MAT201A Homework 3
Fall 2019
Professor Qinglan Xia
Due Date: Wednesday, October 16th at 9:00am

1. Let $C^1([a,b])$ be the space of continuously differentiable functions on $[a,b]$ with the C^1-norm:
$$||f|| = ||f||_\infty + ||f'||_\infty.$$
Show that $C^1([a,b])$ is a Banach space.

2. Suppose $f_n \in C([0,1])$ is a monotone decreasing sequence (i.e. for each $x \in [0,1]$, $(f_n(x))$ is a monotone decreasing sequence of real numbers) that converges pointwise to $f \in C([0,1]$. Prove that f_n converges uniformly to f. This result is called *Dini's monotone convergence theorem.*

3.
   - Prove that $C([0,1])$ with the supremum norm $||\cdot||_\infty$ is separable.
   - Let $B([0,1])$ be the space of all bounded functions on $[0,1]$ with the supremum norm $||\cdot||_\infty$. Show that $B([0,1])$ is not separable.

4. Let $f \in C([0,1]$ be such that $\int_0^1 f(x)x^n dx = 0$ for all integers $n \geq 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

5. Let $P^{even}([a,b])$ be the subspace of polynomial functions on $[a,b]$ containing only even powers:
$$P^{even}([a,b]) = \{f \in C([a,b]) | f(x) = \sum_{i=0}^{n} a_i x^{2i}, n \geq 0, a_0, a_1, \cdots, a_n \in \mathbb{R}\}.$$
   (a) Prove that $P^{even}([0,1])$ is dense in $(C([0,1]), ||\cdot||_{sup})$.
   (b) Is $P^{even}([-1,1])$ dense in $(C([-1,1]), ||\cdot||_{sup})$? Justify your answer.