Lecture 6: Seifert-van Kampen Theorem

Seifert-van Kampen Theorem gives a very effective method to compute fundamental groups of a space from its building blocks.

Theorem (Seifert-van Kampen Theorem, Groupoid version)

Let $X = U \cup V$ where $U, V \subset X$ are open. Then the diagram

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \\
\Pi(V) \arrow[r] & \Pi(X)
\end{tikzcd}$$

is a pushout in the category **Groupoid**.

Proof

Let $\mathcal{C}$ be a groupoid fitting into the commutative diagram

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \\
\Pi(V) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

and we need to show that

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \arrow[ddr] & \\
\Pi(V) \arrow[r] \arrow[drr] & \Pi(X) \arrow[dr, dashed, "\exists ! F"] & \\
 & & \mathcal{C}
\end{tikzcd}$$

Uniqueness: Let $\gamma\colon I \to X$ be a path in X with $x_t = \gamma(t)$. We subdivide I (by its compactness) into

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $\gamma_i := \gamma(t_{i-1}, t_i)$ lies entirely in U or V. Then

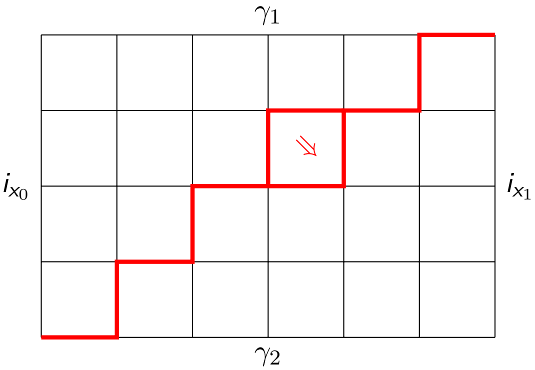
$$F([\gamma]) = F([\gamma_m]) \cdots F([\gamma_1])$$

is determined uniquely in $\mathcal{C}$ as each term is.

Existence: Given a path γ, we can define $F([\gamma])$ using a subdivision of γ, where the result does not depend on the choice of the subdivision (by the definition of pushout). We need to show that this is well-defined on homotopy class.

This follows from a refined double subdivision of $I \times I$, as shown in the picture below. Each square represents a homotopy lying entirely in either U or V and combining them together gives the required homotopy.

$$F(\gamma_1) \simeq F(\gamma_1 \star i_{x_0}) \simeq F(i_{x_1} \star \gamma_2) \simeq F(\gamma_2)$$



$\square$

Theorem (Seifert-van Kampen Theorem, Group version)

Let $X = U \cup V$ where $U, V \subset X$ are open and $U, V, U \cap V$ are path connected. Let $x_0 \in U \cap V$. Then the following diagram

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \\
\pi(V, x_0) \arrow[r] & \pi(X, x_0)
\end{tikzcd}$$

is a pushout in the category **Group**.

Proof

Denote by $\underline{G}$ the groupoid with one object for a group G.

For each $x \in X$, we fix a choice of

$$[\gamma_x] \in \mathrm{Hom}(x_0, x)$$

such that γ_x lies entirely in U when $x \in U$ and γ_x lies entirely in V when $x \in V$. Note that γ_x lies entirely in $U \cap V$ when $x \in U \cap V$. This can be achieved since U, V, $U \cap V$ are all path connected.

Consider the following functors

$$\begin{aligned}\Pi_1(U) &\to \underline{\pi_1(U, x_0)}\\ \Pi_1(V) &\to \underline{\pi_1(V, x_0)}\\ \Pi_1(U\cap V) &\to \underline{\pi_1(U\cap V, x_0)}\\ \gamma &\mapsto \gamma_{x_2}^{-1}\star\gamma\star\gamma_{x_1}, \quad \gamma\in \mathrm{Hom}(x_1, x_2).\end{aligned}$$

These functors are all retracts in **Groupoid**, in other words,

$$\begin{aligned}\underline{\pi_1(U, x_0)} \to \Pi_1(U) &\to \underline{\pi_1(U, x_0)}\\ \underline{\pi_1(V, x_0)} \to \Pi_1(V) &\to \underline{\pi_1(V, x_0)}\\ \underline{\pi_1(U\cap V, x_0)} \to \Pi_1(U\cap V) &\to \underline{\pi_1(U\cap V, x_0)}\end{aligned}$$

are all identity functors.

Suppose there is a group G that fits into the following commutative diagram:

$$\begin{tikzcd}
\Pi_1(U) \arrow[d] & \Pi_1(U \cap V) \arrow[l] \arrow[r] \arrow[d] & \Pi_1(U) \arrow[d] \\
\underline{\pi_1(U, x_0)} \arrow[dr] & \underline{\pi_1(U \cap V, x_0)} \arrow[l] \arrow[r] \arrow[d] & \underline{\pi_1(U, x_0)} \arrow[dl] \\
& \underline{G} &
\end{tikzcd}$$

By Seifert-van Kampen Theorem (Groupoid version),

$$
\begin{tikzcd}
\Pi_1(U) \arrow[d] \arrow[dr] & \Pi_1(U \cap V) \arrow[l] \arrow[r] & \Pi_1(U) \arrow[d] \arrow[dl] \\
\underline{\pi_1(U, x_0)} \arrow[dr] & \Pi_1(X) \arrow[d, dashed, "\exists ! F"] & \underline{\pi_1(U, x_0)} \arrow[dl] \\
& \underline{G} &
\end{tikzcd}
$$

Thus, we obtain a morphism

$$\underline{\pi_1(X, x_0)} \hookrightarrow \Pi_1(X) \xrightarrow{F} \underline{G}$$

which fits into a commutative diagram

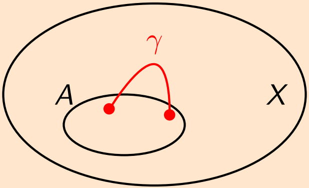
$$\begin{tikzcd}
\underline{\pi_1(U \cap V, x_0)} \arrow[r] \arrow[d] & \underline{\pi_1(U, x_0)} \arrow[d] \arrow[ddr] & \\
\underline{\pi(V, x_0)} \arrow[r] \arrow[drr] & \underline{\pi(X, x_0)} \arrow[dr] & \\
& & \underline{G}
\end{tikzcd}$$

This leads to the required diagram in the category **Group**. □

We also have the relative version.

Definition

Let $A \subset X$, we define $\Pi_1(X, A)$ be the full subcategory of $\Pi_1(X)$ consists of objects in A.



For instance, when $A = \{x_0\}$, we have

$$\Pi_1(X, x_0) = \pi_1(X, x_0).$$

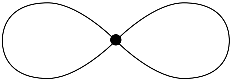
Theorem

Let $X = U \cup V$, U, V be open and $A \subset X$ intersects each path connected components of $U, V, U \cap V$. Then we have a pushout

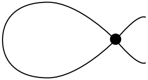
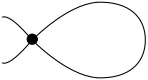
$$\begin{tikzcd}
\Pi_1(U \cap V, A) \arrow[r] \arrow[d] & \Pi_1(U, A) \arrow[d] \\
\Pi_1(V, A) \arrow[r] & \Pi_1(X, A).
\end{tikzcd}$$

Example

For the Figure-8, which is $S^1 \vee S^1$.

$S^1 \vee S^1 =$ 

It can be decomposed into U, V as follows

$U =$  $V =$ 

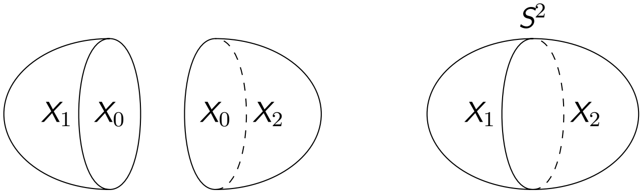
Since U, V are homotopic to S^1, and $U \cap V$ is homotopic to a point, Seifert-van Kampen Theorem implies

$$\pi_1(S^1 \vee S^1) = \pi_1(S^1) \star \pi_1(S^1) = \mathbb{Z} \star \mathbb{Z}.$$

In general, we have $\pi_1(\bigvee_{i=1}^n S^1) = \underbrace{\mathbb{Z} \star \cdots \star \mathbb{Z}}_{n}$.

Example

Consider the 2-sphere $S^2 = D_1 \cup D_2$ where D_i are open disks and $D_0 = D_1 \cap D_2$ is an annulus. D_i is an open neighbourhood of X_i.



Since $\pi_1(D_1) = \pi_1(D_2) = 1, \pi_1(D_0) = \pi_1(S^1) = \mathbb{Z}$, we deduce that

$$\pi_1(S^2) = (1 \star 1)/\mathbb{Z} = 1.$$

Similar argument shows that

$$\pi_1(S^n) = 1, \quad n \geq 2.$$

Example

Let us identiy $X = S^1$ with the unit circle in $\mathbb{R}^2$. Consider

$$U = \{(x, y) \in S^1 \mid y > -1/2\}, \quad V = \{(x, y) \in S^1 \mid y < 1/2\}$$

and $A = \{(\pm 1, 0)\}$. Then we obtain a pushout

$$\begin{tikzcd}
\Pi_1(U \cap V, A) \arrow[r] \arrow[d] & \Pi_1(U, A) \arrow[d] \\
\Pi_1(V, A) \arrow[r] & \Pi_1(S^1, A).
\end{tikzcd}$$

This implies that the groupoid $\Pi_1(S^1, A)$ contains two objects $A_1 = (1,0), A_2 = (-1,0)$ with morphisms

$$\begin{aligned}
\mathrm{Hom}_{\Pi_1(S^1,A)}(A_1, A_1) &= \{(\gamma_-\gamma_+)^n\}_{n\in\mathbb{Z}} \\
\mathrm{Hom}_{\Pi_1(S^1,A)}(A_1, A_2) &= \{(\gamma_+\gamma_-)^n\gamma_+\}_{n\in\mathbb{Z}} \\
\mathrm{Hom}_{\Pi_1(S^1,A)}(A_2, A_1) &= \{(\gamma_-\gamma_+)^n\gamma_-\}_{n\in\mathbb{Z}} \\
\mathrm{Hom}_{\Pi_1(S^1,A)}(A_2, A_2) &= \{(\gamma_+\gamma_-)^n\}_{n\in\mathbb{Z}}.
\end{aligned}$$

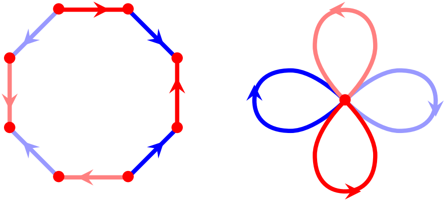
Here γ_+ represent the semi-circle from $(1,0)$ to $(-1,0)$ anti-clockwise, and γ_- represent the semi-circle from $(-1,0)$ to $(1,0)$ anti-clockwise.

Example

Consider the closed orientable surface Σ_g of genus g, which admits a polygon presentation

$$P = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Here is a figure for $g = 2$.



The edges of the polygon form $V_{2g} = \bigvee_{i=1}^{2g} S^1$.

Let U be the interior of the polygon and V be a small open neighbourhood of V_{2g}. $U \cap V$ is an annulus, which is homotopy to S_1 with generator P as above. Thus

$$
\begin{aligned}
\pi_1(\Sigma_g) &= \left(\coprod_{i=1}^{2g} \mathbb{Z}\right) \star 0/\mathbb{Z} \\
&= \langle a_i, b_i \mid i = 1, \ldots, g\rangle \,/\, \left(a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}\right).
\end{aligned}
$$

Example

Using the polygon presentation $P = a^2$, we can similarly compute

$$\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$$

The Jordan Curve Theorem

We give an application of Seifert-van Kampen Theorem to prove the Jordan Curve Theorem. This is an example which sounds totally obvious intuitively, but turns out to be very difficult to prove rigorously.

Definition

A simple closed curve is a subset of $\mathbb{R}^2$ (or S^2) which is homeomorphic to the circle S^1.

Theorem (The Jordan Curve Theorem)

Let C be a simple closed curve in the sphere S^2. Then the complement of C has exactly two connected components.

Proof

We sketch a proof here. Since S^2 is locally path connected, we would not distinguish connected and path connected here.

By an arc, we mean a subset of S^2 which is homeomorphic to the interval I. We first show

if A is an arc in S^2, then $S^2 \backslash A$ is connected.

In fact, assume there are two points $\{a, b\}$ which are disconnected in $S^2\backslash A$. Let us subdivide $A = A_1 \cup A_2$ into two intervals where $A_1 = [0, 1/2], A_2 = [1/2, 1]$ using the homeomorphism $A = [0, 1]$.

We argue that a, b are disconnected in either $S^2\backslash A_1$ or $S^2\backslash A_2$.

Let us choose a set D which contains one point from each connected component of $S^2\backslash A$ and such that $\{a, b\} \subset D$.

Apply Seifert-van Kampen Theorem to

$$V_1 = S^2\backslash A_1, \quad V_2 = S^2\backslash A_2, \quad V_1 \cap V_2 = S^2\backslash A.$$

We obtain a pushout in **Groupoid**

$$\begin{tikzcd} \Pi_1(V_1 \cap V_2, D) \arrow[r] \arrow[d] & \Pi_1(V_2, D) \arrow[d] \\ \Pi_1(V_1, D) \arrow[r] & \Pi_1(Y, D). \end{tikzcd}$$

Here $Y = V_1 \cup V_2$ is the complement of a point in S^2. If $\{a, b\}$ are connected in both V_1 and V_2, then the pushout implies that there exists a nontrivial morphism $a \to a$ in $\Pi_1(X, D)$ by a composition

$$a \stackrel{\text{in } V_1}{\longrightarrow} b \stackrel{\text{in } V_1 \cap V_2}{=\!=} b \stackrel{\text{in } V_2}{\longrightarrow} a$$

But this can not true since Y is contractible. So let us assume a, b are disconnected in $V_1 = S^2 \backslash A_1$. Run the above process replacing A by A_1, and keep doing this, we end up with contradiction in the limit. This proves our claim above for the arc.

Secondly, we show

the complement of C in S^2 is disconnected.

Otherwise, assume $S^2 \backslash C$ is connected. Let us divide $C = A_1 \cup A_2$ into two intervals A_1, A_2 which intersect at two endpoints $\{a, b\}$.

$$U_1 = S^2 \backslash A_1, U_2 = S^2 \backslash A_2, U_1 \cap U_2 = S^2 \backslash C, X = U_1 \cup U_2 = S^2 \backslash \{a, b\}.$$

Since $U_1, U_2, U_1 \cap U_2$ are all connected, Seifert-van Kampen Theorem leads to a pushout in **Group**

$$\begin{tikzcd} \pi_1(U_1 \cap U_2) \arrow[r] \arrow[d] & \pi_1(U_2) \arrow[d] \\ \pi_1(U_1) \arrow[r] & \pi_1(X). \end{tikzcd}$$

Observe $\pi_1(X) = \mathbb{Z}$. We show both $\pi_1(U_i) \to \pi_1(X)$ are trivial. This would lead to a contradiction.

Let us identify $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and assume $a = 0, b = \infty$, so A_1 is parametrized by a path α from 0 to ∞. Let γ be an arbitrary loop in U_1, we need to show γ becomes trivial in X. Let $R > 0$ be sufficient large such that γ is contained in the ball of radius R centered at the origin in $\mathbb{R}^2$. Consider the homotopy

$$F(t, s) = \gamma(t) - \alpha(s), \quad \gamma_s := F(-, s).$$

We have $\gamma_0 = \gamma$. Assume $\alpha(t_0) > R$, then γ_{t_0} lies inside the ball of radius R centered at $\alpha(t_0)$, which is contractible in X. This implies γ is trivial in X. The same argument applies to A_2.

Finally, we show

the complement of C in S^2 has exactly two connected components.

Let $C = A_1 \cup A_2$ and U_1, U_2 as in the previous step. Let D be a set which contains exactly one point from each connected component of $S^2 \backslash C$. We have a pushout in **Groupoid**

$$\begin{tikzcd}
\Pi_1(U_1 \cap U_2, D) \arrow[r] \arrow[d] & \Pi_1(U_2, D) \arrow[d] \\
\Pi_1(U_1, D) \arrow[r] & \Pi_1(X, D).
\end{tikzcd}$$

Assume D contains three points, say a, b, c. Since U_1, U_2 are connected, and points in D are disconnected in $U_1 \cap U_2$, the following two compositions

$$a \xrightarrow{\text{in } U_1} b \overset{\text{in } U_1\cap U_2}{=} b \xrightarrow{\text{in} U_2} a, \quad a \xrightarrow{\text{in } U_1} c \overset{\text{in } U_1\cap U_2}{=} c \xrightarrow{\text{in } U_2} a$$

give two free generators in $\pi_1(X, p)$.

But $\pi_1(X, p) = \mathbb{Z}$. This is a contradiction. $\square$