Differentiablity

Thm (The Increment Theorem)

Suppose the first-order derivatives of $f(x,y)$ are defined on open region R. containing (x_0, y_0), and f_x, f_y are continuous on (x_0, y_0)

Then $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

satisfying $\Delta z = f_x(x_0, y_0)\cdot\Delta x + f_y(x_0, y_0)\cdot\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

in which $\varepsilon_1, \varepsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$

Def A function is ~~and~~ differentiable at (x_0, y_0)

if ① $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and

② $\Delta z = \underbrace{[f_x(x_0, y_0)\cdot\Delta x + f_y(x_0, y_0)\cdot\Delta y]}_{dz} + \boxed{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$

where $\varepsilon_1, \varepsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$

☆. Corollary: If f_x and f_y are continuous throughout the whole region, then $f(x,y)$ is differentiable everywhere.

Remark: 1. Difference between single-variable fun and multi-variable fun.

SVF: the existence of derivative $\Leftrightarrow$ $f(x)$ is differentiable

MVF: the existence of derivative (partial) ⊕ partial derivative is continuous $\Rightarrow$ $f(x,y)$ is differentiable.

existence of partial derivative $\Leftarrow$ $f(x,y)$ is differentiable

eg $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$ $\quad f_x = \dfrac{y(x^2+y^2) - xy\cdot 2x}{(x^2+y^2)^2} = 0$

$\to$ not differentiable.

$\lim (\Delta z - dz) = [f(0+\Delta x, 0+\Delta y) - f(0,0)] - [f_x(0,0)\cdot\Delta x + f_y(0,0)\cdot\Delta y]$

$= (\frac{\Delta x\cdot\Delta y}{\ldots}$ [illegible] $\Delta x \cdot \Delta y$ [illegible] eg. $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

prove: f_x, f_y exist but not continuous at $(0,0)$

BUT $f(x,y)$ is differentiable at $(0,0)$

HINT: By Definition.

Summary: limit $\leq$ continuous $\leq$ differentiable

If $f(x,y)$ is differentiable, then $f(x,y)$ is continuous.

Sec. 14.4. The Chain Rules.

$y = f(g(x))$ $\quad y' = f'\big|_{g(x)} \cdot g'(x)$

- $w = f(x,y)$ $\quad x = x(t),\ y = y(t)$

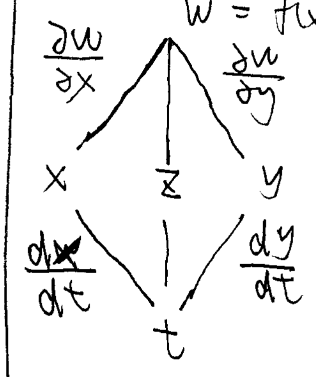
  w is MVF on x, y

  w is SVF on t.

$$\underset{\downarrow\, t_0}{\frac{dw}{dt}} = \underset{\downarrow\, P_0}{\frac{\partial f}{\partial x}} \cdot \underset{\downarrow\, t_0}{\frac{dx}{dt}} + \underset{\downarrow\, P_0}{\frac{\partial f}{\partial y}} \cdot \underset{\downarrow\, t_0}{\frac{dy}{dt}}.$$ chain rules.

If $P_0 = (x_0, y_0)$ $\quad x_0 = x(t_0)$ $\quad y_0 = y(t_0)$

$$\left(\frac{dw}{dt}\right)_{t_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\left(\frac{dy}{dt}\right)_{t_0}$$



$W = f(x,y) \rightarrow$ Dependent Variables.

$\rightarrow$ Intermediate Variables

$\rightarrow$ Independent variables

} branch diagram.

eg: $w = xy \qquad x = \sin t \qquad y = \cos t \qquad \frac{dw}{dt} = ?$

$$\frac{dw}{dt}(t) = \boxed{\frac{dw}{dt}} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$= y\cdot\cos t + x(-\sin t)$$

$$= \cos^2 t - \sin^2 t = \cos 2t$$

$$w = \sin t\cdot\cos t = \tfrac{1}{2}\sin 2t.$$

$$\frac{dw}{dt} = \cancel{\tfrac{1}{2}}\cos 2t\cdot\cancel{2} = \cos 2t.$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t = \cos 2t$$

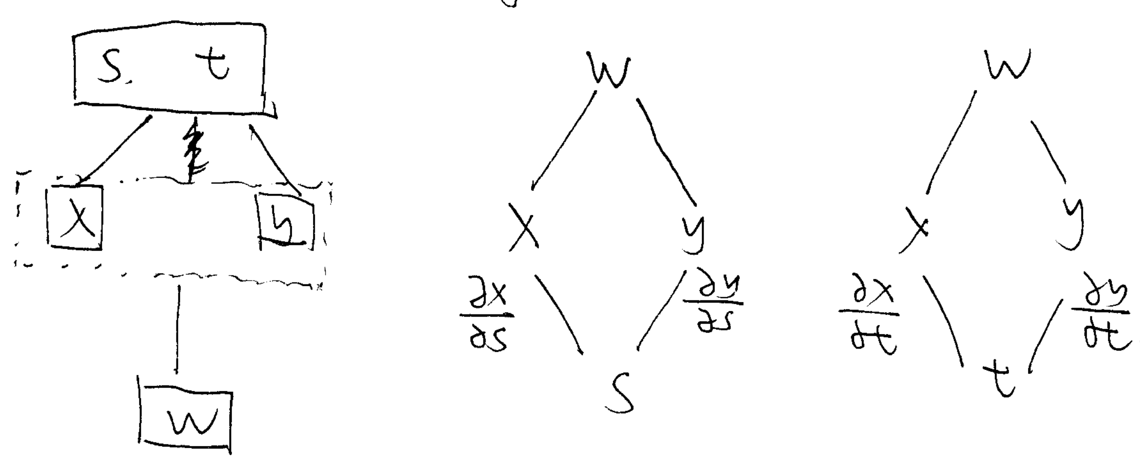
$$2\sin t\cos t = \sin 2t.$$

Thm. $w = f(x,y) \qquad x = g(s,t) \qquad y = h(s,t)$

- $w = f(x,y)$
- w depends on s, t.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}.$$



- Implicit Differentiation.

$F(x,y) = 0 \qquad x^2 + y = 0 \qquad \underline{y = -x^2}$

$$F(x, y(x)) = 0$$

$$0 = \boxed{\frac{dF}{dx}} = \boxed{\frac{\partial F}{\partial x}}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$

$F(x,y)=0 \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$ if $F_y \neq 0$.

eg: $y^2 - x^2 - \sin xy = 0 \qquad \frac{dy}{dx} = ?$

$$2y \cdot \frac{dy}{dx} - 2x - \cos(xy)\cdot\left[y + x\cdot\frac{dy}{dx}\right] = 0$$

$$F(x,y) = y^2 - x^2 - \sin xy$$
$$F_x = -2x - \cos xy \cdot y$$
$$F_y = 2y - \cos xy \cdot x$$
$$\frac{dy}{dx} = -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

Sec. 14.5. Directional Derivatives and Gradient Vectors.

$\vec{u} = u_1\vec{i} + u_2\vec{j} \qquad \vec{u}$ is a unit vector. $|\vec{u}|=1$.

$$\left(\frac{df}{ds}\right)_{\vec{u}_0 P_0} = \lim_{s\to 0}\frac{f(x_0+u_1 s,\ y_0+u_2 s) - f(x_0,y_0)}{s}$$
$$= (D_{\vec{u}} f)_{P_0}$$

$$\left(\frac{df}{ds}\right)_{\vec{u}_0 P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\cdot\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\cdot\frac{dy}{ds}$$
$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0}\cdot u_2$$
$$= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\vec{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\vec{j}\right]\cdot\underbrace{\left[u_1\vec{i} + u_2\vec{j}\right]}_{\vec{u}}$$

gradient $\quad \nabla f = f_x\vec{i} + f_y\vec{j}$

$$\left(\frac{df}{ds}\right)_{\vec{u}_0 P_0} = (D_{\vec{u}} f)_{P_0} = \nabla f\cdot\vec{u} = |\nabla f|\cdot|\vec{u}|\cdot\cos\theta$$

Remark: 1. ∇f is a vector !!!
2. $(D_{\vec{u}} f)_{P_0}$ is a scalar.
3. $\vec{u}$ is a unit vector.
4. when $\vec{u}$ is in the direction of ∇f, $f(x,y)$ increase most

eg. $f(x,y) = xe^y + \cos(xy)$ at $(2,0)$ $\vec{u} = 3\vec{i} - 4\vec{j}$

step 1. normalize $\vec{u}$ $\vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$

step 2. $f_x = e^y - \sin(xy) \cdot y$

$f_y = xe^y - \sin(xy) \cdot x$

step 3. $\nabla f(2,0) = f_x(2,0)\vec{i} + f_y(2,0)\vec{j}$

$= \vec{i} + 2\vec{j}$

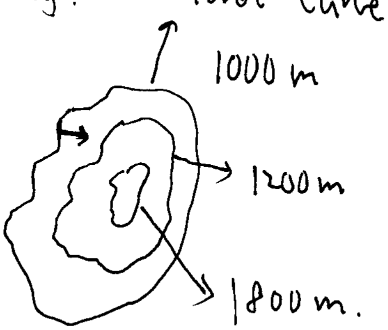
step 4 $D_{\vec{u}} f = \nabla f \cdot \vec{v} = (\vec{i} + 2\vec{j}) \cdot (\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}) = \frac{3}{5} - \frac{8}{5} = -1$

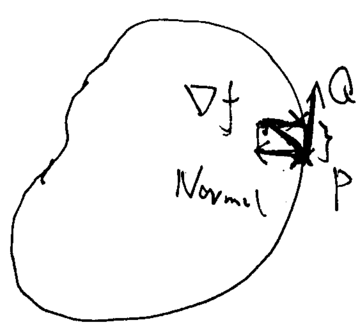
If you choose $\vec{u} = \vec{i} + 2\vec{j}$ / $\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$

$$(D_u f)_{max} = (\vec{i} + 2\vec{j}) \cdot (\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}) = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

$$\| \sqrt{1^2 + 2^2} = \sqrt{5}$$

eg:



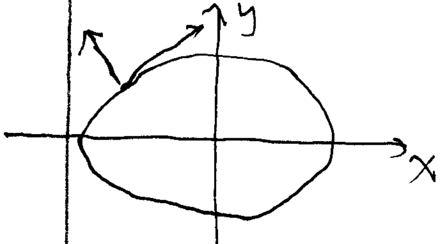


$f(Q) > f(P)$

Contradiction with level curve.

$f(Q) = f(P)$

Thm The gradient of differentiable function $f(x,y)$ is normal to the level curve through (x_0, y_0)



$\frac{x^2}{4} + y^2 = 2$ Find Tangent Vector at $(-2,1)$

$F(x,y) = \frac{x^2}{4} + y^2$

$f_x = \frac{x}{2}$ $\quad$ $f_y = 2y$

$\nabla f(-2,1) = -1\vec{i} + 2\vec{j}$

suppose (x,y) is on tangent vector.

$(x+2, y-1)$ is the direction vector of tangent.

~~[illegible]~~ $(-1, 2) \cdot (x+2, y-1) = 0$

$-(x+2) + 2(y-1) = 0 \rightarrow$ eqn for tangent lines on plane.