08/18 Lines and Planes in Spaces

$P_0(x_0, y_0, z_0)$, there is a Line passing though P_0.

which parallel to vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$P(x, y, z)$ is also locate at L.

$\overrightarrow{P_0P} = (x-x_0, y-y_0, z-z_0) = t\vec{v} = t(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$

$\Rightarrow \begin{cases} x - x_0 = v_1 t \\ y - y_0 = v_2 t \\ z - z_0 = v_3 t \end{cases}$ $\rightarrow$ the standard parametrization of Line though P parallel to $\vec{v}$

$\begin{cases} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ z = z_0 + v_3 t \end{cases}$ $\Leftrightarrow$ $\overrightarrow{OP} = (x, y, z) = \vec{r}(t)$

$\boxed{\vec{r}(t) = \vec{r_0} + \vec{v}t}$ $\rightarrow$ a vector eq.

initial position direction

$\vec{r_0} = \overrightarrow{OP_0} = (x_0, y_0, z_0)$

eg. Parametrize the line segment joining $P(-3, 2, -3)$, and $Q(1, -1, 4)$

$\overrightarrow{PQ} = (4, -3, 7)$

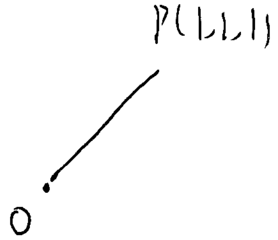
$\begin{cases} x = -3 + 4t \\ y = 2 - 3t \\ z = -3 + 7t \end{cases}$

$t = 0$, $P(-3, 2, -3)$

$t = 1$, $Q(1, -1, 4)$

$\boxed{0 \le t \le 1.}$

eg. A helicopter flies from the origin in the direction of $(1, 1, 1)$ at speed 60 ft/sec. What is position of this helicopter



$$\vec{r}(t) = \vec{r_0} + \vec{v}t$$

$$= \vec{r_0} + \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{direction}} \cdot \underbrace{|\vec{v}|}_{\text{speed}} \underbrace{t}_{\text{time.}}$$

$\vec{r_0} = (0,0,0)$ Unit vecta

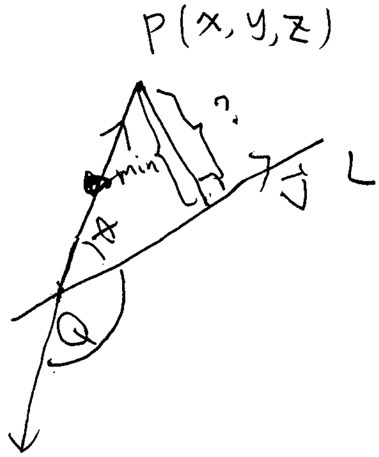
$$\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{1^2+1^2+1^2}} = \underbrace{\frac{\sqrt{3}}{3}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k}}$$

$$\vec{r}(t) = \left(\frac{\sqrt{3}}{3}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k}\right) \cdot 60\,t$$

$$\vec{r}(10) = \left(\frac{\sqrt{3}}{3}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k}\right) \cdot 600$$

$$= (200\sqrt{3},\ 200\sqrt{3},\ 200\sqrt{3}) \text{ (feet)}$$

- Distane between a point and a line.



$\underline{|D|} = |PQ| \cdot \underline{\sin\theta}$ ← introduce cross product.

$$D = \frac{|\vec{QP} \times \vec{v}|}{|\vec{v}|} = \frac{|\vec{QP}|\cancel{|\vec{v}|} \cdot \sin\theta}{\cancel{|\vec{v}|}}$$

$$D = \frac{|\vec{QP} \times \vec{v}|}{|\vec{v}|} = |\vec{QP} \times \vec{v}|$$ when $|\vec{v}|$ is unit vecta

eg. Find the distance $P(1,1,5)$ and L

$x = 1+t \quad y = 3-t \quad z = 2t \rightarrow \vec{v} = (1,-1,2)$

$Q(1,3,0) \quad \underline{\vec{QP} = (0,-2,5)}$

$$d = \frac{|\vec{QP} \times \vec{v}|}{|\vec{v}|} = \frac{\left| \begin{matrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{matrix} \right|}{\sqrt{1^2+1^2+2^2}} = \frac{|1\vec{i} \ominus (5)\vec{j} + 2\vec{k}|}{\sqrt{6}}$$

$$= \frac{\sqrt{1^2+5^2+2^2}}{\sqrt{6}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

II. $Q(x, y, z)$ $\quad P(1,1,5)$

$$d = \boxed{\min_{Q\in L} |PQ| \ \star} = \min_{Q\in L} |(x-1, y-1, z-5)|$$

Larangian multipier Method.

$$= \min_{Q-L} \sqrt{\underset{\substack{\uparrow \\ 1+t}}{(x-1)^2} + \underset{\substack{\uparrow \\ 3-t}}{(y-1)^2} + \underset{\substack{\uparrow \\ 2t}}{(z-5)^2}}$$

$$= \min \sqrt{t^2 + (2-t)^2 + (2t-5)^2}$$

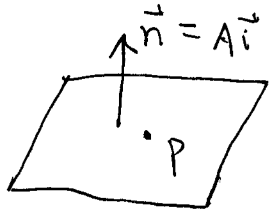
$$= \min \sqrt{6t^2 - 24t + 29}$$

$$= \min \sqrt{6(\underbrace{t^2-4t+4}) + 5}$$

$$= \min \underline{\sqrt{6(t-2)^2 + 5}} \geq \underline{\sqrt{5}} \rightarrow \text{distance}$$

between point and line.

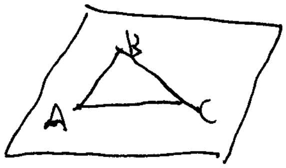
- Equation for planes



$\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ $\quad P(x_0, y_0, z_0)$ $\quad Q(x, y, z)$

$$\vec{n} \cdot \vec{PQ} = 0$$

$$\underline{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0} \quad \text{equation for plane}$$



$A(a_1, a_2, a_3) \quad B(b_1, b_2, b_3) \quad C(c_1, c_2, c_3)$

$\vec{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$

$\vec{AC} = (c_1 - a_1, c_2 - a_2, c_3 - a_3)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} = \overset{m}{\bullet}\vec{i} + \overset{n}{\bullet}\vec{j} + \overset{l}{\bullet}\vec{k}$$

$\cancel{ABC}$ $\quad \vec{n} = (m, n, l)$

$m(x - b_1) + n(y - b_2) + l(z - b_3) = 0 \rightarrow$ plane.

II. $D(x, y, z) \qquad \vec{AD} / \vec{BD} / \vec{CD}$

parameter.

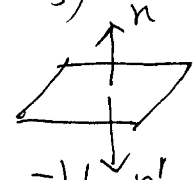
$\vec{AD} = p\underbrace{\vec{AB}} + q\underbrace{\vec{AC}}$

span the plane

$$\begin{cases} x - a_1 = p(b_1 - a_1) + q(\overset{c_1 - a_1}{\cancel{\quad}}) \\ y - a_2 = p(b_2 - a_2) + q(c_2 - a_2) \\ z - a_3 = p(b_3 - a_3) + q(c_3 - a_3) \end{cases}$$

parameterization equation for plane.

$A(0, 0, 1) \quad B(2, 0, 0) \quad C(0, 3, 0)$



$(3, 2, 6)$

I. $\vec{AB} = (2, 0, -1) \qquad \vec{AC} = (0, 3, -1)$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\vec{i} + (-1)(-2)\vec{j} + 6\vec{k}$$

$= 3\vec{i} + 2\vec{j} + 6\vec{k}$

$$3(x-0)+2(y-0)+6(z-1)=0$$

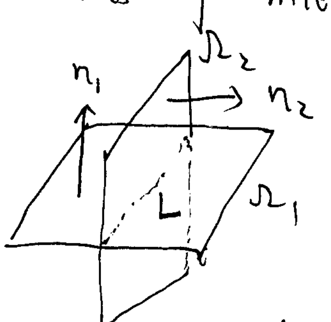
$$3x+2y+6z-6=0$$

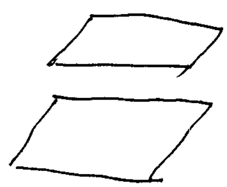
$\cancel{x-0=p}$ $\quad \vec{AD} = p\,\vec{AB} + q\,\vec{AC}$

$$\begin{cases} x-0 = 2p \\ y-0 = 3q \\ z-1 = -p-q \end{cases} \qquad \begin{cases} x = 2p \\ y = 3q \\ z = -p-q+1 \end{cases}$$

$$\underline{3(2p)+2(3q)+6(-p-q+1)-6=0} \quad \checkmark.$$

- Lines of Intersection.





② Find any point on L.

(x_0, y_0, z_0) $\qquad \underline{(3,-1,0)}$

$z=0$

$$A \begin{cases} 3x-6y-2z=15 \\ 2x+y-2z=5 \end{cases} \qquad \begin{cases} 3x-6y=15 \\ 2x+y=5 \end{cases} \Rightarrow \begin{matrix} 3x-6y=15 \\ 12x+6y=30 \end{matrix}$$

$15x=45$

$$\begin{cases} x=3 \\ \cancel{y=5} \\ y=-1 \end{cases}$$

① Find the vector parallel to the intersection L.

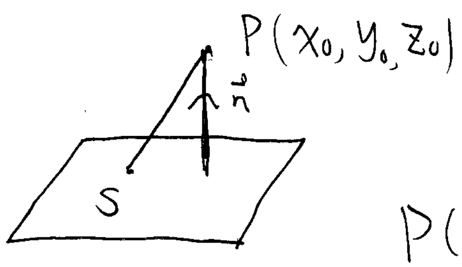
$n_1 \perp L. \qquad n_2 \perp L.$

$\underline{n_1 \times n_2} \perp n_1 \quad , \quad n_1 \times n_2 \perp n_2$

$$L \parallel n_1 \times n_2. = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$(3,-6,-2) \quad (2,1,-2)$

- The distance between point and plane.



$$d = \frac{|\vec{PS} \times \vec{n}|}{|\vec{n}|} = \left| \vec{PS} \times \frac{\vec{n}}{|\vec{n}|} \right|.$$

$P(1,1,3)$ $\underline{3x + 2y + 6z = 6}$

$\underbrace{S\,(0, 3, 0)}$

$\vec{PS} = (-1, 2, -3)$

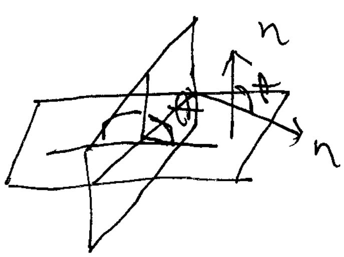
$\vec{n} = (3, 2, 6)$

$$d = \frac{|\vec{PS} \times \vec{n}|}{|\vec{n}|} = \frac{\begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ 3 & 2 & 6 \end{vmatrix}}{}$$

$S = (1, 1, \frac{1}{6})$

$\vec{SP} = (0, 0, \frac{17}{6})$ $d = \underline{\qquad} = \left| \begin{matrix} i & j & k \\ 0 & 0 & \frac{17}{6} \\ 3 & 2 & 6 \end{matrix} \right|.$

- Angles between Planes



θ is acute

$\underbrace{\theta \text{ in } [0, \pi)}$ $\cos\theta$ is $(-1, 1)$

$\cos\theta$ 1

$\cos\frac{\pi}{2}$ 0

$\cos\pi$ -1

eg: $3x - 6y - 2z = 15$ $2x + y - 2z = 5$

$(3, -6, -2)$ $(2, 1, -2)$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)} = \cos^{-1}\left(\frac{6 - 6 + 4}{\qquad}\right)$$