MAT201A Homework 10
Fall 2019
Professor Qinglan Xia
Due Date: Friday, December 6th at 9:00am

1. Let $S = \{e_\alpha : \alpha \in \mathcal{A}\}$ be an orthonormal set in an inner product space X. Show that for any $x, y \in X$,
$$\sum_{\alpha\in\mathcal{A}} |\langle e_\alpha, x\rangle\langle e_\alpha, y\rangle| \leq ||x|| \cdot ||y||.$$

2. Let $\mathcal{H}$ be a separable Hilbert space. Show that every orthonormal set is $\mathcal{H}$ is countable.

3. Let P be an orthogonal projection on a Hilbert space $\mathcal{H}$. Show that

   a.) $I - P$ is also an orthogonal projection.

   b.) $ker(P)$ and $range(P)$ are two closed linear subspaces of $\mathcal{H}$ with $ker(P) = range(P)^\perp$.

4. Let $(X, (\cdot,\cdot))$ be an inner product space. Show that there exists a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ such that X is a dense subset of $\mathcal{H}$ with $\langle x, y\rangle = (x, y)$ for all $x, y \in X$. The Hilbert space $\mathcal{H}$ is called the completion of X.

5. Exercise 6.13 in the textbook, page 146.