Problem 1.

Proof. Since we have

$$\left(\int_0^1 |f(x)|^2\right)^{1/2} \leqslant \left(\int_0^1 (1+x)|f(x)|^2\right)^{1/2} \leqslant \sqrt{2}\left(\int_0^1 |f(x)|^2\right)^{1/2},$$

$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Problem 2.

Proof. Assume $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then the metrics they induced are equivalent as well. By Problem 5 in Homework 5, they induce the identical topology.

Let $a_N = (x_n)_{i=1}^{\infty}$ with $x_i = \frac{1}{i}$ for $i = 1, 2, \cdots, N$ and $x_i = 0$ for $i > N$. For example, $a_1 = (1\ 0\ 0\ \cdots)$, $a_2 = (1\ \frac{1}{2}\ 0\ 0\ \cdots)$, $a_3 = (1\ \frac{1}{2}\ \frac{1}{3}\ 0\ 0\ \cdots)$ and etc...

It is easy to see (a_N) diverges with respect to $\|\cdot\|_1$ but converges with respect to $\|\cdot\|_2$, which yields that they must not have the identical topology. □

Problem 3.

Proof.

1. To show $\|\cdot\|_a$ is a norm:

   - **Well-defined:** For any $f \in C_b([0,\infty))$, there exists a nonnegative constant M, such that $|f(x)| \leqslant M$ for all $x \in [0,\infty)$. Thus

   $$\|f\|_a = \left(\int_0^\infty e^{-ax}|f(x)|^2\right)^{1/2} \leqslant \left(M^2 \int_0^\infty e^{-ax}\mathrm{d}x\right)^{1/2} = \frac{M}{\sqrt{a}} < \infty.$$

   - **Non-negative:** Clearly, $\|\cdot\|_a$ is non-negative, and $\|f\|_a = 0$ if and only if $f \equiv 0$ comes from the fact that e^{-ax} is non-negative and $f(x)$ is continuous.
   - **Homogeneous:** By scaling argument,

   $$\|\lambda f\|_a = \left(\int_0^\infty e^{-ax}|\lambda f(x)|^2\right)^{1/2} = |\lambda|\left(\int_0^\infty e^{-ax}|f(x)|^2\right)^{1/2} = |\lambda|\|f\|_a.$$

   - **Triangle inequality:** By the fact that the normal L^2 norm $\|\cdot\|_2$ is a norm, or by Minkowski inequality,

   $$\|f+g\|_a = \left\|e^{-ax/2}(f+g)\right\|_2 \leqslant \left\|e^{-ax/2}f\right\|_2 + \left\|e^{-ax/2}g\right\|_2 = \|f\|_a + \|g\|_a.$$

2. Since $a > b > 0$, then $e^{-ax} \leqslant e^{-bx}$, which implies that $\|f\|_a \leqslant \|f\|_b$ for any $f \in C_b([0,\infty])$. We now show that $\|\cdot\|_b$ cannot be bounded from above by constant multiple of $\|\cdot\|_a$.

   Let's define

   $$f_N(x) = \begin{cases} e^{bx/2} & x \in [0,N]; \\ e^{bN/2} & x \in (N,\infty). \end{cases}$$

Clearly $1 \leqslant f_N(x) \leqslant e^{bN/2}$, thus $f_N \in C_b([0,\infty))$.

On one hand,

$$\|f_N\|_a = \left(\int_0^\infty e^{-ax}|f(x)|^2 \mathrm{d}x\right)^{1/2} \leqslant \left(\int_0^\infty e^{(b-a)x} \mathrm{d}x\right)^{1/2} = \frac{1}{\sqrt{a-b}} < \infty \text{ for all } N.$$

On the other hand,

$$\|f_N\|_b = \left(\int_0^N 1 \mathrm{d}x + \int_N^\infty e^{bN-bx} \mathrm{d}x\right)^{1/2} \geqslant N.$$

As $N \to \infty$, we can see $\|\cdot\|_a$ and $\|\cdot\|_b$ are not equivalent.

□

Problem 4.

Proof. For a real normed linear space $(X, \|\cdot\|)$, given $x \in X$ and $(e_i)_{i=1}^n$ in X. Without loss of generality, we may assume $(e_i)_{i=1}^n$ are linearly independent. Otherwise, if there is at most k vectors that are linearly independent, up to re-arrangement, we pick $a \in \mathbb{R}^n$ with $a_i = 0$ for all $k+1 \leqslant i \leqslant n$. The minimizing problem in $\mathbb{R}^n$ is equivalent to corresponding minimizing problem in $\mathbb{R}^k$, given k linearly independent vectors.

Let's define $T : \mathbb{R}^n \mapsto \mathbb{R}$ by

$$T(a) = \left\|x - \sum_{i=1}^n a_i e_i\right\| \qquad \text{for all } a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n.$$

Since all norms are equivalent on $\mathbb{R}^n$,

$$\begin{aligned} |T(a) - T(b)| &= \left|\left\|x - \sum_{i=1}^n a_i e_i\right\| - \left\|x - \sum_{i=1}^n b_i e_i\right\|\right| \\ &\leqslant \left\|(x - \sum_{i=1}^n a_i e_i) - (x - \sum_{i=1}^n b_i e_i)\right\| \\ &= \left\|\sum_{i=1}^n (a_i - b_i) e_i\right\| \\ &\leqslant \sum_{i=1}^n |a_i - b_i| \|e_i\| \\ &\leqslant C\|a-b\| \sup_i \|e_i\|. \end{aligned}$$

We have show that T is Lipschitz-continuous, thus T is lower semin-continuous. Furthermore,

$$T(a) = \left\|\sum_{i=1}^n a_i e_i - x\right\| \geqslant \left|\left\|\sum_{i=1}^n a_i e_i\right\| - \|x\|\right| \geqslant \left\|\sum_{i=1}^n a_i e_i\right\| - \|x\|.$$

Once we show that $f(a) = \|\sum_{i=1}^n a_i e_i\|$ is a norm on $\mathbb{R}^n$, the above shows $T(a)$ is coercive. And by Problem 4 in Homework 2, the minimum can be achieved.

f is non-negative, and $f = 0$ if and only if $a = 0$ by linear independence.

$$f(\lambda a) = \left\|\sum_{i=1}^n \lambda a_i e_i\right\| = |\lambda| \left\|\sum_{i=1}^n a_i e_i\right\| = |\lambda| f(a).$$

$$f(a) + f(b) = \left\|\sum_{i=1}^n \lambda a_i e_i\right\| + \left\|\sum_{i=1}^n \lambda b_i e_i\right\| \geqslant \left\|\sum_{i=1}^n \lambda (a_i + b_i) e_i\right\| = f(a+b).$$

$\square$

Problem 5.

Proof. 1. We first prove $\|T\| = 1$.

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |x \int_0^x f(t)\mathrm{d}t| \leqslant \sup_{x\in[0,1]} |x| \int_0^x |f(t)|\mathrm{d}t \leqslant \int_0^1 |f(t)|\mathrm{d}t \leqslant \|f\|_\infty,$$

which yields that

$$\|T\| \leqslant 1.$$

For $g \equiv 1$, $\|g\|_\infty = 1$, thus

$$\|T\| \geqslant \|Tg\|_\infty = \sup_{x\in[0,1]} |x \int_0^x 1\mathrm{d}t| = \sup_{x\in[0,1]} x^2 = 1.$$

2. To show $T^{-1} : \mathrm{ran}(T) \mapsto X$ exists. We first show that T is one-to-one.

Assume there exists $f, g \in C([0,1])$ with $Tf \equiv Tg$ but $f \not\equiv g$. Let x_0 be such that $f(x_0) \neq g(x_0)$. Since $f - g$ is continuous, there exists a $\delta > 0$ such that $f(x) - g(x) \neq 0$ for any $x \in [x_0, x_0 + \delta]$. Without loss of generality, we may assume $f(x) - g(x) > 0$ on $[x_0, x_0 + \delta]$.

$$Tf(x_0) = Tg(x_0) \Longrightarrow x_0 \int_0^{x_0} f(t) - g(t)\mathrm{d}t = 0;$$

$$Tf(x_0 + \delta) = Tg(x_0 + \delta) \Longrightarrow (x_0 + \delta) \int_0^{x_0+\delta} f(t) - g(t)\mathrm{d}t = 0.$$

Since

$$\begin{aligned} 0 &= (x_0 + \delta) \int_0^{x_0+\delta} f(t) - g(t)\mathrm{d}t \\ &= (x_0 + \delta) \int_0^{x_0} f(t) - g(t)\mathrm{d}t + (x_0 + \delta) \int_{x_0}^{x_0+\delta} f(t) - g(t)\mathrm{d}t \\ &= 0 + (x_0 + \delta) \int_{x_0}^{x_0+\delta} f(t) - g(t)\mathrm{d}t \\ &> 0, \end{aligned}$$

contradiction! Hence T is one-to-one, and we are able to define T^{-1} from $\operatorname{ran}(T)$ to X.

Then we show T^{-1} is not bounded. For any $Tf \in \operatorname{ran}(T)$, we define T^{-1} by $T^{-1}(Tf) = f$. To show T^{-1} is unbounded, we might want to find f_n such that $\|Tf_n\|_\infty$ is bounded while $\|f_n\|_\infty \to \infty$.

Let $f_n(x) = nx^{n-1}$ on $[0,1]$, then $\|f_n\|_\infty = n$. But

$$\|Tf_n\|_\infty = \sup_{x\in[0,1]} |Tf(x)| = \sup_{x\in[0,1]} |x \int_0^x nt^{n-1} \mathrm{d}t| = \sup_{x\in[0,1]} |xx^n| = \sup_{x\in[0,1]} |x^{n+1}| = 1.$$

$\square$