Completeness is not a topological property

in complete $\quad$ complete

$\{\frac{1}{n}\} \longleftrightarrow \{n\}$

- Homeomorphic spaces could have different completeness $\quad (\mathbb{R}, d_e) \longleftrightarrow ((0,1), d_e)$

- One topological space that have different metrics inducing its topology is complete under one metric
  is not complete under the other one.

eg space $\{\frac{1}{n} \mid n \in \mathbb{N}\} = X$

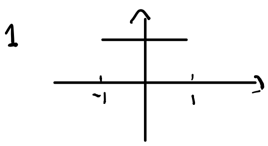
with d_e is not complete
for $0 \notin X$
but induce discrete topology
(every set is open)

with $d_d(x,y) = 1$ if $x \neq y$
is complete
induce the same discrete topology
Every Cauchy sequence will eventually be constant.

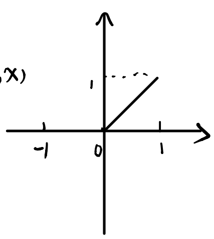
eg $(-1,1)$ $\quad ds := |dx|$

with $d(x,y) = |x-y|$ is not complete

$$d(x,y) = \int_x^y |dx|$$



$d(0,x)$



$(-1,1)$

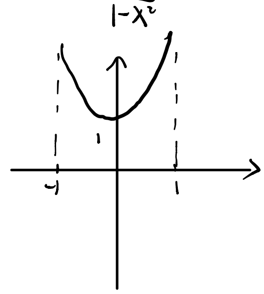
with hyperbolic metric $ds := \frac{|dx|}{1-x^2}$

$d(x,y) \geq 0$

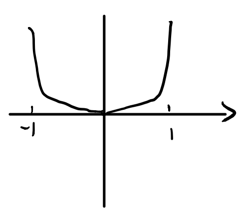
$d(x,y) = 0$ iff $x = y$

$$d(x,y) = \int_x^y \frac{|dx|}{1-x^2}$$

$\frac{1}{1-x^2}$



$d(0,x)$



$\simeq (-\infty, +\infty)$

$-1 < s < 1.$

$$\int_x^y \frac{ds}{1-s^2}$$

$$= \frac{1}{2}\left(\ln|x+1| - \ln|x-1|\right)\Big|_x^y$$

$$= \frac{1}{2}\left(\ln\left|\frac{y+1}{x+1}\right| - \ln\left|\frac{y-1}{x-1}\right|\right)$$

introduce same topology
(prove open ball in terms of a metric is open in terms of the other one)
but complete.
because $(-1,1)$ behavior like $(-\infty, +\infty)$

A metrizable topological space is called complete-metrizable
if there is at least one complete metric inducing its topology.

$(X, \tau) \longleftrightarrow (Y, \tau')$ homeomorphic

- X is not complete-metrizable $\Rightarrow$ Y is not complete-metrizable.

  $\mathbb{Q}$ is not complete-metrizable

- X is complete under some metric but not the other $\Rightarrow$ Y is complete-metrizable but not complete under some metric.

discrete topology on a countable set

$(\mathbb{N}, \lvert x-y \rvert)$	$(\mathbb{N}, \lvert \frac{1}{x}-\frac{1}{y} \rvert) \longleftrightarrow$	$(\frac{1}{\mathbb{N}}, \lvert x-y \rvert)$	$(\frac{1}{\mathbb{N}}, \lvert \frac{1}{x}-\frac{1}{y} \rvert)$
complete	not complete	not complete	complete.

- X is complete under every metric. So is Y.

  discrete topology on a finite set

Recall if f is uniformly continuous, (a_n) is Cauchy, $\{f(a_n)\}$ is Cauchy

$\underbrace{\text{uniformly continuous}}$ homeomorphism preserves completeness

$\downarrow$

generalization

1. two topological vector space V and W

   $f: V \to W$ is uniformly continuous if for any neighborhood B of zero in W,
   there exists a neighborhood A of zero in V such that
   $v_1 - v_2 \in A \Rightarrow f(v_1) - f(v_2) \in B$.

2. generalize to "uniform space"