Problem 1. Dirac delta functional.

Proof.

$$\|T\|_\infty = \sup_{\|f\|_\infty \neq 0} \frac{\|\delta(f)\|}{\|f\|_\infty} = \sup_{\|f\|_\infty \neq 0} \frac{|f(0)|}{\|f\|_\infty} \leqslant 1;$$
$$\|T\|_1 = \sup_{\|f\|_1 \neq 0} \frac{\|\delta(f)\|}{\|f\|_1} = \sup_{\|f\|_1 \neq 0} \frac{|f(0)|}{\|f\|_1} \to \infty,$$

since for $f_n(x) = 1 - nx$ on $[0, \frac{1}{n}]$ and $f_n(x) = 0$ otherwise, $\|f_n\|_1 = \frac{1}{2n} \to 0$ while $f_n(0) = 1$.

Let $f = 1$ on $[0, 1]$, f is continuous function and $\|T\|_\infty$ can achieve 1. □

Problem 2. Schauder basis

Proof. For our convenience, I adopt slightly different notations from the book.

$g_{0,0}(x) = 1$ and $g_{0,1}(x) = x$.

For $k \geqslant 1$ and $1 \leqslant m \leqslant 2^{k-1}$, we define $g_{k,m}(x) = 1 - 2^k|x - \frac{2m-1}{2^k}|$ on $\left[\frac{m-1}{2^{k-1}}, \frac{m}{2^{k-1}}\right]$.

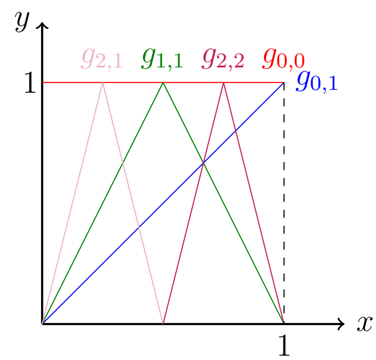


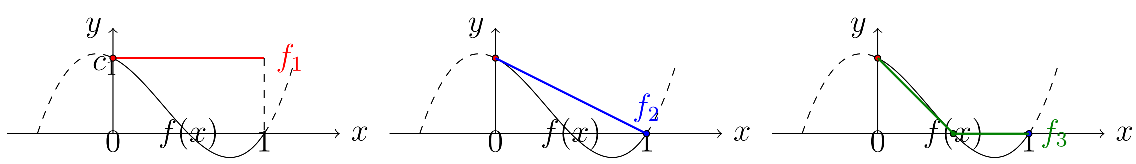
Figure 1: Basis function

Given a $f \in C[0, 1]$, we define f_n recursively.

$f_1(x) = c_1 g_{0,0}(x)$ where $c_1 = f(0)$ such that $f_1(0) = f(0)$, due to the fact $g_{0,0}(0) = 1$.

$f_2(x) = f_1(x) + c_2 g_{0,1}(x)$ where $c_2 = f(1) - f_1(1)$ such that $f_2(1) = f(1)$, due to the fact that $g_{0,1}(1) = 1$. Note that $g_{0,1}(0) = 0$, we have $f_2(0) = f(0)$ as well.

$f_3(x) = f_2(x) + c_3 g_{1,1}(x)$ where $c_3 = f(\frac{1}{2}) - f_2(\frac{1}{2})$ such that $f_3(\frac{1}{2}) = f(\frac{1}{2})$, due to the fact that $g_{1,1}(\frac{1}{2}) = 1$. Note that $g_{1,1}(0) = g_{1,1}(1) = 0$, we have $f_3(0) = f(0)$ and $f_3(1) = f(1)$ as well.



(a) $f_1(x)$ approximation (b) $f_2(x)$ approximation (c) $f_3(x)$ approximation

$f_4(x) = f_3(x) + c_4 g_{2,1}(x)$ where c_4 be such that $f_4(\frac{1}{4}) = f(\frac{1}{4})$. In the meanwhile, $f_4(0) = f(0)$, $f_4(\frac{1}{2}) = f(\frac{1}{2})$ and $f_4(1) = f(1)$. In this way, we recursively define c_n and f_n.

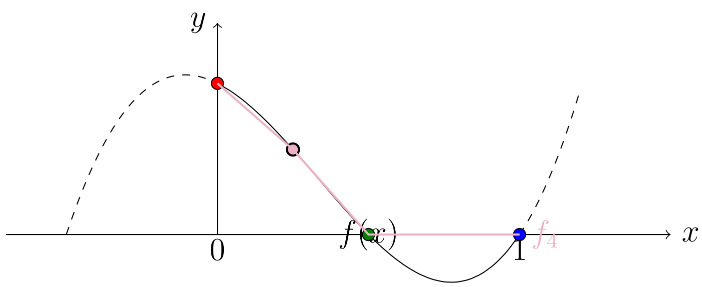


Figure 3: $f_4(x)$ approximation

Furthermore, for $n \geqslant 2$ suppose $f_{n+1}(x) = f_n(x) + c_{n+1}g_{k,m}(x)$, then $n+1 = \frac{k(k-1)}{2} + m + 2$ holds. And $f_{n+1}(x) = f(x)$ for $x = 0, 1, \frac{1}{2}, \cdots, \frac{1}{2^{k-1}}, \cdots, \frac{2^{k-1}-1}{2^{k-1}}, \frac{1}{2^k}, \cdots, \frac{2m-1}{2^k}$.

Last observation is that $f_n(x)$ is piecewise linear function.

We claim $(g_{k,m})$ form a Schauder Basis.

First we note that (c_n) is uniquely determined because $g_{0,0}(x)$ is the only function that is nonzero at $x = 0$, $g_{0,1}$ is the only function that is nonzero at $x = 1$, $g_{1,1}$ is the only function that is nonzero at $x = \frac{1}{2}$, etc... If we have two different (c_n) and (c'_n) that induce (f_n) and (f'_n). If (f_n) and f'_n uniformly converges to f, (f_n) and f'_n pointwisely converges to f. Restriction on points $x = \frac{m}{2^k}$ for $1 \leqslant m \leqslant 2^k$ and all $k \geqslant 1$, leads contradictions.

Second, we show that f_n converges to f pointwisely.

If $x = \frac{m}{2^k}$ for some $1 \leqslant m \leqslant 2^k$ and some $k \geqslant 1$, by construction, there exists a large N such that for $n \geqslant N$, $f_n(x) = f(x)$.

Otherwise, since f is uniformly continuous on $[0, 1]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| \leqslant \delta$ implies $|f(x) - f(y)| \leqslant \frac{\varepsilon}{2}$. We pick K such that $2^{-K} \leqslant \delta$ and partition $[0, 1]$ into $[\frac{m-1}{2^k}, \frac{m}{2^k}]$ for $m = 1, 2, \cdots, 2^k$. By definition, $g_{k+1,m}$ is compactly support on $[\frac{m-1}{2^k}, \frac{m}{2^k}]$.

Suppose $x \in (\frac{m-1}{2^k}, \frac{m}{2^k})$, and let $n + 1 = \frac{k(k-1)}{2} + m + 2$, then $f_{n+1}(x) = f(x)$ on $x = 0, 1, \frac{1}{2}, \cdots, \frac{1}{2^{k-1}}, \cdots, \frac{2^{k-1}-1}{2^{k-1}}, \frac{1}{2^k}, \cdots, \frac{2m-1}{2^k}$, which includes $\frac{m-1}{2^k}$ and $\frac{m}{2^k}$.

$$|f(x) - f_{n+1}(x)| \leqslant |f(x) - f(x_l)| + |f(x) - f(x_r)| \leqslant \varepsilon,$$

where x_l and x_r depend on $n + 1$ being such that $f_{n+1}(x)$ is linear on $[x_l, x_r]$. This comes from the concern that suppose the $\delta = \frac{1}{2}$ and $x \in [0, \frac{1}{2}]$ as shown in graph $f_3(x)$ approximation. We need to show the $|f(x) - f_N(x)| \leqslant \varepsilon$ for $N \geqslant 3$. For $N = 4$ as an example, $f_4(x)$ is no more linear on $[0, \frac{1}{2}]$. Using $|f(x) - f_4(x)| = |f(x) - f_4(0) + f_4(0) - f_4(x)|$ or $|f(x) - f_4(x)| = |f(x) - f_4(\frac{1}{2}) + f_4(\frac{1}{2}) - f_4(x)|$ may not work if f is mixed as partial concave and partial convex in $[0, \frac{1}{2}]$. Hence we need to find an interval $[x_l, x_r]$ depending on N.

Therefore,

$$\|f - f_{n+1}\|_\infty = \max_{1 \leqslant m \leqslant 2^k} \left\{ \max_{x \in [\frac{m-1}{2^k}, \frac{m}{2^k}]} |f(x) - f_{n+1}(x)| \right\} \leqslant \varepsilon.$$

$\square$

Problem 3. Exercise 5.4

Proof. By direct computation, $r(A) = ab$.

By direct computation,

$$A^{2n} = \begin{pmatrix} (ab)^{2n} & 0 \\ 0 & (ab)^{2n} \end{pmatrix}, \qquad A^{2n+1} = \begin{pmatrix} 0 & (ab)^{2n}a^2 \\ (ab)^{2n}b^2 & 0 \end{pmatrix}.$$

By the definition of Euclidean norm and the fact $a > b > 0$, we have

$$\left\|A^{2n}\right\|_2 = a^{2n}b^{2n}, \qquad \left\|A^{2n+1}\right\| = a^{2n+2}b^{2n}.$$

Hence $\left\{\|A^n\|^{1/n}\right\} = \left\{ab, a^{1+1/(2n+1)}b^{1-1/(2n+1)}\right\} \to ab$ as $n \to \infty$, which yields that $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$. □

Problem 4. Exercise 5.5

Proof.

$$|Kf(x)| = |\int_0^1 k(x,y)f(y)\mathrm{d}y| \leqslant \|f\|_\infty \int_0^1 |k(x,y)|\mathrm{d}y,$$

which yields that

$$\|Kf\|_\infty \leqslant \|f\|_\infty \max_{0\leqslant x\leqslant 1} \left\{\int_0^1 |k(x,y)|\mathrm{d}y\right\}.$$

Since k is uniformly continuous on $[0,1]\times[0,1]$, $|k(x,y)| \leqslant M$ for some constant $M \geqslant 0$.

$$\|K\| = \sup \frac{\|Kf\|_\infty}{\|f\|_\infty} \leqslant \max_{0\leqslant x\leqslant 1} \left\{\int_0^1 |k(x,y)|\mathrm{d}y\right\} \leqslant M.$$

To show $\|K\| \geqslant \max_{0\leqslant x\leqslant 1} \left\{\int_0^1 |k(x,y)|\mathrm{d}y\right\}$, we define

$$g_n(x) = \begin{cases} -1 & -1 \leqslant x < -\dfrac{1}{n}; \\ nx & -\dfrac{1}{n} \leqslant x \leqslant \dfrac{1}{n}; \\ 1 & \dfrac{1}{n} < x \leqslant 1. \end{cases}$$

For each $x_0 \in [0,1]$, let $f_(y) = g_n(k(x_0, y))$. Then

$$\begin{aligned}
|Kf_n(x_0)| &= |\int_0^1 k(x_0,y) f_n(y)\mathrm{d}y| \\
&= |\int_0^1 k(x_0,y) g_n(k(x_0,y))\mathrm{d}y| \\
&= |\int_{|k(x_0,y)|\leqslant \frac{1}{n}} k(x_0,y) g_n(k(x_0,y))\mathrm{d}y| + |\int_{|k(x_0,y)|\geqslant \frac{1}{n}} k(x_0,y) g_n(k(x_0,y))\mathrm{d}y| \\
&= |\int_{|k(x_0,y)|\leqslant \frac{1}{n}} n(k(x_0,y))^2\mathrm{d}y| + \int_{|k(x_0,y)|\geqslant \frac{1}{n}} |k(x_0,y)|\mathrm{d}y \\
&= \int_{|k(x_0,y)|\geqslant \frac{1}{n}} |k(x_0,y)|\mathrm{d}y \\
&= \int_0^1 |k(x_0,y)|\mathrm{d}y - \int_{|k(x_0,y)|\leqslant \frac{1}{n}} |k(x_0,y)|\mathrm{d}y \\
&\geqslant \int_0^1 |k(x_0,y)|\mathrm{d}y - \frac{1}{n}.
\end{aligned}$$

Note that $\|f_n\|_\infty = 1$, take maximum on both sides and take n sufficiently large, we have

$$\|K\|_\infty = \sup_{\|f\|_\infty = 1} \|Kf\|_\infty \geqslant \|Kf_n\|_\infty = \max_{x_0\in[0,1]} |Kf_n(x_0)| \geqslant \max_{x_0\in[0,1]} \left\{ \int_0^1 |k(x_0,y)|\mathrm{d}y \right\}.$$

Let $k(x,y) = \sin(2\pi x)$. Then

$$\|K\| = \max_{0\leqslant x\leqslant 1} \left\{ \int_0^1 |\sin(2\pi x)|\mathrm{d}y \right\} = 1;$$

however,

$$\begin{aligned}
K^2 f(x) &= \int_0^1 k(x,y)(Kf(y))\mathrm{d}y \\
&= \int_0^1 k(x,y) \int_0^1 k(y,z) f(z)\mathrm{d}z\mathrm{d}y \\
&= \int_0^1 \int_0^1 \sin(2\pi x)\sin(2\pi y) f(z)\mathrm{d}z\mathrm{d}y \\
&= \sin(2\pi x) \int_0^1 f(z)(\int_0^1 \sin(2\pi y)\mathrm{d}y)\mathrm{d}z \\
&= 0.
\end{aligned}$$

In this case, $\|K^2\| = 0$. □

Problem 5.

Proof. On one hand, if there exist a constant $C > 0$ such that

$$\|x\|_1 \leqslant C\|x\|_2, \forall x \in X,$$

then given any sequence (x_n) that $\|x_n\|_2 \to 0$, we have $\|x_n\|_1 \to 0$.

On the other hand, we prove by contradiction.

Suppose for any $n > 0$, there exists a $x_n \in X$ such that

$$\|x_n\|_1 > n\|x_n\|_2.$$

Let $y_n = \frac{x_n}{\|x_n\|_2\sqrt{n}}$. Since X is linear space, $y_n \in X$. And

$$\|y_n\|_2 = \frac{1}{\sqrt{n}\|x_n\|_2}\|x_n\|_2 = \frac{1}{\sqrt{n}} \to 0.$$

Since $\|\|_2$ is stronger than $\|\|_1$, it implies that $\|y_n\|_1 \to 0$. However,

$$\|y_n\|_1 = \frac{1}{\sqrt{n}\|x_n\|_2}\|x_n\|_1 > \frac{n\|x_n\|_2}{\sqrt{n}\|x_n\|_2} = \sqrt{n} \to \infty,$$

which yields contradiction! $\square$