$(X, \mathcal{M}, \alpha)$ Measure space

Borel measure:

$$\text{open sets} \xrightarrow{\text{generate}} \text{Borel sigma algebra} \longrightarrow \text{Borel measure s.t. Borel sigma algebra measurable.}$$

specific.

Fix σ-algebra $\mathcal{M} = \mathcal{B}$, find measure whose domain could be $\mathcal{B}$.

Borel measure is a collection of measures defined on $\mathcal{B}$.

eg: $X = \mathbb{R}$ $\quad \mu((a,b]) = b-a$, $\lambda^*|_{\mathcal{B}}$ $\Big\}$ $\lambda^*(B) = \mu(B) \quad \forall B \in \mathcal{B}$

f is increasing and $f(a) = f(a+) = \inf_{x>a} f(x) = \lim_{x \downarrow a} f(x)$

Given $F: \mathbb{R} \to \mathbb{R}$ increasing, right-continuous function.

then $\exists$ unique $\mu_F((a,b]) = F(b) - F(a)$

Lebesgue measure

$$\text{set function/outer measure } \lambda^* \xrightarrow{\text{require}} \text{Lebesgue σ-algebra.} \longrightarrow \text{Lebesgue measure.}$$

Collection of E

$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$

$\forall A$

Given λ^*, find largest collection of E (which is Lebesgue-σ-algebra), define $\lambda(E) = \lambda^*(E)$ on $\mathcal{L}$.

Lebesgue measure is a specific measure. (express why Lebesgue measure-zero set not Borel ...)

Remark 1. $\mathcal{B} \subsetneqq \mathcal{L}$

2. Cantor set: uncountable,

$\lambda(C) = 0$, if $\mu((a,b]) = b-a$, Borel set, $\mu(C) = \lambda^*(C) = 0$.

subset of Cantor set must be Lebesgue measure-zero set (complete measure)

may be non-Borel set. (By cantor function)

3. Every Lebesgue measurable set with positive measure contains a non Lebesgue measurable subset.

Measurable function:

$f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is $(\mathcal{M}, \mathcal{N})$ measurable function if $f^{-1}(B) \in \mathcal{M}$ $\forall B \in \mathcal{N}$.

eg: X, Y metric space, every continuous function is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable.

eg: Lebesgue measurable function is $(\mathcal{L}, \mathcal{B})$

Borel measurable function is $(\mathcal{B}, \mathcal{B})$

$$\begin{tikzcd} & (\mathbb{R}, \mathcal{B}) \arrow[dr, "f"] & \\ (\mathbb{R}, \mathcal{L}) \arrow[ur, "g"] \arrow[rr, "f \circ g"'] & & (\mathbb{R}, \mathcal{B}) \end{tikzcd} \qquad \begin{tikzcd} & (\mathbb{R}, \mathcal{B}) \arrow[dr, "f"] & \\ (\mathbb{R}, \mathcal{B}) \arrow[ur, "g"] \arrow[rr, "f \circ g"'] & & (\mathbb{R}, \mathcal{B}) \end{tikzcd}$$

$$(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$$

Composition of Lebesgue measurable function fails.

How about $(\mathcal{L}, \mathcal{L})$-measurable function?

continuous function needs not be $(\mathcal{L}, \mathcal{L})$-measurable (related with Cantor function)

Recall pointwise convergence do not preserve continuity.

$f_n(x) = x^n$ on $[0,1]$ $\qquad \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

But pointwise converge preserves Borel / Lebesgue measurability (larger class of function)

prop: μ is complete iff if f is measurable, $f = g$ μ.a.e then g is measurable.

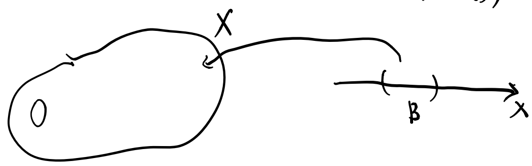
proof: Assume μ is complete.

Since $f(x) = g(x)$ a.e. there exists a measurable set E such that $\mu(E) = 0$ and $f(x) = g(x)$ on E^c

given any $B \in \mathcal{B}$

$$g^{-1}(B) = \left(g^{-1}(B)\cap E\right) \cup \left(g^{-1}(B)\cap E^c\right) = \left(g^{-1}(B)\cap E\right) \cup \left(\underbrace{f^{-1}(B)\setminus E}_{\text{measurable.}}\right)$$

$x \in E^c \Rightarrow f(x) = g(x)$



Since $E \in \Sigma$, $\mu(E) = 0$, μ complete

$g^{-1}(B)\cap E \in \Sigma \implies g^{-1}(B) \in \Sigma$

Assume if f measurable and $f = g$ μ-a.e then g measurable.

Let E be $\mu(E) = 0$, $A \subseteq E$.

$f = 0$, $g = \chi_A$

$f^{-1}(B_0) = X$, $f^{-1}(B_0^c) = \phi$ f measurable.

$f(x) = g(x)$ μ.a.e.

$\Rightarrow g$ measurable $g^{-1}(\{1\}) = A$ measurable.

prop: (X, Σ, μ) complete $\left.\begin{array}{l} f_n : X \to \overline{\mathbb{R}} \text{ measurable} \\ f_n \to f \text{ pointwise } \mu\text{-a.e.} \end{array}\right\} \Rightarrow f$ measurable.