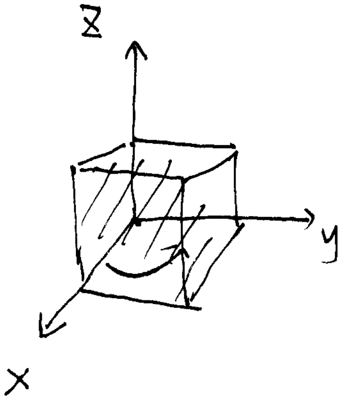
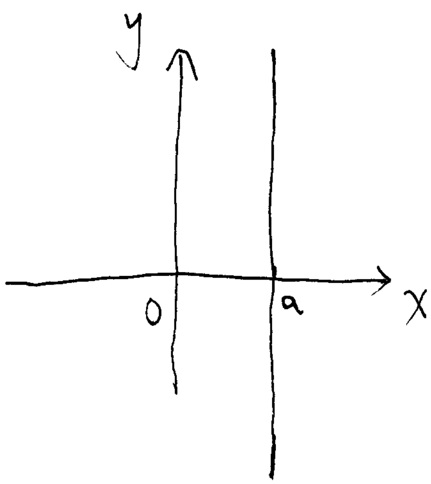
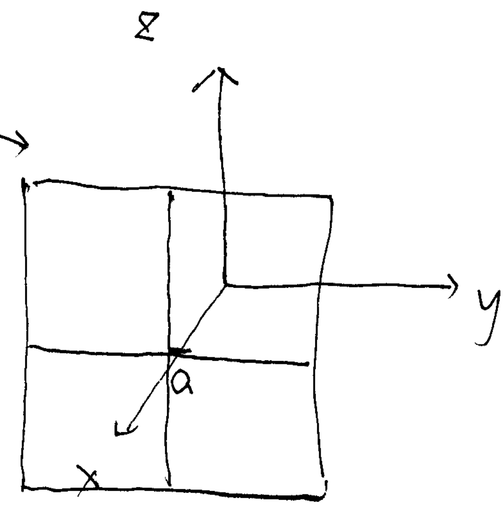
08/16 Chapter 12. Vectors and the Geometry of Spaces.



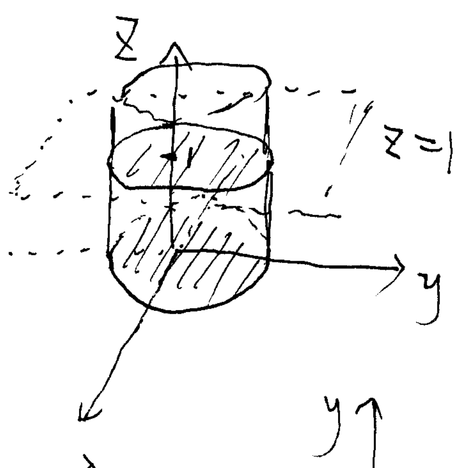
octants.



$x = a$



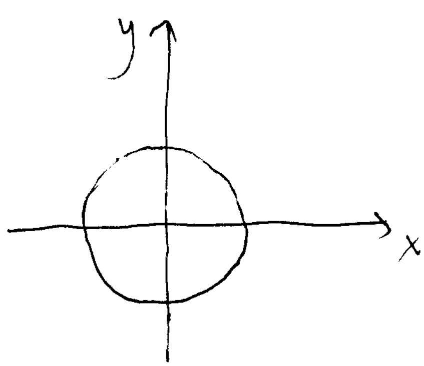
$x^2 + y^2 = 1, \quad z \geq 0$



Step 1. Fix $z = 0$

Step 2. Draw your graph on the plane $z = a$

Step 3. vary z.



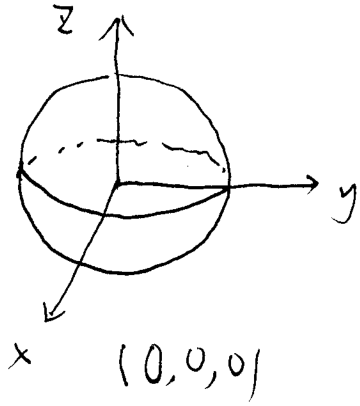
eg. $x^2+y^2+4x+z^2-1=0$ $\qquad x^2+4x+4+y^2+z^2=5$

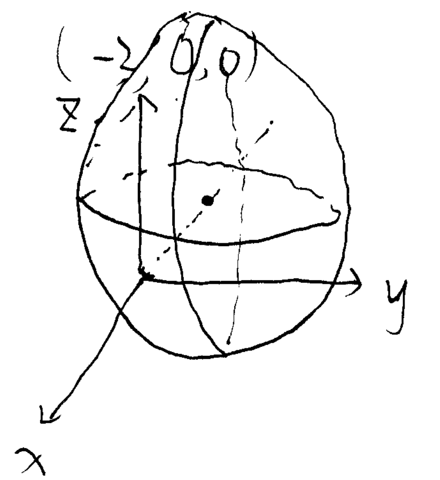
$\updownarrow$ $\qquad \underbrace{(x+2)^2}_{\text{shift}}+y^2+z^2=\underbrace{5}_{\text{radius}}$

$x^2+y^2+z^2=1.$

radius = 1.

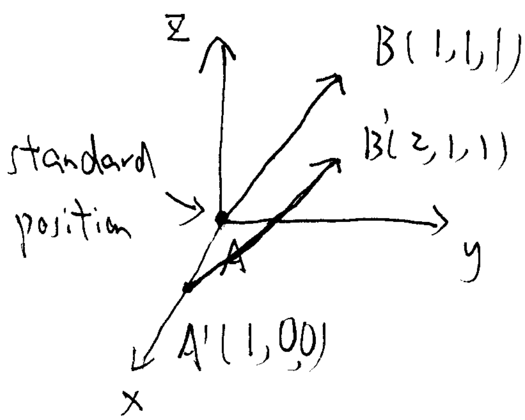
unit sphere





Def: The vector represented by the directed line segment $\overline{AB}$ has initial point A and terminal point B, its length is denoted by $|AB|$

Two vectors are equal if they has the same length and direction.



$\overrightarrow{AB}=(1,1,1)$

$=(1-0,1-0,1-0).$

$\underbrace{|AB|}=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

length is always positive.

$\overrightarrow{A'B'}=(2-1,1-0,1-0)=(1,1,1)=\overrightarrow{AB}$

$\overrightarrow{AC}=(2,2,2)\ //\ \overrightarrow{AB}$ parallel.

In general, $\vec{v} = (v_1, v_2, v_3)$

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$ length / magnitude.

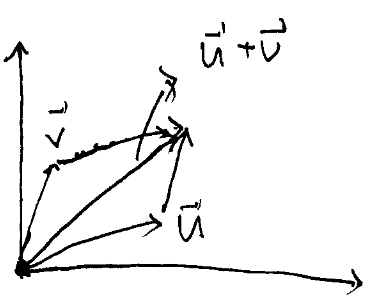
$\dfrac{\vec{v}}{|\vec{v}|}$ → unit vector → which length is always 1.

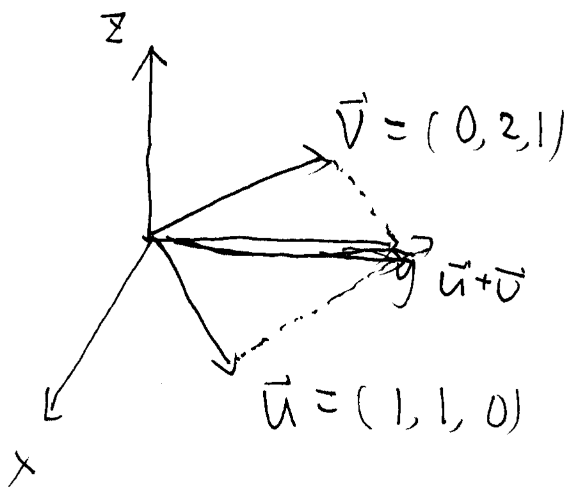
↳ direction

Rules: (1) Addition $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

(2) Scalar Multiplication $k\vec{u} = (ku_1, ku_2, ku_3)$

parallelogram law





$\vec{u} + \vec{v} = (1, 3, 1) = 1\vec{i} + 3\vec{j} + 1 \cdot \vec{k}$

$\vec{i}$ → x-axis unit vector of x-axis

$\vec{j}$ → y-axis

$\vec{k}$ → z-axis

eg: $\vec{A} = 3\vec{i} - 4\vec{j}$, length? direction?

$$|\vec{A}| = \sqrt{3^2 + 4^2 + 0^2} = 5$$

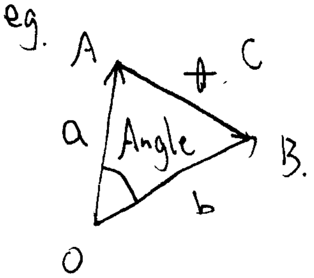
$$\frac{\vec{A}}{|\vec{A}|} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j} = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$ → direction

$$\vec{A} = |\vec{A}| \cdot \frac{\vec{A}}{|\vec{A}|} \quad \text{if } \vec{A} \neq \vec{0} = (0,0,0)$$

eg. Midpoint $A=(a_1, a_2, a_3)$, $B=(b_1, b_2, b_3)$

$\hookrightarrow M\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$

eg.

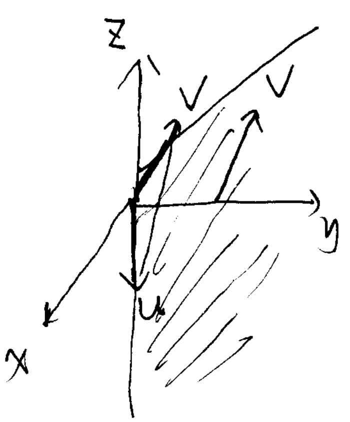


Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cdot\cos\theta.$$

$$\cos\theta = \frac{a^2+b^2-c^2}{2ab} = \frac{|\vec{OA}|^2 + |\vec{OB}|^2 - |\overrightarrow{OB}-\overrightarrow{OA}|^2}{2|\vec{OA}|\cdot|\vec{OB}|}$$

$$\vec{OA} + \vec{AB} = \vec{OB}$$



$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$\vec{OA} = (a_1, a_2, a_3)$$

$$\vec{OB} = (b_1, b_2, b_3)$$

$$\vec{AB} = (b_1-a_1, b_2-a_2, b_3-a_3)$$

$$|\vec{OA}|^2 = a_1^2 + a_2^2 + a_3^2$$

$$|\vec{OB}|^2 = b_1^2 + b_2^2 + b_3^2$$

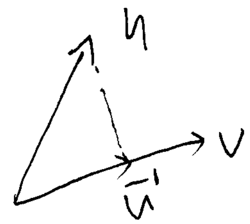
$$|\vec{AB}|^2 = (b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2$$

$$= (a_1^2 - 2a_1b_1 + b_1^2) + (a_2^2 - 2a_2b_2 + b_2^2) + (a_3^2 - 2a_3b_3 + b_3^2)$$

$$\cos\theta = \frac{\cancel{2}(a_1b_1 + a_2b_2 + a_3b_3)}{\cancel{2}|\vec{OA}|\cdot|\vec{OB}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{OA}|\cdot|\vec{OB}|}$$

dot product $\vec{OA}\cdot\vec{OB} = a_1b_1 + a_2b_2 + a_3b_3$

Remark 1. dot product of two vectors is a scalar



$$|\vec{u}'| = |\vec{u}\cdot\cos\theta|$$

$$\frac{\vec{v}}{|\vec{v}|}$$

$$\text{proj}_{\vec{v}}\,\vec{u} = |\vec{u}\cdot\cos\theta|\cdot\frac{\vec{v}}{|\vec{v}|}$$

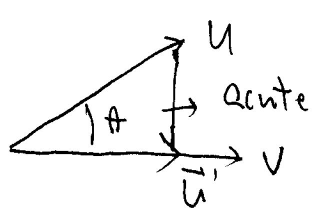
$$= \left|\cancel{\vec{u}}\cdot\left(\frac{\vec{u}\cdot\vec{v}}{\cancel{|\vec{u}|}\cdot|\vec{v}|}\right)\right|\cdot\frac{\vec{v}}{|\vec{v}|}$$

$$= \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)}_{\text{scalar}}\cdot\underbrace{\vec{v}}_{\text{vector}}$$

the scalar component of $\vec{u}$ in the direction of $\vec{v}$

$|\vec{U}|\underbrace{\cos\theta}$ $\qquad \cos(180^\circ) = \cos(\pi) = -1$

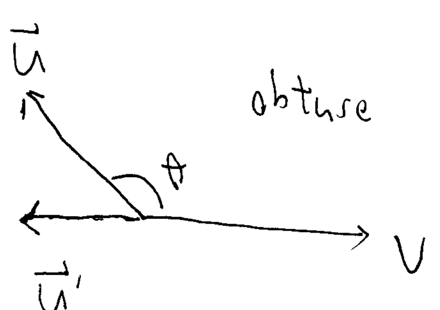
can be positive / negative / zero.



→ Scalar component +

direction same

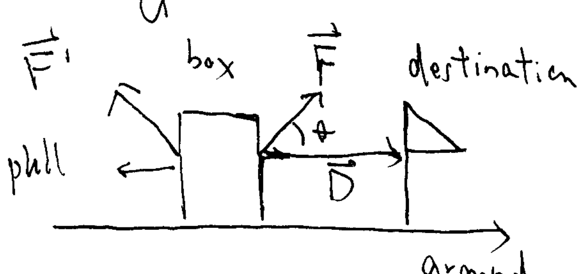
length is same as scalar.



→ scalar component −

direction opposite

length is $(-1)\cdot$ scalar component



$\vec{D}$ ⇒ displacement

$W = \vec{F}\cdot\vec{D} \quad > 0$

$$\cos\theta = \frac{\vec{OA}\cdot\vec{OB}}{|\vec{OA}|\cdot|\vec{OB}|} = \frac{\langle \vec{OA}, \vec{OB}\rangle}{|\vec{OA}|\cdot|\vec{OB}|}$$

$$\theta = \arccos\left(\frac{\vec{OA}\cdot\vec{OB}}{|\vec{OA}|\cdot|\vec{OB}|}\right) \Rightarrow \text{angle.}$$

eg: $\vec{u} = i - 2j - 2k$ $\quad \vec{v} = 6i + 3j - 2k.$

$$\theta = \arccos\left(\frac{6-6+4}{\sqrt{1^2+2^2+2^2}\cdot\sqrt{6^2+3^2+2^2}}\right)$$

$$= \arccos\left(\frac{4}{\sqrt{9}\cdot\sqrt{49}}\right) = \arccos\left(\frac{4}{3\cdot 7}\right)$$

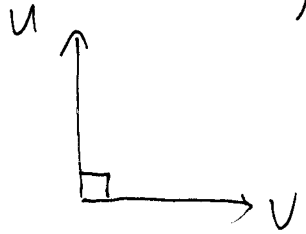
$$= \arccos\left(\frac{4}{21}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right) = \cancel{\frac{1}{\cos\frac{4}{21}}}$$

eg: $u = (3, -2, 0)$ $\quad \vec{v} = (4, 6, 1)$

$$\theta = \arccos\left(\frac{12-12+0}{}\right) = \arccos(0) = 90^\circ$$

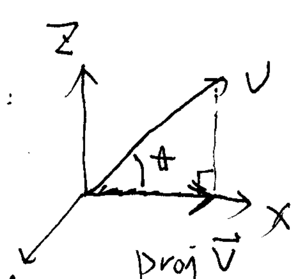
$$\cos 0 = 1 \qquad \cos 90^\circ = 0$$



perpendicular/orthogonal.

$\vec{u}\cdot\vec{v} = 0 \Leftrightarrow \vec{u}$ and $\vec{v}$ are perpendicular.

projection:



$$\text{proj}_x \vec{v} = \vec{v}\cdot\cos\theta$$

$$= \vec{v}\cdot\frac{\vec{v}\cdot\vec{i}}{|\vec{v}|\cdot|\vec{i}|} = \frac{\vec{v}\cdot\vec{i}}{|\vec{v}|}\ \vec{v}$$