MAT201A Homework 1
Fall 2019
Professor Qinglan Xia
Due Date: Wednesday, October 2nd at 9:00am

1. Suppose $(X_1, d_1), (X_2, d_2), \cdots, (X_n, d_n)$ are metric spaces. Let X be the product space

$$X = X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \cdots, x_n) : x_i \in X_i\}.$$

   Then, for any $p \geq 1$, one may define

$$D_p(x, y) := \left(\sum_{i=1}^{n} d_i(x_i, y_i)^p \right)^{1/p}$$

   for any $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in X$.

   Show that (X, D_p) is a metric space. (Hint: You may use the fact that the p-norm $|| \cdot ||_p$ on $\mathbb{R}^n$ is indeed a norm.)

2. Suppose that $f : X \to Y$ is an injective map. (i.e., for any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$). Let d_Y be a metric on Y. Then, one may pull back the metric d_Y on Y to a metric d_X on X by setting

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

   for any $x_1, x_2 \in X$. Show that d_X is indeed a metric on X. This metric is usually denoted by $f^{\#} d_Y$, called the pull-back metric of d_Y.

3. Suppose that (X, d) is a metric space, and $f : [0, \infty) \to [0, \infty)$ is a function such that

   - $f(x) = 0$ iff $x = 0$;
   - f is nondecreasing;
   - f is concave in the sense that $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$.

   Show that d_f is a metric on X, where

$$d_f(x, y) := f(d(x, y)), \forall x, y \in X.$$

4. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$, and d is a metric on X. Show that d is the associated metric of a norm $|| \cdot ||$ on X if and only if d satisfies the following conditions:

   - Translation invariance:

$$d(x + z, y + z) = d(x, y), \forall x, y, z \in X;$$

   - Positive homogeneity:

$$d(\lambda x, \lambda y) = |\lambda| d(x, y), \forall x, y \in X \text{ and } \lambda \in \mathbb{R} \text{ or } \mathbb{C}.$$

5. Prove that a subset F of a metric space X is closed if and only if every convergent sequence of elements in F converges to a limit in F. that is, if $x_n \to x$ and $x_n \in F$ for all n, then $x \in F$.

6. Let (X, d_X), $(Y, d_Y))$, and (Z, d_Z) be metric spaces and $f : X \to Y$, and $g : Y \to Z$ be continuous functions. Show that the composition

$$h = g \circ f : X \to Z,$$

defined by $h(x) = g(f(x))$, is also continuous.