08/02

Sec. 10.1

- Recursion Formula: $\{a_n\}$

eg: $a_n = a_{n-1} + k$

$a_n = f(a_{n-1}, a_{n-2}, \cdots a_1)$

eg: $a_n = a_{n-1}^2 - 2$

suppose $\{a_n\}$ converge, suppose limit L.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(a_{n-1}^2 - 2)$$

$$L = L^2 - 2 \Rightarrow L = ? \qquad L^2 - L - 2 = 0$$

$$(L-2)(L+1) = 0$$

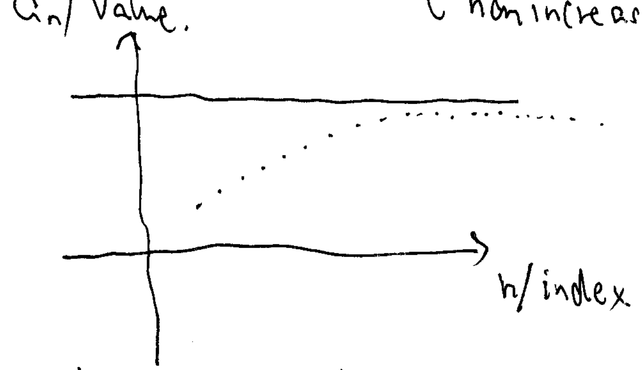
$\{\frac{1}{n}\}_{n=1}^{\infty}$ $\quad M = 1.$

$L = 2$ or $L = -1$. $\Leftarrow$ More Info.

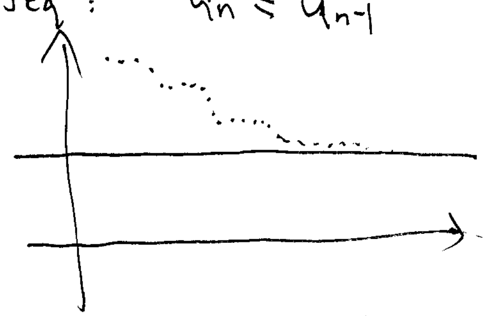
- Bounded Sequence: if $\underline{a_n \leq M}$, (M constant) $\{a_n\}$ is bounded from above.

  \+ if $a_n \geq m$, $\{a_n\}$ is bounded from below.

- Monotonic Sequence: $\begin{cases} \text{nondecreasing seq}: & a_n \geq a_{n-1} \\ \text{nonincreasing seq}: & a_n \leq a_{n-1} \end{cases}$



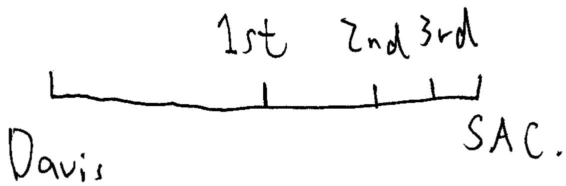
least upper bound



maxium lower bound

10.2 Infinite Series:

$$0.\dot{9} = 0.9 + 0.09 + 0.009 + \cdots = 1$$



$\{a_n\}$, $\underbrace{S_n}_{\text{\# of terms}} = \underbrace{\sum_{k=1}^{n} a_k}$ n-th partial sum.

$\sum a_n \rightarrow$ series. can be regarded as a sequence $\{S_n\}$

$$\{a_n\} = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$$

$$\sum_{n=0}^{\infty} a_n.$$

$$S_{n+1} = \underbrace{\sum_{k=0}^{n}} a_k = \sum_{k=0}^{n} \frac{1}{2^k} = \frac{1\left(1-\frac{1}{2^{n+1}}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^{n+1}}\right)$$

$$\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} 2\left(1-\cancelto{0}{\frac{1}{2^{n+1}}}\right) = 2.$$

$$a_1,\ a_1 r,\ a_1 r^2,\ a_1 r^3,\ \ldots$$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_1 r + \cdots + a_1 r^{n-1}$$

$$r S_n = \quad + a_1 r + a_1 r^2 + \cdots a_1 r^{n-1} + a_1 r^n$$

$$(r-1) S_n = a_1 r^n - a_1$$

first term

index # of terms

if $r \neq 1$, $\underbrace{S_n = \frac{a_1(r^n - 1)}{r-1}}$

if $r = 1$, $a_1, a_1, \ldots a_1$ $\quad \sum_{k=1}^{n} a_k = na_1$

$a_1 \neq 0$, series diverge

$a_1 = 0$, series converge.

Thm: $\{a_k\}_{k=1}^{\infty} = \{a_1 r^{k-1}\}$, $\sum a_k$

$|r|>1$, $\sum a_k$ diverges

$|r|=1$, $\sum a_n$ diverges except for zero trial case

$|r|<1$, $\sum a_k$ converges.

eg. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}} \longrightarrow$ remain.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) \cdots$$

$$= 1 - \cancel{\frac{1}{\infty}}^{\,0} = 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+q)} = \underbrace{\sum \frac{1}{q}\left(\frac{1}{n} - \frac{1}{n+q}\right)} \qquad \frac{1}{n(n+1)(n+2)}$$

Thm $\begin{cases} \text{if } \sum a_n \text{ converges, then } \underline{a_n \to 0} & \text{" prior test"} \\ \text{if } a_n \not\to 0, \text{ then } \sum a_n \text{ diverges.} \end{cases}$

$\sum n^2$, $\sum \frac{n+1}{n}$ $\sum (-1)^{n+1}$ $\sum \frac{-n}{2n+5} \to -\frac{1}{2}$ $\quad \frac{-1}{2+\cancel{\frac{5}{n}}^{\,0}}$

div div div div.

• Rule: Sum / Difference / Constant Multiple Rules.

if $\sum a_n$ converges to A, $\sum b_n$ converges to B.

then $\sum (a_n + b_n) \to A+B$.

eg $\displaystyle\sum \frac{3^{n-1}-1}{6^{n-1}} = \sum\left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}}\right) = \sum\left[\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}\right]$

· Adding / Deleting Terms won't effect convergence or divergence.

16.3. The Integral Test

Thm. A series $\sum a_n$ of non-negative terms converges iff it's partial sums are bounded.

iff $\downarrow$ if and only if

$$\sum a_n \text{ converges} \iff |S_n| \le M.$$

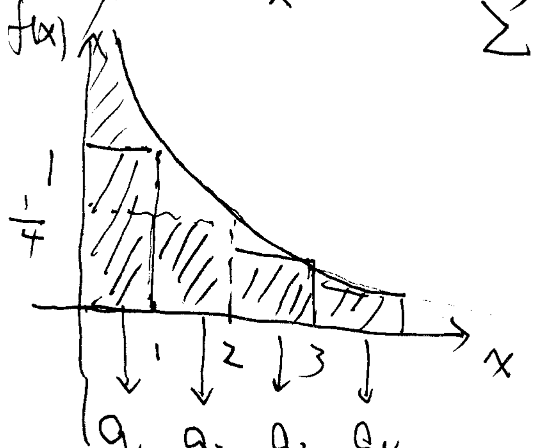
eg: $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$\ge 1 + \frac{1}{2} + \underbrace{2 \cdot \frac{1}{4}}_{\frac{1}{2}} + \underbrace{4 \cdot \frac{1}{8}}_{\frac{1}{2}} \rightarrow \text{diverges.}$$

eg: $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \underbrace{\overset{a_1}{\frac{1}{1}} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}}$$

$f(x) = \frac{1}{x^2}$



$$\sum \frac{1}{n^2} = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2}$$

$$= 1 + \sum_{k=2}^{\infty} f(k)$$

$$\le 1 + \int_1^{\infty} f(x)\,dx$$

$$= 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

$$= 1 + \lim -\frac{1}{x}\Big|^{b} = 1 + \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right)$$

$$\sum_{k=1}^{\infty} f(k) \le \int_0^{\infty} f(x)\,dx$$

$$-\frac{1}{x}\Big|_0^{\infty} \to \infty$$

Thm: (Connection) $\{a_n\}$ positive, $a_n = f(n)$

starting point

$f(x)$ is continuous, positive, decreasing function for $x \geq \overset{\uparrow}{N}$,

$\sum_{N}^{\infty} a_n$ and $\int_N^{\infty} f(x)$ both converge or both diverge

eg: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} p=1 & \text{diverge.} \\ p<1 & \text{diverge} \quad p \neq 1. \\ p>1 & \text{converges.} \uparrow \end{cases}$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \frac{1}{-p+1} \cdot x^{-p+1} \Big|_1^b$$

$$= \lim_{b\to\infty} \frac{b^{-p+1} - 1}{1-p}$$

$p>1$. $\quad \frac{1}{b^{p-1}} \to 0 \qquad \frac{-1}{1-p} = \frac{1}{p-1} \to$ limit.

$p<1$. $\quad b^{1-p} \to \infty \qquad$ diverge.

$p<1 \quad \sum \frac{1}{n^p} \geq \underbrace{\sum \frac{1}{n}}$ diverge

$\Uparrow$

$n^p \leq n.$

eg. $\sum_{n=1}^{\infty} n e^{-n^2} \iff \int_1^{\infty} x \cdot e^{-x^2}\,dx = \int_1^{\infty} \frac{x\,dx}{e^{x^2}}$

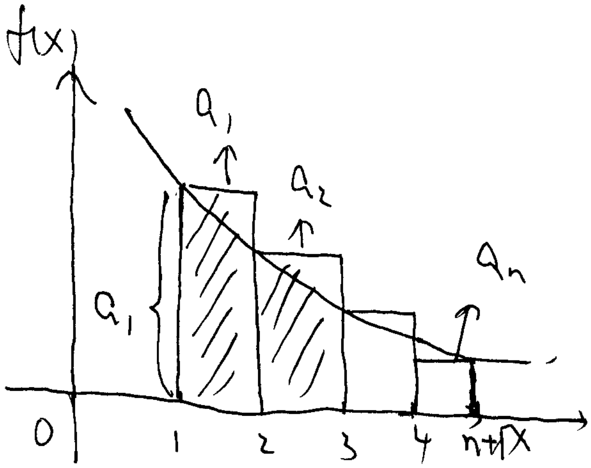
$$\underset{du = 2x\,dx}{\overset{u=x^2}{=\!=}} \frac{1}{2}\int_1^{\infty} \frac{du}{e^u} = \frac{1}{2}\left(-e^{-u}\right)\Big|_1^{\infty} \text{ converges.}$$

Remark: Limits do not have to equal each other.

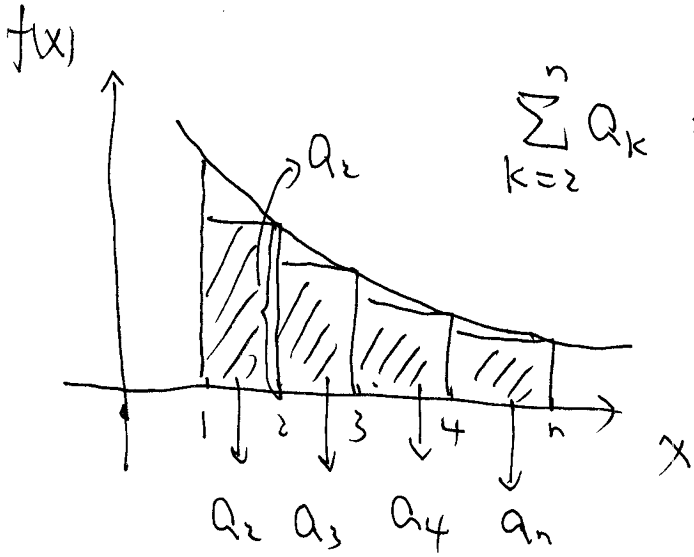
Error Estimation and Remainder.

$$R_n = S - S_n = \sum_{k=n+1}^{\infty} a_k$$

S ↓ Limit, S_n ↓ partial sum



$$\int_1^{n+1} f(x)\,dx \leq \sum_{k=1}^{n} a_k$$



$$\sum_{k=2}^{n} a_k \leq \int_1^n f(x)\,dx$$

$$\int_{n+1}^{\infty} f(x)\,dx \leq \underline{R_n} \leq \int_n^{\infty} f(x)\,dx$$

$$\underline{S_n} + \int_{n+1}^{\infty} f(x)\,dx \leq \underbrace{S_n + R_n}_{\downarrow\ \boxed{S.}} \leq \underline{S_n} + \int_n^{\infty} f(x)\,dx$$