In this homework, we regard $\mu-$integrable as $\int |f| \mathrm{d}\mu = \int f^+ \mathrm{d}\mu + f^- \mathrm{d}\mu < \infty$.

Problem 1.

Proof. Since f is $\mu-$integrable, then $|f|$ is $\mu-$integrable as well by definition.

Consider set $A_n = \{x \in X : f(x) > n\}$ and define $f_n = |f|\mathbb{1}_{A_n}$. Then $f_n \leqslant |f|$. Moreover, since $|f|$ is $\mu-$integrable, then $f_n \to 0$ almost everywhere as $n \to \infty$. By dominated convergence theorem, we have:

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X \lim_{n\to\infty} f_n \mathrm{d}\mu = 0.$$

That is, for every $\varepsilon > 0$, there exists an $N > 0$ such that $\int_X f_n \mathrm{d}\mu \leqslant \frac{\varepsilon}{2}$ for any $n \geqslant N$. Take $n = N$, $\delta = \frac{\varepsilon}{2N}$ and any measurable set E with $\mu(E) \leqslant \delta$, we have:

$$\begin{aligned} 0 < \int_E |f| \mathrm{d}\mu &= \int_{E\cap A_N} |f| \mathrm{d}\mu + \int_{E\setminus A_N} |f| \mathrm{d}\mu \\ &\leqslant \int_E f_N \mathrm{d}\mu + \int_{E\setminus A_N} N \mathrm{d}\mu \\ &\leqslant \frac{\varepsilon}{2} + N\mu(E) \\ &\leqslant \frac{\varepsilon}{2} + N\frac{\varepsilon}{2N} = \varepsilon. \end{aligned}$$

□

Problem 2. L^1 convergence does not imply pointwise convergence.

Proof. Let's define A_n as following measurable subsets of $[0, 1]$:

$$[0,1], \quad [0,\frac{1}{2}], \quad [\frac{1}{2},1], \quad [0,\frac{1}{3}], \quad [\frac{1}{3},\frac{2}{3}], \quad [\frac{2}{3},1], \quad \cdots$$

Then we have

$$\|\mathbb{1}_{A_n} - 0\|_1 = \mathscr{L}(A_n) \to 0.$$

However, for any $x \in [0, 1]$, x belongs infinitely many sets in $\{A_n\}$, which implies that $\mathbb{1}_{A_n} \not\to 0$ for every $x \in [0, 1]$. □

Problem 3.

Proof.

1. Let $\varepsilon > 0$ and M be such that $|g| \leqslant M$. Since $C_c^0(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, that is, there exists a sequence $\{f_n\}$ in $C_c^0(\mathbb{R})$ such that $\int_{\mathbb{R}} |f_n - f| \mathrm{d}x \to 0$ as $n \to \infty$. Assume for $n = N$, $\int_{\mathbb{R}} |f_N - f| \mathrm{d}x \leqslant \frac{\varepsilon}{3M}$. Furthermore, since f_N is continous and compactly supported, then f_N is uniformly continuous on $\mathrm{supp}(f)$. That is, there exists $\delta > 0$ such that for every $|x - y| \leqslant \delta$, $|f_N(x) - f_N(y)| \leqslant \frac{\varepsilon}{3M\mathscr{L}(\mathrm{supp}(f))}$.

$$
\begin{aligned}
&|(f*g)(x)-(f*g)(y)| \\
&= |(f*g)(x)-(f_N*g)(x)+(f_N*g)(x)-(f_N*g)(y)+(f_N*g)(y)-(f*g)(y)| \\
&\leqslant \underbrace{|(f*g)(x)-(f_N*g)(x)|}_{\mathrm{I}}+\underbrace{|(f_N*g)(x)-(f_N*g)(y)|}_{\mathrm{II}}+\underbrace{|(f_N*g)(y)-(f*g)(y)|}_{\mathrm{III}}
\end{aligned}
$$

$$
\mathrm{I} = |\int_{\mathbb{R}}(f(x-y)-f_N(x-y))g(y)\mathrm{d}y| \leqslant M\int_{\mathbb{R}}|f(x-y)-f_N(x-y)|\mathrm{d}y \leqslant M\frac{\varepsilon}{3M}=\frac{\varepsilon}{3};
$$

Based on the same reason, $\mathrm{III} \leqslant \frac{\varepsilon}{3}$.

$$
\begin{aligned}
\mathrm{II} &= |\int_{\mathbb{R}}(f_N(x-z)-f_N(y-z))g(z)\mathrm{d}z| \leqslant M\int_{\mathbb{R}}|f_N(x-z)-f_N(y-z)|\mathrm{d}z \\
&\leqslant M\frac{\varepsilon}{3M\mathscr{L}(\mathrm{supp}(f))}\mathscr{L}(\mathrm{supp}(f))=\frac{\varepsilon}{3}.
\end{aligned}
$$

Combine the above estimations, we get the result.

2. Recall Problem 1 in Homework 2, if f is uniformly continuous and integrable, then $f(x)\to 0$ as $x\to\infty$. In part 1, we prove $(f*g)(x)$ is uniformly continuous, once we prove $(f*g)(x)$ is integrable, then we complete the result. By Tonelli's theroem,

$$
\begin{aligned}
\int_{\mathbb{R}}|(f*g)(x)|\mathrm{d}x &= \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-y)g(y)|\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-y)g(y)|\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}}|g(y)|\left(\int_{\mathbb{R}}|f(x-y)|\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{\mathbb{R}}|g(y)|\left(\int_{\mathbb{R}}|f(x)|\mathrm{d}x\right)\mathrm{d}y \\
&= \|g\|_1\|f\|_1<\infty.
\end{aligned}
$$

□

Problem 4.

Proof. Since $|f_n|\leqslant g$ and f_n converges to f pointwise, then $|f|\leqslant g$. By definition, f_n and f belong to $L^p(X)$. Furthermore,

$$
|f_n-f|^p \leqslant |g-(-g)|^p=(2g)^p.
$$

Apply the dominated convergence theorem on the sequence $h_n(x) = |f_n - f|^p$ and its upper bound $(2g)^p$, we have

$$\begin{aligned}\lim_{n\to\infty} \|f_n - f\|_p = \lim_{n\to\infty} \left(\int_X |f_n - f|^p \mathrm{d}\mu\right)^{1/p} &= \left(\lim_{n\to\infty} \int_X |f_n - f|^p \mathrm{d}\mu\right)^{1/p} \\ = \left(\int_X \lim_{n\to\infty} |f_n - f|^p \mathrm{d}\mu\right)^{1/p} &= \left(\int_X (\lim_{n\to\infty} |f_n - f|)^p \mathrm{d}\mu\right)^{1/p} = 0.\end{aligned}$$

□

Problem 5.

Proof. Given any L^p function f, we may find a sequence of simple function f_n such that $f_n \to f$ a.e. and $|f_n| \leqslant |f|$. By definition, $f_n \in L^p(X, \mu)$. Apply problem 4 for $g = |f|$, we have $f_n \to f$ in $L^p(X, \mu)$. □