**Problem 1.**

*Proof.* To show $\Sigma$ is a $\sigma$-algebra.

- Clearly $\emptyset \in \Sigma$.
- If $E \in \Sigma$, then by definition of $\Sigma$, either $E$ is countable or $X - E$ is countable.

  When $E$ is countable, by definition of $\Sigma$, $X - E \in \Sigma$.

  When $X - E$ is countable, by definition of $\Sigma$, $X - E \in \Sigma$.
- Assume $\{E_i\}_{i=1}^{\infty} \subset \Sigma$, if $\bigcup_{i=1}^{\infty} E_i$ is countable, then $\bigcup_{i=1}^{\infty} E_i \in \Sigma$. Otherwise, there must be a $k$ such that $E_k$ is uncountable, which yields that $X - E_k$ is countable. Note that de Morgan's laws hold for any family of sets:

$$X - \bigcup_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} (X - E_i),$$

  which is at most countable due to $X - E_k$ countable. Thus $\bigcup_{i=1}^{\infty} E_i \in \Sigma$.

$\square$

**Problem 2.**

*Proof.*

a) Let us define $B_1 = A_1$ and $B_{i+1} = A_{i+1} - A_i$ for $i \geqslant 2$. Note that $B_i \in \Sigma$ and $\{B_i\}_{i=1}^{\infty}$ are pairwise disjoint. Moreover, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ and $A_n = \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$. Therefore,

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} B_i) = \lim_{n\to\infty} \mu(A_n).$$

b) Let us define $B_i = A_1 - A_i$. Note that $B_i \in \Sigma$, $B_i \subset B_{i+1}$ for each $i$. By (a), we have:

$$\mu(\bigcup_{i=1}^{\infty} B_i) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_1 - A_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n),$$

which is well-defined if $\mu(A_1) < \infty$. Thus

$$\mu(\bigcap_{i=1}^{\infty} A_i) = \mu(A_1 - \bigcup_{i=1}^{\infty} B_i) = \mu(A_1) - \mu(\bigcup_{i=1}^{\infty} B_i) = \lim_{n\to\infty} \mu(A_n).$$

**Counter-example:** Take the measure space as Lebesgue measure on $\mathbb{R}$. Take $A_i = (i, \infty]$. For each $i$, $\mu(A_i) = \infty$ thus $\lim\limits_{i\to\infty} \mu(A_i) = \infty$ however $\mu(\bigcap_{i=1}^{\infty} A_i) = 0$.

$\square$

**Problem 3.**

*Proof.*

a) To show $\mu$ is a measure on $(X, \Sigma)$:

- Since $\mu_n(\emptyset) = 0$ for all $n$, then $\mu(\emptyset) = \lim_{n\to\infty} \mu_n(\emptyset) = 0$.
- Given a sequence of pairwise disjoint measurable sets $\{E_i\}_{i=1}^{\infty} \subset \Sigma$,

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu_n(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(E_i).$$

By monotone convergence theorem, we can exchange the limit with the summation:

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(E_i) = \sum_{i=1}^{\infty} \lim_{n\to\infty} \mu_n(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

b) To show $\mu$ is a measure on $(X, \Sigma)$:

- Since $\mu_n(\emptyset) = 0$ for all $n$, then $\mu(\emptyset) = \sum_{n=1}^{\infty} \mu_n(\emptyset) = 0$.
- Given a sequence of pairwise disjoint measurable sets $\{E_i\}_{i=1}^{\infty} \subset \Sigma$,

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{n=1}^{\infty} \mu_n(\bigcup_{i=1}^{\infty} E_i) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mu_n(E_i) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_n(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

□

**Problem 4.**

*Proof.* When $f$ is $\Sigma$-measurable, we have $f^{-1}((a, \infty)) \in \Sigma$ for any $a \in \mathbb{R}$. Since Borel set in $\mathbb{R}$ are generated by open intervals. It suffices to show $f^{-1}((a, b)) \in \Sigma$.

Let us recall some facts from set theory, which can be proved in one line by the definition of sets. Given a function $f : X \mapsto Y$, we have:

$$f^{-1}(Y - B) = X - f^{-1}(B) \qquad \text{for any } B \subset Y; \tag{1}$$

$$f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i) \qquad \text{for any } \{B_i\}_{i\in I} \quad \text{for any index set } I; \tag{2}$$

$$f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i) \qquad \text{for any } \{B_i\}_{i\in I} \quad \text{for any index set } I; \tag{3}$$

Since $f^{-1}((b, \infty)) \in \Sigma$, then by (1) we have

$$f^{-1}((-\infty, b]) = f^{-1}((b, \infty)^c) = X - f^{-1}((b, \infty)) \in \Sigma.$$

Now we have both $f^{-1}((a,\infty)) \in \Sigma$ and $f^{-1}((-\infty,b]) \in \Sigma$, by (2):

$$f^{-1}((a,b]) = f^{-1}((a,\infty) \cap (-\infty,b]) = f^{-1}((a,\infty)) \cap f^{-1}((-\infty,b]) \in \Sigma.$$

Note that $(a,b) = \bigcup_{i=1}^{\infty}(a, b-\frac{1}{i}]$, by (3):

$$f^{-1}((a,b)) = f^{-1}(\bigcup_{i=1}^{\infty}(a,b-\frac{1}{i}]) = \bigcup_{i=1}^{\infty} f^{-1}((a,b-\frac{1}{i}]) \in \Sigma,$$

which completes the proof. □

**Problem 5.** To define the equivalence relationship on the space of **measurable function** $f : (X,\Sigma,\mu) \mapsto (\mathbb{R},\mathscr{B})$, we **DO NOT** require $\mu$ is complete.

*Proof.* The reflexive and symmetric properties are clearly satisfied. Now we prove the transitive property.

Let $f, g, h$ be measurable function on $(X,\Sigma,\mu)$ such that $f \sim g$ and $g \sim h$, i.e., there exists a measurable set $X_{fg}$ and $X_{gh}$ of measure zero, such that $f(x) = g(x)$ on $X_{fg}^c$, $f(x) \neq g(x)$ on $X_{fg}$; $g(x) = h(x)$ on $X_{gh}^c$, $g(x) \neq h(x)$ on $X_{gh}$.

As a result, $f(x) = h(x)$ on $(X \setminus X_{fg}) \cap (X \setminus X_{gh}) = X \setminus (X_{fg} \cup X_{gh})$. In the meanwhile, $f(x) \neq h(x)$ on some subset of $X_{fg} \cup X_{gh}$. Note that $X_{fg} \cup X_{gh} \in \Sigma$ and $\mu(X_{fg} \cup X_{gh}) \leqslant \mu(X_{fg}) + \mu(X_{gh}) = 0$.

Since $f, h$ are measurable function, then $f - h$ is measurable function as well, which yields that $(f-h)^{-1}(0) \in \Sigma$. By (1), $X_{fh} := (f-h)^{-1}(0^c) = X - (f-h)^{-1}(0) \in \Sigma$. Moreover, $X_{fh} \subset X_{fg} \cup X_{gh}$ which implies that $\mu(X_{fh}) \leqslant \mu(X_{fg} \cup X_{gh}) = 0$. □

**Remark:** $f = g$ $\mu$-almost everywhere is not necessary to be defined for measurable functions. For functions $f, g : X \mapsto \mathbb{R}$, we say $f = g$ $\mu$-almost everywhere if there exists a set $X_1$ with $\mu(X_1) = 0$ such that:

$f(x) = g(x)$ on $X \setminus X_1$ and $f(x) \neq g(x)$ on $X_1$.

In this case $\{x : f(x) \neq g(x)\}$ is not necessary a measurable set (an element in $\Sigma$).

However, for measurable function $f, g$, the set $X_{fg} = \{x : f(x) \neq g(x)\}$ is always measurable (an element in $\Sigma$).

Furthermore, there is a theorem which might lead this mess as well:

*$\mu$ is complete if and only if the following implication is valid: If $f$ is measurable and $f = g$ $\mu$-almost everywhere, then $g$ is measurable.*

Furtherfurthermore, other form of definition to "almost everywhere" might increase the mess: We say $f = g$ $\mu$-almost everywhere if there exists a set $X_1$ with $\mu(X_1) = 0$ such that: $\{x : f(x) \neq g(x)\} \subset X_1$.

**Problem 6.** Finite measure space has at most countable many atoms.

*Proof.* Consider the set: $X_n = \{x^* \in X : \frac{1}{n} \leqslant \mu(x^*) \leqslant \frac{1}{n-1}\}$. Suppose on contrary there are uncountable number of atoms, then the above set must be uncountable for at least one $n$. Let $X_N$ denote the uncountable set of atoms. Then

$$\mu(X_N) \geqslant \sum_{x \in X_N} \frac{1}{N} \to \infty \quad \text{if } X_N \text{ is uncountable},$$

which contradicts with $\mu(X_N)$ is finite from $\mu$ is finite measure. □