8/30. Chapter 14 Partial Derivatives.

Sec. 14.1 Functions of several variables.

Def. $w = f(\underbrace{x_1, x_2, \cdots x_n}_{\text{independent variables}})$ domain: the space where $x_1, x_2 \cdots x_n$ live.

range: where w lives.

w: dependent variables.

map/function

output. input

eg. $z = f(x,y) = \sqrt{x^2+y^2-1}$

$x^2+y^2-1 \geq 0 \iff \{(x,y) \mid x^2+y^2 \geq 1\}$. $\rightarrow$ domain.

$x=1 \quad y=1 \quad z \in \mathbb{R}$.

eg: $w = \frac{x}{y} \cdot \ln z$ Range is $\mathbb{R}$

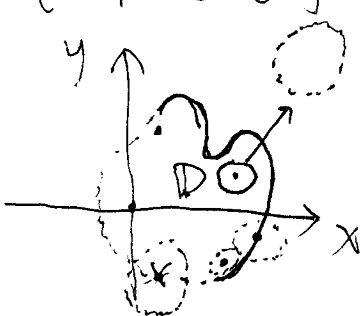
$\{(x,y,z) \mid y \neq 0, z > 0\}$. $\rightarrow$ domain

eg: $z = \frac{1}{x^2+y^2} > 0$ $z > 0$ $z = \frac{1}{2} = \frac{1}{x^2+y^2}$

$x^2+y^2 \neq 0 \iff (x,y) \neq (0,0)$ $\underline{x^2+y^2 = 2}$

$\{(x,y) \mid (x,y) \neq (0,0)\} \rightarrow$ domain

$\{z \mid z > 0\} \rightarrow$ range.



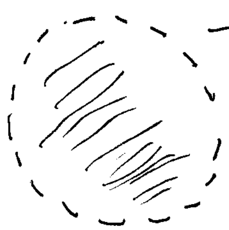
$z = f(x,y)$ In 3D: $w = f(x,y,z)$

interior point of $\mathbb{D}$ if it is the center of some disk with positive radius that lies entirely in $\mathbb{D}$

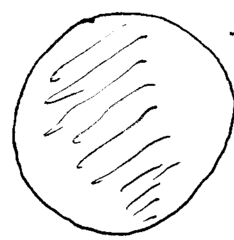
boundary point: if every disk contains point outside of domain and inside of domain as well boundary.

A region is <u>open</u> if it consists ~~of~~ entirely of interior points.

is <u>closed</u> if it consists of all boundary points

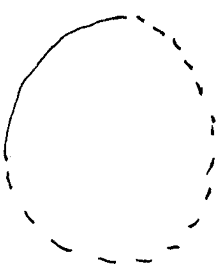


→ open.



→ closed.

empty ϕ : is open and closed.

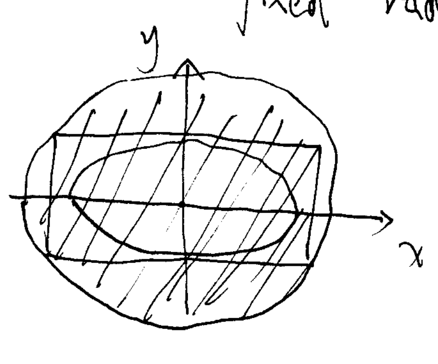


→ not closed.

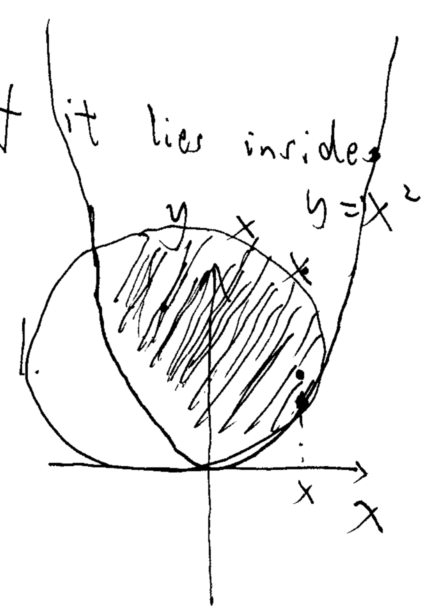
→ not open.

is neither open, nor closed.

Def: A region is <u>bounded</u> if it lies inside a disk of fixed radius.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1.$$



$y - x^2 \geq 0$

$f(x, y) = \sqrt{y - x^2}$.

$y - x^2 \geq 0$

$y = x^2$.

$y \geq x^2$

Region is closed.

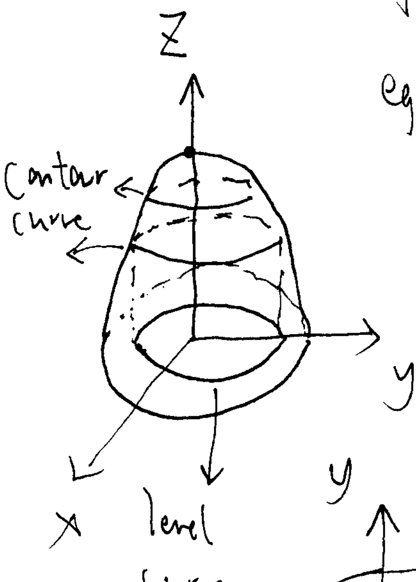
Region is unbounded.

Def: the set of point (x,y) such that $f(x,y)=c$.

is called as a level curve.

Def. $(x, y, f(x,y))$ in space is the graph of $f(x,y)$

Def: $z = f(x,y)$ as surface of $f(x,y)$

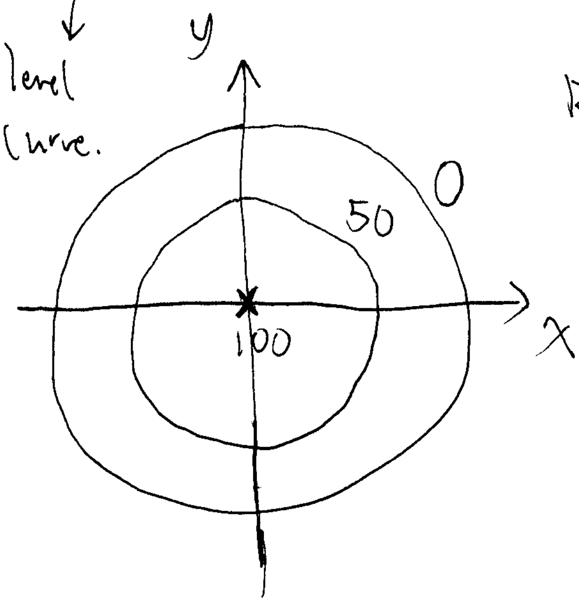


eg: $f(x,y) = 100 - x^2 - y^2$.

$$z = f(x,y) = 0$$

$$x^2+y^2 = 100$$

$$z = 1, 2, 3, \ldots 100.$$



Remark ① level curve is a part of domain

② $f(x,y) = c$

level curve is

$$\{(x,y) \mid f(x,y) = c\}.$$

Def: contour curves: the curve is the intersection of $f(x,y)$ and $z=c$.

$$\{(x,y,c) \mid f(x,y) = c\}.$$

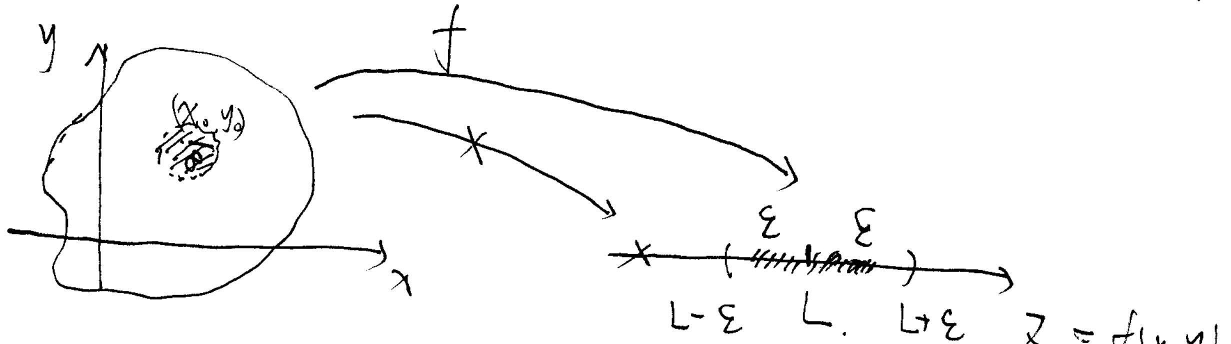
Sec. 14.2 Limit and Continuity

Def: $f(x,y)$ approaches to the limit L as $(x,y) \to (x_0, y_0)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

if $\forall \varepsilon > 0$, there is a δ, such that

when $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$.



Rules: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = M \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L.$

① $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm g(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M \pm L.$

② $\lim_{(x,y)\to(x_0,y_0)} k\, f(x,y) = k \lim_{(x,y)\to(x_0,y_0)} f(x,y)$

③ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \cdot g(x,y) = ML.$

④ $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{M}{L}$ if $L \neq 0$

⑤ $\lim_{(x,y)\to(x_0,y_0)} f(x,y)^n = M^n$ ⑥ $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{M}.$

eg: $\lim\limits_{(x,y)\to(0,1)} \dfrac{x-xy+3}{x^2y+5xy-y^3} = \dfrac{3}{-1} = -3.$

eg. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}} \sim \dfrac{0}{0}$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} = 0$$

eg: $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = ?\ 0 \qquad \dfrac{0}{0}$

$$\sim \frac{4x^3}{x^2+x^2} = \frac{4x^3}{2x^2} = \underline{2x} \to 0$$

Prove: $\forall \varepsilon > 0$, (if) there is $\delta \overset{=\frac{\varepsilon}{4}}{,}$ such that

when $0 < \sqrt{x^2+y^2} < \boxed{\delta}$, then we have

$$\left| \frac{4xy^2}{x^2+y^2} - 0 \right| < \varepsilon. \qquad |x|$$

$$\sqrt{x^2+y^2} < \delta \implies |x| < \delta \quad \leftarrow \quad \overset{|x|}{\overset{\shortparallel}{\sqrt{x^2}}} \leq \sqrt{x^2+y^2} < \delta$$

$|y| < \delta.$ $\boxed{\text{"}\leq\text{"}}$

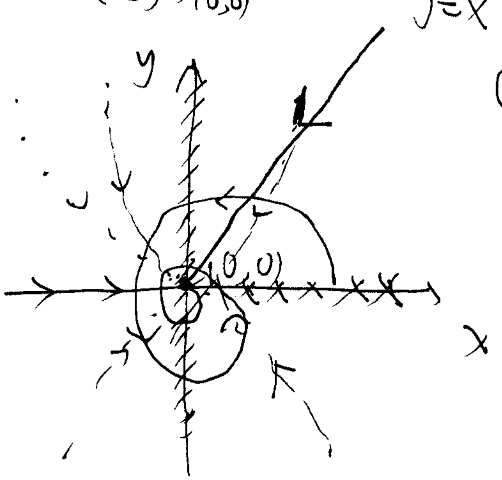
$x^2 \leq x^2+y^2$

$\underline{y^2 \leq x^2+y^2.}$

$$\left| \frac{4xy^2}{x^2+y^2} \right| = \frac{4y^2}{x^2+y^2}|x| \leq 4|x| < 4\delta \qquad < \varepsilon.$$

$\boxed{\delta \leq \frac{\varepsilon}{4}}$ $\boxed{///}$

eg. $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x} = ?$ $\qquad \frac{y}{x}$ domain $x \neq 0$



① pick points on x-axis to approach (0,0)

$$y = 0$$

$$\frac{y}{x} = \frac{0}{x} = 0$$

$$0, 0, 0, 0, \ldots$$

$$\lim_{(x,y)\to(0,0)} \frac{y}{x} = 0 \quad ?$$

③ $y = \frac{1}{2}x$

$-x$

② pick points on $y = x$

$$\frac{y}{x} = 1$$

$$1, 1, 1, 1 \ldots$$

$$\lim_{(x,y)\to(0,0)} \frac{y}{x} = 1 \quad ?$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{y}{x} \text{ does not exist.}$$

Two-Path Test: If a $f(x,y)$ has different limit along two different paths, then the limit at (x_0, y_0) does not exist.

$f(x,y) = \frac{2x^2 y}{x^4 + y^2}$ $\qquad \lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$ $\qquad \frac{2x^2 y}{x^4 + y^2} = \frac{2x^2 m x^2}{x^4 + m^2 x^4}$

$\frac{x^2}{y}$ $\qquad \frac{\cancel{x^2}}{\cancel{x^2} y}$ $\qquad \frac{\cancel{x^4}}{\cancel{x^3}}\ x$

same-order strategy: $y = x^2 \rightarrow y = m x^2$ $\qquad = \frac{2m\cancel{x^4}}{\cancel{x^4} + m^2 \cancel{x^4}}$

$2m$

Follow $\boxed{y = mx^2}$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{2m}{1+m^2} = \underline{\frac{2m}{1+m^2}}$$

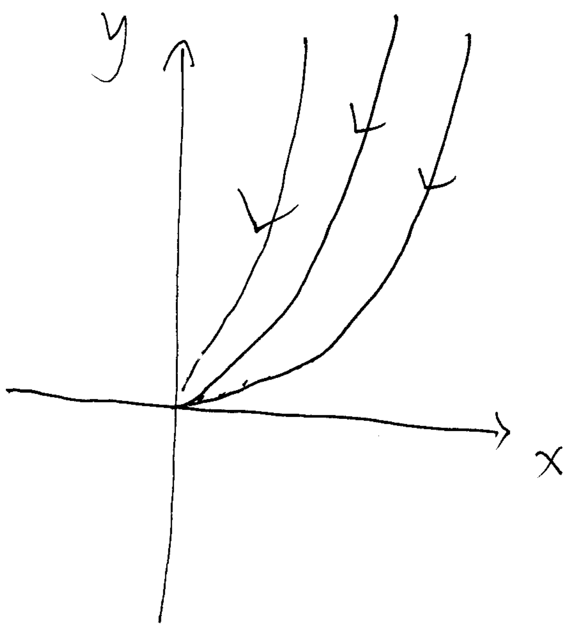
$m = 1$

$$\frac{2}{1+1} = \boxed{1}$$

$m = -1$

$$\frac{-2}{1+1} = \boxed{-1}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$ does not exist



$y = \boxed{\cancel{x}}$

$y = x^2$

$y = \sqrt{x}$

$$\underline{\frac{2m}{m} = 2}$$