In this homework, we regard integrability as $\int f \mathrm{d}\mu < \infty$ and Lebesgue integrability as $\int |f| \mathrm{d}\mu < \infty$.

Problem 1.

Proof.

1. For $n \geqslant 1$, let's define $f_n(x)$ as the graph function of an isosceles triangle in the following way:
$$f_n(x) = \begin{cases} 4n^4(x-n) & \text{on} \quad [n, n+\frac{1}{2n^3}]; \\ -4n^4(x-(n+\frac{1}{n^3})) & \text{on} \quad [n+\frac{1}{2n^3}; n+\frac{1}{n^3}]; \\ 0 & \text{Otherwise.} \end{cases}$$
Next, we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$. And $f(x)$ is a nonnegative continuous function. In addition,
$$\int_{\mathbb{R}} f(x)\mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{2}\frac{1}{n^3}2n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$
which yields that $f(x)$ is Lebesgue integrable.
By construction, $\limsup_{x\to\infty} f(x) = \infty$.

2. Assume f is uniformly continuous, Lebesgue integrable, and for the sake of contradiction, we further assume that $\lim_{x\to\infty} f(x) \neq 0$. Then there exists a $\varepsilon > 0$, such that there exists a sequence of infinite points $\{x_i\}$ with $|f(x_i)| \geqslant \varepsilon$ and $x_i \to \infty$ as $i \to \infty$.
Since $f(x)$ is uniformly continuous, by the triangle inequality we know that $|f(x)|$ is uniformly continuous, as well. Then there exists a $\delta > 0$, such that $|x-y| \leqslant \delta$ implies that $||f(x)|-|f(y)|| \leqslant \frac{\varepsilon}{2}$. As a result, for each i, for any $y \in [x_i-\delta, x_i+\delta]$, we have $|f(y)| \geqslant \frac{\varepsilon}{2}$.
Now, we pick a subsequence $\{x_{i_k}\}$ of $\{x_i\}$ such that $x_{i_{k+1}} - x_{i_k} \geqslant 2\delta$. And $\{x_{i_k}\}$ is a sequence of infinite points too. As a result, $[x_{i_k}-\delta, x_{i_k}+\delta]$ are pairwise disjoint.
$$\int_{\mathbb{R}} |f|\mathrm{d}x > \sum_k \int_{[x_{i_k}-\delta, x_{i_k}+\delta]} |f|\mathrm{d}x \geqslant \sum_k 2\delta\frac{\varepsilon}{2} \to \infty,$$
which yields contradiction.

□

Problem 2.

Proof. Let us define a level set A_M by $A_M := \{x \in \mathbb{R} : |f(x)| \leqslant M\}$. Then $|f(x)|\mathbb{1}_{A_M} \to |f(x)|$ pointwise as $M \to \infty$. Since f is Lebesgue integrable, then by the Lebesgue's dominated convergence theorem, we have:
$$\lim_{M\to\infty} \int_{\mathbb{R}} |f(x)|\mathbb{1}_{A_M}\mathrm{d}x = \int_{\mathbb{R}} |f(x)|\mathrm{d}x.$$

Since $f(x)$ is Lebesgue integrable on $\mathbb{R}$, for every $\varepsilon > 0$, there exists a $M = M(\varepsilon)$, such that

$$\int_{\mathbb{R}\setminus A_M} |f|\mathrm{d}x \leqslant \frac{\varepsilon}{2}.$$

For every $\varepsilon > 0$, there exsits a $\delta = \frac{\varepsilon}{2M}$ such that for every $|x-y| \leqslant \delta$ (we further assume $y > x$ without loss of generality):

$$\begin{aligned}
|F(y) - F(x)| &= |\int_x^y f(x)\mathrm{d}x| \\
&\leqslant \int_x^y |f(x)|\mathrm{d}x \\
&= \int_{[x,y]\cap A_M} |f(x)|\mathrm{d}x + \int_{[x,y]\setminus A_M} |f(x)|\mathrm{d}x \\
&\leqslant M|y-x| + \frac{\varepsilon}{2} \\
&\leqslant \varepsilon,
\end{aligned}$$

which yields that $F(x)$ is uniformly continuous. □

Problem 3.

Proof. Suppose on the contrary, there exists a measurable set E with $\mathscr{L}^d(E) > 0$ such that $f(x) < 0$ on E. Then

$$\int_E f(x)\mathrm{d}x \leqslant \sup_E f(x)\mathscr{L}^d(E) < 0,$$

which yields a contradiction.

For $\int_E f(x)\mathrm{d}x = 0$ for every measurable set E, it can be decomposed into $\int_E f(x)\mathrm{d}x \geqslant 0$ and $\int_E f(x)\mathrm{d}x \leqslant 0$ for every measurable set E. □

Problem 4.

Proof. This problem is **false** if we take $\infty = \infty$.

Counter-example: $f_n(x) = (n+1)x(1-x^2)^n \mathbb{1}_{[0,1]}(x) + \mathbb{1}_{[1,n]}(x)$. It is a sequence of non-negative function. And $f_n(x) \to f(x) = \mathbb{1}_{[1,\infty]}(x)$ pointwise.

$$\int_{\mathbb{R}} f_n(x)\mathrm{d}x = \int_0^1 (n+1)x(1-x^2)^n\mathrm{d}x + \int_1^n 1\mathrm{d}x = -\frac{1}{2}(1-x^2)^{n+1}|_0^1 + (n-1) = n + \frac{1}{2}.$$

If we take $\infty = \infty$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\mathrm{d}x = \int_{\mathbb{R}} f(x)\mathrm{d}x.$$

However, on the measurable set $[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x = \frac{1}{2} \neq 0 = \int_0^1 f(x)\mathrm{d}x.$$

In the following, we assume the notation $\int_X f(x)\mathrm{d}\mu$ has the convention that $\int_X f(x)\mathrm{d}\mu < \infty$.

Let $f(x) = \lim_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$. Since $\{f_n(x)\}$ is a sequence of non-negative function, for every measurable set E, by Fatou's lemma we have

$$\int_E f\mathrm{d}\mu = \int_X f\mathbb{1}_E\mathrm{d}\mu = \int_X \liminf_{n\to\infty} f_n(x)\mathbb{1}_E(x)\mathrm{d}\mu \leqslant \liminf_{n\to\infty}\int_X f_n(x)\mathbb{1}_E(x)\mathrm{d}\mu = \liminf_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu.$$

Since $X \setminus E \in \Sigma$, then

$$\int_{X\setminus E} f\mathrm{d}\mu \leqslant \liminf_{n\to\infty}\int_{X\setminus E} f_n(x)\mathrm{d}\mu.$$

Combine both terms:

$$\begin{aligned}
\int_X f\mathrm{d}\mu &= \int_E f\mathrm{d}\mu + \int_{X\setminus E} f\mathrm{d}\mu \\
&\leqslant \liminf_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu + \liminf_{n\to\infty}\int_{X\setminus E} f_n(x)\mathrm{d}\mu \\
&= \liminf_{n\to\infty}\int_X f_n(x)\mathrm{d}\mu \\
&= \lim_{n\to\infty}\int_X f_n(x)\mathrm{d}\mu \\
&= \int_X f\mathrm{d}\mu,
\end{aligned}$$

which forces that

$$\int_E f\mathrm{d}\mu = \liminf_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu$$

holds for every measurable set E.

On the other hand,

$$\begin{aligned}
\limsup_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu &= \limsup_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu - \int_X f\mathrm{d}\mu + \int_X f\mathrm{d}\mu \\
&= \limsup_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu - \liminf_{n\to\infty}\int_X f_n(x)\mathrm{d}\mu + \int_X f(x)\mathrm{d}\mu \\
&= \int_X f(x)\mathrm{d}\mu - \left(\liminf_{n\to\infty}\int_X f_n(x)\mathrm{d}\mu - \limsup_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu\right) \\
&\leqslant \int_X f(x)\mathrm{d}\mu - \liminf_{n\to\infty}\left(\int_X f_n(x)\mathrm{d}\mu - \int_E f_n(x)\mathrm{d}\mu\right) \\
&= \int_X f(x)\mathrm{d}\mu - \liminf_{n\to\infty}\int_{X\setminus E} f_n(x)\mathrm{d}\mu \\
&= \int_X f(x)\mathrm{d}\mu - \int_{X\setminus E} f(x)\mathrm{d}\mu \\
&= \int_E f(x)\mathrm{d}\mu.
\end{aligned}$$

As a result,

$$\liminf_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu \leqslant \limsup_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu \leqslant \int_E f(x)\mathrm{d}\mu = \liminf_{n\to\infty}\int_E f_n(x),$$

which forces that

$$\int_E f\mathrm{d}\mu = \lim_{n\to\infty}\int_E f_n(x)\mathrm{d}\mu$$

holds for every measurable set E. □

Problem 5.

Proof.

1. Let $E = \{x \in X : \phi(x) \geqslant a\}$, due to ϕ is non-negative, then

$$\int_X \phi(x)\mathrm{d}\mu \geqslant \int_E \phi(x)\mathrm{d}\mu \geqslant a\mu(E)$$

2. Assume otherwise, $\phi(x)$ is infinite on some set E with $\mu(E) > 0$, then

$$\infty = \int_E \phi(x)\mathrm{d}x \leqslant \int_X \phi(x)\mathrm{d}x < \infty,$$

which yields contradiction.

3. Assume otherwise, $\phi(x) > 0$ on some set E with $\mu(E) > 0$, then

$$0 < \int_E \phi(x)\mathrm{d}x \leqslant \int_X \phi(x)\mathrm{d}x = 0,$$

which yields contradiction.

□