Fourier Series of $L^2(\mathbb{T})$ functions:

- $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ forms an ONB of $L^2(\mathbb{T})$

  + $L^2(\mathbb{T})$ is the completion of $C(\mathbb{T})$ w.r.t L^2-norm

$\Rightarrow$ ① $\forall f \in L^2(\mathbb{T})$ $S_N f = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx} \to f(x)$ as $n \to \infty$ w.r.t L^2-norm.

where $\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx} dx$ Fourier coefficients.

② (Carleson 1966) $\forall f \in L^2(\mathbb{T})$ $f(x) = \lim_{N \to \infty} S_N f(x)$ a.e. $x \in \mathbb{T}$

③ $\exists f \in C(\mathbb{T})$ such that $\exists x \in \mathbb{T}$ s.t $f(x) \neq \lim_{N \to \infty} S_N f(x)$ (Stein/Folland)

sawtooth function

Prop 1. If $f \in L^2(\mathbb{T})$, $\hat{f}_n \in l^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$

mean: $\exists g \in C(\mathbb{T})$ s.t $f = g$ a.e in $\mathbb{T}$.

Proof: $S_N f(x) = \frac{1}{\sqrt{2\pi}} \sum_{|n| < N} \hat{f}_n e^{inx}$

then $|f(x) - S_N f(x)| = \frac{1}{\sqrt{2\pi}} \left| \sum_{|n| \geq N} \hat{f}_n e^{inx} \right| \leq \frac{1}{\sqrt{2\pi}} \sum_{|n| \geq N} |\hat{f}_n| < \varepsilon$

$\Rightarrow$ $S_N f \to f$ uniformly

$\Rightarrow$ f is continuous

- Parseval's inequality $\langle f(x), g(x) \rangle_{L^2(\mathbb{T})} = \langle \hat{f}_n, \hat{g}_n \rangle_{l^2(\mathbb{Z})}$

$\mathcal{F}: L^2(\mathbb{T}) \to l^2(\mathbb{Z})$ isometry

Prop 2. (Riemann-Lebesgue Lemma for $L^1(\mathbb{T})$) $f \in L^1(\mathbb{T}) \Rightarrow \hat{f}_n \to 0$

- if $f, g \in L^2(\mathbb{T})$, $f * g \in C(\mathbb{T})$ $\|f*g\|_\infty \leq \|f\|_2 \|g\|_2$ $\leftarrow$ Young's Inequality.

$$\widehat{(f*g)}_n = \frac{1}{\sqrt{2\pi}} \hat{f}_n \hat{g}_n$$

- Approximation identity $(\varphi_n) \subseteq C(\mathbb{T})$ $(-\pi, \pi)$

1). $\varphi_n(x) \geq 0$ 2). $\int_{\mathbb{T}} \varphi_n(x)\,dx = 1, \forall n$ 3). $\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} \varphi_n(x)\,dx = 0, \quad \forall \delta > 0.$

$\forall f \in C(\mathbb{T})$, $\varphi_n * f \to f$ uniformly as $n \to +\infty$

$$\langle \varphi_n, f \rangle = \int_{-\pi}^{\pi} f(x)\varphi_n(x)\,dx = \underbrace{\int_{|x|\leq\delta} f(x)\varphi_n(x)\,dx}_{I_{n,\delta}} + \underbrace{\int_{|x|>\delta} f(x)\varphi_n(x)\,dx}_{J_{n,\delta}}$$

$$|J_{n,\delta}| \leq \int_{|x|>\delta} \|f(x)\|_\infty \varphi_n(x)\,dx \to 0 \quad \text{as } n \to \infty$$

$$\left(\min_{|x|\leq\delta} f\right) \int_{|x|\leq\delta} \varphi_n(x)\,dx \leq I_{n,\delta} \leq \left(\max_{|x|\leq\delta} f\right) \int_{|x|\leq\delta} \varphi_n(x)\,dx$$

$n \to \infty$ $\min_{|x|\leq\delta} f(x) \leq I_{n,\delta} \leq \max_{|x|\leq\delta} f(x)$

$\delta \to 0$ $\lim_{\delta\to 0} \lim_{n\to\infty} I_{n,\delta} = f(0)$

$\lim_{n\to\infty} \langle \varphi_n, f \rangle = f(0)$ $\lim_{n\to\infty} \varphi_n = \delta$ Dirac Delta measure.

Let $D_N(x) = \frac{1}{2\pi} \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$ $S_N f = D_N * f$

Dirichlet kernel $D_N(x) \not\geq 0 \Rightarrow D_N$ not approximate identity

$S_N f = D_N * f \not\to f$ uniformly

$T_N = \frac{1}{N+1} \sum_{i=0}^{N} S_i$ $\quad T_N f = F_N * f \to f$ uniformly (pointwisely)

$$F_N = \frac{1}{2\pi(N+1)} \left(\frac{\sin(\frac{(N+1)x}{2})}{\sin(\frac{x}{2})} \right)^2$$

- Young's inequality $f, g \in L^1(\mathbb{T})$ then $\|f * g\|_1 \le \|f\|_1 \|g\|_1$

proof: $\int_{\mathbb{T}} \left| \int_{\mathbb{T}} f(x-y) g(y) \, dy \right| dx \le \int_{\mathbb{T}} \int_{\mathbb{T}} |f(x-y) g(y)| \, dy \, dx$

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} |f(x-y)| \, dx \, |g(y)| \, dy = \int_{\mathbb{T}} \int_{\mathbb{T}} |f(z)| \, dz \, |g(y)| \, dy = \|f\|_1 \|g\|_1$$

As a result $* : L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to L^1(\mathbb{T})$, $(f*g)*h = f*(g*h)$.

$L^1(\mathbb{T})$ with $(*)$ is a Banach algebra

$f*g = g*f$, it's a commutative Banach algebra.

$(L^p(\mathbb{T}), *)$ are also Banach algebras

$(L^1(\mathbb{R}), *)$ is Banach algebra, but $(L^p(\mathbb{R}), *)$ not

$(L^\infty(\mathbb{T}), \cdot)$ is Banach algebra, but $(L^p(\mathbb{T}), \cdot)$ not.

$f, g \in L^2(\mathbb{T}) \Rightarrow f \cdot g \in L^1(\mathbb{T})$ $\qquad (L^2(\mathbb{T}), \cdot)$ is not an algebra.

eg: $f_\varepsilon(x) = \frac{1}{x^{\frac{1}{3}}} \mathbb{1}_{[\varepsilon, 1]}$ $\qquad \|f_\varepsilon\|_2 = \left(\int_\varepsilon^1 \frac{1}{x^{\frac{2}{3}}} dx \right)^{\frac{1}{2}} = \left(3 x^{\frac{1}{3}} \Big|_\varepsilon^1 \right)^{\frac{1}{2}}$

$$= \left(3(1 - \varepsilon^{\frac{1}{3}}) \right)^{\frac{1}{2}} < \sqrt{3}$$

$$\|f_\varepsilon g_\varepsilon\|_2 = \left(\int_\varepsilon^1 \frac{1}{x^{\frac{4}{3}}} dx \right)^{\frac{1}{2}} = \sqrt{3} \left(\frac{1}{\sqrt[3]{\varepsilon}} - 1 \right)^{\frac{1}{2}} \to +\infty$$