Problem 1. Induced metric of a space, which is the product space of metric spaces.

Proof. We want to show

- **Non-negativity:** for any $p \geqslant 1$ and $i = 1, \cdots, n$, since $d_i(x_i, y_i) \geqslant 0$ for any $x_i, y_i \in X_i$, then we have
$$D_p(x, y) = \left(\sum_{i=1}^{n} d_i(x_i, y_i)^p\right)^{1/p} \geqslant 0;$$
and by non-negativity,
$$D_p(x, y) = 0 \Leftrightarrow \forall i,\ d_i(x_i, y_i) = 0 \Leftrightarrow \forall i,\ x_i = y_i \Leftrightarrow x = y.$$

- **Symmetry:** for $i = 1, \cdots, n$, we know $d_i(x_i, y_i) = d_i(y_i, x_i)$, then we have
$$D_p(x, y) = \left(\sum_{i=1}^{n} d_i(x_i, y_i)^p\right)^{1/p} = \left(\sum_{i=1}^{n} d_i(y_i, x_i)^p\right)^{1/p} = D_p(y, x).$$

- **Triangle inequality:** Recall Minkowski's inequality for $p \geqslant 1$:
$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}, \tag{1}$$
for any numbers $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, thus take $a_i = d_i(x_i, y_i)$ and $b_i = d_i(y_i, z_i)$
$$\begin{aligned} D_p(x, y) + D_p(y, z) &= \left(\sum_{i=1}^{n} d_i(x_i, y_i)^p\right)^{1/p} + \left(\sum_{i=1}^{n} d_i(y_i, z_i)^p\right)^{1/p} \\ &\overset{(1)}{\geqslant} \left(\sum_{i=1}^{n} \left(d_i(x_i, y_i) + d_i(y_i, z_i)\right)^p\right)^{1/p} \\ &\geqslant \left(\sum_{i=1}^{n} d_i(x_i, z_i)^p\right)^{1/p} = D_p(x, z), \end{aligned}$$
since $d_i(x_i, y_i) + d_i(y_i, z_i) \geqslant d_i(x_i, z_i)$ and $x \mapsto x^p$ is increasing.

□

Problem 2. Induced metric of pre-image space under injection. (pull-back metric)

Proof. Let $y_i = f(x_i) \in Y$. On the other hand, due to injection, for any $y_i \in Y$, there exists a unique x_i such that $f(x_i) = y_i$. And we want to show

- **Non-negativity:**
$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = d_Y(y_1, y_2) \geqslant 0,$$
and $d_X(x_1, x_2) = 0 \Leftrightarrow d_Y(y_1, y_2) = 0 \Leftrightarrow y_1 = y_2 \Leftrightarrow x_1 = x_2$.

- **Symmetry:**
$$d_X(x_1,x_2) = d_Y(y_1,y_2) = d_Y(y_2,y_1) = d_X(x_2,x_1).$$
- **Triangle inequality:**
$$d_X(x_1,x_2) + d_X(x_2,x_3) = d_Y(y_1,y_2) + d_Y(y_2,y_3) \geqslant d_Y(y_1,y_3) = d_X(x_1,x_3).$$

$\square$

Problem 3.

Proof. We want to show

- **Non-negativity:** Since $f(0) = 0$ and f is non-decreasing, $f(x) \geqslant 0$ on $[0,\infty)$. And $d(x,y) \geqslant 0$, thus $f(d(x,y))$ is well-defined.

  $d_f(x,y) = f(d(x,y)) \geqslant 0$ and $d_f(x,y) = 0 \Leftrightarrow f(d(x,y)) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$.
- **Symmetry:**
$$d_f(x,y) = f(d(x,y)) = f(d(y,x)) = d_f(y,x).$$
- **Triangle inequality:**
$$d_f(x,y)+d_f(y,z) = f(d(x,y))+f(d(y,z)) \geqslant f(d(x,y)+d(y,z)) \geqslant f(d(x,z)) = d_f(x,z).$$

$\square$

Problem 4. Metric induced by norm.

Proof. First, assume d is associated metric of norm, i.e., $d(x,y) = \|x-y\|,\ \forall x,y \in X$. Then

$$d(x+z,y+z) = \|x+z-(y+z)\| = \|x-y\| = d(x,y)$$
$$d(\lambda x,\lambda y) = \|\lambda x - \lambda y\| = |\lambda|\|x-y\| = |\lambda| d(x,y).$$

Second, assume d is a metric satisfying translation invariance and positive homogeneity, then we define a function $f : X \mapsto \mathbb{R}$ by $f(x) = d(x,0)$, since $0 \in X$. We verify that:

- **Non-negativity:** $f(x) = d(x,0) \geqslant 0$. And $f(x) = 0 \Leftrightarrow d(x,0) = 0 \Leftrightarrow x = 0$.
- **Homogeneity:** $f(\lambda x) = d(\lambda x, 0) = |\lambda| d(x,0)$, by positive homogeneity of d.
- **Triangle inequality:** $f(x) + f(y) = d(x,0) + d(y,0) = d(x,0) + d(y-y, 0-y) = d(x,0) + d(0,-y) \geqslant d(x,-y) = d(x+y,-y+y) = d(x+y,0) = f(x+y)$.

$\square$

Problem 5. Alternative characterization of closed sets in a metric space.

Proof. "$\Rightarrow$:" Assume $F \subset X$ is closed and $x_n \to x$ for $x_n \in F$, i.e., for any $\varepsilon > 0$, there exists an integer $N > 0$ such that for $n \geqslant N$, $d(x_n, x) < \varepsilon$. Define $B(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$. Then for $n \geqslant N$, $x_n \in B(x, \varepsilon)$.

Prove by contradiction. Assume there exists a sequence $\{y_n\}$ in F and $y_n \to y$, but $y \notin F$. Since F is closed, $X \backslash F$ is open and $y \in X \backslash F$. There exist a $\varepsilon > 0$ such that $B(y, \varepsilon) \subset X \backslash F$. On the other hand, there exists a $N' > 0$ such that for $n \geqslant N'$, $y_n \in B(y, \varepsilon) \subset X \backslash F$, contradicted with the fact that every $y_n \in F$.

"$\Leftarrow$:" Assume for every sequence $\{x_n\}$ in F, if $x_n \to x$, then $x \in F$. We want to show F is closed.

Prove by contradiction. Assume F is not closed, then $X \backslash F$ is not open. Therefore $X \backslash F$ is not empty and there exists a $x \in X \backslash F$ such that for every $\varepsilon = \frac{1}{n} > 0$, $B(x, \frac{1}{n}) \not\subset X \backslash F$, i.e., there exists a $y_n \in B(x, \frac{1}{n})$ but $y_n \in F$.

By construction, the sequence $\{y_n\}$ in F converges to x but $x \in X \backslash F$, contradiction! $\square$

Problem 6. Composition of continuous functions

Proof. For any $x_0 \in X$, let $y_0 = f(x_0)$ and $y = f(x)$. For every $\varepsilon > 0$, since $g : Y \mapsto Z$ is continuous, there exists a $\theta = \theta(\varepsilon)$ such that if $d_Y(y, y_0) < \theta$, we have $d_Z(g(y), g(y_0)) < \varepsilon$.

For above x_0 and θ, since $f : X \mapsto Y$ is continuous, there exists a $\delta = \delta(\theta) > 0$ such that if $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \theta$.

Therefore, for any $x_0 \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ defined above such that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) = d_Y(y, y_0) < \theta$, thus we have $d_Z(h(x), h(x_0)) = d_Z(g(f(x)), g(f(x_0))) = d_Z(g(y), g(y_0)) < \varepsilon$. $\square$