Problem 1. $C^1([a,b])$ is Banach space.

Proof. Given a Cauchy sequence (f_n) in $C^1([a,b])$, i.e., for any $\varepsilon > 0$, there exists a N such that $n, m \geqslant N$, we have
$$\|f_n - f_m\| = \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty \leqslant \varepsilon,$$
which yields (f_n) and (f_n') are Cauchy sequence in $C([a,b])$. Since $C([a,b])$ is complete, then there exist f and g in $C([a,b])$ such that
$$\|f_n - f\|_\infty \to 0 \qquad \text{and} \qquad \|f_n' - g\|_\infty \to 0,$$
which implies for any $x \in [a,b]$, $\lim_{n\to\infty} f_n(x) = f(x)$ and $\lim_{n\to\infty} f_n'(x) = g(x)$. What left to prove is $f' = g$.

For each n, by the fundamental theorem of calculus, we have
$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\mathrm{d}t.$$

By the uniform convergence of f_n', we have
$$|\int_a^x f_n'(t)\mathrm{d}t - \int_a^x g(t)\mathrm{d}t| \leqslant \int_a^x |f_n'(t) - g(t)|\mathrm{d}t \leqslant \varepsilon|x-a| \to 0.$$

Therefore,
$$f(x) - f(a) = \lim_{n\to\infty} f_n(x) - f_n(a) = \lim_{n\to\infty} \int_a^x f_n'(t)\mathrm{d}t = \int_a^x g(t)\mathrm{d}t.$$

□

Problem 2. Dini's monotone convergence theorem.

Proof. Let $g_n(x) = f_n(x) - f(x) \in C([0,1])$ and $U_n = \{x \in [0,1] : g_n(x) < \varepsilon\}$ is open in $[0,1]$. Furthermore, $U_1 \subset U_2 \subset \cdots$ by the monotone decreasing.

For each $x \in [0,1]$, since $g_n(x) \to 0$ as $n \to \infty$, so each x is in some U_n. That is, $[0,1] \subset \bigcup U_n$. Then there exists a finite subcover such that $[0,1] \subset \bigcup_{k=1}^N U_{n_k} = U_{n_N} \subset U_n \subset [0,1]$, for all $n \geqslant n_N$.

For any $\varepsilon > 0$, there exists a n_N such that for $n \geqslant n_N$ we have $f_n(x) - f(x) < \varepsilon$ for all $x \in [0,1]$. That is, f_n converges uniformly to f. □

Problem 3.

Proof.

- Step 1 : Set of polynomials with rational coefficients is countable.

  Let $\mathbb{Q}[x]$ denote the set of polynomials with rational coefficients. Define a map $f : \mathbb{Q}[x] \mapsto \mathbb{Q}$ by $f(a_0 + a_1 x + \cdots a_n x^n) = 2^{a_0} 3^{a_1} \cdots p_n^{a_n}$, where p_n is the $n-$th odd prime. Since f is injective and $\mathbb{Q}$ is countable, $\mathbb{Q}[x]$ is countable.

  Step 2: Set of polynomials with rational coefficients is dense in the set of polynomials with respect to $\|\|_\infty$, which is from $\mathbb{Q}$ is dense in $\mathbb{R}$ and take supremum.

  Step 3: Set of polynomials with rational coefficients is one countable dense subset in $C[0,1]$, since by Weierstrass approximation, the set of polynomials is dense in $C([0,1])$.

- Let $f_x(y) = 1$ if $y \neq x$ and $f_x(y) = 0$ if $y = x$.

  Thus $(f_x)_{x\in[0,1]}$ is bounded and thus is a subspace of $B([0,1])$. Note $\|f_x - f_y\|_\infty = 1$ if $x \neq y$. Thus (f_x) is not separable.

  Recall every subspace of a separable metric space is separable. Thus it is impossible to have $B([0,1])$ separable.

□

Problem 4.

Proof. Since the set of polynomials is dense in $C([0,1])$, for any $\varepsilon > 0$, there exist a polynomial $p(x)$ such that $\|f - p\|_\infty < \varepsilon$, thus $|f(x) - p(x)| \leqslant \|f - p\|_\infty \leqslant \varepsilon$ for any $x \in [0,1]$.

Since $\int_0^1 f(x)x^n \mathrm{d}x = 0$, then $\int_0^1 f(x)p(x)\mathrm{d}x = 0$. So

$$\begin{aligned}\int_0^1 f(x)f(x)\mathrm{d}x &= \int_0^1 f(x)f(x)\mathrm{d}x - \int_0^1 f(x)p(x)\mathrm{d}x = \int_0^1 f(x)(f(x) - p(x))\mathrm{d}x \\ &\leqslant \int_0^1 |f(x)||f(x) - p(x)|\mathrm{d}x \\ &\leqslant \varepsilon \int_0^1 |f(x)|\mathrm{d}x \to 0,\end{aligned}$$

which yields that $f(x) = 0$ on $[0,1]$. □

Problem 5.

Proof.

- For any $f \in C([0,1])$, $f \circ \sqrt{\cdot} \in C([0,1])$. Since the set of polynomial is dense in $C([0,1])$, there exists a polynomial $p(x)$ such that $|f(\sqrt{x}) - p(x)| \leqslant \left\| f \circ \sqrt{\ } - p \right\|_\infty \leqslant \varepsilon$ for any $x \in [0,1]$.

  Let $y = \sqrt{x} \in [0,1]$, then $f(\sqrt{x}) = f(y)$ and $p(x) = p(y^2) = q(y)$ for some even power polynomial $q(y) \in P^{\text{even}}([0,1])$. Thus

  $$\|f - q\|_\infty = \sup_{y\in[0,1]} |f(y) - q(y)| = \sup_{x\in[0,1]} |f(\sqrt{x}) - p(x)| \leqslant \varepsilon,$$

  which implies that $P^{\text{even}}([0,1])$ is dense in $(C([0,1]), \|\cdot\|_\infty)$.

- No, it is not.

  For example, let $f(x) = x$ on $[-1,1]$. Suppose there is a polynomial $q(x) \in P^{\text{even}}([-1,1])$ such that $\|f - q\|_\infty \leqslant \varepsilon$, i.e., $|q(x) - x| \leqslant \varepsilon < 1$ for any $x \in [-1,1]$. Take $\varepsilon = \frac{1}{2}$, then for any $x \in [-1,1]$, $-\frac{1}{2} + x < q(x) < \frac{1}{2} + x$. So $-\frac{3}{2} < q(-1) < -\frac{1}{2}$ and $\frac{1}{2} < q(1) < \frac{3}{2}$, contradicted with $q(1) = q(-1)$.

□