Problem 1.

Proof. **NOTE:** From the proof to Theorem 6.24 (Bessel's inequality) for the first conclusion, we in fact do not require that the space is Hilbert. It works for just an inner product space with an orthonormal set. If we denote $\|\cdot\|$ the norm induced by the inner product, thus we have

$$\sum_{\alpha \in \mathcal{A}} |\langle e_\alpha, x\rangle|^2 \leqslant \|x\|^2 < \infty \qquad \text{and} \qquad \sum_{\alpha \in A} |\langle e_\alpha, y\rangle|^2 \leqslant \|y\|^2 < \infty.$$

Recall any uncountable sum of positive real numbers always diverges, thus though $\mathcal{A}$ may be uncountable, there are at most countable elements in $\mathcal{A}$ such that $|\langle e_\alpha, x\rangle| \neq 0$. Without loss of generality, we use the index set I of countable many elements to replace the index set $\mathcal{A}$ in the following.

Let $x_i = \langle e_i, x\rangle$ and $y_i = \langle e_i, y\rangle$. Then $(x_i), (y_i) \in \ell^2$. With the inner product $\langle\langle \cdot, \cdot\rangle\rangle$ of ℓ^2, by Cauchy-Schwarz inequality, we have

$$|\langle\langle (x_i), (y_i)\rangle\rangle| \leqslant \|(x_i)\|_2 \|(y_i)\|_2.$$

Thus

$$\sum_{i=1}^{\infty} |\langle e_i, x\rangle \langle e_i, y\rangle| = \sum_{i=1}^{\infty} |x_i y_i| = |\langle\langle (x_i), (y_i)\rangle\rangle| \leqslant \|(x_i)\|_2 \|(y_i)\|_2 \leqslant \|x\|\|y\|.$$

□

Problem 2. Every orthonormal set of a separable Hilbert space is countable.

Proof. Assume there is an uncountable orthonormal set M in $\mathcal{H}$, and pick any two different elements $x, y \in M$, note $\langle x, y\rangle = 0$ then

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle} = \sqrt{\langle x, x\rangle + \langle y, y\rangle} = \sqrt{2}.$$

Therefore balls $B(x, \frac{\sqrt{2}}{3})$ and $B(y, \frac{\sqrt{2}}{3})$ are disjoint. For the uncountable orthonormal set M, there are uncountable balls in $\mathcal{H}$ of radius $\frac{\sqrt{2}}{3}$ that are pairwise disjoint. This will contradict with the assumption that there exists a countable dense set of $\mathcal{H}$. □

Problem 3.

Proof.

- Suppose P is an orthogonal projection, it is trivial to show that $(I - P)^2 = I - P$ by the fact $P^2 = P$. It remains to show that $\langle x, (I - P)y\rangle = \langle (I - P)x, y\rangle$ holds for any $x, y \in \mathcal{H}$.

$$\begin{aligned}
\langle x, (I - P)y\rangle &= \langle x, y - Py\rangle \\
&= \langle x, y\rangle - \langle x, Py\rangle \\
&= \langle x, y\rangle - \langle Px, y\rangle \\
&= \langle x - Px, y\rangle \\
&= \langle (I - P)x, y\rangle.
\end{aligned}$$

- We first show that P is bounded linear operator.

$$\|Px\| = \frac{\langle Px, Px\rangle}{\|Px\|} = \frac{\langle x, P^2x\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|} \leqslant \|x\|,$$

  which yields that $\|P\| \leqslant 1$. Based on the same reason, since $I - P$ is an orthogonal projection, $\|I-P\| \leqslant 1$.

  By Theorem 5.25, $\mathrm{Ker}(P)$ is closed. Based on the same reason, $\mathrm{Ker}(I-P)$ is closed, as well.

  Once we show $\mathrm{Ran}(P) = \mathrm{Ker}(I-P)$, we complete the proof.

  For any $x \in \mathrm{Ker}(I-P)$, $(I-P)x = 0$ implies that $x = Px \in \mathrm{Ran}(P)$.

  For any $x \in \mathrm{Ran}(P)$, there exists a $y \in \mathcal{H}$ such that $Py = x$. Then $x = Py = P^2y = P(Py) = Px$. Thus $x \in \mathrm{Ker}(I-P)$.

□

Problem 4. Inner-product completion.

Proof. Let $(X, (\cdot,\cdot))$ denote an inner-product space and metric $d(x,y) = \|x-y\| = \sqrt{(x-y, x-y)}$.

By theorem 1.52 and definition 1.51, there is a completion $(\bar{X}, \bar{d})$ of (X, d) such that $(\bar{X}, \bar{d})$ is complete, X is dense in $\bar{X}$ and $\bar{d}(x,y) = d(x,y)$ for any $x, y \in X$. Now we define an inner product $\langle\cdot,\cdot\rangle$ on $(\bar{X}, \bar{d})$. Once we show $\langle z, w\rangle$ is an inner product for any $z, w \in \bar{X}$ and $\langle x, y\rangle = (x,y)$ for any $x, y \in X$, we complete the proof.

Since X is dense in $\bar{X}$, for any $z, w \in \bar{X}$, there exists $(x_n), (y_n)$ in X such that $x_n \to z$ and $y_n \to w$. Now we define

$$\langle z, w\rangle := \lim_{n\to\infty} (x_n, y_n).$$

First, it is well-defined since $(\cdot,\cdot)$ is continuous function that maps Cauchy sequence to Cauchy sequence, the limit exists.

Second, we show that it does not depend on the choice of Cauchy sequence. Withous loss of generality, we fix (y_n) and assume there are $(x_n), (x'_n)$ that converge to z, then

$$\lim_{n\to\infty} |(x_n, y_n) - (x'_n, y_n)| = \lim_{n\to\infty} |(x_n - x'_n, y_n)| \leqslant \lim_{n\to\infty} \|x_n - x'_n\|\|y_n\| \to 0.$$

Third, we show that $\langle z, w\rangle$ is an inner product. All the properties of inner product follow from the facts that $(\cdot,\cdot)$ is an inner product and $(\cdot,\cdot)$ is continuous which allows you switch the order of limit with $(\cdot,\cdot)$.

Last, $\langle x, y\rangle = (x,y)$ for any $x, y \in X$ holds trivially for our construction.

□

Problem 5.

Proof. First $T_n(x) = \cos(n\theta) \in \mathcal{H}$ for

$$\int_{-1}^{1} \frac{|T_n(x)|^2}{\sqrt{1-x^2}}\mathrm{d}x = \int_{\pi}^{0} \frac{\cos^2(n\theta)}{\sqrt{1-\cos^2(\theta)}}(-\sin(\theta))\mathrm{d}\theta = \int_0^{\pi} \cos^2(n\theta)\mathrm{d}\theta = \begin{cases} \frac{\pi}{2} & n \geqslant 1; \\ \pi & n = 0. \end{cases}$$

Thus $\|T_0\| = \sqrt{\pi}$ and $\|T_n\| = \sqrt{\frac{\pi}{2}}$ for $n \geqslant 1$.

Second, we show $(T_n, T_m) = 0$ for $m \neq n$.

$$(T_n, T_m) = \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\mathrm{d}x = \int_{\pi}^{0} \frac{\cos(n\theta)\cos(m\theta)}{\sqrt{1-\cos^2(\theta)}}(-\sin(\theta))\mathrm{d}\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\mathrm{d}\theta = 0.$$

$\square$