Reproducing Kernel Hilbert space

Hilbert space of functions.

$(C[0,1], \| \|_2) \xrightarrow{\text{complete}} L^2[0,1]$ Hilbert but not RKHS.

$\|u\|_{H^{-1}} = \max\{\int f(x)\,du : f \in H^1, \|\nabla f\|_{L^2}^2 \le 1\}$ is not well-defined for positive measure

$\downarrow$

dual norm $\overset{\text{exist}}{=} \int k(x,y)\,du(x)\,du(y)$

$\delta_t(f) = \int f(x)\,d\delta_t = f(t)$

1° $f(x)+c$ .. $\max \to +\infty$

2° $\int_{\mathbb{R}^d} f(x)\,du$ is not well-defined for singular measure

$d>1$ unless $f(x)$ is continuous/smooth enough.

Francesco Maggi: Set of finite perimeter and geometric variational problems

Frank Morgan: Geometric measure theory: a beginner's guide

Mariano Giaquinta/ Giuseppe Modica/ Jiří Souček: Cartesian Currents in the Calculus of Variations.

(X, Σ, u) measure spaces, not a space of measures.

$u: \underset{\substack{\uparrow \\ \text{domain}}}{\Sigma} \to [0,+\infty) / \mathbb{R} / \mathbb{R}^n$

① $u(\emptyset) = 0$

② σ-additivity: $A_n \in \Sigma$ disjoint

property: $u(A) \le u(B)$ if $A \subset B$ $(\in \Sigma)$

prop: σ-subadditivity + additivity $\Rightarrow$ σ-additivity

$\downarrow$

$A \in \Sigma$, $A_n \in \Sigma$

$A \subset \bigcup_{n=1}^{\infty} A_n \Rightarrow u(A) \le \sum_{n=1}^{\infty} u(A_n)$

disjoint $A_1, A_2 \in \Sigma$

$u(A_1 \cup A_2) = u(A_1) + u(A_2)$

pairwise disjoint $A_n \in \Sigma$

$$u\left(\bigcup_{n=1}^{\infty} A_n\right) \overset{\text{sub}}{\le} \sum_{n=1}^{\infty} u(A_n) \overset{\text{limit}}{=} \lim_{k\to\infty} \sum_{n=1}^{k} u(A_n) \overset{\text{additiv}}{=} \lim_{k\to\infty} u\left(\bigcup_{n=1}^{k} A_n\right) \le u\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$u\left(\bigcup_{n=1}^{\infty} A_n\right) = u\left(\left(\bigcup_{n=1}^{k} A_n\right) \cup \left(\bigcup_{n=k+1}^{\infty} A_n\right)\right) = u\left(\bigcup_{n=1}^{k} A_n\right) + u\left(\bigcup_{n=k+1}^{\infty} A_n\right) \underset{\text{positive.}}{\ge} u\left(\bigcup_{n=1}^{k} A_n\right)$$

Why Σ: Lebesgue measure (outer) $L'(E) = \inf_{\mathcal{F}} \sum_{F \in \mathcal{F}} r(F)$

$\mathcal{F}$: countable covering of interval (not assume open/closed)

r: length of interval.

L' is not σ-additive on power set of $(0,1)$ $2^{(0,1)}$ (Vitali's example)

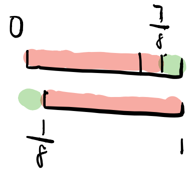
$\simeq$ equivalence relation if $x-y \in$ rational number

$E \subset (0,1)$ containing exactly one element from each equivalence class.

$\{x_n\}_{n\in\mathbb{N}} \in \mathbb{Q} \cap (0,1)$

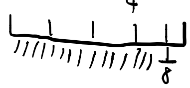
$E_n = \left(x_n + (E \cap (0, 1-x_n))\right) \cup \left((x_n - 1) + (E \cap [1-x_n, 1))\right)$

$\frac{3.14}{4}$ $\frac{\pi}{4} \in E$ $x_n = \frac{1}{8}$



$\frac{1}{8} + \frac{\pi}{4}$

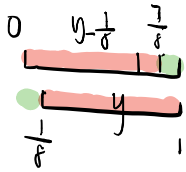
$\frac{\pi}{4}$

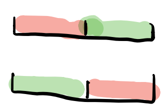


assume $y \in E_n \cap E_m$

$n \leq m$ $n = \frac{1}{8}$ $m = \frac{1}{2}$

E_n





if $y - \frac{1}{8} \in [0, \frac{1}{2})$

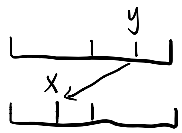
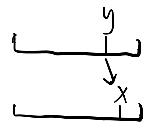
$y - \frac{1}{8} \in E$ then $y - \frac{1}{8} + \frac{1}{2} \in E$ if $y \in E_m$

$y - \frac{1}{8} \in [0, \frac{7}{8})$ otherwise $y - \frac{1}{8} \in [\frac{1}{2}, 1)$

$y - \frac{1}{8} - \frac{1}{2} \in E$ if $y \in E_m$

① $\{E_n\}$ disjoint

② $|E_n| = |E \cap (0, 1-x_n)| + |E \cap [1-x_n, 1)| = |E|$

③ $(0,1) = \bigcup_{n\in\mathbb{N}} E_n$  $x_n - 1 + y = x$  $x_n + y = x$

$[x] = y > \frac{1}{2}$

$1 = (0,1) = \sum_n |E|$ $\quad |E| \to$ not define.

Borel measure:

open sets $\xrightarrow{\text{generate}}$ Borel sigma algebra $\longrightarrow$ Borel measure u s.t. Borel sigma algebra measurable.

Lebesgue measure

set function / outer measure λ^* $\xrightarrow{\text{require}}$ Lebesgue σ-algebra. (collection of E) $\longrightarrow$ Lebesgue measure.

$\lambda^*(A) = \lambda^*(A\cap E) + \lambda^*(A\cap E^c)$

$\forall A$

every Borel set is Lebesgue measurable, Lebesgue

u is a measure, its total variation $|u|$ on $A \in \Sigma$:

$|u|(A) = \sup\left\{ \sum_{n=1}^{\infty} |u(A_n)| : A_n \in \Sigma \text{ pairwise disjoint}, A = \bigcup_{n=1}^{\infty} A_n \right\}$

prop: $|u|$ is a positive finite measure.

$u(\phi) = 0$

σ-additivity $\Leftarrow$ σ-sub + additivity

Assume $|u|(X) = \infty$, then there exists a countable partition