08/10. Sec 10.8 Taylor and Maclaurin Series

suppose $f(x) = \sum_{0}^{\infty} a_n (x-a)^n$ then $\sum_{0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

if we can expand.

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

$$f'(a) = a_1 \qquad a_1 = f'(a)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-a)^{n-2}$$

$$f''(a) = 2 \cdot 1 \cdot a_2 \qquad a_2 = \frac{f''(a)}{2!}$$

$$\vdots$$

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Def: $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + \frac{f'(a)}{1!} (x-a)^1 + \frac{f''(a)}{2!} (x-a)^2 + \frac{f^{(3)}(a)}{3!} (x-a)^3 + \cdots$

↓

Taylor Series generated by f at $x = a$.

Remark: 1 f can be differentiated of any order

domain of f includes a.

$a = 0$, $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x-0)^k = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \cdots$

↳ Maclaurin Se...

eg. $\underset{\displaystyle f(x)}{\underset{\downarrow}{\frac{1}{1-x}}} = \underbrace{\sum x^n}$

$f(x) = \frac{1}{1-x} = (1-x)^{-1}$

$f'(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$

$f^{(k)}(x) = k!\,(1-x)^{-k-1}$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$

$f^{(3)}(x) = -3 \cdot 2\,(1-x)^{-3}(-1) = 3!\,(1-x)^{-4}$

$a_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1$

$f(x) = \frac{1}{1-x} = \sum a_n x^n = \underbrace{\sum x^n}$

eg. $\ln(1+t) = \underbrace{\sum_{1}^{\infty} (-1)^{n+1} \frac{t^n}{n}} = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$

$f(t) = \ln(1+t)$ Taylor series at $t=0$

$f'(t) = \frac{1}{1+t} = (1+t)^{-1}$

$f''(t) = -(1+t)^{-2}$

$f^{(3)}(t) = 2(1+t)^{-3} = \underbrace{(-1)^2}\overset{2!}{\boxed{2}}(1+t)^{-3}$

$f^{(k)}(t) = (-1)^{k-1}(k-1)!\,(1+t)^{-k}$

$a_n = \frac{f^{(n)}(a)}{n!} = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}$

$\ln(1+t) \sim \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot t^n}$

$(-1)^{n-1} = (-1)^{n+1}$

$\ln(1+t) \sim \sum^{\infty} (-1)^{n-1}\, t^n$

eg. estimate $\ln 2$ by Taylor Series.

$$\ln(1+t) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n$$

$$t=1 \qquad \underline{\ln 2} \sim \underline{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}} = \underbrace{\frac{1}{1} - \frac{1}{2} + \frac{1}{3}}_{} = \frac{5}{6}$$

? remain question.

Def: $P_n(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$ ~~Tayly~~ Taylor Polynomial of order $\underline{\underline{n}}$.

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

eg: $f(x) = e^x$ at $a=0$

$$f^{(n)}(x) = e^x$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

$$e^x \sim \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} x^n}_{} = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$\downarrow$

~~Talya~~ Taylor Series.

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k \quad \text{Taylor Polynomial.}$$

$$e^x \sim 1$$

$$\sim 1+x$$

$$\sim 1 + x + \frac{1}{2}x^2$$

$$\sim 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

eg. $f(x) = \cos x$ at $x=0$

$$f'(x) = -\sin x \qquad f^{(1+4k)} = -\sin x \qquad a_{4k+1} = \frac{f^{(4k+1)}(0)}{(4k+1)!} = 0$$

$$f^{(2)}(x) = -\cos x \qquad f^{(2+4k)} = -\cos x \qquad a_{4k+2} = \frac{-1}{(4k+2)!}$$

$$f^{(3)}(x) = \sin x \qquad f^{(3+4k)} = \sin x \qquad a_{4k+3} = 0$$

$$f^{(4)}(x) = \underbrace{\cos x} \qquad f^{(4k)} = \cos x \qquad a_{4k+4} = \frac{1}{(4k+4)!}$$

$$f^{(5)}(x) = -\sin x$$

$$\cos x \sim \sum a_n x^n$$

$$f(0) + \underline{\frac{f'(0)}{1!} x} + \frac{f''(0)}{2!} x^2 + \cdots$$

$$1 + 0 - \frac{1}{2!} x^2 + 0 + \frac{1}{4!} x^4$$

$$\cos x \sim 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$\cos x \sim 1$$
$$\sim 1 - \frac{1}{2} x^2$$
$$\sim 1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4$$

Sec. 10.9 Convergence of Taylor Series.

Thm 23. Taylor's Thm. If f and its first n-th order derivatives $f', f'', \cdots f^{(n)}$ are continuous on the closed interval between a and b, $f^{(n)}$ is differentiable on (a,b), then there exist a number c ($a<c<b$) such that

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \underline{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}.$$

$$\underline{f(b)} = P_n(b) + R_n(c)$$

$$R_n(c) = \frac{f^{(n)}(c)}{(n+1)!}(b-a)^{n+1} \qquad a < c < b$$

$$|R_n(c)| = |\underbrace{f(b) - P_n(b)}_{\text{error.}}|$$

(a,b)

Taylor's Formula: if f has derivatives of all orders on open interval I,

$$(*) \quad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{2!}(x-a)^n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad a < c < b$$

$$f(x) = P_n(x) + R_n(x)$$

Def: $P_n(x) \to f(x)$ if $R_n(x) \to 0$ as $n \to +\infty$

converges

$$f(x) \overset{\text{"}\;\text{"}}{=} \sum_{n=0}^{\infty} \frac{f^{(k)}(a)}{n!}(x-a)^n$$

eg. $$e^x = \underbrace{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}_{P_n(x)} + \overbrace{\boxed{\cdots\cdots}}^{R_n(x)}$$

$$e^1 = \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!}}_{2+\boxed{}} = e \;\; (= 2.6)$$

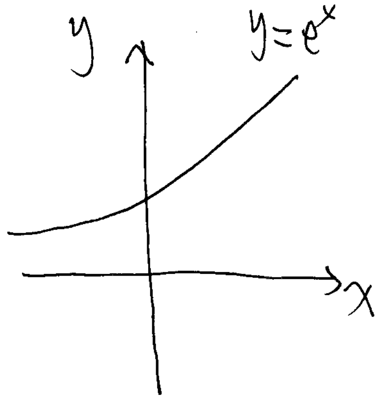
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \qquad 0 < c < x$$

if $\lim_{n\to\infty} R_n(x) = 0$, then $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$f(x) = e^x \qquad f^{(n+1)}(x) = e^x$$

$$x > 0 \qquad 0 < c < x$$

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \qquad e^c < e^x$$



$$|R_n(x)| \leq e^x \frac{x^{n+1}}{(n+1)!}$$

$$x < 0 \qquad x < c < 0$$

$$|R_n(x)| \leq \frac{x^{n+1}}{(n+1)!}$$

if we can $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$ for every x

then we have $\lim_{n\to\infty} |R_n(x)| = 0 \Rightarrow$ converges.

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \ ? \iff \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

fixed x, we can find M, $|x| < M$,

then $n > M$, $\frac{x^n}{n!} \leq \frac{x^n}{1 \cdot 2 \cdots M \cdot M^{n-M}}$

$$= \left(\frac{M^M}{M!}\right) \cdot \frac{x^n}{M^n}$$

$$0 \leq \left|\frac{x^n}{n!}\right| \leq \underbrace{\frac{M^M}{M!} \left|\frac{x}{M}\right|^n} \qquad \lim_{n\to\infty} \frac{M^M}{M!} \left|\frac{x}{M}\right|^n = 0$$

$$\left|\frac{x}{M}\right| < 1$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$