Problem 1.

Proof. To prove $\mathcal{F}$ is precompact, we need to show

- $\mathcal{F}$ is bounded with respect $\|\|_\infty$, which is true since $\|f\|_\infty \leqslant M$ for any $f \in \mathcal{F}$.
- $\mathcal{F}$ is equicontinuous. For any $f \in \mathcal{F}$, by fundamental theorem of calculus, we have

  $$f(x) - f(a) = \int_a^x f'(t)\mathrm{d}t.$$

  For any $\varepsilon > 0$, there exists a $\delta = \frac{\varepsilon}{N}$, such that for $|x-y| < \delta$,

  $$|f(x) - f(y)| = |\int_y^x f'(t)\mathrm{d}t| \leqslant \|f'\|_\infty |x-y| < N \cdot \frac{\varepsilon}{N} = \varepsilon.$$

- $\mathcal{F}$ is not closed. For example, consider $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$ on $[-1,1]$. It is differentiable and bounded by $M = 2$, its derivative is bounded by $N = 1$. However f_n uniformly converges to $f = |x|$, which is not differentiable everywhere on $[-1,1]$.

$\square$

Problem 2. Cauchy inequality or Hölder's inequality

Proof. Let's consider a polynomial $P(\lambda) = \int_a^b (f(x) + \lambda)^2 \mathrm{d}x$. If $f(x)$ is constant on $[a,b]$, the inequality holds naturally. Otherwise $P(\lambda) > 0$.

Note that $P(\lambda) = \int_a^b (\lambda^2 + 2\lambda f(x) + f(x)^2)\mathrm{d}x = \lambda^2 (\int_a^b \mathrm{d}x) + \lambda (2\int_a^b f(x)\mathrm{d}x) + (\int_a^b f(x)^2 \mathrm{d}x)$ is a 2nd-order polynomial with no roots! Therefore

$$(2\int_a^b f(x)\mathrm{d}x)^2 \leqslant 4(\int_a^b \mathrm{d}x)(\int_a^b f(x)^2\mathrm{d}x).$$

$\square$

Problem 3.

Proof. To prove $\mathcal{F}_M$ is precompact, we need to show

- $\mathcal{F}_M$ is bounded. By the fundamental theorem of calculus, $f(x) - f(a) = \int_a^x f'(t)\mathrm{d}t$. Thus

  $$|f(x)| \leqslant |f(a)| + |\int_a^x f'(t)\mathrm{d}t| \leqslant |f(a)| + |x-a|^{1/2} (\int_a^x f'(t)^2 \mathrm{d}t)^{1/2} \leqslant |b-a|^{1/2} M.$$

- $\mathcal{F}_M$ is equicontinuous. For any $f \in \mathcal{F}_M$, for any $\varepsilon > 0$, there exists a $\delta = \frac{\varepsilon^2}{M}$ such that for $|x-y| < \delta$,

  $$|f(x) - f(y)| = |\int_y^x f'(t)\mathrm{d}t| \leqslant |x-y|^{1/2} (\int_y^x f'(t)^2\mathrm{d}t)^{1/2} \leqslant \delta^{1/2} M^{1/2} \leqslant \varepsilon$$

□

Problem 4. Union of topologies and intersection of topologies.

Proof. $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology.

- $X \in \mathcal{T}_1$ and $X \in \mathcal{T}_2$ imply that $X \in \mathcal{T}_1 \cap \mathcal{T}_2$; $\emptyset \in \mathcal{T}_1$ and $\emptyset \in \mathcal{T}_2$ imply that $\emptyset \in \mathcal{T}_1 \cap \mathcal{T}_2$.
- Let $\{G_\alpha\}_\alpha$ be a collection of subsets such that $G_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$ for any α, thus $G_\alpha \in \mathcal{T}_1$ which implies $\bigcup_\alpha G_\alpha \in \mathcal{T}_1$ and $G_\alpha \in \mathcal{T}_2$ which implies $\bigcup_\alpha G_\alpha \in \mathcal{T}_2$. Therefore, $\bigcup_\alpha G_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$.
- Let $\{G_i\}_{i=1}^n$ be a collection of subsets for some n such that $G_i \in \mathcal{T}_1 \cap \mathcal{T}_2$ for any $1 \leqslant i \leqslant n$. Thus $G_i \in \mathcal{T}_1$ which implies $\bigcap_{i=1}^n G_i \in \mathcal{T}_1$ and $G_i \in \mathcal{T}_2$ which implies $\bigcap_{i=1}^n G_i \in \mathcal{T}_2$. Therefore, $\bigcap_{i=1}^n G_i \in \mathcal{T}_1 \cap \mathcal{T}_2$.

$\mathcal{T}_1 \cup \mathcal{T}_2$ is not necessary a topology. For example, $X = \{1, 2, 3\}$, let $\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{2\}, X\}$ are two topologies on X. However $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{1\}, \{2\}, X\}$ is no more a topology because $\{1, 2\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. □

Problem 5. Equivalent definition of base.

Proof.

- "$\Rightarrow$ " Assume $\mathcal{B}$ is a base for some topology $\mathcal{T}$.

  Since $X \in \mathcal{T}$, by definition of base,there is a collection such that $X = \bigcup_\alpha B_\alpha$.

  For any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, since $\mathcal{B} \subset \mathcal{T}$, $B_1, B_2 \in \mathcal{T}$. Thus $B_1 \cap B_2 \in \mathcal{T}$.

  By the definition of base, there exists a collection of set $B_\alpha \in \mathcal{B}$ such that $B_1 \cap B_2 = \bigcup_\alpha B_\alpha$.

  $x \in B_1 \cap B_2$ implies $x \in \bigcup_\alpha B_\alpha$. Thus there exist a α_0 such that $x \in B_{\alpha_0} \subset B_1 \cap B_2$, where $B_{\alpha_0} \in \mathcal{B}$.

- "$\Leftarrow$" Assume $\mathcal{B}$ is a collection satisfying those properties. We prove $\mathcal{B}$ is a base for some topology $\mathcal{T}$, or equivalently we prove $\mathcal{B}$ can induce a topology.

  Let $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$. We prove $\mathcal{T}$ is a topology.

  - $\emptyset, X \in \mathcal{T}$.
  - Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of subsets such that $G_\alpha \in \mathcal{T}$ for each $\alpha \in \mathcal{A}$. Then $G_\alpha = \bigcup_{\beta(\alpha)} B_\beta$. Thus $\bigcup_{\alpha \in \mathcal{A}} G_\alpha = \bigcup_\beta B_\beta \in \mathcal{T}$.
  - Let $\{G_i\}_{i=1}^n$ be a collection of subsets such that $G_i \in T$ for $1 \leqslant i \leqslant n$. Then $G_i = \bigcup_{\beta(i)} B_\beta$. By mathematical induction, we just need to prove $G_1 \cap G_2 \in \mathcal{T}$. Assume $G_1 \cap G_2 \neq \emptyset$, for each $x \in G_1 \cap G_2$, there exist a B_1 and B_2 such that $x \in B_1 \subset G_1$ and $x \in B_2 \subset G_2$. For $x \in B_1 \cap B_2$, there exists a $W_x \in \mathcal{B}$ such that $x \in W_x \subset B_1 \cap B_2$. Therefore $G_1 \cap G_2 = \bigcup_x W_x \in \mathcal{T}$.

□

Problem 6.

Proof. Let $\mathcal{T}$ be the metric topology associated with the metric space X. Say one of its base $\mathcal{B} = \{B_1, B_2, \cdots, B_n\}$. Though union, intersection and complement, the induced topology has finite number of elements. However X is Hausdorff, if X has infinite points, every set $X \backslash \{x\}$ is open, which leads the topology has infinite number of elements, contradiction! $\square$