separable Hilbert space

$\downarrow$ countable orthonormal basis

$\downarrow$ complete $\langle \cdot, \cdot \rangle$

Def

A function $k: X \times X \to \mathbb{R}$ is a kernel

if $\exists\ \mathcal{H}$ and $\varphi: X \to \mathcal{H}$ s.t

$k(x, x') = \langle \phi(x), \phi(x') \rangle_{\mathcal{H}}$

property 1 $\quad k(x, x') = k(x', x)$

property 2. $\quad k_1 + k_2$, ak are kernels on X.

property 3. $\left.\begin{array}{l} A: X \to Y \\ k \text{ is kernel on } Y \end{array}\right\} \Rightarrow k \circ A$ is kernel on X.

4 $\quad k_1$ on X_1, k_2 on $\Rightarrow k_1 \times k_2$ on $X_1 \times X_2$

eg: $\phi(x) = (\phi_1(x), \phi_2(x) \cdots \phi_n(x)) \in l_2(X)$

$$k(x, x') = \sum_{i=1}^{\infty} \phi_i(x)\, \phi_i(x')$$

eg: $k(x, x') = \exp\left(-\frac{\|x - x'\|^2}{\sigma^2}\right)$

Def: $k: X \times X \to \mathbb{R}$ is positive definite if

$\forall\, n \geq 1$, $\forall\, a \in \mathbb{R}^n$, $\forall\, (x_1, \cdots x_n) \in X^n$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j\, k(x_i, x_j) \geq 0$$

$$[a_1, \cdots a_n]_{1\times n} \begin{bmatrix} k(x_1,x_1) & k(x_1,x_2) & \cdots & k(x_1,x_n) \\ \vdots & & & \vdots \\ k(x_n,x_1) & & \cdots & k(x_n,x_n) \end{bmatrix}_{n\times n} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1}$$

property 5 $\quad$ kernel is positive definite.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j\, k(x_i, x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle a_i \phi(x_i), a_j \phi(x_j) \rangle_{\mathcal{H}} = \left\|\sum_{i=1}^{n} a_i \phi(x_i)\right\|_{\mathcal{H}}^2 \geq 0$$

Def: An evaluation functional over $\mathcal{H}$ of functions on X is a linear functional

$\mathcal{F}_x: \mathcal{H} \to \mathbb{R}$ that $\mathcal{F}(f) = f(x)$ $\forall f \in \mathcal{H}$.

Def: $\mathcal{H}$ is a reproducing kernel Hilbert space if the evaluation functional are bounded.

$\forall\, x \in X$, $\exists\, M > 0$ such that

$|\mathcal{F}_x[f]| = |f(x)| \leq M \|f\|_{\mathcal{H}}$ $\quad \forall f \in \mathcal{H}$.

eg: $L_2[a,b]$ is Hilbert but not RKHS.

local values are arbitrary large.

Recall: Riesz Representation Thm

If φ is a bounded linear functional on $\mathcal{H}$

then there exists a unique $y \in \mathcal{H}$ s.t $\varphi(x) = \langle y, x \rangle \quad \forall x \in \mathcal{H}$.

Thm [Reproducing property]

$\forall F_x \in (\text{RKHS})', \ \exists k_x \in \mathcal{H}$ s.t

$F_x(f) = f(x) = \langle k_x, f \rangle_{\mathcal{H}}. \quad \forall f \in \mathcal{H}. \qquad \|F_x\|^2_{dual} = \langle k_x, k_x \rangle_{\mathcal{H}}$

$k_x(y) = F_y(k_x) = \langle k_y, k_x \rangle_{\mathcal{H}}$

Therefore we define $k: X \times X \to \mathbb{R}$ is (reproducing) kernel of $\mathcal{H}$ by

$k(x,y) = k_x(y)$

Thm A RKHS defines a reproducing kernel.

A reproducing kernel defines a unique RKHS.

$f(x) = \sum_{i=1}^{s} a_i k_{x_i}(x) \qquad g(x) = \sum_{j=1}^{s'} B_j k_{x_j}(x)$

$\langle f, f \rangle_{\mathcal{H}} = \sum_{i=1}^{s} \sum_{j=1}^{s'} a_i \beta_j k(x_i, x_j)$

$\|F_f\|^2_{dual} = \int_{X \times X} k(x,y)\, da(x)\, da(y)$

$\mathcal{M}(X)$: finite signed measure on metric space X.

By Jordan decomposition thm: $\forall u \in \mathcal{M}, \quad u = u^+ - u^-$ unique, where $u^+, u^- \in \mathcal{M}_+(X)$

at finite one is finite.

$\|u\|_{TV} = u_+(X) + u_-(X)$

The space of bounded variation finite signed measure on Σ is a Banach space with $\| \ \|_{TV}$.

Given X compact metric space

$Y = (C(X), \| \ \|_\infty)$, then $Y' = (\mathcal{M}(X), \| \ \|_{TV})$ Convergence in $\mathcal{M}$ is weak-*

$\|u\|_B = \max_f \{ \int f(x) du : f \in B \} \qquad B = \{ f : \|f\| \le 1 \}$

$\|u\|_{TV} = \max_f \{ \int f(x) du : \|f\|_\infty \le 1 \}$ unit ball in Y

$Y = H' \qquad \|u\|^2_{H^{-1}} = \max \{ \int f(x) du : \|\nabla f\|^2_{L^2} \le 1 \} = \int_{X \times X} k(x,y)\, du(x)\, du(y)$

$k(x,y) = \exp(-\frac{\|x-y\|^2}{2\sigma^2})$

$\|u\|^2_G = \int_{X \times X} k(x,y)\, du(x)\, du(y)$