08/31. Sec 14.2

$$\lim_{(x,y)\to(x_0,y_0)} \overset{?}{=} \lim_{x\to x_0}\lim_{y\to y_0}$$

$$\overset{?}{=} \lim_{y\to y_0}\lim_{x\to x_0}$$

eg: $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & x\neq 0,\ y\neq 0 \\ 0 & x=0,\ y\neq 0 \\ 0 & y=0,\ x\neq 0 \end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y) = ?$

$$\lim_{x\to 0}\Big(\underbrace{\lim_{y\to 0} f(x,y)}_{\text{Fixed } x}\Big) = \underbrace{\lim_{x\to 0}\lim_{y\to 0}\big(\underbrace{x\sin\tfrac{1}{y}}_{\text{no limit}} + \cancelto{0}{y\sin\tfrac{1}{x}}\big)}_{\text{does not exist}}$$

$$\lim_{y\to 0}\lim_{x\to 0}\underbrace{\big(\cancelto{0}{x\sin\tfrac{1}{y}} + \underbrace{y\sin\tfrac{1}{x}}_{\text{no limit}}\big)}_{\text{limit does not exist}}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\left(x\sin\frac{1}{y} + y\sin\frac{1}{x}\right)$$

$$\lim_{(x,y)\to(0,0)}\left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| \leq \left|x\sin\frac{1}{y}\right| + \left|y\sin\frac{1}{x}\right| \leq \lim_{(x,y)\to(0,0)} |x|+|y| = 0$$

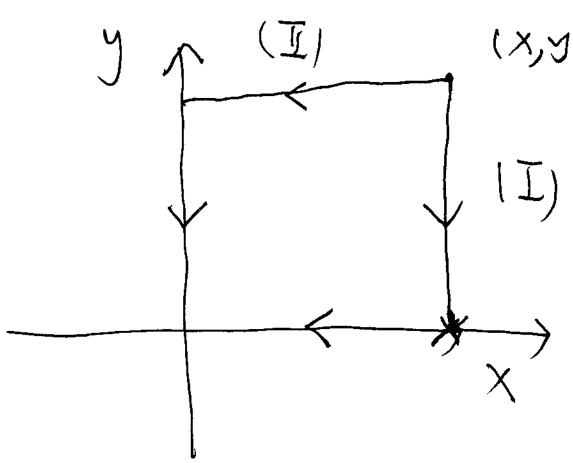
Recall: $|a+b| \leq |a|+|b|$.

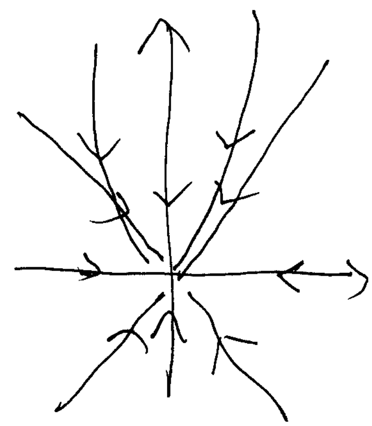
$$\lim_{(x,y)\to(0,0)} x\sin\frac{1}{y} + y\sin\frac{1}{x} = 0. \quad \blacksquare$$

eg. $f(x,y) = \dfrac{x^2 - y^2 + x^3 + y^3}{x^2+y^2}$ $\quad \lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

(I) $\lim\limits_{x\to 0} \lim\limits_{y\to 0} \dfrac{x^2 - y^2 + x^3 + y^3}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{x^2 + x^3}{x^2} = \lim\limits_{x\to 0} (1+x) = 1.$

(II) $\lim\limits_{y\to 0} \lim\limits_{x\to 0} \dfrac{x^2 - y^2 + x^3 + y^3}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{-y^2 + y^3}{y^2} = \lim\limits_{y\to 0} (-1+y) = -1.$





By two-path Test, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

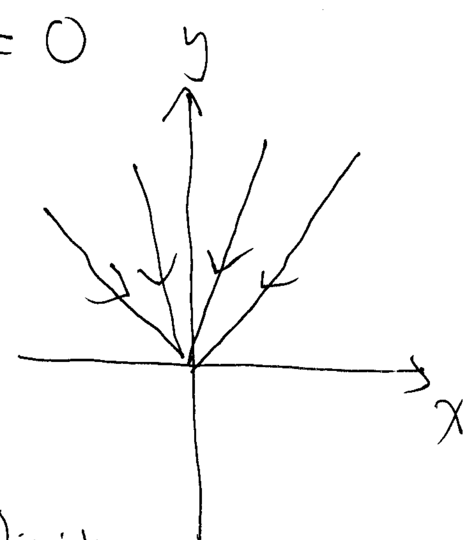
eg. $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad \lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$ $\quad 0?$

(I) $\lim\limits_{x\to 0} \lim\limits_{y\to 0} \dfrac{xy}{x^2+y^2} = \lim \dfrac{0}{x^2} = 0$

(II) $\lim\limits_{y\to 0} \lim\limits_{x\to 0} \dfrac{xy}{x^2+y^2} = \lim \dfrac{0}{y^2} = 0$

$y = mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x \cdot mx}{x^2 + m^2x^2} = \frac{m}{1+m^2}$$



For different m, you get different limits.

$\lim\limits_{(x,y)\to(x_0,y_0)}$	$\lim \lim$
exist	does not exist
does not exist	exist / different
exist / does not exist	exist / same

$$\lim_{(x,y)\to(1,1)} xy = ?$$

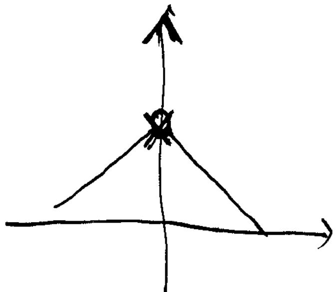
Thm: If $\lim\limits_{x\to x_0} \lim\limits_{y\to y_0} \neq \lim\limits_{y\to y_0} \lim\limits_{x\to x_0}$, then $\lim\limits_{(x,y)\to(x_0,y_0)}$ does not exist.

Remark: If $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad y = mx$

$$\frac{mx^2}{x^2+m^2x^2} = \boxed{\frac{m}{1+m^2}} \qquad \frac{\cancel{m}}{2\cancel{m}} = \boxed{\frac{1}{2}}$$

Def: A function $f(x,y)$ is continuous at (x_0, y_0) if

1. $f(x,y)$ is defined at (x_0, y_0)
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$



eg: $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is discontinuous at $(0,0)$

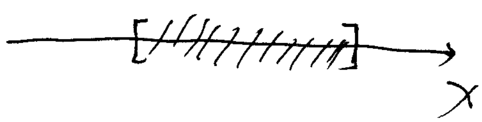
Thm: Continuity of Compositions.

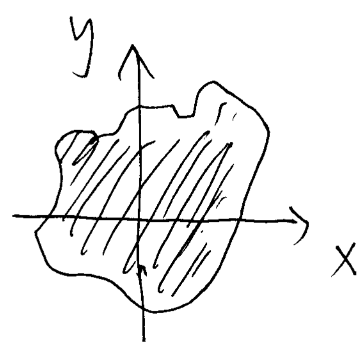
If f is continuous at (x_0, y_0), g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f = g(f(x,y))$ is continuous at (x_0, y_0)

$$\underline{e^{x-y}} = g(f(x,y))$$

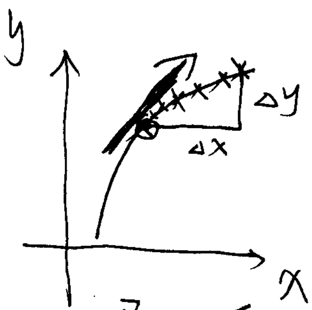
$$\begin{cases} g(z) = \underline{e^{z}} \\ \underline{f(x,y) = x-y} \end{cases}$$

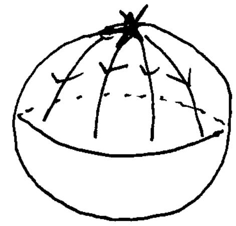
Thm: If $f(x,y)$ is continuous on some bounded / closed region, then the $f(x,y)$ has $\underline{\max}$ / $\underline{\min}$ on this region.

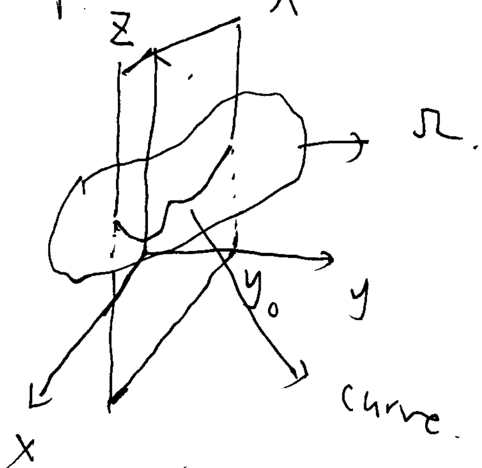




Sec. 14.3. Partial Derivatives.







The partial derivative $f(x,y)$ w.r.t x

is $$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

The partial derivative $f(x,y)$ w.r.t y.

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}$$

$\frac{\partial f}{\partial x}$, f_x, z_x, $\frac{\partial z}{\partial x}$.

eg: $f(x,y) = x^2 + 3xy + y - 3$.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,0)} = 2x + 3y + 0 + 0 \Big|_{(1,0)} = 2.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,0)} = 0 + 3x + 1 + 0 \Big|_{(1,0)} = 4.$$

eg: $f(x,y) = y \sin xy$

$$\frac{\partial f}{\partial y} = 1 \cdot \sin xy + y \cdot \cos xy \cdot (x)$$

$$= \sin xy + xy \cdot \cos xy.$$

eg: $\underline{yz - \ln z = x+y}$ $\quad z = f(x,y)$

$\frac{\partial z}{\partial x} = ?$ $\quad z = f(x,y)$

$\underline{yz = x+y} \rightarrow z = \frac{x+y}{y}$

$$\frac{\partial (yz)}{\partial x} - \frac{\partial (\ln z)}{\partial x} = \frac{\partial}{\partial x}(x+y)$$

$$y \cdot \frac{\partial z}{\partial x} - \frac{1}{z} \cdot \frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right) \frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}}$$

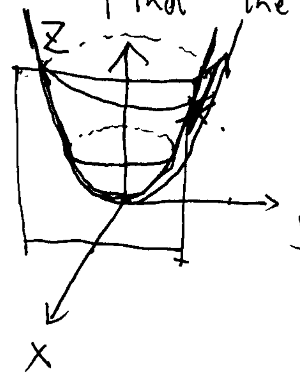
$\frac{\partial z}{\partial y}$ $\quad \frac{\partial (yz)}{\partial y} - \frac{\partial (\ln z)}{\partial y} = \frac{\partial}{\partial y}(x+y)$

$$1 \cdot z + y \cdot \frac{\partial z}{\partial y} - \frac{1}{z} \cdot \frac{\partial z}{\partial y} = 0 + 1$$

$$\left(y - \frac{1}{z}\right) \frac{\partial z}{\partial y} = 1 - z \qquad \frac{\partial z}{\partial y} = \frac{1-z}{y - \frac{1}{z}}$$

eg: The plane $x=1$ intersects with $z = x^2 + y^2$ in a parabola

Find the <u>slope</u> of the tangent at $(1,2,5)$



$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y \Big|_{(1,2,5)} = \underline{4}$$

f_x, f_y

$f_{xx}, f_{xy}, f_{yx}, f_{yy}$

$$f_{xy} = (f_x)_y \qquad \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$$

$$f(x,y) = x\cos y + yx^2$$

$$f_x = \cos y + 2yx \qquad f_y = -x\sin y + x^2$$

$$f_{xx} = 2y \qquad \underline{f_{xy}} = \underline{-\sin y + 2x} \qquad f_{yx} = \underline{-\sin y + 2x} \qquad f_{yy} = -x\cos y$$

mixed.

Thm (The Mixed Derivative Thm) If $f(x,y)$ and derivatives f_x, f_y, f_{xy}, f_{yx} are defined on open region containing (x_0, y_0) and all are continuous at (x_0, y_0),

Then we have $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

Differentiability: $\Delta y = f'(x_0)\cdot\Delta x + \varepsilon\,\Delta x$

$\varepsilon \to 0$ when $\Delta x \to 0$

$y = f(x)$ is differentiable at $x = x_0$