Chapter 10. Infinite Sequences and Series.   08/01

Sec. 10.1. Sequences:

$\{0, 1, 2, 3, 4 \dots\}$

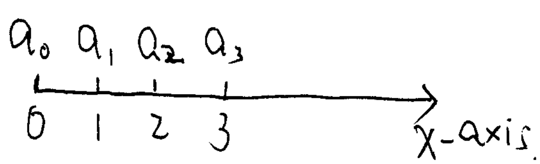
$\{0, 2, 4, \dots\}$

$\{1, 1, 2, 3, 5, 8 \dots\}$.

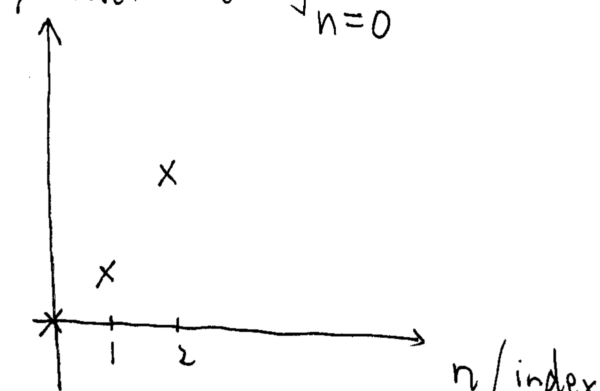
$\{a_n\} \to$ index. $= \{\sqrt{n}\}_{n=1}^{\infty}$

$\downarrow$

term $\{n\}$.



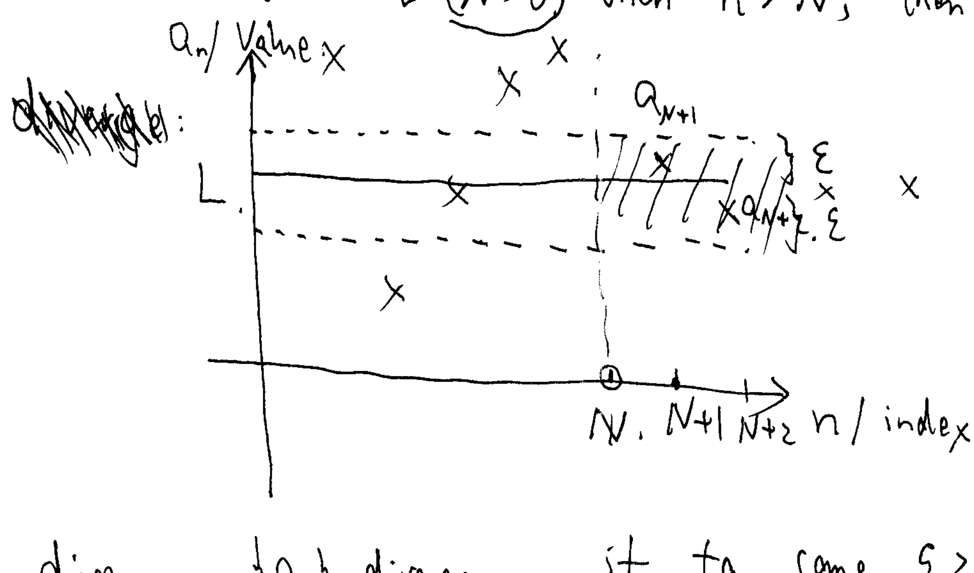
$\{n^2\}_{n=0}^{\infty}$



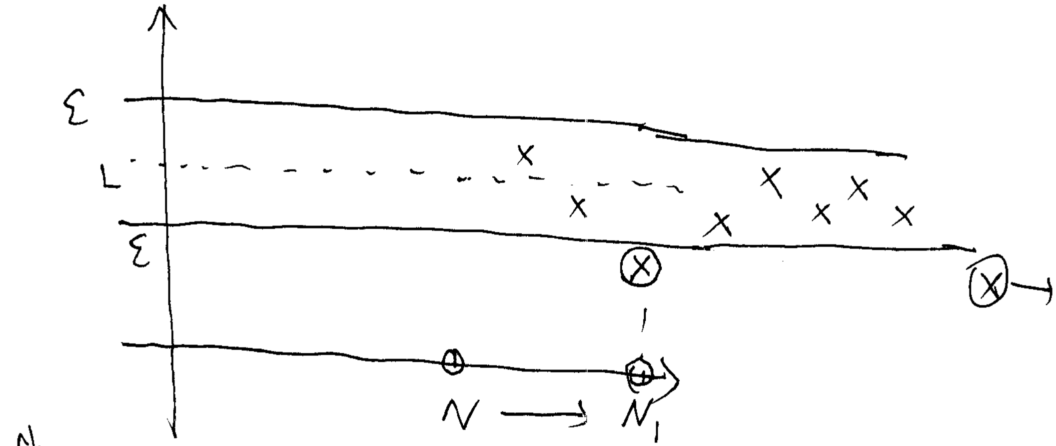
"end" $\to$ limit.

limit. $\forall$

Converge: $\{a_n\}$ converges to L, if for fixed $\varepsilon > 0$, there exists a number $N > 0$ when $n > N$, then $|a_n - L| < \varepsilon$



diverge: $\{a_n\}$ diverges, if for some $\varepsilon > 0$, there does not exist any $N > 0$, such that when $n > N$, $|a_n - L| < \varepsilon$



Show :

eg 1. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \to 0$

$0 < \frac{1}{n} < \frac{1}{n-1} < \frac{1}{n-2} < \cdots < \frac{1}{1}.$

Proof: $\forall\, \varepsilon > 0$, target: find a N, such that $n > N$,

$$|a_n - 0| < \varepsilon$$

$$\downarrow$$

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \to \text{equivalent.}$$

$$\Updownarrow$$

$$\frac{1}{\varepsilon} < n$$

$$N > \frac{1}{\varepsilon}$$

$\forall\, \varepsilon > 0$, there exist a N, $N > \frac{1}{\varepsilon}$,

$$\downarrow$$

$$N = \left[\frac{1}{\varepsilon}\right] + 1.$$

$$\downarrow$$

integer $\le \frac{1}{\varepsilon}$

when $n > N$, $|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\left[\frac{1}{\varepsilon}\right]+1}$

$< \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$

$$\downarrow$$

$$\boxed{\text{when } n > N \quad |a_n - 0| < \varepsilon}$$

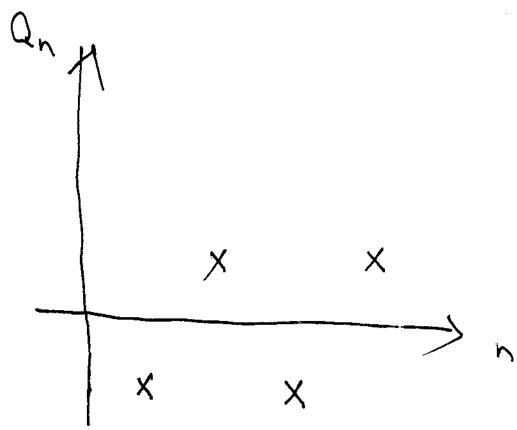
eg 2. $\left\{ \frac{n-1}{n} \right\}_{n=1}^{\infty}$

$$\frac{n-1}{n} = \frac{n}{n} - \frac{1}{n} = 1 - \frac{1}{n\uparrow} \downarrow \qquad \uparrow$$

$a_n = \frac{n-1}{n} < 1$. $\quad a_n$ increasing

$L = 1$.

eg 3. $\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \cdots\}$



Proof: suppose there is a limit L,

for some $\varepsilon = \frac{1}{2}$.

if there is a N, when $n > \boxed{N}$, $|a_n - L| < \frac{1}{2}$.

$$|(-1)^n - L| < \frac{1}{2}$$

$$-\frac{1}{2} < (-1)^n - L < \frac{1}{2}$$

$$(-1)^n - \frac{1}{2} < L < (-1)^n + \frac{1}{2}$$

$n = 1 + N$: $\quad -\frac{3}{2} < L < -\frac{1}{2}$

$n = 2 + N$: $\quad \frac{1}{2} < L < \frac{3}{2}$

$\Rightarrow$ L does not exist.

Rules: If $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, $\Rightarrow$ prior condition

$\{a_n\} \to A$

Then: ① $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A+B$ Sum

② $\lim_{n\to\infty}(a_n-b_n) = A-B$ Difference.

③ $\lim_{n\to\infty} k a_n = kA$. constant multiple.

④ $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = A\cdot B$ product.

⑤ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$ when $B \neq 0$, quotient.

eg: $\{a_n\} = \{\frac{1}{n}\} \to 0$ $\{b_n\} = \{-\frac{1}{n}\} \to 0$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} -a_n \overset{③}{=} -\lim_{n\to\infty} a_n = 0$$

$$\lim_{n\to\infty}(a_n+b_n) = 0$$

☆ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{-\frac{1}{n}} = \lim_{n\to\infty} -1 = -1$

eg: $\{a_n\} = \{n\}$ $\{b_n\} = \{-n\}$

☆ $\underbrace{\lim_{n\to\infty}(a_n+b_n)}_{\downarrow\ c_n = 0} = \lim_{n\to\infty}(\underbrace{n-n}_{\downarrow\ 0}) = 0$

eg: $\lim\limits_{n\to\infty} \frac{n^2+1}{2n^2-1} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n^2}}{2-\frac{1}{n^2}} \underset{\uparrow}{=} \frac{1}{2}$

check.

Apply Quotient Rules.

check: ① $\lim\limits_{n\to\infty} 1+\frac{1}{n^2} = 1.$

② $\lim\limits_{n\to\infty} 2-\frac{1}{n^2} = 2 \neq 0$

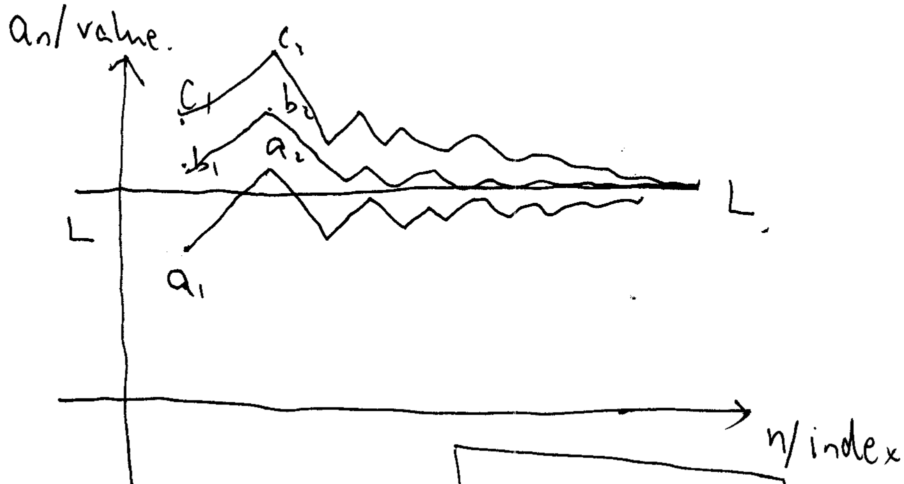
Thm: The Sandwich Theorem:

if $\forall\ n>0$, $\{a_n\}\{b_n\}\{c_n\}$, $\underline{a_n \leq b_n \leq c_n}$

$\forall \downarrow$ for any/ for all

if $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} c_n = L$,

then $\lim\limits_{n\to\infty} b_n = L.$



eg: $\left\{ \frac{\cos n}{n} \right\}_{n=1}^{\infty}$

$\boxed{\lim\limits_{n\to\infty} \frac{\cos n}{n} = 0}$

$-1 \leq \cos n \leq 1$

$|\cos n| \leq 1.$

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

$$\downarrow \quad\quad \downarrow \quad\quad \downarrow$$

$$0 \quad\quad 0 \quad\quad 0$$

Thm: (Connection between function and seq)

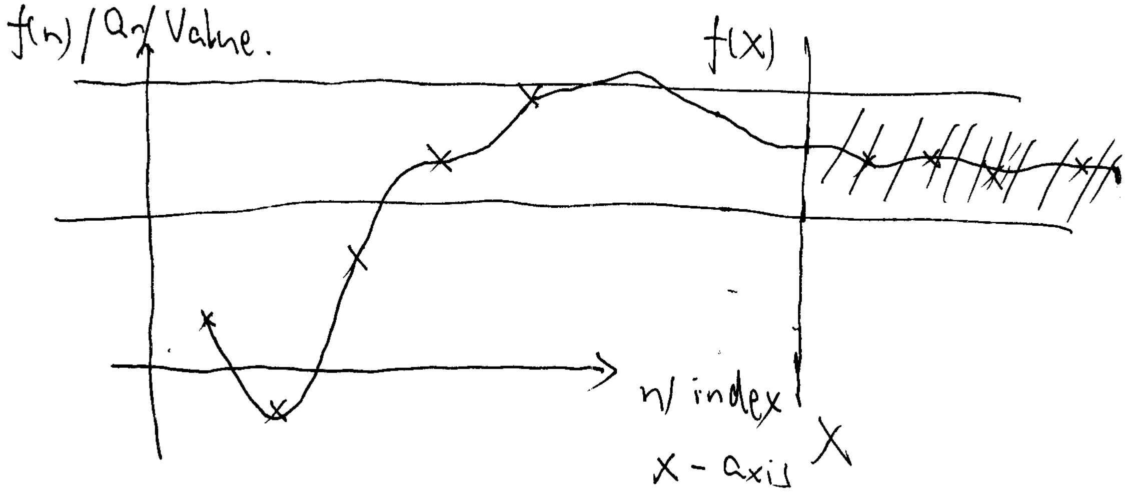
If $\{a_n\} \to L$, f is continous at any a_n,

Then $\{f(a_n)\} \to f(L)$

~~eg: $\lim\limits_{n\to\infty} \frac{\ln n}{\infty}$~~

Thm: $f(x)$ is defined on $x \geq n_0$, $\underline{a_n = f(n)}$ $\forall n$.

if $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = \underline{L}$.



eg. $\lim\limits_{n\to\infty} \frac{\ln n}{n} = ?\ 0$

$\downarrow$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$f(x) = \frac{\ln x}{x}$ $\qquad$ L'Hospital Rule.