10.5. Absolute Convergence / The Ratio Test / The Root Test 08/04

. The ratio test

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$ (a) $\rho < 1 \Rightarrow$ absolute converge $\Rightarrow$ Converge.

(b) $\rho > 1 \Rightarrow$ diverges

(c) $\underbrace{\rho = 1}$ $\Rightarrow$ inconclusive

Proof: $\forall \varepsilon > 0$, $\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon$ when $n > N$.

$$\rho - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon$$

(a) $\rho < 1$. Let ε small enough such that $\rho + \varepsilon < 1$.

$$|a_{n+1}| < (\rho+\varepsilon)^1 \cdot |a_n| < (\rho+\varepsilon)^2 |a_{n-1}|$$
$$< (\rho+\varepsilon)^n \cdot |a_1|$$

$$0 < \underbrace{\sum |a_k|}_{\downarrow \text{ Converges.}} \leq \sum |a_1| \cdot \underbrace{(\rho+\varepsilon)^{k-1}} \sim \sum |a_1| r^{k-1}$$

$r < 1$. $\downarrow$ Converge

eg: based on (c)

$\sum \frac{1}{n}$ $\sum \frac{1}{n^2}$

$\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1$. $\sum \frac{1}{n}$ diverge.

$\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{n^2}{n^2+2n+1}\right| = 1$ $\sum \frac{1}{n^2}$ converges.

eg. $\sum \frac{2^n+5}{3^n}$ $\rightarrow$ absolutely converges / converges.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{3(2^n+5)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+\cancel{\frac{5}{2^{n+1}}}}{\frac{3}{2}+\cancel{\frac{15}{2^{n+1}}}}\right| = \frac{2}{3} < 1.$$

eg. $\sum \frac{(2n)!}{n!\,n!} = \sum \frac{(2n)\cdots(n+1)\cancel{n\cdots 1}}{n!\,\cancel{n!}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2(n+1))!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(2n)!\,(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\overset{2}{\cancel{(2n+2)}}(2n+1)}{\cancel{(n+1)}(n+1)}\right| = \lim_{n\to\infty}\left|\frac{4n+2}{n+1}\right| = 4 > 1$$

diverge.

eg: $a_n = \begin{cases} \frac{n}{2^n} & n \text{ odd} \\ \frac{1}{2^n} & n \text{ even} \end{cases}$ $\quad \sum a_n$

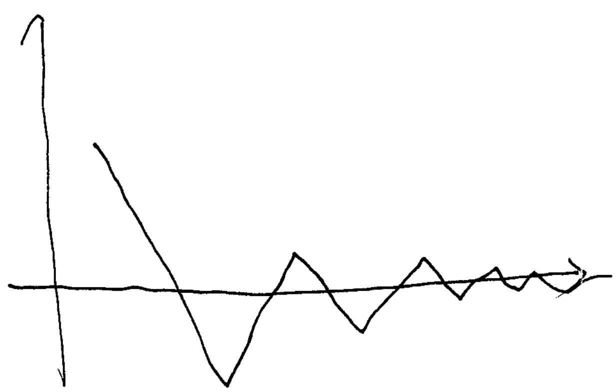
n odd $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \left|\frac{1}{2n}\right| \rightarrow 0$

n even $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{n+1}{2^{n+1}}}{\frac{1}{2^n}}\right| = \left|\frac{n+1}{2}\right| \rightarrow +\infty$

$n = 3$ odd. $\left|\dfrac{a_4}{a_3}\right| = \left|\dfrac{\frac{1}{2^4}}{\frac{3}{2^3}}\right|$

$$a_n = \begin{cases} \dfrac{n}{2^n} & n \text{ odd} \\ \dfrac{1}{2^n} & n \text{ even} \end{cases}$$

$n = 4$ even $\left|\dfrac{a_5}{a_4}\right| = \left|\dfrac{\frac{5}{2^5}}{\frac{1}{2^4}}\right|$



- The Root Test

$\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \rho$

(1) converge $\rho < 1$

(2) diverge $\rho > 1$

(3) inconclusive $\rho = 1$

proof: $\forall \varepsilon > 0$, $\left|\sqrt[n]{|a_n|} - P\right| < \varepsilon$, when $n > N$.

$$P - \varepsilon < \sqrt[n]{|a_n|} < P + \varepsilon$$

(1) $P < 1$, ε small enough such that $P + \varepsilon < 1$.

$$|a_n| < \underbrace{(P+\varepsilon)^n}$$

$\sum |a_n| < \sum (P+\varepsilon)^n \to$ Geometric Series eg. $\sum \frac{2^n}{n^3}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^3}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^3} = 2 > 1 \quad \text{diverge}$$

$$L = \lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\ln L = \ln \sqrt[n]{n} = \frac{\ln n}{n} \qquad \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

$$\ln L = 0 \qquad L = e^{\ln L} = e^0 = 1$$

10.6 Alternating Series and Conditional Convergence.

$$\sum \frac{1}{n} \qquad \sum \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

Alternating Series Test: $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.

if: ① $u_n > 0$ ② $u_n \geq u_{n+1}$ ③ $u_n \to 0$

Remark: 1. In general, $\sum u_n$ converges $\Rightarrow u_n \to 0$

if $u_n \to 0 \not\Rightarrow \sum u_n$ converges.

Proof: $$S_{2m} = \underbrace{(u_1 - u_2)}_{\geq 0} + \underbrace{(u_3 - u_4)}_{\geq 0} + \underbrace{(u_5 - u_6)}_{\geq 0} + \cdots + \underbrace{(u_{2m-1} - u_{2m})}_{\geq 0}$$

$$S_{2m} = u_1 - \underbrace{(u_2 - u_3)}_{\geq 0} - \underbrace{(u_3 - u_4)}_{\geq 0} \cdots - \underbrace{(u_{2m-2} - u_{2m-1})}_{\geq 0} - u_{2m}$$

$$\leq u_1$$

$$\lim_{m\to\infty} S_{2m} = L. \qquad \lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} (S_{2m} + \underbrace{(-1)^{2m+2} u_{m+1}}_{0}) = L.$$

estimate the remainder of Alternating Series:

$$|L - S_n| = \cancel{\left|\sum_{k=n+1}^{\infty} u_k\right| \le |u_{n+1}|} \quad \left|\sum_{k=n+1}^{\infty} (-1)^{k+1} u_k\right| \le u_{n+1}$$

$$S_n - u_{n+1} < L < S_n + u_{n+1}$$

- conditional convergence: series converges but not absolutely converges.

$$\sum \frac{(-1)^{n+1}}{n}$$

$$\sum \frac{(-1)^{n+1}}{n^p} = \begin{cases} p > 1 & \text{absolutely convergence.} \\ p \le 1 & \text{conditional convergence.} \end{cases}$$

- $1 + 2 = 2 + 1$

$\sum a_n \overset{?}{=} \sum b_n$, $\{b_n\}$ is an arrangement of $\{a_n\}$

$$L = \sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}$$

$$2L = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5}$$

suppose we can re-arrange w/o changing sum

$$= 1 - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

$$= L$$

$$2L = L \Rightarrow L = 0$$

But for absolute convergent series, you are free to re-arrange the order and $\sum a_n = \sum b_n$

Summary of Tests.