MAT 201B Homework 7
Winter 2020
Professor Qinglan Xia
Due Date: Friday, February 28 at 9:00am

1. Let (x_n) and (y_n) be two sequences in a Hilbert space $\mathcal{H}$. If $x_n \rightharpoonup x_0$ and $y_n \to y_0$ in $\mathcal{H}$, show that
$$\langle x_n, y_n \rangle \to \langle x_0, y_0 \rangle.$$
2. Exercise 8.19 in the textbook "Applied Analysis", page 214.
3. Exercise 8.20 in the textbook "Applied Analysis", page 214.
4. Let $\{e_k\}_{k=1}^{\infty}$ ba an orthonormal set in a Hilbert space $\mathcal{H}$.
   (a) Show that this gives an example of a bounded and closed set which is not compact.
   (b) If $\{c_k\}_{k=1}^{\infty}$ is a sequence of positive real numbers with $\sum (c_k)^2 < \infty$, show that the set
   $$Q = \left\{ \sum_{k=1}^{\infty} a_k e_k \ : \ |a_k| \le c_k \right\}$$
   is compact in $\mathcal{H}$. The set Q is called the Hilbert cube when $c_k = \frac{1}{k}$.
5. Let $\mathcal{H}$ be a Hilbert space.
   (a) Let S be a weakly dense subset of a Hilbert space $\mathcal{H}$. Show that the span of S is $\mathcal{H}$.
   (b) Show that a Hilbert space is (strongly) separable if and only if it is weakly separable.