Problem 1. Exercise 6.1

Proof. Let C denote the closed convex subset, let $a = \inf_{x\in C} \|x\|^2$ and let (x_n) be a minimizing sequence in C such that $\|x_n\|^2 \to a$ as $n \to \infty$. That is, for any $\varepsilon > 0$, there exists an N such that for any $n \geqslant N$, $|\|x_n\|^2 - a| \leqslant \varepsilon$.

For any $m \geqslant n \geqslant N$, since C is convex, $\frac{x_m+x_n}{2} \in C$ and $\left\|\frac{x_m+x_n}{2}\right\|^2 \geqslant a$.

By by the parallelogram equality, we have

$$\begin{aligned}\|x_m - x_n\|^2 &= 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \\ &\leqslant 2\|x_m\|^2 + 2\|x_n\|^2 - 4a \\ &= 2(\|x_m\|^2 - a) + 2(\|x_n\|^2 - a) \\ &\leqslant 4\varepsilon,\end{aligned}$$

which yields that (x_n) is a Cauchy sequence in C. Furthermore, C is a closed subspace of a complete space. C is complete and (x_n) converges in C.

Assume that there exists $x \neq y$ such that $\|x\|^2 = a$ and $\|y\|^2 = a$. Once again apply the parallelogram equality,

$$\begin{aligned}\|x - y\|^2 &= 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 \\ &\leqslant 2a + 2a - 4a = 0,\end{aligned}$$

which implies that $x = y$. □

Problem 2. Exercise 6.2

Proof.

1. For $v = u - \bar{u}$, $v \in N$. Thus

$$\inf_{n\in N} \|u - n\| \leqslant \|u - v\| = \|\bar{u}\| = |\bar{u}|.$$

   On the other hand, for any $n \in N$:

$$\|u - n\|_\infty \geqslant |\int_0^1 (u(x) - n(x))\mathrm{d}x| = |\int_0^1 u(x)\mathrm{d}x - \int_0^1 n(x)\mathrm{d}x| = |\int_0^1 u(x)\mathrm{d}x| = |\bar{u}|.$$

   Combine both sides, we have shown that $\inf_{n\in N} \|u - n\| = |\bar{u}|$.

2. For any $m \in M$:

$$\|u - m\|_\infty \geqslant |\int_0^1 (u(x) - m(x))\mathrm{d}x| = |\int_0^1 x\mathrm{d}x| = \frac{1}{2},$$

   which implies that $\inf_{m\in M} \|u - m\| \geqslant \frac{1}{2}$.

Let's define a sequence $(m_k)_{k=3}^{\infty}$ of piecewise continuous function in M by:

$$m_k(x) = \begin{cases} -\dfrac{kx}{2}, & 0 \leqslant x \leqslant \dfrac{1}{k}; \\ \dfrac{k}{k-2}(x - \dfrac{1}{2}), & \dfrac{1}{k} \leqslant x \leqslant \dfrac{1}{2}; \\ x - \dfrac{1}{2}, & \dfrac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

The idea is to cut the area of $m_k(x)$ by $x-$axis to get two triangles of same area. From this construction,

$$\inf_{m \in M} \|u - m\| \leqslant \|u - m_k\|_\infty = \sup_{x \in [0,1]} |x - m_k(x)| = \frac{1}{k} + \frac{1}{2} \to \frac{1}{2} \qquad \text{as} \qquad k \to \infty.$$

Combine both sides, $\inf_{m \in M} \|u - m\| = \frac{1}{2}$. However, it cannot be achieved for some element in M. Assume otherwise, then $m(x) \geqslant x - \frac{1}{2}$ for $x \in [0, 1]$. And:

$$0 = \int_0^1 m(x)\mathrm{d}x \geqslant \int_0^1 (x - \frac{1}{2})\mathrm{d}x = 0,$$

which yields that $m = x - \frac{1}{2}$, but $m(0) \neq 0$ which yields contradiction!

□

Problem 3. Theorem 6.9

Proof. Suppose X is an inner product space, we have

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y \rangle + \langle x - y, x - y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= 2\|x\|^2 + 2\|y\|^2. \end{aligned} \tag{1}$$

Suppose X is a normed space satisfying (1). Let's define

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2 \right),$$

and prove it is an inner product.

- Non-negative and positive definite

$$\begin{aligned} \langle x, x \rangle &= \frac{1}{4} \left(\|2x\|^2 - i\|(1 + i)x\|^2 + i\|(1 - i)x\|^2 \right) \\ &= \frac{1}{4}\|x\|^2 \left(4 - i(1 + i)^2 + i(1 - i)^2 \right) \\ &= \|x\|^2 \geqslant 0. \end{aligned}$$

Positive define comes from the positive define of norm for free.

- Hermitian symmetric

$$\begin{aligned}
\langle y, x\rangle &= \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2 - i\|y+ix\|^2 + i\|y-ix\|^2\right)\\
&= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i|i|^2\|x-iy\|^2 + i|-i|^2\|x+iy\|^2\right)\\
&= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)\\
&= \overline{\langle x, y\rangle}.
\end{aligned}$$

- Liner in the second argument

$$\begin{aligned}
\langle x, y\rangle + \langle x, z\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)\\
&\quad + \frac{1}{4}\left(\|x+z\|^2 - \|x-z\|^2 - i\|x+iz\|^2 + i\|x-iz\|^2\right)\\
&= \frac{1}{4}(\|x+y\|^2 + \|x+z\|^2) - \frac{1}{4}(\|x-y\|^2 + \|x-z\|^2)\\
&\quad - \frac{i}{4}(\|x+iy\|^2 + \|x+iz\|^2) + \frac{i}{4}(\|x-iy\|^2 + \|x-iz\|^2)\\
&= \frac{1}{8}(\|2x+(y+z)\|^2 + \|y-z\|^2) - \frac{1}{8}(\|2x-(y+z)\|^2 + \|y-z\|^2)\\
&\quad - \frac{i}{8}(\|2x+i(y+z)\|^2 + \|i(y-z)\|^2) + \frac{i}{8}(\|2x-i(y+z)\|^2 + \|i(y-z)\|^2)\\
&= \frac{1}{2}\langle 2x, y+z\rangle\\
&= \langle x, y+z\rangle.
\end{aligned}$$

□

Problem 4.

Proof. Suppose there exists such an inner product that derives the norm $\|\cdot\|_\infty$, then it satisfies the Parallelogram Law. Take $f(x) = x$ and $g(x) = 1-x$, then we have

$$\begin{aligned}
\|f+g\|_\infty^2 + \|f-g\|_\infty^2 &= 2\|f\|_\infty^2 + 2\|g\|_\infty^2\\
\|1\|_\infty^2 + \|2x-1\|_\infty^2 &= 2\|x\|_\infty^2 + 2\|1-x\|_\infty^2\\
1+1 &= 2(1+1),
\end{aligned}$$

which yields contradiction! □

Problem 5.

Proof. On one hand, we pick any $x \in \overline{\mathrm{span}(M)}$, there exists a sequence (x_n) converges to x with respect to norm. By the continuity of inner product, for any $y \in M^\perp$, we have

$$\langle x, y\rangle = \langle \lim_{n\to\infty} x_n, y\rangle = \lim_{n\to\infty}\langle x_n, y\rangle = 0,$$

which implies that $x \in (M^\perp)^\perp$. We have shown that $\overline{\operatorname{span}(M)} \subset (M^\perp)^\perp$.

On the other hand, we first prove a lemma: $B \subset A$ implies that $A^\perp \subset B^\perp$.

For any $x \in A^\perp$ and every $b \in B \subset A$, we have $\langle x, b\rangle = 0$ for all $b \in B$, that is, $x \in B^\perp$.

Hence $M \subset \overline{\operatorname{span}(M)}$ implies that $\overline{\operatorname{span}(M)}^\perp \subset M^\perp$.

Apply that lemma again, $\overline{\operatorname{span}(M)}^\perp \subset M^\perp$ implies that $(M^\perp)^\perp \subset (\overline{\operatorname{span}(M)}^\perp)^\perp = \overline{\operatorname{span}(M)}$.

$\square$

Problem 6.

Proof. To show R is bounded, we note that for any $z \in \mathcal{H}$:

$$\frac{\|Rz\|}{\|z\|} = \frac{\|\langle y, z\rangle x\|}{\|z\|} = \frac{|\langle y, z\rangle|\|x\|}{\|z\|} \leqslant \frac{\|y\|\|z\|\|x\|}{\|z\|} = \|x\|\|y\| < \infty$$

To show $\|R\| = \|x\|\|y\|$, we note that for $z = \frac{y}{\|y\|}$:

$$\|Rz\| = \left\|\langle y, \frac{y}{\|y\|}\rangle x\right\| = \frac{|\langle y, y\rangle|}{\|y\|}\|x\| = \frac{\|y\|^2}{\|y\|}\|x\| = \|x\|\|y\|,$$

which implies that $\|R\| \geqslant \|x\|\|y\|$. $\square$

HAPPY THANKSGIVING HOLIDAYS!