MAT 21C Calculus
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Summer Session 2, 2016
Midterm Exam
08/23/2016
Time Limit: 100 Minutes

Name (PRINT): ______________________

ID: ______________________

This exam contains 12 pages (including this cover page) and 5 questions.
Total of points is 160.
No notes, books, or calculators allowed.
Show all your work unless otherwise indicated. Unsupported answers get partial credits.
Last page is your scratch paper. Raise your hand if you need more.
Circle your answers if possible.

Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	40	
2	20	
3	20	
4	40	
5	40	
Total:	160	

I agree to adhere to the UCD Code of Academic Conduct.

Name (Signed): ______________________

1. (40 points) Questions (a) to (j) are True-or-False or multiple choice problems. For each problem, choose the best ONE option. Please write your answers in the following table. Answers in other places will not be graded.

Table 1: Questions (a)-(j) Solution Table

a	b	c	d	e	f	g	h	i	j
B	B	A	D	A	C	D	D	B	A

(a) (4 points) True or False: If $\sum_{n=1}^{\infty} a_n$ converges then so does $\sum_{n=1}^{\infty} |a_n|$.

A. True

B. False

(b) (4 points) True or False: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} -a_n$ converges conditionally.

A. True

B. False

(c) (4 points) True or False: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

A. True

B. False

(d) (4 points) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} (a_n)^2$

A. converges absolutely

B. converges conditionally

C. diverges

D. is inconclusive

(e) (4 points) If $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$

A. converges absolutely

B. converges conditionally

C. diverges

D. is inconclusive

(f) (4 points) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$. Choose ∞ if the series diverges to infinity.

A. 0
B. $\frac{1}{2}$
C. 1
D. ∞

(g) (4 points) Two planes $x + y - z = 3$ and $x + z = 1$ intersect in a line L. Which one of the following parametric representation describes this line?

A. $x = 2 + t, y = -2t, z = 1 - t$
B. $x = 2 + t, y = 2t, z = -1 - t$
C. $x = 2 - t, y = 2t, z = -1 - t$
D. $x = 2 - t, y = 2t, z = -1 + t$

(h) (4 points) Find an equation of the plane containing the point $(1, -2, 3)$ and parallel to both vectors $\mathbf{u} = \langle 3, 0, 2 \rangle$ and $\mathbf{v} = \langle 0, -1, 1 \rangle$.

A. $2x - 3y - 3z = -13$
B. $2x + 3y - 3z = -13$
C. $2x + 3y - 3z = -1$
D. $2x - 3y - 3z = -1$

(i) (4 points) Compute the distance from the origin $(0, 0, 0)$ to the plane $x + \frac{1}{2}y - z = 3$.

A. 1
B. 2
C. 3
D. 4

(j) (4 points) Choose the Maclaurin series, i.e. Taylor series at $x = 0$ generated by function $f(x) = xe^{2x}$.

A. $\sum_{n=0}^{\infty} \frac{1}{n!} 2^n x^{n+1}$
B. $\sum_{n=1}^{\infty} \frac{1}{n!} 2^n x^{n+1}$
C. $\sum_{n=0}^{\infty} \frac{1}{n!} 1^n x^{2n+1}$
D. $\sum_{n=1}^{\infty} \frac{1}{n!} (-1)^n x^{2n+1}$

2. (20 points) Determine the limit of each sequence. If it diverges, say so. If it diverges to ∞ or $-\infty$, say so. Show all your work.

(a) (5 points) $\lim_{n\to\infty} \frac{\sin 2n}{n}$.

$\because -\frac{1}{n} \leq \frac{\sin 2n}{n} \leq \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By Sandwich Thm $\lim_{n\to\infty} \frac{\sin 2n}{n} = 0$

(b) (5 points) $\lim_{n\to\infty} \left(1+\frac{2}{5n}\right)^n$.

$\because \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$

$a = \frac{2}{5}$

$\therefore \lim_{n\to\infty} \left(1+\frac{2}{5n}\right)^n = e^{\frac{2}{5}}$

(c) (5 points) $\lim_{n\to\infty} n - \sqrt{n^2-n}$.

$= \lim_{n\to\infty} \frac{(n-\sqrt{n^2-n})(n+\sqrt{n^2-n})}{n+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}}$

$= \lim_{n\to\infty} \frac{n}{n+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}} = \frac{1}{2}.$

(d) (5 points) $\lim_{n\to\infty} \sqrt[n]{n!}$.

By ratio test $\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} = (n+1)^{\frac{1}{n+1}} \cdot (n!)^{\frac{n}{n+1}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$

Consider $a_1 = 1$ $a_2 = \sqrt{2}$

$\therefore \lim_{n\to\infty} a_n = \infty.$

3. (20 points) Determine if the series converges. If yes, state whether it converges absolutely or conditionally. If no, state whether it diverges to $\pm\infty$ or just diverges. If you plan to apply any test, show your prior check on those conditions of tests.

(a) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$.

$\because \dfrac{\ln n}{n^3} \geq 0$ for any n

$\therefore$ if converges, then converges absolutely.

$\because \dfrac{\ln n}{n^3} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, $\sum \dfrac{1}{n^2}$ converges

$\therefore$ By Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges.

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges absolutely.

(b) (10 points) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$.

$\because$ ① $\dfrac{1}{\ln n} > 0$ for $n \geq 2$

② $\dfrac{1}{\ln n} > \dfrac{1}{\ln(n+1)}$ for $n \geq 2$

③ $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$

$\therefore$ By Alternating Series Test, $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ converges.

$\displaystyle\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\because \ln n < n \quad \therefore \dfrac{1}{\ln n} > \dfrac{1}{n}$

$\therefore \displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$ which diverges.

$\therefore \displaystyle\sum^{\infty} (-1)^n \frac{1}{\ }$ Converges conditionally

4. (40 points) Now we discover something about $\sin(x)$ function.

   (a) (10 points) Use $\varepsilon - \delta$ to prove $\lim_{x\to 0} \sin x = 0$.

Prove: $\forall \varepsilon > 0$, we pick $\delta = \varepsilon > 0$,

when $|x| < \delta$,

$|\sin x - 0| = |\sin x| < |x| < \delta = \varepsilon$

By definition of limit, $\lim_{x\to 0} \sin x = 0$

   (b) (10 points) Find Taylor series generated by $\sin x$ at $x = 0$. Write the general term.

$f(x) = \sin x \qquad a = 0$

$f'(x) = \cos x \qquad f'(a) = 1$

$f''(x) = -\sin x \qquad f''(a) = 0$

$f^{(3)}(x) = -\cos x \qquad f^{(3)}(a) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(a) = 0$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(c) (10 points) Use the Remainder Estimation Theorem to show the Taylor series you got converges to $\sin x$ for every value of x. (*hint:* Check it satisfies the conditions in the Remainder Estimation Theorem first.)

$f = \sin x$ has derivatives of all orders.

and $|f^{(n+1)}(x)| \leq 1$ for all n.

$\therefore \quad |R_n(x)| \leq 1 \cdot \dfrac{x^{n+1}}{(n+1)!}$

$\because \quad \lim\limits_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0$ for every value of x

$\therefore \quad \lim\limits_{n\to\infty} |R_n(x)| = 0$ for every value of x.

By Reminder Estimation Theorem,

Taylor series converges to $\sin x$ for every x.

i.e., $\quad \sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(d) (10 points) Use Term-by-Term Differentiation Theorem on $\sin x$ to get the Taylor Series generated by $\cos x$. What is the interval of convergence of the Taylor Series generated by $\cos x$ by Term-by-Term Differentiation Theorem? (*hint:* Check it satisfies the conditions in the Term-by-Term Differentiation Theorem first.)

$f(x) = \sin x$ has derivatives of all orders.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$ for every value of x.

then by term-by-term differentiation theorem

$$(\sin x)' = \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n+1}}{(2n+1)!} \right)'$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

for every value of x.

5. (40 points) Finally we talk something magic about π.

   (a) (10 points) Liu Hui, a Chinese mathematician in the 3rd century, first made the best known approximation to π, to an accuracy of seven decimal places. He thought the area of regular polygon can approach to the area of unit circle well as $n \to \infty$.

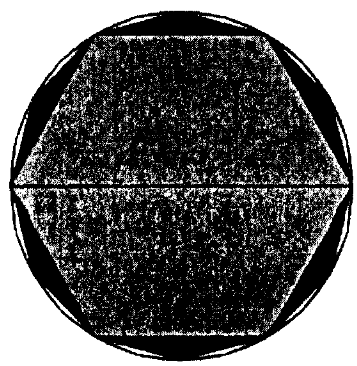

Figure 1: regular polygon with $n = 6, 12, 24$. cited from *wikipedia*

If we know for unit circle,

$$R = 1 = \frac{s}{2\sin\frac{\pi}{n}} = \frac{a}{\cos\frac{\pi}{n}},$$

$$\Rightarrow \quad s = 2\sin\frac{\pi}{n}, \quad a = \cos\frac{\pi}{n}$$

where s and a are corresponding lengths in regular n-sided polygon. You can see the following graph as an example, when $n = 5$, i.e., pentagon.

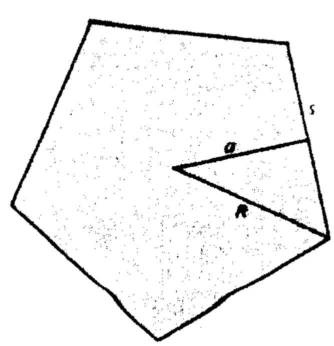

Figure 2: pentagon cited from *wikipedia*

If the area A of regular n-sided polygon is $A = \frac{1}{2}nsa$, compute $\lim_{n\to\infty} A$. Compare the limit you got with the area of unit circle.

$$\lim_{n\to\infty} A = \lim_{n\to\infty} \frac{1}{2}n \cdot 2\sin\frac{\pi}{n} \cdot \cos\frac{\pi}{n} = \lim_{n\to\infty} \frac{1}{2}n \cdot \sin\frac{2\pi}{n}$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{\sin\frac{2\pi}{n}}{\frac{1}{n}} = \frac{1}{2} \cdot 2\pi = \pi.$$

area of unit circle $= \pi \cdot 1^2 = \pi$ the same!

(b) (10 points) Use $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ to get the Taylor Series of $\tan^{-1} x$. You may not check any condition in any theorem you may apply.

$$\because \tan^{-1} x = \int_0^x \frac{1}{1+s^2}\, ds$$

$$= \int_0^x (1 - s^2 + s^4 - s^6 + \cdots)\, ds$$

$$= s - \frac{1}{3}s^3 + \frac{1}{5}s^5 - \frac{1}{7}s^7 + \cdots \Big|_0^x$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

(c) (10 points) Use the Taylor Series of $\tan^{-1} x$ to get a series of π.

plug $x = 1$

$$\tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1}$$

$$\therefore \pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

(d) (10 points) By the Alternating Series Estimation Theorem, $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, if R_n is the nth remainder, then $|R_n| \leq u_{n+1}$. Find the condition for n such that $|R_n|$ in the series of π satisfies $|R_n| \leq 0.1$.

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1} = \sum_{n=1}^{\infty}(-1)^{n+1}u_n$$

$$\therefore \quad u_n = \frac{4}{2n-1}$$

$$\therefore \quad |R_n| \leq u_{n+1} = \frac{4}{2n+1} \leq 0.1$$

$$2n+1 \geq 40$$

$$2n \geq 39$$

$$n \geq \frac{39}{2}$$

$\because$ n is integer

$$\therefore \quad n \geq 20$$

SCRATCH PAGE