08/11

Thm (The Remainder Estimation Thm)

If there is a positive number M, such that $|f^{(n+1)}(s)| \leq M$

for $a \leq s \leq x$

then the remainder $R_n(x)$ satisfies $|R_n(x)| \leq \underline{M} \dfrac{|x-a|^{n+1}}{(n+1)!}$ $(*)$

Furthermore, if $\overset{(*)}{\text{this}}$ inequality holds for every n, then ~~the~~ Taylor Series

Converges to $f(x)$

Remark: $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$

eg: show that Taylor Series generated by $f(x) = \sin x$ at $x=0$

Converges to $f(x)$ for all x.

Step 1. $f^{(1)}(x) = \cos x \rightarrow 1$

$f^{(2)}(x) = -\sin x$

$f^{(3)}(x) = -\cos x \longrightarrow -1$

$f^{(4)}(x) = \sin x$

$$\sin x \sim 0 + 1 \cdot x + \frac{-1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots$$

$$+ \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \underline{R_{2k+1}(x)}$$

$$R_{2k+1}(x) = \frac{f^{(2k+2)}(c)}{(2k+2)!} x^{2k+2}$$

c is in the neighbor of 0 and

$$|R_{2k+1}(x)| \leqslant 1 \cdot \frac{x^{2k+2}}{(2k+2)!}$$

$$\lim_{k\to\infty} \frac{x^{2k+2}}{(2k+2)!} = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} R_{2k+1} = 0$$

$$\sin x = \overset{u_0}{x} - \frac{\overset{u_1}{x^3}}{3!} + \frac{\overset{u_2}{x^5}}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots$$

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \left| \frac{\dfrac{x^{2k+3}}{(2k+3)!}}{\dfrac{x^{2k+1}}{(2k+1)!}} \right| = \left| \frac{x^2}{(2k+3)(2k+1)} \right| \to 0$$

converge for all x ($R = +\infty$)

eg: $\underline{e^x}\,\underline{\cos x}$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)$$

$$= \left(1 - \cancel{\frac{x^2}{2!}} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots\right)$$

$$+ \left(\cancel{\frac{x^2}{2!}} - \frac{x^4}{2!\,2!} + \cdots\right) + \left(\frac{x^3}{3!} - \frac{x^5}{3!\,2!} + \cdots\right)$$

$$= 1 + x + \left(-\frac{1}{2} + \frac{1}{6}\right)x^3 + \cdots$$

eg: For what value of x can we replace $\underline{\sin x}$ by $\underline{x - \frac{x^3}{3!}}$ with error less then 3×10^{-4} ?

$$\sin x = x - \frac{x^3}{3!} + \underline{\frac{x^5}{5!} \cdots\cdots}$$

$$\left|\frac{x^5}{5!}\right| < 3 \times 10^{-4} \qquad |x^5| < 5! \times 3 \times 10^{-4}$$

$$\Longrightarrow \quad -0.514 < x < 0.514$$

Sec. 10.10 The Binomial Series and Application.

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

$$f(x) = (1+x)^m \quad \text{at } x = 0$$

$$f'(x) = m(1+x)^{m-1} \rightarrow m$$

$$f''(x) = m(m-1)(1+x)^{m-2} \rightarrow m(m-1)$$

m is positive integer, the series is finite.

Otherwise, the series is infinite.

$$\lim_{k\to\infty} \left|\frac{m-k}{k}x\right| = |x| < 1 \quad \text{Converges. } R = 1.$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

k → non-nega integer

$$\binom{-1}{1} = \frac{-1}{1!} = -1 \qquad \binom{-1.5}{2} = \frac{-1.5\,(-2.5)}{2!}$$

$$\binom{-1}{2} = \frac{-1\cdot(-2)}{2!} = 1$$

$$\binom{-1}{k} = \frac{-1(-2)\cdots(-k)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k$$

$$(1+x)^{-1} = \frac{1}{1+x} = 1 + \sum_{k=1}^{\infty} \binom{-1}{k} x^k$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k x^k$$

$$= 1 - x + x^2 - x^3 \cdots$$

eg. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$\uparrow$ Integration

$$\frac{d}{dx}\tan^{-1}x = \underbrace{\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots}$$

$x=1$. $\quad \tan^{-1}1 = \frac{\pi}{4} \Leftarrow \tan\frac{\pi}{4} = 1$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

eg. $\lim\limits_{x\to1} \frac{\ln x}{x-1} = \lim\limits_{x\to1} \frac{\frac{1}{x}}{1} = 1.$

$\uparrow$ L-Hospital Rule.

$\ln x \longrightarrow$ polynomial at $x=1$

$$\ln x \sim f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \cdots$$

$\frac{1}{x} = x^{-1}$

$$\underbrace{0 + (x-1) - \frac{1}{2!}(x-1)^2}_{x=1.1} + \underbrace{\frac{2}{3!}(x-1)^3}_{0.1^3} + \cdots$$

$\downarrow$

$-x^{-2}$

$2x^{-3}$

0.1^2

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{0+(x-1)-\frac{1}{2!}(x-1)^2+R_3(x)}{x-1} = 1+\lim_{x\to 1} -\frac{1}{2!}\overset{\to 0}{\cancel{(x-1)}}+\cdots$$

$$= 1.$$

$$R_3(x) \ll (x-1)^2 \ll x-1$$

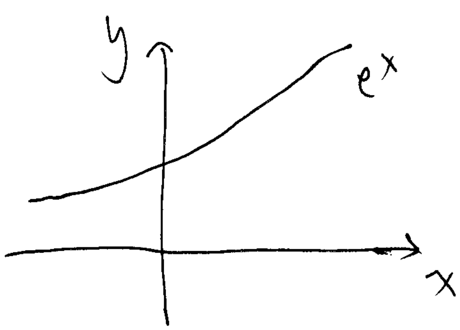
$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{x-1+R_2(x)}{x-1} = 1$$

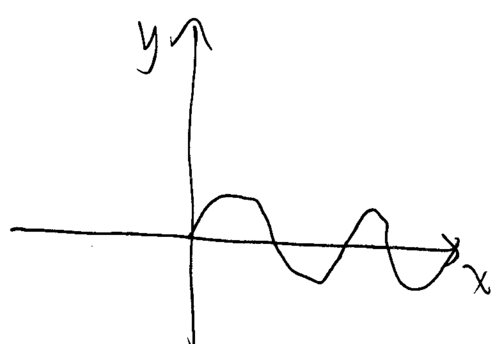
eg: $\sin x-\cos x = \sin\frac{x}{2}\cos\frac{x}{2}\cdots\cdots$ X

$$\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right) - \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\right)$$

eg: $i=\sqrt{-1}$ $\qquad i^2=-1$

$$e^{i\theta} = \cos\theta + i\cdot\sin\theta \longrightarrow \text{Euler's Identity.}$$





$$e^x = 1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots \qquad i^2=-1$$

$$e^{i\theta} = 1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\cdots$$

$$= \underbrace{\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)} + i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\right)$$

$$= \cos\theta + i\sin\theta$$