**MAT 201B Homework 1**
**Winter 2020**
Professor Qinglan Xia
Due Date: Wednesday, January 15th at 9:00am

1. Let $X$ be an uncountable set. Show that

$$\Sigma = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable } \}$$

is a $\sigma-$algebra on $X$.

2. Let $(X, \Sigma, \mu)$ be a measure space, and $(A_i)$ be a sequence of measurable sets in $\Sigma$.

(a.) If $(A_i)$ is increasing in the sense that

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots,$$

then show that

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i).$$

(b.) If $(A_i)$ is decreasing in the sense that

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_i \supseteq A_{i+1} \supseteq \cdots$$

and $\mu(A_1) < +\infty$, then show that

$$\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i).$$

Give a counterexample to show that this result need not be true if $\mu(A_i)$ is infinite for every $A_i$.

3. Let $(X, \Sigma)$ be a measuable space.

(a.) Show that if $(\mu_n)$ is an increasing sequence of measures on $(X, \Sigma)$ (in the sense that $\mu_n(A) \leq \mu_{n+1}(A)$ for each $A \in \Sigma$ and $n$), then the formula

$$\mu(A) := \lim_{n \to \infty} \mu_n(A)$$

defines a measure on $(X, \Sigma)$.

(b.) Show that if $(\mu_n)$ is an arbitrary sequence of measures on $(X, \Sigma)$, then the formula

$$\mu(A) := \sum_n \mu_n(A)$$

defines a measure on $(X, \Sigma)$.

4. Let $(X, \Sigma)$ be a measuable space, and $f : X \to \mathbb{R}$ be a real-valued function on $X$. Show that for any Borel set $A \subseteq \mathbb{R}$, the set $f^{-1}(A) := \{x \in X : f(x) \in A\}$ is $\Sigma-$measurable whenever $f$ is $\Sigma-$measurable.

5. Prove that $f \sim g$ if and only if $f = g$ pointwise-a.e. defines an equivalence relation on the space of all measuable functions.

6. Suppose $\mu$ is a finite measure on a set $X$. A point $x^* \in X$ is called an atom of $\mu$ if $\mu(\{x^*\}) > 0$. Show that $\mu$ has at most countable many atoms.