08/09. Sec. 10.7 Power Series

Def: A power series about $x=0$ is $\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$

about $x=a$ is $\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$

$\{a_n\} \to \sum a_n$

eg. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

$$= \lim_{n\to\infty} \frac{1-x^n}{1-x} = \begin{cases} \text{converges} & |x|<1 \to \frac{1}{1-x} \\ \text{diverges} & |x|>1 \\ \text{diverges.} & |x|=1 \end{cases}$$

$$\frac{1}{1-x} \triangleq \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \underline{|x|<1}$$

$$\downarrow$$

$$=$$

eg. $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n$

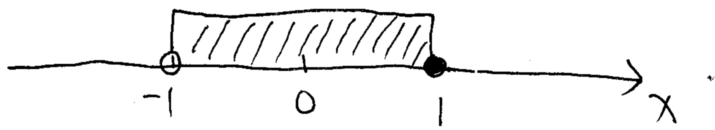
$$= \lim_{n\to\infty} \frac{1-\left(-\frac{1}{2}(x-2)\right)^n}{1-\left(-\frac{1}{2}(x-2)\right)} = \begin{cases} \text{converge} & \left|\frac{x-2}{2}\right| < 1 \to \text{Limit.} \\ \text{diverges} & \left|\frac{x-2}{2}\right| > 1 \end{cases}$$

further discuss endpoints

eg. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\qquad \sum C_n x^n \qquad C_n = \frac{(-1)^{n-1}}{n}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \left|\frac{n}{n+1} \cdot x\right| \to |x| \begin{cases} <1 & \text{absolutely conv.} \\ >1 & \text{diverge.} \end{cases}$$

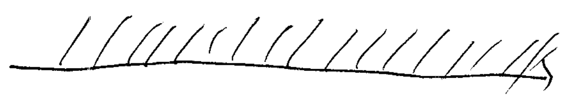
$|x| = 1$ $\begin{cases} x = 1 & \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \to \underline{\text{converge}} \text{ Apply Alternating Series Test} \\ x = -1 & \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} \to \text{diverge} \end{cases}$



eg. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x}{n+1} \right| = 0$$

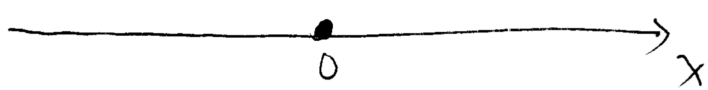
absolutely converges for $x \in \mathbb{R}$ (in)



eg. $\sum n!\, x^n$

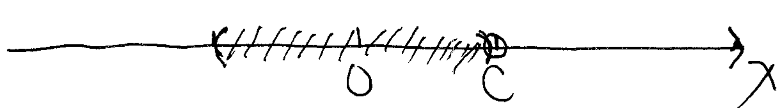
$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} \left| \underline{(n+1) \cdot x} \right| = \infty > 1$$

diverges for all $x$ except $x = 0$



Thm. if $\sum a_n x^n$ converges at $x = c \neq 0$,

then it converges absolutely for all $x$, with $|x| < |c|$.



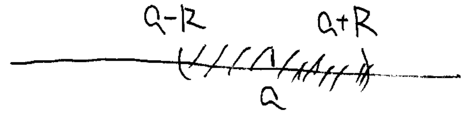
if .... diverges at $x = d \neq 0$

then it diverges for all $|x| > |d|$.

~~Corollary~~ Corollary to $\sum C_n(x-a)^n$

① There is a positive number, $R$ such that $\sum C_n(x-a)^n$ converge $|x-a|<R$

diverge $|x-a|>R$




② converges for all $x$ $(R=\infty)$

③ only converges at $x=a$ / diverges otherwise $(R=0)$

$R \rightarrow$ radius of convergence

☆ Summary: How to test a power series for convergences.

Step 1. Use Ratio/Root Test to find the radius of convergence and interval of convergences. (absolute convergence)

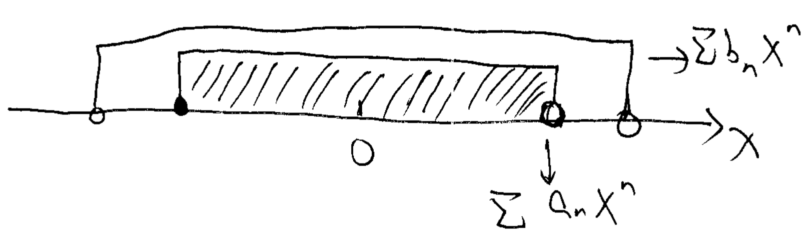
Step 2. For endpoints, apply Comparison test / Integral test / Alternating Series Test

Step 3. complete the interval of convergence.

- Add/Subtract Rules: $\sum a_n x^n$, $\sum b_n x^n$,

  on their intersection of interval of convergences.

$$\sum a_n x^n \pm \sum b_n x^n = \sum (a_n \pm b_n) x^n$$




- Multiplication $\sum a_n x^n$ $\sum b_n x^n$

Thm: $\sum a_n(x-a)^n$ absolutely converges $|x-a|<R$

if $f(x) = \underline{\sum a_n(x-a)^n}$ differentiate terms by terms

then $f'(x) = \underbrace{\sum a_n \cdot n(x-a)^{n-1}}_{\text{limit}}$ at $|x-a|<R$

eg $f(x) = \underline{\frac{1}{1-x}} = \sum_{n=0}^{\infty} x^n \qquad -1<x<1$

$$\left(\frac{1}{1-x}\right)' = \underline{\sum_{n=1}^{\infty} n x^{n-1}} \qquad -1<x<1$$

$$\| $$

$$\frac{1}{(1-x)^2}$$

$$\lim_{n\to\infty}\left|\frac{n x^{n-1}}{(n-1)x^{n-2}}\right| = \lim_{n\to\infty}\left|\frac{n}{n-1}\cdot x\right| = |x| < 1$$

eg. $\underline{f(x)} = \underline{\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}} \rightarrow$ ~~absolutely~~ converges

$$\underline{f'(x)} \neq \underline{\sum_{n=1}^{\infty} \boxed{\frac{n!\cos(n!x)}{n^2}}} \not\to 0 \quad \text{diverges.}$$

Thm: $\sum C_n(x-a)^n$ absolutely converges $|x-a|<R$

$$f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$$

$|x-a|<R$

$$\int_a^x f(s)\,ds = \underline{\sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1}} + \underline{C}$$

Intergration term by term

$$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

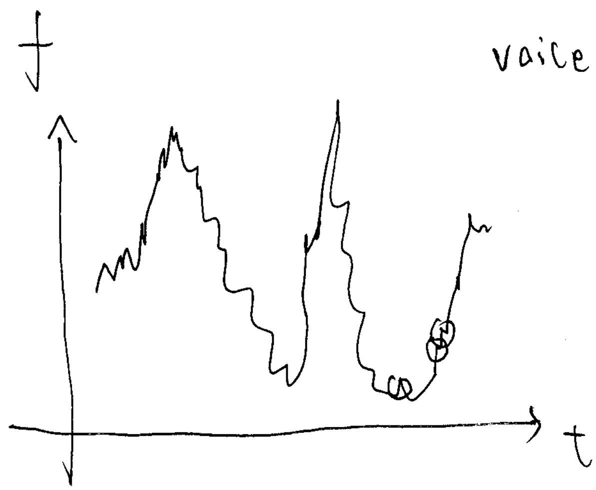
Thm: if $\sum a_n x^n$ and $\sum b_n x^n$ both absolutely converges at $|x|<R$

then $\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$ at $|x|<R$.

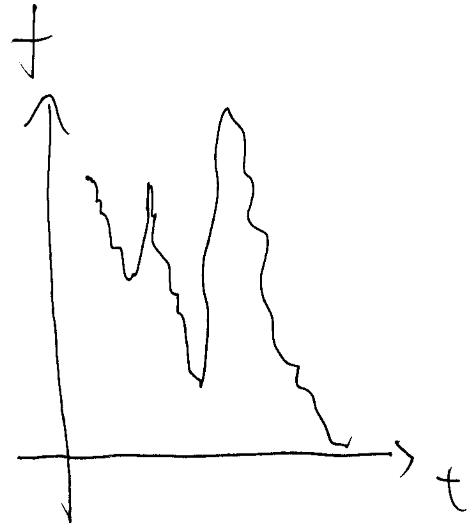
Remark 1: $(\underbrace{1+2}_{3})(\underbrace{3+4}_{7}) \quad 21 \quad \rightarrow \quad c_0 + c_1 x + c_2 x^2$

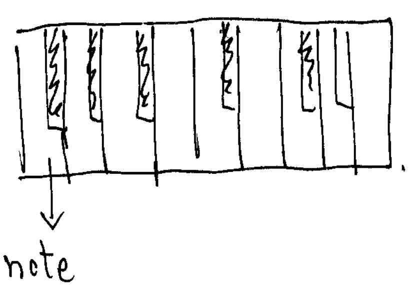
only for absolute convergence.

eg.




voice




chord :

Property :

Thm: $\sum a_n x^n$ absolutely converges for $|x|<R$,

then $\sum a_n (f(x))^n$ absolutely converges for $|f(x)|<R$.

$$\sum_{n=0}^{\infty} (4x^2)^n \Leftrightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x|<1 \qquad \rightarrow |x| < \frac{1}{2}.$$

$$\sum_{n=0}^{\infty} (4x^2)^n = \frac{1}{1-4x^2} \quad \text{for } |4x^2|<1 \Rightarrow |x^2| < \frac{1}{4}$$

eg: $\underline{\sum_{n=0}^{\infty} (-1)^n x^{2n}} = \frac{1}{1-(-x^2)} = \underline{\frac{1}{1+x^2}}$ at $-1 < x < 1$.

$$\underline{\int \frac{1}{1+x^2}\, dx}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1} x + C.$$

$$1 \cdot \frac{x}{1} + (-)\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$\underline{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots} = \underline{\tan^{-1} x} + C$$

eg. $\underline{\frac{1}{1+t}} = 1 - t + t^2 - t^3 + \cdots$

$$\underline{\ln(1+t)+C} = \underline{t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots}$$