MAT201A Homework 2
Fall 2019
Professor Qinglan Xia
Due Date: Wednesday, October 9th at 9:00am

1. Let (x_n) be a Cauchy sequence in a metric space (X, d). Prove that
   (a) If (x_n) has a convergent subsequence, then it converges itself.
   (b) For any sequence (ϵ_n) of positive numbers, there exists a subsequence (x_{n_k}) of (x_n) such that
   $$d(x_{n_k}, x_{n_{k+1}}) \leq \epsilon_k$$
   for all k.

2. Suppose x and y are any two points in a metric space (X, d). Let r and R be two positive numbers. Show that
   (a) If $d(x, y) \geq R + r$, then the open balls $B_R(x)$ and $B_r(y)$ are disjoint;
   (b) If $d(x, y) \leq R - r$, then $B_r(y) \subseteq B_R(x)$;
   (c) the converse statements to (a) and (b) are not always true (give counterexamples).

3. We first introduce a definition: Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is called Lipschitz continuous if there exists $C \geq 0$ such that
   $$d_Y(f(x), f(y)) \leq C d_X(x, y), \forall x, y \in X.$$
   The smallest possible constant for the above inequality is called the Lipschitz constant $Lip(f)$ of f.

   Let $\tilde{X}$ be a metric space and X a dense subset of $\tilde{X}$. let Y be a complete metric space. Then,
   (a) For any Lipschitz map $f : X \to Y$, prove that there exists a unique continuous map $\tilde{f} : \tilde{X} \to Y$ such that $\tilde{f}|_X = f$. Moreover, $\tilde{f}$ is Lipschitz and $Lip(\tilde{f}) = Lip(f)$.
   (b) Show that one cannot replace "Lipschitz" in (a) by "continuous".

4. A function $f : \mathbb{R}^n \to \mathbb{R}$ is coercive if
   $$\lim_{||x|| \to \infty} f(x) = \infty.$$
   Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous and coercive, then f is bounded from below and attains its infimum.

5. Suppose $S \subseteq X$ is an $\epsilon - net$ of A contained in a metric space X. Show that there exists a subset K of A such that K is a 2ϵ-net of A and the cardinality of K is no greater than the cardinality of S.

6. Prove that

    a) Any subset of a totally bounded set is totally bounded.

    b) Any totally bounded subset of a metric space is bounded. Is it true that any bounded set must be totally bounded? Prove or disprove it.

    c) In $\mathbb{R}^n$, any bounded set is totally bounded.

7. Define the distance between two subsets A and B of a metric space (X, d) by

$$dist(A, B) := \inf\{d(a, b) : a \in A, b \in B\}.$$

(Warning, this kind of distance does not satisfy the triangle inequality!)

Prove that if A and B are compact, then there exist $x \in A$ and $y \in B$ such that

$$d(x, y) = dist(A, B).$$