Resolvent: $R_\lambda = (\lambda I - A)^{-1}$ $\leftarrow$ Remark notation

Self-adjoint operators:

If $A \in B(\mathcal{H})$ is self-adjoint, then

- $\|A\| = \sup\limits_{\|x\|=1} |\langle x, Ax \rangle|$
- $\|A^2\| = \|A\|^2$
- $\sigma(A) \subseteq \mathbb{R}$, more specifically, $\sigma(A) \subseteq [-\|A\|, \|A\|]$

  $r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\} = \lim\limits_{n\to+\infty} \|A^n\|^{\frac{1}{n}} = \|A\|$
- $\sigma_r(A) = \phi$
- $\forall \lambda \neq \mu \in \sigma_p(A)$, associated eigenvectors $x \perp y$.

  (provide with orthonormal basis)

  Note that $A$ may not have eigenvalues
- $\langle Ax, x \rangle$ is real, introduce a partial order on the set of self-adjoint operators:

  $A_1 \leq A_2 \iff \langle A_1 x, x \rangle \leq \langle A_2 x, x \rangle \ \forall x \in \mathcal{H}$

treat self-adjoint operators like real numbers.

eg: $M: L^2[0,1] \to L^2[0,1]$

$Mf(x) = xf(x)$

1° bounded $\quad \|Mf(x)\|^2 = \int_0^1 (xf(x))^2 dx \leq \int_0^1 f(x)^2 dx = \|f\|^2$

2° self-adjoint

$\langle Mf, g\rangle = \int_0^1 \overline{xf(x)}\, g(x)\, dx = \int_0^1 \overline{f(x)}\, x g(x)\, dx = \langle f, Mg\rangle$

$\sigma(M) \subseteq [-\|M\|, \|M\|] \subseteq [-1, 1]$

$\sigma_r(M) = \phi$

$Mf = \lambda f \iff xf(x) = \lambda f(x) \iff f = 0$ a.e.

$\sigma_p(M) = 0$

$M\delta_\lambda(x) = \lambda \delta_\lambda(x)$ however $\delta_\lambda(x) \notin L^2([0,1])$

eg: orthogonal projection.

Compact operator $K$

- iff $\forall (x_n) \subseteq \mathcal{H}$ bounded. $(Kx_n)$ precompact
- iff $K$ maps weakly convergent sequences to strong convergent sequences.

Compact operator behaves like a regulator which produces finite dimensional behavior.

- The class of compact operators is a closed subspace of $B(\mathcal{H})$
- $SK$, $KS$ are compact if $S$ bounded $K$ compact
- $K^*$ is compact.

Prop: $K \in B(\mathcal{H})$ compact, $(T_n) \subseteq B(\mathcal{H})$,

$\downarrow$ separable

$T \in B(\mathcal{H})$ s.t $\forall x \in \mathcal{H}$, $T_n x \to Tx$

Then $T_n K \to TK$.

proof: Suppose $\|T_n K - TK\| \not\to 0$, $\exists\, \delta > 0$, $\{T_{n_j} K\}$ s.t

$$\|T_{n_j} K - TK\| > \delta$$

Choose unit vector $(X_{n_i}) \subseteq \mathcal{H}$, s.t

$$\|(T_{n_j} K - TK)(X_{n_i})\| > \delta$$

Since $K$ compact, $\exists\, X_{n_j}$ s.t $KX_{n_j} \to y$

Then $$\delta < \|(T_{n_j} K - TK)(X_{n_j})\| \leq \|(T_{n_j} - T)(KX_{n_j} - y)\| + \underbrace{\|(T_{n_j} - T)y\|}_{\to 0}$$

$$\leq 2\|T_n\| \, \underbrace{\|KX_{n_j} - y\|}_{\to 0}$$

eg: bounded Operators of finite rank are compact operator.

its image is a finite dimensional subspace

proposition: the set of finite rank bounded operators is dense in the space of compact operators on a separable Hilbert space

Remark: 1. Not true for Banach space, even for separable Banach space.

Proof: $\{\phi_n\}$ be ONB of $H$, $H_n = \mathrm{span}\{\phi_k\}_{k=1}^{n}$

$P_n : H \to H$

$$P_n x = \sum_{j=1}^{n} \langle x, \phi_j \rangle \phi_j \to x \quad \forall x \in H.$$

if $K$ is compact, then $P_n K \to K$

$R(P_n K) \subseteq R(P_n) = H_n$ finite dimensional.

Thm [spectrum thm for self-adjoint compact operators]

$$K = \sum_k \lambda_k P_k$$, where $\lambda_k$ are real eigenvalues.

$P_k$ orthogonal projection onto finite-dim eigenspace of eigenvectors with $\lambda_k$

[Converse] $K = \sum \lambda_k P_k$ is self-adjoint, compact

[generalization] $f$ real, continuous with $f(0)=0$, $K = \sum f(\lambda_k) P_k$ is self-adjoint, compact.

Prelim: $X \in \mathcal{B}(\mathcal{H})$ self-adjoint, compact, and $\frac{1}{3}X^3 - X^2 + \frac{2}{3}X = 0$

then $X$ can be written as sum of two finite-rank

orthogonal projections.

proof: $f(x) = \frac{1}{3}x^3 - x^2 + \frac{2}{3}x$ , $f(0) = 0$

$$0 = f(X) = f\left(\sum_k \lambda_k P_k\right) = \sum_k f(\lambda_k) P_k$$

$$\frac{1}{3}\lambda_k^3 - \lambda_k^2 + \frac{2}{3}\lambda_k = \frac{1}{3}\lambda_k(\lambda_k - 1)(\lambda_k - 2)$$

$\lambda_1 = 1 \quad \lambda_2 = 2$

$X = P_1 + 2P_2$