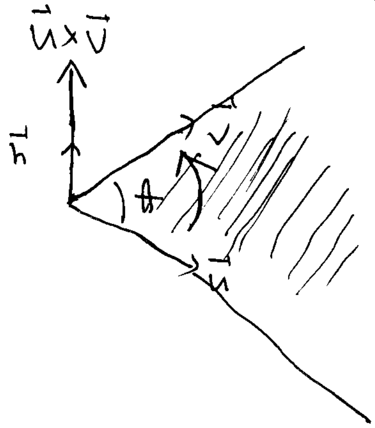
08/17. Sec 12.4 The cross product

Def: $\vec{u} \times \vec{v} = \underbrace{(|\vec{u}| \cdot |\vec{v}| \cdot \sin\theta)}_{\text{"length"}} \vec{n}$



"length" → scale

$\vec{n}$ → unit normal vecta.

Remark: The cross product generates a vecta rather than a scale.

Remark 2: $\theta = 0, \pi,$

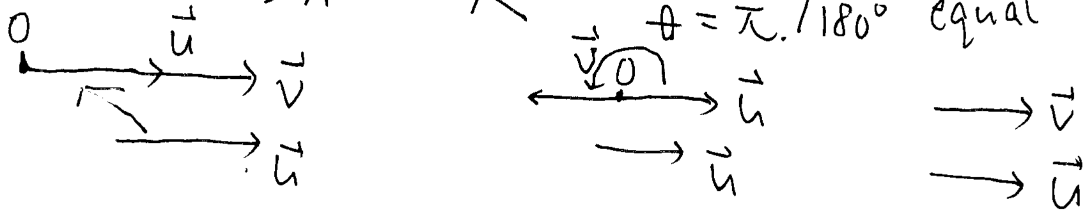
$\underline{\vec{u} \times \vec{v} = 0 \cdot \vec{n} = \vec{0}}$

if and only if

Nonzero vectas $\vec{u}$ and $\vec{v}$ are parallel iff $\underline{\vec{u} \times \vec{v} = \vec{0}}$.

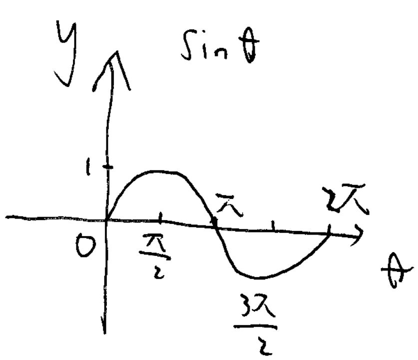
$\theta = 0$ → parallel

$\theta = \pi / 180°$ equal

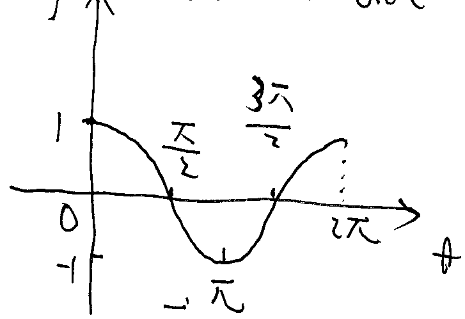


Recall: $\vec{u} \cdot \vec{v} = 0$ iff $\vec{u} \perp \vec{v}$

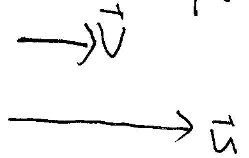
$[0, 2\pi)$

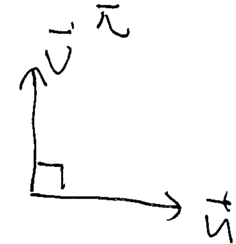


$\cos\theta$ → dot product.



parallel



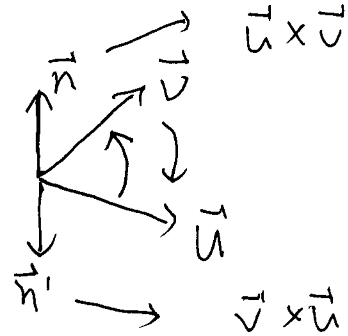


Rules:

1. $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$

2. $\vec{u} \times (\vec{v} + \vec{w}) = \overbrace{\underbrace{\vec{u} \times \vec{v}}\ \underbrace{+}_{\text{addition of vector.}}\ \underbrace{\vec{u} \times \vec{w}}}^{\text{vectors}}$

3. ☆ $\vec{v} \times \vec{u} = -1(\vec{u} \times \vec{v})$



"Cross product is not communicate."

"Order makes a difference".

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

4. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

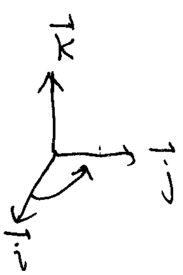
5. $\vec{0} \times \vec{u} = \vec{0}$

6. ☆ $\vec{u} \times (\vec{v} \times \vec{w}) = \underbrace{(\vec{u} \cdot \vec{w})}_{\text{scale}} \vec{v} - \underbrace{(\vec{u} \cdot \vec{v})}_{\text{scale.}} \cdot \vec{w}$

$\neq (\vec{u} \times \vec{v}) \times \vec{w}$

$\vec{u} \times (\vec{v} \times \vec{w})$: is perpend to the plane generated by $\vec{v}$ and $\vec{w}$

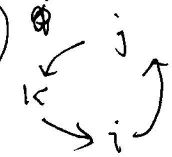
$(\vec{u} \times \vec{v}) \times \vec{w}$: locate at the plane generated by $\vec{v}$ and $\vec{w}$



$\vec{i} \times \vec{j} = \vec{k} = -(\vec{j} \times \vec{i})$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

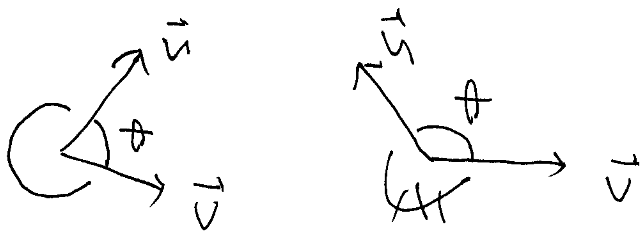


$\vec{i} \times \vec{i} = \vec{0}$

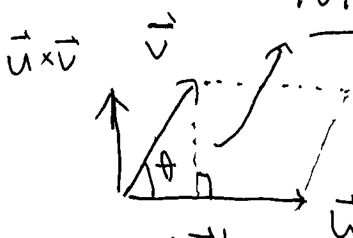
$\vec{j} \times \vec{j} = \vec{0}$

$\vec{k} \times \vec{k} = \vec{0}$

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}||\sin\theta||\vec{n}| = \underline{|\vec{u}||\vec{v}|\cdot\sin\theta}$$



$\underline{|\vec{v}|\cdot\sin\theta} \to$ height



$|\vec{u}| \to$ base

$$|\vec{u}\times\vec{v}| = \underbrace{|\vec{u}||\vec{v}|\sin\theta}_{\downarrow}$$

area of parellelogram generated by $\vec{u},\vec{v}$.

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$$

$$\vec{v} = v_1 i + v_2 j + v_3 j$$

$$\vec{u}\times\vec{v} = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$

$$= \cancel{(u_1\vec{i}\times v_1\vec{i})}^{0} + (\underline{u_1\vec{i}\times v_2\vec{j}}) + (u_1\vec{i}\times v_3\vec{k}) + \cdots$$

$$u_1v_2\vec{k} \quad + \quad u_1v_3(-\vec{j})$$

$$= u_1v_2\vec{k} - u_1v_3\vec{j} - u_2v_1\vec{k} + u_2v_3\vec{i} \qquad k \to i \to j$$

$$u_3v_1\vec{j} \quad - \quad u_3v_2\vec{i}$$

$$= (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

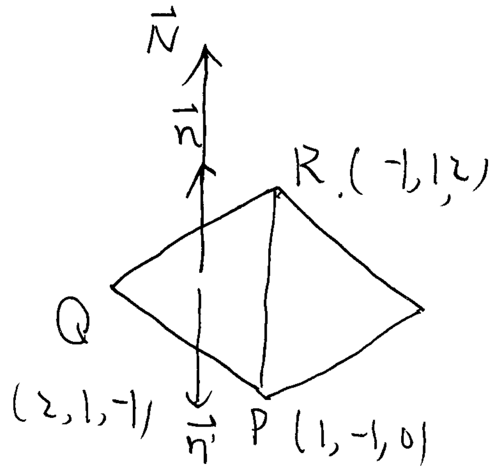
$$= \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}\vec{i} - \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}\vec{j} + \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\vec{k}$$

$$\vec{u}\times\vec{v} = \begin{pmatrix} i & \cancel{j} & k \\ u_1 & \boxed{u_2 \quad u_3} \end{pmatrix}$$

$$\vec{u} \times \vec{v} = \begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

eg. $P(1,-1,0)$ $Q(2,1,-1)$ $R(-1,1,2)$

Find a vector perpendicular to the plane.



$$\vec{PQ} = (1, 2, -1)$$

$$\vec{PR} = (-2, 2, 2)$$

$$\vec{PQ} \times \vec{PR} = (i - 2j - k)(-2i + 2j + 2k)$$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \vec{k}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= 6\vec{i} - 0\vec{j} + 6\vec{k} = \underbrace{6\vec{i} + 6\vec{k}}$$

unit normal vector $\leftarrow \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{k} = \frac{6}{\sqrt{6^2+6^2}}\vec{i} + \frac{6}{\sqrt{6^2+6^2}}\vec{k}$

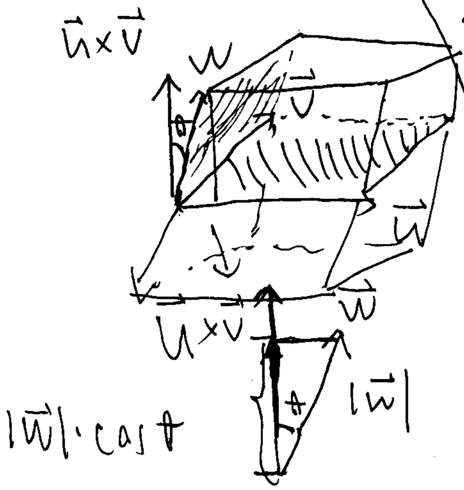
What is the area of this triangle? $\frac{1}{2}\left|6\vec{i} + 6\vec{k}\right| = \frac{1}{2}\sqrt{6^2+6^2} = 3\sqrt{2}$

The triple / box scalar product:

$(\vec{u}\times\vec{v})\cdot\vec{w} \rightarrow$ scalar.

angle between $\vec{u}\times\vec{v}$ and $\vec{w}$

$$|(\vec{u}\times\vec{v})\cdot\vec{w}| = \underbrace{|\vec{u}\times\vec{v}|}_{\text{Area of Base plane}}\cdot\underbrace{|\vec{w}|\cdot|\cos\theta|}_{\text{Height}}$$

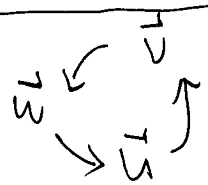


$\rightarrow$ volume of parallelogram-base box / parallelepiped

$$\underbrace{|(\vec{v}\times\vec{u})\cdot\vec{w}|} = |-(\vec{u}\times\vec{v})\cdot\vec{w}| = |(\vec{u}\times\vec{v})\cdot\vec{w}|$$

$$(\vec{v}\times\vec{u})\cdot\vec{w} = -(\vec{u}\times\vec{v})\cdot\vec{w}$$

Property: $(\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{v}\times\vec{w})\cdot\vec{u} = (\vec{w}\times\vec{u})\cdot\vec{v}$



$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \rightarrow \text{scalar.}$$

eg. $\vec{u} = \vec{i} + 2\vec{j} - \vec{k} \qquad \vec{v} = -\vec{i} + 3\vec{k} \qquad \vec{w} = 7\vec{j} - 4\vec{k}$

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ \end{vmatrix} =$$