Problem 1. Cauchy sequence and its subsequence

Proof. (a) Let (x_{n_k}) denote a convergent subsequence of (x_n), let $x_{n_k} \to x$ as $k \to \infty$, i.e for any $\varepsilon > 0$, there exists a K such that $\forall k \geqslant K$, we have $d(x_{n_k}, x) < \varepsilon$.

Since (x_n) is Cauchy sequence, for any $\varepsilon > 0$, there exists a N such that for any $n, m \geqslant N$, we have $d(x_n, x_m) < \varepsilon$.

For any $\varepsilon > 0$, pick $N_0 = \max(N, n_K)$, for any $n, m \geqslant N_0$, we have

$$d(x_n, x) \leqslant d(x_n, x_m) + d(x_m, x) \leqslant \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $x_n \to x$ as $n \to \infty$.

(b) For ε_1, since (x_n) is Cauchy sequence, there exists a N_1 such that for any $m > n \geqslant N_1$, we have $d(x_n, x_m) \leqslant \varepsilon_1$. We then define $n_1 = N_1$. For any $n_2 > n_1$, we have $d(x_{n_1}, x_{n_2}) \leqslant \varepsilon_1$.

For each ε_k, suppose $n_k = N_k$, where for any $m > n_k \geqslant N_k$, $d(x_{n_k}, x_m) \leqslant \varepsilon_k$, we then define recursively n_{k+1}. For ε_{k+1}, there exists a N_{k+1}, such that for $m > n \geqslant N_{k+1}$, we have $d(x_n, x_m) \leqslant \varepsilon_{k+1}$.

If $N_{k+1} > N_k$, then we define $n_{k+1} = N_{k+1}$. Since $n_{k+1} > n_k = N_k$, $d(x_{n_k}, x_{n_{k+1}}) \leqslant \varepsilon_k$. And for any $m > n_{k+1} = N_{k+1}$, $d(x_{n_{k+1}}, x_m) \leqslant \varepsilon_{k+1}$.

If $N_{k+1} \leqslant N_k$, then we define $n_{k+1} = N_k + 1$. Since $n_{k+1} > n_k = N_k$, $d(x_{n_k}, x_{n_{k+1}}) \leqslant \varepsilon_k$. And for any $m > n_{k+1} = N_k + 1 > N_{k+1}$, $d(x_{n_{k+1}}, x_m) \leqslant \varepsilon_{k+1}$.

□

Problem 2.

Proof. (a) Assume there is a point z in both $B_R(x)$ and $B_r(y)$, i.e., $d(x, z) < R$ and $d(y, z) < r$. Thus $R + r > d(x, z) + d(y, z) \geqslant d(x, y) \geqslant R + r$, contradiction!

(b) Let $z \in B_r(y)$, i.e., $d(y, z) < r$. Then $d(z, x) \leqslant d(z, y) + d(y, x) < r + R - r = R$, i.e., $z \in B_R(x)$.

(c) 1. Given $\mathbb{R}^2$ associated with metric $d(x, y) = \frac{e(x,y)}{1+e(x,y)}$, where $e(x, y)$ denotes the Euclidean metric. For $x = (0, 0), y = (5, 0)$ and $R = \frac{2}{3}, r = \frac{2}{3}$. We notice that

$$B_{2/3}(x) = \left\{ z \in \mathbb{R}^2 : d(z, x) < \frac{2}{3} \right\} = \left\{ z \in \mathbb{R}^2 : e(z, x) < 2 \right\};$$
$$B_{2/3}(y) = \left\{ z \in \mathbb{R}^2 : d(z, y) < \frac{2}{3} \right\} = \left\{ z \in \mathbb{R}^2 : e(z, y) < 2 \right\},$$

which yields that $B_{2/3}(x)$ and $B_{2/3}(y)$ are disjoint. However $d(x, y) \leqslant 1 < \frac{2}{3} + \frac{2}{3}$.

2. Given $\mathbb{R}^2$ associated with discrete metric $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. For $x = (0, 0), y = (5, 0)$ and $R = r = 2$. We note that $B_2(x) = \mathbb{R}^2$ so that $B_2(y) \subset B_2(x) = \mathbb{R}^2$, however $d(x, y) = 1 > 0 = 2 - 2$.

□

Problem 3. Extension of Lipschitz functions on a dense set.

Proof. (a) **[Existence]** For any $x \in \tilde{X}\backslash X$, since X is a dense subset, so there is a sequence (x_n) in X such that $x_n \to x$. We define $\tilde{f}(x) := \lim_{n\to\infty} f(x_n)$. For any $x \in X$, we define $\tilde{f}(x) = f(x)$. It satisfies $\tilde{f}|_X = f$ obviously.

First we want to show $\tilde{f}(x)$ is well-defined on $\tilde{X}$.

For any $x \in \tilde{X}\backslash X$, since $x_n \to x$, for any $\varepsilon > 0$, there exists a N such that for any $m, n \geqslant N$, we have

$$d_{\tilde{X}}(x_n, x_m) \leqslant d_{\tilde{X}}(x_n, x) + d_{\tilde{X}}(x, x_m) \leqslant 2\varepsilon,$$

Furthermore, due to the fact that f is Lipschitz continuous function, we have

$$d_Y(f(x_n), f(x_m)) \leqslant Lip(f) d_{\tilde{X}}(x_n, x_m) \leqslant 2Lip(f)\varepsilon,$$

which yields that $(f(x_n))$ is a Cauchy sequence in (Y, d_Y). Since Y is complete, $(f(x_n))$ converges in Y.

On the other hand, assume that there exists a different sequence (y_m) in X that $y_m \to x$. For any ε, there exists N, M such that for $n \geqslant N$ and $m \geqslant M$, we have $d(x_n, x) \leqslant \varepsilon$ and $d(y_m, x) \leqslant \varepsilon$. Then

$$d_Y(f(x_n), f(x_m)) \leqslant Lip(f) d_{\tilde{X}}(x_n, y_m) \leqslant Lip(f) \left(d_{\tilde{X}}(x_n, x) + d_{\tilde{X}}(x, y_m)\right) \leqslant 2Lip(f)\varepsilon,$$

which shows that the limit $\tilde{f}(x)$ does not depend the choice of sequences.

Second we want to show $\tilde{f}(x)$ is Lipschitz and $Lip(\tilde{f}) = Lip(f)$. (which implies it is continuous, as well.)

We show $d_Y(y, \cdot) : Y \mapsto \mathbb{R}$ is continuous with respect to y. Then the same holds for $d_Y(\cdot, y)$ and for $d_{\tilde{X}}$. In fact,

$$-d_Y(y, y_0) \leqslant d_Y(y, \cdot) - d_Y(y_0, \cdot) \leqslant d_Y(y, y_0).$$

Furthermore, we have

$$\lim_n d_Y(y_n, \cdot) = d_Y(\lim y_n, \cdot).$$

For any $x, y \in \tilde{X}$, assume that $x_n \to x$ and $y_n \to y$ with $(x_n), (y_n)$ in X.

$$\begin{aligned} d_Y(\tilde{f}(x), \tilde{f}(y)) &= d_Y(\lim_n f(x_n), \lim_n f(y_n)) = \lim_n d_Y(f(x_n), f(y_n)) \\ &\leqslant \lim_n Lip(f) d_{\tilde{X}}(x_n, y_n) = Lip(f) d_{\tilde{X}}(\lim_n x_n, \lim_n y_n) = Lip(f) d_{\tilde{X}}(x, y), \end{aligned}$$

which implies that $\tilde{f}$ is Lipschitz and $Lip(\tilde{f}) = Lip(f)$.

[Uniqueness] Suppose there is another continuous map $g : \tilde{X} \mapsto Y$ such that $g|_X = f$. Then there exists a $x \in \tilde{X}\backslash X$ such that $g(x) \neq \tilde{f}(x)$. Since g is continuous, then for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $d_{\tilde{X}}(x_n, x) < \delta$ implies $d_Y(g(x), g(x_n)) < \varepsilon$. Note that $d_Y(g(x), g(x_n)) = d_Y(g(x), f(x_n))$ and Cauchy sequence $(f(x_n))$ in Y has the unique limit. Contradiction!

(b) **[Sharpness]** Let $X = (-\infty, 0) \cup (0, \infty) \subset \tilde{X} = \mathbb{R}$ with Euclidean metric e. Then X is a dense subset of $\tilde{X}$. Let $f(x) = \frac{1}{|x|}$ on X. Then $f(x) : (X, e) \mapsto (\mathbb{R}, e)$ is a continuous function, however there is no way to extend f from X to $\tilde{X}$ continuously.

□

Problem 4. "lower semi-continuous + coercive ⇒ minimum"

Proof. For any $M > 0$, there exists a $R > 0$ such that for $\|x\| \geqslant R$ implies $f(x) \geqslant M$. Thus

$$\inf_{x \in \mathbb{R}^d} f(x) = \inf(\inf_{\|x\| > R} f(x), \inf_{\|x\| \leqslant R} f(x)) \geqslant \inf(M, \inf_{\|x\| \leqslant R} f(x)).$$

Suppose $f(x)$ on $B_R(0) := \{x : \|x\| \leqslant R\}$ is not bounded from below, i.e., for any $n > 0$, there exist a $x_n \in B_R(0)$ such that $f(x_n) < -n$. Since $B_R(x)$ is compact, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ in $B_R(0)$. By lower semi-continuity, we have

$$f(x) \leqslant \liminf_{k \to \infty} f(x_{n_k}) \to -\infty,$$

which makes f is not defined at x, contradiction! Thus there exists a $B \in \mathbb{R}$ such that $\inf_{x \in B_R(0)} f(x) = B$. Then

$$\inf_{x \in \mathbb{R}^d} f(x) \geqslant \min(M, B),$$

which implies f is bounded from below on $\mathbb{R}^d$.

Let (x_n) denote a minimizing sequence such that $f(x_n) \to \inf_{x \in \mathbb{R}^d} f(x)$ (why exists?). Due to coercivity of f, (x_n) is bounded, thus there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x$. By lower semi-continuity of f, we have

$$f(x) \leqslant \liminf_{k \to \infty} f(x_{n_k}) = \inf_{x \in \mathbb{R}^d} f(x).$$

By the definition of inf, we know x is the minimizer. □

Problem 5.

Proof. Suppose S is an $\varepsilon-$net of A, then

$$A \subset \bigcup_{s \in S} B_\varepsilon(s) = \left(\bigcup_{s \in S \cap A} B_\varepsilon(s) \right) \cup \left(\bigcup_{s' \in S \backslash A} B_\varepsilon(s') \right).$$

Note that for any $s' \in S \backslash A$, if $B_\varepsilon(s') \cap A = \emptyset$, then $S \backslash \{s'\}$ is still an $\varepsilon-$net of A.

Let $S' = \{s \in S \backslash A : B_\varepsilon(s) \cap A \neq \emptyset\}$. For any $s' \in S'$, we pick any $k \in B_\varepsilon(s') \cap A$, thus $d(k, s') < \varepsilon = 2\varepsilon - \varepsilon$. By Problem 2b, $B_\varepsilon(s') \subset B_{2\varepsilon}(k)$. In this way, we induce a map from set S' to set K' such that $\bigcup_{s' \in S'} B_\varepsilon(s') \subset \bigcup_{k \in K'} B_{2\varepsilon}(k)$ with cardinality $|K'| \leqslant |S'|$.

Now we define $K := \{s : s \in S \cap A\} \cup K'$. By construction, $K \subset A$ and K is a $2\varepsilon-$net of A, with cardinality $|K| \leqslant |S \cap A| + |K'| \leqslant |S|$. □

Problem 6.

Proof.

1. Assume A has a finite $\varepsilon-$net X_ε for every $\varepsilon > 0$, then for any subset B of A, we have

$$B \subset A \subset \bigcup_{x \in X_\varepsilon} B_\varepsilon(x),$$

i.e., X_ε is also a finite $\varepsilon-$net of B.

2. Assume A is nonempty and totally bounded in (X, d), i.e., for every $\varepsilon > 0$, there exist a finite $\varepsilon-$net $Y_\varepsilon \subset X$ of A.

Define $D := \max_{(y_1,y_2) \in Y_\varepsilon \times Y_\varepsilon} d(y_1, y_2)$. Since Y_ε is finite, D exists and is finite.

Pick $x \in A$ and for any $a \in A$, assume $x \in B_\varepsilon(y_1)$ and $a \in B_\varepsilon(y_2)$ for some $y_1, y_2 \in Y_\varepsilon$. Then

$$d(x, a) \leqslant d(x, y_1) + d(y_1, y_2) + d(y_2, a) \leqslant 2\varepsilon + D.$$

We prove A is bounded.

Take $\mathbb{N}$ with discrete metric $d(x, y) = 1$ when $x \neq y$ and $d(x, y) = 0$ when $x = y$. Then $\mathbb{N}$ is bounded because for any integer $n \in \mathbb{N}$, $d(0, n) \leqslant 1$.

However $\mathbb{N}$ is not totally bounded, because for $\varepsilon \leqslant 1$, there does not exist a finite $\varepsilon-$net X_ε.

3. We assume $\mathbb{R}^n$ equipped with the associated metric d of Euclidean norm or other equivalent norm.

Let $A \subset \mathbb{R}^n$ is bounded, i.e., there exists a $r > 0$ and $x \in A$ such that for any $a \in A$, $d(x, a) \leqslant r$.

For any small $\varepsilon > 0$, define $X_\varepsilon = \{x + \sum_{j=1}^n k_j \cdot e_j : -(\lfloor \frac{r}{2\varepsilon} \rfloor + 1) \leqslant k_j \leqslant (\lfloor \frac{r}{2\varepsilon} \rfloor + 1)\}$, which is finite. Furthermore, we have

$$A \subset \bigcup_{x \in X_\varepsilon} B_\varepsilon(x),$$

which implies X_ε is a finite $\varepsilon-$net.

□

Problem 7.

Proof. Since d is continuous with respect to each argument, let (a_n, b_n) be a minimizing sequence such that $d(a_n, b_n) \to \mathrm{dist}(A, B)$, by lower-semi continuity of d. Since A and B are compact, there exists a subsequence (a_{n_k}, b_{n_k}) such that $(a_{n_k}, b_{n_k}) \to (a, b)$. By the lower semi-continuity of d, we have

$$d(a, b) \leqslant \liminf_k d(a_{n_k}, b_{n_k}) = \mathrm{dist}(A, B).$$

□