MAT201A Homework 4
Fall 2019
Professor Qinglan Xia
Due Date: Wednesday, October 23th at 9:00am

1. Let $M > 0$ and $N > 0$ be constants. Define

$$\mathcal{F} = \{f \in C^1([a,b]) \mid ||f||_\infty \leq M, ||f'||_\infty \leq N\}.$$

Show that $\mathcal{F}$ is a precompact but not compact subset of $(C([a,b]), ||\cdot||_\infty)$.

2. Let $f \in C([a,b])$. Prove that

$$|\int_a^b f(x)dx| \leq |b-a|^{1/2}\left(\int_a^b f(x)^2 dx\right)^{1/2}.$$

3. For $M > 0$, define $\mathcal{F}_M \subset C([a,b])$ as follows:

$$\mathcal{F}_M := \{f \in C([a,b]) \mid f' \in C([a,b]), f(a) = f(b) = 0, \text{ and } \int_a^b (f'(x))^2 dx \leq M\}.$$

Prove that $\mathcal{F}_M$ is precompact in $(C([a,b]), ||\cdot||_\infty)$.

4. Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a nonempty set X. Show that $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology. Given an example to show that $\mathcal{T}_1 \cup \mathcal{T}_2$ may fail to be a topology on X.

5. Let $\mathcal{B}$ be a collection of subsets of a nonempty set X. Then, $\mathcal{B}$ is a base for some topology $\mathcal{T}$ of X if and only if
   (a) $X = \bigcup_{B \in \mathcal{B}} B$;
   (b) For any $B_1, B_2 \in \mathcal{B}$, and any $x \in B_1 \cap B_2$, there exists $W_x \in \mathcal{B}$ such that

$$x \in W_x \subseteq B_1 \cap B_2.$$

6. Suppose X is a metric space. If one of its base has only finitely many elements, show that X must be a metric space with only finitely many points.