sequentially compact iff it's complete + totally bounded

Every total bounded metric space (thus compact metric space) is separable.

(idea) A space fails to be separable $\Rightarrow$ A space fails to be compact.

A metric space is separable if there exists a countable dense set (A is dense w.r.t B if $B \subset \bar{A}$)

(A is dense in B, B is dense in C $\Rightarrow$ A is dense in C)

eg $P^{even}[0,1]$ is dense in $P[0,1]$; $P[0,1]$ is dense in $C[0,1]$

(idea) $\exists$ dense set $\xrightarrow{\text{Find}}$ countable dense set.

countable $\rightarrow \mathbb{Q} / \mathbb{N}$ / the set of all finite subsets of natural number

uncountable $\rightarrow [0,1]$ / $\mathbb{R} \setminus \mathbb{Q}$

the set of all infinite sequences of natural numbers

the set of all subsets of natural numbers

Cantor set

eg l^∞ is not separable.

I is a subset of $\mathbb{N}$

$e_I \in l^\infty$ defined by $(e_I)_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$

$d_\infty(e_I, e_J) = 1$ if $I \neq J$

$\mathcal{B} = \{B(e_I, \frac{1}{2}) : I \subseteq \mathbb{N}\}$ is uncountably infinite disjoint open ball in l^∞

Let S be dense subset

each ball in $\mathcal{B}$ must contain at least one $s \in S$; these elements are distinct $\Big\} \Rightarrow$ S must be uncountably infinite.

eg: c_0 is separable

Let S be the subset of c_0 with rational entries of which at most finitely many nonzero.

S is countable.

$\forall\ x = (x_1, x_2 \ldots) \in c_0$ $\quad \lim_{n\to\infty} x_n = 0$ $\quad \exists N \quad n \geq N \quad |x_n| \leq \varepsilon$

$y = (y_1, y_2 \cdot y_N, 0 \ldots)$ $\quad |x_i - y_i| \leq \varepsilon$ $\quad$ ($\mathbb{Q}$ is dense in $\mathbb{R}$)

$d_\infty(x, y) \leq \varepsilon$ $\quad$ dense.

Remark: separable could be defined as a topological property (countable dense set)

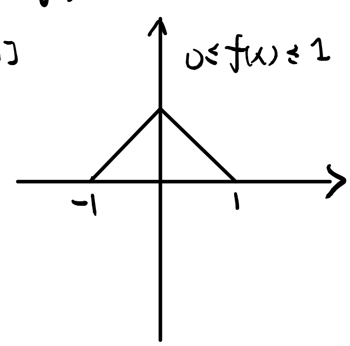
A subspace of a separable space need not be separable

A subspace of a separable metric space is separable.

eg: $C[K]$ is separable, (complete), but not compact.

$f(x) = \max(1-|x|, 0)$

$C[a,1]$ $\quad 0 \le f(x) \le 1$



$f_n(x) = f\left(2n(n+1)(x-\frac{1}{n})\right)$

$x = \frac{1}{n} + \frac{1}{2n(n+1)} \le \frac{1}{n} + \frac{1}{2n^2} = \frac{2n+1}{2n^2} \le 1$

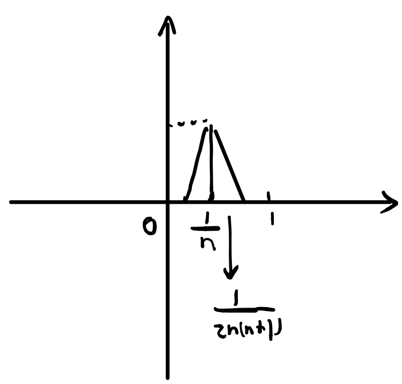
$1 - |-2(n+1)|$

supported on

$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$r_n = \frac{1}{2n(n+1)}$

$r_{n+1} < \frac{1}{2n(n+1)}$



$f_n(x)$ disjoint $\quad \|f_n\| = 1$

$\|f_n - f_m\|_\infty = 1$ no convergent subsequence.

also show unit closed ball in $C[0,1]$ is not compact.

put (f_n) into infinite disjoint ball with radius 1

A space fail to be compact either too big at infinity

or too big locally

(too many directions in which one can go away from a given point)

1° $\underbrace{\text{Discrete metric}}$ space is compact iff it's finite (runs off to infinity)

$\hookrightarrow (X, d_d)$

$(\mathbb{N}, d(x,y) = |x-y|)$ $(\mathbb{Z}, d(x,y) = |x-y|)$

(Every set is open) (finest topology)

2° sequence space c_0 $x = (x_1, x_2 \ldots)$ with $\|x\|_\infty = \sup_i |x_i|$

$\lim_{n\to\infty} x_n = 0$

closed unit ball is not compact.

$e_i \in c_0$ $\|e_i\|_\infty = 1$ $\|e_i - e_j\|_\infty = 1$ $\forall i \neq j$

$\{e_i\}$ no convergent subsequence.

Remark 1. Completeness → no holes   Compactness → small

math.ucla.edu/~tao/preprints/compactness.pdf.

2. Compactness is sort of topological generalization of finiteness.

topology : how something behaves on an open set

compact space : there are only finitely many possible behaviors

A finite f function : $A \to \mathbb{R}$ finite choice → max/min/bounded

A compact; f continuous : $A \to \mathbb{R}$ ⟶ max/min/bounded.