10.1

16. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}$

$\Rightarrow 1, -\frac{1}{2^2}, \frac{1}{3^2}, -\frac{1}{4^2}, \frac{1}{5^2}$

$\Rightarrow (-1)^0 \cdot \frac{1}{1^2}, (-1)^1 \frac{1}{2^2}, (-1)^2 \cdot \frac{1}{3^2}, (-1)^3 \cdot \frac{1}{4^2}$

The n th term is $(-1)^{n-1} \cdot \frac{1}{n^2}$.

23 Notice $1 = 1$

$2 = 1 \times 2$

$6 = 1 \times 2 \times 3$

$24 = 4!$

$120 = 5!$

$a_n = \frac{2+3n}{n!}$

28 converge.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right)$$

$$= 1 + \lim_{n\to\infty} \frac{(-1)^n}{n}$$

$$= 1.$$

31 converge.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-5 + \frac{1}{n^4}}{1 + \frac{8}{n}} = \frac{-5+0}{1+0} = -5$$

46. converges.

since $0 \le \sin^2 n \le 1$

we have $\frac{0}{2^n} \le a_n \le \frac{1}{2^n}$

$\Rightarrow \lim_{n\to\infty} \frac{0}{2^n} \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{1}{2^n}$

$\Rightarrow 0 \le \lim_{n\to\infty} a_n \le 0$

$\Rightarrow \lim a_n = 0$

54.* converge.

Notice that $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n-1})^n}$

$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n-1})^{n-1}\cdot(1+\frac{1}{n-1})}$

$= \frac{1}{e\cdot 1} = \frac{1}{e}$

58.* converge.

consider $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln(n+4)}{n+4}$

$~~= \lim_{n\to\infty} \frac{1}{n+4}~~$

$= \lim_{n\to\infty} \frac{1}{n+4} = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 1.$

87* converge.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}}$

$= \lim_{n\to\infty} \frac{n}{n+\sqrt{n^2-n}}$

$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}}$

$= \frac{1}{2}.$

101a* since $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$

$f(x) = x^2 - 2$.

$x_0 = 1 \Rightarrow x_1 = \frac{3}{2}$

And we know $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \geq 2 \cdot \sqrt{\frac{1}{2}} = \sqrt{2}$. $(n \geq 0)$

It's easy to show when $n \geq 1$.

$x_{n+1} \leq x_n$

$\Leftrightarrow \quad \frac{1}{x_n} \leq \frac{x_n}{2}$

$\Leftrightarrow \quad 2 \leq x_n^2$

(since when $n \geq 1$ $\quad x_n^2 \geq (\sqrt{2})^2 = 2$.)

Hence $\{x_n\}_{n=1}^{\infty}$ is a monotonously decreasing sequence with lower bound $\sqrt{2}$.

$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \left(\frac{x_n}{2} + \frac{1}{x_n}\right)$

$\Rightarrow \quad \bar{x} = \bar{x}/2 + 1/\bar{x}$

$\Rightarrow \quad \bar{x} = \sqrt{2}$.

125*. For $\forall\ \varepsilon > 0$.

$\frac{n}{n+1} > 1 - \varepsilon$.

$\Leftrightarrow \quad \varepsilon > \frac{1}{n+1}$

$\Leftrightarrow \quad \frac{1}{\varepsilon} - 1 < n$

That is when n is larger than $\left(\frac{1}{\varepsilon} - 1\right)$, $\frac{n}{n+1}$ exceeds $1 - \varepsilon$.

10.2.

18. converge.

$$\text{Sum} = \left(\frac{-2}{3}\right)^2 \cdot \left(\frac{1}{1-\frac{-2}{3}}\right)$$

$$= \frac{4}{15}$$

28. Diverge.

$$a_n = \frac{n(n+1)}{(n+2)(n+3)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{(1+\frac{2}{n})(1+\frac{3}{n})}$$

$$= \frac{1}{1}$$

$$= 1 \neq 0.$$

36*. Converge.

$$\sum_{n=1}^{k}\left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$$

$$= \frac{3}{1} - \frac{3}{(k+1)^2}$$

$$= 3\left(1 - \left(\frac{1}{k+1}\right)^2\right)$$

$$\lim_{k\to\infty} 3\left(1 - \frac{1}{(k+1)^2}\right) = 3.$$

56*. $\sum_{n=1}^{\infty} \ln\frac{1}{3^n} = \sum_{n=1}^{\infty} -n\ln 3.$

Since $\ln 3 \neq 0$.

It's divergent.

60*: $a_n = \left(1-\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} a_n = \frac{1}{e} \neq 0.$$

It's divergent.

10.3.

6*. We can see $a_n = f(n) = \frac{1}{n(\ln n)^2}$ is a continuous bounded decreasing function. We can apply integral test.

$$\int_2^{+\infty} \frac{1}{n(\ln n)^2} \cdot dn = -\frac{1}{\ln n}\Big|_2^{+\infty} < \infty$$

It's convergent.

33*. consider $a_n = n \cdot \sin\frac{1}{n}$.

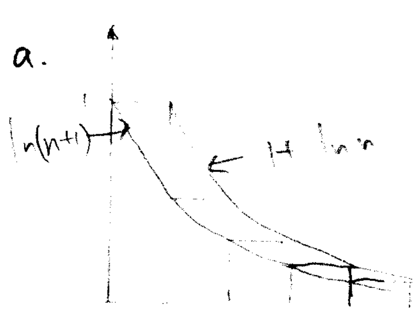
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \cdot \sin\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$$

$$= 1 \neq 0$$

$\Rightarrow$ It's divergent.

43. a.



57.* a. From 43. we already know

$$\ln(n+1) \le 1+\frac{1}{2}+\cdots+\frac{1}{n} \le 1+\ln n.$$

b. $\int_n^{n+1} \frac{1}{x}\, dx \geqslant \int_n^{n+1} \frac{1}{n+1}\, dx = 1\cdot\frac{1}{n+1} = \frac{1}{n+1}$

$\Rightarrow \frac{1}{n+1} \le \ln(n+1) - \ln n.$

In order to show $a_n = 1+\frac{1}{2}+\cdots+\frac{1}{n} - \ln n \downarrow$,

It suffices to show $a_{n+1} \le a_n$ for $\forall n \geq 0$.

$\Leftrightarrow 1+\frac{1}{2}+\cdots+\frac{1}{n+1} - \ln(n+1) \le 1+\frac{1}{2}+\cdots+\frac{1}{n} - \ln n$

$\Leftrightarrow \frac{1}{n+1} \le \ln(n+1) - \ln n.$

This is what we have done above.

10.4.

1. we know $\frac{1}{n^2+30} \le \frac{1}{(n-1)^2}$ when $n \to \infty$

~~and~~ and $\sum \frac{1}{(n-1)^2}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+30}$ converges.

5*. $\frac{\cos^2 n}{n^{\frac{3}{2}}} \le \frac{1}{n^{\frac{3}{2}}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{\frac{3}{2}}}$ converges

10. Use limit comparison.

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}} = 1$$

and we know $\sum \frac{1}{\sqrt{n}}$ diverges

So. $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$ diverges.

15*. we know $\frac{1}{\ln n} > \frac{1}{n}$ when $n \to \infty$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\ln n}$ diverges.

17. Diverges

Compare $\frac{1}{2\sqrt{n}+\sqrt[3]{n}}$ with $\frac{1}{\sqrt{n}}$.

20. Converge

$$\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2} = 2\cdot\sum_{n=1}^{\infty}\left(\frac{\cos n}{n}\right)^2 \le 2\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Converges.

64 Diverges

$$\sum_{n=2}^{\infty}\sqrt{\frac{\ln n}{n}} \geq \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$$

Diverge

10.5.

1. $a_n = \frac{2^n}{n!}$

$$r = \frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} < 1 \quad \text{when } n \to \infty$$

$\Rightarrow$ converges.

6. $a_n = \frac{3^{n+2}}{\ln n}$

$$r = \frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+3}}{\ln(n+1)}}{\frac{3^{n+2}}{\ln n}} = \frac{3\ln n}{\ln(n+1)} > 1 \quad \text{when } n \to \infty$$

Diverges.

12*. $a_n = \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

$$r_n = \sqrt[n]{a_n} = \ln\left(e^2 + \frac{1}{n}\right) \cdot \sqrt[n]{\ln\left(e^2 + \frac{1}{n}\right)}$$

$$\lim_{n\to\infty} r_n = 2 \cdot 1 = 2.$$

Diverge.

15*. $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$$

$$= \frac{1}{e} < 1$$

Converge.

18. converge.

By ratio test

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot e^{-(n+1)}}{n^2 \cdot e^{-n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot e^{-1}$$

$$= e^{-1} < 1.$$

29 Diverge.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{(1-n)}{n^2}.$$

$$a_n = \frac{1-n}{n^2}$$

Compare it with $\frac{1}{n}$.

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1-n}{n^2}} = (-1)$$

$\Rightarrow$ Diverge.

31. $$\sum_{n=1}^{\infty} \frac{\ln n}{n} = 0 + \frac{\ln 2}{2} + \sum_{n=3}^{\infty} \frac{\ln n}{n}$$

$$\geq 0 + \frac{\ln 2}{2} + \sum_{n=3}^{\infty} \frac{1}{n}$$

↑

Diverges

45*. Converge

By ratio test. $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+\sin n}{n}$

$= 0 < 1.$

10.6.

4. converge.

$$a_n = (-1)^n \cdot \frac{4}{(\ln n)^2}$$

is alternating

and $|a_n|$ is decreasing to 0.

20. $a_n = \frac{n!}{2^n} \cdot (-1)^{n+1}$

$$|a_n| = \frac{n!}{2^n}$$

$$r = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{2} \Rightarrow \infty$$

So the series diverges.

28. ~~conditionally d~~

conditionally converge.

First. It's alternating and decreasing to 0.

Second: $|a_n| = \frac{1}{n\ln n}$

By Intergral test.

$\sum_{n=1}^{\infty} \frac{1}{n \cdot \ln n}$ diverges.

30.* conditionally converge.

$f(n) = \frac{\ln n}{n - \ln n}$ is decreasing.

and $\sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n} \geq \sum_{n=1}^{\infty} \frac{\ln n}{n}$

$\uparrow$

Diverges.

49*.

$$\Sigma = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$\text{Error} = \left|\Sigma - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right)\right|$$

$$= \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$\frac{1}{5} - \frac{1}{6} \leq \text{Error} \leq \frac{1}{5}$$

$$\Rightarrow \frac{1}{30} \leq \text{Error} \leq \frac{1}{5}$$

62*.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{1}{n} - \frac{1}{n+1}$$

The series converges.

$\sum_{k=1}^{2n+1} = 1$ always. The sum is 1