Problem 1. Metric space is first countable. Separable metric space is second countable.

Proof.

- We show that $\mathcal{B} = \left\{B_{1/n}(x) : n \in \mathbb{N}\right\}$ is a countable neighborhood base for each x in X.

  For any neighborhood V of x, there exists an open set G such that $x \in G \subset V$.

  By the definition of open set in metric space, there exist a ball $B_\varepsilon(x) \subset G$. Let n be such that $\frac{1}{n} \leqslant \varepsilon$, then $x \in B_{1/n}(x) \subset B_\varepsilon(x) \subset V$, i.e., there exists a $B_{1/n}(x) \in \mathcal{B}$ such that $B_{1/n}(x) \subset V$. So $\mathcal{B}$ is a countable neighborhood base.

- If X is separable, we assume A is a countable dense set of X. Then $\mathcal{B} = \left\{B_{1/n}(x) : x \in A, n \in \mathbb{N}\right\}$. Since the Cartesian product of two countable sets A and B is countable, then $\mathcal{B}$ is countable.

  Let G be an open set in X, for each $x \in G$ there exists a $n = n(x)$ such that $x \in B_{1/n(x)}(x) \subset G$. Furthermore, if $x \in A$, $B_{n(x)}(x) \in \mathcal{B}$. Otherwise, we pick $y \in A$ such that $d(x, y) \leqslant 1/4n(x)$, thus $B_{1/2n(x)}(y) \in \mathcal{B}$ and $x \in B_{1/2n(x)}(y) \subset B_{1/n(x)}(x) \subset G$.

  In summary, for each $x \in A$, we are able to find a $B_x \in \mathcal{B}$ such that $x \in B_x \subset G$. Then $G = \bigcup_x B_x$.

□

Problem 2.

Proof. We first show $\mathcal{T}$ is a topology.

- $\emptyset, X \in \mathcal{T}$;

- Let $\{G_\alpha\}$ be a collection of subsets such that $G_\alpha \in \mathcal{T}$, i.e., $X \backslash G_\alpha$ is countable. Then $X \backslash \bigcup_\alpha G_\alpha$ is countable. Thus $\bigcup_\alpha G_\alpha \in \mathcal{T}$.

- Let $\{G_i\}_{i=1}^n$ be a collection of subsets such that $G_i \in \mathcal{T}$, i.e., $X \backslash G_i$ is countable. $X \backslash \bigcap_{i=1}^n G_i = \bigcup_{i=1}^n (X \backslash G_i)$ and $\bigcup_{i=1}^n (X \backslash G_i)$ is countable. Thus $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

It is not first countable. Assume for each x, there is a countable neighborhood base $\mathcal{N}$. For each neighborhood $W_i \in \mathcal{N}$ of x, there exists an open set G such that $x \in G \subset W$ and $X \backslash G$ is countable, thus $X \backslash W_i$ is countable.

Then $\bigcup_i (X \backslash W_i) = X \backslash \bigcap_i W_i$ is countable. Thus $\bigcap_i W_i$ is an open set with $x \in W_i$. We claim $\cap_i W_i = \{x\}$. Since for any $y \neq x$, $X \backslash \{y\}$ is a neighborhood of x. Then there exist a $W \in \mathcal{N}$ such that $W \subset X \backslash \{y\}$, then $y \notin W$ which implies that $y \notin \bigcap_i W_i$.

$\bigcup_i W_i$ is countable and $X \backslash \bigcup_i W_i$ is countable, thus X is countable, contradiction! □

Problem 3.

Proof. Let $x \in X \backslash K$, then for each $y \in K$, there are disjoint neighborhood W_y of x and V_y of y. We pick open set $U_y \subset V_y$, and $\bigcup_y U_y$ is an open cover of K. Since K is compact, there exists a finite open cover $\bigcup_{i=1}^n U_{y_i}$.

Without loss of generality, we can pick W_{y_i} as open. Let $U = \bigcup_{i=1}^n U_{y_i}$ and $W = \bigcap_{i=1}^n W_{y_i}$. Then U and W are open, $U \cap W = \emptyset$ with $K \subset U$ and $x \in W$. Therefore, $x \in W \subset X \backslash K$.

By the definition, $X \backslash K$ is open and K is closed.

Take $X = \{1, 2\}$ and $\mathcal{T} = \{\emptyset, X\}$. X is not Hausdorff. $K = \{1\}$ is compact but not closed. □

Problem 4. Homeomorphic spaces can have different completeness.

Proof. $(\mathbb{R}, d_e)$ and $((0,1), d_e)$ are homeomorphic where d_e is Euclidean distance. But $\mathbb{R}$ is complete while $(0,1)$ is not complete. □

Problem 5.

Proof.

1. Metric topology is the collection of subsets that are the union of open balls. Let $\mathcal{T}_1$ denote the metric topology induced by d_1 and $\mathcal{T}_2$ denote the metric topology induced by d_2. We just need to show an open ball $B_1(x,r) = \{y \in X : d_1(x,y) < r\}$ is open with respect to $\mathcal{T}_2$ and vice versa.

   We need to show for every $y \in B_1(x_0, r)$, there is a $B_2(y,R) = \{z \in X : d_2(z,y) < R\}$ such that $B_2(y,R) \subset B_1(x,r)$.

   Let $0 < R < c(r - d_1(x,y))$, then for any $z \in B_2(y,R)$,

   $$d_1(z,x) \leqslant d_1(z,y) + d_1(y,x) \leqslant \frac{d_2(z,y)}{c} + d_1(z,y) < \frac{R}{c} + d_1(x,y) = r,$$

   which implies that $z \in B_1(x,r)$.

2. Let's define $d_b(x,y) = \frac{d(x,y)}{1+d(x,y)}$. It is clear that $d_b(x,y) \leqslant 1$. And we need to show $d_b(x,y)$ is a metric.

   It is obviously that $d_b(x,y) \geqslant 0$, $d_b(x,y) = d_b(y,x)$ and $d_b(x,y) = 0$ if and only if $x = y$.

   $$\begin{aligned} d_b(x,z) &= \frac{d(x,z)}{1+d(x,z)} \leqslant \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \leqslant \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \\ &\leqslant d_b(x,y) + d_b(y,z). \end{aligned}$$

   Since $d_b = f(d)$ where $f(x) = \frac{x}{1+x}$ is one-to-one, onto and continuous function. And $f^{-1}(x) = \frac{x}{1-x}$ is also one-to-one, onto and continous function.

   So (X,d) and (X,d_b) are homeomorphic, $\mathcal{T}_d = \mathcal{T}_{d_b}$.

3. The above example is when d and d_b are not equivalent. Since $d_b(x,y) = \frac{d(x,y)}{1+d(x,y)} \leqslant d(x,y)$, however there does not exist a c such that $d_b(x,y) \geqslant cd(x,y)$.

□