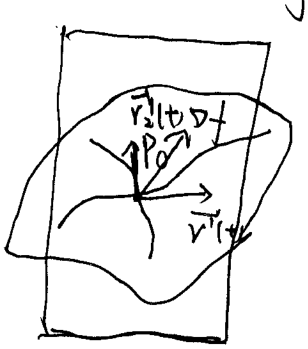
09/06

5 Problems
① Multi-Choice 40
② Computation P 20 < 2 partial qs
③ Geometric P 20 < 2 partial qs.
40 ④ Extrema Value. 14.7 < 2 partial qs.
40 ⑤ Comprehensive Q < 3 partial questions.

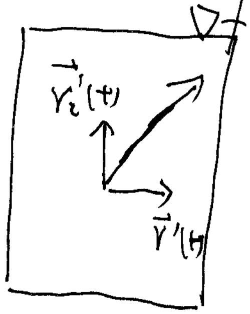
Sec. 14.6. Tangent Plane and Differentials.



$\underline{f(x,y,z) = C}$ $\vec{r}(t) = (x(t), y(t), z(t))$

$f(\vec{r}(t)) = C.$

$\frac{df(\vec{r}(t))}{dt} = 0$ $\underline{\nabla f \cdot \vec{r}'(t) = 0}$



$\nabla f = \vec{n}$ $P_0(x_0, y_0, z_0)$ $\nabla f = (f_x, f_y, f_z)$

$$\underline{f_x(P_0)}(x - x_0) + \underline{f_y(P_0)}(y - y_0) + \underline{f_z(P_0)}(z - z_0) = 0$$

eqn for tangent plane of the level surface at P_0

$$\begin{cases} x = x_0 + f_x(P_0)\cdot t \\ y = y_0 + f_y(P_0)\cdot t \\ z = z_0 + f_z(P_0)\cdot t \end{cases} \rightarrow$$ Parameter eqn for normal lines.

$z = f(x,y)$ (x, y, z) $f(x,y) - z = 0 \Leftrightarrow F(x,y,z) = 0$

$F_x = f_x$, $F_y = f_y$, $F_z = -1$ $F(x,y,z) = f(x,y) - z$

eg. $f(x,y,z) = \boxed{x^2+y^2+z-9=0}$ at $(1,2,4)$

$\nabla f = (2x,\ 2y,\ 1)\Big|_{(1,2,4)} = (2,\ 4,\ 1) \to$ normal vector.

$2(x-1) + 4(y-2) + 1\cdot(z-4) = 0 \to$ tangent plane.

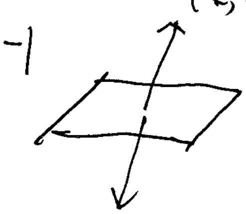
same surface

$$\begin{cases} x = 1+2t \\ y = 2+4t \\ z = 4+t \end{cases} \to \text{normal line.}$$

$z = \underbrace{-x^2 - y^2 + 9} = f(x,y)$

$f_x = -2x$ $\qquad f_y = -2y$ $\qquad f_z = -1$

$(-2,\ -4,\ -1)$

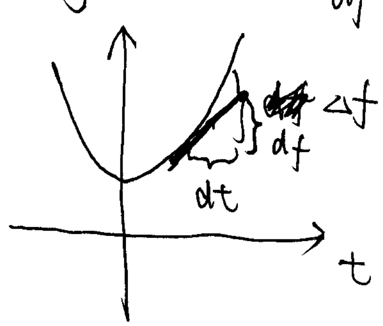


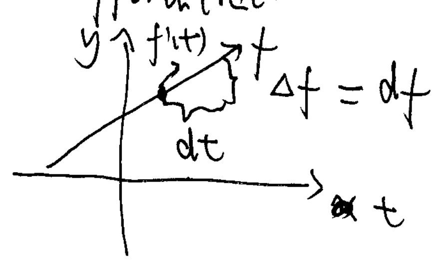
• Estimate the change in f in a direction $\vec{u}$

$\underline{f_x},\ \underline{f_y}$ $= \nabla f \cdot \vec{u} = \underbrace{D_{\vec{u}} f}$

Recall: 1D: $\dfrac{df}{dt} = f'(t)$ $\qquad \underset{\downarrow}{df} = f'(t)\cdot dt$

$df \neq \Delta f$ $\qquad$ differential. $\qquad \neq \Delta f$

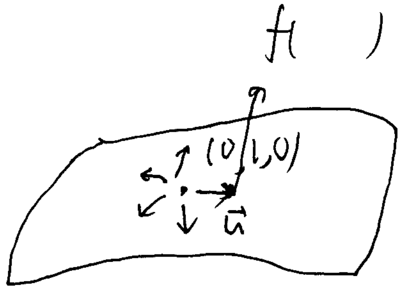




Multi-variable: $df = \underbrace{(\nabla f \cdot \vec{u})}_{D_{\vec{u}} f}\, dt$

eg. $f(x,y,z) = y\sin x + 2yz$.

How change if move from $(0,1,0)$ by 0.1 unit along $\vec{u}\,(2,1,-2)$



$$\nabla f\Big|_{(0,1,0)} = (y\cos x,\ \sin x + 2z,\ 2y)\Big|_{(0,1,0)}$$

$$= (1,\ 0,\ 2)$$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{2\vec{i} + \vec{j} - 2\vec{k}}{\sqrt{2^2+1^2+2^2}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$$

$$D_{\vec{u}}f = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|} = (1,0,2)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}$$

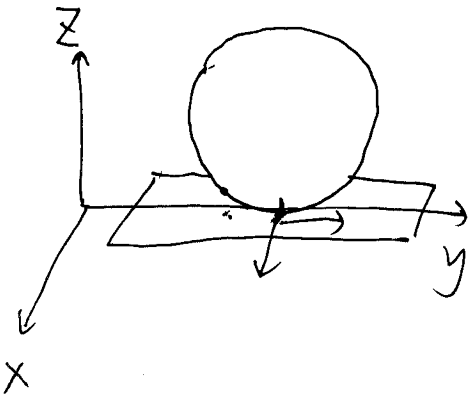
$$\underline{df} = D_{\vec{u}}f \cdot dt = -\frac{2}{3}\cdot 0.1 = -\frac{2}{30} = \underline{-\frac{1}{15}}$$

$$\Delta f = f(0+0.1\cdot 2,\ 1+1\cdot 0.1,\ 0-2\cdot 0.1) - f(0,1,0)$$

$$= f(0.2,\ 1.1,\ -0.2) - f(0,1,0)$$

- Linearization of $f(x,y)$ at (x_0, y_0)

$$\underline{L(x,y)} = f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$$



eg: $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at $(2,3)$

$$f_x = 2x - y\,\Big|_{(2,3)} = 1$$

$$f_y = y - x\,\Big|_{(2,3)} = 1$$

$$f(2,3) = 4 - 6 + \frac{1}{2}9 + 3 = 5.5$$

$$\underline{L(x,y) = 5.5 + (x-2) + (y-3)}$$

1D. Recall Taylor's Series

$$f(x) \sim \cancel{f(x)}\ \underline{f(x_0) + f'(x_0)(x-x_0)} + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$f(x_0) + f'(x_0)(x-x_0) \rightarrow \text{Linearization of } f.$$

- Error Estimation:

If f has continuous first/ second partial derivatives in an open set containing a rectangle R centered at (x_0, y_0), $|f_{xx}| \leq M$, $|f_{yy}| \leq M$, $|f_{xy}| \leq M$.

If M is the upper bound for $|f_{xx}|$, $|f_{xy}|$ and $|f_{yy}|$

then $|E(x,y)| = |f(x,y) - L(x,y)| \leq \frac{1}{2} M[\underbrace{(x-x_0)^2 + (y-y_0)^2}_{= D^2}]$

$$\text{Distance} = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

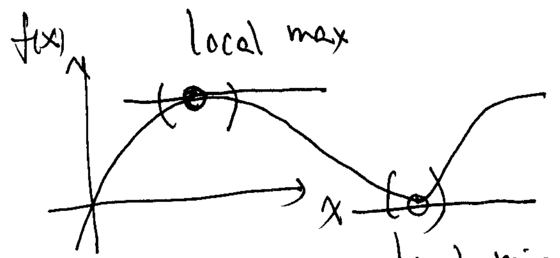
- Remark: Compare Error Estimation in Linearization and Taylor's Thm.

Definition: $df = f_x(x_0, y_0) \cdot dx + f_y(x_0, y_0) \cdot dy$ in the linearization of f is called the total differential of f.

$$L(x,y) = f(x_0, y_0) + \underbrace{f_x(x_0, y_0) \cdot dx + f_y(x_0, y_0)\, dy}_{df}$$

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \Delta f$$

14.7. Extreme Values and Saddle points



$f'(x) = 0$

local max / min:

We say $f(x_0, y_0)$ is a local (relative) max / min

if an a open disk centered at (x_0, y_0)

$\forall\ (x,y)$ is in the open disk, $f(x,y) \leq f(x_0, y_0) \rightarrow$ max

$f(x,y) \geq f(x_0, y_0) \rightarrow$ min.

First Derivative Test: If $f(x,y)$ has a local maxium / minium at (a,b) iff its first partial derivative exist there, and $f_x(a,b) = 0$, $f_y(a,b) = 0$

Def: An ~~the~~ interior point of the domain is called as the critical point if $f_x = 0$ and $f_y = 0$ or either f_x or f_y does not exist.

Second Derivative Test: Suppose $f(x,y)$ and its first / second partial derivatives are continuous at the disk centered at (a,b) where $f_x(a,b) = f_y(a,b) = 0$

Then: (1) $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,b) \rightarrow$ local max

(2) $f_{xx} > 0$, and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,b) \rightarrow$ local min

(3) $f_{xx} f_{yy} - f_{xy}^2 < 0 \rightarrow$ saddle point

(4) $f_{xx} f_{yy} - f_{xy}^2 = 0 \rightarrow$ inconclusive.

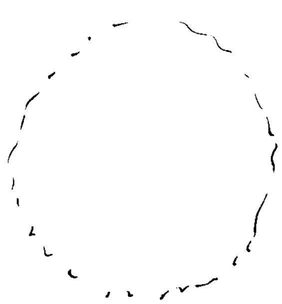
$$\begin{vmatrix} \boxed{f_{xx}}^{>0} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0 \rightarrow \text{local min.}$$

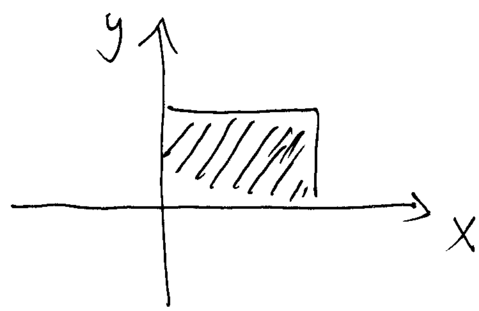
$$\begin{vmatrix} <0 & \\ & \end{vmatrix} > 0 \rightarrow \text{local max}$$

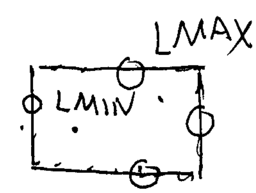
$$\begin{vmatrix} & \\ & \end{vmatrix} < 0 \rightarrow \text{saddle point}$$

$$\begin{vmatrix} & \\ & \end{vmatrix} = 0 \rightarrow \text{inconclusive.}$$

Absolute Max/Min. → bounded domain.





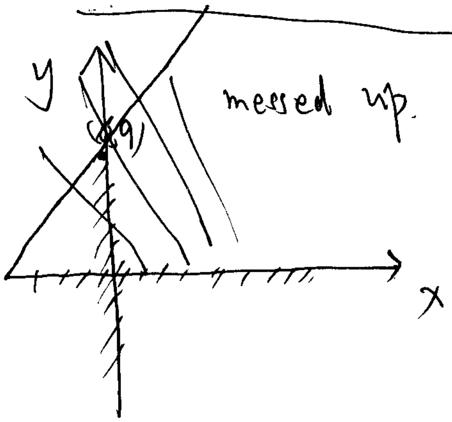


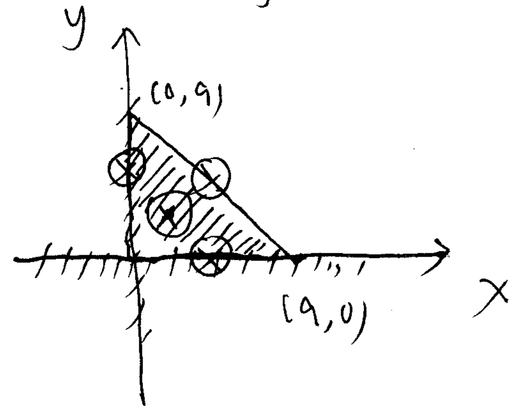
Step 1. List the interior points of R where f may have local max/min.
↳ critical points. — [Apply First Derivative Test / Apply Second Derivative Test]

Step 2 List the boundary points of R where f may be local max/min.

Step 3. Compare these points to find the greatest/smallest.

eg. $f(x,y) = 2+2x+2y-x^2-y^2$, bounded by $x=0$, $y=0$, $y=9-x$.





① $f_x = 2-2x = 0$

$f_y = 2-2y = 0$ $\Rightarrow \begin{cases} x=1 \\ y=1 \end{cases}$ $(1,1)$

$\begin{cases} xy - 2y^2 = 0 \\ xy - 2x^2 = 0 \end{cases}$ $\quad xy = 2y^2$

if $y=0$ $\quad 2y^2=0 \quad xy=0$

$0 = 2x^2 \Rightarrow x=0 \quad (0,0)$

$\downarrow$

$xy = 2x^2 \quad$ if $y \neq 0 \quad x = 2y$

$y = \frac{x}{2}$

$x \cdot \frac{x}{2} = 2x^2 \qquad \frac{1}{2}x^2 = 2x^2$

② $x=0 \quad f(0,y) = 2+2y-y^2 = -(y^2-2y+1)+3$

$0 \le y \le 9 \qquad = -(y-1)^2+3 \qquad \boxed{(0,1)}$

② $y=0$ $\qquad \boxed{}$

$y=9-x \quad f(x,9-x) = 2+2x+2(9-x)-x^2-(9-x)^2 \qquad \boxed{}$

③