• Metric(d): $X \times X \to \mathbb{R}$ $\quad$ X is not necessary linear space

binary-relationship $\quad$ Algebra structure is not necessary.

Eg: $w: [0,1] \to X$ is a curve valued in the metric space (X,d)

define metric derivative of w at t denoted by $|w'|(t)$

$$|w'|(t) := \lim_{h \to 0} \frac{d(w(t+h), w(t))}{|h|}$$

Eg: $\mathbb{P}_1 = \{$ Probability measure space $\}$ on X $\quad u, v$

$d(u,v) = W(u,v)$ $\quad$ Wasserstein distance.

Eg: To have same linear even differential structure

Eg: $\mathbb{R}^2$ $\quad e(x,y) = \|x-y\|_{l^2}$ $\quad$ $d(x,y) = \begin{cases} e(x,y) & x, y \text{ lie on same ray though origin} \\ e(x,0) + e(0,y) & \text{if not} \end{cases}$

$\geq e(x,y)$

Prove metric.

$d(x,y) + d(y,z) \geq d(x,z)$

if x, z on the same ray $\quad d(x,z) = e(x,z) \leq e(x,y) + e(y,z)$

$x\,z$ not $\begin{cases} y \text{ not} & d(x,y) + d(y,z) = d(x,0) + 2d(y,0) + d(0,z) \\ y \text{ in either} & d(x,y) + d(y,z) = d(x,y) + d(0,y) + d(0,z) \end{cases}$

$\geq e(x,y) + e(0,y) + e(0,z)$

$\geq e(x,0) + e(0,z)$

• $\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |b_i|^p\right)^{\frac{1}{p}} \quad p \geq 1$

· $(X,d) \to (X, d^P)$ $0<P\leq 1$

$$f(d(x,y)+d(y,z)) = (d(x,y)+d(y,z))^P \leq (d(x,y))^P + (d(y,z))^P$$ sub-additivity

$\leq$ non-decreasing

$f(d(x,z))$

Eg d induce open ball $B_r(a) = \{d(x,a)<r\}$

open set G if $\forall x \in G$, $\exists r>0$ s.t $B_r(x) \subset G$.

Eg. F is closed iff $\forall$ convergent sequence of elements in F converges to a limit in F.

open in $(\mathbb{R}^2, d)$ not in $(\mathbb{R}^2, e)$

$(100,101)$ $\forall x \in (100,101)$

$y=0$

$B_r(X) = \{d(x,y)<1\}$

$r<1$

Q4 $e(x,y) < d(x,y) < 1$

$d(x,y) = d(x,0) + d(0,y) > \underline{100}$

not unique.

· Norm: $\| \ \|: X \to \mathbb{R}$ more conditions

$$\begin{tikzcd} d(x,y) \arrow[r, bend left, "d(x,0)"] & \|x\| \arrow[l, bend left, "\|x-y\|"] \end{tikzcd}$$

always

$d(x,y) = \frac{|x-y|}{1+|x-y|} \leq 1$

$d(x,0)$ is not a norm.

- Convergence w.r.t metric $\xrightarrow{\text{induce}}$ topogology (define open set)

  ↑ open ball

  Metrization theorem

  (Nagata-Smirnov metrization thm)

- For normed linear space we $\|\,\|$ define d. (induced metric)

  Convergence w.r.t norm.

- all norms on a finite-dimensional linear space lead to exactly the same notion of convergence. (Not true for infinite-dimension)

  eg $\mathbb{R}^n$ $\|\ \|_1, \|\ \|_2, \|\ \|_\infty$ (dimension of space not defined)

  $(\mathbb{P}, W_2) \rightarrow$ weak * convergence. $\leftarrow (\mathbb{P}, d_\alpha)$

  (l_2, d_2) $\{(a_1, a_2, \dots a_n)\}$ $e_n = (\dots, \underset{n}{1}, \dots)$

  $d_2(x-y) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{\frac{1}{2}}$

  $d_\infty(x-y) = \sup_n |x_n - y_n|.$

$\exists A, B$ s.t $A\|v\|_\alpha \le \|v\|_\beta \le B\|v\|_\alpha$ lead same Convergent

$\{v_n\}$ converges w.r.t $\|\ \|_\alpha$ $\exists v$ $\|v_n - v\|_\alpha = 0$

$A\|v_n - v\|_\alpha \le \|v_n - v\|_\beta \le B\|v_n - v\|_\alpha$

$\nearrow 0$

$\|\ \|_1$ and $\|\ \|_\infty$ on $C[0,1]$ $\|f\|_1 = \int_0^1 |f(x)|\,dx$

not equivalent $\|f\|_\infty = \max_{[0,1]} |f(x)|.$

$e_i = (\frac{1}{i}, \ldots \frac{1}{i}, \ldots)$ $\qquad \|e_i\|_1 = 1$

$\qquad\qquad\qquad\qquad\qquad\qquad \|e_i\|_\infty = \frac{1}{i} \to 0$

For real numbers, $\begin{cases} \text{Cauchy sequence is convergent} \\ \text{Every convergent sequence is a Cauchy sequence.} \end{cases}$

Strong Convergent $\Longleftrightarrow \langle x, d \rangle$

$(1, \ldots) \in l^2$

$\|x\| \leq \liminf_{n\to\infty} \|x_n\|$ $\qquad 0 \leq \liminf_{n\to\infty} \|e_n\| = 2$