Problem 1.

Proof. Since $T(x)$ is even function while $S(x)$ is odd function. We may use the orthogonal basis, $\{1, \cos(nx), \sin(nx), \cdots\}$.

a) $T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$.

$$a_0 = \frac{1}{\pi}\int_{\mathbb{T}} |x|\mathrm{d}x = \frac{2}{\pi}\frac{1}{2}x^2|_0^\pi = \pi$$
$$a_n = \frac{1}{\pi}\int_{\mathbb{T}} |x|\cos(nx)\mathrm{d}x = \frac{2}{\pi}\int_0^\pi x\cos(nx)\mathrm{d}x = \frac{2((-1)^n - 1)}{n^2\pi}.$$

Thus,

$$T(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2}\cos((2n-1)x).$$

$S(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$.

$$b_n = \frac{1}{\pi}\int_{\mathbb{T}} S(x)\sin(nx)\mathrm{d}x = \frac{2}{\pi}\int_0^\pi \sin(nx)\mathrm{d}x = \left.\frac{-2\cos(nx)}{n\pi}\right|_0^\pi = \frac{-2((-1)^n - 1)}{n\pi}.$$

Thus,

$$S(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin((2n-1)x).$$

b) Note that $T(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \hat{T}_n e^{inx}$, thus $\frac{1}{\sqrt{2\pi}}\hat{T}_0 = \frac{\pi}{2}$, $\frac{2}{\sqrt{2\pi}}(\hat{T}_{2n-1} + \hat{T}_{-2n+1}) = \frac{4}{\pi(2n-1)^2}$ and $\hat{T}_{2n-1} = \hat{T}_{-2n+1}$.

$$\begin{aligned}\|T\|^2_{H^1(\mathbb{T})} &= \sum_{n=-\infty}^{\infty}(1+n^2)\overline{\hat{T}_n}\hat{T}_n = (\hat{T}_0)^2 + 2\sum_{n=1}^{\infty}(1+(2n-1)^2)(|\hat{T}_{2n-1}|)^2 \\ &\leqslant \frac{\pi^3}{2} + 2\sum_{n=1}^{\infty} 2(2n-1)^2\frac{2}{\pi(2n-1)^4} \\ &= \frac{\pi^3}{2} + 8\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} < \infty,\end{aligned}$$

which yields that $T \in H^1(\mathbb{T})$.

Recall the differentiation theorem for Fourier series:

Assume that $f(x)$ is continuous on (π, π) and $f'(x)$ is piecewise continuous on (π, π). Then the Fourier series of $f(x)$ can be differentiated term-by-term. And the equal sign can be interpreted as pointwise limit.

As a result, the Fourier series of $T(x)$ can be differentiated term-by-term. A quick check that $T'(x) = S(x)$. Even if sometimes we cannot differentiate term-by-term, we can verify this through weak derivatives. In this case, the equal sign is in weak sense.

c) Based on a similar observation, $\frac{2i}{\sqrt{2\pi}}(\hat{S}_{2n-1} - \hat{S}_{-2n+1}) = \frac{4}{(2n-1)\pi}$ and $\hat{S}_{2n-1} = -\hat{S}_{-2n+1}$. Then

$$\begin{aligned}\|S\|^2_{H^1(\mathbb{T})} = \sum_{n=-\infty}^{\infty}(1+n^2)\overline{\hat{S}_n}\hat{S}_n &= 2\sum_{n=1}^{\infty}(1+(2n-1)^2)(|S_{2n-1}|)^2 \\ &\leqslant 2\sum_{n=1}^{\infty}2(2n-1)^2\frac{2}{\pi(2n-1)^2} \to \infty.\end{aligned}$$

□

Problem 2.

Proof. We follow closely to the proof of Lemma 7.8 in the textbook.

Let

$$S_N(x) := \sum_{|n_i|\leqslant N, n\in\mathbb{Z}^d} a_n e^{in\cdot x}.$$

Then

$$\begin{aligned}\|S_N - S_M\|_\infty &\leqslant \sum_{N<|n_i|\leqslant M}|a_n| \\ &= \sum_{N<|n_i|\leqslant M}|n|^k|a_n|\frac{1}{|n|^k} \\ &\leqslant \left[\sum_{N<|n_i|\leqslant M}|n|^{2k}|a_n|^2\right]^{1/2}\left[\sum_{N<|n_i|\leqslant M}\frac{1}{|n|^{2k}}\right]^{1/2} \\ &\leqslant \left[\sum_{|n_i|>N}\frac{1}{|n|^{2k}}\right]^{1/2}\left[\sum_{n\in\mathbb{Z}^d}|n|^{2k}|a_n|^2\right]^{1/2},\end{aligned}$$

where we note that

$$\begin{aligned}\sum_{|n_i|>N}\frac{1}{|x|^{2k}} &\leqslant \int_{\mathbb{R}^d\setminus B_N(0)}\frac{1}{|n|^{2k}}\mathrm{d}x \\ &= \int_N^\infty\int_{S^{d-1}}\frac{1}{r^{2k}}r^{d-1}\mathrm{d}S\mathrm{d}r \\ &= |S^{d-1}|\int_N^\infty\frac{1}{r^{2k-d+1}}\mathrm{d}r \\ &\leqslant C(d)\frac{1}{N^{2k-d}}.\end{aligned}$$

For $k > \frac{d}{2}$ and $\sum_{n\in\mathbb{Z}^d}|n|^{2k}|a_n|^2 < \infty$, we know that $\|S_N - S_M\|_\infty \to 0$ as $N \to \infty$. As a result, S_N converges to f uniformly and S_N is continuous, thus f is continuous as well. □

Problem 3.

Proof. Note that for domain other than $\mathbb{T}$, we define $H^1([a,b])$ by its weak derivatives belongs to $L^2([a,b])$.

Let's define a function $\bar{g}:[0,\pi]\mapsto\mathbb{R}$ by

$$\bar{g}(x) := f(\frac{(b-a)x}{\pi}+a).$$

Then we extend $\bar{g}$ to $g:\mathbb{T}\mapsto\mathbb{R}$ by odd extention with the observation that $g(-\pi)=-\bar{g}(\pi)=-f(b)=0=g(\pi)$, $\int_{\mathbb{T}} g(x)\mathrm{d}x=0$ and $g'(x)=f'(\frac{b-a}{\pi}x+a)\frac{b-a}{\pi}$ (Chain Rule of weak derivatives).

Let's do the following change of variables $y=\frac{b-a}{\pi}x+a$, thus $\mathrm{d}y=\frac{b-a}{\pi}\mathrm{d}x$.

$$\begin{aligned}
\int_a^b |f(y)|^2\mathrm{d}y &= \int_0^\pi |f(\frac{b-a}{\pi}x+a)|^2\frac{b-a}{\pi}\mathrm{d}x\\
&= \frac{b-a}{\pi}\int_0^\pi |\bar{g}(x)|^2\mathrm{d}x\\
&= \frac{b-a}{2\pi}\int_{\mathbb{T}}|g(x)|^2\mathrm{d}x\\
&\leqslant \frac{b-a}{2\pi}\int_{\mathbb{T}}|g'(x)|^2\mathrm{d}x\\
&= \frac{b-a}{\pi}\int_0^\pi |\bar{g}'(x)|^2\mathrm{d}x\\
&= \frac{b-a}{\pi}\int_0^\pi |f'(\frac{b-a}{\pi}x+a)|^2\frac{(b-a)^2}{\pi^2}\mathrm{d}x\\
&= \frac{(b-a)^2}{\pi^2}\int_a^b |f'(y)|\mathrm{d}y.
\end{aligned}$$

□

Problem 4.

Proof. We note that $\Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}$. By the maximum principle, the solution to the Dirichlet problem is unique. We claim that $u(r,\theta)=2+\frac{3r\sin\theta}{a}$ is a solution.

First $u(a,\theta)=2+3\sin\theta$ satisfying the boundary condition.

Second $u_r=\frac{3\sin\theta}{a}$, $u_{rr}=0$; $u_\theta=\frac{3r\cos\theta}{a}$ and $u_{\theta\theta}=-\frac{3r\sin\theta}{a}$. Thus

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0+\frac{3\sin\theta}{ar}-\frac{3\sin\theta}{ar}=0.$$

□

Problem 5. Poisson kernel is an approximation identity.

Proof. First,

$$P_r(\theta)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}=\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}.$$

Non-negativity:

$$P_r(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2} \geqslant \frac{1}{2\pi}\frac{1-r^2}{1+2r+r^2} = \frac{1}{2\pi}\frac{1-r^2}{(1+r)^2} \geqslant 0,$$

for $0 \leqslant r < 1$.

Unit Mass:

$$\int_{-\pi}^{\pi} P_r(\theta)\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}\mathrm{d}\theta = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}\int_{-\pi}^{\pi} e^{in\theta}\mathrm{d}\theta = 1.$$

Mass concentrated around zero:

$$\begin{aligned}\int_{\delta\leqslant|\theta|\leqslant\pi} P_r(\theta)\mathrm{d}\theta &= \frac{1}{\pi}\int_{\delta}^{\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}\mathrm{d}\theta \leqslant \frac{1}{\pi}\int_{\delta}^{\pi}\frac{1-r^2}{1-2r\cos\delta+r^2}\mathrm{d}\theta \\ &= \frac{(1-r^2)}{1-2r\cos\delta+r^2}(\pi-\delta) \to \frac{1-r}{1-\cos\delta}(\pi-\delta) \to 0,\end{aligned}$$

as $r \to 1$. $\square$