MAT201A Homework 5
Fall 2019
Professor Qinglan Xia
Due Date: Monday, October 28th at 9:00am

1. Let X be a metric space with its metric topology $\mathcal{T}$. Show that

   a) X is first countable.

   b) When X is separable, show that X is second countable.

2. Let X be an uncountable set. Define

$$\mathcal{T} = \{A \subseteq X \mid \text{ the complement } X \setminus A \text{ is countable } \} \cup \{\emptyset\}.$$

   Show that $(X, \mathcal{T})$ is a topological space. Is it first countable? Justify your answer.

3. Suppose that K is a compact subset of a Hausdorff space. Prove that K is closed. Show that this result need not be true if X is not Hausdorff.

4. Give an example of two metric spaces (X_1, d_1) and (X_2, d_2), such that X_1 and X_2 are homeomorphic as topological spaces but X_1 is a complete metric space while X_2 is not.

5. Two metrics, d_1 and d_2, on the same space X are called equivalent if there exist constants $c, C > 0$ such that

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y), \forall x, y \in X.$$

   a) Show that the topologies on X defined by two equivalent metrics are identical.

   b) Let (X, d) be a metric space. Show that there exists a metric d_b with the property that $d_b(x, y) \leq 1$ for all $x, y \in X$, and such that the topology on X derived from the metric d_b is the same as the one derived from the metric d.

   c) Give an example of the situation described in part b) with metrics d and d_b that are not equivalent.