $||\cdot||_1$ and $||\cdot||_2$ are equivalent norms in $C([0,1])$.

2. Let X be the space of all sequences of real numbers with only finitely many nonzero terms. Consider the following two norms on X:

$$||(x_n)||_1 := \sum_{n=1}^{\infty} |x_n| \quad \text{and } ||(x_n)||_2 := \sqrt{\sum_{n=1}^{\infty} |x_n|^2}.$$

Are the norms $||\cdot||_1||$ and $||\cdot||_2$ equivalent? Justify your answer.

3. Let $X = C_b([0,\infty))$ be the space of all bounded and continuous functions on $[0,\infty)$. For any $a > 0$, define

$$||f||_a := \left(\int_0^\infty e^{-ax}|f(x)|^2 \right), \quad \forall f \in X.$$

(a) Show that $||\cdot||_a$ is a norm on X.

(b) For any $a > b > 0$, show that $||\cdot||_a$ and $||\cdot||_b$ are not equivalent norms on X.

4. Let $e_1, e_2, \cdots, e_n$ be any given vectors in a real linear space X, and let $||\cdot||$ be a norm on X. Show that for any $x \in X$, there exists $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$ such that

$$||x - \sum_{i=1}^{n} \lambda_i e_i|| = \min_{(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n} ||x - \sum_{i=1}^{n} a_i e_i||.$$

5. Let $X = (C([0,1]), ||\cdot||_\infty)$. Define $T : X \to X$ by

$$(Tf)(x) = x\int_0^x f(t)dt, \quad \forall f \in X.$$

Show that $T \in \mathcal{B}(X)$ and compute $||T||$. Also prove that the inverse $T^{-1} : ran(T) \to X$ exists but is not bounded.