Different Modes of Convergence / topology

$T_2 \subseteq T_1$ T_1 is stronger than T_2
$\Leftrightarrow$ $x_n \to x$ w.r.t T_1 $\Rightarrow$ $x_n \to x$ w.r.t T_2

X, Y normed linear space

on $B(X,Y)$ ① norm topology (uniform topology) induced by $\|\cdot\|_{op}$ $\|T_n - T\|_{op} \to 0$
$\cup$
② strong topology $T_n \to T$ strongly if $T_n x \to Tx$ $\|T_n x - Tx\|_Y \to 0$ $\forall x$
$\cup$
③ weak topology

nonzero element can have zero norm.

on X ① norm topology (strong topology) induced by $\| \ \|_X$ $\|x_n - x\|_X \to 0$
$\cup$
② weak topology : $x_n \to x$ if $f(x_n) \to f(x)$ $\forall f \in X^*$ generated by semi-norm $\|x\|_\lambda := |\lambda(x)|$ for all $\lambda \in V^*$ sub
$\cup$ $\hookrightarrow$ weakest topology that make bdd linear functionals continuous.
③ W-weak topology : $W \subseteq X^*$, the weakest topology that $f(x_n) \to f(x)$, $\forall f \in W \subseteq X^*$

weak = W-weak iff $W = X^*$

weak = strong iff X is finite-dim

if X is infinite dim $T_W \subsetneq T_S$ strictly weaker

want to show open (closed) sets in the strong topology are not open in the weak topology.

- eg unit sphere $S = \{\|x\| = 1\}$ is closed w.r.t norm, not closed w.r.t weak topology $\sigma(X, X^*)$

$$\bar{S}^{\sigma(X,X^*)} = B_X = \{\|x\| \le 1\} \qquad \bar{S} = S$$

Thm Let $x_0 \in X$. A basis of neighborhood of x_0 for the weak topology is given by the collection of the sets of the form

(identity a basis of neighborhood for $\sigma(X, X^*)$)

$$W_{\varepsilon, f_1 \cdots f_n} = \{x \in X \mid \forall i \in \{1, \cdots n\} \ \ |f_i(x) - f_i(x_0)| < \varepsilon\}$$
$n \in \mathbb{N}$, $\varepsilon > 0$, $f_1, \cdots f_n \in X^*$

Recall: A collection $\mathcal{N}$ of neighborhoods of a point $x \in X$ is a neighborhood basis if $\forall$ neighborhood V of x, $\exists W \in \mathcal{N}$ s.t $W \subset V$.

$x \in \bar{S}^{\sigma(X,X^*)} \Leftrightarrow \forall$ neighborhood of x contains a point of S other rather x

For simplicity, we just show $0 \in \bar{S}^{\sigma(E,E^*)}$

Let O be any weak neighborhood of 0
$\exists \varepsilon > 0$, n and $f_1, f_2 \cdots f_n \in X^*$ such that
$W_{\varepsilon, f_1, f_2 \cdots f_n} \subset O$

$\Phi: X \to \mathbb{R}^n$

$x \to (f_1(x), f_2(x) \cdots f_n(x))$ is linear

$\ker \Phi = \bigcap_{i=1}^{n} \ker f_i$

By rank-nullity theorem $\quad \operatorname{Dim} \ker \Phi + \underbrace{\operatorname{Dim} \operatorname{Im} \Phi}_{\leq n} = \operatorname{Dim} X = \infty \qquad \left(\begin{array}{l} \text{if } \infty + \infty = \infty \\ n + \infty = \infty \end{array} \right)$

$\Rightarrow \ker \Phi$ is infinite dimension

$\exists\, x \neq 0 \in \ker \Phi \qquad f_i(x) = 0 \quad \forall\, i = 1 \cdots n$

$\forall\, \lambda \quad f_i(\lambda x) = 0 < \varepsilon$

$|f_i(\lambda x) - f_i(0)| = 0 < \varepsilon \Rightarrow \lambda x \in W \subset O$

take $\lambda = \frac{1}{\|x\|} \qquad \frac{x}{\|x\|} \in O \quad \frac{x}{\|x\|} \in S$

$\forall$ weakly open neighborhood O of 0, it intersect with S

$\Rightarrow 0$ is in the weak closure of S.

- $U = \{ \|x\| < 1 \}$ is open w.r.t norm, but not open w.r.t the weak topology.

Remark 1. the weak topology is strictly coarser than the strong topology for infinite-dimensional space.

However, there exists infinite dimensional space that every weakly convergent sequence is strongly convergent. (Schur's property)

l^1 space

Because two metric spaces with the same convergent sequence has identical topology.

But two topological spaces with the same convergent sequences need not have identical sequences.

Because infinite-dimension spaces with weak topology is not metrizable.

2. $x_n \rightharpoonup x$ in X then $\quad \|x_n\|$ is bounded

$\|x\| \leq \liminf_{n} \|x_n\|$

3. One cannot always conclude that there exists sequence converging weakly to elements in the closure.

Let $\{y^{(n)}\} \subset l^1 \rightharpoonup 0$ in l^1 but $\exists \varepsilon > 0$ s.t $\|y^{(n)}\| \geq 3\varepsilon$

$(l^1)^* = l^\infty$

$l^1 \subset (l^\infty)^*$