10.7

2. Radius $= 1$

Interval $= (-6, -4)$

when $x \in (-6, -4)$

converge absolutely.

5*. when $\left|\left(\frac{x-2}{10}\right)\right| < 1$,

it converges.

$\iff -8 < x < 12$

when $x \in (-8, 12)$

converge absolutely.

12*. Consider $\cancel{\lim_{n\to\infty}\left(\frac{3^n x^n}{n!}\right)^{\frac{1}{n}}}$

$\cancel{= \lim_{n\to\infty} \frac{3x}{\sqrt[n]{n!}}}$

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}\cdot x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}} = \lim_{n\to\infty} \frac{3x}{n+1} = 0$$

It means Radius $\to \infty$.

when $x \in \mathbb{R}$.

The series converges absolutely.

15. when $|x| > 1$. $\lim_{n\to\infty} \frac{x^n}{\sqrt{n^2+5}} = \infty$.

diverge!

when $|x| = 1$ $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$ diverges

$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ converges conditionally.

when $|x| < 1$.

$\sum_{n=0}^{\infty} \left|\frac{x^n}{\sqrt{n^2+3}}\right| < \sum_{n=0}^{\infty} |x|^n < \infty$.

converges absolutely.

27. when $|x+2| > 2$ It diverges

when $|x+2| = 2$ $x = 0$ or -4

$\Rightarrow x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges Conditionally.

$x = -4$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

when $|x+2| < 2$. $\Rightarrow -4 < x < 0$

$\sum \frac{(-1)^{n+1}\cdot(x+2)^n}{n \cdot 2^n}$ converges abs.

37. consider

$$\lim \frac{\frac{n!}{3\cdot 6\cdots 3n}}{\frac{(n+1)!}{3\cdot 6\cdots 3(n+1)}} = \lim_{n\to\infty} \frac{3(n+1)}{n+1} = 3.$$

$r=3.$

56 a). $\frac{d}{dx}\cdot e^x = 0+1+x+\frac{x^2}{2!}+\cdots = e^x$

b). $\int e^x\cdot dx = x+\frac{x^2}{2!}+\cdots+C$

c). $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$

$$e^x\cdot e^{-x} = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}\cdots\right)$$
$$\times\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\frac{x^6}{6!}\cdots\right)$$

$= 1$

57: $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}$

Notice $\ln|\sec x| = \int \tan x\cdot dx = \frac{x^2}{2}+\frac{x^4}{12}+\frac{2x^6}{90}+\frac{17\cdot x^8}{8\times 315}+\frac{62\cdot x^{10}}{28350}+\cdots$

It converges when $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

b). $\sec^2 x = (\sec x)(\sec x)$

$$= \left(1+\frac{x^2}{2}+\frac{5x^4}{24}+\cdots\right)\left(1+\frac{x^2}{2}+\frac{5x^4}{24}+\cdots\right)$$

$$= 1+x^2+\frac{2x^4}{3}+\frac{17x^6}{45}+\cdots$$

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\sec^2 x = \frac{d}{dx}\tan x = \frac{d}{dx}\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots\right)$$

$$= 1+x^2+\frac{2x^4}{3}+\frac{17x^6}{45}+\cdots$$

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

10.8.

1. $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\cdot f^{(k)}(0)$

$$= 1+2x+\frac{4x^2}{2}+\frac{8x^3}{6}+\cdots$$

$$= 1+2x+2x^2+\frac{4x^3}{3}+\cdots$$

7. $f(a) = \frac{\sqrt{2}}{2}$ $\quad f^{(1)}(a) = \cos a = \frac{\sqrt{2}}{2}$

$f^{(2)}(a) = -\sin a = -\frac{\sqrt{2}}{2}$ $\quad f^{(3)}(a) = -\cos a = -\frac{\sqrt{2}}{2}$

$$f(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\left(x-\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\cdot\left(x-\frac{\sqrt{2}}{2}\right)^2\Big/2 + \left(-\frac{\sqrt{2}}{2}\right)\cdot\left(x-\frac{\sqrt{2}}{2}\right)^3\Big/6 + \cdots$$

14. $f(x) = \dfrac{2+x}{1-x} = -1 + \dfrac{3}{1-x}$

$= -1 + 3(1 + x + x^2 + \cdots)$

$= 2 + 3x + 3x^2 + 3x^3 + \cdots$

19*: $\cosh x = \dfrac{e^x + e^{-x}}{2}$

since $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$

$e^{-x} = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots$

$\cosh x = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$

26. $f(a) = -3 - 1 - 2 + 1 - 2$

$= -7$

$f'(a) = 15x^4 - 4x^3 + 6x^2 + 2x \big|_{x=-1}$

$= 15 + 4 + 6 - 2$

$= 23$

$f''(a) = 20x^3 - 12x^2 + 12x + 2 \big|_{x=-1}$

$= -20 - 12 - 12 + 2$

$= -42$

$f'''(a) = 60x^2 - 24x + 12 \big|_{x=-1}$

$= 60 + 24 + 12$

$= 96$

$$f^{(4)}(a) = 120x - 24\Big|_{x=-1}$$

$$= -120 - 24$$

$$= -144$$

$$f^{(5)}(a) = 120$$

$$\therefore f(x) = -7 + 23 \cdot \frac{(x+1)}{1} + (-42) \cdot \frac{(x+1)^2}{2!} + 96 \cdot \frac{(x+1)^3}{3!} - 144 \cdot \frac{(x+1)^4}{4!}$$

$$+ 120 \cdot \frac{(x+1)^5}{5!}$$

33. $$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\frac{2}{1-x} = 2 + 2x + 2x^2 + 2x^3.$$

$$\Rightarrow f(x) = -1 - 2x - \frac{x^2}{2} - 2x^2 + \cdots$$

$$= -1 - 2x - \frac{5}{2}x^2 + \cdots$$

when $x \in (-1, 1)$, the series converges absly.

10.9.

11. $x \cdot e^x = x \cdot (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots$

12. $x^2 \cdot \sin x = x^2 \cdot (x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots)$

$= (x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} + \cdots)$

29. $e^x \cdot \sin x = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots)(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots)$

$= x + x^2 + (\frac{1}{2} - \frac{1}{3!}) x^3 + \cdots$

$= x + x^2 + \frac{1}{3} x^3 + \cdots$

37. $\sin x = x - \frac{x^3}{6} + \frac{\xi^5}{120}$.

$\Rightarrow \left| \sin x - (x - \frac{x^3}{6}) \right| = \frac{\xi^5}{120} < 5 \cdot 10^{-4}$

$\Rightarrow \xi < 0.5697$

$\Rightarrow$ when x takes any values in $(-0.5697, 0.5697)$, we can approximate $\sin x$ with $x - \frac{x^3}{6}$.

41.

$$\left|e^x - 1 - x - \frac{x^2}{2}\right| = \frac{e^c}{6}\cdot x^3 \qquad (|c|<|x|)$$

$$\frac{e^c}{6}\cdot x^3 < \frac{e^x}{6}\cdot x^3 < \frac{e^{0.1}}{6}\cdot(0.1)^3 \simeq 1.8\times 10^{-4}.$$

43. $$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2}\left(1-\left(1-\frac{4x^2}{2}+\frac{16x^4}{4!}-\frac{2^6x^6}{6!}+\dots\right)\right)$$

$$= \frac{4x^2}{4} - \frac{16x^4}{2\cdot 4!} + \frac{2^6x^6}{2\cdot 6!} - \dots \quad = x^2 - \frac{x^4}{\cancel{3}} + \frac{2}{45}x^6$$

$$2\cdot\sin x\cdot\cos x = 2\cdot\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\dots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right)$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \dots$$

On the other hand: $$(\sin^2 x)' = 2\sin x\cdot\cos x = \cancel{2\sin}\ \sin 2x$$

$$= 2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{5!}$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5$$

44.

$$\cos^2 x = \cos 2x + \sin^2 x$$

$$= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4}{45}x^6 + \cdots + x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \cdots$$

$$\cancel{= 1 - x^2 - \frac{2}{45}x^6}$$

$$= 1 - x^2 + \frac{x^4}{3} - \frac{2}{45}x^6 + \cdots$$

47. If $f'' \le 0$. then $f(x) = f(a) + \frac{f''(c_2)}{2}(x-a)^2$.

the coefficient in front of $(x-a)^2$ is non-positive

$\Rightarrow$ $f(x) \le f(a)$ locally.

$\Rightarrow$ $x = a$ is local maximum.

10.10. $f(x) = (1-x)^{-\frac{1}{2}}$.

$f(0) = 1$

$f'(0) = \frac{1}{2} \cdot (1-x)^{-\frac{3}{2}}$

$= \frac{1}{2}$

$f''(0) = \frac{1}{2} \cdot \frac{3}{2}$

$f'''(0) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}$.

$$\Rightarrow f(x) = 1 + \frac{1}{2} \cdot x + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2!} \cdot x^2 + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{1}{3!} x^3 + \cdots$$

11. $(1+x)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$

15. $\int_0^{0.2} \sin x^2 \cdot dx = \int_0^{0.2} \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{5!} - \cdots \right) dx$

$$= \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \cdots \Big|_0^{0.2}.$$

$$= \frac{(0.2)^3}{3} - \frac{(0.2)^7}{7 \cdot 3!}$$

$$\frac{(0.2)^3}{3} = 2.66 \times 10^{-3}$$

$$\frac{(0.2)^7}{7 \cdot 3!} = 3.05 \times 10^{-7}$$

$$\Rightarrow \int_0^{0.2} \sin x^2 \cdot dx = 2.66 \times 10^{-3}$$

with error $< 3.05 \times 10^{-7}$.

23. ~~cot~~ $\int_0^1 \left(\cos t^2 - \left(1 - \frac{t^4}{2} + \frac{t^8}{4!}\right)\right) \cdot dt.$

$$\approx \int_0^1 R(x) \cdot dt$$

$$= \int_0^1 \cos \xi^2 \frac{t^{12}}{6!} \cdot dt \qquad |\xi| < |t|$$

$$\leq \int_0^1 \frac{t^{12}}{6!} \cdot dt = 1.06 \times 10^{-4}$$

26.
$$F(x) = \int_0^x t^2 \cdot \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{24} - \frac{t^{10}}{120} + \cdots\right) \cdot dt.$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!} + \int_0^x t^2 \cdot R(t) \cdot dt$$

$$\int_0^x t^2 R(t) \cdot dt = \frac{1}{13} \cdot \frac{1}{5!} \approx 0.00064.$$

$$\Rightarrow F(x) \approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!}$$

29 we have $e^x - (1+x) = \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$$\Rightarrow \lim_{x \to 0} \frac{e^x - (1+x)}{x^2} = \lim_{x \to 0} \frac{1}{2} + \frac{x}{6} + \cdots$$

$$= \frac{1}{2}.$$

30. $$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2} + \cdots - \left(1 - x + \frac{x^2}{2} - \cdots\right)}{x}$$

$$= 2.$$

58. It suffices to show

$$(1+x)\cdot f'(x) = m\cdot f(x)$$

$$\Leftrightarrow (1+x)\cdot \sum_{k=1}^{\infty} C_m^k\cdot k\cdot x^{k-1} = m + \sum_{k=1}^{\infty} m\cdot C_m^k\cdot x^k.$$

$$LHS = \sum_{k=1}^{\infty} C_m^k\cdot k\cdot x^{k-1} + \sum_{k=1}^{\infty} C_m^k\cdot k\cdot x^k$$

$$= m + \sum_{k=1}^{\infty}\left(C_m^{k+1}\cdot(k+1)\cdot x^k + C_m^k\cdot k\cdot x^k\right)$$

$$= m + \sum_{k=1}^{\infty}\left(C_m^{k+1}\cdot(k+1) + C_m^k\cdot k\right)\cdot x^k$$

$$= m + \sum_{k=1}^{\infty}\left(\frac{(m-k)\,m!}{k!\,(m-k)!} + \frac{k\cdot m!}{k!\,(m-k)!}\right)\cdot x^k$$

$$= m + \sum_{k=1}^{\infty} m\cdot C_m^k\cdot x^k$$

$$= RHS.$$

b). $$g'(x) = -m\,(1+x)^{-(m+1)}\cdot f(x) + f'(x)\,(1+x)^{-m}$$

$$= -m\,(1+x)^{-(m+1)}\,f(x) + \frac{m\cdot f(x)}{1+x}\,(1+x)^{-m}$$

$$= 0.$$

c). Since $g'(x) = 0$. on. $x\in(-1,1)$ we have $g(x) = g(0) = \text{Const}$ on $x\in(-1,1)$

$g(0) = f(0) = 1.$

$\Rightarrow (1+x)^{-m}\cdot f(x) = 1 \quad \Rightarrow f(x) = (1+x)^m$ on $x\in(-1,1)$

61. Notice $\frac{1}{(1+x)^2} = \left(-\frac{1}{1+x}\right)'$

$$-\frac{1}{1+x} = -1 + x - x^2 + x^3 - \cdots \qquad x \in (-1, 1)$$

$$\Rightarrow \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \cdots \qquad x \in (-1, 1)$$