$A \in B(\mathcal{H})$ operator: $\mathcal{H} \to \mathcal{H}$ $\qquad$ $\ker(A-\lambda I) = \{0\}$ $\qquad$ $\text{range}(A-\lambda I) = \mathcal{H}$.

resolvent set $\quad \rho(A) = \{\lambda \in \mathbb{C} \mid A-\lambda I \text{ is } 1\text{-}1 \text{ and onto}\}$

"good"

$\iff \rho(A) = \{\lambda \in \mathbb{C} \mid A-\lambda I \text{ is invertible}\}$

spectrum $\quad \sigma(A) = \mathbb{C} \setminus \rho(A)$

$\iff \sigma(A) = \{\lambda \in \mathbb{C} \mid A-\lambda I \text{ is not invertiable}\}$

$$\begin{cases} A-\lambda I \text{ is not } 1\text{-}1 & \text{point spectrum (eigenvalue)} \\ A-\lambda I \text{ is } 1\text{-}1 \text{ but not onto} & \nearrow \overline{\text{ran}(A-\lambda I)} = \mathcal{H} \quad \text{Continuous spectrum} \\ \text{(only in infinite-dim)} & \searrow \overline{\text{ran}(A-\lambda I)} \neq \mathcal{H} \quad \text{residual spectrum} \end{cases}$$

spectral radius: $\quad r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$

$$= \lim_{n \to +\infty} \|A^n\|^{\frac{1}{n}}.$$

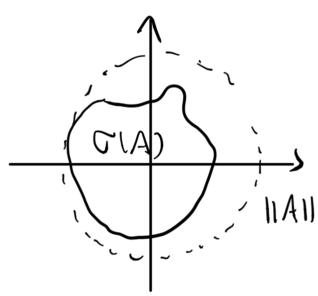
Property: 1. $\forall \lambda \in \rho(A)$, $A-\lambda I$, $(A-\lambda I)^{-1}$ are 1-1, onto, $\in \mathcal{B}(\mathcal{H})$

$R_A: \rho(A) \longrightarrow \mathcal{B}(\mathcal{H})$

$\lambda \to (A-\lambda I)^{-1}$ Neumann series.

2. If $|\lambda| > \|A\|$, then $\lambda \in \rho(A)$

$\rho(A)$ is open while $\sigma(A)$ is compact, nonempty.



Prelim $A \in B(\mathcal{H})$

(a) if $u \in \rho(A)$ and $|v-u| \leq \frac{1}{\|R_A(u)\|}$, then $v \in \rho(A)$

and $R_A(v) = [1-(v-u)R_A(u)]^{-1} R_A(u)$

proof: $A-vI = A-uI+uI-vI = \underbrace{(A-uI)}_{\text{invertible}}[\underbrace{1-(v-u)(A-uI)^{-1}}_{\text{TS invertible.}}]$

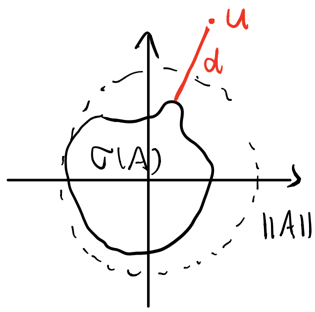
Recall if $\|K\| < 1 \Rightarrow I-k$ invertible

$(I-k)^{-1} = I+k+k^2+\cdots$

Since $|v-u| \, \|R_u(A)\| < 1$

$(A-vI)^{-1} = [1-(v-u)R_A(u)]^{-1} \cdot R_u(A)$

(b) if $u \in \rho(A)$, then $\|R_u(A)\| \geq \frac{1}{d(u,\sigma(A))}$



u closes to $\sigma(A)$

$A-uI$ more singular

unlikely invertible

$(A-uI)^{-1}$ behave bad.

suppose not $\|R_u(A)\| \cdot d(u,\sigma(A)) < 1$

then $\exists \lambda \in \sigma(A)$ s.t $\|R_u(A)\| \cdot |u-\lambda| < 1$

By (a) $\lambda \in \rho(A)$ contradiction.

Prelim $T \in \mathcal{B}(\mathcal{H})$ and $T^* = -T$, $T^2 = -1$, $T \neq \pm i 1$.

define: $P = \frac{1}{2}(1 + iT)$, $Q = \frac{1}{2}(1 - iT)$

(a) Show that P, Q are athogonal projections

proof: $P^2 = \frac{1}{4}(1 + 2iT - T^2) = \frac{1}{4}(2 + 2iT) = \frac{1}{2}(1 + iT) = P$.

$\forall x, y \in \mathcal{H}$,

$$\langle Px, y \rangle = \langle \tfrac{1}{2}(1 + iT)x, y \rangle$$
$$= \langle x, \tfrac{1}{2}(1 - iT^*)y \rangle$$
$$= \langle x, \tfrac{1}{2}(1 + iT)y \rangle = \langle x, Py \rangle$$

(b) Classify $\sigma(T)$

First compute $\|T\|$?

$TT^* = -T^2 = 1$ $T^*T = -T^2 = 1 \Rightarrow T$ is unitary

$\Rightarrow \|T\| = \|T^*\| = 1$.

if $|\lambda| > 1 \Rightarrow \lambda \in \rho(T)$ $(T - \lambda I = \lambda(\frac{T}{\lambda} - 1))$

if $|\lambda| < 1 \Rightarrow \|\lambda T^*\| < 1$ $T - \lambda I = -\underbrace{T}_{\text{invertible}}(\underbrace{\lambda T^* - 1}_{\text{invertible}})$

$\Rightarrow \lambda \in \rho(T)$

$\Rightarrow \sigma(T) \subseteq \{|\lambda| = 1\}$.

for $\lambda = a+ib \in \sigma(T)$

$\forall x \in \mathcal{H}$, $\|(T-\lambda I)x\|^2 = \langle (T-\lambda I)x, (T-\lambda I)x \rangle$

$= \langle Tx - ax - ibx, Tx - ax - ibx \rangle$

$= \|Tx - ibx\|^2 + \|ax\|^2 - \langle Tx - ibx, +ax \rangle$

$- \langle ax, Tx - ibx \rangle$

$= \|Tx - ibx\|^2 + \|ax\|^2 - \langle Tx - ibx, ax \rangle$

$- \langle (\underset{-T}{\underset{\shortparallel}{T^*}} + ib)x, ax \rangle$

for $a \neq 0$ $\geq a^2\|x\|^2$ $\Longrightarrow$ $(T-\lambda I)$ is 1-1

closed range.

for $a \neq 0$ $\lambda \in \sigma(T)$ $\operatorname{ran}(T-\lambda I) \neq \mathcal{H}$ $\Rightarrow$ $\lambda \in \sigma_r(T)$

for $a = 0$ $\lambda = \pm i$

since $T \neq \pm iI$ $T \pm iI \neq 0$ $\Longleftrightarrow$ $iT \pm I \neq 0$

thus $P = \frac{1}{2}(I + iT) \neq 0$ $Q \neq 0$

but $PQ = \frac{1}{4}(I - iT + iT + T^2) = 0$

$\operatorname{ran}\{I + iT\} \neq 0$ $\operatorname{ran}\{I - iT\} \neq \{0\}$

but $\operatorname{ran}(I - iT) \perp \operatorname{ran}(I + iT)$

$$PQz = Px = 0 \qquad x \in \ker P$$
$$x \in \operatorname{ran} Q.$$

$\{0\} \neq \operatorname{ran}(1+iT) = \ker(1-iT) \Rightarrow 1-iT$ is not 1-1

$\Rightarrow \sigma_p(T) = \{\pm i\}$

However, Recall $\lambda \in \sigma_r(T) \Rightarrow \bar{\lambda} \in \sigma_p(T^*)$

$\Rightarrow -\bar{\lambda} \in \sigma_p(T)$

$\exists\, x \neq 0$ s.t $-\bar{\lambda}x = Tx$

$$-\bar{\lambda}\langle x, x\rangle = \langle x, -\bar{\lambda}x\rangle = \langle x, Tx\rangle = -\langle Tx, x\rangle$$
$$= -\langle -\bar{\lambda}x, x\rangle = \lambda\langle x, x\rangle$$

$\Rightarrow a = 0$ in fact $\sigma_r(T) = \phi$

and $\sigma(T) \subseteq \{\pm i\}$

Logic: Assume $\lambda = a + bi \in \sigma(T)$

$a \neq 0 \Rightarrow \lambda \in \sigma_r(T)$

$a = 0 \Rightarrow \lambda \in \sigma_p(T)$

However for $a \neq 0$, the assumption does not hold.