Problem 1. Exercise 5.6

Proof. 1. For any nonzero $x \in X$, let $Y = \{kx : k \in \mathbb{R}\}$. Then Y is a linear subspace of X.

We define a functional $\psi : Y \mapsto \mathbb{R}$ by $\psi(kx) = k\|x\|$.

It is linear since $\psi(kx + lx) = (k + l)\|x\| = \psi(kx) + \psi(lx)$.

It is bounded and $\|\psi\| = 1$.

By Hahn-Banach theorem, there exist a bounded linear function $\phi : X \mapsto \mathbb{R}$ such that $\|\phi\| = \|\psi\| = 1$. Moreover, since $x \in Y$, $\phi(x) = \psi(1 \cdot x) = \|x\|$.

For $x = 0 \in X$, we pick any ψ_y with nonzero y defined as above. The corresponding $\phi_y(0) = \|0\| = 0$ with $\|\phi_y\| = 1$.

2. Assume $x \neq y$, let $z = x - y$ and z is nonzero. We define $Z = \{kz : k \in \mathbb{R}\}$ and a bounded linear functional $\psi : Z \mapsto \mathbb{R}$ by $\psi(kz) = k\|z\|$. It can be extended to a bounded linear functional $\phi : X \mapsto \mathbb{R}$ such that $\phi(z) = \phi(x - y) = \|x - y\| > 0$, however $\phi(x - y) = \phi(x) - \phi(y) = 0$. Contradiction!

□

Problem 2. Lower semi-continuity of weak convergence.

Proof. From Problem 1, we know that there exists a bounded linear functional $\phi \in X^*$ such that $\phi(x) = \|x\|$ and $\|\phi\| = 1$.

$x_n \rightharpoonup x$ implies that $\phi(x_n) \to \phi(x)$. Thus

$$\|x\| = \phi(x) = |\phi(x)| = \lim_{n\to\infty} |\phi(x_n)| \leqslant \liminf_{n\to\infty} \|\phi\|\|x_n\| = \liminf_{n\to\infty} \|x_n\|.$$

□

Problem 3. Isometric embedding of a separable Banach space in ℓ^∞.

Proof. Let X denote the separable Banach space and Y denote its countable dense subset. For any $x \in X$, let (x_n) denote the sequence in Y that converges to x. By Exercise 5.6, there exists a sequence (ϕ_n) in X^* such that $\|\phi_n\| = 1$ and $\phi_n(x_n) = \|x_n\|$. Now we define the isometric embedding $J : X \mapsto \ell^\infty$ by

$$J(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_n(x), \cdots)$$

.

J is well-defined since $|\phi_k(x)| \leqslant \|\phi_k\|\|x\| = \|x\|$ for all k.

J is linear and we will prove $\|J(x)\|_\infty = \|x\|$ to show it is an isometry. Especially we prove that $\|J(x)\|_\infty \geqslant \|x\|$. By the continuity of norm and continuity of linear functional ϕ_n, we have

$$\begin{aligned}\|x\| &= \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \phi_n(x_n) = \lim_{n\to\infty} \phi_n(x_n - x) + \phi_n(x) \\ &\leqslant \lim_{n\to\infty} \|\phi_n\|\|x_n - x\| + \sup_n \phi_n(x) = \|J(x)\|_\infty.\end{aligned}$$

□

Remark: Thanks Matthew Corbelli for pointing this out. I think the assumption of completeness is not necessary in this proof. It might suggest that there is an isometric isomorphism between any separable Banach space with a closed linear subspace of ℓ^∞. However, we just need an embedding in this problem.

Problem 4. Exercise 5.7

Proof.

$$\begin{aligned} Kf(x) &= \int_0^1 \sin \pi(x-y) f(y)\mathrm{d}y \\ &= \int_0^1 \left(\sin \pi x \cos \pi y - \cos \pi x \sin \pi y\right) f(y)\mathrm{d}y \\ &= \sin \pi x \int_0^1 \cos \pi y f(y)\mathrm{d}y - \cos \pi x \int_0^1 \sin \pi y f(y)\mathrm{d}y \\ &= a(f) \sin \pi x + b(f) \cos \pi x, \end{aligned}$$

where $a(f) = \int_0^1 \cos \pi y f(y)\mathrm{d}y$ and $b(f) = -\int_0^1 \sin \pi y f(y)\mathrm{d}y$.

Thus the range $\mathrm{ran}(K) \subset \mathrm{span}\left\{\sin \pi x, \cos \pi x\right\}$.

On the other hand, since for $f \equiv 1$, $Kf(x) = -\frac{2}{\pi} \cos \pi x$; for $f = \cos \pi x$, $Kf(x) = \frac{1}{2} \sin \pi x$. Thus for $f(x) = 2a \cos \pi x - \frac{b\pi}{2}$, $Kf(x) = a \sin \pi x + b \cos \pi x$, which yields that $\mathrm{ran}(K) = \mathrm{span}\left\{\sin \pi x, \cos \pi x\right\}$.

$$Kf(x) = 0 \iff \begin{cases} \displaystyle\int_0^1 \cos \pi y f(y)\mathrm{d}y = 0; \\ \displaystyle\int_0^1 \sin \pi y f(y)\mathrm{d}y = 0. \end{cases}$$

$$\mathrm{ker(K)} = \left\{ f \in C[0,1] : \int_0^1 \cos \pi y f(y)\mathrm{d}y = 0 \quad \text{and} \quad \int_0^1 \sin \pi y f(y)\mathrm{d}y = 0. \right\}$$ □

Problem 5. Exercise 5.11

Proof.

$$0 \leqslant \left| \|T_n\| - \|T\| \right| \leqslant \|T_n - T\| \to 0.$$

□

Problem 6. Weak limit is unique.

Proof. This is the result from Problem 1 question b. □

Problem 7. Exercise 5.17

Proof.

- $I - K$ is one-to-one.

  For any $x \neq y$, $(I-K)(x-y) = (x-y) - K(x-y)$.

  And $\|K(x-y)\| \leqslant \|K\|\|x-y\| \leqslant \|x-y\|$. Thus

$$\|(I-K)(x-y)\| = \|(x-y) - K(x-y)\| \geqslant |\|x-y\| - \|K(x-y)\|| > 0.$$

- $I - K$ is onto.

  For any $y \in X$, we define $x_n = \left(I + \sum_{i=1}^{n-1} K^i\right) y$. Now we show that (x_n) is a Cauchy sequence in the Banach space X.

$$\|x_n - x_m\| = \left\| \sum_{n}^{m-1} K^i y \right\| \leqslant \sum_{n}^{m-1} \left\|K^i\right\| \|y\| \leqslant \sum_{n}^{m-1} \|K\|^i \|y\| \leqslant \frac{\|K\|^n}{1 - \|K\|} \|y\| \leqslant \varepsilon,$$

  for large enough N and $m > n \geqslant N$.

  Let $x_n \to x$ with respect to $\|\cdot\|$. By the continuity of $I - K$, we have

$$(I-K)x = \lim_{n\to\infty} (I-K)x_n = \lim_{n\to\infty} (I-K)(I + \sum_{i=1}^{n-1} K^i)y = \lim_{n\to\infty} (I - K^n)y = y,$$

  since $\lim_{n\to\infty} \|K^n y\| = 0$.

  In this way, for any $y \in X$, we can find a $x \in X$ such that $(I-K)x = y$, that is, $I - K$ is onto.

- $(I-K)^{-1} = I + K + K^2 + K^3 + \cdots$ and the right hand side converges uniformly.

$$\left\| I - (I-K)(I + \sum_{i=1}^{n-1} K^i) \right\| = \|I - (I - K^n)\| = \|K^n\| \leqslant \|K\|^n \to 0,$$

  which implies that

$$(I-K)(I + \sum_{i=1}^{n-1} K^i) \to I \qquad \text{with respect to } \|\cdot\|.$$

  Based on a similar process, one can also show that:

$$(I + \sum_{i=1}^{n-1} K^i)(I-K) \to I \qquad \text{with respect to } \|\cdot\|.$$

  Therefore, we have $(I-K)^{-1} = I + K + K^2 + K^3 + \cdots$.

$\square$