Problem 1.

Proof. $x_n \rightharpoonup x_0$ implies that $\langle x_n, y\rangle \to \langle x_0, y\rangle$ for all $y \in \mathcal{H}$. Furthermore, every weakly convergence sequence is uniformly bounded, i.e., $\|x_n\| \leqslant M$.

$y_n \to y_0$ implies that $\|y_n - y_0\| \leqslant \frac{\varepsilon}{2M}$. Then by the Cauchy Schwarz inequality:

$$\begin{aligned}
|\langle x_n, y_n\rangle - \langle x_0, y_0\rangle| &\leqslant |\langle x_n, y_n\rangle - \langle x_n, y_0\rangle| + |\langle x_n, y_0\rangle - \langle x_0, y_0\rangle| \\
&\leqslant |\langle x_n, y_n - y_0\rangle| + \frac{\varepsilon}{2} \\
&\leqslant \|x_n\|\|y_n - y_0\| + \frac{\varepsilon}{2} \\
&\leqslant M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

□

Problem 2. A strong lower-semicontinuous convex function is weakly lower-semicontinuous as well.

Proof. First, we prove the following equivalent definitions of lower semicontinuity:

Claim: Given a function $f : (X, \mathcal{T}) \mapsto [-\infty, \infty]$ is said to be lower semicontinuous if and only if

- $f^{-1}[-\infty, t]$ is closed ($f^{-1}(t, \infty] \in \mathcal{T}$) for any $t \in \mathbb{R}$;
- for all $x \in X$ and every sequence x_n converges to x with respect to topology $\mathcal{T}$, we have

$$f(x) \leqslant \liminf_{n\to\infty} f(x_n).$$

Assume that x_n converges to x w.r.t $\mathcal{T}$ and pick any t such that $t < f(x)$, then $x \in f^{-1}(t, \infty]$ and as $f^{-1}(t, \infty]$ is open, there exists a N such that for all $n \geqslant N$, $x_n \in f^{-1}(t, \infty]$, that is, $f(x_n) > t$, which implies that $\liminf_{n\to\infty} f(x_n) \geqslant t$. Since this is true for all $t < f(x)$, hence we have $f(x) \leqslant \liminf_{n\to\infty} f(x_n)$.

Assume that for all $x \in X$ and x_n converges to x w.r.t $\mathcal{T}$, we have $f(x) \leqslant \liminf_{n\to\infty} f(x_n)$. Given $t \in \mathbb{R}$, let $A = f^{-1}[-\infty, t]$. For any $x \in \bar{A}$, there exists a sequence $(x_n) \subset A$ that converges to x w.r.t $\mathcal{T}$, then $f(x) \leqslant \liminf_{n\to\infty} f(x_n) \leqslant t$, which shows that $x \in A$ as well. Then A is closed.

As a result, for any $t \in \mathbb{R}$, since f is strong lower-semicontinuous, $f^{-1}[-\infty, t]$ is closed w.r.t strong topology.

Since f is convex, for any $x, y \in f^{-1}[-\infty, t]$, we have

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y) \leqslant t,$$

which yields that $\alpha x + (1-\alpha)y \in f^{-1}[-\infty, t]$. This shows that $f^{-1}[-\infty, t]$ is convex.

Now, we prove that any convex closed set K in Hilbert space is weakly closed. (Another version of Mazur's theorem).

Let $x_0 \in \mathcal{H} \setminus K$, by Hahn-Banach theorem, there exists a bounded linear functional $T_y \in \mathcal{H}^*$ such that:

$$\langle y, x_0 \rangle = T_y(x_0) < c = \inf_{x \in K} T_y(x) = \inf_{x \in K} \langle y, x \rangle.$$

Then x_0 must be in the weakly interior of $\mathcal{H} \setminus K$, which shows that $\mathcal{H} \setminus K$ is weakly open, thus K is weakly closed.

As a result, $f^{-1}[-\infty, t]$ is weakly closed. By the equivalent definition, f is weakly lower-semicontinuous. □

Problem 3.

Proof. By Riesz representation theorem, let $\phi(x) = \langle y, x \rangle$ for some fixed y.

f **is strongly lower-semicontinuous:** $f(x)$ is continuous with respect to norm topology in fact.

f **is coercive:**

$$f(x) = \frac{1}{2}\|x\|^2 - \langle y, x \rangle \geqslant \frac{1}{2}\|x\|^2 - \|y\|\|x\| = \|x\| \left(\frac{1}{2}\|x\| - \|y\| \right) \to \infty, \text{ as } \|x\| \to \infty.$$

f **is strictly convex:** Let $x_1 \neq x_2$ and $\alpha \in [0, 1]$, recall that norm is strictly convex,

$$\begin{aligned}
f(\alpha x_1 + (1-\alpha)x_2) &= \frac{1}{2}\|\alpha x_1 + (1-\alpha)x_2\|^2 - \langle y, \alpha x_1 + (1-\alpha)x_2 \rangle \\
&< \frac{1}{2}\left(\alpha\|x_1\| + (1-\alpha)\|x_2\|\right)^2 - \alpha\langle y, x_1 \rangle - (1-\alpha)\langle y, x_2 \rangle \\
&= \frac{\alpha^2}{2}\|x_1\|^2 + \frac{2\alpha(1-\alpha)}{2}\|x_1\|\|x_2\| + \frac{(1-\alpha)^2}{2}\|x_2\|^2 - \alpha\langle y, x_1 \rangle - (1-\alpha)\langle y, x_2 \rangle \\
&< \left(\frac{\alpha^2}{2} + \frac{\alpha(1-\alpha)}{2} \right) \|x_1\|^2 - \alpha\langle y, x_1 \rangle \\
&\qquad + \left(\frac{(1-\alpha)^2}{2} - \frac{\alpha(1-\alpha)}{2} \right) \|x_2\|^2 - (1-\alpha)\langle y, x_2 \rangle \\
&= \alpha f(x_1) + (1-\alpha) f(x_2).
\end{aligned}$$

Apply Theorem 8.50, we conclude that f attains its infimum at a unique point $\bar{x}$. □

Problem 4.

Proof.

a) Unit closed balls in any infinite dimensional Hilbert space is not compact.

b) It is obvious that Q is a closed subset of a complete space ℓ_2, thus Q is complete as well. What we need to show is Q is totally bounded.

Since $(c_k) \in \ell_2$, there exists an $N > 0$, such that $\sum_{k>N}(c_k)^2 \leqslant \frac{\varepsilon}{2}$, which implies that $\sum_{k>N}(a_k)^2 \leqslant \frac{\varepsilon}{2}$ for any $(a_k) \in Q$.

Now define $Q^N = \{(a_k) \in Q : a_k = 0 \text{ for } k \geqslant N+1\}$, then Q^N is a closed bounded subset of $\mathbb{R}^N$, which is compact. As a result, there exists an $\frac{\varepsilon}{2}$-net A of Q^N. Note that A is an ε-net of Q, then Q is totally bounded.

□

Problem 5.

Proof.

a) Recall Corollary 6.15, we can decompose $\mathcal{H} = [S] \oplus [S]^\perp$. Once we show that $[S]^\perp = \{0\}$, we draw that $[S] = \mathcal{H}$.

For any $x \in \mathcal{H}$, there exists $(x_n) \subset S$ such that $x_n \rightharpoonup x$. For any $y \in [S]^\perp \subset \mathcal{H}$, there exist $(x_n) \subset S$ such that $x_n \rightharpoonup y$. Thus,

$$|\langle y, y \rangle| = \lim_{n\to\infty} |\langle x_n, y \rangle| = \lim_{n\to\infty} 0 = 0,$$

which yields that $y = 0$. As a result, $[S]^\perp = \{0\}$ and $[S] = \mathcal{H}$.

b) Assume $\mathcal{H}$ is weakly separable, then there is a countable subset S which is weakly dense. By part a), we know $[S] = \mathcal{H}$. Now let $[S]_{\mathbb{Q}}$ denote the closed linear span of S with rational coefficients. By the density argument, we know that $[S]_{\mathbb{Q}}$ is in fact dense in $\mathcal{H}$. As $[S]_{\mathbb{Q}}$ is countable, $\mathcal{H}$ is separable.

Assume $\mathcal{H}$ is strongly separable, then there is a countable subset S which is dense in $\mathcal{H}$. Since the strong convergence implies weak convergence, S must be weakly dense as well. As S is countable, S is weakly separable.

□