MAT 201B Homework 2
Winter 2020
Professor Qinglan Xia
Due Date: Wednesday, January 22th at 9:00am

1. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

   (a.) Find a nonnegative continuous function f on $\mathbb{R}$ so that f is Lebesgue integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

   (b). However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{x\to\infty} f(x) = 0$.

2. Suppose that f is Lebesgue integrable on $\mathbb{R}$. Show that

$$F(x) := \int_{-\infty}^{x} f(t)dt$$

   is uniformly continuous on $\mathbb{R}$.

3. Suppose that f is a real-valued Lebesgue integrable on $\mathbb{R}^n$. If $\int_E f(x)dx \geq 0$ for every measuable set E, then $f(x) \geq 0$ for a.e. x. If $\int_E f(x)dx = 0$ for every measuable set E, then $f(x) = 0$ for a.e. x.

4. Let (f_n) be a sequence of non-negative μ-integrable functions on X with f beings its pointwise limit. If

$$\lim_{k\to\infty} \int_X f_k(x)d\mu = \int_X f(x)d\mu,$$

   show that

$$\lim_{k\to\infty} \int_E f_k(x)d\mu = \int_E f(x)d\mu,$$

   for every measuable subset E of X.

5. Let (X, Σ, μ) be a measure sapce, and $\varphi : X \to [0, \infty]$ be $\Sigma-$measurable. Show that

   (a.) For any $a > 0$,

$$\mu(\{x \in X \ : \ \varphi(x) \geq a\}) \leq \frac{1}{a} \int_X \varphi d\mu.$$

   (b.) If φ is $\mu-$integrable, then $\varphi(x)$ is finite μ-a.e. x.

   (c.) If $\int_X \varphi d\mu = 0$, then $\varphi(x) = 0$ for μ-a.e. x.