

sequentially compact iff it's complete + totally bounded

Every total bounded metric space (thus compact metric space) is separable.

(idea) A space fails to be separable  $\Rightarrow$  A space fails to be compact.

A metric space is separable if there exists a countable dense set (A is dense wrt B if  $B \subset \bar{A}$ )

(A is dense in B, B is dense in C  $\Rightarrow$  A is dense in C)

eg  $P^{\text{fin}}[0,1]$  is dense in  $P[0,1]$ ;  $P[0,1]$  is dense in  $C[0,1]$

(idea)  $\exists$  dense set  $\xrightarrow{\text{Find}}$  countable dense set.

Countable  $\rightarrow \mathbb{Q}/\mathbb{N}$  / the set of all finite subsets of natural numbers

Uncountable  $\rightarrow [0,1] / \mathbb{R} \times \mathbb{Q}$

the set of all infinite sequences of natural numbers

the set of all subsets of natural numbers  
Cantor set

eg  $l^\infty$  is not separable.

$I$  is a subset of  $\mathbb{N}$

$e_I \in l^\infty$  defined by  $(e_I)_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$

$d_\infty(e_I, e_J) = 1$  if  $I \neq J$

$\mathcal{B} = \{B(e_I, \frac{1}{2}) : I \subseteq \mathbb{N}\}$  is uncountably infinite disjoint open ball in  $l^\infty$

Let  $S$  be dense subset

each ball in  $\mathcal{B}$  must contain at least one  $s \in S$  }  $\Rightarrow$   $S$  must be uncountably infinite.  
these elements are distinct

eg:  $C_0$  is separable

Let  $S$  be the subset of  $C_0$  with rational entries of which at most finitely many nonzero.

$S$  is countable.

$\forall x = (x_1, x_2, \dots) \in C_0 \quad \lim_{n \rightarrow \infty} x_n = 0 \quad \exists N \quad n \geq N \quad |x_n| \leq \epsilon$

$y = (y_1, y_2, y_3, \dots)$   $|x_i - y_i| \leq \epsilon$  ( $\mathbb{Q}$  is dense in  $\mathbb{R}$ )

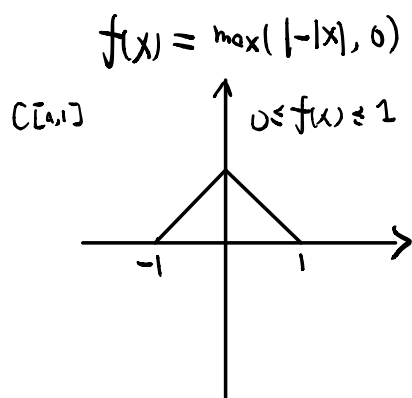
$d_\infty(x, y) \leq \epsilon$  dense.

Remark: separable could be defined as a topological property (countable dense set)

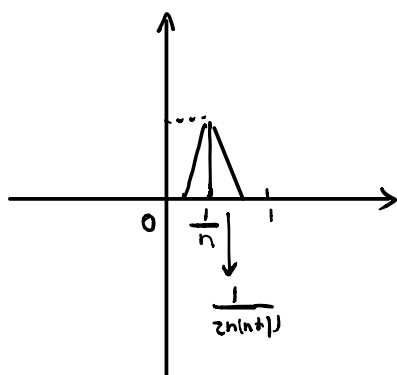
A subspace of a separable space need not be separable

A subspace of a separable metric space is separable.

eg:  $C[0,1]$  is separable, (complete), but not compact.



$$f_n(x) = f(2n(n+1)(x - \frac{1}{n}))$$



$f_n(x)$  disjoint

$$\|f_n - f_m\|_\infty = 1 \quad \text{no convergent subsequence.}$$

$$x = \frac{1}{n} + \frac{1}{2n(n+1)} \leq \frac{1}{n} + \frac{1}{2n^2} = \frac{2n+1}{2n^2} \leq 1$$

$$1 - \frac{1}{2n(n+1)}$$

supported on

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$r_n = \frac{1}{2n(n+1)}$$

$$r_{n+1} < \frac{1}{2n(n+1)}$$

$$\|f_n\| = 1$$

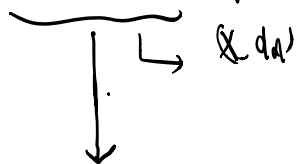
also show unit closed ball in  $C[0,1]$  is not compact.

put  $(f_n)$  into infinite disjoint ball with radius 1

A space fail to be compact either too big at infinity  
or too big locally

(too many directions in which one can go  
away from a given point)

1° Discrete metric space is compact iff it's finite (runs off to infinity)



$$(\mathbb{N}, d(x,y) = |x-y|) \quad (\mathbb{Z}, d(x,y) = |x-y|)$$

(Every set is open) (finite topology)

2° sequence space  $C_0$   $X = (x_1, x_2, \dots)$  with  $\|X\|_\infty = \sup_i |x_i|$   
 $\lim_{n \rightarrow \infty} x_n = 0$

closed unit ball is not compact.

$$e_i \in C_0 \quad \|e_i\|_\infty = 1 \quad \|e_i - e_j\|_\infty = 1 \quad \forall i \neq j$$

$\{e_i\}$  no convergent subsequence.

Remark 1. Completeness  $\rightarrow$  no holes    Compactness  $\rightarrow$  small

[math.uchc.edu/~ntao/preprints/compactness.pdf](http://math.uchc.edu/~ntao/preprints/compactness.pdf).

2. Compactness is sort of topological generalization of finiteness.

Topology: how something behaves on an open set

compact space: there are only finitely many possible behaviors

A finite  $f$  function:  $A \rightarrow \mathbb{R}$  finite choice  $\rightarrow$  max/min/bounded

A compact;  $f$  continuous:  $A \rightarrow \mathbb{R} \rightarrow$  max/min/bounded.