Problem 1.

Proof. Without loss of genearlity, we may assume that $\|A\| \neq 0$. First, we note that both AA^* and A^*A are bounded self-adjoint operators,

$$\|A^*A\| = \sup_{\|x\|=1} |\langle x, A^*Ax\rangle| = \sup_{\|x\|=1} |\langle Ax, Ax\rangle| = \sup_{\|x\|=1} \|Ax\|^2 = \left(\sup_{\|x\|=1} \|Ax\|\right)^2 = \|A\|^2.$$

Switch A^* with A, we have

$$\|AA^*\| = \|A^*\|^2.$$

By Theorem 5.37,

$$\begin{cases} \|A^*\|^2 = \|AA^*\| \leqslant \|A\|\|A^*\| \\ \|A\|^2 = \|A^*A\| \leqslant \|A^*\|\|A\| \end{cases} \implies \begin{cases} \|A^*\| \leqslant \|A\| \\ \|A\| \leqslant \|A^*\| \end{cases}$$

which yields that $\|A\| = \|A^*\|$. As a result,

$$\|AA^*\| = \|A^*A\| = \|A\|^2 = \|A^*\|^2.$$

□

Problem 2.

Proof. Note that $\phi_y(x) = \langle y, x\rangle$. Let's first prove $\langle \phi_x, \phi_y\rangle_{\mathcal{H}^*} := \langle y, x\rangle_{\mathcal{H}}$ is indeed an inner product. Let ϕ_x, ϕ_y, ϕ_z be continuous linear functional on $\mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$.

(a) Note that for any $a \in \mathcal{H}$, we have

$$\phi_x(a) + \phi_y(a) = \langle x, a\rangle + \langle y, a\rangle = \langle x + y, a\rangle = \phi_{x+y}(a);$$
$$\phi_{\lambda x}(a) = \langle \lambda x, a\rangle = \bar{\lambda}\langle a, x\rangle = \bar{\lambda}\phi_x(a).$$

Thus,

$$\langle \phi_x, \lambda\phi_y + \mu\phi_z\rangle = \langle \phi_x, \phi_{\bar{\lambda}y+\bar{\mu}z}\rangle = \langle \bar{\lambda}y + \bar{\mu}z, x\rangle = \lambda\langle y, x\rangle + \mu\langle z, x\rangle = \lambda\langle \phi_x, \phi_y\rangle + \mu\langle \phi_x, \phi_z\rangle.$$

(b)

$$\langle \phi_y, \phi_x\rangle = \langle x, y\rangle = \overline{\langle y, x\rangle} = \overline{\phi_x, \phi_y}.$$

(c)

$$\langle \phi_x, \phi_x\rangle = \langle x, x\rangle \geqslant 0,$$

where equal sign is achieved only when ϕ_x is zero operator.

Then we prove the space H^* is complete. Note that

$$\|\phi_x\|^2 = \langle \phi_x, \phi_x\rangle = \langle x, x\rangle = \|x\|^2,$$

then the completeness of H^* follows from Riesz representation theorem.

□

Problem 3.

Proof. This directly follows from Riesz representation theorem that $T : \mathcal{H} \mapsto \mathcal{H}^*$ with $T(x) = \phi_x$ is an isometry. □

Problem 4.

Proof. By Proposition 5.30, we know $\operatorname{ran}(A)$ is closed and $\ker(A) = 0$. We guarantee the uniqueness of solution.

Furthermore, by Theorem 8.17 and Theorem 8.18, since $A = A^*$, the equation $Ax = y$ has a solution if y is orthogonal to $\ker(A)$, which is true since $\langle 0, y\rangle = 0$ for every $y \in \mathcal{H}$. □

Problem 5.

Proof. a Since for all $x, y \in \mathcal{H}$, we have

$$\langle x, By\rangle = \langle x, Ay\rangle = \langle A^* x, y\rangle.$$

Then by definition and the uniqueness of the adjoint of bounded operator, $B^* = A^*$ which implies that $B = A$.

b For any $x, y \in H$, we have

$$\begin{aligned}
0 &= \langle x + y, (A - B)(x + y)\rangle \\
&= \langle x, (A - B)x\rangle + \langle y, (A - B)y\rangle + \langle x, (A - B)y\rangle + \langle y, (A - B)x\rangle \\
&= \langle x, (A - B)y\rangle + \langle y, (A - B)x\rangle. \\
0 &= \langle x + iy, (A - B)(x + iy)\rangle \\
&= \langle x, (A - B)x\rangle + \langle iy, (A - B)iy\rangle + \langle x, (A - B)iy\rangle + \langle iy, (A - B)x\rangle \\
&= i\left(\langle x, (A - B)y\rangle - \langle y, (A - B)x\rangle.\right)
\end{aligned}$$

As a result, $\langle x, (A - B)y\rangle = 0$ for any x, y. Thus by part a, we get $A = B$.

A real Hilbert space, as shown in the above, it won't has such a property due to lack of cancellation. for example we take $L^2 : [0, 1] \mapsto \mathbb{C}$ with inner product:

$$\langle f, g\rangle = \operatorname{Re}\left(\int_0^1 \overline{f(x)} g(x) \mathrm{d}x\right)$$

Take $A = \mathbb{1}$ and $B = i$, then both

$$\langle f, Af\rangle = \operatorname{Re}\left(\int_0^1 |f(x)|^2 \mathrm{d}x\right) = \langle f, Bf\rangle.$$

□