Operator Norm: X, Y normed linear space

$T: X \to Y$ bdd linear map

$$\|T\| = \inf\{M \mid \|Tx\| \le M\|x\|\}$$

$$= \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \le 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$$

- A linear map is bounded iff it's continuous

Thm: $T: X \to Y$ linear, if X is finite-dim, then T is continuous.

Proof: $\dim(X) = n$

then $\dim(TX) \le n$ $\quad \dim(TX) = k$

$$x = \sum_{i=1}^{n} c_i x_i$$

$$T(x) = \sum_{i=1}^{n} c_i f(x_i) \qquad \|T(x)\| \le \sum_{i=1}^{n} |c_i| \|f(x_i)\| \le CM\|x\| \Rightarrow \text{bounded (continuous)}$$

Let $M = \sup_i \|f(x_i)\|$, $\sum_{i=1}^{n} |c_i| \le C\|x\|$ (any two norms on finite dimension are equivalent)

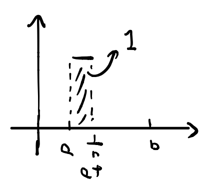
- Compute operator norm can be hard.

$\forall f \in L^1[a,b]$ $\quad T: L^1[a,b] \to L^1[a,b]$ by $(Tf)(x) = \int_a^x f(t)\,dt$.

$$\|Tf\|_{L^1} = \int_a^b \left|\int_a^x f(t)\,dt\right| dx \le \int_a^b \int_a^x |f(t)|\,dt\,dx \le \int_a^b |f(t)|\,dt \cdot (b-a) = (b-a)\|f\|_{L^1}$$

$\|T\| \le b-a$.

$$f_n(x) = \begin{cases} n & [a, \frac{1}{n}+a] \\ 0 & (a+\frac{1}{n}, b] \end{cases}$$



$$Tf_n(x) = \begin{cases} \int_a^x n\,dt & \text{if } x \in [a, a+\frac{1}{n}] \\ \int_a^{a+\frac{1}{n}} n\,dt & \text{if } x \in [a+\frac{1}{n}, b] \end{cases} = \begin{cases} n(x-a) \\ 1 \end{cases}$$

$$\|Tf_n\|_{L^1} = \int_a^{a+\frac{1}{n}} n(x-a)\,dx + \int_{a+\frac{1}{n}}^{b} 1\,dx = \frac{1}{2n} + \left(b-a-\frac{1}{n}\right) = b-a-\frac{1}{2n}$$

$$\|T\| \ge \sup_n \|Tf_n\|_{L^1} = b-a.$$

Let X and Y be normed linear space

$B(X,Y)$ = the space of all continuous / bounded linear mapping from X to Y

- $\|X\|_{X\to Y} = \inf\{M : \|Tx\|_Y \le M\|x\|_X\}$ is a norm of $B(X,Y)$ Normed space.
- $B(X,Y)$ is vector space.
- Let X and Y be normed linear space. Y complete

  Then $B(X,Y)$ is Banach.

  (T_n) Cauchy $\qquad T_nx \to Tx$ convergent sequence is bounded

  $\|T_n x - T_m x\|_Y \le \|T_n - T_m\|_{X\to Y}\|x\|_X$

  $T_n x \to y = Tx \qquad T$ is linear Need T bounded $\|T\| = \sup\limits_{\|x\|=1} \frac{\|Tx\|}{\|x\|} = \sup\limits_{\|x\|=1}\|Tx\|$

  (Cauchy sequence in normed space is bounded)

  + $\|\ \|$ is continuous function

  $T_n \to T \Rightarrow \|T\| \le \liminf \|T_n\|$

  1° Define Limit T $\qquad$ 2° $T \in B(X,Y)$ $\qquad$ 3° $T_n \to T$ w.r.t $\|\ \|$

- Do not depend on X is complete or not.

  $T : A \overset{\text{dense}}{\subset} X \to \underset{\text{Banach}}{Y}$ is continuous linear mapping

  $\exists\ \tilde{T} : X \to Y$ unique extention.

  $Id : B(A,Y) \to B(X,Y)$ is an isometric isomorphism (preserve norm)

  $\qquad T \to \tilde{T}$

- Every incomplete normed linear space X can be isometrically embedded as a dense linear subspace of its completion $\hat{X}$.
- $\hat{X}$ completion contain X as a dense subset.

  $Id : \quad B(X,Y) \to B(\hat{X},Y)$

  $\qquad T \Leftrightarrow \tilde{T}$ bijection / isometric

What's the problem of unbounded linear map?

I) Let $l^2 = \{x \mid \sum_{i=1}^{\infty} (x_i)^2 < \infty\}$ $A = \mathrm{diag}(1, 2, 3, \dots)$

$\|x\|_{l^2} = \left(\sum_{i=1}^{\infty} (x_i)^2\right)^{\frac{1}{2}}$

$(Ax)_k = k x_k$

1° $A: l^2 \to l^2$ is not true $\exists x \in l^2: Ax \notin l^2$ (unbounded operator cannot be defined on the whole Banach space)

subspace

$A: M \to l^2$ $M = \{x \in l^2 \mid \sum_{k=1}^{\infty} k^2 |x_k|^2 < \infty\}$ $\|\ \|_{l^2}$

A is unbounded

$e_n \in M$

$\|e_n\|_{l^2} = 1$ $\|Ae_n\|_{l^2} = n$

2° M is dense in l^2 (at most finitely many nonzero sequence already dense in l^2)

$\forall x \in l^2, \exists y \in l^2$ s.t $\begin{cases} \sum_{k=1}^{\infty} |x_k - y_k|^2 \le \varepsilon \\ \sum_{k=1}^{\infty} k^2 y_k^2 < \infty \end{cases}$ given $\sum_{k=1}^{\infty} x_k^2 < \infty$

$C[0,1]$ D

$D: C^1 \subseteq C[0,1] \to C[0,1]$ unbounded.

$f \to f'$

$f_n = x^n$ $f_n' = n x_n^{n-1}$

$\|f_n\|_{\infty} = 1$ $\|Df_n\|_{\infty} = n$

3° $T: X \to Y$ is closed iff $(x_n) \in D(T)$ with $x_n \to x$ in X and $Tx_n \to y$ in Y then $x \in D(T)$ with $y = Tx$ banach banach

Closed Graph Thm imply that if $T: X \to Y$ is closed and $D(T) = X$, then T is bounded.

Thus for closed, densely defined operators, $D(T) \neq X$ is equivalent with unboundedness

II). $T: X \to Y$ unbounded

$\hookrightarrow D(T)$ s.t $\overline{D(T)} = X$ $D(T)$ is dense in X

no extension like bounded linear mapping.