Problem 1.

Proof. By direct computation,

$$(\hat{f_N})_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} \frac{e^{-inx}}{2\pi} \sum_{k=-N}^{N} e^{ikx} \mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{2\pi}} & N \geqslant n; \\ 0 & N < n. \end{cases}$$

Apply Theorem 7.5 in the textbook, for fixed n, there exists sufficiently large N, such that

$$(\widehat{f_N * g})_n = \sqrt{2\pi}(\hat{f_N})_n \hat{g}_n = \hat{g}_n.$$

Apparently, the sequence $\left\{(\widehat{f_N * g})_n\right\}_N$ in ℓ^2 converges to $\hat{g}_n$ with respect to $\|\cdot\|_\ell^2$ norm, by Parseval's identity, the sequence $\{f_N * g\}$ converges to g with respect to $\|\cdot\|_{L^2}$ norm. □

Problem 2.

Proof. We note that $f_n = f_{n-1} * f$. We first compute $(\hat{f})_k$.

$$\begin{aligned}(\hat{f})_k &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} a(2\mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x) - 1)e^{ikx}\mathrm{d}x = \frac{a}{\sqrt{2\pi}} \left(\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\pi}^{-\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} \right) e^{ikx}\mathrm{d}x \right) \\ &= \begin{cases} \dfrac{a}{\sqrt{2\pi}} \dfrac{4\sin(\frac{k\pi}{2})}{k} & k \neq 0; \\ 0 & k = 0. \end{cases}\end{aligned}$$

As a result, by the mathematical induction, we find that

$$(\hat{f}_n)_k = \begin{cases} \dfrac{4^n a^n \sin^n(\frac{k\pi}{2})}{k^n\sqrt{2\pi}} & k \neq 0; \\ 0 & k = 0. \end{cases}$$

For any $n \geqslant 1$, to have $f_n \in L^2(\mathbb{T})$, we need

$$\sum_{k=-\infty}^{\infty} \frac{16^n a^{2n} \sin^{2n} \frac{k\pi}{2}}{2\pi k^{2n}} = \sum_{k=1}^{\infty} \frac{16^n a^{2n} \sin^{2n} \frac{k\pi}{2}}{\pi k^{2n}} < \infty.$$

To pass the limit $n \to \infty$, we need to require $|a| \leqslant \frac{1}{4}$; otherwise the series diverge.

For $|a| < \frac{1}{4}$, $(\hat{f}_n)_k \to 0$ as $n \to \infty$ for each k. Then $f_n \to g \equiv 0$, which is excluded since g is nonzero.

For $a = -\frac{1}{4}$, $(\hat{f}_n)_k = \frac{(-1)^n \sin^n(\frac{k\pi}{2})}{\sqrt{2\pi}k^n}$ diverges as $n \to \infty$ for each k.

For $a = \frac{1}{4}$, $(\hat{f}_n)_1 = \frac{1^n}{\sqrt{2\pi}1^n} = \frac{1}{\sqrt{2\pi}}$ as $n \to \infty$, $(\hat{f}_n)_{-1} = \frac{(-1)^n}{\sqrt{2\pi}(-1)^n} = \frac{1}{\sqrt{2\pi}}$ as $n \to \infty$ and $(\hat{f}_n)_k = \frac{\sin^n(\frac{k\pi}{2})}{\sqrt{2\pi}k^n} \to 0$ as $n \to \infty$ for all $|k| \geqslant 2$. As the result, the limit function:

$$g(x) = \frac{1}{\sqrt{2\pi}}(\frac{1}{\sqrt{2\pi}}e^{ix} + \frac{1}{\sqrt{2\pi}}e^{-ix}) = \frac{e^{ix} + e^{-ix}}{2\pi} = \frac{\cos(x)}{\pi}.$$

□

Problem 3.

Proof. Since $g(x) = f(x-\tau) = f(x)$, apply Fourier transform on both sides, we have

$$\hat{g}_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} f(x-\tau)e^{-inx}\mathrm{d}x \overset{y=x-\tau}{=} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} f(y)e^{-in(y+\tau)}\mathrm{d}y = e^{-in\tau}\hat{f}_n = \hat{f}_n.$$

Since $\frac{\tau}{n}$ is an irrational number, $e^{-in\tau} = 1$ only when $n = 0$. As a result, $\hat{f}_n = 0$ for all $n \neq 0$. Therefore, f must be a constant function. □

Problem 4.

Proof.

a) Note that $(\hat{S_N})_k = \hat{f}_k$ for $k \leqslant N$ and $(\hat{S_N})_k = 0$ for $k \geqslant N+1$. Now we compute the Fourier coefficients for D_N.

First, we recall a trigonometric identity:

$$\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} = 1 + 2\sum_{n=1}^{N}\cos(nx) = \sum_{n=-N}^{N} e^{inx}.$$

Then:

$$\begin{aligned}(\hat{D_N})_k &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} \frac{1}{2\pi}\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}e^{-ikx}\mathrm{d}x = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{T}}\sum_{n=-N}^{N} e^{inx}e^{-ikx}\mathrm{d}x.\\ &= \begin{cases} \frac{1}{\sqrt{2\pi}} & k \leqslant N; \\ 0 & k \geqslant N+1. \end{cases}\end{aligned}$$

Since $\sqrt{2\pi}(\hat{D_N})_k\hat{f}_k = (\hat{S_N})_k$, then $S_N = D_N * f$.

b) $N = 0$ goes to the case in part a). For $N \geqslant 1$, we recall a trigonometric identity:

$$\sum_{k=0}^{N}\frac{\sin((k+\frac{1}{2})x)}{\sin(\frac{x}{2})} = \left(\frac{\sin(\frac{(N+1)x}{2})}{\sin(\frac{x}{2})}\right)^2.$$

Then $F_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_N(x)$. Thus

$$\sqrt{2\pi}(\hat{F_N})_k\hat{f}_k = \frac{\sqrt{2\pi}}{N+1}\sum_{k=0}^{N}(\hat{D_N})_k\hat{f}_k = \frac{1}{N+1}\sum_{k=0}^{N}(\hat{S_N})_k = (\hat{T_N})_k,$$

which yields that $T_N = F_N * f$.

c) (D_N) are not approximation identities since they are not necessary nonnegative. From the book by Folland or Stein and Shakarchi, there exists continuous function f that $S_N = D_N * f$ does not convergence to f pointwise, thus we do not expect further for uniform convergence.

Now we prove (F_N) are indeed approximation identities. As a result, $T_N = F_N * f$ converges to f uniformly, as a property of approximation identities.

(Nonnegative): It is obvious that (F_N) are nonnegative.

(Unit mass):

$$\int_{-\pi}^{\pi} \frac{1}{2\pi(N+1)} \left(\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \right)^2 \mathrm{d}x = \frac{1}{2\pi(N+1)} \sum_{k=0}^{N} \int_{-\pi}^{\pi} \frac{\sin((k+\frac{1}{2})x)}{\sin(\frac{x}{2})} \mathrm{d}x$$
$$= \frac{1}{2\pi(N+1)} \sum_{k=0}^{N} \int_{-\pi}^{\pi} (\sum_{j=-k}^{k} e^{ijx}) \mathrm{d}x = \frac{1}{2\pi(N+1)} \sum_{k=0}^{N} \int_{-\pi}^{\pi} 1 \mathrm{d}x = 1.$$

(Mass stays around zero): Fix $0 < \delta \leqslant \pi$,

$$\begin{aligned} 0 \leqslant \int_{\delta \leqslant |x| \leqslant \pi} \frac{1}{2\pi(N+1)} \left(\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \right)^2 \mathrm{d}x &= \frac{1}{\pi(N+1)} \int_{\delta \leqslant x \leqslant \pi} \left(\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \right)^2 \mathrm{d}x \\ &\leqslant \frac{1}{\pi(N+1)} \int_{\delta \leqslant x \leqslant \pi} \left(\frac{1}{\sin(\frac{\delta}{2})} \right)^2 \mathrm{d}x \\ &= \frac{\pi - \delta}{\pi(N+1)\sin^2(\frac{\delta}{2})}, \end{aligned}$$

take $N \to \infty$ on both sides, by sandwich's theorem, we have

$$\lim_{N\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} F_N(x) \mathrm{d}x = 0.$$

□

Problem 5.

Proof. a) **Orthonormality:**

$$\begin{aligned} \langle e_m, e_n \rangle_{L^2[0,\pi]} &= \int_0^{\pi} \overline{e_m} e_n \mathrm{d}x = \int_0^{\pi} \frac{2}{\pi} \sin(mx)\sin(nx) \mathrm{d}x \\ &= \frac{1}{\pi} \int_0^{\pi} (\cos((m-n)x) - \cos((m+n)x)) \mathrm{d}x \\ &= \begin{cases} m = n: & \frac{1}{\pi} \int_0^{\pi} (\cos(0) - \cos(2mx)) \mathrm{d}x \\ m \neq n: & \frac{1}{\pi} \left(\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n} \right) \Big|_0^{\pi} \end{cases} \\ &= \begin{cases} m = n: & 1 \\ m \neq n: & 0 \end{cases} \end{aligned}$$

Completeness: Let $f \in L^2([0,\pi])$, we define $\bar{f} \in L^2([-\pi,\pi])$ by odd extension:

$$\bar{f}(x) = \begin{cases} f(x) & x > 0; \\ 0 & x = 0; \\ -f(-x) & x < 0. \end{cases}$$

Since $\bar{f} \in L^2([-\pi,\pi])$, and $\frac{1}{\sqrt{2\pi}}e^{inx}$ is an orthonormal basis of $L^2([-\pi,\pi])$, then there exists a sequence of coefficients, such that

$$\bar{f}(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

holds with respect to $\|\cdot\|_{L^2([-\pi,\pi])}$. Note that $\bar{f}$ is an odd function, then the above right hand side is indeed in the form of $\sum_{n=1}^{\infty} b_n \sin(nx)$. Since

$$\left\| f - \sum_{n=1}^{N} b_n \sin(nx) \right\|_{L^2([0,\pi])} \leqslant \left\| \bar{f} - \sum_{n=1}^{N} b_n \sin(nx) \right\|_{L^2([-\pi,\pi])} \to 0, \quad \text{as} \quad N \to \infty,$$

which yields that (e_n) is complete orthonormal basis.

b) **Orthonormality:** For $m \neq 0$,

$$\begin{aligned} \langle f_0, f_0 \rangle &= \int_0^{\pi} \frac{1}{\pi} \mathrm{d}x = 1; \\ \langle f_0, f_m \rangle &= \int_0^{\pi} \frac{2}{\pi} \cos(mx) \mathrm{d}x = \frac{2}{m\pi} \sin(mx)|_0^{\pi} = 0. \end{aligned}$$

For $m, n \neq 0$,

$$\begin{aligned} \langle f_m, f_n \rangle &= \int_0^{\pi} \frac{2}{\pi} \cos(mx)\cos(nx) \mathrm{d}x = \frac{1}{\pi} \int_0^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \mathrm{d}x \\ &= \begin{cases} m = n: & \frac{1}{\pi} \int_0^{\pi} (\cos(2mx) + \cos(0)) \mathrm{d}x; \\ m \neq n: & \frac{1}{\pi} \left(\frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n} \right) \Big|_0^{\pi}. \end{cases} \\ &= \begin{cases} m = n: & 1; \\ m \neq n: & 0. \end{cases} \end{aligned}$$

Completeness: Let $f \in L^2([0,\pi])$, we define $\tilde{f} \in L^2([-\pi,\pi])$ by even extension:

$$\tilde{f}(x) = \begin{cases} f(x) & x \geqslant 0; \\ f(-x) & x < 0. \end{cases}$$

Since $\tilde{f} \in L^2([-\pi,\pi])$, and $\frac{1}{\sqrt{2\pi}}e^{inx}$ is an orthonormal basis of $L^2([-\pi,\pi])$, then there exists a sequence of coefficients, such that

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

holds with respect to $\|\cdot\|_{L^2([-\pi,\pi])}$. Note that $\tilde{f}$ is an even function, then the above right hand side is indeed in the form of $\sum_{n=0}^{\infty} b_n \cos(nx)$. Since

$$\left\| f - \sum_{n=0}^{N} b_n \cos(nx) \right\|_{L^2([0,\pi])} \leqslant \left\| \tilde{f} - \sum_{n=0}^{N} b_n \cos(nx) \right\|_{L^2([-\pi,\pi])} \to 0, \quad \text{as} \quad N \to \infty,$$

which yields that (f_n) is complete orthonormal basis.

$\square$