08/03. Section 10.4. $\underbrace{\sum \frac{1}{n}}$ $\underbrace{\sum \frac{1}{n^p}}$ $p>1$.

- Comparison Test : if $0 \le d_n \le a_n \le c_n$, non-negative term

① if $\underbrace{\sum c_n \text{ converges}}$, then $\sum a_n$ converges

② if $\underbrace{\sum d_n \text{ diverges}}$, then $\sum a_n$ diverges.

$$\sum d_n \to \infty$$

$$\underbrace{\infty \le \sum a_n} \to \text{diverges.}$$

eg: $0 \le \frac{1}{n^3} \le \frac{1}{n^2} \le \frac{1}{n}$

$$\frac{1}{n^3} \downarrow d_n \qquad \frac{1}{n^2} \downarrow a_n \qquad \frac{1}{n} \downarrow c_n$$

$\sum \frac{1}{n}$ diverges.

$$0 \le \underbrace{\sum \frac{1}{n^3}}_{\downarrow \text{ converges. } L.} \le \underbrace{\sum \frac{1}{n^2}}_{\downarrow\ \underbrace{\sum a_n}\ \text{converge}} \le \underbrace{\sum \frac{1}{n}}_{\downarrow\ \infty\ \downarrow\ \text{diverges.}}$$

converges. $\overset{2}{\shortparallel}$

eg. $1+\sum_{k=1}^{\infty}\left(\cancel{\ }\frac{1}{2^k+\sqrt{k}}\right) = \sum_{k=0}^{\infty}\frac{1}{2^k+\sqrt{k}} \le \boxed{\sum_{k=0}^{\infty}\frac{1}{2^k}} = \lim_{n\to\infty}\frac{1(1-\frac{1}{2^n})}{1-\frac{1}{2}}$

$\Downarrow$ converges.

$$\underbrace{\sum \frac{1}{2^k+\sqrt{k}}}_{\downarrow\ ?} \le \underbrace{\sum_{k=0}^{\infty}\frac{1}{\sqrt{k}}}_{\downarrow\ \text{diverges}}$$

eg. $\sum_{n=0}^{\infty}\frac{1}{n!} = 1+\frac{1}{2!}+\frac{1}{3\times2\times1}+\underline{\frac{1}{4\times3\times2\times1}}$ $\ge\le$ $\sum \frac{1}{n^p}$

$\le 1 + 1 + \frac{1}{2^2} + \frac{1}{[illegible]} + \frac{1}{[illegible]} \to$ converges.

$1+\frac{1}{2!}+\frac{1}{3\times2\times1}+\frac{1}{4\times3\times2\times1}$

$\leq \underline{1+\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{3^2}\cdots+\frac{1}{3^n}} \rightarrow$ converge.

$\leq \boxed{1+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}} \rightarrow$ diverges.

is not suitable for test.

- Limit Comparison Test: $a_n>0$, $b_n>0$.

① if $\lim_{n\to+\infty}\frac{a_n}{b_n}=C>0$, then $\sum a_n$ and $\sum b_n$ both converges or both diverges.

$\Rightarrow a_n << b_n$

② if $\lim_{n\to+\infty}\frac{a_n}{b_n}=0$, $\sum b_n$ converges $\Rightarrow \underline{\sum a_n \text{ converges}}$.

b_n

③ if $\lim_{n\to+\infty}\frac{a_n}{b_n}=\infty$, $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges.

a_n $\quad \sum\frac{1}{n}$ $\quad$ sequence $\underline{\{\frac{1}{n}\}\rightarrow 0}$

series $\{\sum\frac{1}{n}\}$ diverge.

eg: $\sum\frac{2n+1}{(n+1)^2}=\sum\frac{2n+1}{\underline{n^2+2n+1}} \rightarrow \frac{2n}{n^2+2n}=\frac{2}{n+2}\sim\frac{2}{n}$

$\sum b_n=\sum\frac{1}{n}$ diverge.

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim\frac{\frac{2n+1}{(n+1)^2}}{\frac{1}{n}}=\lim\frac{(2n+1)n}{n^2+2n+1}=\lim_{n\to\infty}\frac{2n^2+n}{n^2+2n+1}$

$=\lim_{n\to\infty}\frac{2+\cancel{\frac{1}{n}}}{1+\cancel{\frac{2}{n}}+\cancel{\frac{1}{n^2}}}=2.$

0

$\Rightarrow \sum\frac{2n+1}{(n+1)^2}$ diverges.

$$\sum \frac{1+n\ln n}{n^2+5} \sim \sum \frac{n\ln n}{n^2} = \sum \underbrace{\frac{\ln n}{n}} \sim \frac{n^b}{n} = n^{b-1}$$

$$= \underbrace{\frac{1}{n^{1-b}}}$$

diverge.

Recall: $\ln n < n < n^{1+a}$

$\downarrow$

n^b

$0<b<1.$

$$\boxed{\int \frac{1+x\ln x}{x^2+5}\,dx} \overset{x}{\leq} \int \frac{1+x\ln x}{x^2}\,dx \qquad \frac{1+x\ln x}{x^2+5}$$

$\downarrow$

$0<x<1$

$\ln x<0$

$b_n = \frac{1}{n}.$

$$\lim_{n\to\infty} \frac{\frac{1+n\ln n}{n^2+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+n^2\ln n}{n^2+5} = \cancel{\lim 1+} \quad \lim_{n\to\infty} \frac{\overset{0}{\cancel{\frac{1}{n}}} + \ln n \to \infty}{1+\frac{5}{n^2}\to 0} \to \infty$$

$\downarrow$

OR: L'Hospital Rule

Remark: $\ln n < n^b < n^1 < n^{1+a}$

$\downarrow$

$0<b<1 \qquad 0<a.$

$$\lim_{n\to\infty} \frac{\ln n}{n^b} = \lim_{n\to\infty} \frac{\frac{1}{n}}{bn^{b-1}} = \lim_{n\to\infty} \frac{1}{bn^b} = 0$$

$\ln n \ll n^b$

10.5 Absolute Convergence / The Ratio and Root Test

$\sum a_n$ absolutely converges if $\sum |a_n|$ converges.

$$\sum a_n \leq \underbrace{\sum |a_n|}$$

$\underbrace{\sum \frac{1}{n} \text{ diverges.}}$ $\underbrace{\sum (-1)^{n+1} \frac{1}{n}}$ converges.

- Absolute Convergence Test: $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof: $\underbrace{-\sum |a_n|}_{-L,} \leq \sum a_n \leq \underbrace{\sum |a_n|}_{L}$

eg: $\underbrace{\sum \frac{\sin n}{n^2}}$ $\longrightarrow$ $\sum \frac{|\sin n|}{n^2}$ $\qquad -1 \leq |\sin n| \leq 1$

0

$\downarrow$

not non-negative series.

$$\boxed{0} \; \cancel{\sum} \leq \sum \frac{|\sin n|}{n^2} \leq \underbrace{\sum \frac{1}{n^2}}$$

$\downarrow$

converge $\qquad$ converges $\to L$.

If we can show $\sum \left|\frac{\sin n}{n^2}\right|$ converges (i.e. $\sum \frac{\sin n}{n^2}$ absolutely converges)

then we have $\sum \frac{\sin n}{n^2}$ converges.

$$-1 \leq (-1)^n \leq 1$$

$\sum (-1)^n$ diverge