# Bounded linear operators on Hilbert space $\mathcal{H}$.

- Orthogonal Projection $P^2 = P$ $P = P^*$ ← Hilbert structure

  $\forall P$ non-zero OP, $\|P\| = 1$

  $\mathcal{H} = \underbrace{\operatorname{ran} P}_{\text{closed.}} \oplus \ker P$, $\operatorname{ran} P \perp \ker P$

  ↘ orthogonal direct sum

- Adjoint operator $A \in \mathcal{B}(\mathcal{H})$ (by Riesz Representation)

  $\forall x, y \in \mathcal{H}$. $\langle Ax, y \rangle = \langle x, A^* y \rangle$ $\Rightarrow \exists A^*$ of $A$

  $\|A\| = \|A^*\|$

  if $A^*A = AA^* \Rightarrow A$ is normal

  if $A^* = A \Rightarrow A$ is self-adjoint ($\Rightarrow A$ is normal)

  if $A^* = A^{-1}$ ($A^*A = AA^* = Id$) $\Rightarrow A$ is unitary/orthogonal.
  ($\Rightarrow A$ is normal)

eg: H is finite-dim / separable.

NO $\Rightarrow$ A normal matrix $\Rightarrow$ diagonalizable $A = \overset{\text{unitary matrix}}{U} \Lambda U^*$

self-adjoint operator $\Rightarrow$ Symmetric / Hermitian matrix

unitary / orthogonal operator $\Rightarrow$ rotation / reflection matrix.

Prop: $A, B \in \mathcal{L}(\mathcal{H})$ if $\forall x, y \in \mathcal{H}$, $\langle Ax, y\rangle = \langle x, By\rangle$

then $A \in \mathcal{B}(\mathcal{H})$

Proof: Recall closed graph thm

if $A : \overset{\text{Banach}}{X} \to \overset{\text{Banach}}{Y}$ is closed and $D(A) = X$,

then $A$ is bounded.

Let $(x_n) \subseteq \mathcal{H}$, $x_0, y_0 \in \mathcal{H}$ s.t $x_n \to x_0$, $Ax_n \to y_0$

We want to show that $Ax_0 = y_0$ or $\|Ax_0 - y_0\| = 0$

$$\langle y_0, y\rangle = \langle \lim_{n\to\infty} Ax_n, y\rangle = \lim_{n\to+\infty} \langle Ax_n, y\rangle = \lim_{n\to+\infty} \langle x_n, By\rangle$$

$$= \langle x_0, By\rangle = \langle Ax_0, y\rangle$$

$$\Rightarrow \langle y_0 - Ax_0, y\rangle = 0 \quad \forall y$$

pick $y = y_0 - Ax_0$ $\square$

Prelim $\mathcal{H} = \{ f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} e^{-|x|} |f(x)|^2 \, dx < \infty \}$ with

$$\langle f, g\rangle = \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} g(x) \, dx$$

let $T \in \mathcal{B}(\mathcal{H})$ s.t $Tf(x) = f(x+1)$

compute $T^*$, find $\|T\|$

Soln: $\langle Tf, g\rangle = \int_{\mathbb{R}} e^{-|x|}\, \overline{f(x+1)}\, g(x)\, dx$

$$= \int_{\mathbb{R}} e^{-|x-1|}\, \overline{f(x)}\, g(x-1)\, dx$$

$$= \int_{\mathbb{R}} e^{-|x|}\, \overline{f(x)}\, e^{|x|-|x-1|}\, g(x-1)\, dx$$

$$= \langle f, T^* g\rangle$$

where $T^* g = e^{|x|-|x-1|}\, g(x-1)$

$$\|Tf\|^2 = \int_{\mathbb{R}} e^{-|x|}\, |f(x+1)|^2\, dx = \int_{\mathbb{R}} e^{-|x-1|}\, |f(x)|^2\, dx$$

$$= \int_{\mathbb{R}} e^{-|x|}\, |f(x)|^2\, e^{|x|-|x-1|}\, dx \qquad |x|-|x-1| = \begin{cases} 1 & x \geq 1 \\ 2x-1 & 0 \leq x \leq 1 \\ -1 & x < 0 \end{cases}$$

$$\leq \left(\sup_{x \in \mathbb{R}} e^{|x|-|x-1|}\right) \|f\|^2 = e\, \|f\|^2$$

$\Rightarrow \|T\| \leq \sqrt{e}.$

Let $f_n(x) = \sqrt{n}\, e^{\frac{|x|}{2}} \chi_{[1, 1+\frac{1}{n})}(x)$

$$\|f_n\|^2 = \int_{\mathbb{R}} e^{-|x|}\, n \cdot e^{|x|} \chi_{[1, 1+\frac{1}{n})}(x)\, dx = 1.$$

$$\|Tf_n\|^2 = \int_{\mathbb{R}} e^{-|x|}\, n \cdot e^{|x|} \left(\chi_{[1, 1+\frac{1}{n}]}(x)\right)^2 e^{|x|-|x-1|} dx$$

$$= n \int_1^{1+\frac{1}{n}} e^{|x|-|x-1|} dx$$

$$= n \cdot e \cdot \frac{1}{n} = e.$$

Prop: Multiplication operator: $\phi \in C([0,1])$

$\Phi : L^2[0,1] \to L^2[0,1]$ by

$\Phi(f) = \phi f$

(a) Show $\Phi$ is linear bounded operator.

$$\|\Phi(f)\|_{L^2}^2 = \int_0^1 |\phi(x) f(x)|^2 dx \leq \|\phi\|_\infty^2 \|f\|_{L^2}^2$$

$\Rightarrow \quad \|\Phi\| \leq \|\Phi\|_\infty$

(b) when $\Phi$ unitary? $\forall f, g \in L^2([0,1])$

$$\langle \Phi f, \Phi g\rangle = \int_0^1 \overline{\phi f}\, \phi g = \int_0^1 |\phi|^2 \bar{f} g = \int_0^1 \bar{f} g = \langle f, g\rangle$$

$$|\phi(x)|^2 = 1 \quad \forall x \in [0,1] \implies \phi(x) = e^{i\theta(x)} \quad \theta : [0,1] \to \mathbb{R}$$

(c) when $\Phi$ self-adjoint.

$$\langle \Phi f, g\rangle = \langle f, \Phi g\rangle$$

$$\int_0^1 \overline{\phi(x) f(x)}\, g(x)\, dx = \int_0^1 \overline{f(x)}\, \phi(x)\, g(x)\, dx \qquad \forall f, g$$

$$\implies \phi = \bar{\phi}.$$