Recall if X is a Banach space

① $(x_n) \subseteq X$ $\quad x_n \to x$ strongly / uniformly / in norm topology

if $\|x_n - x\| \to 0$ as $n \to +\infty$

② $(x_n) \subseteq X$, $\quad x_n \rightharpoonup x$ weakly

if $\forall \varphi \in X^*$, $|\varphi(x_n) - \varphi(x)| \to 0$ as $n \to +\infty$

Hilbert space $\mathcal{H}$

① $(x_n) \subseteq \mathcal{H}$ $\quad x_n \to x$ strongly / uniformly / in norm topology

if $\|x_n - x\| \to 0$ as $n \to \infty$

$|\langle x_n - x, x_n - x \rangle| \to 0$

② $(x_n) \subseteq \mathcal{H}$ $\quad x_n \rightharpoonup x$ weakly

if $\forall y \in \mathcal{H}$, $|\langle x_n, y \rangle - \langle x, y \rangle| \to 0$ as $n \to \infty$

$$\| $$

$|\langle x_n - x, y \rangle| \to 0$ as $n \to +\infty$

[Uniform Boundedness] $\{\varphi_n\} \subseteq X^*$, X Banach

if $\forall x \in X$, $|\varphi_n(x)| \leq M_x$

then $\|\varphi_n\| \leq M$

[Radon-Risez Thm] if $u_n \rightharpoonup u$ in $\mathcal{H}$ then $\left(\begin{array}{l}\text{normed space} \\ \text{Radon-Risez Thm}\end{array}\right)$

$u_n \to u$ iff $\lim_{n\to\infty} \|u_n\| = \|u\|$

Defn: If $\mathcal{H}$ is a Hilbert space, $(x_n) \subseteq \mathcal{H}$ is weakly Cauchy if $\forall y \in \mathcal{H}$, $(\langle x_n, y\rangle) \subseteq \mathbb{C}$ is Cauchy.

Prop. Every weakly Cauchy sequence of elements in $\mathcal{H}$ weakly converges to some element of $\mathcal{H}$.

Proof: Since $(\langle x_n, y\rangle)$ is Cauchy and $\mathbb{C}$ is complete.

Then define $\varphi(y) = \lim_{n\to\infty} \varphi_n(y) := \lim_{n\to\infty} \langle x_n, y\rangle$

$(\varphi_n) \subseteq \mathcal{H}^*$, apply uniform boundedness thm

φ is bounded. $\in \mathcal{H}^*$

By Risez-Representation thm, $\exists x \in \mathcal{H}$, s.t

$\varphi(y) = \langle x, y\rangle$ s.t $\langle x_n, y\rangle \to \langle x, y\rangle$

i.e $x_n \rightharpoonup x$.

Prop $\mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$ compact operator with closed range.

$(x_n) \subseteq \mathcal{H}$ converges weakly to $x \in \mathcal{H}$

(a) (Ax_n) converges weakly

(b) (Ax_n) converges strongly.

Pf (a) $\forall y$, $\langle Ax_n - Ax, y\rangle = \langle A(x_n - x), y\rangle = \langle x_n - x, A^* y\rangle \to 0$ $\boxed{\emptyset}$

(b) $x_n \rightharpoonup x \Rightarrow (x_n)$ is bounded (Every weakly convergence sequence in Banach space is bounded)

By definition of compact operator

$\Rightarrow (Ax_n)$ is precompact. + closed range

$\forall$ convergent subsequence $(Ax_{n_k}) \subseteq (Ax_n)$ $Ax_{n_k} \to y$

$$\forall z \in \mathcal{H}, \quad \langle y, z\rangle = \lim_{k\to\infty} \langle Ax_{n_k}, z\rangle = \lim_{k\to\infty} (x_{n_k}, A^* z)$$

$$= (x, A^* z) = (Ax, z)$$

$$\Rightarrow \quad y = Ax$$

All convergent subsequences of (Ax_n) converges to Ax.

Recall: (x_n) is a sequence in a compact subset of a metric space X with the property that every convergent subsequence has the same limit $x \in X$. Prove that $x_n \to x$ as $n \to +\infty$

Proof: Suppose $(x_n) \not\to x$:

$\exists\, \varepsilon_0 > 0$, s.t $\forall N \in \mathbb{N}$, $\exists\, n > N$ s.t $d(x_n, x) \geq \varepsilon_0$

Construct $\{x_{n_k}\}$ subsequence

Due to compactness, extract a subsequence of (x_{n_k})

s.t $x_{n_{k_j}} \to x$ contradiction!

Prop: Prelim

$(A_n) \subseteq B(\mathcal{H})$ $\quad A_n \rightharpoonup A$ $\quad \left(\forall x, y \in \mathcal{H},\ |\langle A_n x, y\rangle - \langle Ax, y\rangle| \to 0\right)$

$\forall x \in \mathcal{H}$, $\|A_n x\| \to \|Ax\|$

then $A_n \to A$ strongly in $B(\mathcal{H}, \mathbb{R})$ (or $A_n \xrightarrow{*} A$ in $\mathcal{H}^*$)

Proof: $\forall x \in \mathcal{H}$ fixed, WTS $\|A_n x - Ax\| \to 0$

$\|A_n x - Ax\|^2 = \langle A_n x - Ax, A_n x - Ax\rangle$

$= \|A_n x\|^2 - \langle A_n x, Ax\rangle - \langle Ax, A_n x\rangle + \|Ax\|^2$

$= \|A_n x\|^2 - \langle (A_n - A)x, Ax\rangle - \langle Ax, (A_n - A)x\rangle - \|Ax\|^2$

$= (\|A_n x\|^2 - \|Ax\|^2) - \langle (A_n - A)x, Ax\rangle - \langle Ax, (A_n - A)x\rangle$

$\to 0$