Problem 1.

Proof. Since $R_\lambda^{-1} = \lambda I - A$ and $R_\mu^{-1} = \mu I - A$, thus:

$$R_\lambda - R_\mu = R_\lambda(\mu I - A)R_\mu - R_\lambda(\lambda I - A)R_\mu = R_\lambda(\mu I - \lambda I)R_\mu = R_\lambda(\mu - \lambda)R_\mu = (\mu - \lambda)R_\lambda R_\mu.$$

Note: As a result, $R_\lambda R_\mu = R_\mu R_\lambda$. □

Problem 2.

Proof.

- $P = 0$: the only spectrum such that $P - \lambda I = -\lambda I$ not invertible is zero.
- $P = I$: the only spectrum such that $P - \lambda I = (1 - \lambda)I$ not invertible is one.
- Otherwise, let P denote an orthogonal projection. Then P is not invertible. This is due to the fact $P^2 = P$. Indeed, $P(P - I) = 0$ and if P is invertible, then $P = I$. Now we show that for any $\lambda \notin \{0, 1\}$, $P - \lambda I$ is invertible.

$$\begin{aligned}
-\frac{1}{\lambda}\left(I + \frac{1}{\lambda - 1}P\right)(P - \lambda I) &= -\frac{1}{\lambda}\left(P - \lambda I + \frac{1}{\lambda - 1}P^2 - \frac{\lambda}{\lambda - 1}P\right) \\
&= -\frac{1}{\lambda}\left(P - \lambda I + \frac{1}{\lambda - 1}P - \frac{\lambda}{\lambda - 1}P\right) \\
&= -\frac{1}{\lambda}(P - \lambda I - P) \\
&= I,
\end{aligned}$$

  which shows that $\lambda \in \rho(P)$.

  For $\lambda = 0$, $P - \lambda I = P$ is not invertible, then $0 \in \sigma(P)$.

  For $\lambda = 0$, $P - I$ is also a orthogonal projection. As a result, $P - I$ is not invertible as well. Then $1 \in \sigma(P)$.

□

Problem 3.

Proof. Since A is nonnegative,

$$\langle Ax, x\rangle = \overline{\langle Ax, x\rangle} = \langle x, Ax\rangle,$$

which yields that A is self-adjoint. By lemma 9.13, $\sigma(A) \subseteq [-\|A\|, \|A\|]$. So what we need to prove is that there is no negative spectrum.

Let $\lambda \in \sigma(A)$, λ is real and $\lambda < 0$, we plan to show that $A - \lambda I$ is bounded from below. Since A is nonnegative,

$$0 \leqslant \langle Ax, x\rangle = \langle (A - \lambda I)x, x\rangle + \lambda\langle x, x\rangle,$$

which implies that

$$|\lambda|\|x\|^2 = -\lambda\langle x, x\rangle \leqslant \langle (A-\lambda I)x, x\rangle \leqslant \|(A-\lambda I)x\|\|x\|.$$

Apply the Proposition 5.30, we have $A-\lambda I$ is one-to-one and has closed range. As a result, λ is not an eigenvalue since $A-\lambda I$ is one-to-one; λ cannot belong to the continuous spectrum, either. (Otherwise, $A-\lambda I$ is one-to-one and onto.) Furthermore, apply Corollary 9.14, the residua spectrum of a bounded, self-adjoint operator is empty. This leads to a contradiction! □

Problem 4.

Proof.

(a)

$$\begin{aligned}\langle g, Kf\rangle &= \int_0^1 \overline{g(x)}\int_0^x f(y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^1\int_0^x f(y)\overline{g(x)}\mathrm{d}y\mathrm{d}x \\ &= \int_0^1\int_y^1 \overline{g(x)}f(y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \overline{\left(\int_y^1 g(x)\mathrm{d}x\right)}f(y)\mathrm{d}y \\ &= \langle K^*g, f\rangle,\end{aligned}$$

where $K^*g(y) = \int_y^1 g(x)\mathrm{d}x$.

(b) For any $f\in L^2([0,1])$, we have

$$\begin{aligned}\|Kf\|_2^2 = \langle Kf, Kf\rangle &= \int_0^1\left(\int_0^x f(y)\mathrm{d}y\right)^2\mathrm{d}x \\ &= \int_0^1\left(\int_0^x \sqrt{\cos(\frac{\pi y}{2})}\frac{f(y)}{\sqrt{\cos(\frac{\pi y}{2})}}\mathrm{d}y\right)^2\mathrm{d}x \\ &\leqslant \int_0^1\left(\int_0^x \cos(\frac{\pi y}{2})\mathrm{d}y\cdot\int_0^x\frac{f^2(y)}{\cos(\frac{\pi y}{2})}\mathrm{d}y\right)\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^1\sin(\frac{\pi x}{2})\int_0^x\frac{f^2(y)}{\cos(\frac{\pi y}{2})}\mathrm{d}y\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^1\frac{f^2(y)}{\cos(\frac{\pi y}{2})}\int_y^1\sin(\frac{\pi x}{2})\mathrm{d}x\mathrm{d}y \\ &= \frac{2^2}{\pi^2}\int_0^1|f(y)|^2\mathrm{d}y = \frac{2^2}{\pi^2}\|f\|_2^2.\end{aligned}$$

This shows that $\|K\|\leqslant\frac{2}{\pi}$. And the equal sign can be achieved by $f(y)=\cos(\frac{\pi y}{2})$ due to the Cauchy–Schwarz inequality.

(c) First,we prove by mathematical induction that

$$K^n f(x) = \frac{1}{(n-1)!}\int_0^x f(y)(x-y)^{n-1}\mathrm{d}y.$$

For $n=1$, it holds. Assume that it holds for $n=k$, now for $n=k+1$:

$$\begin{aligned} K^{k+1}f(x) &= \int_0^x K^k f(y)\mathrm{d}y \\ &= \int_0^x \frac{1}{(k-1)!}\int_0^y f(z)(y-z)^{k-1}\mathrm{d}z\mathrm{d}y \\ &= \frac{1}{(k-1)!}\int_0^x f(z)\int_z^x (y-z)^{k-1}\mathrm{d}y\mathrm{d}z \\ &= \frac{1}{k!}\int_0^x f(z)(x-z)^{k-1}\mathrm{d}z. \end{aligned}$$

Then, we have:

$$\begin{aligned} \|K^n f\|_2^2 &= \int_0^1 \left(\frac{1}{(n-1)!}\int_0^x f(y)(x-y)^{n-1}\mathrm{d}y\right)^2 \mathrm{d}x \\ &\leqslant \frac{1}{[(n-1)!]^2}\int_0^1 \left(\int_0^x (x-y)^{2n-2}\mathrm{d}y\right)\left(\int_0^x |f(y)|^2\mathrm{d}y\right)\mathrm{d}x \\ &\leqslant \frac{\|f\|_2^2}{[(n-1)!]^2}, \end{aligned}$$

which yields that

$$\left\|K^2\right\|^{\frac{1}{n}} \leqslant \frac{1}{[(n-1)!]^{\frac{2}{n}}} \to 0,$$

as suggested by the Stirling approximation bounds

$$\sqrt{2\pi}n^{n+0.5}e^{-n} \leqslant n! \leqslant en^{n+0.5}e^{-n}.$$

(d) Since the operator has a nonempty spectrum and $r(K)=0$, then $0\in\sigma(K)$.

First, suppose that $0\in\sigma_p(K)$, there exists a nonzero $g\in L^2([0,1])$ such that

$$Kg(x) = \int_0^x g(y)\mathrm{d}y = 0,$$

for every $x\in[0,1]$. Then we conclude with $g(x)=0$ almost everywhere. Thus $0\notin\sigma_p(K)$.

Then, suppose that $0\in\sigma_r(K)$, then by Proposition 9.12, $0\in\sigma_p(K^*)$. By the analogous argument as the first part, we know it is impossible, either.

As a result, $0\in\sigma_c(K)$.

□

Problem 5.

Proof. Suppose it is not true and $\lambda^n \in \rho(A^n)$, then $A^n - \lambda^n I$ is invertible.

$$A^n - \lambda^n I = A^n - (\lambda I)^n = (A - \lambda I)(A^{n-1} + \lambda A^{n-2} + \cdots + \lambda^{n-1}).$$

Suppose K is the inverse of $A^n - \lambda^n I$, then

$$(A - \lambda I)(A^{n-1} + \lambda A^{n-2} + \cdots + \lambda^{n-1})K = I,$$

which yields that $A - \lambda I$ is invertible, contradiction! $\square$