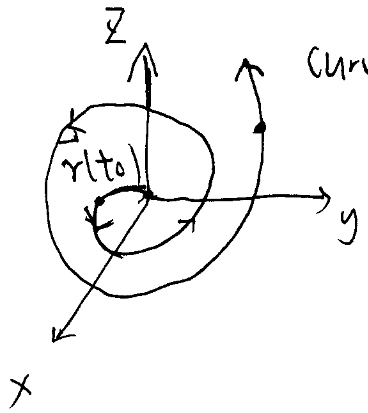
08/24 Chapter 13. Vector-Valued functions and motion in space.



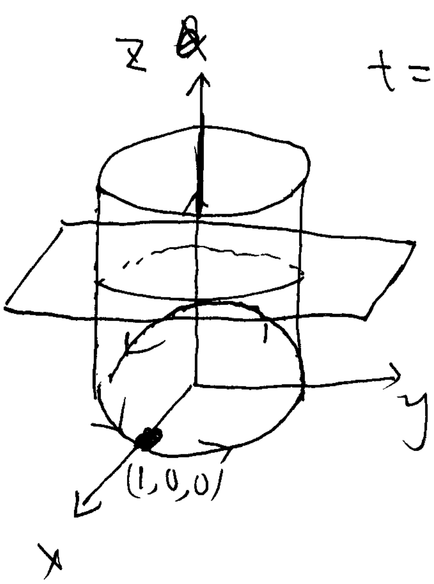
curve $\rightarrow \vec{r}(t) = (x(t), y(t), z(t))$

$$\vec{r}(t) = \underline{x(t)}\,\vec{i} + \underline{y(t)}\,\vec{j} + \underline{z(t)}\,\vec{k}$$

vector-valued function　　component functions.

eg: $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$

$$= \underbrace{(\cos t, \sin t, t)}$$



$t = 0$　　$\underline{\vec{r}(0) = (1, 0, 0)}$

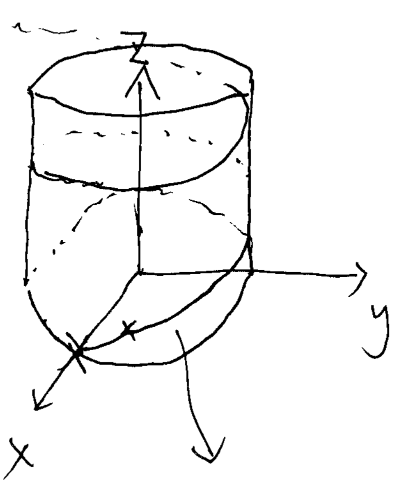
$$\vec{r}\left(\tfrac{\pi}{2}\right) = \left(0, 1, \tfrac{\pi}{2}\right)$$

$\cos^2 t + \sin^2 t = 1.$

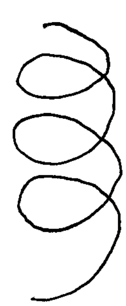
$x^2(t) + y^2(t) = 1.$　for all time

what is $x^2 + y^2 = 1.$?

$Z = 0$



$(\cos t, \sin t, 0) \leftarrow (\cos t, \sin t, \underline{t})$

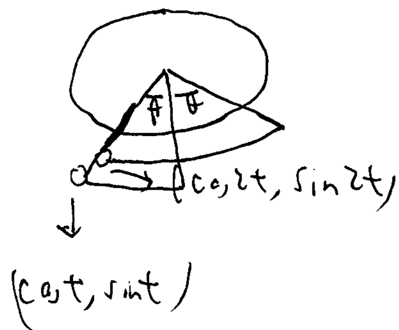


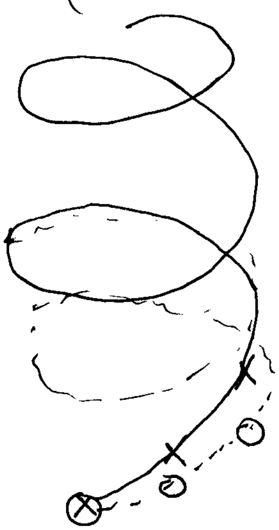
curve called as helix

$\vec{r}_1 \ (\cos 2t, \sin 2t, \underline{\underline{t}})$

$t = t_0$ $\qquad \vec{r}_2 \ (\cos t, \sin t, t)$

$\hookrightarrow$ $\vec{r}_1$ and $\vec{r}_2$ locate at $z = t_0$ plane.





$\vec{r}_1 \ (\underline{\cos 2t}, \ \underline{\sin 2t}, \ \underline{t})$

$\vec{r}_3 \ (\underline{\cos 2t}, \ \underline{\sin 2t}, \ \underline{\tfrac{1}{2} t})$

Def: Let $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$ with domain D

we say $\lim\limits_{t \to t_0} \vec{r}(t) = \vec{L} = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$

if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that when $|t - t_0| < \delta$,

length $\leftarrow$ $\underline{|\vec{r}(t) - \vec{L}|} < \varepsilon$. $\qquad$ at $t = t_0$

position , if and only if

$\lim\limits_{t \to t_0} \vec{r}(t) = \vec{L}$ iff $\lim\limits_{t \to t_0} x(t) = x_0$, $\lim\limits_{t \to t_0} y(t) = y_0$, $\lim\limits_{t \to t_0} z(t) = z_0$

eg $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \rightarrow \frac{1}{t-\frac{\pi}{3}}$

$$\lim_{t\to\frac{\pi}{3}} \vec{r}(t) = \left(\lim_{t\to\frac{\pi}{3}} \cos t, \ \lim_{t\to\frac{\pi}{3}} \sin t, \ \lim_{t\to\frac{\pi}{3}} t \right)$$

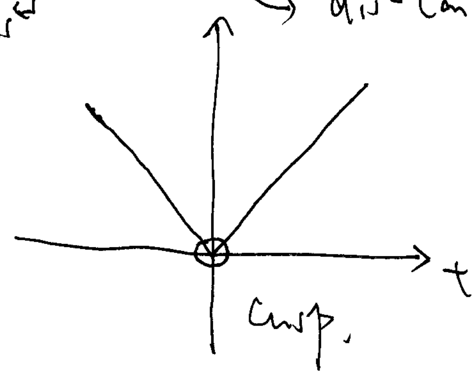
$$= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right)$$

Def: $\vec{r}(t)$ is continuous at $t = t_0$,

if $\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$

$\vec{r}(t)$ is a continuous fun if $\vec{r}(t)$ is continuous on any t.

eg: $\vec{r}(t) = \underbrace{\cos t}_{\text{continuous}}\, \vec{i} + \underbrace{\sin t}_{\text{continuous}}\, \vec{j} + \underbrace{|t|}_{\text{dis-continuous at } t=0}\, \vec{k}$



$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

$$\Delta\vec{r}(t) = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$= \big(x(t+\Delta t) - x(t)\big)\vec{i} + \big(y(t+\Delta t) - y(t)\big)\vec{j} + \big(z(t+\Delta t) - z(t)\big)\vec{k}$$

$$\lim_{\Delta t\to 0} \frac{\Delta\vec{r}(t)}{\Delta t} = \lim_{\Delta t\to 0} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t} \right) \vec{i} + \lim \ \vec{j} + \lim \ \vec{k}$$

$\downarrow$

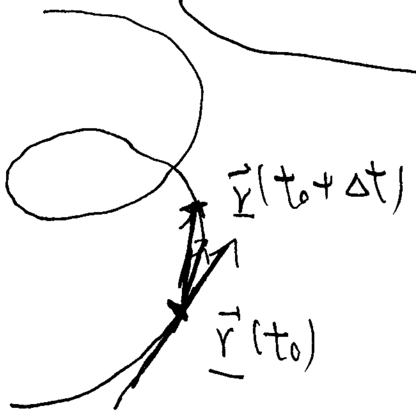
scale

$$= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

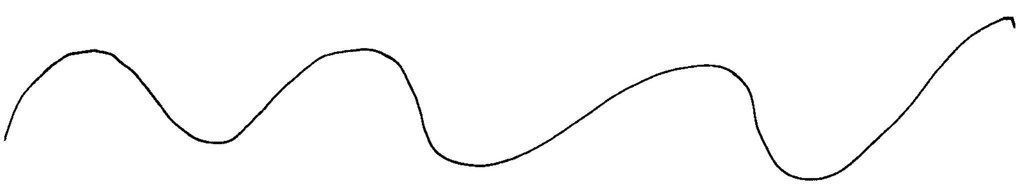
def: $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is differentable / has derivatives at t if $x(t)$, $y(t)$, $z(t)$ has derivatives at t.

$$\underline{\vec{r}'(t)} = \frac{d\vec{r}}{dt} = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$$

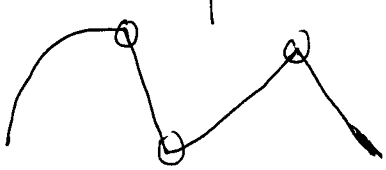
$\longrightarrow$ the direction of tangent line at $t = t_0$



smooth.

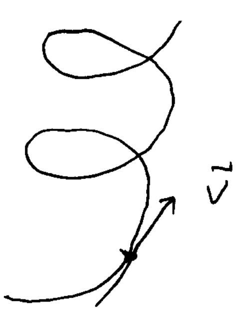


cusp



if curve is smooth if $\underline{\frac{d\vec{r}}{dt}}$ is continuous and nonzero.

Def: velocity vector: $\vec{v}(t) = \frac{d\vec{r}}{dt}$, where $\vec{r}(t)$ represent position of a particle.



velocity: $\vec{v}(t) = \frac{d\vec{r}}{dt}$

speed: $|\vec{v}| \geq 0$

acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

$\frac{\vec{v}}{|\vec{v}|} \rightarrow$ the direction of motion.

Rules: a) $\frac{d}{dt}\vec{c} = \vec{0}$

b) $\frac{d}{dt}(c\vec{u}) = c\frac{d}{dt}\vec{u}$

$$\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\cdot\frac{d\vec{u}}{dt}$$

c). $\frac{d}{dt}(\vec{u}\pm\vec{v}) = \frac{d}{dt}\vec{u} \pm \frac{d}{dt}\vec{v}$

d) $\underbrace{\frac{d}{dt}(\underline{\vec{u}}\cdot\underline{\vec{v}})}_{\text{scale differentiation}} = \underline{\vec{u}'}\cdot\underline{\vec{v}} + \underline{\vec{u}}\cdot\underline{\vec{v}'}$ dot product rule.

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} \quad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3)$$

$$= \underline{u_1'v_1} + u_1v_1' + \underline{u_2'v_2} + u_2v_2' + \underline{u_3'v_3} + u_3v_3'$$

$$= \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$$

e) $\frac{d}{dt}(\underbrace{\vec{u}\times\vec{v}}_{\text{vecta}}) = \underbrace{\vec{u}'\times\vec{v}}_{\text{vecta}} + \underbrace{\vec{u}\times\vec{v}'}_{\text{vecta.}}$

$$\frac{d}{dt}(\vec{u}\times\vec{v}) = \lim_{\Delta t\to 0}\frac{\vec{u}(t+\Delta t)\times\vec{v}(t+\Delta t) - \vec{u}(t)\times\vec{v}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{\vec{u}(t+\Delta t)\times\vec{v}(t+\Delta t) - \vec{u}(t+\Delta t)\times\vec{v}(t) + \vec{u}(t+\Delta t)\times\vec{v}(t) - \vec{u}(t)\times\vec{v}(t)}{\Delta t}$$

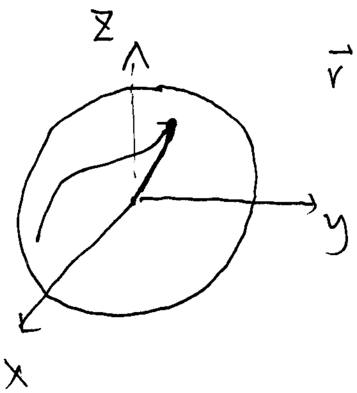
$$= \lim_{\Delta t\to 0}\frac{\vec{u}(t+\Delta t)\times(\vec{v}(t+\Delta t) - \vec{v}(t))}{\Delta t} = \vec{u}(t)\times\vec{v}'(t)$$

$$+ \lim \frac{(\vec{u}(t+\Delta t) - \vec{u}(t))\times\vec{v}(t)}{} \qquad + \vec{u}'(t)\times\vec{v}(t)$$

f). $\frac{d}{dt}\left(\vec{u}(f(t))\right) = \underline{f'(t)} \cdot \underline{u'(f(t))}$

$$\parallel$$

$$u_1(f(t))\,\vec{i} + u_2(f(t))\,\vec{j} + u_3(f(t))\,\vec{k}$$



$\vec{r}(t)$ sphere with R.

$$|\vec{r}(t)| = R$$

$$\vec{r}(t)\cdot\vec{r}(t) = R^2$$

$$\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = 0$$

$$\vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

$$2\,\vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

$$\vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

$$\downarrow$$

$$\vec{r}(t) \perp \vec{r}\,'(t)$$

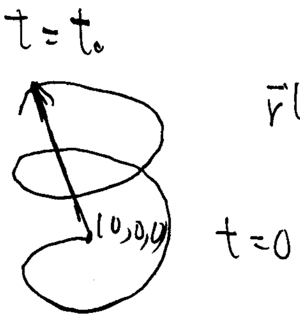
Thm: if $\vec{r}(t)$ is differentiable and of constant length,

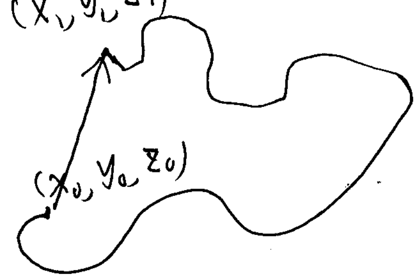
then $\underline{\vec{r}(t)}\cdot\underline{\frac{d\vec{r}}{dt}} = 0$

Sec 13.2. Integrals of vector function.

Def: the indefinite integral of $\vec{r}$, if ~~the~~ $\vec{R}$ is the antiderivative of $\vec{r}$, then $\int \vec{r}(t)\,dt = \vec{R}(t) + C$.

Def: definite integral: $\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\vec{i} + \left(\int_a^b y(t)\,dt\right)\vec{j} + \left(\int_a^b z(t)\,dt\right)\vec{k}$





$$\int_0^{\pi} (\cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k})\,dt$$

$$= \left(\int_0^{\pi} \cos t\,dt\right)\vec{i} + \left(\int_0^{\pi} \sin t\,dt\right)\vec{j} + \left(\int_0^{\pi} t\,dt\right)\vec{k}$$

$$= \sin t\Big|_0^{\pi}\,\vec{i} + -\cos t\Big|_0^{\pi}\,\vec{j} + \tfrac{1}{2}t^2\Big|_0^{\pi}\,\vec{k}$$

$$= \left(0,\ -(-1-1),\ \tfrac{1}{2}\pi^2\right)$$

$$= \left(0,\ 2,\ \tfrac{1}{2}\pi^2\right)$$