# A Crash Course on QFT and Symmetry

(version 2025.7)

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**Abstract**

This is the lecture note of the mini-course "Quantum Field Theory" for the 2025 summer school on geometry and topology at Sichuan University.

# Contents

I would like to begin by apologizing for the hastily prepared nature of this note. It was initiated to accompany the mini-course on quantum field theory that I taught during the 2025 summer school at Sichuan University. This was an intensive program featuring 30 hours of lectures over 10 days. These materials were compiled on the fly, serving as a survival guide for both myself and the students. The topics covered reflect my personal perspective and are by no means exhaustive. I sincerely apologize for any omissions in references and for potential errors or typos that may remain. Nevertheless, I hope these notes may prove useful to someone.

I would like to thank Prof Bin Zhang for inviting me into this program, and thank Han Liu and Qiyuan Chen to help type an early version of my lectures. I will polish and update this note later when I get more time. New versions will be updated on my homepage: https://sili-math.github.io/.

# 1 Introduction

## 1.1 Action Principle

A physics system is usually described (though not always) by a map

$$S\colon \mathcal{E} \longrightarrow k \qquad k = \mathbb{R} \text{ or } \mathbb{C}$$

where $\mathcal{E}$ is called the *space of fields* and the functional $S$ is called the *action*. The space $\mathcal{E}$ of fields is usually infinite-dimensional. Here are some typical examples.

- Scalar Field Theory. $\mathcal{E} = C^\infty(X)$. A possible action looks like

$$S[\phi] = \frac{1}{2}\int_X \left(|\nabla\phi|^2 + m^2\phi^2\right) \qquad \phi \in \mathcal{E}$$

- Gauge Theory. $\mathcal{E} = \{\text{connection } A \text{ on vector bundles}\}$. In Yang-Mills theory,

$$YM[A] = \frac{1}{4}\int Tr(F \wedge *F)$$

where $F = dA + \frac{1}{2}[A, A]$ is the curvature for $A$. In Chern-Simons theory

$$CS[A] = \int_M Tr\left(\frac{1}{2}A \wedge dA + \frac{1}{6}A \wedge [A, A]\right)$$

where $M$ is a three dimensional manifold.

- $\sigma$-model. $\mathcal{E} = \{\text{maps } \Sigma \to X\}$. The fields are maps between two manifolds.
- Gravity. $\mathcal{E} = \{\text{metrics on X}\}$. The fields are metrics on a manifold $X$.

Sometimes we can reduce problems in the infinite dimensional quantum fields to finite-dimensional geometry via methods such that cut-off, discretization, localization, etc.. Such reduction has led to many profound mathematical predictions.

In classical physics, the equation of motion describes the extremal configurations of the action. They can be viewed as the critical points of $S$

$$Crit(S) = \{\delta S = 0\}$$

in terms of variational method. Many interesting equations arise in this way, such that Laplace equation, Yang-Mills equation, etc.

In quantum physics, we are interested in quantum correlation of observables. Roughly speaking, an observable $\mathcal{O}$ is a function on the space $\mathcal{E}$ of fields. One popular approach to quantum correlation takes the following form

$$\langle \mathcal{O} \rangle \propto \int_{\mathcal{E}} e^{\frac{i}{\hbar}S}\mathcal{O}$$

which is called the Feynman path integral. This is an infinite dimensional integral where the measure is mostly unknown in mathematical rigor. One example where such infinite dimensional integral theory is known is the Wiener integral for the Brownian motion. This can be viewed as quantum mechanics with imaginary time and has been a fundamental concept in stochastic calculus. In general, the mathematical construction of such integrals of fields is one of the most challenging problem in quantum field theory.

On the other hand, we can obtain partial information in terms of asymptotic analysis (in $\hbar$). This is called perturbative theory which has solid foundation in terms of renormalization theory. We will explain in details the construction of perturbative renormalization theory in this note.

## 1.2 Observable Algebra

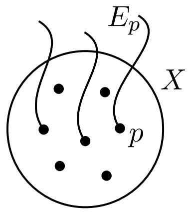
Observables form rich algebraic structures in quanatum field theory (QFT). To illustrate this, consider QFT on a spacetime manifold $X$, and $\mathcal{E}$ is the space of sections of a vector bundle $E$, denoted $\mathcal{E} = \Gamma(X, E)$. We want to understand

$$\int_{\mathcal{E}} (-).$$

- When $X =$ point, $E =$ vector space, say, $\mathbb{R}^n$. Then the space of fields $\mathcal{E} = \mathbb{R}^n$ and $\int_{\mathcal{E}}$ leads to the usual calculus that we learned in high school.
- When $\dim X > 0$ (i.e. $X$ is not a point) and there is a fiber $E_p$ (for example, a linear space) at each point $p \in X$. But

$$\mathcal{E} \neq \coprod_{p \in X} E_p.$$

The topology of $X$ makes a difference which leads to new algebraic structures.




Roughly speaking, **observables** are functions on fields, which we denote by $\mathscr{O}(\mathcal{E})$. For example, distributions are *linear observables.* The new structures come from the following fact. Given an open subset $U \subset X$, we can talk about observables $\mathrm{Obs}(U)$ supported in $U$

$$\mathrm{Obs}(U) = \text{functions on } \mathcal{E}(U) = \Gamma(U, E).$$

**Example 1.1.** $\mathcal{E} = C^\infty(X),\ p \in X$. *Consider*

$$\begin{aligned} \mathcal{O}_1 : \mathcal{E} &\to \mathbb{R} \\ f &\to \mathcal{O}_1(f) = f(p)^3. \end{aligned}$$

$\mathcal{O}_1$ *is an observable supported in any open neighborhood of* $p$.

Given disjoint open subsets $U_i \subset V$ contained in an open set $V$, such that the disjoint union is $\coprod_i U_i \subset V$, we have a map (factorization product) for observables:

$$\bigotimes_i \mathrm{Obs}(U_i) \to \mathrm{Obs}(V).$$

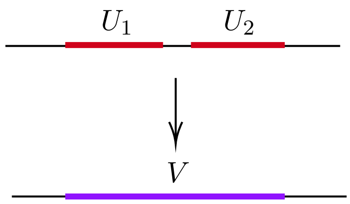
Intuitively, if $\mathcal{E}(U)$ is restricted to $\mathcal{E}(U_i)$, then dually $\mathscr{O}(\mathcal{E}(U_i)) \to \mathscr{O}(\mathcal{E}(U))$. Naively, we can *multiply* those *functions*, then we get

$$\bigotimes_i \mathrm{Obs}(U_i) \to \mathrm{Obs}(V).$$

This, however, requires further *quantum corrections* in which fields in $U_i$'s may "talk" to each other. This algebraic structure is called the **operator product expansion (OPE)** in physics or **factorization product** in mathematics. Such mathematical theory of observable algebras in QFT are developed in the works of:

- Beilinson-Drinfeld [5]
  - developed **factorization algebra** to formulate 2d chiral conformal field theory (CFT), and introduced the notion of **chiral homology**.
- Costello-Gwilliam [11]
  - constructed factorization algebras from perturbative renormalization theory, in the **Batalin-Vilkovisky (BV) formalism** [3].

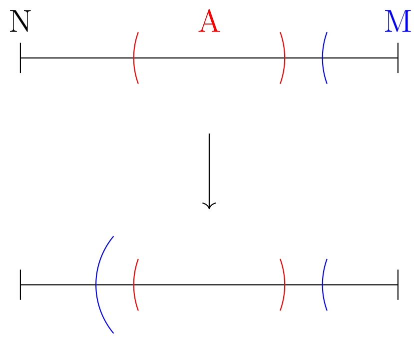
**Example 1.2** (Topological Quantum Mechanics (TQM)). *In topological QFT,* $\mathrm{Obs}(U)$ *only depends on the topology of* $U$. *Consider* $\dim X = 1$ *(a 1d QFT is a quantum mechanics) and* $\mathrm{Obs}(U) = A$ *for any contractible open interval* $U$. *Now consider two open intervals* $U_1, U_2$ *on* $X$ *and embed them into a larger interval* $V$.




*In topological theory, the factorization product does not depend on the location and size. This leads to a map*

$$\mathrm{Obs}(U_1) \otimes \mathrm{Obs}(U_2) \to \mathrm{Obs}(V) \quad \textit{or} \quad A \otimes A \to A.$$

*We find an* ***associative algebra****! We can also add boundary.*

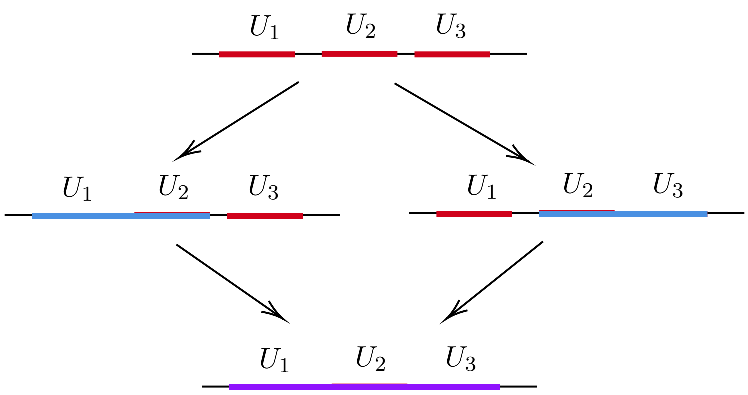



*Then we get* $A \otimes M \longrightarrow M$, *which implies that* $M$ *is a left* $A$*-module. Similarly,* $N$ *is a right* $A$*-module. Above all, quantum mechanics studies operator algebras and their representation (modules).*

The appearing algebraic structure in this example has a homological interpretation as follows. Consider two operators being placed at two different points on a line. When one operator approaches closely the other, the algebraic structure of the topology of the line comes from the homology group

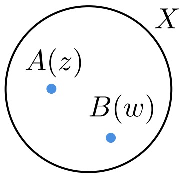
$$H_\bullet(\mathbb{R} - \{0\}) = H_0(\mathbb{R} - \{0\}) = \mathbb{Z}_{left} \oplus \mathbb{Z}_{right}$$

which comprises the left and right multiplications. *Associativity* comes from a further consistency:




$$\Longrightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c), \quad a, b, c \in \mathrm{Obs}(U_i),\ i = 1, 2, 3.$$

**Example 1.3** (2d Chiral QFT)**.** *Let* $\dim X = 2$*. The factorization product of a two-dimensional (2d) chiral theory is* ***holomorphic****.*




*In holomorphic coordinates, when an operator approach (or wind around) the other on* $X$*, the winding number can be expressed via their Fourier/Laurent modes:*

$$A(z)B(w) = \sum_{n\in\mathbb{Z}} \frac{\left(A_{(n)}B\right)(w)}{(z-w)^{n+1}}.$$

*We find* $\infty$*-ly many "products" (or binary operations)* $\left\{A_{(n)}B\right\}$*. In this case, the observable algebra gives rise to a* ***vertex operator algebra****.*

These examples play an fundamental role to guide us through the general constructions in quantum field theory, as we will see along this note.

# 2 Cohomology and Integration

## 2.1 De Rham Cohomology

We discuss some algebraic and geometric aspects of "integration" that will help us to understand quantum field theory constructions later.

**Definition 2.1.** A cochain complex over ring R is a sequence of R-module maps

$$... \to C^{n-1} \overset{d_{n-1}}{\to} C^n \overset{d_n}{\to} C^{n+1} \to ...$$

such that $d_n d_{n+1} = 0$. We denote it by $(C^\bullet, d)$, where $d_n = d|_{C^n}$ and $d^2 = 0$.

Condition $d_n d_{n+1} = 0$ implies

$$Im(d_{n-1}) \subset Ker(d_n).$$

Thus we can define the $R$-module

$$H^n(C^*, d) := \frac{Ker(d_n)}{Im(d_{n-1})}$$

which is called the **n-th cohomology group** of $(C^*, d)$. This is the dual notion of chain complex and homology group.

### De Rham Complex

Let us now consider a geometric example in calculus. Let $U \subset \mathbb{R}^n$ be an open set, and $\{x^1, ...x^n\}$ be its coordinates. Introduce

$$\Omega^0(U) = \{\text{smooth functions on U}\} = C^\infty(U)$$
$$\Omega^1(U) = \{\text{smooth 1-forms on U}\}.$$

An element $\alpha \in \Omega^1(U)$ can be written as

$$\alpha = \sum_{i=1}^n \alpha_i(x) dx^i \in \Omega^1(U), \quad \alpha_i \in C^\infty(U).$$

The total differential gives a map

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U)$$
$$f \mapsto df = \sum_i \partial_i f dx^i$$

which can be extended to a cochain complex

$$\Omega^0(U) \overset{d}{\to} \Omega^1(U) \overset{d}{\to} \Omega^2(U) \to ... \overset{d}{\to} \Omega^n(U)$$

where

$$\Omega^p(U) = \{\text{smooth } p\text{-forms on U}\}.$$

An element $\xi \in \Omega^p(U)$ takes the form

$$\xi = \sum_{i_1,...i_p} \xi_{i_1...i_p}(x) dx^{i_1} \wedge ... \wedge dx^{i_p}$$

with $\xi_{i_1\dots i_p}(x) \in C^\infty(U)$ and the skew-symmetry property

$$dx^i \wedge dx^j = -dx^j \wedge dx^i, \quad \forall i, j$$

In particular, $dx^i \wedge dx^i = 0$. Thus we can also write

$$\Omega^p(U) = \bigoplus_{1\leq i_1<\dots<i_p\leq n} C^\infty(U) dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

The de Rham differential $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is defined by

$$d(\xi) = \sum_{i_1,\dots i_p,k} \partial_{x^k} \xi_{i_1\dots i_p} dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

Check: $d^2 = 0$. Thus

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \to \dots \to \Omega^n(U)$$

defines a cochain complex called the de Rham complex and denoted by $(\Omega^*(U), d)$.

**Definition 2.2.** A $p$-form $\alpha \in \Omega^p$ is called

- closed if $d\alpha = 0$
- exact if $\alpha = d\beta$ for some $p - 1$ form $\beta$.

The quotient space

$$H^P_{dR}(U) = H^P(\Omega^*, d) = \frac{\text{closed p-form}}{\text{exact p-form}}$$

is called the $p$-th **de Rham cohomology**.

One important property is that the de Rham differential $d$ is **intrinsic**: it does not depend on the choice of coordinates. Therefore we can define $(\Omega^*(M), d)$ on a smooth manifold $M$ by gluing the above local construction. Then the de Rham cohomology on manifold

$$H^p(M) = H^p_{dR}(M)$$

is well-defined. We can also consider $(\Omega^*_c(M), d)$ where

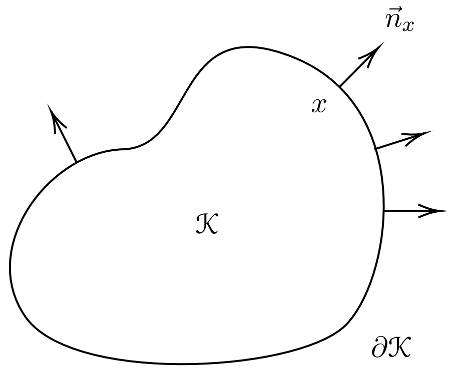
$$\Omega^*_c(M) = \{p\text{-form on M with compact support}\}.$$

Then we can define **de Rham cohomology with compact support**

$$H^p_c(M) = H^p(\Omega^*_c(M), d).$$

## Stokes' Theorem

Let $\mathcal{K} \subset \mathbb{R}^n$ be a $p$-dimensional surface. Let $\partial\mathcal{K}$ be the boundary of $\mathcal{K}$. The orientation of $\mathcal{K}$ determines an orientation of $\partial\mathcal{K}$ as follows. Let $x \in \mathcal{K}$ and $\vec{n}_x$ be an outward normal vector field on $\partial\mathcal{K}$, i.e., $\vec{n}_x$ is tangent at $\mathcal{K}$ and directed to the exterior of $\mathcal{K}$.




Then an oriented parametrization $\tau^1, \cdots, \tau^{p-1}$ of $\partial\mathcal{K}$ is such that $\vec{n}_x, \frac{\partial}{\partial\tau^1}, \cdots, \frac{\partial}{\partial\tau^{p-1}}$ gives an orientation of $T_x\mathcal{K}$. In terms of this induced orientation, we have

**Theorem 2.3** (Stokes' Theorem)**.**

$$\int_{\mathcal{K}} d\alpha = \int_{\partial\mathcal{K}} \alpha.$$

*Here $\alpha$ is a smooth $(p-1)$-form.*

**Example 2.4.** *Consider the case $n = p = 2$. Let*

$$\alpha = \alpha_1(x)dx^1 + \alpha_2(x)dx^2. \qquad d\alpha = \left(\frac{\partial\alpha_2}{\partial x^1} - \frac{\partial\alpha_1}{\partial x^2}\right) dx^1 \wedge dx^2.$$

*Then*

$$\iint_{\mathcal{K}} \left(\frac{\partial\alpha_2}{\partial x^1} - \frac{\partial\alpha_1}{\partial x^2}\right) dx^1 \wedge dx^2 = \int_{\partial\mathcal{K}} \alpha_1(x)dx^1 + \alpha_2(x)dx^2.$$

*This gives Green's Theorem.*

**Example 2.5.** *Consider the case $n = p = 3$. Let*

$$\alpha = \alpha_1(x)dx^2 \wedge dx^3 + \alpha_2(x)dx^3 \wedge dx^1 + \alpha_3(x)dx^1 \wedge dx^2.$$

$$d\alpha = \left(\frac{\partial\alpha_1}{\partial x^1} + \frac{\partial\alpha_2}{\partial x^2} + \frac{\partial\alpha_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3.$$

*Then*

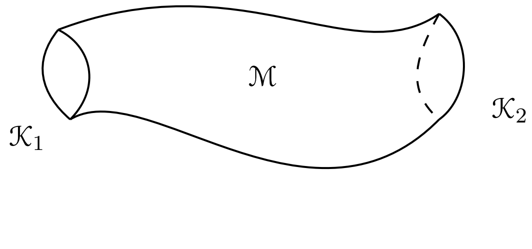
$$\iiint_K \left(\frac{\partial\alpha_1}{\partial x^1} + \frac{\partial\alpha_2}{\partial x^2} + \frac{\partial\alpha_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 = \iint_{\partial K} \alpha_1(x)dx^2 \wedge dx^3 + \alpha_2(x)dx^3 \wedge dx^1 + \alpha_3(x)dx^1 \wedge dx^2.$$

*This is the integral theorem of Gauss.*

As an application of Stokes' Theorem, consider two regions $\mathcal{K}_1$ and $\mathcal{K}_2$ which are the boundary of a region $\mathcal{M}$ as illustrated in the figure below.

$$\partial\mathcal{M} = \mathcal{K}_2 - \mathcal{K}_1.$$

Here the $\pm$ sign is about the orientation.



Let $\alpha$ be a closed form

$$d\alpha = 0.$$

Then

$$\int_{\mathcal{K}_2} \alpha - \int_{\mathcal{K}_1} \alpha = \int_{\partial \mathcal{M}} \alpha = \int_{\mathcal{M}} d\alpha = 0.$$

So

$$\int_{\mathcal{K}_1} \alpha = \int_{\mathcal{K}_2} \alpha.$$

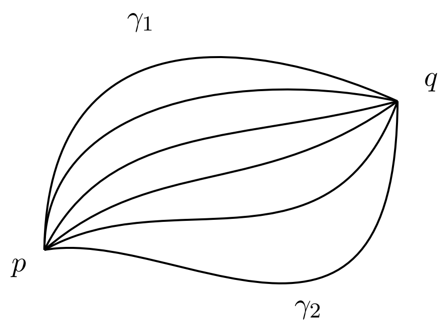
In particular, $\int_{\mathcal{K}} \alpha$ will be invariant under any smooth deformation of $\mathcal{K}$ if $d\alpha = 0$.

**Example 2.6.** *Let $\alpha \in \Omega^1(\mathbb{R}^n)$ and $d\alpha = 0$. Let $p, q \in \mathbb{R}^n$ be two points. Define*

$$\int_p^q \alpha = \int_\gamma \alpha$$

*by choosing any smooth path $\gamma$ connecting $p$ and $q$. It does not depend on the choice of $\gamma$ since any such path $\gamma_1$ can be continuously deformed to another $\gamma_2$. Then*

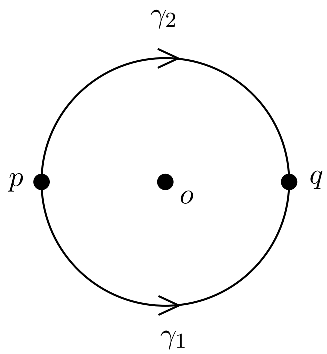
$$\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha \qquad \text{by Stokes' Theorem.}$$




**Example 2.7.** *Consider $\mathcal{U} = \mathbb{R}^2 - \{0\}$ and*

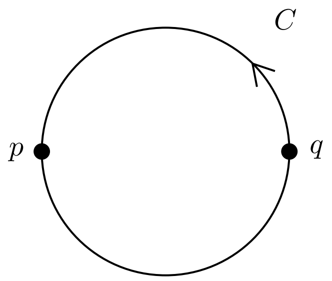
$$\alpha = \frac{xdy - ydx}{x^2 + y^2}, \qquad d\alpha = 0.$$

*We consider two paths from $p = -1$ to $q = 1$ by following the upper and lower semicircle.*



*Since there is a hole in the middle, $\gamma_1$ can not be deformed to $\gamma_2$ inside the region $\mathfrak{U}$. This can be also seen by computing*

$$\int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha = \oint_C \alpha.$$




*In terms of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$,*

$$\alpha = d\theta, \qquad \oint_C \alpha = \int_0^{2\pi} d\theta = 2\pi \neq 0.$$

## 2.2 Cohomology v.s. Integration

**Theorem 2.8** (Poincaré Lemma)**.**

$$H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p > 0 \end{cases} \qquad H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & p = n \\ 0, & p \neq n \end{cases}$$

*A generator of $H^0(\mathbb{R}^n)$ is the constant function $f \equiv 1$. A generator of $H_c^n(\mathbb{R}^n)$ is a compactly supported form $\alpha = f(x)d^n x$, where $f \in C_c^\infty(\mathbb{R}^n)$, such that $\int_{\mathbb{R}^n} \alpha = 1$.*

Observe that any $n$-form $\alpha \in \Omega_c^n(\mathbb{R}^n)$ is closed. On the other hand, if $\alpha = d\beta$ for $\beta \in \Omega_c^{n-1}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} d\beta = \int_K d\beta = \int_{\partial K} \beta = 0$$

where K is a large enough ball. The integration map

$$\begin{aligned} \int_{\mathbb{R}^n} : H_c^n(\mathbb{R}^n) &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \int \alpha \end{aligned}$$

is the isomorphism described in the Poincaré Lemma. Viewed differently, the map

$$\begin{aligned} \Omega_c^n(\mathbb{R}^n) &\longrightarrow H_c^n(\mathbb{R}^n) \overset{Poincar}{\cong} \mathbb{R} \\ \alpha &\longmapsto [\alpha] \end{aligned}$$

sending a top $n$-form to its cohomology class determines precisely the integration (up to a rescaling constant). Thus

$$\int_{\mathbb{R}^n} \iff H_c^n$$

i.e. "integration" can be modeled by a homological operation. This viewpoint will play an important role in our later discussion on perturbative quantization.

The rest of this subsection is devoted to prove the Poincaré Lemma.

### Cartan Formula

We first introduce some useful notions in differential geometry. Let

$$\mathrm{Vect}(M) = \text{smooth vector fields on a manifold } M.$$

A vector field $V \in \mathrm{Vect}(M)$ can be expressed in coordinate as

$$V(x) = \sum_i V^i(x) \frac{\partial}{\partial x^i}$$

We can view $V$ as a 1st-order differential operator

$$\begin{aligned} V : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto V(f) = \sum_i V^i(x) \frac{\partial f}{\partial x^i} \end{aligned}$$

which satisfies the leibniz rule (i.e. defines a derivation): $V(fg) = V(f)g + fV(g)$.

$\mathrm{Vect}(M)$ forms a Lie algebra with bracket

$$\begin{aligned} [-,-] : \mathrm{Vect}(M) \times \mathrm{Vect}(M) &\longrightarrow \mathrm{Vect}(M) \\ (V, W) &\longmapsto [V, W] \end{aligned}$$

This bracket satisfies the following properties:

- As differential operators

$$[V, W](f) := V(W(f)) - W(V(f))$$

- Explicitly in coordinates

$$[V, W] = \sum_i (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i}$$

- Jacobi-Identity

$$[V, [W, T]] + [W, [T, V]] + [T, [V, W]] = 0$$

We can view $\mathrm{Vect}(M)$ formally as the Lie algebra of the group $\mathrm{Diff}(M)$ of diffeomorphisms of $M$. Assume $V$ generates a flow on $M$

$$\begin{aligned} \varphi : (-\epsilon, \epsilon) \times M &\longrightarrow M \\ (t, x) &\longmapsto \varphi(t, x) \end{aligned}$$

satisfying $\varphi(0, x) = x$ and $\frac{\partial \varphi(t,x)}{\partial t} = V(\varphi)$.

Denote

$$\begin{aligned} \varphi_t : M &\longrightarrow M \\ x &\longmapsto \varphi(t, x) \end{aligned}$$

which satisfies the group property: $\varphi_0 = Id, \quad \varphi_t \varphi_s = \varphi_{t+s}, \quad \varphi^{-1} = \varphi_{-t}$. Then

$$V(f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(\varphi_t(x))$$

represents the infinitesimal change of $f$ along the flow generated by $V$.

This formalism can be generalized to differential forms. Recall a smooth map

$$\begin{aligned} \phi : X &\longrightarrow Y \\ x &\longmapsto y^i = \phi^i(x) \end{aligned}$$

allows us to define the "pull-back" map

$$\phi^* : \Omega^p(Y) \longrightarrow \Omega^p(X)$$

by sending $\alpha = \sum \alpha_{i_1..i_p}(y) dy^{i_1} \wedge ... \wedge dy^{i_p}$ to

$$\phi^*(\alpha) = \sum \alpha_{i_1...i_p}(\phi(x)) d\phi^{i_1}(x) \wedge ... d\phi^{i_p}(x).$$

In particular, the flow $\varphi_t$ generated by $V$ leads to

$$\varphi_t^* : \Omega^p(M) \to \Omega^p(M).$$

**Definition 2.9.** We define the Lie derivative $\mathscr{L}_V$ on differential forms by

$$\begin{aligned} \mathscr{L}_V : \Omega^p &\to \Omega^p \\ \alpha &\mapsto \mathscr{L}_V(\alpha) := \left.\frac{d}{dt}\right|_{t=0} \partial_t^*(\alpha) \end{aligned}$$

On the other hand, the natural pairing between vector fields and 1-forms leads to the notion of interior product.

**Definition 2.10.** We define the interior product $\iota_V$ on differential forms

$$\begin{aligned} \iota_V : \Omega^p &\to \Omega^{p-1} \\ \alpha &\mapsto \iota_V \alpha \end{aligned}$$

by

$$\begin{aligned} \iota_V(\alpha_{i_1...i_p}(x) dx^{i_1} \wedge ... dx^{i_p}) := & \sum \alpha_{i_1...i_p} \iota_V(dx^{i_1}) \wedge ... \wedge dx^{i_p} \\ & - \sum \alpha_{i_1...i_p} dx^{i_1} \wedge \iota_V(dx^{i_2}) \wedge ... \wedge dx^{i_p} \\ & + ... + (-1)^{p-1} \sum \alpha_{i_1...i_p} dx^{i_1} \wedge ... \wedge \iota_V(dx^{i_p}). \end{aligned}$$

Here for $V = \sum_i V^i(x) \frac{\partial}{\partial x^i}$, $\iota_V(dx^i) = V^i(x)$.

Now we have three operators

$$\begin{aligned} d : \Omega^p &\to \Omega^{p+1} \\ \mathscr{L}_V : \Omega^p &\to \Omega^p \\ \iota_V : \Omega^p &\to \Omega^{p-1} \end{aligned}$$

There are related by **Cartan's Magic Formula**:

$$\mathscr{L}_V = d\iota_V + \iota_V d.$$

The above relation can be further extended as follows.

**Proposition 2.11.** *Let $V, W \in \mathrm{Vect}(\mathcal{U})$. Then as operators on $\Omega^\bullet(\mathcal{U})$,*

- $[\mathscr{L}_V, \mathscr{L}_W] := \mathscr{L}_V \mathscr{L}_W - \mathscr{L}_W \mathscr{L}_V = \mathscr{L}_{[V,W]}$.
- $\iota_V \iota_W + \iota_W \iota_V = 0$.

- $[\mathscr{L}_V, d] := \mathscr{L}_V d - d\mathscr{L}_V = 0$.
- $\iota_V d + d\iota_V = \mathscr{L}_V$.
- $[\mathscr{L}_V, \iota_W] := \mathscr{L}_V \iota_W - \iota_W \mathscr{L}_V = \iota_{[V,W]}$.

*Proof of Poincaré Lemma.* We prove the case for $H^p(\mathbb{R}^n)$ and leave the case with compact support to the reader. Consider the diffeomorphims

$$\begin{aligned} \varphi_t : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (x^1, ...x^n) &\longmapsto (e^t x^1, ..., e^t x^n) \end{aligned}$$

which is the flow generated by the vector field $V = \sum x^i \frac{\partial}{\partial x^i}$. Observe $\varphi_0 = Id$ and $\varphi_{-\infty}$ is the constant map

$$\varphi_{-\infty} : \mathbb{R}^n \longrightarrow \{0\} \in \mathbb{R}^n$$

Assume $p > 0$ and let $\alpha \in \Omega^p(\mathbb{R}^n)$ be a closed form: $d\alpha = 0$. Since

$$\varphi_0^*(\alpha) = \alpha, \quad \varphi_{-\infty}^*(\alpha) = 0$$

we have

$$\begin{aligned} \alpha =& \varphi_0^*(\alpha) - \varphi_{-\infty}^*(\alpha) \\ =& \int_{-\infty}^0 \frac{d}{dt}(\varphi_t^*(\alpha))dt \\ =& \int_{-\infty}^0 \mathscr{L}_V(\varphi_t^*(\alpha))dt \\ =& \int_{-\infty}^0 (d\iota_V + \iota_V d)(\varphi_t^*(\alpha))dt. \end{aligned}$$

Here we have used Cartan's formula.

Since $d\varphi_t^*(\alpha) = \varphi_t^*(d\alpha) = 0$ by closedness of $\alpha$, we can further simplify

$$\alpha = d\left(\int_{-\infty}^0 \iota_V(\varphi_t^*(\alpha)dt\right) = d\beta$$

where $\beta = \int_{-\infty}^0 \iota_V(\varphi_t^*(\alpha))dt$. This shows $H^p(\mathbb{R}^n) = 0$ for $p > 0$.. For $p = 0$

$$f \in H^0(\mathbb{R}^n) \quad \Longrightarrow \quad df = 0 \quad \Longrightarrow \quad f = const.$$

Thus $H^0(\mathbb{R}^n) = \mathbb{R}$. $\square$

# 3 Classical Field Theory

## 3.1 Equation of Motion

Consider a classical field theory described by an action functional $S$

$$S : \mathcal{E} \longrightarrow \mathbb{R}/\mathbb{C}.$$

Typically, $\mathcal{E}$ will be the space of smooth sections of a vector bundle $E$ over a manifold $X$. We assume $X$ is $n$-dimensional and oriented, then $S$ will be a *local functional* with a **Lagrangian density** $\mathscr{L}$ valued in $n$-forms on $X$,

$$S[\phi] = \int_X \mathscr{L}(\phi), \quad \phi \in \mathcal{E}.$$

The Lagrangian density is built up by jets of $\phi$, i.e. by $\phi$ and its derivatives.

**Example 3.1** (Mechanics on Phase Space)**.** *In Hamiltonian mechanics, we can describe a field $\phi$ by a path on the phase space $\mathbb{R}^{2m}$ parametrized by time $t$,*

$$\phi = \{q^i(t), p_i(t)\} : \mathbb{R} \longrightarrow \mathbb{R}^{2m}$$

*Thus the space of fields is*

$$\mathcal{E} = C^\infty(\mathbb{R})^{\oplus 2m}$$

*can be identified with the space of smooth sections of a trivial bundle over $X = \mathbb{R}$. The action $S$ is constructed from a Hamiltonian function $H$ on the phase space,*

$$\begin{aligned} S[\phi] &= \int p_i dq^i - H(q,p)dt \\ &= \int \big(p_i(t)\dot{q}^i(t) - H(q,p)\big)dt. \end{aligned}$$

Since a path here is a 1-dim object, we can view quantum mechanics as a 1-dim quantum field theory.

**Example 3.2** (Scalar Field Theory)**.** *Let $(X, g)$ be a Riemannian manifold and $\mathcal{E}$ be the space of smooth functions $C^\infty(X)$. Consider the action*

$$S[\phi] = \int_X \big(\frac{1}{2}|\nabla\phi|^2 + V(\phi)\big)dVol_g,$$

*where $|\nabla\phi|^2 = \partial_i\phi\partial_j\phi g^{ij}$, with a potential function $V$ and the volume form $dVol_g = \sqrt{g}\ d^n x$. Assume $X$ has no boundary, then we can also write*

$$S[\phi] = \int_X \big(-\frac{1}{2}\phi\Delta\phi + V(\phi)\big)dVol_g,$$

*where $\Delta$ is the **Laplace-Beltrami operator** ($-\Delta$ is a positive operator)*

$$\Delta\phi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\phi).$$

*For $X = \mathbb{R}^n$ with standard metric $g = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$,*

$$\Delta = \sum_i \frac{\partial^2}{\partial x^i \partial x^i}.$$

**Example 3.3** (Chern-Simons Theory)**.** *Let $X$ be a 3-fold and $\mathfrak{g}$ be a semi-simple Lie algebra with the non-degenerate trace pairing (Killing form)* $\mathrm{Tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. *For simplicity, consider a trivial bundle and*

$$\mathcal{E} = \Omega^1(X, \mathfrak{g})$$

*is the space of $\mathfrak{g}$-valued 1-forms. The Chern-Simons action is*

$$CS[A] = \int_X \mathrm{Tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A]\right), \quad A \in \mathcal{E}$$

*where $[-,-]$ is the generalized Lie bracket defined by*

$$[\alpha \otimes e_1, \beta \otimes e_2] = (\alpha \wedge \beta) \otimes [e_1, e_2], \quad \text{for } \alpha, \beta \in \Omega^1(X), e_i \in \mathfrak{g}.$$

*Remark* 3.4. See [14] for topological classifications of Chern-Simons in general.

We assume $X$ has no boundary in the following discussion. The classical solutions are critical points of $S$,

$$Crit(S) = \{\delta S = 0\} \subset \mathcal{E}.$$

Consider an arbitrary variation $\delta\phi$. We can always use integration by parts to write the variation of the action as

$$\delta S = \int_X EL(\phi) \delta\phi.$$

Since $\delta\phi$ is arbitrary, we obtain the classial equation of motion

$$Crit(S) = \{\phi : EL(\phi) = 0\}$$

which is also called the **Euler-Lagrangian equation**.

**Example 3.5** (Mechanics on Phase Space)**.** *The action functional is*

$$S[q, p] = \int \left(p\dot{q} - H(q, p)\right) dt.$$

*Its variation is computed by*

$$\begin{aligned}
\delta S &= \int \left[\delta p \dot{q} + p \frac{d}{dt}(\delta q) - \left(\frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p\right)\right] dt \\
&= \int \left(\dot{q}\delta p - \dot{p}\delta q - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p\right) dt \\
&= \int \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q.
\end{aligned}$$

*The classical equation of motion is the Hamilton's equations*

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q}. \end{cases}$$

**Example 3.6** (Scalar Field Theory)**.**

$$S[\phi] = \int_X \left(\frac{1}{2}|\nabla\phi|^2 + V(\phi)\right) dVol_g,$$

*whose variation leads to*

$$\begin{aligned}\delta S &= \int_X \big(\nabla\phi\cdot\nabla\delta\phi + V'(\phi)\delta\phi\big)dVol_g\\ &= \int_X \big(-(\Delta\phi)\delta\phi + V'(\phi)\delta\phi\big)dVol_g.\end{aligned}$$

*The classical equation of motion is*

$$-\Delta\phi + V'(\phi) = 0.$$

**Example 3.7** (Chern-Simons Theory)**.**

$$CS[A] = \int_X \mathrm{Tr}\,\big(\frac{1}{2}A\wedge dA + \frac{1}{6}A\wedge[A,A]\big).$$

*Its variation gives*

$$\begin{aligned}\delta CS &= \int_X \mathrm{Tr}\,\big(\delta A\wedge dA + \frac{1}{2}\delta A\wedge[A,A]\big)\\ &= \int_X \mathrm{Tr}(\delta A\wedge F_A),\end{aligned}$$

*where* $F_A = dA + \frac{1}{2}[A,A]$ *is the curvature of* $A$*. The classical equation of motion*

$$F_A = 0$$

*describes flat connections.*

## 3.2 Noether's Theorem

### Symmetry

Assume we have a classical theory

$$S : \mathcal{E} \longrightarrow \mathbb{R}/\mathbb{C}.$$

By a (classical) symmetry, we mean a group $G$ acting on $\mathcal{E}$ such that $S$ is $G$-invariant

$$S[g(\phi)] = S[\phi] \quad \forall g\in G.$$

There are two types of classical symmetries, *discrete* symmetries with discrete group and *continuous* symmetries with Lie groups.

**Example 3.8.** *Consider mechanics on a phase space with action*

$$S[q,p] = \int \big(p\dot{q} - H(q,p)\big)dt.$$

*Here the Hamiltonian* $H$ *is time-independent. There is a smooth action of Lie group* $\mathbb{R}$ *on* $\mathcal{E}$ *by time translation. For* $a\in\mathbb{R}$*, we define* $\varphi_a : \mathcal{E}\to\mathcal{E}$ *as*

$$\{q(t),p(t)\} \longmapsto \{q(t+a),p(t+a)\}.$$

*For any* $a\in\mathbb{R}$*,* $S$ *is invariant under* $\varphi_a$*. This gives a continuous symmetry.*

*Consider a particular Hamiltonian*

$$H(q,p) = \frac{1}{2}p^2 + V(q),$$

*where* $V$ *is an even function. Then we have a discrete* $\mathbb{Z}_2$*-symmetry. Denote by* $\sigma$ *the generator of* $\mathbb{Z}_2$*. Then this symmetry is given by*

$$\sigma : (q,p) \to (-q,-p).$$

### Classical Noether's Theorem

Assume we have a continuous symmetry by a connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Suppose the infinitesimal transformation takes the form

$$\delta^G_\epsilon \phi = \epsilon^\alpha \mathfrak{X}_\alpha(\phi)$$

with infinitesimal constants $\epsilon^\alpha$ parametrizing elements in $\mathfrak{g}$.

**Example 3.9.** *The infinitesimal transformation of time translation is*

$$\delta_\epsilon \phi = \phi(t+\epsilon) - \phi(t) = \epsilon\dot{\phi}(t),$$

*and* $\mathfrak{X}(\phi) = \dot{\phi}$.

Assume $X$ is $n$-dim oriented manifold and we have a Lagrangian density on $X$

$$S = \int_X \mathscr{L}(\phi).$$

$G$-invariance of $S$ can be expressed at the infinitesimal level by

$$\delta^G_\epsilon \mathscr{L} = \epsilon^\alpha d\kappa_\alpha$$

where $\kappa_\alpha$'s are $(n-1)$-forms. Note that $\kappa_\alpha$ is only defined up to $d$-closed forms.

On the other hand, for arbitrary variation

$$\delta \mathscr{L} = EL(\phi)\delta\phi + d\gamma$$

where $d\gamma$ is an exact $n$-form that arises from integration by parts. Plug into the special variation by symmetry $\delta^G_\epsilon$, we can write

$$\begin{aligned}\delta^G_\epsilon \mathscr{L} &= EL(\phi)\ \epsilon^\alpha \mathfrak{X}_\alpha(\phi) + d(\epsilon^\alpha \gamma_\alpha)\\ &= \epsilon^\alpha\big(EL(\phi)\ \mathfrak{X}(\phi) + d\gamma_\alpha\big),\end{aligned}$$

for some $\gamma_\alpha$ that is determined by $\gamma$ and $\mathfrak{X}_\alpha$.

Comparing the above two expressions on $\delta^G_\epsilon \mathscr{L}$, we obtain

$$\epsilon^\alpha\big(d(\gamma_\alpha - \kappa_\alpha) + EL(\phi)\ \mathfrak{X}_\alpha(\phi)\big) = 0.$$

Since $\epsilon^\alpha$ are arbitrary, we find

$$dJ_\alpha + EL\ \mathfrak{X} = 0, \qquad \text{where} \quad J_\alpha = \gamma_\alpha - \kappa_\alpha.$$

In particular, at the configuration satisfying the equation of motion $EL(\phi) = 0$,

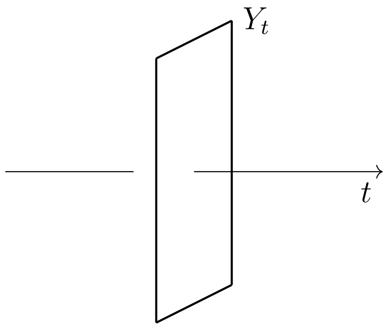
$$dJ_\alpha\big|_{Crit(S)} = 0.$$

$J_\alpha$ leads to conserved quantities as follows. Let $Y_1, Y_2 \subset X$ be two codim=1 hypersurfaces which are homologous by $\Sigma$. Stokes' theorem implies that for $\phi \in Crit(S)$

$$\int_{Y_1} J_\alpha(\phi) - \int_{Y_2} J_\alpha(\phi) = \int_{\partial\Sigma} J_\alpha = \int_\Sigma dJ_\alpha = 0.$$

In particular, if $X$ is the spacetime and $Y_t$ is a time slice at $t$, then the integral $\int_{Y_t} J_\alpha$ is independent of $t$ assuming $\phi$ satisfies the equation of motion. $J_\alpha$ is called the **conserved current** and the spatial integral $\int_{Y_t} J_\alpha$ is called the **conserved charge**.

The above argument shows that continuous symmetries correspond to conserved currents. This is the content of Noether's theorem.




**Example 3.10.** *Consider mechanics on the phase space*

$$S = \int \big(p\dot{q} - H(q,p)\big)dt.$$

*The variation of $S$ is computed to be*

$$\delta\mathscr{L} = (\dot{q} - \partial_p H)\delta p - (\dot{p} + \partial_q H)\delta q + \frac{d}{dt}(p\delta q)$$

*For the infinitesimal time translation*

$$\delta_\epsilon q = \epsilon\dot{q}, \quad \delta_\epsilon p = \epsilon\dot{p}$$

*we have*

$$\delta_\epsilon \mathscr{L} = \epsilon\frac{d}{dt}\mathscr{L}.$$

*In terms of the notions above, we find*

$$\gamma = p\dot{q}, \kappa = \mathscr{L}.$$

*Noether's theorem says that*

$$J = \gamma - \kappa = p\dot{q} - \mathscr{L} = H.$$

*is conserved if we impose the Hamilton's equation. This fact can be directly verified*

$$\frac{d}{dt}H = \partial_p H\dot{p} + \partial_q\dot{q} = -\partial_p H\partial_q H + \partial_q H\partial_p H = 0.$$

*We arrive at the result: time-translation symmetry implies energy conservation.*

There is an equivalent way to derive the conserved current as follows. We promote the constant $\epsilon^\alpha$ to be a function $\widetilde{\epsilon}^\alpha(x)$ on $X$ and consider the variation

$$\delta^G_{\widetilde{\epsilon}}\phi = \widetilde{\epsilon}^\alpha(x)\mathfrak{X}_a(\phi).$$

Using integration by parts and the $G$-invariance of $S$, we can always write

$$\delta^G_{\widetilde{\epsilon}} S = -\int \widetilde{\epsilon}^\alpha d\widetilde{J}_\alpha$$

which vanishes when $\widetilde{\epsilon}^\alpha = \epsilon^\alpha$ is a constant.

On the equation of motion, $\delta S = 0$ for any variation $\delta\phi$. Applied to the special variation $\delta^G_{\widetilde{\epsilon}}\phi$ and using the arbitrarines of $\widetilde{\epsilon}(x)$, we find $d\widetilde{J}_\alpha = 0$ on the equation of motion. This is the Noether current.

Comparing with the previous computation and using $dJ_\alpha + EL\ \mathfrak{X} = 0$,

$$\delta^G_{\widetilde{\epsilon}} S = \int EL\ \delta^G_{\widetilde{\epsilon}}\phi = \int \widetilde{\epsilon}^\alpha EL\ \mathfrak{X}_\alpha = -\int \widetilde{\epsilon}^\alpha dJ_\alpha.$$

Thus $J_\alpha = \widetilde{J}_\alpha$ modulo $d$-closed forms.

**Example 3.11.** *We use the second method to compute the time translation for*

$$S = \int \big(p\dot{q} - H(q,p)\big)dt.$$

*The promoted infinitesimal transformation is*

$$\delta_{\widetilde{\epsilon}}\phi = \widetilde{\epsilon}\dot{\phi}$$

*Plugging in to the variation,*

$$\delta_{\widetilde{\epsilon}}S = \int \widetilde{\epsilon}\frac{d}{dt}\mathscr{L} + \dot{\widetilde{\epsilon}}p\dot{q} = -\int \widetilde{\epsilon}\frac{d}{dt}(p\dot{q} - \mathscr{L}) = \int \widetilde{\epsilon}\frac{d}{dt}H.$$

*We find again $\tilde{J} = H$.*

### Spontaneous Symmetry Breaking

Assume $G$ is a symmetry, then there is a $G$-action on the space of classical solutions. However, given $\phi \in Crit(S)$ a classical solution, $\phi$ may not be $G$-fixed itself. In this case we say the symmetry is *spontaneously broken.*

**Example 3.12.** *Consider*

$$S = \int (p\dot{q} - H)dt, \qquad H = \frac{1}{2}p^2 + V(q)$$

*where $V(q) = V(-q)$. We have a $\mathbb{Z}_2$-symmetry. The equation of motion is*

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = p \\ \dot{p} = -\frac{\partial H}{\partial q} = -V'(q). \end{cases}$$

*Assume $V'(a) = 0$. Then the path*

$$\begin{cases} q(t) = a \\ p(t) = 0 \end{cases}$$

*solves the equation of motion, but not preserved by $\mathbb{Z}_2$.*

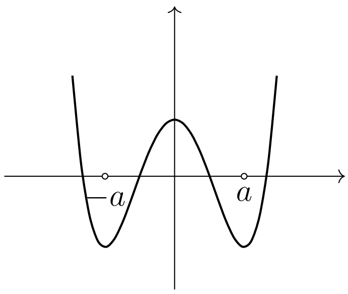



Figure 1: Spontaneous Symmetry Breaking

## 3.3 Gauge Symmetry

Gauge symmetries are symmetries by *local* transformations. Let $X$ be the space-time manifold. Roughly speaking, locality here says the set of infinitesimal gauge transformations is a $C^\infty(X)$-module. The notion of gauge transformations was introduced by H. Weyl.

**Example 3.13** (Maxwell's Equations)**.**

$$\begin{cases} \nabla\cdot\vec{\mathbf{E}} = \frac{1}{\varepsilon_0}\rho \\ \nabla\cdot\vec{\mathbf{B}} = 0 \\ \nabla\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \\ \nabla\times\vec{\mathbf{B}} = \mu_0(\vec{\mathbf{j}} + \varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t}), \end{cases}$$

*where* $\vec{\mathbf{E}} = (E_x, E_y, E_z)$ *and* $\vec{\mathbf{B}} = (B_x, B_y, B_z)$ *are the electric field and the magnetic field while* $\vec{\mathbf{j}} = (j_x, j_y, j_z)$ *is the current density. Introduce a 2-form on* $\mathbb{R}^{3,1}$

$$F = (E_x dx + E_y dy + E_z dz)\wedge dt + B_x dy\wedge dz + B_y dz\wedge dx + B_z dx\wedge dy$$

*and 1-form*

$$J = \frac{1}{\varepsilon_0}\rho dt - \mu_0(j_x dx + j_y dy + j_z dz).$$

*Then Maxwell's equations can be rewritten in a compact form as*

$$\begin{cases} dF = 0 \\ d(*F) = *J, \end{cases}$$

*where*

$$* : \Omega^p(\mathbb{R}^{3,1}) \to \Omega^{4-p}(\mathbb{R}^{3,1})$$

*is the Hodge star operator with respect to the Minkowski metric. The compatibility condition* $d(*J) = 0$ *turns out to be the charge conservation.*

*On the contractible space* $\mathbb{R}^{3,1}$, $dF = 0$ *implies* $F = dA$ *for a potential 1-form* $A$*, which can be decomposed as scalar potential and vector potential*

$$A = -\phi dt + \vec{\mathbf{A}} = -\phi dt + A_x dx + A_y dy + A_z dz.$$

*Then we can rewrite the Maxwell's equations as*

$$\begin{cases} \vec{\mathbf{E}} = -\nabla\phi - \partial_t\vec{\mathbf{A}} \\ \vec{\mathbf{B}} = \nabla\times\vec{\mathbf{A}}. \end{cases}$$

*The gauge transformation is parametrized by an arbitary 0-form* $\chi$ *which transforms* $A$ *to* $A + d\chi$ *and preserves* $F = dA$*.*

*Introduce the action functional*

$$S[A] = \frac{1}{\mu_0}\int_{\mathbb{R}^{3,1}} \frac{1}{2}F\wedge *F + *J\wedge A.$$

*Under the variation* $A \to A + \delta A$*,*

$$\begin{aligned} \delta S &= \frac{1}{\mu_0}\int_{\mathbb{R}^{3,1}} \delta F\wedge *F + *J\wedge\delta A \\ &= \frac{1}{\mu_0}\int_{\mathbb{R}^{3,1}} d\delta A\wedge *F - \delta A\wedge *J \\ &= \frac{1}{\mu_0}\int_{\mathbb{R}^{3,1}} \delta A\wedge\big(d(*F) - *J\big). \end{aligned}$$

*The equation of motion is read off by*

$$d(*F) - *J = 0,$$

*which is precisely Maxwell's equations.*

*Under the gauge transformation* $A \to A + d\chi$,

$$\delta_\chi S = \frac{1}{\mu_0}\int_{\mathbb{R}^{3,1}} *J \wedge d\chi = \frac{1}{\mu_0}\int_{\mathbb{R}^{3,1}} d(*J) \wedge \chi = 0.$$

*The last equality follows from the charge conservation* $d(*J) = 0$.

# 4 Algebraic Aspects of Symmetry

## 4.1 Chevalley-Eilenberg Complex

We discuss algebraic structures underlying gauge theories. Let $G$ be the Lie group and $V$ be the corresponding G-representation space. In gauge theories, $G$ will be $\infty$-dim Lie group of gauge transformations, and $V$ represents the matter fields.

To simplify the presentation and illustrate the main structures, we will focus on the toy model in this section where $G$ is a finite dimensional connected Lie group with Lie algebra $\mathfrak{g}$ and V is a finite dimensional representation. Let

$$\mathfrak{g}^\vee = \mathrm{Hom}(\mathfrak{g}, k)$$

be the linear dual of $\mathfrak{g}$ (the base field $k = \mathbb{R}$ *or* $\mathbb{C}$). Consider the exterior algebra

$$\wedge^\bullet \mathfrak{g}^\vee = \bigoplus_p \wedge^p \mathfrak{g}^\vee.$$

Let $\{e_1, ...e_m\}$ be a basis of $\mathfrak{g}$ where $m = \dim\ \mathfrak{g}$, and $\{c^1, ...c^m\}$ be the dual basis of $\mathfrak{g}^\vee$. Then we explicitly write

$$\wedge^\bullet \mathfrak{g}^\vee = k[c^\alpha], \qquad \text{where} \quad c^\alpha c^\beta = -c^\beta c^\alpha.$$

The algebraic structure is given by wedge product. Let

$$[-,-] : \wedge^2 g \longrightarrow \mathfrak{g}$$

be the Lie bracket. Its dual gives a map

$$d_{\mathrm{CE}} : \mathfrak{g}^\vee \longrightarrow \wedge^2 \mathfrak{g}^\vee$$

which extends to a unique map

$$d_{\mathrm{CE}} : \wedge^\bullet \mathfrak{g}^\vee \longrightarrow \wedge^\bullet \mathfrak{g}^\vee$$

determined by the following conditions

1) on generators, $d_{\mathrm{CE}} : \mathfrak{g}^\vee \longrightarrow \wedge^2 \mathfrak{g}^\vee$ as above.
2) graded Leibnitz rule

$$d_{\mathrm{CE}}(a \wedge b) = (d_{\mathrm{CE}} a) \wedge b + (-1)^p a \wedge d_{\mathrm{CE}} b, \quad \text{where} \quad a \in \wedge^p \mathfrak{g}^\vee.$$

Explicitly in terms of the above basis

$$[e_\alpha, e_\beta] = \sum_\gamma f^\gamma_{\alpha\beta} e_\gamma$$

where $f^\gamma_{\alpha\beta}$ is the structure constant. Dually, we have

$$d_{\mathrm{CE}} c^\gamma = \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta \text{ on generators.}$$

Then the action of $d_{\mathrm{CE}}$ on a general element is

$$\begin{aligned} d_{\mathrm{CE}}(c^{\alpha_1} c^{\alpha_2} \cdots c^{\alpha_p}) =& (d_{\mathrm{CE}} c^{\alpha_1}) \cdots \gamma^{\alpha_p} - c^{\alpha_1} (d_{\mathrm{CE}} c^{\alpha_2}) \cdots c^{\alpha_p} \\ &+ ... + (-1)^{p-1} c^{\alpha_1} \cdots c^{\alpha_{p-1}} (d_{\mathrm{CE}} c^{\alpha_p}). \end{aligned}$$

**Proposition 4.1.** $d_{\mathrm{CE}} : \wedge^\bullet \mathfrak{g}^\vee \longrightarrow \wedge^\bullet \mathfrak{g}^\vee$ *satisfies*

$$d_{\mathrm{CE}}^2 = 0$$

*which is equivalent to the Jacobi-identity of* $[-,-]$.

*Proof:* Exercise. □

**Definition 4.2.** $C^\bullet(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^\vee, d_{\mathrm{CE}})$ defines a cochain complex where

$$C^p(\mathfrak{g}) = \wedge^p \mathfrak{g}^\vee, \quad \text{and} \quad d_{\mathrm{CE}} : C^p(\mathfrak{g}) \longrightarrow C^{p+1}(\mathfrak{g}).$$

This is called the **Chevalley-Eilenberg cochain complex**.

*Remark* 4.3. In physics (gauge theory) terminology, $c^\alpha$ is called the **ghost field** and $d_{\mathrm{CE}}$ is called the **BRST** differential.

We can extend this to incorporate representations. Let V be a $\mathfrak{g}$-module with

$$\rho : \mathfrak{g} \otimes V \longrightarrow V.$$

Consider the free $\wedge^\bullet \mathfrak{g}^\vee$-module $\wedge^* \mathfrak{g}^\vee \otimes V$. There exists a unique map

$$d_{\mathrm{CE}} : \wedge^p g^\vee \otimes V \longrightarrow \wedge^{p+1} \mathfrak{g}^\vee \otimes V$$

determined by the following conditions

1) $d_{CE} : V \longrightarrow \mathfrak{g}^\vee \otimes V$ is the dual of the representation map $\rho : \mathfrak{g} \otimes V \longrightarrow V$.
2) For $a \in \wedge^p g^\vee, m \in V$,

$$d_{\mathrm{CE}}(a \otimes m) = d_{\mathrm{CE}}(a) \otimes m + (-1)^p a \wedge d_{\mathrm{CE}}(m).$$

**Proposition 4.4.** $d_{\mathrm{CE}} : \wedge^\bullet \mathfrak{g}^\vee \otimes V \longrightarrow \wedge^\bullet \mathfrak{g}^\vee \otimes V$ *satisfies*

$$d_{\mathrm{CE}}^2 = 0.$$

*This is equivalent to* $V$ *being a* $\mathfrak{g}$*-representation.*

*Proof:* Exercise. □

**Definition 4.5.** The **Chevalley-Eilenberg cochain complex** associated to the $\mathfrak{g}$-module $V$ is

$$C^\bullet(\mathfrak{g}, V) := (\wedge^\bullet \mathfrak{g}^\vee \otimes V, d_{\mathrm{CE}})$$

Note that $C^\bullet(\mathfrak{g}, V)$ is a $C^\bullet(\mathfrak{g})$-module:

$$\begin{aligned} C^p(\mathfrak{g}) \otimes C^q(\mathfrak{g}, V) &\to C^{p+q}(\mathfrak{g}, V) \\ a \otimes u &\to a \wedge u \end{aligned}$$

which is compatible with the differential

$$d_{\mathrm{CE}}(a \wedge u) = (d_{\mathrm{CE}} a) \wedge u + (-1)^p a \wedge d_{\mathrm{CE}}(u).$$

In other word, $C^\bullet(\mathfrak{g}, V)$ is a dg-module over $C^\bullet(\mathfrak{g})$.

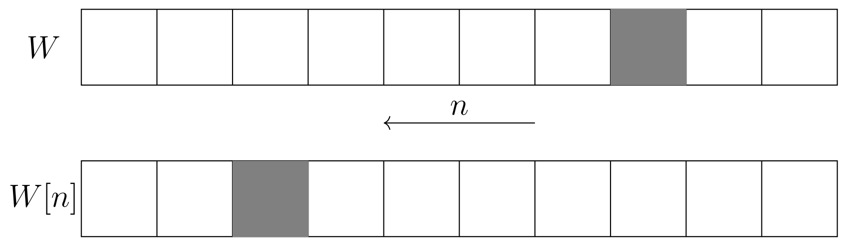
## 4.2 Differential Graded Lie Algebra (DGLA)

We will work with the category of graded vector spaces. Let

$$W = \bigoplus_{m \in \mathbb{Z}} W_m$$

be a $\mathbb{Z}-$graded vector space, where $W_m$ is the degree $m$ component. For $a \in W_m$, we write $|a| = m$. We will frequently use the following operations on graded spaces.

- $W[n]$ denotes the $\mathbb{Z}$-graded space with $W[n]_m{:=}W_{n+m}$ (degree shift by n).




- $W^\vee$ denotes the linear dual with

$$(W^\vee)_m := \mathrm{Hom}(W_{-m}, k) = (W_{-m})^\vee$$

- Given two $\mathbb{Z}$-graded vector spaces V,W

$$(V \otimes W)_n := \bigoplus_{i+j=n} (V_i \otimes W_j)$$
$$\mathrm{Hom}(V, W)_n := \bigoplus_{i} \mathrm{Hom}(V_i, W_{i+n})$$

- $\mathrm{Sym}^m(V) := V^{\otimes m}/\sim$ where

$$a \otimes b \sim (-1)^{|a||b|} b \otimes a.$$

  This is called **m-th graded symmetric tensor**. We will write

$$\mathrm{Sym}(V) := \bigoplus_{m \geq 0} \mathrm{Sym}^m(V), \quad \widehat{\mathrm{Sym}}(V) := \prod_{m \geq 0} \mathrm{Sym}^m(V).$$

- $\bigwedge^m(V) := V^{\otimes m}/\sim$ where

$$a \otimes b \sim -(-1)^{|a||b|} b \otimes a.$$

  This is called **m-th graded skew-symmetric tensor**.

**Proposition 4.6.** $\bigwedge^k(V[1]) \cong \mathrm{Sym}^k(V)[k]$

*Proof:* This is an useful and helpful exercise. □

**Definition 4.7.** A **graded Lie algebra** is a $\mathbb{Z}-$graded space

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} g_m$$

together with a bilinear map

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying the following conditions:

1) graded bracket: $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$
2) graded skew-symmetry:

$$[a,b] = -(-1)^{\alpha\beta}[b,a], \quad \forall a \in g_\alpha, b \in g_\beta$$

3) graded Jacobi Identity:

$$[[a,b],c] = [a,[b,c]] - (-1)^{\alpha\beta}[b,[a,c]], \quad \forall a \in g_\alpha, b \in g_\beta, c \in g_\gamma$$

**Definition 4.8.** A differential graded Lie algebra (**DGLA**) is a graded Lie algebra $\mathfrak{g}$ together with a deg $= 1$ linear map $d$ (i.e. $d : \mathfrak{g}_m \to \mathfrak{g}_{m+1}$) satisfying

1) $d^2 = 0$, so $(\mathfrak{g}, d)$ is a cochain complex.
2) graded Leibniz rule:

$$d[a,b] = [da,b] + (-1)^\alpha [a,db], \quad \forall a \in g_\alpha, b \in g_\beta$$

**Example 4.9.** *An ordinary Lie algebra is a DGLA with*

- $\mathfrak{g} = \mathfrak{g}_0$ *concentrated in deg=0*
- *d=0*

Thus DGLA is a natural generalization of Lie algebra.

**Example 4.10.** *Let $X$ be a manifold, $\mathfrak{g}$ be a Lie algebra. Let $(\Omega^\bullet(x), d)$ be the de Rham complex. Then $(\Omega^\bullet(x) \otimes \mathfrak{g}, d, [-,-])$ is a DGLA.*

- $\Omega^p(x) \otimes \mathfrak{g}$ *is the* deg $= p$ *component.*
- $d : \Omega^p(x) \otimes \mathfrak{g} \to \Omega^{p+1}(x) \otimes \mathfrak{g}$ *is the de Rham differential*

$$d(\alpha \otimes h) = d\alpha \otimes h, \quad \textit{for } \alpha \in \Omega^p, h \in \mathfrak{g}.$$

- *the bracket is induced from the Lie bracket $[-,-]_g$ on $\mathfrak{g}$*

$$[\alpha_1 \otimes h_1, \alpha_2 \otimes h_2] := (\alpha_1 \wedge \alpha_2) \otimes [h_1, h_2]_g, \forall \alpha_1, \alpha_2 \in \Omega^*, h_1, h_2 \in \mathfrak{g}.$$

*This example is related to the Chern-Simons theory.*

**Example 4.11.** *Let $X$ be a complex manifold. Let $(\Omega^{0,\bullet}, \bar\partial)$ be the Dolbeault complex. Let $T_X^{1,0}$ denote the bundle of (1,0)-vector fields. Then*

$$\left(\Omega^{0,\bullet}\left(X, T_X^{1,0}\right), \bar\partial, [-,-]\right)$$

*is a DGLA. Explicitly, let $z^i$ be local holomorphic coordinates. An element $\alpha \in \Omega^{0,p}\left(X, T_X^{1,0}\right)$ can be written as*

$$\alpha = \sum_{i,J} \alpha_J^i d\bar z^J \otimes \frac{\partial}{\partial z^i}.$$

*Here $J = \{j_1 < ... < j_p\}$ is a multi-index and $d\bar z^J = d\bar z^{j_1} \wedge ... \wedge d\bar z^{j_p}$. Then the differential is*

$$\bar\partial\alpha = \sum \bar\partial(\alpha_J^i) \wedge d\bar z^J \otimes \frac{\partial}{\partial z^i} = \sum_l \frac{\partial \alpha_J^i}{\partial \bar z^l} d\bar z^l \wedge d\bar z^J \otimes \frac{\partial}{\partial z^i}$$

*Given two elements*

$$\alpha = \sum \alpha^i_{\bar{J}} d\bar{z}^J \otimes \frac{\partial}{\partial z^i}, \quad \beta = \sum \beta^i_{\bar{M}} d\bar{z}^M \frac{\partial}{\partial z^i}$$

*their bracket is given by*

$$[\alpha, \beta] = \sum \left( \alpha^i_{\bar{J}} \frac{\partial \beta^i_{\bar{M}}}{\partial z^j} - \beta^i_{\bar{M}} \frac{\partial \alpha^i_{\bar{J}}}{\partial z^j} \right) d\bar{z}^J \wedge d\bar{z}^M \otimes \frac{\partial}{\partial z^i}.$$

*On* $\deg = 0$ *component, this is precisely the usual Lie bracket on* $(1, 0)$ *vector fields.*

*This DGLA is related to the deformation of complex structures. In physics, this is related to the B-twisted topological string field theory (Kodaira-Spencer gravity).*

Chevally-Eilenberg complex construction can be naturally extended to DGLA. Let $(\mathfrak{g}, d, [-,-])$ be a DGLA. Define the graded space

$$C^\bullet(\mathfrak{g}) := \mathrm{Sym}(\mathfrak{g}^\vee[-1]) = \bigoplus_m \mathrm{Sym}^m(\mathfrak{g}^\vee[-1]).$$

Since $\mathfrak{g}^\vee[-1] = (\mathfrak{g}[1])^\vee$, we can think about $C^\bullet(\mathfrak{g})$ as (polynomial) functions on $\mathfrak{g}[1]$. When $\mathfrak{g}$ is an ordinary Lie algebra, Proposition 4.6 impiles

$$C^k(\mathfrak{g}) = \mathrm{Sym}^k(g^\vee[-1]) \simeq \wedge^k \mathfrak{g}^\vee[-k]$$

which is $\wedge^k \mathfrak{g}^\vee$ sitting at $\deg = k$. Let

- the dual of $d : \mathfrak{g} \longrightarrow \mathfrak{g}$ be

$$d_1 : \mathfrak{g}^\vee[-1] \longrightarrow \mathfrak{g}^\vee[-1]$$

- the dual of the bracket $[-,-] : \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ be

$$d_2 : \mathfrak{g}^\vee[-1] \longrightarrow \mathrm{Sym}^2(\mathfrak{g}^\vee[-1]) \simeq \wedge^2 \mathfrak{g}^\vee[-2]$$

Note that both $d_1$ and $d_2$ have $\deg = 1$. Since $C^\bullet(\mathfrak{g})$ is freely generated by $\mathfrak{g}^\vee[-1]$, we can extend $d_1$ and $d_2$ to $C^\bullet(\mathfrak{g})$ by

- on generators $\mathfrak{g}^\vee[-1]$, defined above.
- graded Leibniz rule: for $d = d_1$ *or* $d_2$

$$d(a \wedge b) = (da) \wedge b + (-1)^{|a|} a \wedge db$$

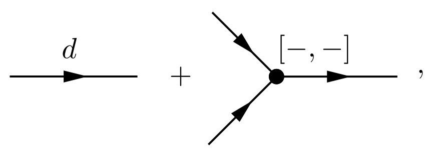
Define the Chevalley-Eilenberg differential

$$d_{\mathrm{CE}} = d_1 + d_2 : C^\bullet(\mathfrak{g}) \longrightarrow C^\bullet(\mathfrak{g})$$

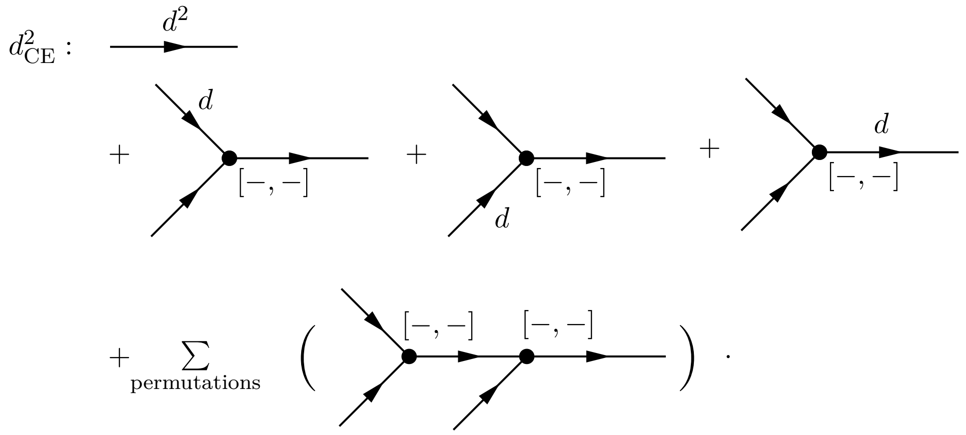
**Proposition 4.12.** $d^2_{\mathrm{CE}} = 0$. *The complex* $(C^\bullet(\mathfrak{g}), d_{\mathrm{CE}})$ *is also called Chevally-Eilenberg cochain complex.*

*Sketch of proof.* We illustrate why this is true and leave the details to readers. In fact, if we represent

$d_{\mathrm{CE}}$ :



then we have




We can “see” that $d_{CE}^2 = 0$ is equivalent to the defining properties of DGLA:

- $d^2 = 0$,
- Leibniz rule: $d\,[-,-] = [d(-),-] \pm [-,d(-)]$,
- Jacobi identity.

$\square$

The generalization to $\mathfrak{g}$-module is straightforward.

**Definition 4.13.** Let $\mathfrak{g}$ be a DGLA. A $\mathfrak{g}$-module is a cochain complex $(M, d_M)$ together with bilinear map $\mathfrak{g} \otimes M \xrightarrow{\cdot} M$ satisfying

- $\mathfrak{g}_\alpha \otimes M_\beta \longrightarrow M_{\alpha+\beta}$
- $a \cdot (b \cdot m) - (-1)^{\alpha\beta} b \cdot (a \cdot m) = [a,b] \cdot m, \quad \forall a \in \mathfrak{g}_\alpha, b \in \mathfrak{g}_\beta, m \in M$
- $d_m(a \cdot m) = (da) \cdot m + (-1)^\alpha a \cdot d_M m, \quad \forall a \in \mathfrak{g}_\alpha, m \in M$

Equivalently, let us define the graded space

$$\mathfrak{g}_M := \mathfrak{g} \oplus M$$

equipped with the ifferential $d \oplus d_M$ and the bracket $[-,-]$ where

- on $\mathfrak{g} \otimes \mathfrak{g}$, $[-,-]$ is the bracket on $\mathfrak{g}$: $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.
- on $\mathfrak{g} \otimes M$, $[-,-]$ is the $\mathfrak{g}$-module action: $[-,-] : \mathfrak{g} \otimes M \to M$.
- on $M \otimes M$, $[-,-] = 0$.

Then $M$ is $\mathfrak{g}$-module $\iff$ $\mathfrak{g}_M$ is DGLA.

The corresponding Chevally-Eilenberg complex is

$$C^\bullet(\mathfrak{g}, M) := (\mathrm{Sym}(\mathfrak{g}^\vee[-1]) \otimes M, d_{\mathrm{CE}})$$

We leave the description of $d_{\mathrm{CE}}$ to readers.

## 4.3 Homotopy Lie algebra ($L_\infty$-algebra)

Given a graded vector space $V$, we consider a $\deg = 1$ derivation on $\mathrm{Sym}(V)$:

$$\delta : \mathrm{Sym}(V) \to \mathrm{Sym}(V),$$

which satisfies the graded Leibniz rule

$$\delta(a \otimes b) = (\delta a) \otimes b + (-1)^{|a|} a \otimes \delta b.$$

Such $\delta$ is completely determined by how $\delta$ acts on the generator

$$\delta : V \to \mathrm{Sym}(V).$$

We can decompose

$$\delta = \delta_0 + \delta_1 + \delta_2 + \cdots ,$$

where $\delta_k : V \to \mathrm{Sym}^k(V)$. For DGLA, we have

$$\delta = d_{\mathrm{CE}} = d_1 + d_2 = \delta_1 + \delta_2.$$

This is a derivation where only $\delta_1$, $\delta_2$ are nontrivial. It is natural to generalize this operator by encoding all possible components $\delta_k$. This leads to the $L_\infty$-*algebra*.

**Definition 4.14.** An $L_\infty$**-algebra** is a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ with a collection of multilinear maps

$$\ell_n : \bigwedge^n \mathfrak{g} \to \mathfrak{g}, \quad \deg(\ell_n) = 2 - n \quad (n \geq 1)$$

satisfying the following $L_\infty$-relation

$$\sum_{k=1}^{n} \pm \ell_{n-k+1} \left( \ell_k \left( \cdots \right), \cdots \right) = 0 \quad \forall n.$$

The complicated $L_\infty$-relation can be understood as follows. Let

$$\delta_n : \mathfrak{g}^\vee[-1] \to \mathrm{Sym}^n \left( \mathfrak{g}^\vee[-1] \right) \simeq \bigwedge^n \mathfrak{g}^\vee[-n]$$

denote the dual of $\ell_n : \bigwedge^n \mathfrak{g} \to \mathfrak{g}$. Note that

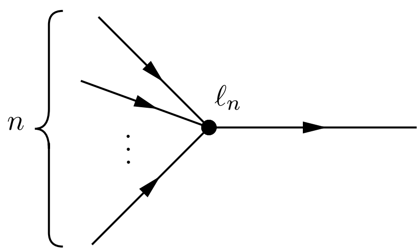
$$\deg(\ell_n) = 2 - n \;\Leftrightarrow\; \deg(\delta_n) = 1.$$

Let $\delta = \sum\limits_{n \geq 1} \delta_n = \delta_1 + \delta_2 + \cdots$ which defines a derivation on $C^\bullet(\mathfrak{g}) = \widehat{\mathrm{Sym}}\left(\mathfrak{g}^\vee[-1]\right)$ via the graded Leibniz rule. (Here we use the (graded) formal power series ring $\widehat{\mathrm{Sym}}\left(\mathfrak{g}^\vee[-1]\right)$ so that $\delta$ is defined.) Then

$$L_\infty\text{-relations for } \{\ell_n\}_{n\geq 1} \;\Leftrightarrow\; \delta^2 = 0 .$$

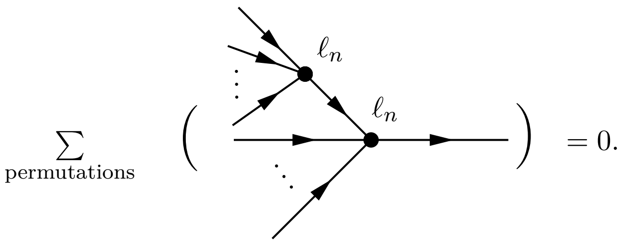
$(C^\bullet(\mathfrak{g}), \delta)$ is the Chevalley-Eilenberg complex for $\mathfrak{g}$.

If we represent each $\delta_n$ as a graph



,

then $\delta^2 = 0$ can be pictured as




As we will see, this has a natural interpretation via Feynman diagram technique. See [27] and its references for a nice review on $L_\infty$-algebra and its relation in QFT.

# 5 Perturbative Expansion

In quantum field theory, we are interested in quantum correlations for observables $\mathcal{O}$, which is an $\infty$-dimensional integral

$$\int_{\mathcal{E}} e^{\frac{iS}{\hbar}} \mathcal{O}[D\phi].$$

The rigorous construction of such measure is not known in general. Nevertheless the $\hbar \to 0$ asymptotic theory is established as perturbative renormalization method.

## 5.1 Wick's Theorem

### Gaussian Integral

We start with the finite dimensional situation. Let

$$Q(x) = \sum_{i,j=1}^{n} Q_{ij} x^i x^j$$

be a positive definite quadratic form, i.e., the matrix $(Q_{ij})$ is positive definite. Consider the following volume form on $\mathbb{R}^n$

$$\Omega = \sqrt{\det Q} \prod_{i=1}^{n} \frac{dx^i}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}Q(x)}, \qquad \hbar > 0.$$

By the Gaussian integral $\int_{\mathbb{R}} dx\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$, it is easy to find

$$\int_{\mathbb{R}^n} \Omega = 1.$$

Thus $\Omega$ defines a probability density on $\mathbb{R}^n$.

Consider for simplicity a polynomial function $f(x) \in \mathbb{R}[x^i]$. Define the following expectation value with respect to the probability density $\Omega$:

$$\langle f(x) \rangle_x := \frac{\int_{\mathbb{R}^n} \Omega f(x)}{\int_{\mathbb{R}^n} \Omega} = \int_{\mathbb{R}^n} \Omega f(x).$$

The subscript $x$ indicates the integration variable. This expectation value defines a map

$$\langle - \rangle_x : \mathbb{R}[x^i] \longrightarrow \mathbb{R}.$$

To compute this expectation map, let us consider the auxiliary integral

$$\begin{aligned} Z[J] :=& \left\langle e^{\sum_i x^i J_i} \right\rangle_x \qquad\qquad J = (J_1, \cdots, J_n) \\ =& \sqrt{\det Q} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{dx^i}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}Q(x) + \sum_i x^i J_i} \end{aligned}$$

Completing the square, we find

$$\frac{1}{2\hbar}Q(x) - \sum_i x^i J_i = \frac{1}{2\hbar} Q(x^i - \hbar \sum_j (Q^{-1})^{ij} J_j) - \frac{\hbar}{2} Q^{-1}(J).$$

Here $(Q^{-1})^{ij}$ is the inverse matrix of $Q_{ij}$

$$\sum_k Q_{ik}(Q^{-1})^{kj} = \delta_i{}^j$$

and $Q^{-1}(J)$ is the quadratic expression

$$Q^{-1}(J) = \sum_{i,j} (Q^{-1})^{ij} J_i J_j.$$

Thus

$$Z[J] = e^{\frac{\hbar}{2}Q^{-1}(J)} \int_{\mathbb{R}^n} \Omega = e^{\frac{\hbar}{2}Q^{-1}(J)}.$$

Now for a polynomial $f(x)$, Taylor series expansion at $x=0$ gives

$$f(x) = e^{\sum_i x^i \frac{\partial}{\partial a^i}} f(a)\Big|_{a=0}.$$

Using this formula, we find

$$\begin{aligned}\langle f(x)\rangle_x &= \left\langle e^{\sum_i x^i \frac{\partial}{\partial a^i}} f(a)\Big|_{a=0} \right\rangle_x \\ &\xlongequal{J=\frac{\partial}{\partial a}} Z\left[\frac{\partial}{\partial a}\right] f(a)\Big|_{a=0} = e^{\frac{\hbar}{2}Q^{-1}\left(\frac{\partial}{\partial a}\right)} f(a)\Big|_{a=0}\end{aligned}$$

where $Q^{-1}\left(\frac{\partial}{\partial a}\right) = \sum_{i,j} (Q^{-1})^{ij} \frac{\partial}{\partial a^i}\frac{\partial}{\partial a^j}$ is a second-order differential operator.

### Wick's Theorem

The obtained formula

$$\langle f(x)\rangle_x = e^{\frac{\hbar}{2}Q^{-1}\left(\frac{\partial}{\partial a}\right)} f(a)\Big|_{a=0}$$

has a combinatorial interpretation as follows. Let

$$i_1, i_2, \cdots, i_{2m} \in \{1, 2, \cdots, n\}$$

be $2m$ indices. We consider the monomial $x^{i_1}x^{i_2}\cdots x^{i_{2m}}$ of degree $2m$ obtained from the index set. Let us compute its expectation from the above formula:

$$\left\langle x^{i_1}x^{i_2}\cdots x^{i_{2m}}\right\rangle_x = e^{\frac{\hbar}{2}Q^{-1}\left(\frac{\partial}{\partial a}\right)} a^{i_1}a^{i_2}\cdots a^{i_{2m}}\Big|_{a=0} = \frac{\hbar^m}{m!}\left[\frac{1}{2}\sum_{i,j}(Q^{-1})^{ij}\frac{\partial}{\partial a^i}\frac{\partial}{\partial a^j}\right]^m a^{i_1}a^{i_2}\cdots a^{i_{2m}} \quad (*)$$

To compute this value, let

$$P(2m) = \text{ set of partitions of } \{1, 2, \cdots, 2m\} \text{ into unordered } m \text{ pairs.}$$

An element $\sigma \in P(2m)$ can be described by a permutation $\sigma \in S_{2m}$ such that

$$\sigma(1) < \sigma(2), \quad \sigma(3) < \sigma(4), \quad \cdots, \quad \sigma(2m-1) < \sigma(2m)$$

and

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2m-3) < \sigma(2m-1).$$

Then we identify this permutation

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & 2m-1 & 2m \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(2m-1) & \sigma(2m) \end{bmatrix}$$

with an element of $P(2m)$ by pairing

$$\underbrace{\sigma(1)\sigma(2)}_{\text{pair}} \qquad \underbrace{\sigma(3)\sigma(4)}_{\text{pair}} \qquad \cdots \qquad \underbrace{\sigma(2m-1)\sigma(2m)}_{\text{pair}}$$

In the above formula (*), it is computed to become the following sum

$$\boxed{\left\langle x^{i_1} x^{i_2} \cdots x^{i_{2m}} \right\rangle_x = \hbar^m \sum_{\sigma \in P(2m)} (Q^{-1})^{\sigma(1)\sigma(2)} (Q^{-1})^{\sigma(3)\sigma(4)} \cdots (Q^{-1})^{\sigma(2m-1)\sigma(2m)}}.$$

This is called the **Wick's Theorem**.

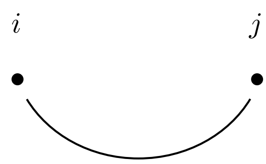
We can also draw each sum as


For each edge connecting $i$ and $j$, we assign the factor $\hbar(Q^{-1})^{ij}$ to it.

**Example 5.1** (Two-point function)**.** *The two-point function*

$$\left\langle x^i x^j \right\rangle_x = \hbar (Q^{-1})^{ij}$$

*is given by the inverse matrix of* $Q$*.* $(Q^{-1})^{ij}$ *is also called the "**propagator**".*




**Example 5.2.** *The following four-point function is computed by*

$$\begin{aligned} \left\langle x^i x^j x^k x^l \right\rangle_x =& ijkl + ijkl + ijkl \\ =& \hbar^2 \left[ (Q^{-1})^{ij} (Q^{-1})^{kl} + (Q^{-1})^{ik} (Q^{-1})^{jl} + (Q^{-1})^{il} (Q^{-1})^{jk} \right]. \end{aligned}$$

We can also add "background" by considering the shift $x \to x + a$

$$\langle f(x+a) \rangle_x = \int_{\mathbb{R}^n} \Omega\, f(x+a)$$

which is now a function of $a$. A similar argument shows

$$\langle f(x+a) \rangle_x = e^{\frac{\hbar}{2} Q^{-1}\left(\frac{\partial}{\partial a}\right)} f(a).$$

If $f(x) = x^{i_1} x^{i_2} \cdots x^{i_k}$, then Wick's Theorem in this case is a sum over partial pairings


We assign the propagator to the paired indices and assign $a^i$ to unpaired index $i$.

**Example 5.3.** *Let $f(x) = x^i x^j x^k x^l$. Then*

$$\begin{aligned}\langle f(x+a)\rangle_x =& ijkl + \overbrace{ij}kl + \overbrace{ijk}l + \overbrace{ijkl} + i\overbrace{jk}l + i\overbrace{jkl} + ij\overbrace{kl} + \overbrace{ij}\,\overbrace{kl} + ijkl + ijkl \\ =& a^i a^j a^k a^l + \hbar\Big[(Q^{-1})^{ij}a^k a^l + (Q^{-1})^{ik}a^j a^l + (Q^{-1})^{il}a^j a^k \\ &+ (Q^{-1})^{jk}a^i a^l + (Q^{-1})^{jl}a^i a^k + (Q^{-1})^{kl}a^i a^j\Big] \\ &+ \hbar^2\Big[(Q^{-1})^{ij}(Q^{-1})^{kl} + (Q^{-1})^{ik}(Q^{-1})^{jl} + (Q^{-1})^{il}(Q^{-1})^{jk}\Big].\end{aligned}$$

## 5.2 Feynman Graph Expansion

Now we consider integrals of the form

$$\sqrt{\det Q}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\frac{dx^i}{\sqrt{2\pi\hbar}}\, e^{-\frac{1}{\hbar}S(x)}$$

where

$$S(x) = \frac{1}{2}Q(x) - \frac{\lambda}{3!}I(x).$$

Here $Q(x) = \sum\limits_{i,j} Q_{ij}x^i x^j$ is a positive quadratic as before, and

$$I(x) = \sum_{i,j,k} I_{ijk}x^i x^j x^k$$

is a cubic polynomial, called the "**interaction**". The constants $I_{ijk}$ parametrize the cubics. The constant $\lambda$ is called the "**coupling constant**".

Since the cubic approaches both $\pm\infty$ and grows faster than quadratic, the above integral is simply "divergent". There are essentially two ways out to make sense of it:

① Complexify $x^i$ to complex variables $z^i$ and change the integration contour

$$\mathbb{R}^n \subset \mathbb{C}^n \qquad \Longrightarrow \qquad \Gamma \subset \mathbb{C}^n$$

to some other contour $\Gamma$ such that the integration becomes convergent. "Airy integral" is such an example. This method is usually referred to as the non-perturbative method.

② Treat $\lambda$ as a perturbative parameter and compute the asymptotic series. This method is usually referred to as the perturbative method.

We will focus on the perturbative method ② here. Let us rewrite

$$\text{“}\sqrt{\det Q}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\frac{dx^i}{\sqrt{2\pi\hbar}}\, e^{-\frac{1}{\hbar}S(x)}\text{”} = \text{“}\int_{\mathbb{R}^n}\Omega\, e^{\frac{\lambda}{3!\hbar}I(x)}\text{”} = \text{“}\left\langle e^{\frac{\lambda}{3!\hbar}I(x)}\right\rangle_x\text{”}.$$

Here $\Omega = \sqrt{\det Q} \prod_{i=1}^{n} \frac{dx^i}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}Q(x)}$ is the probability density as before. Now we can redefine the above divergent integral as a power series by

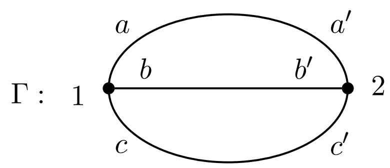
$$\sum_{m=0}^{\infty} \frac{\lambda^m}{m!\hbar^m} \left\langle \left(\frac{1}{3!} I(x)\right)^m \right\rangle_x .$$

Here each term $\left\langle \left(\frac{1}{3!}I(x)\right)^m \right\rangle_x$ has a well-defined value which can be computed by Wick's Theorem. This will lead to a combinatorial graph formula for this power series.

**Definition 5.4.** By a graph $\gamma$, we refer to the following data

- $V(\gamma)$ = set of vertices
- $HE(\Gamma)$ = set of half-edges
- $i_\Gamma : HE(\Gamma) \to V(\gamma)$ incidence map
- $E(\Gamma)$: a perfect matching on $HE(\Gamma)$ into pairs of two elements. Each pair is called an "edge".
- For each $v \in V(\Gamma)$, $\#\left\{i_\Gamma^{-1}(v)\right\}$ is called the valency of $v$.

**Example 5.5.** *Consider the following* $\Theta$ *graph*




*Here* $V = \{1, 2\}$, $HE = \{a, b, c, a', b', c'\}$, $E = \{(a, a'), (b, b'), (c, c')\}$. *The incidence map is*

$$\begin{aligned} i_\Gamma : HE &\longrightarrow V \\ \{a, b, c\} &\longmapsto 1 \\ \{a', b', c'\} &\longmapsto 2 \end{aligned}$$

*Both vertex* 1 *and* 2 *have valency* 3.

**Definition 5.6.** A graph isomorphism between two graphs $\Gamma$ and $\Gamma'$ is a pair of bijections

$$\begin{aligned} \sigma_V : V(\Gamma) &\longrightarrow V(\Gamma') \\ \sigma_{HE} : HE(\Gamma) &\longrightarrow HE(\Gamma') \end{aligned}$$

which are compatible with incident maps (the following diagram commutes)

$$\begin{tikzcd} HE(\Gamma) \arrow[r, "\sigma_{HE}"] \arrow[d, "i_\Gamma"] & HE(\Gamma') \arrow[d, "i'_\Gamma"] \\ V(\Gamma) \arrow[r, "\sigma_V"] & V(\Gamma') \end{tikzcd}$$

and compatible with edgesfor any $a, b \in HE(\Gamma)$,

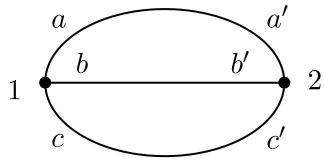
$$(a, b) \in E(\Gamma) \qquad \Longleftrightarrow \qquad (\sigma_{HE}(a), \sigma_{HE}(b)) \in E(\Gamma').$$

An automorphism of $\Gamma$ is an isomorphism $\sigma : \Gamma \to \Gamma$. Denote

$$\mathrm{Aut}(\Gamma) = \text{Group of automorphisms of } \Gamma.$$

**Example 5.7.** *Consider*

$\Gamma :$




*Here* $HE = \{a, b, c, a', b', c'\}$, $V = \{1, 2\}$, $E = \{(a, a'), (b, b'), (c, c')\}$. *The incidence map is as above. Here are two examples of automorphisms of* $\Gamma$

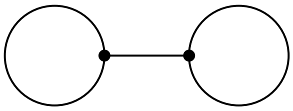
① $\sigma_V : (1, 2) \longrightarrow (2, 1)$
$\sigma_{HE} : (a, b, c, a', b', c') \longrightarrow (a', b', c', a, b, c)$

② $\sigma_V : (1, 2) \longrightarrow (1, 2)$
$\sigma_{HE} : (a, b, c, a', b', c') \longrightarrow (b, c, a, b', c', a')$

*Similarly, we find* $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times S_3$. *Here,* $\mathbb{Z}_2$ *corresponds to permutation of the two vertices, and* $S_3$ *correponds to permutation of the three edges.*

**Example 5.8.** *Consider*

$\Gamma :$

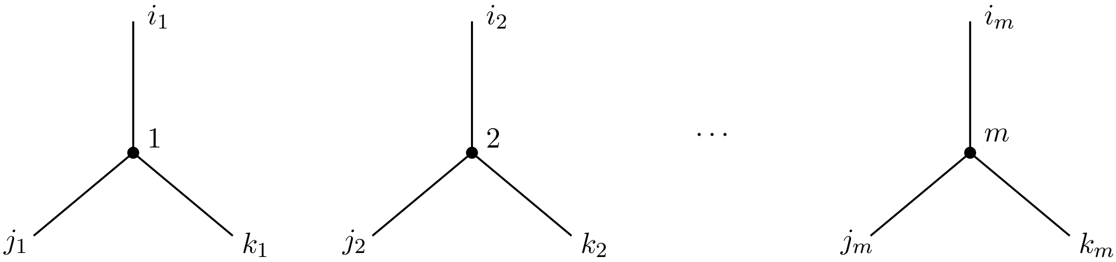


*We find* $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now let us consider the computation of

$$\langle I(x)^m \rangle_x = \langle \underbrace{I(x)I(x)\cdots I(x)}_{m} \rangle_x$$

Each $I(x)$ gives a vertex with 3 half-edges



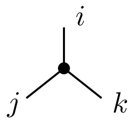

| | |
|---|---|
| Vertex set | $V = \{1, 2, \cdots, m\}$ |
| Half-edge set | $HE = \{i_1, j_1, k_1, i_2, j_2, k_2, , \cdots, i_m, j_m, k_m, \}$ |
| Incidence map | $i : \{i_s, j_s, k_s\} \longrightarrow \{s\}$ |

By Wick's Theorem, $\langle I(x)^m \rangle_x$ is a sum over perfect matching $E$

$$\langle I(x)^m \rangle_x = \sum_{E \in Y} \hbar^{|E(\Gamma_E)|} \omega_{\Gamma_E}.$$

Here $Y = \{\text{perfect matchings of } HE\}$ and $\Gamma_E$ is the graph by assigning the pairing $E$ to the vertex set $V$ and the half-edge set $HE$.

For each graph $\Gamma$, $\omega_\Gamma$ is the number by assigning

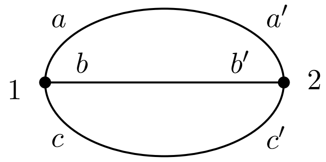
① each vertex 

$\Longrightarrow \quad I_{ijk}$

② each edge $i$ —— $j$ $\Longrightarrow \quad (Q^{-1})^{ij}$

③ sum over indices

**Example 5.9.** *Consider*

$\Gamma:$ 


$$\omega_\Gamma = \sum_{\substack{i_1,j_1,k_1\\ i_2,j_2,k_2}} I_{i_1 j_1 k_1} I_{i_2 j_2 k_2} (Q^{-1})^{i_1 i_2} (Q^{-1})^{j_1 j_2} (Q^{-1})^{k_1 k_2}$$

It is clear that

$$\omega_\Gamma = \omega_{\Gamma'} \quad \text{if } \Gamma \text{ is isomorphic to } \Gamma'.$$

All graphs here are trivalent (valency 3 for each vertex). The set of bijections

$$\sigma_V : V \longrightarrow V$$
$$\sigma_{HE} : HE \longrightarrow HE$$

that preserve the incidence map is

$$G = \underbrace{S_m}_{\text{permuting vertices}} \times \underbrace{(S_3)^m}_{\text{permuting half-edges}}.$$

We have a natural $G$-action on $Y$ with orbits

$$Y/G = \{\text{isomorphic class of trivalent graphs consisting of } m\text{-vertices}\}.$$

Thus

$$\begin{aligned} \frac{1}{m!(3!)^m} \langle I(x)^m \rangle_x &= \frac{1}{|G|} \sum_{\Gamma \in Y/G} \frac{|G|}{|\operatorname{Aut}(\Gamma)|} \omega_\Gamma \hbar^{|E(\Gamma)|} \\ &= \sum_{\substack{\Gamma: \text{ trivalent graph}\\ \text{with } m \text{ vertices}}} \frac{\omega_\Gamma}{|\operatorname{Aut}(\Gamma)|} \hbar^{|E(\Gamma)|} \end{aligned}$$

We can also refine this formula by grouping disconnected graphs into connected isomorphic ones. This leads to the following Feynman graph expansion formula (details left to the reader).

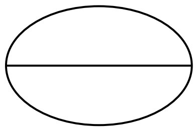
**Proposition 5.10** (Feynman Graph Expansion Formula)**.** *We have the following combinatorial graph expansion formula for the asymptotic power series*

$$\begin{aligned} \left\langle e^{\frac{\lambda}{3!\hbar} I(x)} \right\rangle_x &:= \sum_{m=0}^{\infty} \frac{\lambda^m}{m! \hbar^m} \left\langle \left( \frac{1}{3!} I(x) \right)^m \right\rangle_x \\ &= \sum_{\Gamma: \text{ trivalent}} \hbar^{|E(\Gamma)| - |V(\Gamma)|} \lambda^{|V(\Gamma)|} \frac{\omega_\Gamma}{|\operatorname{Aut}(\Gamma)|} \\ &= \exp\left( \sum_{\substack{\Gamma: \text{ connected}\\ \text{trivalent}}} \lambda^{|V(\Gamma)|} \hbar^{l(\Gamma)-1} \frac{\omega_\Gamma}{|\operatorname{Aut}(\Gamma)|} \right). \end{aligned}$$

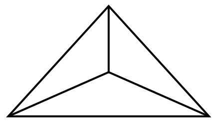
*Here for a connected graph* $\Gamma$*,*

$$l(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1 = 1 - \chi(\Gamma)$$

*is called the loop number of* $\Gamma$*.*



$l(\Gamma) = 3 - 2 + 1 = 2$



$l(\Gamma) = 6 - 4 + 1 = 3$

In general, if the interaction contains terms of all possible degrees

$$\frac{\lambda_3}{3!} I_3(x) + \frac{\lambda_4}{4!} I_4(x) + \cdots + \frac{\lambda_m}{m!} I_m(x) + \cdots$$

then the series expension will have all possible graphs, where each vertex of valency $m$ is associated the value $I_m$ similarly. The general Feynman graph expansion formula reads

$$\left\langle e^{\frac{1}{\hbar}\left(\sum\limits_{m\geq 3} \frac{\lambda_m}{m!} I_m(x)\right)} \right\rangle_x \xlongequal{\text{asymptotic}} \exp\left( \sum_{\substack{\Gamma:\text{ connected}\\ \text{graph}}} \left( \prod_{m\geq 3} \lambda_m^{|V_m(\Gamma)|} \right) \hbar^{l(\Gamma)-1} \frac{\omega_\Gamma}{|\operatorname{Aut}(\Gamma)|} \right)$$

where $V_m(\Gamma)$ is the set of vertices with valency $m$.

## 5.3 Green's Function

Now we discuss the infinite dimensional case. To illustrate the basic idea, let us consider the scalar field theory for $\mathcal{E} = C_c^\infty(\mathbb{R}^d)$ with action functional

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^d} \big( \sum_i (\partial_i \phi)^2 + m^2\phi^2 \big) dx^d, \quad \phi \in \mathcal{E}.$$

We are interested in the correlation function

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{E}} [D\phi]\ e^{-\frac{1}{\hbar}S[\phi]} \mathcal{O}(\phi)}{\int_{\mathcal{E}} [D\phi]\ e^{-\frac{1}{\hbar}S[\phi]}}$$

where $\mathcal{O}$ is a function on $\mathcal{E}$ called an "**observable**". For example we want to compute

$$\langle \phi(y_1)\cdots\phi(y_{2s}) \rangle, \quad \text{where} \quad y_i \in \mathbb{R}^d.$$

Although we do not have a general measure theory for such infinite dimensional integral, we can mimic the finite dimensional situation to obtain a combinatorial expected result. Let us write

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^d} \phi D \phi\ dx^d$$

where

$$D = -\sum_{i=1}^d \frac{\partial^2}{\partial x^i \partial x^i} + m^2.$$

This is quadratic in $\phi$, similar to $Q(x)$ in Gaussian integral.

To be precise, we can compare the finite and the infinite dimensional situation

| **Finite Dim** | **$\infty$-Dim** |
| --- | --- |
| $\mathbb{R}^n$ | $\mathcal{E} = C_c^\infty(\mathbb{R}^d)$ |
| $x^i$ | $\phi(x)$ |
| $i$ | a point $x \in \mathbb{R}^d$ |
| $\sum_i$ | $\int_{\mathbb{R}^d} dx$ |
| $\frac{1}{2}Q(x) = \frac{1}{2}\sum_{i,j} Q_{ij}x^i x^j$ | $S[\phi] = \frac{1}{2}\int \phi D \phi$ |
| $\langle x^i x^j \rangle = (A^{-1})^{ij}$ | $\langle \phi(x)\phi(y)\rangle = G(x,y)$ |
| $\delta_{ij}$ | $\delta(x-y)$ |

The function $G(x,y)$ represents the inverse of $D$ on $x$,

$$D_x G(x,y) = \delta(x-y).$$

Actually, if we define the following operator on smooth $f$

$$(Gf)(x) = \int dy\ G(x,y) f(y).$$

Then

$$\begin{aligned} D(Gf)(x) &= \int dy\ D_x G(x,y) f(y) \\ &= \int dy\ \delta(x-y) f(y) = f(x). \end{aligned}$$

Such function exists and is called the **Green's function**.

**Example 5.11.** *Consider the standard Laplacian $D = -\sum_i \frac{\partial^2}{\partial x^i \partial x^i}$ on $\mathbb{R}^d$. For $d \geq 3$, the Green's function takes the form*

$$G(x,y) \propto \frac{1}{\|x-y\|^{d-2}}.$$

In particular, $G(x,y)$ becomes singular when $x \to y$. This reflects the nature of infinite dimension and is the origin of "**UV divergence**" in quantum field theory.

We use the Wick's theorem to compute

$$\langle \phi(y_1)\cdots\phi(y_{2s})\rangle = \hbar^s \sum_{\sigma \in P(2s)} G(y_{\sigma(1)}, y_{\sigma(2)}) G(y_{\sigma(3)}, y_{\sigma(4)}) \cdots G(y_{\sigma(2s-1)}, y_{\sigma(2s)}),$$

which is visualized by sum over perfect pairings. Each edge from $y$ to $z$ represents the Green's function $G(y,z)$ which plays the role of the propagator.

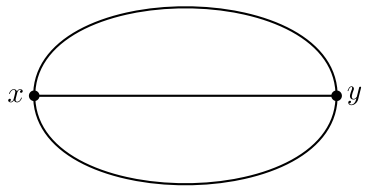
Now we can move on to consider the interacting theory. For instance, we can consider the $\phi^3$ theory by adding to $S$ the term

$$I_3(\phi) = \frac{\lambda}{3!}\int \phi^3,$$

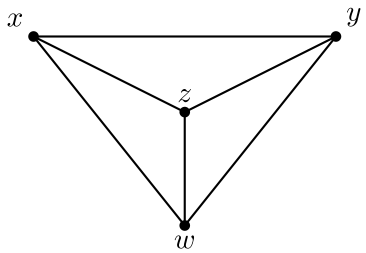
or the $\phi^4$ theory by adding

$$I_4 = \frac{\lambda}{4!}\int \phi^4.$$

We will have similar Feynman diagram expansions. Here are two examples of trivalent graphs and the integrals they represent.




$$\int_{(\mathbb{R}^d)^2} dx\, dy\, G(x,y)^3$$




$$\int_{(\mathbb{R}^d)^4} dxdydzdw\ G(x,y)G(y,w)G(w,x)G(x,z)G(y,z)G(w,z)$$

However, due to the singularity of $G(x,y)$ on the diagonal, such graph integral will be divergent in general. This is the **UV divergence**. The "renormalization theory" studies how to deal with such divergence to obtain a meanful finite answer.

## 5.4 Canonical Quantization

Before we move on to study the UV divergence in general, let us first understand how such consideration leads to the canonical quantization in quantum mechanics.

Consider mechanics on the phase space $\mathbb{R}^2$ coordinated by $(x,p)$, with the symplectic form $\omega = dx \wedge dp$. Classically, the set of functions on phase space has a Poisson structure $\{-,-\}$ defined by

$$\{f(x,p), g(x,p)\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

Quantum mechanically, $x$ and $p$ are quantized to be operators $\hat{x}$ and $\hat{p}$ satisfying the canonical commutation relation

$$[\hat{x},\hat{p}] = i\hbar.$$

Algebraically, we can view it as a deformation of the function ring $A$ on the phase space. Denote by $A^\hbar = A[\hbar]$ the deformed ring with the deformed product $*$ called the *Moyal product*. For $f, g \in A$, we have (asymptotically in $\hbar$)

$$\begin{aligned} f(x,p) * g(x,p) &= f(x,p) e^{\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)} g(x,p) \\ &= f(x,p)g(x,p) + \frac{i\hbar}{2}\{f,g\} + O(\hbar^2). \end{aligned}$$

One can check that

① $*$ is associative:

$$f * (g * h) = (f * g) * h, \quad \forall f, g, h \in A;$$

② The canonical commutation relation holds:

$$[x,p]_* = x * p - p * x = i\hbar.$$

The Moyal product $*$ has a Feynman diagram interpretation as follows. Introduce two fields describing mechanics on the phase space $\mathbb{R}^2$

$$\gamma(t) = \big(\mathbb{X}(t), \mathbb{P}(t)\big) : \mathbb{R}_t \to \mathbb{R}^2$$

and consider the free action

$$S_0[\gamma] = \int_{\mathbb{R}} \mathbb{P}(t) d\mathbb{X}(t) = \int_{\mathbb{R}} \mathbb{P}(t)\dot{\mathbb{X}}(t) dt = \int_{\mathbb{R}} \mathbb{P}(t)\frac{d}{dt}\mathbb{X}(t).$$

We are interested in the expectation of the observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int [D\gamma] e^{\frac{i}{\hbar}S_0[\gamma]}\mathcal{O}(\gamma)}{\int [D\gamma] e^{\frac{i}{\hbar}S[\gamma]}}.$$

Let us write (ignore the boundary behavior so far)

$$\begin{aligned} S_0[\gamma] =& \frac{1}{2}\int \left(\mathbb{P}\dot{\mathbb{X}} - \mathbb{X}\dot{\mathbb{P}}\right) dt \\ =& \frac{1}{2}\int \begin{bmatrix} \mathbb{X} & \mathbb{P} \end{bmatrix} \begin{bmatrix} & -\frac{d}{dt} \\ \frac{d}{dt} & \end{bmatrix} \begin{bmatrix} \mathbb{X} \\ \mathbb{P} \end{bmatrix} dt \end{aligned}$$

Thus

$$e^{\frac{i}{\hbar}S_0[\gamma]} = \exp\left(-\frac{1}{2\hbar}\left(\frac{1}{i}\int \begin{bmatrix} \mathbb{X}(t) & \mathbb{P}(t) \end{bmatrix} \begin{bmatrix} 0 & -\frac{d}{dt} \\ \frac{d}{dt} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{X}(t) \\ \mathbb{P}(t) \end{bmatrix} dt\right)\right).$$

Comparing with the finite-dimensional situation

$$e^{-\frac{1}{2\hbar}\sum\limits_{i,j} Q_{ij}x^i x^j}$$

the analogue of the inverse matrix $Q^{-1}$ is

$$i\begin{bmatrix} 0 & -\frac{d}{dt} \\ \frac{d}{dt} & 0 \end{bmatrix}^{-1}$$

The Green's function of $\frac{d}{dt}$ is a function $G(t_1, t_2)$ satisfying

$$\frac{\partial}{\partial t_1} G(t_1, t_2) = \delta(t_1 - t_2).$$

Here $\delta(t_1 - t_2)$ can be viewed as the infinite-dimensional analogue of $\delta_{ij}$.

In terms of $G(t_1, t_2)$, we have

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial t_1} \\ \frac{\partial}{\partial t_1} & 0 \end{bmatrix} \begin{bmatrix} 0 & G(t_1, t_2) \\ -G(t_1, t_2) & 0 \end{bmatrix} = \begin{bmatrix} \delta(t_1 - t_2) & 0 \\ 0 & \delta(t_1 - t_2) \end{bmatrix}$$

The right hand side can be viewed as the $(t_1, t_2)$-entry of the infinite-dimensional identity matrix. Thus the $(t_1, t_2)$-entry of the above inverse can be written as

$$i \begin{bmatrix} 0 & -\frac{d}{dt} \\ \frac{d}{dt} & 0 \end{bmatrix}^{-1}_{t_1, t_2} = \begin{bmatrix} 0 & iG(t_1, t_2) \\ -iG(t_1, t_2) & 0 \end{bmatrix}.$$

Then we could "define" the two-point functions in terms of this inverse by

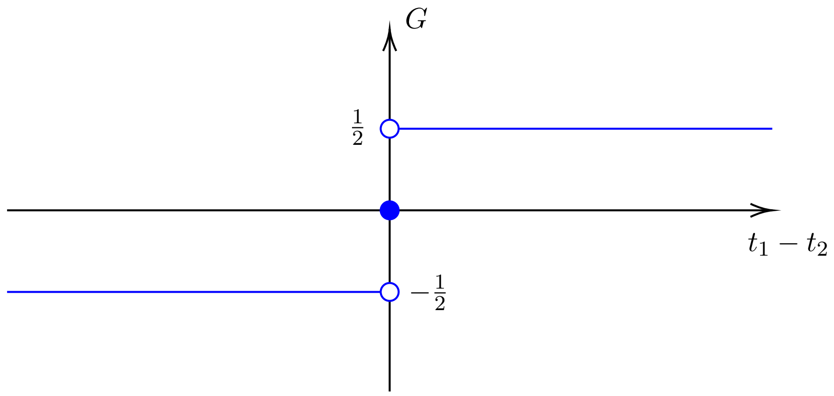
$$\begin{cases} \langle \mathbb{X}(t_1)\mathbb{P}(t_2)\rangle_\gamma := i\hbar G(t_1, t_2) \\ \langle \mathbb{P}(t_1)\mathbb{X}(t_2)\rangle_\gamma := -i\hbar G(t_1, t_2) \\ \langle \mathbb{X}(t_1)\mathbb{X}(t_2)\rangle_\gamma = \langle \mathbb{P}(t_1)\mathbb{P}(t_2)\rangle_\gamma = 0 \end{cases}$$

We are left to solve $G(t_1, t_2)$. The above expression asks for $G(t_1, t_2)$ such that

$$G(t_1, t_2) = -G(t_2, t_1).$$

Such Green's function of $\frac{d}{dt}$ is explicitly given by

$$G(t_1, t_2) = \frac{1}{2}\operatorname{sgn}(t_1 - t_2) = \begin{cases} \frac{1}{2} & t_1 > t_2 \\ 0 & t_1 = t_2 \\ -\frac{1}{2} & t_1 < t_2 \end{cases}$$




In fact, for any compactly supported test function $f(t)$

$$-\int_{\mathbb{R}} dt_1\, G(t_1, t_2) f'(t_1) = \frac{1}{2}\int_{-\infty}^{t_2} dt_1\, f'(t_1) - \frac{1}{2}\int_{t_2}^{\infty} dt_1\, f'(t_1) = \frac{1}{2} f(t_2) - \frac{1}{2}\left(-f(t_2)\right) = f(t_2).$$

So

$$\frac{\partial}{\partial t_1} G(t_1, t_2) = \delta(t_1 - t_2)$$

holds as a distributional equation.

In summary, we are interested in the following two-point function

$$\langle \mathbb{X}(t_1)\mathbb{P}(t_2)\rangle_\gamma \text{``}=\text{''} \frac{\int [D\gamma]\, e^{\frac{i}{\hbar}S_0[\gamma]}\, \mathbb{X}(t_1)\mathbb{P}(t_2)}{\int [D\gamma]\, e^{\frac{i}{\hbar}S_0[\gamma]}}$$

Although we do not know a prior the precise information about the path integral measure, we can still derive a reasonable result in comparison with the finite dimensional Gaussian integral

$$\langle \mathbb{X}(t_1)\mathbb{P}(t_2)\rangle_\gamma := i\hbar G(t_1,t_2) = \frac{i\hbar}{2}\operatorname{sgn}(t_1-t_2).$$

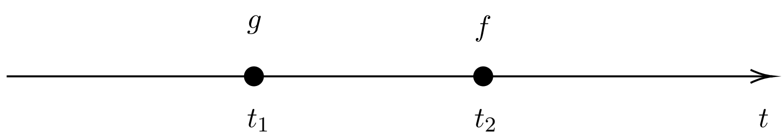
In the following, we use this formula to define our two-point function.

Note that this two-point function, or the propagator, does not depend on the precise value of $t_1, t_2$, but only on the relative position

$$\langle \mathbb{X}(t_1)\mathbb{P}(t_2)\rangle_\gamma = \begin{cases} \frac{i\hbar}{2} & t_1 > t_2 \\ 0 & t_1 = t_2 \\ -\frac{i\hbar}{2} & t_1 < t_2 \end{cases}$$

This indicates a topological nature of this model, which is indeed the case.

Now let $f(x,p)$ and $g(x,p)$ be two polynomials . We plot them on the time line as




We define a new function on the phase space by the following correlation

$$\langle f(x+\mathbb{X}(t_2), p+\mathbb{P}(t_2)) g(x+\mathbb{X}(t_1), p+\mathbb{P}(t_1))\rangle_\gamma .$$

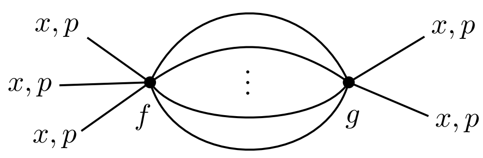
Here we perturb the variables $x, p$ by quantum fluctuations $\mathbb{X}(t), \mathbb{P}(t)$ at time $t_1$ for $g$ and at time $t_2$ for $f$. Equivalently, we can treat $x$ and $p$ as "background" shift. This correlation function depends on the background variable $\{x, p\}$ and defines a function on the phase space.

Let us apply Wick's Theorem to compute this correlation. We represent each $f$ and $g$ as a vertex whose valency is the polynomial degree

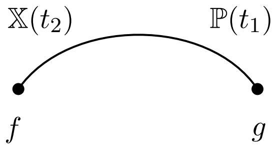

Then Wick's Theorem says

$$\langle f(x+\mathbb{X}(t_2), p+\mathbb{P}(t_2)) g(x+\mathbb{X}(t_1), p+\mathbb{P}(t_1))\rangle_\gamma$$

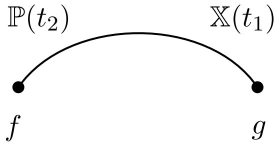
$$= \sum$$




Here for each unpaired half-edge, we assign $x$ or $p$ as in the original vertex. For paired half-edges




$$\Longrightarrow \qquad \langle \mathbb{X}(t_2)\mathbb{P}(t_1)\rangle_\gamma = \frac{i}{2}\hbar$$



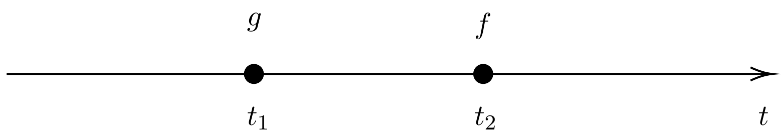

$$\Longrightarrow \qquad \langle \mathbb{P}(t_2)\mathbb{X}(t_1)\rangle_\gamma = -\frac{i}{2}\hbar$$

For a graph with $m$ propagators, there will be an automorphism group $S_m$ by permuting the edges, contributing $\frac{1}{m!}$ in the Feynman graph expansion. Thus we find

$$\begin{aligned}&\langle f(x+\mathbb{X}(t_2), p+\mathbb{P}(t_2))g(x+\mathbb{X}(t_1), p+\mathbb{P}(t_1))\rangle_\gamma\\ =&f(x,p)e^{\frac{i}{2}\hbar\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)}g(x,p) = (f *_\hbar g)(x,p)\end{aligned}$$

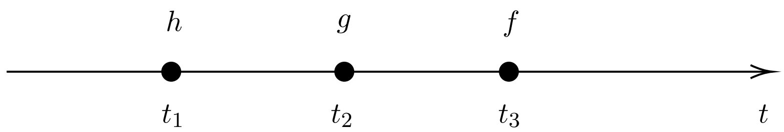
which is precisely the Moyal product.

We can think about




as inserting an operator $g$ at time $t_1$, and then inserting another operator $f$ at a later time $t_2$. The total effect is the composition of two operators, which is represented quantum mechanically by the Moyal product $f *_\hbar g$.

This picture also gives a simple explanation of the associativity of the Moyal product $*_\hbar$. Given three functions $f, g, h$, we insert them in time order




Consider the correlation

$$\langle f(x+\mathbb{X}(t_3), p+\mathbb{P}(t_3))g(x+\mathbb{X}(t_2), p+\mathbb{P}(t_2))h(x+\mathbb{X}(t_1), p+\mathbb{P}(t_1))\rangle_\gamma .$$

This value is invariant under local deformations of the time positions and only the order is relevant. By our previous computation

$$(f *_\hbar g) *_\hbar h = \lim_{t_1\to t_2^-}\lim_{t_3\to t_2^+}\langle f(x+\mathbb{X}(t_3), p+\mathbb{P}(t_3))g(x+\mathbb{X}(t_2), p+\mathbb{P}(t_2))h(x+\mathbb{X}(t_1), p+\mathbb{P}(t_1))\rangle_\gamma$$

$$f *_\hbar (g *_\hbar h) = \lim_{t_3\to t_2^+}\lim_{t_1\to t_2^-}\langle f(x+\mathbb{X}(t_3), p+\mathbb{P}(t_3))g(x+\mathbb{X}(t_2), p+\mathbb{P}(t_2))h(x+\mathbb{X}(t_1), p+\mathbb{P}(t_1))\rangle_\gamma$$

The topological nature of the above correlation implies

$$(f *_\hbar g) *_\hbar h = f *_\hbar (g *_\hbar h) .$$

## 5.5 Counter-term and Renormalization

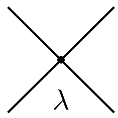
In general, Feynman graph integrals are divergent due to singularities of the propagator. To make sense of them and extract meaningful values, the method of counter-term plays an important role. We illustrate the basic idea of counter-term through the renormalization of $\phi^4$-theory on $\mathbb{R}^4$. The action is

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} d^4x\ \phi D\phi + \frac{\lambda}{4!}\int_{\mathbb{R}^4} d^4x\ \phi^4, \quad \phi \in C_c^\infty(\mathbb{R}^4)$$

where $D = -\sum_{i=1}^{4} \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}$ and $\lambda$ is the coupling constant. The propagator $D^{-1}$ is given by the Green's function

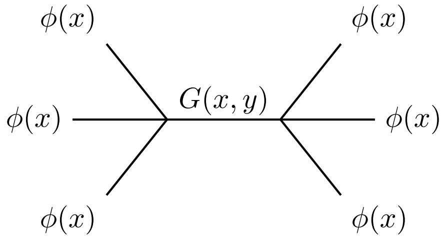
$$G(x,y) = \frac{1}{|x-y|^2}, \quad x,y \in \mathbb{R}^4.$$

The interaction term gives a quadrivalent vertex.




### Tree diagrams

Let us start with a tree diagram

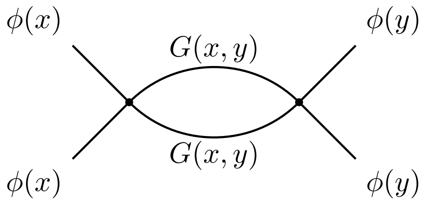



$$= \lambda^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4y\ \phi(x)^3\phi(y)^3\ G(x,y)$$
$$= \lambda^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4y\ \phi(x)^3\phi(y)^3\ \frac{1}{|x-y|^2}.$$

This is convergent for any compactly supported input $\phi$. In fact, all tree diagrams are convergent. The only potential divergence is when we allow arbitrary $\phi$ without decay properly at infinity. This is called *IR divergence* due to non-compactness of the space-time manifold, which is not as serious as UV divergence.

### One-loop diagrams

Consider a one-loop diagram



$$= \hbar\lambda^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4y\ \phi(x)^2\phi(y)^2\ G(x,y)^2$$

$$= \hbar\lambda^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4y\ \phi(x)^2\phi(y)^2\ \frac{1}{|x-y|^4}.$$

This is in general a divergent integral near $x = y$. The celebrated idea of renormalization gives a solution following a three-step strategy.

(1) Regularize $G$ by a cut-off parameter $\epsilon$;

(2) Require the coupling constant (here is $\lambda$) to depend on the cut-off $\epsilon$;

(3) Choose (1) and (2) in a proper way so that the computation has a finite value when the cut-off is removed.

A popular regularization in physics in the momentum cut-off. Let $k$ be the Fourier transform of $x$, then we can regularize the Green's function $G(x, y)$ with a cut-off $[\Lambda_0, \Lambda_1]$ of $|k|$,

$$G(x,y) = \int_{\mathbb{R}^4} d^4k\ \frac{e^{ik(x-y)}}{k^2} \rightsquigarrow \int_{\Lambda_0\leq|k|\leq\Lambda_1} d^4k\ \frac{e^{ik(x-y)}}{k^2}.$$

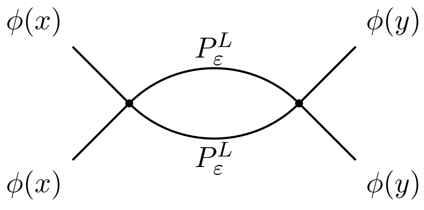
The momentum cut-off is natural and most used in physics. However it is not convenient on general manifold. We can also use the *heat kernel cut-off*. For the Laplacian $D$, the Green's function is a time integral of the **heat kernel** $e^{-tD}$,

$$G(x,y) = \int_0^\infty e^{-tD}dt.$$

We choose a UV-cut-off $\varepsilon$ and a IR cut-off $L$. Then the regularized propagator is

$$P_\varepsilon^L = \int_\varepsilon^L e^{-tD}dt = \int_\varepsilon^L \frac{dt}{(2\pi t)^2} e^{-\frac{|x-y|^2}{4t}}.$$

The regularized one-loop diagram becomes (details left to readers)

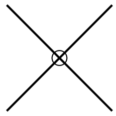



$$\begin{aligned}
&= \hbar\lambda^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4y\ \phi(x)^2\phi(y)^2\ P_\varepsilon^L(x,y)^2 \\
&= \hbar\lambda^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4y\ \phi(x)^2\phi(x+y)^2 \int_\varepsilon^L \int_\varepsilon^L \frac{dt_1}{(2\pi t_1)^2}\frac{dt_2}{(2\pi t_2)^2} e^{-\frac{|y|^2}{4}(\frac{1}{t_1}+\frac{1}{t_2})} \\
&= -\hbar\lambda^2 \frac{\log\varepsilon}{\pi^2} \int_{\mathbb{R}^4} d^4x\ \phi(x)^4 + \text{terms smooth as } \varepsilon \to 0.
\end{aligned}$$

Then we can correct the action $S$ by adding a counter-term $I^{CT}(\varepsilon)$ by

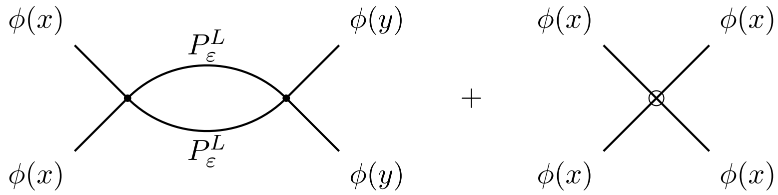
$$S + I^{CT}(\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^4} d^4x\ \phi D\phi + \frac{\lambda}{4!}\int_{\mathbb{R}^4} d^4x\ \phi^4 + \frac{\log\varepsilon}{4!\pi^2}\hbar\lambda^2\int_{\mathbb{R}^4} d^4x\ \phi^4.$$

The counter-term gives a new one-loop vertex,



$$\hbar\lambda^2\frac{\log\varepsilon}{\pi^2}.$$

Now if we evaluate the one-loop diagram with 4-external inputs, we find two contributions




and the limit $\varepsilon \to 0$ exists for this sum by construction.

From the action, the effect is to correct the coupling constant by

$$\lambda \longrightarrow \lambda + \hbar\lambda^2\frac{\log\varepsilon}{\pi^2}.$$

In fact, it can shown that by adding higher $\hbar$-order $\varepsilon$-dependent term

$$\lambda \longrightarrow \lambda + \hbar\lambda^2\frac{\log\varepsilon}{\pi^2} + \hbar^2(-) + \cdots,$$

the UV divergence of all Feynman diagrams cancel with each other! This is an example of renormalizable theory.

In general, we need to add many new terms into the counter-term $I^{CT}(\varepsilon)$ to cancel all UV divergence at all loops, even possibley infinite many different types of Lagrangians. If we only need finite terms as

$$I^{CT}(\varepsilon) = \lambda_1(\varepsilon)\int \mathscr{L}_1 + \lambda_2(\varepsilon)\int \mathscr{L}_2 + \cdots + \lambda_N(\varepsilon)\int \mathscr{L}_N$$

and adjust finite coefficients $\lambda_1(\varepsilon), \cdots, \lambda_N(\varepsilon)$ to cancel all divergences, the theory is called "**renormalizable**" in physics. Note that the counter-terms are not unique. The reason for this notion of renormalizability is that we can measure the unknown coupling constants by finite experiments, and then use the theory to make new predictions that can be further tested. If we have infinite many coupling constants, then there is no prediction ability of the theory, though it is still meaningful mathematically. The above example of $\phi^4$-theory on $\mathbb{R}^4$ is renormalizable.

# 6 BRST-BV Formalism

We discuss the BRST-BV formalism of classical and quantum gauge theories. In this section we focus on the finite dimensional toy models to illustrate the main structures. In the next section, we discuss how to incorporate renormalization.

## 6.1 Quotient and Ghost

Consider the finite dimensional toy model of gauge theory

$$f : V \longrightarrow \mathbb{C}$$

with connected Lie group $G$ acting on $V$ such that $f$ is G-invariant. Equivalently, $f$ is a function on $V/G$ and we are interested in the integral on orbit space

$$\int_{V/G} e^{\frac{1}{\hbar}f}(-).$$

There are typically two approaches.

① Find a subspace $Y \subset V$ such that points of $Y$ represent orbits of $G$ and construct a form $\xi_Y$ associated to Y such that

$$\int_V e^{if/\hbar}\delta(Y)\xi_Y(-)$$

is invariant under continuous deformation of $Y$. $Y$ is called the gauge fixing condition. The celebrated Faddeev-Popov procedure is of this type.

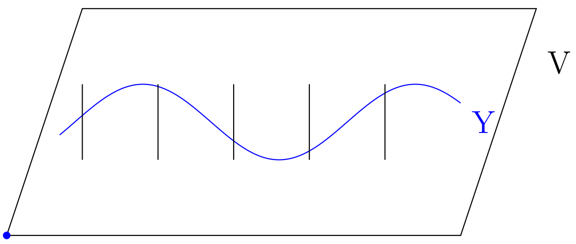



Figure 2: gauge fixing slice: $Y \subset V$

This construction works well when the quotient $V/G$ is not very singular. In general if we have "bad" orbit space $V/G$, new issues will appear. This is a typical problem in geometry, and the lesson is that we should take the derived object to resolve the singularity. This leads to a different approach.

② BRST-BV approach. Let $\mathcal{O}(V)$ represent function on $V$. Then functions

$$\mathcal{O}(V/G) = \mathcal{O}(V)^G$$

on $V/G$ can be naturally identified as $G$-invariant functions on $V$. Assume $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$. Each $u \in \mathfrak{g}$ generates a vector field $X_u$ on $V$ and $G$-invariant function can be expressed infinitesimally as

$$\mathcal{O}(V)^G = \{\varphi \in \mathcal{O}(V) | X_u(\varphi) = 0, \forall u \in \mathfrak{g}\}$$

Let $\{e_1, \cdots, e_m\}$ be a basis of $\mathfrak{g}$ and $\{c^1, \cdots, c^m\}$ be the dual basis of $\mathfrak{g}^\vee$. Let $X_\alpha := X_{e_\alpha}$. Recall the Chevalley-Eilenberg differential (BRST)

$$\begin{aligned} d_{\mathrm{CE}} : \mathcal{O}(V) &\longrightarrow \mathfrak{g}^\vee \otimes \mathcal{O}(V) \\ \varphi &\longmapsto \sum_\alpha c^\alpha \otimes X_\alpha(\varphi) \end{aligned}$$

We have a natural exact sequence

$$0 \longrightarrow \mathcal{O}(V)^G \longrightarrow \mathcal{O}(V) \longrightarrow \mathfrak{g}^\vee \otimes \mathcal{O}(V).$$

Now it is clear that the full Chevellay-Eilenberg complex

$$C^\bullet(\mathfrak{g}, \mathcal{O}(V)) = (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}(V), d_{\mathrm{CE}})$$

is a natural derived replacement of $\mathcal{O}(V)^G$. The anti-commutative variables $c^i$ in $\mathfrak{g}^\vee$ are called **ghost fields** as we have seen before. Since

$$\wedge^\bullet \mathfrak{g}^\vee = \mathrm{Sym}^\bullet(\mathfrak{g}^\vee[-1]) = \mathrm{Sym}^\bullet((\mathfrak{g}[1])^\vee),$$

we can think about $\wedge^\bullet \mathfrak{g}^\vee$ as a free polynomial ring on the (odd) graded space $\mathfrak{g}[-1]$. Thus

$$C^\bullet(\mathfrak{g}, \mathcal{O}(V)) = \text{functions on } \; V \times \mathfrak{g}[-1]$$

This is a dg space since $C^\bullet(\mathfrak{g}, \mathcal{O}(V))$ is a dg algebra with Chevalley-Eilenberg differential

$$d_{\mathrm{CE}} = \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta \frac{\partial}{\partial c^\gamma} + c^\alpha X_\alpha$$

We should think about $V \times \mathfrak{g}[-1]$ as the derived $G-$quotient of $V$.

Our next goal is to build up integration theory on $V \times \mathfrak{g}[-1]$

$$\int_{V \times \mathfrak{g}[1]} e^{f/\hbar}$$

in such a way that allows natural generalizations to $\infty-$dim case of QFT.

## 6.2 Polyvector Fields

We now move on to discuss a different view on integration. Let $M$ be an oriented n-dim manifold (or simply $\mathbb{R}^n$). Let $\Omega$ be a volume form on $M$. This defines

$$\begin{aligned} \int : C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_M \Omega f \end{aligned}$$

Geometrically, the volume form $\Omega$ leads to the divergence operator

$$\mathrm{Div} : \mathrm{Vect}(M) \longrightarrow C^\infty(M)$$

by requiring

$$\int \Omega\, \mathrm{Div}(X)\rho = -\int \Omega X(\rho), \quad \forall \rho \in C^\infty_c(M).$$

Explicitly,

$$\mathrm{Div}(X) = \frac{\mathscr{L}_X \Omega}{\Omega} = \frac{d\iota_X \Omega}{\Omega}.$$

**Definition 6.1.** We define the polyvector fields

$$\mathrm{PV}(M) = \bigoplus_k \mathrm{PV}^k(M), \quad \text{where} \quad \mathrm{PV}^k(M) := \Gamma(M, \wedge^k T_M), \quad k \geq 0$$

We write $|\alpha| = k$ for $\alpha \in \mathrm{PV}^k(M)$.

Note that differential forms are

$$\Omega^k(M) = \Gamma(M, \wedge^k T_M^\vee).$$

Thus polyvector fields can be viewed as a dual notion of differential forms. $\mathrm{PV}(M)$ carries a similar algebraic structure by exterior product

$$\mathrm{PV}^k(M) \otimes \mathrm{PV}^l(M) \xrightarrow{\wedge} \mathrm{PV}^{k+l}(M)$$

which is graded commutative

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha.$$

When $k = 1$, $\mathrm{PV}^1(M) = \mathrm{Vect}(M)$ are precisely vector fields.

The volume form $\Omega$ allows us to identify polyvector fields with forms by

$$\begin{aligned} \Gamma : \mathrm{PV}^k(M) &\xrightarrow{\Omega} \Omega^{n-k}(M) \\ \mu &\longmapsto \mu \lrcorner \Omega \end{aligned}$$

Here for $\mu = X_1 \wedge X_2 \wedge ... \wedge X_k, X_i \in \mathrm{Vect}(M)$,

$$\mu \lrcorner \Omega = \iota_{X_1} \iota_{X_2} ... \iota_{X_k} \Omega.$$

Since $\Omega$ is a volume form, $\Gamma$ gives a linear isomorphism $\mathrm{PV}^k \xrightarrow{\simeq} \Omega^{n-k}$.

We can use $\Gamma$ to transform the de Rham complex $(\Omega^\bullet(M), d)$ to $\mathrm{PV}(M)$

$$\begin{tikzcd} \Omega^0(M) \arrow[r, "d"] \arrow[d, "\simeq"'] & \Omega^1(M) \arrow[r, "d"] \arrow[d, "\simeq"'] & \cdots \arrow[r, "d"] & \Omega^n(M) \arrow[d, "\simeq"] \\ \mathrm{PV}^n(M) \arrow[r, "\Delta"'] & \mathrm{PV}^{n-1}(M) \arrow[r, "\Delta"'] & \cdots \arrow[r, "\Delta"'] & \mathrm{PV}^0(M) \end{tikzcd}$$

<u>Check</u>: $\Delta : \mathrm{PV}^1(M) = \mathrm{Vect}(M) \longrightarrow \mathrm{PV}^0(M) = C^\infty(M)$ is precisely the divergence operator associated to $\Omega$.

In general,

$$\Delta : \mathrm{PV}^k(M) \longrightarrow \mathrm{PV}^{k-1}(M)$$

can be viewed ad the generalized divergence operator on polyvector fields. Note that $d^2 = 0$ implies

$$\Delta^2 = 0.$$

In physics, $\Delta$ is called the **Batalin-Vilkovisky** (BV) operator .

To see how $\Delta$ looks like, we work locally on $U$ and choose local coordinates $\{x^1, ...x^n\}$. Assume

$$\Omega = e^{\rho(x)} dx^1 \wedge ... \wedge x^n.$$

Locally

$$\mathrm{PV}(U) = C^\infty(U)[\partial_1, ...\partial_n]$$

where $\partial_i$'s are anti-commutating $\partial_i \wedge \partial_j = -\partial_j \wedge \partial_i$.

For convenience, let us introduce Grassmann variable $\theta_i$ to represent

$$\theta_i \equiv \partial_j, \quad \theta_i\theta_j = -\theta_j\theta_i.$$

Then $\mu \in \mathrm{PV}(U)$ can be written as

$$\mu = \mu(x^i, \theta_i) = \sum_I \mu_I(x^i)\theta^I.$$

Let $\frac{\partial}{\partial\theta_i}$ be the derivative with respect to $\theta_i$ defined by

$$\frac{\partial g}{\partial \theta_i} = 0, \qquad \frac{\partial(\theta_i g)}{\partial\theta_i} = g$$

Here $g$ does not contain $\theta_i$. These operators are also anti-commutative:

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} = -\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}.$$

For example

$$\begin{aligned}
&\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}(\theta_1\theta_2) = -\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}(\theta_2\theta_1) = -\frac{\partial\theta_1}{\partial\theta_1} = -1\\
&\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1}(\theta_1\theta_2) = 1
\end{aligned}$$

**Proposition 6.2.** *Under the above assumption and notations, we have*

$$\Delta = \sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial\theta_i} + \sum_i \partial_i\rho\frac{\partial}{\partial\theta_i}.$$

*Proof:* Exercise. □

In particular, for a vector field

$$\mathfrak{X} = \sum_i V^i(x)\frac{\partial}{\partial x^i} = \sum_i V^i(x)\theta_i$$

we have

$$\mathrm{Div}(\mathfrak{X}) = \Delta\mathfrak{X} = \sum_i (\partial_i V^i + \partial_i\rho V^i).$$

This is precisely the formula of divergence.

Note that the first term $\sum \frac{\partial}{\partial x^i}\frac{\partial}{\partial\theta_i}$ looks like a Laplacian, but it is **NOT**. $\theta_i$ are odd variables. Nevertheless, $\Delta$ is sometimes called BV Laplacian, or odd Laplacian.

Recall the homological view on integration

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] \arrow[d, "\simeq"'] & \Omega^1(M) \arrow[r, "d"] \arrow[d, "\simeq"'] & \cdots \arrow[r, "d"] & \Omega^n(M) \arrow[d, "\simeq"'] \arrow[dr, "\int_{dR}"] & \\
 & & & & \mathbb{R} \\
\mathrm{PV}^n(M) \arrow[r, "\Delta"'] & \mathrm{PV}^{n-1}(M) \arrow[r, "\Delta"'] & \cdots \arrow[r, "\Delta"'] & \mathrm{PV}^0(M) \arrow[ur, dashed, "\int_{BV}"'] &
\end{tikzcd}$$

From this diagram, we find

$$\int_{dR} \sim H^n_{dR} \qquad \int_{BV} \sim H^0_\Delta$$

In quantum field theory, we have to deal with $\infty$-dim geometry, that is $n = \infty$. In this case $H^n_{dR}$ becomes meaningless, while $H^0_\Delta$ becomes better and the problem is reduced to constructing $\Delta$ in QFT. We will see how to do this in perturbative renormalization theory.

## 6.3 Batalin-Vilkovisky (BV) Algebra

**Definition 6.3.** A **BV algebra** is a pair $(A, \Delta)$ where

① $A$ is a graded commutative associative algebra: $a \cdot b = (-1)^{|a||b|} b \cdot a, \forall a, b \in A$

② $\Delta : A \longrightarrow A$ is a "2nd order" operator of $\deg = 1$ and satisfies $\Delta^2 = 0$.

Here "2nd order" means the following. Consider the failure of $\Delta$ being a derivation by the so-called "**BV bracket**"

$$\{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{|a|} a \Delta b.$$

Then $\{-, -\}$ satisfies Leibniz rule in both $a$ and $b$ components. Thus

$$\{-, -\} : A \times A \longrightarrow A$$

is of $\deg = 1$ and satisfies

① Graded symmetry

$$\{a, b\} = (-1)^{|a||b|} \{b, a\}$$

② Graded Leibniz rule

$$\{a, bc\} = (-1)^{(1+|a|)|b|} b \{a, c\} + \{a, b\} c$$

③ By construction, $\{-, -\}$ is also compatible with respect to $\Delta$

$$\Delta \{a, b\} = - \{\Delta a, b\} - (-1)^{|a|} \{a, \Delta b\}$$

If we have an additional

$$Q : A \longrightarrow A, \quad \deg = 1$$

which is a derivation, squares zero $Q^2 = 0$, and $\Delta-$compatible

$$Q\Delta + \Delta Q = 0.$$

Then the triple $\{A, Q, \Delta\}$ is called a **DGBV** algebra.

**Example 6.4.** *Let $\Omega$ be a volume form on $M$. Let $\Delta$ be the associated divergence operator on* $\mathrm{PV}(M)$*. Then*

$$(\mathrm{PV}(M), \Delta)$$

*forms a BV algebra. The associated BV bracket*

$$\{-, -\} : \mathrm{PV}^k(M) \times \mathrm{PV}^l(M) \longrightarrow \mathrm{PV}^{k+l-1}(M)$$

*does not depend on the choice of $\Omega$ and $\{-, -\}=$ SchoutenNijenhuis bracket (up to a sign) (in fact $\{\alpha, \beta\} = -(-1)^{|\alpha|} \{\alpha, \beta\}_{SN}$).*

*In local coordinates, assume*

$$\Omega = e^{\rho(x)} dx^1 ... dx^n$$

*Then*

$$\Delta = \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \theta^i} + \sum_i \partial_i \rho \frac{\partial}{\partial \theta^i}.$$

*We find*

$$\{x^i, \theta_j\} = \delta^i_j, \quad \{x^i, x^j\} = \{\theta_i, \theta_j\} = 0$$

*This is the odd analogue of Poisson bracket.*

*Remark* 6.5. This example illustrates an important viewpoint on quantization which amounts to constructing the $\infty$-dim volume/ measure.

$$\{-,-\} \xrightarrow[\text{quantization }(\Omega)]{} \Delta$$

classical quantum

**Example 6.6.** *Let $X$ be a complex manifold and $\Omega$ be a holomorphic volume form. Such a pair $(X,\Omega)$ is called Calabi-Yau manifold. We can define the complex analogue of polyvector fields by*

$$\mathrm{PV}^{k,l}(X) = \Omega^{0,l}(X, \wedge^k T_X^{1,0})$$

*Let $\Delta$ be the divergence operator associated with $\Omega$*

$$\Delta : \mathrm{PV}^{k,l} \longrightarrow \mathrm{PV}^{k-1,l}$$

*Let $\bar\partial$ be the Dolbeault differential*

$$\bar\partial : \mathrm{PV}^{k,l} \longrightarrow \mathrm{PV}^{k,l+1}$$

*Then $(PV^{-\bullet,\bullet}, \bar\partial, \Delta)$ is a DGBV. Here the degree convention is that $\mathrm{PV}^{k,l}$ has degree $l-k$. This DGBV controls the deformation theory of $X$.*

**Example 6.7.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let*

$$A = \mathrm{Sym}(\mathfrak{g}^\vee[-1]) \otimes \mathrm{Sym}(\mathfrak{g}[2]) = \mathbb{C}[e_\alpha, c^\alpha].$$

*Here*

- $\{e_\alpha\}$ *is a basis of* $\mathfrak{g}$, $\deg(e_\alpha) = -2$. *They are commuting*
$$e_\alpha e_\beta = e_\beta e_\alpha.$$
- $\{c^\alpha\}$ *is the dual basis of* $\mathfrak{g}$, $\deg(c^\alpha) = 1$. *They are anti-commuting*
$$c^\alpha c^\beta = -c^\beta c^\alpha.$$
- $\Delta = \sum_\alpha \frac{\partial}{\partial c^\alpha}\frac{\partial}{\partial e_\alpha}$, *and* $\{c^\alpha, e_\beta\} = \delta^\alpha_\beta$

*Then $(A,\Delta)$ defines a BV algebra.*

*Comparing with the above polyvector construction*

$$\begin{array}{ccc} \deg = 1 & c^\alpha \leftrightsquigarrow x^i & \deg = 0 \\ \deg = -2 & e_\alpha \leftrightsquigarrow \theta_i & \deg = -1 \end{array}$$

*We can view $A$ as polyvector fields on the odd space $\mathfrak{g}[1]$*

$$A = \mathrm{PV}(\mathfrak{g}[1]) = \mathcal{O}(\mathfrak{g}[1] \oplus \mathfrak{g}^\vee[-2])$$

**Example 6.8.** *In general, let $W$ be a graded vector space. Denote*

$$T^*W[-1] = W \oplus W^\vee[-1].$$

*Then*

$$\mathrm{PV}(W) := \mathcal{O}(T^*W[-1]) = \mathrm{Sym}(W^\vee) \otimes \mathrm{Sym}(W[1])$$

*with a canonical BV bracket $\{-,-\}$ of $\deg = 1$. In the special cases*

- $W = \mathbb{R}^n \Longrightarrow \mathrm{PV}(\mathbb{R}^n)$
- $W = \mathfrak{g}[1] \Longrightarrow \mathrm{PV}(\mathfrak{g}[1])$

*In particular, to understand $\int_{V\times\mathfrak{g}[1]}$, we will need to explore*

$$\mathrm{PV}(V \times \mathfrak{g}[1])$$

## 6.4 Classical BV Master Equation

We now discuss the so-called BV master equation, which is a fundamental relation for classical and quantum gauge theories.

**Definition 6.9** (Classical BV Master Equation)**.** Let $(A, \Delta)$ be a BV algebra, $\{-,-\}$ be the BV bracket. A degree 0 element $S_0 \in A$ is said to satisfy **Classical BV Master Equation** if

$$\{S_0, S_0\} = 0 \quad \text{(CME)}.$$

$S_0$ induced an operator on $A$ by

$$\delta = \{S_0, -\} : \ A \longrightarrow A.$$

Since the BV bracket has degree 1 and $S_0$ has degree 0, the operator $\delta$ has degree 1. And $\delta$ is a derivative by the Leibniz rule for $\{-,-\}$.

**Proposition 6.10.** *The CME* $\{S_0, S_0\} = 0$ *implies*

$$\delta^2 = 0.$$

*Proof:* By Jacobi identity

$$\big\{\{S_0, S_0\}, -\big\} = -2\{S_0, \{S_0, -\}\big\} = -2\delta^2.$$

□

**Example 6.11** (Symmetry)**.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Denote by $\{e_\alpha\}$ a basis of $\mathfrak{g}$ and $\{c^\alpha\}$ the dual basis of $\mathfrak{g}^\vee$. Let $(A, \Delta)$ be the BV algebra of polyvector fields on $\mathfrak{g}[1]$*

$$A = \mathrm{Sym}(\mathfrak{g}^\vee[-1]) \otimes \mathrm{Sym}(\mathfrak{g}[2]) = \mathbb{C}[e_\alpha, c^\alpha], \quad \deg(c^\alpha) = 1, \deg(e_\alpha = -2)$$

*and*

$$\Delta = \sum_\alpha \frac{\partial}{\partial c^\alpha} \frac{\partial}{\partial e_\alpha}, \quad \{c^\alpha, e_\alpha\} = \delta^\alpha{}_\beta.$$

*The Lie bracket on the chosen basis is*

$$[e_\alpha, e_\beta] = \sum_\gamma f^\gamma_{\alpha\beta} e_\gamma$$

*where $f^\gamma_{\alpha\beta}$'s are structure coefficients of $\mathfrak{g}$. Then the degree 0 element*

$$I = \frac{1}{2} \sum_{\alpha,\beta,\gamma} f^\gamma_{\alpha\beta} c^\alpha c^\beta e_\gamma \in A.$$

*satisfies the classical master equation*

$$\{I, I\} = 0.$$

*Let us consider the induced differential*

$$\delta = \{I, -\} : A \longrightarrow A.$$

*As $A = \mathbb{C}[c^\alpha, e_\alpha]$ is a free algebra, we only need to know how $\delta$ applies to generators. It is not hard to see that*

$$\delta(c^\gamma) = \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta$$
$$\delta(e_\alpha) = \frac{1}{2} f^\gamma_{\alpha\beta} c^\beta e_\gamma.$$

*Thus the derivation $\delta$ is given by the vector field*

$$\delta = \frac{1}{2} \sum_{\alpha,\beta,\gamma} f^\gamma_{\alpha,\beta} c^\alpha c^\beta \frac{\partial}{\partial c_\gamma} + \sum_{\alpha,\beta,\gamma} f^\gamma_{\alpha\beta} c^\beta e_\gamma \frac{\partial}{\partial e_\alpha}.$$

*If we restrict $\delta$ to $\mathbb{C}[c^\alpha] = \mathrm{Sym}(\mathfrak{g}^\vee[-1])$, $\delta = d_{\mathrm{CE}}$. Thus*

$$\big(\mathrm{Sym}(\mathfrak{g}^\vee[-1]), \delta\big) = C^\bullet(\mathfrak{g}).$$

*On the full BV algebra $A$, we have*

$$(A, \delta) = C^\bullet\big(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[2])\big)$$

*where $\mathfrak{g}$ acts on $\mathrm{Sym}(\mathfrak{g}[2]))$ via the adjoint representation.*

**Example 6.12** (Symmetry+Matter)**.** *Let $G$ be a Lie group acting on a manifold $X$ coordinated by $\{x^i\}$ and $\{e_\alpha\}$ be a basis of the Lie algebra $\mathfrak{g}$. The infinitesimal action gives a Lie algebra morphism*

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(x)$$
$$e_\alpha \longmapsto V_\alpha = V^i_\alpha(x) \frac{\partial}{\partial x^i}.$$

*Consider a $G$-invariant function*

$$f : X \longrightarrow \mathbb{C}$$

*which models a gauge invariant classical action. Consider*

$$A = \mathrm{PV}(X \times \mathfrak{g}[1]) = \mathrm{PV}(X) \otimes \mathrm{PV}(g[1]) = \mathbb{C}[x^i, \theta_i, e_\alpha, c^\alpha]$$

*where $\theta_i = \partial_i$ and $\{c^\alpha\}$ is the dual basis. We summarize the variables in the table.*

| | $c^\alpha$ | $x^i$ | $\theta_i$ | $e_\alpha$ |
|---|---|---|---|---|
| Degree | 1 | 0 | -1 | -2 |
| Physics | ghost | (matter) field | anti-field | anti-ghost |

*We can extend the function $I = \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta e_\gamma$ by incorporating $x$ and the $G$-invariant function $f$ by*

$$S_0 = f(x) + c^\alpha V^i_\alpha(x)\theta_i + \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta e_\gamma.$$

**Proposition 6.13.** *$S_0$ satisfies the classical master equation.*

*Proof:* Exercise. □

*Thus any gauge invariant action (i.e. $f$) extends naturally to a solution of the classical master equation (i.e. $S_0$). This is the universality of the BV formalism.*

*Let us now look at the induced differential*

$$\delta = \{S_0, -\}.$$

*Write $S_0 = f + J$ with*

$$J = c^\alpha V^i_\alpha(x)\theta_i + \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta e_\gamma.$$

*A few moment thought on $J$ reveals that $\{J, -\}$ is the Chevalley-Eilenberg differential on $C^\bullet\big(\mathfrak{g}, \mathrm{PV}(X) \otimes \mathrm{Sym}(\mathfrak{g}[2])\big)$. Here $\mathfrak{g}$ acts on $\mathrm{PV}(X)$ via Lie derivative and acts on $\mathrm{Sym}(\mathfrak{g}[1])$ via adjoint representation. Then*

$$\delta = d_{\mathrm{CE}} + \{f, -\}$$

*where $d_{\mathrm{CE}}$ represents the symmetry and*

$$\{f, -\} = \sum_i \partial_i f \frac{\partial}{\partial \theta_i}$$

*encodes the critical points of $f$, i.e., the equation of motion.*

## 6.5 Quantum BV Master Equation

Now we move on to discuss the quantum theory.

**Definition 6.14.** Let $(A, \Delta)$ be a BV algebra. An degree 0 element

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots \in A[[\hbar]]$$

is said to satisfy the **quantum (BV) master equation** (QME) if

$$\hbar \Delta S + \frac{1}{2}\{S, S\} = 0.$$

Here $\hbar$ is viewed as a formal variable.

In the classical limit $\hbar \to 0$, the leading $\hbar$-order of QME becomes

$$\frac{1}{2}\{S_0, S_0\} = 0$$

which is precisely the classical master equation.

**Proposition 6.15.** *The quantum master equation is equivalent to*

$$\Delta(e^{\frac{1}{\hbar}S}) = 0.$$

*Proof:* This follows from the "2nd order" property of $\Delta$. We leave the details to the reader. □

This expression gives the precise physical meaning of QME as a condition of quantum gauge consistency. In calculus, if $\omega$ is a closed differential form $d\omega = 0$, then the integral

$$\int_\gamma \omega$$

over a closed cycle $\gamma$ is invariant under continuous deformations of $\gamma$.

An analogue holds in BV theory. If $\Delta(e^{\frac{S}{\hbar}}) = 0$, then a notion of BV integration

$$\int_{\mathscr{L}} e^{\frac{S}{\hbar}}$$

is invariant under continuous deformations of $\mathscr{L}$. Here $\mathscr{L}$ is a super Lagrangian subspace related to the gauge fixing condition. Thus if $S$ satisfies QME, then

$$\int_{\mathscr{L}} e^{\frac{S}{\hbar}}$$

is invariant under continuous change of gauge fixing conditions. This is the precise form of the quantum gauge consistency.

Assume $S$ satisfies QME. Define the quantum BRST operator

$$\delta^{\hbar} = \hbar\Delta + \{S, -\}.$$

**Proposition 6.16.** *$\delta^{\hbar}$ satisfies $(\delta^{\hbar})^2 = 0$.*

*Proof:* Let $\delta_S = \{S, -\}$. Using

$$\Delta\{S, -\} = -\{\Delta S, -\} - \{S, \Delta(-)\}$$

we have

$$\Delta\delta_S + \delta_S\Delta = -\{\Delta S, -\}.$$

Thus

$$\begin{aligned}
(\delta^{\hbar})^2 &= (\hbar\Delta)^2 + \hbar(\Delta\delta_S + \delta_S\Delta) + \delta_S^2 \\
&= -\{\hbar\Delta S, -\} - \frac{1}{2}\{\{S, S\}, -\} \\
&= -\Big\{\hbar\Delta S + \frac{1}{2}\{S, S\}, -\Big\} = 0
\end{aligned}$$

□

It is clear that $\lim_{\hbar\to 0} \delta^{\hbar} = \delta = \{S_0, -\}$ gives the classical BRST operator. Thus $\delta^{\hbar}$ can be viewed as the quantization of $\delta$. The differential $\delta$ and $\delta^{\hbar}$ capture the classical and quantum gauge transformation.

**Definition 6.17.** Let $(A, \Delta)$ be a BV algebra. Assume $S = S_0 + \hbar S_1 + \cdots$ solves the quantum master equation (hence $S_0$ solves CME)

① The cohomology

$$\mathrm{Obs}^c = H^\bullet(A, \{S_0, -\})$$

is called the space of **classical observables**.

② The cohomology

$$\mathrm{Obs}^q = H^\bullet(A[[\hbar]], \hbar\Delta + \{S, -\})$$

is called the space of **quantum observables**.

They represent classical and quantum gauge invariant objects modulo gauge equivalence.

**Example 6.18** (Singularity Theory/ Landau-Ginzburg B-model)**.** *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with an isolated critical point at the origin. Consider the BV algebra $(A, \Delta)$ of polynomial holomorphic polyvector fields*

$$A = \mathbb{C}[z^i, \theta_i], \quad \theta_i\theta_j = -\theta_j\theta_i,$$
$$\Delta = \sum_i \frac{\partial}{\partial z^i}\frac{\partial}{\partial \theta_i}$$

*Then*

- *$f = f(z^i)$ solves the QME since $f$ does not depend on $\theta_i$'s*

$$\Delta\left(e^{\frac{1}{\hbar}f(z)}\right) = 0.$$

- *Classical observables*

$$Obs^c = H^\bullet(A, \{f, -\}) = Jac(f) = \mathbb{C}[z^i]/(\partial_i f)$$

*Here $Jac(f)$ is the Jacobi ring and $(\partial_i f)$ is the ideal generated by $\partial_i f$.*

- *Quantum observables*

$$Obs^q = H^\bullet(A[[\hbar]], \hbar\Delta + \{f, -\}) \simeq \Omega^n[[\hbar]]/(\hbar d + df\wedge)\ \Omega^{n-1}[[\hbar]]$$

*This is called the (formal) Brieskorn lattice which plays an important role in singularity theory.*

- *BV integration models the oscillatory integral*

$$\int_{\mathscr{L}} e^{\frac{1}{\hbar}f}\mathcal{O}$$

*where $\mathscr{L}$ is a Lefschetz thimble.*

*Both the Brieskorn lattice and the oscillatory integral are deeply related to Hodge structures. In fact, the $\hbar$-adic filtration is precisely the Hodge filtration. Thus quantum field theory can be viewed as a version of $\infty$-dim Hodge theory.*

## 6.6 Deformation-Obstruction Theory

Now we address the following natural question: starting with a solution $S_0$ of the CME, can we always quantize it to a solution $S = S_0 + O(\hbar)$ of the QME?

The answer is **NO** in general. The observable complex precisely controls this deformation theory and captures the obstruction.

Let $(A, \Delta)$ be a BV algebra. Assume $S_0$ solves the CME and $\delta = \{S_0, -\}$. We are looking for $S_1, S_2, \cdots \in A$ such that

$$S = S_0 + S_1\hbar + S_2\hbar^2 + \cdots$$

solves the QME. We analyze this order by order in $\hbar$.

### $\hbar^1$-order

The $\hbar^1$-term of the QME is

$$\hbar\Delta S_0 + \hbar\{S_0, S_1\} = 0$$

i.e. we are looking for $S_1$ solving

$$\delta S_1 = -\Delta S_0.$$

Since $\delta^2 = 0$, a necessary condition for the solvability of $S_1$ is

$$\delta(\Delta S_0) = 0.$$

This indeed holds since

$$\{S_0, \Delta S_0\} = -\frac{1}{2}\Delta\{S_0, S_0\} = 0$$

by CME. Let $\mathcal{O}_1 = \Delta S_0$. Since

$$\delta\mathcal{O}_1 = 0,$$

$\mathcal{O}_1$ represents a cohomology clas $[\mathcal{O}_1] \in H^1(A, \delta)$. The above equation says that $S_1$ can be solved if and only if $\mathcal{O}_1$ is $\delta$-exact.

**Proposition 6.19.** *$S_1$ can be solved if and only if $[\mathcal{O}_1] = 0$ in $H^1(A, \delta)$.*

Now assume $[\mathcal{O}_1] = 0$. Let $S_1$ and $\widetilde{S_1}$ be two solutions at $\hbar^1$-order. Then

$$\delta S_1 = \delta\widetilde{S_1} = -\mathcal{O}_1$$

thus $\widetilde{S^1} - S^1$ is $\delta$-closed. On the other hand, for any $J \in A_0$, the solution $S_1 + \delta J$ and $S_1$ are gauge equivalent in a suitable sense. Thus

$$\{\text{solutions of } S_1 \text{ modulo gauge equivalence}\} = \begin{cases} \varnothing & \text{if } [\mathcal{O}_1] \neq 0, \\ H^0(A, \delta) & \text{if } [\mathcal{O}_1] = 0. \end{cases}$$

The $\delta$-cohomology class $[\mathcal{O}_1]$ is called the **obstruction class** at $\hbar^1$-order.

### $\hbar^{>1}$-order

Assume we have found

$$S_{<k} := S_0 + \hbar S_1 + \cdots + \hbar^{k-1}S_{k-1}$$

which solves the QME up to order $\hbar^{k-1}$

$$\hbar\Delta S_{<k} + \frac{1}{2}\{S_{<k}, S_{<k}\} = O(\hbar^k).$$

Let us consider the problem of solving $S_k$. The above equation can be equivalently written as

$$\Delta e^{\frac{1}{\hbar}S_{<k}} = O(\hbar^{k-2})e^{\frac{1}{\hbar}S_{<k}}.$$

Let us write

$$\Delta e^{\frac{S_{<k}}{\hbar}} = (\hbar^{k-2}\mathcal{O}_k + O(\hbar^{k-1}))e^{\frac{S_{<k}}{\hbar}}$$

where $\mathcal{O}_k$ is the leading term in $O(\hbar^{k-2})$. Equivalently, we have

$$\hbar\Delta S_{<k} + \frac{1}{2}\{S_{<k}, S_{<k}\} = \hbar^k\mathcal{O}_k + O(\hbar^{k+1}).$$

**Claim.** *$\mathcal{O}_k$ is $\delta$-closed.*

*Proof:* Apply $\Delta$ to the defining equation of $\mathcal{O}_k$, we find

$$\begin{aligned}
&\Delta\big((\hbar^{k-2}\mathcal{O}_k + O(\hbar^{k-1}))e^{\frac{1}{\hbar}S_{<k}}\big) = \Delta^2 e^{\frac{1}{\hbar}S_{<k}} = 0 \\
\Longrightarrow\ &\hbar^{k-3}\big(\delta\mathcal{O}_k + O(\hbar)\big)e^{\frac{1}{\hbar}S_{<k}} = 0 \\
\Longrightarrow\ &\delta\mathcal{O}_k = 0.
\end{aligned}$$

□

The question for $S_k$ is to solve the QME at order $\hbar^k$:

$$\hbar\Delta(S_{<k} + \hbar^k S_k) + \frac{1}{2}\{S_{<k} + \hbar^k S_k, S_{<k} + \hbar^k S_k\} = O(\hbar^{k+1}).$$

This is equivalent to

$$\hbar^k\{S_0, S_k\} + \hbar^k\mathcal{O}_k = 0$$

i.e.

$$\delta S_k = -\mathcal{O}_k.$$

We are in the same situation as before. The solvability of $S_k$ is the same as asking whether $\mathcal{O}_k$ is $\delta$-exact or not. Thus

$$\{\text{solutions of } S_k \text{ modulo gauge equivalence}\} = \begin{cases} \varnothing & \text{if } [\mathcal{O}_k] \neq 0, \\ H^0(A,\delta) & \text{if } [\mathcal{O}_k] = 0. \end{cases}$$

Here $[\mathcal{O}_k] \in H^1(A,\delta)$ is the obstruction class (gauge anomaly) for solving the QME up to $\hbar^k$. The space $H^1(A,\delta)$ is the **obstruction space** and $H^0(A,\delta)$ is the **tangent space** or the **deformation space**. The above analysis leads to the following powerful theorem on the abstract nonsense of constructing quantization.

**Theorem 6.20.** *If $H^1(A,\delta) = 0$, then $S_0$ can be quantized to a solution $S = S_0 + \hbar S_1 + \cdots$ of the QME.*

# 7 Effective Theory of BV Quantization

## 7.1 (-1)-shifted Symplectic Structure

We shall show that the geometry of classical field theory and its quantization has a universal description in terms of (-1)-shifted symplectic structure. This is particularly convenient to quantize gauge theories in the BV framework.

We start with the finite dimensional toy model. Let $(V, Q, \omega)$ be a finite dimensional dg symplectic space. Here

- V is a finite dim graded space
- $Q : V \longrightarrow V$ differential, $\deg Q = 1$ and $Q^2 = 0$.
- $\omega : \wedge^2 V \longrightarrow \mathbb{R}$ non-degenerate pairing of deg$=-1$, that is,
$$\omega(a, b) = 0, \text{ unless } |a| + |b| = 1.$$
- $\omega$ is Q-compatible $Q(\omega) = 0$, i.e.,
$$\omega(Q(a), b) + (-1)^{|a|}\omega(a, Q(b)) = 0.$$

The non-degeneracy of $\omega$ leads to linear isomorphisms

$$\omega : V^\vee \xrightarrow{\sim} V[1]$$
$$\Longrightarrow \wedge^2(V^\vee) \xrightarrow{\sim} \wedge^2(V[1]) \simeq Sym^2(V)[2]$$
$$\omega \longleftrightarrow K[2]$$

Here $K = \omega^{-1} \in Sym^2(V)$ is the Poisson Kernel and

$$\deg(K) = 1, \quad Q(K) = 0.$$

We obtain a triple $(A, Q, \Delta)$ as follows

- $A = \mathcal{O}(V) := \widehat{\mathrm{Sym}}(V^\vee)$ (formal power series on V)
- $Q : A \longrightarrow A$ derivation induced dually from $Q : V \longrightarrow V$
- BV operator
$$\Delta = \Delta_K : A \longrightarrow A$$
by contracting with the Poisson Kernel $K$
$$\Delta_K : \mathrm{Sym}^m(V) \longrightarrow \mathrm{Sym}^{m-2}(V).$$
Explicitly, for $\alpha_i \in V^\vee$
$$\Delta_K(\alpha_1 \otimes \cdots \otimes \alpha_m) = \sum_{i<j} \pm\langle K, \alpha_i \otimes \alpha_j\rangle \alpha_1 \otimes \cdots \widehat{\alpha_i} \otimes \cdots \widehat{\alpha_i} \otimes \cdots \otimes \alpha_m.$$
- Since $K$ is $Q-$closed, we have
$$[Q, \Delta_K] := Q\Delta_K + \Delta_K Q = 0$$

Therefore $(A, Q, \Delta)$ defines a **DGBV**. Given such a DGBV, we can talk about

- Classical master equation:
$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0 \qquad \text{for} \quad I_0 \in A, \ \deg(I_0) = 0.$$
Then the classical BRST $\delta = Q + \{I_0, -\}$ satisfies $\delta^2 = 0$.
- Quantum master equation:
$$QI + \hbar I + \frac{1}{2}\{I, I\} = 0 \Longleftrightarrow (Q + \hbar\Delta)e^{I/\hbar} = 0, \quad \text{for} \quad I = I_0 + \hbar I_1 + ... \in A[[\hbar]].$$
Then the quantum BRST $\delta^\hbar = Q + \hbar\Delta + \{I, -\}$ satisfies $(\delta^\hbar)^2 = 0.$

### Classical Field Theory

Now we discuss the QFT situation. A classical field theory can be organized into $\infty-$dimensional (-1)-shifted dg symplectic space

$$(\mathcal{E}, Q, \omega)$$

- $\mathcal{E} = \Gamma(X, E^{\bullet})$ the space of fields. Here $E^{\bullet}$ is a graded vector bundle on $X$.
- $(\mathcal{E}, Q)$ elliptic complex

  $$\ldots \longrightarrow \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^{0} \xrightarrow{Q} \mathcal{E}^{1} \xrightarrow{Q} \ldots$$

  For example, $Q = \bar{\partial}$ or $d$.
- $\omega$: local (-1)-symplectic pairing

  $$\omega(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle, \quad \forall \alpha, \beta \in \mathcal{E}$$

  and compatible with Q.

**Example 7.1** (Chern-Simons Theory)**.** *Let $X$ be a* dim $= 3$ *manifold, and $\mathfrak{g}$ be a Lie algebra with trace pairing $Tr : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$. The space of fields is*

$$\mathcal{E} = \Omega^{\bullet}(X, \mathfrak{g})[1].$$

*The degree shifting* [1] *gives the following intepretation.*

| | $\Omega^0(X, \mathfrak{g})$ | $\Omega^1(X, \mathfrak{g})$ | $\Omega^2(X, \mathfrak{g})$ | $\Omega^3(X, \mathfrak{g})$ |
|---|---|---|---|---|
| *deg* | *-1* | *0* | *1* | *2* |
| | $c$ | $A$ | $A^{\vee}$ | $c^{\vee}$ |
| | *ghost* | *field* | *anti-field* | *anti-ghost* |

*The $Q = d$ is the de Rham differential. The $(-1)$-symplectic pairing is*

$$\omega(\alpha, \beta) = \int_X Tr(\alpha \wedge \beta), \quad \alpha, \beta \in \mathcal{E}$$

*which pairs* 0*-forms with* 3*-forms and pairs* 1*-forms with* 2*-forms.*

**Example 7.2** (Scalar Field Theory in BV formalism)**.** *The field complex $\mathcal{E}$ is*

$$C^{\infty}(M) \xrightarrow{Q = \Delta + m^2} C^{\infty}(M)$$

$$\deg = 0 \qquad\qquad \deg = 1$$

$$\phi \qquad\qquad \phi^{\vee}$$

*The $(-1)$-symplectic pairing is*

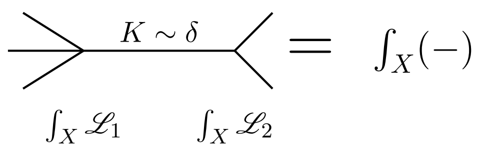
$$\omega(\phi, \phi^{\vee}) = \int_M \phi \phi^{\vee}.$$

### UV Problem

Let us now perform the same construction of DGBV algebra following the toy model. We first need the notion of "functions" $\mathcal{O}(V) = \widehat{\mathrm{Sym}}(V^\vee)$ on $V$.

- linear function: we have to take a continuous dual and
$$\mathcal{E}^\vee = \mathrm{Hom}_X(\mathcal{E}, \mathbb{R})$$
is given by distributions.
- $(\mathcal{E}^\vee)^{\otimes n} = \mathrm{Hom}_{X\times\ldots\times X}(\mathcal{E}^{\otimes n}, \mathbb{R})$ are distributions on $X^n$. Here
$$\mathcal{E}^{\otimes n} = \Gamma(X^n, E^{\boxtimes n})$$
is the completed tensor product. Thus
$$\mathrm{Sym}^m(\mathcal{E}^\vee) := (\mathcal{E}^\vee)^{\otimes m}/S_m$$
is well-defined by distributions on $X^m$. As a result, we can form
$$\mathcal{O}(\mathcal{E}) = \prod_{m\geq 0} \mathrm{Sym}^m(\mathcal{E}^\vee)$$
representing (formal) functions on $\mathcal{E}$.
- $Q : \mathcal{E} \longrightarrow \mathcal{E}$ induces duality $Q : \mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee$ on distributions, and gives rise to
$$Q : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E}).$$
- BV operator: Let $K = \omega^{-1}$ be the Poisson kernel as above. Since
$$\omega = \int \langle -, - \rangle$$
is an integral, its inverse $K$ is a $\delta$-function distributions supported on the diagonal of $X \times X$. Thus $K$ is NOT a smooth element in $\mathrm{Sym}^2(\mathcal{E})$, but a distributional section. As a result, the naive BV operator
$$\Delta_K : \mathrm{Sym}^m(\mathcal{E}^\vee) \to \mathrm{Sym}^{m-2}(\mathcal{E}^\vee)$$
is ill-defined since we can not pair two distributions. This is essentially the Ultra-Violet problem. Renormalization is needed in the quantum theory!

Before we move on to discuss the issue of renormalization, we first explain that the classical theory is actually well-behaved. Let $\mathcal{O}_{loc}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E})$ denote the subspace of local functionals, i.e., those by integrals of lagrangian densities
$$\mathcal{O}_{loc}(\mathcal{E}) = \left\{ \int_X \mathscr{L}(\ldots) \right\}$$
Although the BV operator $\Delta_K$ is ill-defined, the associated BV bracket $\{-,-\}$ is actually well-defined on local functionals.
$$\{-,-\} : \mathcal{O}_{loc}(\mathcal{E}) \otimes \mathcal{O}_{loc}(\mathcal{E}) \longrightarrow \mathcal{O}_{loc}(\mathcal{E})$$




$$\Longrightarrow \begin{cases} \text{CME makes sense for local functions} \\ \text{QME needs renormalization} \end{cases}$$

**Example 7.3** (Chern-Simons theory). $\mathcal{E} = \Omega^\bullet(X, \mathfrak{g})[1]$

| | $\Omega^0(X,\mathfrak{g})$ | $\Omega^1(X,\mathfrak{g})$ | $\Omega^2(X,\mathfrak{g})$ | $\Omega^3(X,\mathfrak{g})$ |
|---|---|---|---|---|
| *deg* | *-1* | *0* | *1* | *2* |
| | $c$ | $A$ | $A^\vee$ | $c^\vee$ |
| | *ghost* | *field* | *anti-field* | *anti-ghost* |

*Let* $\mathcal{A} = C + A + A^\vee + C^\vee \in \mathcal{E}$ *denote the master field collecting all components. Then the BV Chern-Simons action is*

$$CS[\mathcal{A}] = \int_X Tr(\frac{1}{2}\mathcal{A} \wedge d\mathcal{A} + \frac{1}{6}\mathcal{A} \wedge [\mathcal{A}, \mathcal{A}]).$$

*This takes the same form as ordinary Chern-Simons except that we have expand* $\mathcal{A}$ *to get terms containing different components. The first quadratic term is denoted by* $S_{free}$*, the free part. The second cubic term is denoted by* $I$*, the interaction part.*

**Proposition 7.4.** *CS satisfies the following classical master equation*

$$\{CS, CS\} = 0.$$

*Proof:* This follows from the general argument that classical gauge theory is organized into a solution of classical master equation. □

*If we separate the free part and interaction*

$$CS = S_{free} + I$$

*It is easy to see that*

$$\{S_{free}, -\} = d\ (= Q)$$

*which precisely correspons to the de Rham differential. Thus*

$$\begin{aligned} &\{CS, CS\} = 0 \\ \Leftrightarrow &\frac{1}{2}\{S_{free}, S_{free}\} + \{S_{free}, I\} + \frac{1}{2}\{I, I\} = 0 \\ \Leftrightarrow &QI + \frac{1}{2}\{I, I\} = 0 \end{aligned}$$

*This is precisely the form of classical master equation in our DGBV.*

## 7.2 Effective Renormalization

Assume we have a classical field theory $(\mathcal{E} = \Gamma(X, E^\bullet), Q, \omega)$ with classical local functional $I_0$ (interaction) satisfying CME

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0.$$

As we explained before, quantization asks for

$$I_0 \longrightarrow I = I_0 + \hbar I_1 + \hbar I_2 + ... \in \mathcal{O}(\mathcal{E})[[\hbar]]$$

satisfying QME

$$\text{“}QI_0 + \frac{1}{2}\{I_0, I_0\} + \hbar\Delta I = 0\text{”}.$$

<u>Problem:</u> $\Delta I$ is <u>NOT</u> well-defined. In the following, we explain Costello's homotopic renormalization theory to solve this problem.

### Toy Model

To motivate the construction, let us look back again at the toy model where $(V, Q, \omega)$ is finite dim (-1)-shifted dg symplectic space. The Poisson kernel

$$K_0 \in \mathrm{Sym}^2(V)$$

has $\deg(K_0) = 1$ and satisfies $Q(K_0) = 0$. This allows us to construction the BV operator $\Delta_0$ by contracting with $K_0$ and obtain the DGBV triple $(A, Q, \Delta_0)$.

Let us now consider the change of $K_0$ by chain homotopy. Let

$$P \in \mathrm{Sym}^2(V), \quad \deg(P) = 0.$$

Define

$$K_P = K_0 + Q(P) = K_0 + (Q \otimes 1 + 1 \otimes Q)P.$$

We again have

① $K_P \in \mathrm{Sym}^2(V), \deg(K_P) = 1$

② $Q(K_P) = 0$

Thus we can construct a new BV operator

$$\Delta_P = \text{contraction with } K_P$$

such that $(\mathcal{O}(V), Q, \Delta_P)$ forms a new DGBV.

To see the relation with the original DGBV, denote

$$\partial_P : \mathrm{Sym}^m(V^\vee) \longrightarrow \mathrm{Sym}^{m-2}(V^\vee)$$

where $\partial_P$ is a 2nd order operator of contracting with $P \in \mathrm{Sym}^2(V)$

**Proposition 7.5.** *The following diagram commutes*

$$\begin{tikzcd}
\mathcal{O}(V)[[\hbar]] \arrow[r, "e^{\hbar\partial_P}"] \arrow[d, "Q+\hbar\Delta_0"] & \mathcal{O}(V)[[\hbar]] \arrow[d, "Q+\hbar\Delta_P"] \\
\mathcal{O}(V)[[\hbar]] \arrow[r, "e^{\hbar\partial_P}"] & \mathcal{O}(V)[[\hbar]]
\end{tikzcd}$$

*i.e.*

$$(Q + \hbar\Delta_P)e^{\hbar\partial_P} = e^{\hbar\partial_P}(Q + \hbar\Delta_0).$$

*Proof:* This follows from the chain homotopy relation $K_P = K_0 + Q(P)$. □

**Corollary 7.6.** *Assume $I \in \mathcal{O}(V)[[\hbar]]$ satisfies QME*

$$(Q + \hbar\Delta_0)e^{I/\hbar} = 0$$

*in the DGBV $(\mathcal{O}(V), Q, \Delta_0)$. Then $\tilde{I} \in \mathcal{O}(V)[[\hbar]]$ satisfies QME*

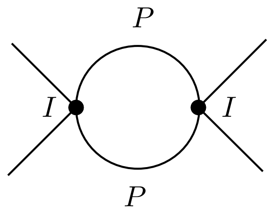
$$(Q + \hbar\Delta_P)e^{\tilde{I}/\hbar} = 0$$

*in the DGBV $(\mathcal{O}(V), Q, \Delta_P)$. Here $\tilde{I}$ is related to $I$ by*

$$e^{\tilde{I}/\hbar} = e^{\hbar\partial_P}e^{I/\hbar}$$

As we have seen in the discussion of Feynman graph, the operator $e^{\hbar\partial_P}$ plays the role of integration with respect to the Gaussian measure. The relation $e^{\tilde{I}/\hbar} = e^{\hbar\partial_P}e^{I/\hbar}$ can be read via Wick's Theorem as

$$\tilde{I} = \sum_{\text{connected graphs}} \left( \quad \right) .$$




Here $I$ serves as vertices and $P$ for the propagator. Thus, Feynman diagrams give the required chain homotopy between different DGBV's.

## Back to QFT

Now consider the QFT set-up $(\mathcal{E} = \Gamma(X, E^\bullet), Q, \omega)$. The problem is that the Poisson kernel $K_0 = \omega^{-1}$ is a $\delta$-function distribution which leads to a singular BV operator. Nevertheless we know $K_0$ is $Q$-closed

$$Q(K_0) = 0.$$

Costello's approach: Using elliptic regularity

$$H^\bullet(\text{Distribution}, Q) = H^\bullet(\text{Smooth}, Q).$$

Thus we can replace $K_0$ by a smooth object in its $Q$-cohomology class by

$$K_0 = K_r + Q(P_r).$$

Here $K_r$ is smooth while $P_r$ (called parametrix) is singular. Define

$$\Delta_r : \text{BV operator associated with } K_r$$

Since $K_r \in \mathrm{Sym}^2(\mathbb{E})$ is now smooth,

$$\Delta_r \curvearrowright \mathcal{O}(\mathcal{E}) \text{ is well-defined.}$$

**Definition 7.7.** The DGBV $(\mathcal{O}(\mathcal{E}), Q, \Delta_r)$ will be called the **effective DGBV** with respect to the regularization $r$.

Let $r'$ be another regularization with parametrix $P_{r'}$

$$K_0 = K_{r'} + Q(P_{r'}).$$

Then the two regularized Poisson kernel differs by a chain homotopy

$$K_{r'} - K_r = Q(P_r^{r'})$$

where $P_r^{r'} \in \mathrm{Sym}^2(\mathcal{E})$ is smooth. Let

$$\partial_{P_r^{r'}} : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$$

be the 2nd order operator of contracting with the smooth kernel $P_r^{r'}$.

The same argument as in the toy model gives the chain homotopy

$$\big(\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta_r\big) \xrightarrow[\substack{\uparrow \\ \text{Homotopy RG flow}}]{\exp\left(\hbar\partial_{P_r^{r'}}\right)} \big(\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta'_r\big)$$

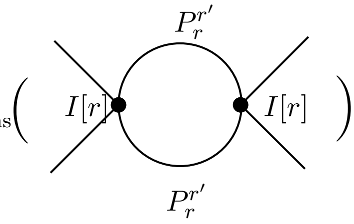
**Definition 7.8** (Costello)**.** An effective perturbative quantization of $I_0$ (which satisfies CME) a family

$$I[r] \in \mathcal{O}(\mathcal{E})[[\hbar]]$$

for each choice of regularization $r$ satisfying

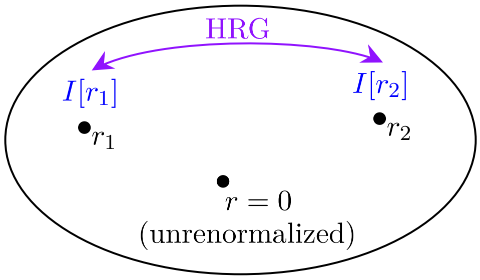
- Effective QME
$$(Q + \hbar\Delta_r)e^{I[r]/\hbar} = 0.$$
- Homotopy RG flow
$$e^{I[r']/\hbar} = e^{\hbar\partial_{P_r^{r'}}} e^{I[r]/\hbar}$$
which is equivalent to the Feynman diagram expansion

$$I[r'] = \sum_{\text{connected graphs}}$$




- $I[r]$ is asymptotic local when $r \longrightarrow 0$ and has the classical limit
$$\lim_{r\to 0} I_0[r] = I_0$$

Here is a pictue to illustrate what is going on. The situation is very similar to how residue is defined in algebraic geometry: we need to perturb the singularity and define residue at the deformed configuration, and show that all local deformations give the same answers. Here we use all "nearby" regularizations to define the unrenormalized point.




In practice, here are steps for constructing perturbative quantization.

① Construct counter-term $I^\varepsilon \in \hbar\mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$ such that

$$e^{I[r]^{Naive}/\hbar} := \lim_{\varepsilon\to 0}(e^{\hbar P_\varepsilon^r} e^{(I_0+I^\varepsilon)/\hbar}) \ \textit{exists}$$

Then this naive family $\left\{I[r]^{Naive}\right\}_r$ satisfies HRG by construction.

② The choice of counter-terms is not unique. We need to further correct $I^\varepsilon$ such that $e^{I[r]/\hbar}$ satisfies QME.

① is always possible by the method of counter-term. ② is NOT always possible: obstruction may exist which is called "**gauge anomaly**" in physics terminology. Just as we have is the discussion of deformation-obstruction theory, similar story applies here and the gauge anomaly lies in

$$H^1(\mathcal{O}_{loc}(\mathcal{E}), Q + \{I_0, -\}).$$

## 7.3 Heat Kernel Regularization

There are many ways of regularizations. One method that connects to geometry is the heat kernel regularization. Typically, fixing a choice of metric, we have

- the adjoint of the elliptic operator $Q : \mathcal{E} \to \mathcal{E}$, denoted as $Q^\dagger : \mathcal{E} \to \mathcal{E}$,
- a generalized Laplacian, $\left[Q, Q^\dagger\right] = QQ^\dagger + Q^\dagger Q$.

Thus we can define a heat operator $e^{-L[Q,Q^\dagger]}$ for $L > 0$. Let $K_L \in \mathrm{Sym}^2(\mathcal{E})$ be the kernel of the heat operator by

$$\left(e^{-L[Q,Q^\dagger]}\alpha\right)(x) = \int dy\, \langle K_L(x,y), \alpha(y)\rangle \quad \text{for } \alpha \in \mathcal{E}.$$

Here $\langle -, - \rangle$ is the pairing from $\omega$. Note that

- $K_0 = \lim\limits_{L\to 0} K_L$ is the $\delta$-function distribution $\omega^{-1}$,
- $K_L \in \mathrm{Sym}^2(\mathcal{E})$ is smooth for $L > 0$.

Let $P_L$ be the kernel of the operator $\int_0^L dt\, Q^\dagger e^{-t[Q,Q^\dagger]}$. Explicitly, we have

$$P_L = \int_0^L dt \left(Q^\dagger \otimes 1\right) K_t.$$

The operator equation

$$\left[Q, \int_0^L dt\, Q^\dagger e^{-t[Q,Q^\dagger]}\right] = \int_0^L dt \left[Q, Q^\dagger\right] e^{-t[Q,Q^\dagger]} = 1 - e^{-L[Q,Q^\dagger]}$$

can be translated into the kernel equation:

$$K_0 - K_L = (Q \otimes 1 + 1 \otimes Q) P_L$$

or simply written as

$$K_0 - K_L = Q\left(P_L\right).$$

We can use $K_L$ to define the effective QME.

For $0 < \varepsilon < L$, similarly the operator equation is

$$\left[Q, \int_\varepsilon^L dt\, Q^\dagger e^{-t[Q,Q^\dagger]}\right] = e^{-\varepsilon[Q,Q^\dagger]} - e^{-L[Q,Q^\dagger]}$$

or

$$K_\varepsilon - K_L = (Q \otimes 1 + 1 \otimes Q) P_\varepsilon^L,$$

where $P_\varepsilon^L = \int_\varepsilon^L dt \left(Q^\dagger \otimes 1\right) K_t$ is called the *regularized propagator*. Now we can use $P_\varepsilon^L$ to connect the effective QME at $\varepsilon$ with the effective QME at $L$ via the HRG.


*Remark* 7.9. $P_0^\infty = \int_0^\infty dt \left(Q^\dagger \otimes 1\right) K_t$ is the *full propagator*. At $t = 0$, one will encounter ultraviolet (UV) divergence since there exists a singularity for the full propagator. On a non-compact manifold, one will encounter infrared (IR) divergence at $t = \infty$.

Consider the case when $X$ is compact. Let

$$\mathbb{H} = \left\{ \varphi \in \mathcal{E} \mid \left[Q, Q^\dagger\right] \varphi = 0 \right\} = \left\{ \varphi \in \mathcal{E} \mid Q\varphi = Q^\dagger \varphi = 0 \right\} \simeq H^\bullet\left(\mathcal{E}, Q\right).$$

$\mathbb{H}$ is called the space of **harmonics** (or the **zero modes**), which is a finite-dimensional space (by Hodge theory). Then we have

$$\begin{array}{c} \infty - \text{dimensional } (-1) - \text{symplectic geometry } (\mathcal{E}, Q, \omega) \\ \Big\downarrow{\scriptstyle L\to\infty} \\ \text{finite-dimensional } (-1) - \text{symplectic geometry } (\mathbb{H}, \omega_H = \omega|_{\mathbb{H}}) \end{array}$$

The BV operator $\Delta_H$ associated with $\omega_H^{-1}$ is $\Delta_H = \Delta_\infty$. On the complete story of BV quantization, as depicted in the following diagram,

$$\overset{0}{\bullet}\!\!\xrightarrow{\qquad\qquad\qquad\qquad\qquad} L = \infty$$

$I[\infty]$ solves the QME for $(\mathscr{O}(\mathbb{H}), \Delta_H)$ at $L = \infty$. This is an interesting point where we will find some finite-dimensional geometric data.

## 7.4 UV Finite Theory

In the BV formalism, the classical master equation

$$QI_0 + \frac{1}{2}\{I_0, I_0\}$$

is quantized to the quantum master equation

$$\text{“}QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0\text{”}.$$

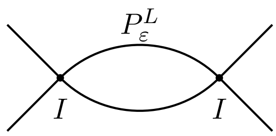
As we explained as above, this naive quantum master equation is ill-defined for local $I \in \mathcal{O}_{loc}(\mathcal{E})$, and we have to use regularization to formulate the renormalized quantum master equation

$$QI[r] + \hbar\Delta_r I[r] + \frac{1}{2}\{I, I\}_r = 0.$$

If the effective action at regularization $r$ can be found as

$$e^{I[r]/\hbar} = \lim_{\varepsilon\to 0} e^{\hbar\partial_{P_\varepsilon^r}} e^{I/\hbar}$$

for $I \in \mathcal{O}_{loc}(\varepsilon)[[\hbar]]$, i.e., the $\varepsilon$-dependent counter-term is **<u>NOT</u>** needed, we say the theory is UV finite. That is, for all regularized Feynman diagrams

$\lim_{\varepsilon\to 0}$ 

exist.

In this way, we can consider the limit

$$I[r] \to I, \qquad r \to 0,$$

and the $r \to 0$ limit of the renormalized quantum master equation

$$QI + \frac{1}{2}\{I, I\} + \cdots = 0$$

will have a local expression that deforms the CME. It is expected to be the form

$$l_1^\hbar I + \frac{1}{2} l_2^\hbar (I, I) + \frac{1}{3!} l_3^\hbar (I, I, I) + \cdots = 0$$

where $\{l_1^\hbar, l_2^\hbar, \cdots\}$ defines a quantized $L_\infty$-algebra which can be viewed as traded from $\Delta$ in terms of the renormalization procedure.

There are two main classes of UV finite theories.

① Topological theory (Chern-Simons type) where $\mathcal{E}$ is of the form of de Rham complex. See Kontsevich [25] and Axelrod-Singer [2].

② Holomorphic theory where $\mathcal{E}$ is of the form of Dolbeault complex. See Li [28, 29] for $\dim_{\mathbb{C}} = 1$ and Wang [39] for higher dim.

It is an extremely interesting question to figure out $\{l_1^\hbar, l_2^\hbar, \cdots\}$ in these examples. In Section 9 and Section 10, we explain the simplest example in each of these two categories to illustrate their rich structures.

# 8 Factorization Algebra

## 8.1 Prefactorization Algebra

Let $M$ be a topological space. We denote $\mathrm{Open}(M)$ to be the category where objects are open subsets of $M$ and morphisms are open embeddings

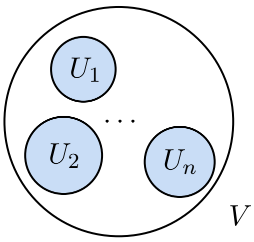
$$U \longrightarrow V.$$

**Definition 8.1.** A prefactorization algebra $\mathscr{F}$ on a topological space $M$ valued in $k$-vector sapce is a functor

$$\begin{aligned} \mathcal{F} : \mathrm{Open}(M) &\longrightarrow \mathrm{Vect}_k \\ U &\longmapsto \mathcal{F}(U) \end{aligned}$$

together with maps

$$m_V^{U_1,\cdots,U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V)$$

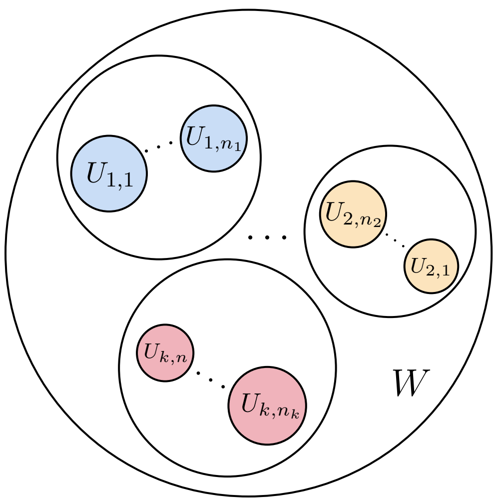



for every finite collection of pairwise disjoint open subsets $U_i \subset V$ (i.e. $U_1 \sqcup \cdots \sqcup U_n \subset V$) such that the following properties hold

① For $U \subset V$, $m_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is the morphism of the functor $\mathcal{F}$.

② Compatibility: If $U_{i,1} \sqcup \cdots \sqcup U_{i,n_i} \subset V_i$ and $V_1 \sqcup \cdots \sqcup V_k \subset W$, then the following diagram commutes

$$\begin{tikzcd} \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_{i,j}) \arrow[rr] \arrow[dr] & & \bigotimes_{i=1}^{k} \mathcal{F}(V_i) \arrow[dl] \\ & \mathcal{F}(W) & \end{tikzcd}$$



By definition, $\mathcal{F}(\varnothing)$ on the emptyset is a commutative algebra. We say $\mathcal{F}$ is a unital prefactorization algebra if $\mathcal{F}(\varnothing)$ is a unital commutative algerba.

**Example 8.2.** *Let $A$ be a unital associative algebra. We define a prefactorization algebra $A^{fact}$ on $\mathbb{R}$ as follows.*

- *For open interval $(a,b)$, we set*

$$A^{fact}\big((a,b)\big) = A.$$

- *For open set $\mathcal{U} = \bigsqcup_{j\in J} I_j$ where $I_j$'s are open intervals, we set*

$$A^{fact}(\mathcal{U}) = \operatorname*{colim}_{\textit{finite } K\subset J} \bigotimes_{\alpha\in K} A.$$

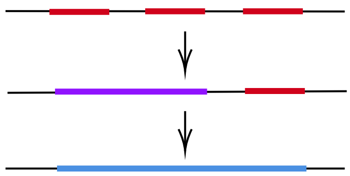
  *Here the colimit is taken with respect to the direct system*

$$\bigotimes_{K} A \longrightarrow \bigotimes_{\tilde{K}} A$$

  *for any finite indices $K \subset \tilde{K}$ by tensoring the unit $1 \in A$*

$$\bigotimes_{\alpha\in K} a_k \longrightarrow \Big(\bigotimes_{\alpha\in K} a_k\Big) \otimes \Big(\bigotimes_{\beta\in\tilde{K}-K} 1\Big).$$

- *The structure maps $\{m_V^{U_1,\cdots,U_n}\}$ are given by multiplication in $A$.*



$$\begin{array}{c} a_1\otimes a_2\otimes a_3 \in A\otimes A\otimes A \\ \downarrow \\ a_1a_2\otimes a_3 \in A\otimes A \\ \downarrow \\ a_1a_2a_3 \in A \end{array}$$

**Example 8.3.** *Let $A$ be a unital associative algebra. Let $M$ be a left $A$-module and $N$ be a right $A$-module. We can associative a prefactorization on*

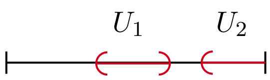
$$X = [0,1]$$

*as follows.*

- *For $U = \bigsqcup_{j\in J} I_j \subset (0,1)$ where $I_j$'s are open intervals, we associate*

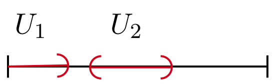
$$\mathcal{F}(U) = \operatorname*{colim}_{\textit{finite } K\subset J} \bigotimes_{\alpha\in K} A.$$

- *For $U = U_1 \sqcup U_2$ where $U_1 \subset (0,a)$, $U_2 = (a,1]$*



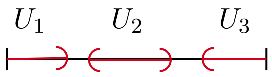

$$\mathcal{F}(U) = \mathcal{F}(U_1)\otimes M.$$

- *For $U = U_1 \sqcup U_2$ where $U_2 \subset (b,1)$, $U_1 = [0,b)$*




$$\mathcal{F}(U) = N\otimes \mathcal{F}(U_2).$$

- *For $U = U_1 \sqcup U_2 \sqcup U_3$ where $U_1 = [0, a)$, $U_2 \subset (a, b)$, $U_3 = (b, 1]$*




$$\mathcal{F}(U) = N \otimes \mathcal{F}(U_2) \otimes M.$$

**Example 8.4.** *Let $E \to X$ be a vector bundle on $X$. For each open set $U \subset X$, we associate*

$$\mathcal{F}(U) = \mathrm{Sym}\left(\Gamma_c(U, E)\right).$$

*Here $\Gamma_c(U, E)$ are compactly supported sections of $E$ inside $U$. Using extension by zero, we have*

$$\Gamma_c(U, E) \hookrightarrow \Gamma_c(V, E)$$

*for $U \subset V$, which induces*

$$\mathcal{F}(U) \hookrightarrow \mathcal{F}(V).$$

*For finite pairwise disjoint $U_1 \sqcup \cdots \sqcup U_n$,*

$$\Gamma_c(\bigsqcup_i U_i, E) = \bigoplus_{i=1}^n \Gamma_c(U_i, E),$$

*and*

$$\mathrm{Sym}\left(\Gamma_c(\bigsqcup_i U_i, E)\right) = \mathrm{Sym}\left(\bigoplus_{i=1}^n \Gamma_c(U_i, E)\right) = \bigotimes_{i=1}^n \mathrm{Sym}\left(\Gamma_c(U_i, E)\right).$$

*This shows*

$$\mathcal{F}(\bigsqcup_{i=1}^n U_i) = \bigotimes_{i=1}^n \mathcal{F}(U_i)$$

*and so we obtain a prefactorization algebra.*

**Definition 8.5.** A morphism of prefactorization algebras

$$\phi : \mathcal{F} \longrightarrow \mathcal{G}$$

consists of maps $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open $U \subset M$ which are compatible with structure maps, i.e., the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \arrow[r, "\phi_{U_1} \otimes \cdots \otimes \phi_{U_n}"] \arrow[d] & \mathcal{G}(U_1) \otimes \cdots \otimes \mathcal{G}(U_n) \arrow[d] \\
\mathcal{F}(V) \arrow[r, "\phi_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

Thus prefactorizations algebras form a category.

## 8.2 Factorization Algebra

**Definition 8.6.** Let U be an open set in M. An open cover $\mathfrak{U} = \{U_i \subset U\}_{i \in I}$ of U is called a Weiss cover if for any finite collection of points $\{x_1, \ldots, x_k\}$ in U, there is an open set $U_j \in \mathfrak{U}$ such that $\{x_1, \ldots, x_k\} \subseteq U_j$.

**Proposition 8.7.** *If $\mathfrak{U} = \{U_i\}$ is a Weiss cover of $U$, then for any $n > 0$,*

$$\{U_i^n\}_{i \in I}$$

*is an open cover of $U^n = U \times ... \times U$.*

*Proof:* For any $(x_1, \cdots, x_n) \in U^n$, there exists $U_i$, s.t.

$$\{x_1, \cdots, x_k\} \subset U_i \Longrightarrow (x_1, \cdots, x_n) \in U_i^n$$

$\square$

**Example 8.8.** *The collection $\{U$ with finite points deleted$\}$ is a Weiss cover of $U$.*

**Definition 8.9.** A prefactorization algebra $\mathcal{F}$ on M is called a factorization algebra if for every open $U \subset M$ and every Weiss cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of $U$, the following sequence is exact:

$$\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \longrightarrow \bigoplus_{k} \mathcal{F}(U_k) \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

Recall that a sheaf $\mathcal{E}$ of $k$-vector spaces on M is a functor

$$\mathcal{E} : \mathrm{Open}(M)^{op} \longrightarrow \mathrm{Vect}_k$$

satisfying the gluing condition: for any open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of $U$,

$$0 \longrightarrow \mathcal{E}(U) \longrightarrow \bigoplus_{i \in I} \mathcal{E}(U_i) \longrightarrow \bigoplus_{i,j} \mathcal{E}(U_i \cap U_j)$$

forms an exact sequence. This means that a section $s$ of $\mathcal{E}(U)$ can be uniquely specified as sections $s_i$ on each $U_i$ that agree on the overlaps $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Dualizing the arrows, we arrive at the notion of cosheaf.

Thus a prefactorization algebra $\mathcal{F}$ is a factorization algebra if it is a cosheaf with respect to the Weiss topology.

**Example 8.10.** *Let $E \longrightarrow X$ be a vector bundle on $X$. We denote $\mathcal{E}_c$ the cosheaf*

$$\mathcal{E}_c(U) = \Gamma_c(U, E)$$

*which assigns compactly sections on $U$. Denote*

$$\mathcal{F} = \mathrm{Sym}(\mathcal{E}_c)$$

*the prefactorization algebra where*

$$\mathcal{F}(U) = \mathrm{Sym}(\mathcal{E}_c(U))$$

*We show that $\mathcal{F}$ is a factorization algebra. Let $U \subset M$ be open and*

$$\mathfrak{U} = \{U_i \subset U\}_{i \in I}$$

*be a Weiss cover of $U$. We need to show*

$$\bigoplus_{i,j \in I} \mathrm{Sym}(\mathcal{E}_c(U_i \cap U_j)) \longrightarrow \bigoplus_{i \in I} Sym(\mathcal{E}_c(U_i)) \longrightarrow \mathrm{Sym}(\mathcal{E}_c(U)) \longrightarrow 0$$

*is exact. Observe that*

$$\mathcal{E}_c(U)^{\hat{\otimes} m} = \mathcal{E}_c^{\boxtimes m}(U^m)$$

*where $\mathcal{E}_c^{\boxtimes m}$ is the cosheaf on $U^m$ of compactly supported sections of $E^{\boxtimes m}$. Decomposing into each tensor component, we need to show*

$$\bigoplus_{i,j \in I} \mathcal{E}_c^{\boxtimes m}(U_i \cap U_j)) \longrightarrow \bigoplus_{i \in I} \mathcal{E}_c^{\boxtimes m}(U_i)) \longrightarrow \mathcal{E}_c^{\boxtimes}(U) \longrightarrow 0$$

*is exact. Since $\{U_i\}_{i \in I}$ is a Weiss cover, $\{U_i^m\}_{i \in I}$ form an open cover of $U$ and*

$$(U_i \cap U_j)^m = U_i^m \cap U_j^m$$

*Thus the exactness follows from the cosheaf property of $\mathcal{E}_c^{\boxtimes m}$ on $U^m$.*

More generally, we can define prefactorization algebra $\mathcal{F}$ valued in a cochain complex. Thus $\mathcal{F}(U)$ is a cochain complex for each $U \subset M$. For any cover $\mathfrak{U} = \{U_i \subset U\}_{i \in I}$, we can similarly define Čech cochain

$$C^p(\mathfrak{U}, \mathcal{F}) := \bigoplus_{j_0,,,j_p \ distinct} \mathcal{F}(U_{j_0} \cap ... \cap U_{j_p})$$

and the Čech complex as the total complex of

$$\check{\mathrm{C}}(\mathfrak{U}, \mathcal{F}) := \bigoplus_{p=0}^{\infty} C^p(\mathfrak{U}, \mathcal{F})$$

**Definition 8.11.** A homotopy factorization algebra is a prefactorization algebra $\mathcal{F}$ on M such that for every open $U \subset X$ and Weiss cover $\mathfrak{U}$ of U, the natural map

$$\check{\mathrm{C}}(\mathfrak{U}, \mathcal{F}) \longrightarrow \mathcal{F}(U)$$

is a quasi-isomorphism. Thus, $\mathcal{F}$ is a homotopy cosheaf with respect to the Weiss topology.

In Costello-Gwillian [11], it is shown that perturbative BV quantization leads to (homotopy) factorization algebras. Such factorization algebra describes "quantum observables", and the factorization property is the local-to-global property.

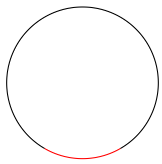
**Example 8.12.** *As an example, consider topological quantum mechanics on the phase space* $\mathbb{R}^{2n}$ *(see Section 9 for details). The space of fields is*

$$\mathcal{E} = \Omega^{\bullet}(S^1) \otimes \mathbb{R}^{2n}.$$

*Given* $U \subset S^1$*, observables* $\mathrm{Obs}(U)$ *inside* $U$ *are functions on*

$$\mathcal{E}(U) = \Omega^{\bullet}(U) \otimes \mathbb{R}^{2n}.$$

- *Local observable: for* $U$ *a small interval,*



$U$

$\Omega^{\bullet}(U) \simeq \mathbb{R}$ *are quasi-isomorhic by Poincaré's Lemma. Thus*

$$\mathrm{Obs}(U) = O(\mathcal{E}(U)) \simeq \mathscr{O}(\mathbb{R}^{2n}) = \mathbb{R}[x^i, p_i]$$

- *Global observable: for* $U = S^1$ *which has a nontrivial homology*

$$\Omega(S^1) \simeq H^{\bullet}(S^1) = \mathbb{R} \oplus \mathbb{R}[-1]$$

*Thus global observables on* $S^1$ *can be modeled by*

$$\mathrm{Obs}(S^1) \simeq \mathscr{O}(\Omega(S^1) \otimes \mathbb{R}^{2n}) \simeq \mathscr{O}(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}[-1]) \simeq \mathbb{R}[x^i, p_i, dx^i, dp_i].$$

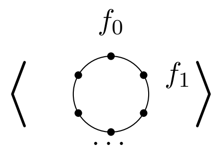
*The relation between local observable* $\mathrm{Obs}(U) = \mathbb{R}[x^i, p_i]$ *and global observable* $\mathrm{Obs}(S^1) = \mathbb{R}[x^i, p_i, dx^i, dp_i]$ *is precisely the Hochschild Homology (by Hochschild-Kostant-Rosenberg (HKR) Theorem).*

$$S^1 - \textit{factorization Homology} = \textit{Hochschild Homology}$$

*At the quantum level, local observables become the Weyl algebra*

$$\mathcal{A}^{\hbar} = (\mathcal{A}[[\hbar]], \star) \qquad \star = \textit{Moyal product}$$

*Then quantum correlation functions in this model (see Definition 9.10)*




*leads to an explicit chain map (Gui-Li-Xu [24] and Li-Wang-Yang [32])*

$$(C_{\bullet}((\mathcal{A}[[\hbar]], b) \longrightarrow (\Omega^{\bullet}_{\mathcal{A}}[[\hbar]], \hbar\Delta).$$

*Here* $(C_{\bullet}((\mathcal{A}[[\hbar]], b)$ *is the Hochschild chain complex of the Weyl algebra* $\mathcal{A}^{\hbar}$*.* $\Delta = \mathscr{L}_{\omega^{-1}}$ *is the Lie derivative with respect to the Poisson bi-vector* $\omega^{-1} = \sum_i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_i}$*.*

*This chain map becomes a quasi-isomorphism if we invert* $\hbar$

$$(C_{\bullet}((\mathcal{A}((\hbar)), b) \longrightarrow (\Omega^{\bullet}_{\mathcal{A}}((\hbar)), \hbar\Delta)$$

*which can be viewed as the quantum HKR map.*

# 9 Topological Quantum Mechanics

In this section we study the example of topological quantum mechanics and illustrate its connection with deformation quantization and algebraic index theorem.

## 9.1 Deformation quantization

The method of deformation quantization was developed in the series of papers by Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer [4]. The space of the real-valued (or complex-valued) functions on a phase space admits two algebraic structures: a structure of *associative algebra* given by the usual product of functions and a structure of Lie algebra given by the *Poisson bracket.* The study of the properties of the deformations (in a suitable sense) of these two structures gives a new invariant approach for quantum mechanics.

$$\begin{tikzcd}
\text{Poisson algebra} \arrow[r, bend left, "\text{quantization}"] & \text{Associative algebra} \arrow[l, bend left, "\hbar\to 0"]
\end{tikzcd}$$

This is essentially the quantization method in quantum mechanics, in which a function $f$ on the classical phase space is quantized to an operator $\widehat{f}$.

**Definition 9.1.** A **Poisson manifold** is a pair $(X, P)$, where $X$ is a smooth manifold, and $P \in \Gamma(X, \bigwedge^2 TX)$ satisfying $\{P, P\}_{\mathrm{SN}} = 0$.

Here $\{-,-\}_{\mathrm{SN}}$ is the *Schouten-Nijenhuis bracket.* $P$ is called the **Poisson tensor/bi-vector**. In local coordinates, we can write

$$P = \sum_{i,j} P^{ij}(x)\partial_i \wedge \partial_j.$$

It defines a *Poisson bracket* $\{-,-\}_P$ on $C^\infty(x)$:

$$\{f, g\}_P \coloneqq \sum_{i,j} P^{ij}\partial_i f \partial_j g, \quad \forall f, g \in C^\infty(x).$$

Here $\{P, P\}_{\mathrm{SN}} = 0$ implies that $\{-,-\}_P$ satisfies Jacobi identity. Hence $\{-,-\}_P$ naturally defines the Poisson algebra $(C^\infty(x), \{-,-\}_P)$.

**Example 9.2.** *Let $(X, \omega)$ be a symplectic manifold, where $\omega = \frac{1}{2}\sum_{i,j}\omega_{ij}dx^i \wedge dx^j$ is the symplectic 2-form. Let*

$$P = \omega^{-1} = \frac{1}{2}\sum_{i,j}\omega^{ij}\partial_i \wedge \partial_j,$$

*where $(\omega^{ij})$ is the inverse of $(\omega_{ij})$. Then*

$$d\omega = 0 \Leftrightarrow \{P, P\}_{SN} = 0.$$

*Hence $(X, \omega^{-1})$ defines a Poisson manifold.*

**Definition 9.3.** A **star-product** on a Poisson manifold $(X, P)$ is a $\mathbb{R}[[\hbar]]$-bilinear map

$$\begin{aligned} C^\infty(X)[[\hbar]] \times C^\infty(X)[[\hbar]] &\to C^\infty(x)[[\hbar]] \\ f \times g \quad &\mapsto f \star g = \sum_{k \geq 0} \hbar^k c_k(f, g) \end{aligned}$$

such that

(1) $\star$ is associative: $(f \star g) \star h = f \star (g \star h)$,

(2) $f \star g = fg + \mathcal{O}(\hbar), \quad \forall f, g \in C^\infty(x)$,

(3) $\frac{1}{2}(f \star g - g \star f) = \hbar \{f, g\} + \mathcal{O}(\hbar^2), \quad \forall f, g \in C^\infty(X)$,

(4) $c_k : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ is a bidifferential operator.

Then $(C^\infty(X)[[\hbar]], \star)$ is called a **deformation quantization** of $(X, P)$.

The definition of deformation quantization is purely algebraic. The *existence* of deformation quantization is highly nontrivial. DeWilde-Lecomte [13] obtained the general existence of deformation quantization on symplectic manifolds via cohomological method in 1983. Independently, Fedosov [17] presented another beautiful approach via differential geometric method in 1985. More generally, Kontsevich [26] gave the complete solution for general Poisson manifold.

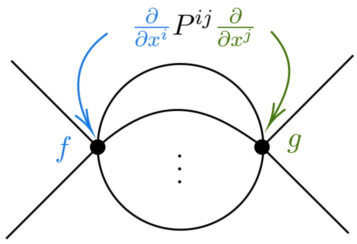
**Example 9.4.** *Let $X = \mathbb{R}^{2n}$, with Poisson tensor*

$$P = \frac{1}{2} \sum_{i,j} P^{ij} \partial_i \wedge \partial_j$$

*Here $P^{ij}$ are constants. Given $f(x), g(x)$, define the Moyal product $\star$ by*

$$(f \star g)(x) = \exp\left( \frac{\hbar}{2} \sum_{i,j} P^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) \Bigg|_{y=z=x} f(y) g(z)$$

*or pictorially,*




*Then $\star$ defines a deformation quantization.*

*Remark* 9.5. If $P^{ij} \neq$ constant, then the above formula does not work.

The BV structure naturally arises as follows. Let $(V, \omega)$ be a linear symplectic space, where $V \simeq \mathbb{R}^{2n}$ and $\omega : \bigwedge^2 V \to \mathbb{R}$ is a symplectic pairing. Write

$$\widehat{\mathscr{O}}(V) := \widehat{\operatorname{Sym}}(V^\vee) = \prod_{k \geq 0} \operatorname{Sym}^k(V^\vee),$$

where $V^\vee = \operatorname{Hom}(V, \mathbb{R})$ is the linear dual of $V$. Then the Moyal-Weyl product defines an associative algebra $\left(\widehat{\mathscr{O}}(V)[[\hbar]], \star\right)$, called the **(formal) Weyl algebra**.

## 9.2 Fedosov quantization

We will focus on *symplectic manifolds* now. Fedosov [17] gave a simple and geometric construction of deformation quantization on symplectic cases. On a symplectic manifold $(X, \omega)$, the tangent plane $T_pX$ at each point $p \in X$ is a linear symplectic space. Quantum fluctuations deform the algebra of functions on $T_pX$ to the associated *Weyl algebra*. These pointwise Weyl algebras form a vector bundle — the *Weyl bundle* $\mathcal{W}(X)$ on $X$.

**Definition 9.6.** Let $(X, \omega)$ be a symplectic manifold. We define the **Weyl bundle**

$$\mathcal{W}(X) := \prod_{k \geq 0} \mathrm{Sym}^k \left(T^*X\right)[[\hbar]].$$

So at each point $p \in X$, its fiber is

$$\mathcal{W}(X)|_p = \widehat{\mathscr{O}}\left(T_pX\right)[[\hbar]].$$

Here $\widehat{\mathscr{O}}$ refers to formal power series functions.

A local section of $\mathcal{W}(X)$ is

$$\sigma(x, y) = \sum_{k,l \geq 0} \hbar^k a_{k,i_1 \cdots i_l}(x) y^{i_1} \cdots y^{i_l},$$

where $\{x\}$ are the base coordinates and $\{y\}$ are the fiber coordinates; $a_{k,i_1\cdots i_l}(x)$ are smooth functions. Since $\left(T_pX, \omega|_{T_pX}\right)$ is linear symplectic, we have a fiberwise Moyal product, still denoted by $\star$. Thus $(\mathcal{W}(X), \star)$ defines the $\infty$-dimensional bundle of algebras.

Let $\nabla$ be a connection on $TX$ which is torsion-free and compatible with $\omega$ (i.e. $\nabla\omega = 0$). Such connection is called a **symplectic connection** (which always exists and is not unique). $\nabla$ induces a connection on all tensors. In particular, it defines a connection on $\mathcal{W}(X)$, still denoted by $\nabla$. Its curvature is

$$\nabla^2 \sigma = \frac{1}{\hbar}\left[R_\nabla, \sigma\right]_\star, \quad \forall \sigma \in \Gamma(X, \mathcal{W}(X))$$

where

$$R_\nabla = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l \in \Omega^2(X, \mathcal{W}(X))$$

is a 2-form valued in the Weyl bundle $\mathcal{W}(X)$; $R_{ijkl} = \omega_{im} R^m{}_{jkl}$ is a curvature form contracted with the symplectic pairing.

Given a sequence of closed 2-forms $\{\omega_k\}_{k \geq 1}$ on $X$, Fedosov shows that there exists a unique (up to gauge) connection on $\mathcal{W}(X)$ of the form $\nabla + \frac{1}{\hbar}[\gamma, -]_\star$, where $\gamma \in \Omega^1(X, \mathcal{W}(X))$ is a $\mathcal{W}(X)$-valued 1-form, satisfying some initial conditions and the equation

$$\nabla\gamma + \frac{1}{2\hbar}[\gamma, \gamma]_\star + R_\nabla = \omega_\hbar \qquad \text{(Fedosov equation)}$$

where $\omega_\hbar = -\omega + \sum_{k \geq 1} \hbar^k \omega_k$. Let

$$D = \nabla + \frac{1}{\hbar}[\gamma, -]_\star$$

be the **Fedosov connection**. Then the Fedosov equation implies

$$D^2 = \frac{1}{\hbar}[\omega_\hbar, -]_\star = 0$$

where $\omega_\hbar$ is constant along each fiber, and thus a central term. So we obtain a flat connection $D$ on $\mathcal{W}(X)$. As we will see, Fedosov connection has the geometric interpretation of quantum master equation [21, 24].

Let $\mathcal{W}_D(X) := \{\sigma \in \Gamma(X, \mathcal{W}(X)) \mid D\sigma = 0\}$ be the space of flat sections. Then $(\mathcal{W}_D(X), \star)$ is an associative algebra. Let

$$\sigma : \mathcal{W}_D(X) \to C^\infty(X)[[\hbar]]$$

be a **symbol map** by sending $y \mapsto 0$. Then $\sigma$ is an isomorphism, and

$$f \star g \mapsto \sigma\left(\sigma^{-1}(f) \star \sigma^{-1}(g)\right)$$

defines a deformation quantization. $\omega_\hbar$ is the corresponding **characteristic class** (or **moduli**).

## 9.3 Algebraic Index Theorem

Given a deformation quantization $(C^\infty(X)[[\hbar]], \star)$ on a symplectic manifold with characteristic class $\omega_\hbar$, there exists a unique **trace map**

$$\mathrm{Tr} : C^\infty(X)[[\hbar]] \to \mathbb{R}((\hbar))$$

satisfying a normalization condition and the trace property:

$$\mathrm{Tr}\,(f \star g) = \mathrm{Tr}\,(g \star f)\,.$$

Then the index is obtained as the partition function of the theory, which can be formulated as

$$\mathrm{Index} = \mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar}\widehat{A}(X),$$

where $\widehat{A}(X)$ is the *(formal) $\widehat{A}$-genus* of $X$. This is the simplest version of **algebraic index theorem** formulated by Fedosov [18] and Nest-Tsygan [37] as the algebraic analogue of *Atiyah-Singer index theorem.*

We can similarly construct a *deformation quantization* for $C^\infty(X, \mathrm{End}(E))[[\hbar]]$ and construct the trace map, then we have

$$\mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar}\,\mathrm{Ch}(E)\widehat{A}(X),$$

where $\mathrm{Ch}(E)$ is the Chern character of the vector bundle $E$ over $X$.

### Relation with QFT

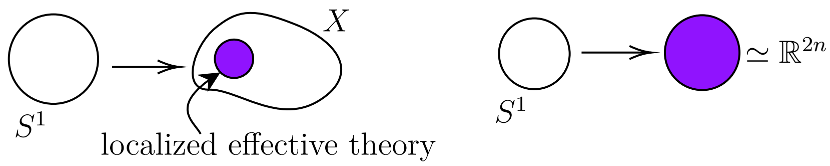
In supersymmetric (SUSY) QFT, *localization* often appears, in which the path integral on $\mathcal{E}$ is often localized effectively to an equivalent integral on a finite-dimensional space $M \subset \mathcal{E}$ describing some interesting moduli space:

$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_M (-)\,.$$

In topological QM, we find

$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \overset{\hbar\to 0}{=} \int_X (-)\,,$$

where $\mathrm{Map}(S^1, X)$ is a loop space, and $\int_X$ indicates the localization to a *constant map*. The path integral will be captured exactly by an effective theory in the formal neighborhood of constant maps inside the full mapping space. Exact semi-classical approximation in $\hbar \to 0$ allows us to reduce the path integral into a meaningful integral on the moduli space of constant maps, i.e., $X$. The left-hand side usually gives a physics presentation of the analytic index of certain elliptic operator; the right-hand side will end up with integrals of various curvature forms representing the topological index.




Geometrically, a loop space is mapped to a localized neighborhood of a point in $X$ (specified by the constant map), where a localized effective theory exists. Locally, by the choice of Darboux coordinates, $X$ can be thought of as a standard phase space, $\mathbb{R}^{2n}$. The loop spaces are then glued together on $X$ as a family of effective field theory. This can be done rigorously within the framework of effective BV quantization [24]:

- effective action $\rightsquigarrow \gamma$,
- QME $\rightsquigarrow$ Fedosov equation,
- BV integral $\rightsquigarrow$ trace map,
- partition function $\rightsquigarrow$ algebraic index.

## 9.4 Local Theory

In this section we study topological quantum mechanics in terms of the effective renormalization method, and use it to prove the algebraic index theorem. We follow the presentations in [21, 24].

### Local model

Let's consider the standard phase space $(V, \omega)$, where $V \simeq \mathbb{R}^{2n}$ with coordinates

$$(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n}) = (q^1, \cdots, q^n, p_1, \cdots, p_n)$$

and

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

Let $S^1_{dR}$ be the (local ring) space with underlying topology of a circle, $S^1$, and a structure sheaf $\mathscr{O}(S^1_{dR}) = \Omega^\bullet_{S^1}$, which is a dg (differential graded) ring of differential forms, with the de Rham differential operator $d$.

Consider the *local model* describing the space of maps

$$\varphi : S^1_{dR} \to V \simeq \mathbb{R}^{2n}.$$

Such a $\varphi$ can be identified with an element in $\Omega^\bullet_{S^1} \otimes V$. Explicitly, let $\theta$ be the coordinates on $S^1$ (with the identification $\theta \sim \theta + 1$). The space of maps can then be written as

$$\{\varphi\} = \left\{\mathbb{P}_i(\theta), \mathbb{Q}^i(\theta)\right\}_{i=1,\cdots,n}, \quad \mathbb{P}_i, \mathbb{Q}^i \in \Omega^\bullet_{S^1}.$$

Writing in form component,

$$\mathbb{P}_i(\theta) = p_i(\theta) + \eta_i(\theta)d\theta, \quad \mathbb{Q}^i(\theta) = q^i(\theta) + \xi^i(\theta)d\theta.$$

So the space of fields is

$$\mathcal{E} = \Omega^{\bullet}_{S^1} \otimes V$$

The triple $\left(\Omega^{\bullet}_{S^1} \otimes V, d, \int_{S^1} \langle -, - \rangle_\omega\right)$ is an $\infty$-dimensional $(-1)$-dg symplectic space. The action is the free one:

$$\begin{aligned} S[\varphi] &:= \int_{S^1} \langle \varphi, d\varphi \rangle_\omega \\ &= \sum_i \int_{S^1} \mathbb{P}_i d\mathbb{Q}^i = \sum_i \int_{S^1} p_i(\theta) dq^i(\theta). \end{aligned}$$

*Remark* 9.7. This is the first-order formalism of topological quantum mechanics along the line of the *AKSZ* construction [1].

### Propagator

Let us choose the standard flat metric on $S^1$. Let $d^*$ be the adjoint of $d$. The Laplacian is

$$[d, d^*] = -\left(\frac{d}{d\theta}\right)^2.$$

Let

$$\Pi = \omega^{-1} = \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} = \frac{1}{2} \sum_i \left( \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i} - \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i} \right) \in \bigwedge\nolimits^2 V$$

be the Poisson bivector (or Poisson kernel). Let

$$h_t(\theta_1, \theta_2) = \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} e^{-\frac{(\theta_1 - \theta_2 + n)^2}{4t}}$$

be the standard heat kernel on $S^1$. Then the regularized propagator is

$$P_\varepsilon^L = \int_\varepsilon^L \partial_{\theta_1} h_t(\theta_1, \theta_2)\, dt \otimes \Pi \ \in \mathcal{E} \otimes \mathcal{E},$$

where $\int_\varepsilon^L \partial_{\theta_1} h_t(\theta_1, \theta_2)\, dt \in C^\infty(S^1 \times S^1)$ and $\Pi \in V \otimes V$. Let us denote

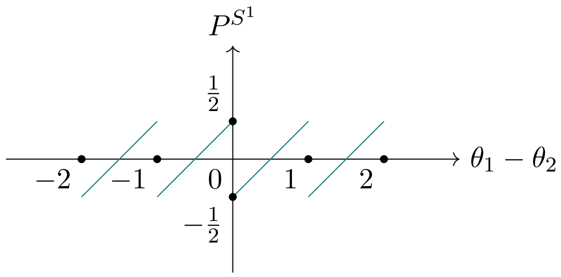
$$P^{S^1}(\theta_1, \theta_2) = \int_0^\infty \partial_{\theta_1} h_t(\theta_1, \theta_2)\, dt.$$

Then the full propagator is given by

$$P_0^\infty = P^{S^1} \otimes \Pi.$$

**Proposition 9.8.** *$P^{S^1}(\theta_1, \theta_2)$ is the following periodic function of $\theta_1 - \theta_2 \in \mathbb{R}/\mathbb{Z}$ where*

$$P^{S^1}(\theta_1, \theta_2) = \theta_1 - \theta_2 - \frac{1}{2} \quad \textit{if } 0 < \theta_1 - \theta_2 < 1.$$




$P^{S^1}$ *is NOT a smooth function on* $S^1 \times S^1$ *(as expected), but it is* bounded.

### Correlation map

Let us denote the formal Weyl algebra

$$\mathcal{W}_{2n} = \left(\mathbb{R}[[p_i, q^i]]((\hbar)), \star\right) ,$$

and the formal Weyl subalgebra

$$\mathcal{W}_{2n}^{+} = \left(\mathbb{R}[[p_i, q^i]][[\hbar]], \star\right) ,$$

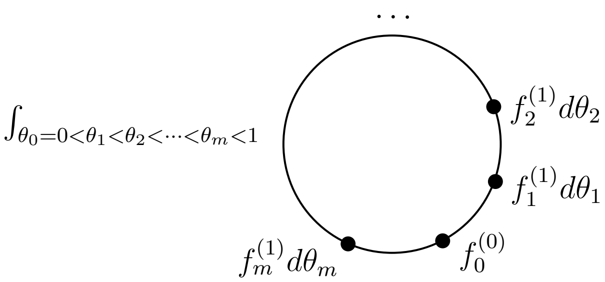
where $\star$ is the Moyal-Weyl product. We can identify the formal Weyl subalgebra as (formal) functions on $V$ (via deformation quantization):

$$\mathcal{W}_{2n}^{+} \simeq \left(\widehat{\mathscr{O}}(V)[[\hbar]], \star\right).$$

Given $f_0, f_1, \cdots, f_m \in \mathcal{W}_{2n}$, we define $\mathcal{O}_{f_0, f_1, \cdots, f_m} \in \mathscr{O}(\mathcal{E})((\hbar))$ by

$$\mathcal{O}_{f_0, f_1, \cdots, f_m}[\varphi] := \int_{0<\theta_1<\theta_2<\cdots<\theta_m<1} d\theta_1 d\theta_2 \cdots d\theta_m f_0^{(0)}(\varphi(\theta_0)) f_1^{(1)}(\varphi(\theta_1)) \cdots f_m^{(1)}(\varphi(\theta_m)).$$

Here $\varphi \in \Omega^{\bullet}_{S^1} \otimes V$. $f(\varphi(\theta)) = f(\mathbb{P}_i(\theta), \mathbb{Q}^i(\theta)) \in \Omega^{\bullet}_{S^1}$ and we decompose it as $f(\varphi(\theta)) = f^{(0)}(\varphi(\theta)) + f^{(1)}(\varphi(\theta)) d\theta$.




*Remark* 9.9. $f^{(1)}(\varphi)$ is the **topological descent** of $f^{(0)}(\varphi)$ in the sense of Witten.

Now let us apply the HRG flow, $\exp\left(\hbar P_0^{\infty}\right)\left(\mathcal{O}_{f_0, f_1, \cdots, f_m}\right)$. Since $P_0^{\infty}$ is bounded, it is convergent and well-defined! This is the *UV finite property*. As we have discussed, at $L = \infty$, we can view it as defining a function on zero modes

$$\mathbb{H} = H^{\bullet}\left(\Omega^{\bullet}_{S^1} \otimes V, d\right) = H^{\bullet}(S^1) \otimes V = V \oplus V d\theta.$$

We have $\widehat{\mathscr{O}}(\mathbb{H}) = \widehat{\Omega}_{2n}^{-\bullet}$ forms on $V$.

**Definition 9.10.** We define the following correlation map:

$$\langle \cdots \rangle_{free} : \mathcal{W}_{2n} \otimes \cdots \otimes \mathcal{W}_{2n} \to \widehat{\Omega}^{-\bullet}_{2n}((\hbar))$$

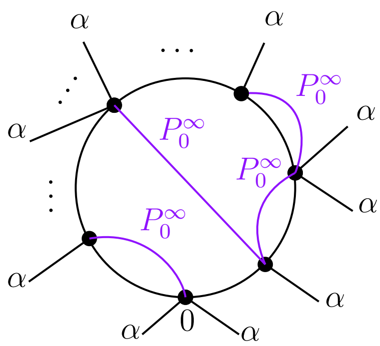
by

$$\langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{free} := \exp\left(\hbar P_0^\infty\right)\left(\mathcal{O}_{f_0, f_1, \cdots, f_m}\right)\big|_{\mathbb{H}}.$$

In the path integral perspective, this is

$$\langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{free}(\alpha) = \int_{\operatorname{Im} d^* \subset \mathcal{E}} [D\varphi]\, e^{-S[\varphi+\alpha]/\hbar} \mathcal{O}_{f_0, f_1, \cdots, f_m}[\varphi + \alpha], \quad \alpha \in \mathbb{H} = H^\bullet(S^1) \otimes V.$$

Here the zero modes $\alpha$ is viewed as the background field. It can also be represented as a Feynman diagram as follows.



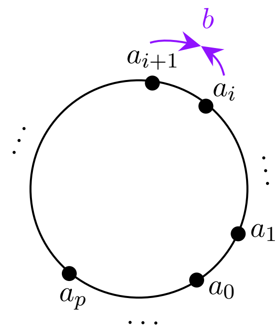

## (Cyclic) Hochschild complex reviewed

Let $A$ be a unital associative algebra and $\overline{A} := A/(\mathbb{C} \cdot 1)$. Let $C_{-p}(A) := A \otimes \overline{A}^{\otimes p}$ be the cyclic $p$-chains. It carries a natural **Hochschild differential**

$$b : C_{-p}(A) \to C_{-p+1}(A), \quad p \geq 1$$

by

$$b(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p.$$




Then the associativity implies $b \circ b = 0$. Thus, $(C_{-\bullet}(A), b)$ defines the **Hochschild chain complex**.

We can also define the **Connes operator**:

$$B : C_{-p}(A) \to C_{-p-1}(A)$$

by

$$B(a_0\otimes\cdots\otimes a_p)=1\otimes a_0\otimes\cdots\otimes a_p+\sum_{i=1}^{p}(-1)^{pi}1\otimes a_i\otimes\cdots\otimes a_p\otimes a_0\otimes\cdots\otimes a_{i-1}.$$

We have the following relations:

$$b^2=0,\quad B^2=0,\quad [b,B]=bB+Bb=0.$$

Let $u$ be a formal variable of $\deg = 2$. Then $(b+uB)^2=0$. This defines a complex

$$CC_{-\bullet}^{per}(A)=\left(C_{-\bullet}(A)[u,u^{-1}],b+uB\right),$$

called the **periodic cyclic complex**.

## Back to Correlation map

It is not hard to see via type reason that

$$\langle\cdots\rangle_{free}:C_{-p}\left(\mathcal{W}_{2n}\right)\to\widehat{\Omega}_{2n}^{-p}((\hbar)),$$

i.e. $\langle f_0\otimes f_1\otimes\cdots\otimes f_p\rangle_{free}$ is a $p$-form. Recall that $\widehat{\Omega}_{2n}^{-\bullet}$ is equipped with a BV operator $\Delta=\mathcal{L}_{\omega^{-1}}=\mathcal{L}_{\Pi}$.

**Proposition 9.11** ([24])**.**

$$\langle b(-)\rangle_{free}=\hbar\Delta\left\langle\cdots\right\rangle_{free},$$
$$\langle B(-)\rangle_{free}=d_{2n}\left\langle\cdots\right\rangle_{free}.$$

*Here $d_{2n}:\widehat{\Omega}_{2n}^{-\bullet}\to\widehat{\Omega}_{2n}^{-(\bullet+1)}$ is the de Rham differential.*

In other words, the **correlation map**:

$$\langle\cdots\rangle_{free}:C_{-\bullet}(\mathcal{W}_{2n})\to\widehat{\Omega}_{2n}^{-\bullet}((\hbar))$$

intertwines $b$ with $\hbar\Delta$ and $B$ with $d_{2n}$. We can combine the above two to get

$$\langle\cdots\rangle_{free}:CC_{-\bullet}^{per}(\mathcal{W}_{2n})\to\widehat{\Omega}_{2n}^{-\bullet}((\hbar))[u,u^{-1}]$$

which intertwines $b+uB$ with $\hbar\Delta+ud_{2n}$.

## BV integral on zero modes

We can define a *BV integration* map on the BV algebra $\left(\widehat{\Omega}_{2n}^{-\bullet},\Delta\right)$ which is only non-zero on top forms $\widehat{\Omega}_{2n}^{-2n}$ and sends

$$\beta\in\widehat{\Omega}_{2n}^{-2n}\mapsto\left.\frac{\hbar^n}{n!}\iota_{\Pi}^{n}\beta\right|_{p=q=0}.$$

This is the **Berezin integral** over the purely fermionic superLagrangian. We can extend this BV integration to an $S^1$-equivariant version by

$$\int_{BV}:\widehat{\Omega}_{2n}^{-\bullet}[u,u^{-1}]\to\mathbb{R}((\hbar))[u,u^{-1}],\quad\beta\mapsto\left.\left(u^ne^{\hbar\iota_\Pi/u}\beta\right)\right|_{p=q=0}.$$

Then it has the following property

$$\int_{BV}\left(\hbar\Delta+ud_{2n}\right)(-)=0$$

*Remark* 9.12. For $\beta \in \widehat{\Omega}^{-\bullet}_{2n}$, the equivariant limit

$$\lim_{u\to\infty}\int_{BV}\beta = \frac{\hbar^n}{n!}\iota_\Pi^n\beta\Big|_{p=q=0}$$

gives back the Berezin integral.

Combining the above maps, we define

$$\mathrm{Tr} := \int_{BV}\circ\langle\cdots\rangle_{free} : CC^{per}_{-\bullet}(\mathcal{W}_{2n}) \to \mathbb{R}((\hbar))[u,u^{-1}]$$

which satisfies the following equation:

$$\mathrm{Tr}\left((b+uB)(-)\right) = 0.$$

Therefore Tr descends to **periodic cyclic homology**. This is essentially the Feigin-Felder-Shoikhet formula.

### Quantum Master Equation

We can generalize slightly by considering a graded vector space $V$ with a deg $= 0$ symplectic pairing $\omega$. We still have the canonical quantization $\left(\widehat{\mathscr{O}}(V)[[\hbar]],\star\right)$ by Moayl product and similarly can define the BV algebra of forms $\left(\widehat{\Omega}^{-\bullet}_V, \Delta = \mathcal{L}_{\omega^{-1}}\right)$. The same trace map gives

$$\langle\cdots\rangle_{free} : C_{-\bullet}\left(\widehat{\mathscr{O}}(V)[[\hbar]]\right) \to \widehat{\Omega}^{-\bullet}_V((\hbar)), \quad b\mapsto \hbar\Delta.$$

Given $\gamma\in\widehat{\mathscr{O}}(V)[[\hbar]]$, $\deg(\gamma)=1$, it defines an action functional:

$$I_\gamma = \int_{S^1}\gamma(\varphi) \quad \forall\varphi\in\Omega^\bullet(S^1)\otimes V.$$

Let us treat $I_\gamma$ as an **interaction** and consider

$$\underbrace{\frac{1}{2}\int_{S^1}\langle\varphi,d\varphi\rangle}_{\text{free part}} + \underbrace{\int_{S^1}\gamma(\varphi)}_{I_\gamma}.$$

Then we run the HRG flow to get

$$e^{\frac{1}{\hbar}I_\gamma[\infty]} := e^{\hbar\partial_{P_0^\infty}}e^{\frac{1}{\hbar}I_\gamma}$$

which is well-defined since $P_0^\infty$ is bounded.

Let us now analyze the QME. By construction,

$$e^{\frac{1}{\hbar}I_\gamma[\infty]} = \left\langle 1\otimes e^{\gamma/\hbar}\right\rangle_{free}.$$

Assume $\gamma\star\gamma = \frac{1}{2}[\gamma,\gamma]_\star = 0$. Then

$$\hbar\Delta e^{\frac{1}{\hbar}I_\gamma[\infty]} = \left\langle b\left(1\otimes e^{\gamma/\hbar}\right)\right\rangle_{free} = 0.$$

**Proposition 9.13** ([21])**.** *If $[\gamma,\gamma]_\star = 0$, then the local interaction $I_\gamma = \int_{S^1}\gamma(\varphi)$ defines a family of solutions of effective QME $I_\gamma[L]$ at scale $L>0$ by*

$$e^{\frac{1}{\hbar}I_\gamma[L]} := \lim_{\varepsilon\to\infty} e^{\hbar\partial_{P_\varepsilon^L}}e^{\frac{1}{\hbar}I_\gamma}.$$

## 9.5 Global Theory

Recall in Local Theory, we have discussed the first-order formalism of TQM such that in a local model with maps $\varphi : \Omega^{\bullet}_{S^1} \to V \simeq \mathbb{R}^{2n}$, the correlation map

$$\langle \cdots \rangle_{free} : C_{-\bullet}(\mathcal{W}_{2n}) \to \widehat{\Omega}^{-\bullet}_{2n}((\hbar))$$

intertwines $b$ with $\hbar\Delta$ and $B$ with $d_{2n}$.

In this section, we are going to glue this construction to a symplectic manifold and establish the algebraic index to universal Lie algebra cohomology computations. The basic idea is to *glue* the local model $\Sigma \to T^{Model} \subset X$ [24].

### Gluing via Gelfand-Kazhdan formal geometry

**Definition 9.14.** A **Harish-Chandra pair** is a pair $(\mathfrak{g}, K)$, where $\mathfrak{g}$ is a Lie algebra, $K$ is a Lie group, with

- an action of $K$ on $\mathfrak{g}$: $K \xrightarrow{\rho} \mathrm{Aut}(\mathfrak{g})$,
- a natural embedding: $\mathrm{Lie}(K) \xhookrightarrow{i} \mathfrak{g}$, where $\mathrm{Lie}(K)$ is the Lie algebra associated with $K$,

such that they are compatible:

$$\begin{tikzcd} \mathrm{Lie}(K) \arrow[r, hook, "i"] \arrow[dr, "d\rho"'] & \mathfrak{g} \arrow[d, "adjoint"] \\ & \mathrm{Der}(\mathfrak{g}) \end{tikzcd}$$

**Definition 9.15.** A $(\mathfrak{g}, K)$**-module** is a vector space $V$ with

- an action of $K$ on $V$: $K \xrightarrow{\varphi} \mathrm{GL}(V)$,
- a Lie algebra morphism: $\mathfrak{g} \to \mathrm{End}(V)$,

such that they are compatible:

$$\begin{tikzcd} \mathrm{Lie}(K) \arrow[r, hook, "i"] \arrow[dr, "d\varphi"'] & \mathfrak{g} \arrow[d] \\ & \mathrm{End}(V) \end{tikzcd}$$

**Definition 9.16.** A **flat** $(\mathfrak{g}, K)$**-bundle** over X is

- a principal $K$-bundle $P \xrightarrow{\pi} X$,
- a $K$-equivariant $\mathfrak{g}$-valued 1-form $\gamma \in \Omega^1(P, \mathfrak{g})$ on $P$,

satisfying the following conditions:

(1) $\forall a \in \mathrm{Lie}(K)$, let $\xi_a \in \mathrm{Vect}(P)$ generated by $a$. Then we have the contraction $\gamma(\xi_a) = a$ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(K) \arrow[r] \arrow[dr, hook, "i"'] & \mathrm{Vect}(P) \arrow[d, "\gamma"] \\ & & \mathfrak{g} \end{tikzcd}$$

(2) $\gamma$ satisfies the Maurer-Cartan equation

$$d\gamma + \frac{1}{2}[\gamma,\gamma] = 0,$$

where $d$ is the de Rham differential on $P$, and $[-,-]$ is the Lie bracket in $\mathfrak{g}$.

Given a flat $(\mathfrak{g},K)$-bundle $P \to X$ and $(\mathfrak{g},K)$-module $V$, let

$$\Omega^\bullet(P,V) := \Omega^\bullet(P) \otimes V$$

denote differential forms on $P$ valued in $V$. It carries a connection

$$\nabla^\gamma = d + \gamma : \Omega^\bullet(P,V) \to \Omega^{\bullet+1}(P,V)$$

which is *flat* by the Maurer-Cartan equation. The group $K$ acts on $\Omega^\bullet(P)$ and $V$, and hence inducing a natural action on $\Omega^\bullet(P,V)$.

Let

$$V_P := P \times_K V$$

be the vector bundle on $X$ associated to the $K$-representation $V$. Let $\Omega^\bullet(X;V_P)$ be differential forms on $X$ valued in the bundle $V_P \to X$. Similar to the usual principal bundle case, $\nabla^\gamma$ induces a flat connection on $V_P \to X$. This defines a (de Rham) chain complex $(\Omega^\bullet(X;V_P), \nabla^\gamma)$, and $H^\bullet(X;V_P)$ denotes the corresponding de Rham cohomology.

Next we discuss how to descend Lie algebra cohomologies to geometric objects on $X$.

**Definition 9.17.** Let $V$ be a $(\mathfrak{g},K)$-module. Define the $(\mathfrak{g},K)$ **relative Lie algebra cochain complex** $(C^\bullet_{\mathrm{Lie}}(\mathfrak{g},K;V), \partial_{Lie})$ by

$$C^p_{\mathrm{Lie}}(\mathfrak{g},K;V) = \mathrm{Hom}_K\left(\bigwedge\nolimits^p(\mathfrak{g}/\operatorname{Lie}(K)), V\right).$$

Here $\mathrm{Hom}_K$ means $K$-equivariant linear maps. $\partial_{\mathrm{Lie}}$ is the Chevalley-Eilenberg differential if we view $C^p_{\mathrm{Lie}}(\mathfrak{g},K;V)$ as a subspace of the Lie algebra cochain $C^p_{\mathrm{Lie}}(\mathfrak{g};V)$. Explicitly, for $\alpha \in C^p_{\mathrm{Lie}}(\mathfrak{g},K;V)$,

$$\begin{aligned}(\partial_{\mathrm{Lie}}\alpha)(a_1 \wedge \cdots \wedge a_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i-1} a_i \cdot \alpha(a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_{p+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j] \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_{p+1}).\end{aligned}$$

The corresponding cohomology is $H^\bullet_{\mathrm{Lie}}(\mathfrak{g},K;V)$.

Given a $(\mathfrak{g},K)$-module $V$ and flat $(\mathfrak{g},K)$-bundle $P \to X$ with the flat connection $\gamma \in \Omega^1(P,\mathfrak{g})$. We can define the **descent map** from the $(\mathfrak{g},K)$ relative Lie algebra cochain complex to $V$-valued de Rham complex on $P$ by

$$\mathrm{desc} : (C^\bullet_{Lie}(\mathfrak{g},K;V), \partial_{Lie}) \to (\Omega^\bullet(X;V_P), \nabla^\gamma), \quad \alpha \mapsto \alpha(\gamma,\cdots,\gamma)$$

inducing the cohomology descent map

$$\mathrm{desc} : H^\bullet_{Lie}(\mathfrak{g},K;V) \to H^\bullet(X;V_P).$$

### Fedosov connection revisited

Recall the (formal) Weyl algebras

$$\mathcal{W}_{2n} = \mathbb{R}[[p_i, q^i]]((\hbar)), \quad \mathcal{W}_{2n}^+ = \mathbb{R}[[p_i, q^i]][[\hbar]]$$

with the induced Lie algebra structure such that the Lie bracket is defined by

$$[f, g] := \frac{1}{\hbar}[f, g]_\star = \frac{1}{\hbar}(f \star g - g \star f).$$

Let $\mathrm{Sp}_{2n}$ be the symplectic group of linear transformations preserving the Poisson bivector $\Pi$. It acts on Weyl algebras by inner automorphisms. We can identify the Lie algebra $\mathfrak{sp}_{2n}$ of $\mathrm{Sp}_{2n}$ with the quadratic polynomial in $\mathbb{R}[p_i, q^i]$, and $\mathfrak{sp}_{2n}$ is a Lie subalgebra of $\mathcal{W}_{2n}^+$. The action $\mathrm{Sp}_{2n} \curvearrowright \mathbb{R}^{2n}$ induces $\mathrm{Sp}_{2n} \curvearrowright \mathcal{W}_{2n}^+$. Hence, $\left(\mathcal{W}_{2n}^+, \mathrm{Sp}_{2n}\right)$ and $(\mathcal{W}_{2n}, \mathrm{Sp}_{2n})$ are Harish-Chandra pairs.

Let $(X, \omega)$ be a symplectic manifold, and $F_{\mathrm{Sp}}(X)$ be the symplectic frame bundle. We have the Weyl bundles

$$\mathcal{W}_X^+ = F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \mathcal{W}_{2n}^+, \quad \mathcal{W}_X = F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \mathcal{W}_{2n}.$$

Consider the Harish-Chandra pair

$$(\bar{\mathfrak{g}}, K) = (\mathfrak{g}/Z(\mathfrak{g}), \mathrm{Sp}_{2n}),$$

where $\mathfrak{g} = \mathcal{W}_{2n}^+$, and $Z(\mathfrak{g}) = \mathbb{R}[[\hbar]]$ is the center of $\mathfrak{g}$, $Z(\mathfrak{g}) \cap \mathfrak{sp}_{2n} = 0$. Fedosov constructed a flat $(\bar{\mathfrak{g}}, K)$-bundle $F_{\mathrm{Sp}}(X) \to X$ and $H^0(X; \mathcal{W}_X^+)$ gives a *deformation quantization*. Choose the trivial $(\bar{\mathfrak{g}}, K)$-module $\mathbb{R}((\hbar))$. Then

$$\mathrm{desc} : C^\bullet_{\mathrm{Lie}}\left(\overline{\mathcal{W}_{2n}^+}, \mathfrak{sp}_{2n}; \mathbb{R}((\hbar))\right) \to \Omega_X^\bullet((\hbar)).$$

This is the **Gelfand-Fuks map**. Here

$$C^\bullet_{\mathrm{Lie}}\left(\overline{\mathcal{W}_{2n}^+}, \mathfrak{sp}_{2n}; \mathbb{R}((\hbar))\right) \simeq C^\bullet_{Lie}\left(\mathcal{W}_{2n}^+, \mathfrak{sp}_{2n} \oplus Z(\mathcal{W}_{2n}^+); \mathbb{R}((\hbar))\right).$$

### Characteristic classes

Let us review the Chern-Weil construction of *characteristic classes* in Lie algebra cohomology. They will descent to the usual characteristic forms via the Gelfand-Fuks map.

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ be its Lie subalgebra. Let the projection map

$$\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$$

be the $\mathfrak{h}$-equivariant splitting of the embedding $\mathfrak{h} \subset \mathfrak{g}$. In general pr is not a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$. The *failure* of pr being a Lie algebra homomorphism gives $R \in \mathrm{Hom}\left(\bigwedge^2 \mathfrak{g}, \mathfrak{h}\right)$ by

$$R(\alpha, \beta) = [\mathrm{pr}(\alpha), \mathrm{pr}(\beta)]_{\mathfrak{h}} - \mathrm{pr}\,[\alpha, \beta]_{\mathfrak{g}}, \quad \alpha, \beta \in \mathfrak{g}.$$

The $\mathfrak{h}$-equivariance of pr implies that $R \in \mathrm{Hom}_{\mathfrak{h}}\left(\bigwedge^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right)$. $R$ is called the **curvature form**. Let $\mathrm{Sym}^m(\mathfrak{h}^\vee)^{\mathfrak{h}}$ be $\mathfrak{h}$-invariant polynomials on $\mathfrak{h}$ of homogeneous degree $\deg = m$. Given $P \in \mathrm{Sym}^m(\mathfrak{h}^\vee)^{\mathfrak{h}}$, we can associate a cochain $P(R) \in C^{2m}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$ by the composition

$$P(R) : \bigwedge^{2m} \mathfrak{g} \xrightarrow{\bigwedge^m R} \mathrm{Sym}^m(\mathfrak{h}) \xrightarrow{P} \mathbb{R}.$$

It can be checked that $\partial_{\mathrm{Lie}} P(R) = 0$, defining a cohomology class $[P(R)]$ in $H^{2m}(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$ which does not depend on the choice of pr. Therefore we have the analogue of Chern-Weil characteristic map

$$\chi : \mathrm{Sym}^\bullet(\mathfrak{h}^\vee)^{\mathfrak{h}} \to H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}), \quad P \mapsto \chi(P) := [P(R)] \,.$$

Now we apply the above construction to the case where

$$\mathfrak{g} = \mathcal{W}_{2n}^+, \quad \mathfrak{h} = \mathfrak{sp}_{2n} \oplus Z(\mathfrak{g}).$$

Any element $f$ in $\mathfrak{g} = \mathcal{W}_{2n}^+$ can be uniquely written as a polynomial $f = f(y^i, \hbar)$, with coordinates $(y^1, \cdots, y^n, y^{n+1}, \cdots, y^{2n}) = (p_1, \cdots, p_n, q^1, \cdots, q^n)$. Define the $\mathfrak{h}$-equivariant projections

$$\begin{aligned} \mathrm{pr}_1(f) &= \frac{1}{2} \sum_{i,j} \partial_i \partial_j f \bigg|_{y=\hbar=0} y^i y^j \in \mathfrak{sp}_{2n}, \\ \mathrm{pr}_3(f) &= f|_{y=0} \in Z(\mathfrak{g}). \end{aligned}$$

We obtain the corresponding curvature

$$\begin{aligned} R_1 &:= [\mathrm{pr}_1(-), \mathrm{pr}_1(-)] - \mathrm{pr}_1\,[-,-] \ \in \mathrm{Hom}(\textstyle\bigwedge^2 \mathfrak{g}, \mathfrak{sp}_{2n}), \\ R_3 &:= -\,\mathrm{pr}_3\,[-,-] \ \in \mathrm{Hom}(\textstyle\bigwedge^2 \mathfrak{g}, \mathbb{R}[[\hbar]]). \end{aligned}$$

*Remark* 9.18. A more general case can be considered when we incorporate vector bundles, where $\mathfrak{g} = \mathcal{W}_{2n}^+ + \hbar\left(\mathfrak{gl}\left(\mathcal{W}_{2n}^+\right)\right)$, $\mathfrak{h} = \mathfrak{sp}_{2n} \oplus \hbar\mathfrak{gl} \oplus Z(\mathfrak{g})$. There the extra projection $\mathrm{pr}_2$ and its corresponding curvature $R_2$ are defined as elements in $\hbar\mathfrak{gl}$ and $\mathrm{Hom}\left(\bigwedge^2, \mathfrak{gl}\right)$, respectively. It is worthwhile to point out that all the Homs here are only $\mathbb{R}$-linear map, but not $\mathbb{R}[[\hbar]]$-linear, although $\mathfrak{g}$ is a $\mathbb{R}[[\hbar]]$-module.

We now define the **$\widehat{A}$-genus**

$$\widehat{A}(\mathfrak{sp}_{2n}) := \left[\det\left(\frac{R_1/2}{\sinh\left(R_1/2\right)}\right)^{\frac{1}{2}}\right] \in H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}).$$

**Proposition 9.19.** *Under the descent map* desc : $H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}((\hbar))) \to H^\bullet(X)((\hbar))$ *via the Fedosov connection, we have*

$$\begin{aligned} \mathrm{desc}\left(\widehat{A}(\mathfrak{sp}_{2n})\right) &= \widehat{A}(X), \\ \mathrm{desc}\,(R_3) &= \omega_\hbar - \hbar\omega. \end{aligned}$$

### Universal trace map

Recall that using $\Omega^\bullet_{S^1} \to \mathbb{R}^{2n}$, we have obtained

$$\mathrm{Tr} = \int_{BV} \circ \langle - \rangle_{free} : CC_{-\bullet}^{per}(\mathcal{W}_{2n}) \to \mathbb{K} := \mathbb{R}((\hbar))[u, u^{-1}].$$

Let us write

$$\mathrm{Tr} \in \mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}\right).$$

This is a $\left(\mathcal{W}_{2n}^+, \mathrm{Sp}_{2n}\right)$-module. Via the flat $\left(\mathcal{W}_{2n}^+, \mathrm{Sp}_{2n}\right)$-bundle $F_{\mathrm{Sp}}(X) \to X$, we obtain the associated bundle

$$E^{per} := F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}\right)$$

with induced flat connection $\nabla^\gamma$.

Recall the Weyl bundle $\mathcal{W}(X) = F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \mathcal{W}_{2n}$ with flat connection $\nabla^\gamma$. We would like to glue Tr on $X$. Let us denote $\delta$ for the differential on $\mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}\right)$ induced from $b + uB$. So

$$\delta\,\mathrm{Tr} = \mathrm{Tr}\left((b+uB)(-)\right) = 0.$$

We can view Tr as defining an element in

$$C_{Lie}^0\left(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}\right)\right),$$

where we take

$$\mathfrak{g} = \mathcal{W}_{2n}^+/Z(\mathcal{W}_{2n}^+), \quad \mathfrak{h} = \mathfrak{sp}_{2n}.$$

However, Tr is NOT $\mathfrak{g}$-invariant, i.e. $\partial_{\mathrm{Lie}}\,\mathrm{Tr} \neq 0$. In other words, Tr is NOT a map of $(\mathfrak{g}, \mathrm{Sp}_{2n})$-module. So Tr can not be glued directly.

It is observed that $\partial_{\mathrm{Lie}}\,\mathrm{Tr} = \delta(-)$. It turns out that we have a canonical way to lift Tr to

$$\widehat{\mathrm{Tr}} \in C_{Lie}^\bullet\left(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}\right)\right)$$

such that

$$\widehat{\mathrm{Tr}} = \mathrm{Tr} + \text{terms in } C_{Lie}^{>0}\left(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}\right)\right)$$

and satisfying the coupled cocycle condition

$$(\partial_{Lie} + \delta)\,\widehat{\mathrm{Tr}} = 0.$$

$\widehat{\mathrm{Tr}}$ is called the **universal trace map**. Let us insert $1 \in \mathcal{W}_{2n}$, then $\widehat{\mathrm{Tr}}(1)$ is $\partial_{Lie}$-closed, which defines the **universal index**, $\left[\widehat{\mathrm{Tr}}(1)\right] \in H_{Lie}^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$.

**Theorem 9.20** (Universal algebraic index theorem)**.**

$$\left[\widehat{\mathrm{Tr}}(1)\right] = u^n e^{-R_3/(u\hbar)} \widehat{A}(\mathfrak{sp}_{2n})_u,$$

*where for* $A = \sum_{p \text{ even}} A_p$, $A_P \in H^p(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$,

$$A_u = \sum_p u^{-p/2} A_p.$$

This theorem is developed in the works of Feigin-Tsygan [20], Feigin-Felder-Shoikhet [19], Bressler-Nest-Tsygan [8], and many others. This can be naturally generalized to the bundle case [24].

Now we apply the Gelfand-Fuks (descent) map on $\widehat{\mathrm{Tr}}$, such that

$$\begin{array}{c} C_{Lie}^\bullet\left(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}\right)\right) \\ \Big\downarrow{\scriptstyle \mathrm{desc}} \\ \Omega^\bullet\left(X, \mathrm{Hom}_{\mathbb{K}}\left(CC_{-\bullet}^{per}(\mathcal{W}(X)), \mathbb{K}\right)\right) \end{array}$$

Let $\mathcal{W}_D(X)$ be the space of flat sections of $\mathcal{W}(X)$ that gives a *deformation quantization.* Then

$$\mathrm{desc}(\widehat{\mathrm{Tr}}) : CC_{-\bullet}^{per}(\mathcal{W}_D(X)) \to \Omega^\bullet(X)((\hbar))[u, u^{-1}], \quad b + uB \mapsto d_X.$$

In particular, it defines a *trace map* in deformation quantization by

$$f \in \mathcal{W}_D(X) \mapsto \int_X \operatorname{desc}(\widehat{\operatorname{Tr}})(f) \in \mathbb{R}((\hbar)).$$

We can show that $\int_X \operatorname{desc}(\widehat{\operatorname{Tr}})(f)$ does not involve $u$. By the *universal algebraic index theorem*, we have

$$\int_X \operatorname{desc}(\widehat{\operatorname{Tr}})(1) = \int_X e^{-\omega_\hbar/\hbar} \widehat{A}(X).$$

This gives the **algebraic index theorem**.

## Construction of universal trace map $\widehat{\operatorname{Tr}}$

We have the following relations.

$$\begin{tikzcd}
\boxed{\text{background symmetry}} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \boxed{\text{connection form}} \arrow[dl, leftrightarrow] \\
& \boxed{\text{interaction}} &
\end{tikzcd}$$

Let $\Theta : \mathfrak{g} \to \mathcal{W}_{2n}^+/Z(\mathcal{W}_{2n}^+) = \mathfrak{g}$ be the canonical identity map. For each $f \in \mathcal{W}_{2n}^+/Z(\mathcal{W}_{2n}^+)$, we have defined the local functional on $\mathcal{E} = \Omega^\bullet(S^1) \otimes \mathbb{R}^{2n}$ by

$$I_f(\varphi) = \int_{S^1} f(\varphi), \quad \varphi \in \mathcal{E}.$$

Then $\Theta$ gives a map

$$I_\Theta : \mathfrak{g} \to \mathcal{O}_{loc}(\mathcal{E}), \quad f \mapsto I_{\Theta(f)}.$$

We can view this map as

$$I_\Theta \in C^1(\mathfrak{g}, \mathcal{O}_{loc}(\mathcal{E})) = \mathfrak{g}^\vee \otimes \mathcal{O}_{loc}(\mathcal{E}).$$

Now we can construct $\widehat{\operatorname{Tr}} \in C^\bullet_{Lie}\left(\mathfrak{g}, \mathfrak{h}; \operatorname{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)\right)$ by

$$\begin{aligned}
\widehat{\operatorname{Tr}}\left(f_0 \otimes f_1 \otimes \cdots \otimes f_m\right) &:= \int_{BV} \exp\left(\hbar P_0^\infty\right)\left(\mathcal{O}_{f_0, f_1, \cdots, f_m} e^{\frac{1}{\hbar} I_\Theta}\right) \in C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K}), \quad f_i \in \mathcal{W}_{2n} \\
\text{“} &= \int_{BV} \int_{\operatorname{Im} d^* \subset \mathcal{E}} e^{-\frac{1}{2\hbar}\int_{S^1}\langle \varphi, d\varphi\rangle + \frac{1}{\hbar} I_\Theta} \mathcal{O}_{f_0, f_1, \cdots, f_m} \text{”}.
\end{aligned}$$

## Computation of index

The Weyl algebra $\mathcal{W}_{2n}$ can be viewed as a family of associative algebras parameterized by $\hbar$. This leads to the **Gauss-Manin-Getzler connection** $\nabla_{\hbar\partial_\hbar} \curvearrowright CC^{per}_{-\bullet}(\mathcal{W}_{2n})$. The calculation of index consists of the following steps:

(1) *Feynman diagram computation* implies

$$\widehat{\operatorname{Tr}}(1) = u^n e^{-R_3/(u\hbar)} \left( \underbrace{\widehat{A}(\mathfrak{sp}_{2n})_u}_{1-\text{loop computation}} + \mathcal{O}(\hbar) \right).$$

(2) Computation of *Gauss-Manin-Getzler connection* shows $\nabla_{\hbar\partial_\hbar}\left(e^{R_3/(u\hbar)}\widehat{\operatorname{Tr}}(1)\right)$ is $\partial_{Lie}$-exact.

(3) Combining (1) and (2), we find

$$\left[\widehat{\operatorname{Tr}}(1)\right] = \left[u^n e^{-R_3/(u\hbar)} \widehat{A}(\mathfrak{sp}_{2n})_u\right] \in H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K}).$$

# 10 Two-dim Chiral QFT

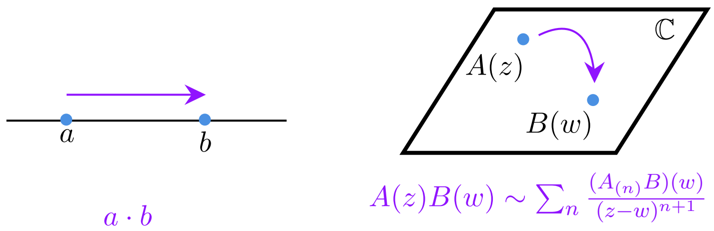
We have discussed the first-order formalism of topological QM, where the fields are differential forms $\Omega^\bullet(S^1, V)$ on $S^1$ valued in the vector bundle $V$ with the de Rham differential $d$. Here $d$ being part of the BRST operator implies that "translation is homologically trivial." This defines a topological theory.

We will now consider 2d chiral models where the fields are complex differential forms, $\Omega^{0,\bullet}(\Sigma, h)$ with the Dolbeault differential $\bar{\partial}$. The Dolbeault differential being part of the BRST operator implies that "anti-holomorphic translation is homologically trivial," which in turn defines a chiral (or holomorphic) theory.

In topological QM, the theory is *UV finite*. The general consideration in Section 7.4 applies and we find that the renormalized QME is traded to Moyal commutator and Fedosov equation. We will see that 2d chiral theory is also *UV finite* and we have a similar geometric result for QME [29].

## 10.1 Vertex algebra

As discussed in Section 1.2, in 1d topological theory we have associative algebra defined by the fusion $a \cdot b$; in 2d chiral theory we have (chiral) vertex algebra defined by $A_{(n)}B$. The algebras are found when one operator approaches another either on a line (for 1d) or on a plane (for 2d).




On a plane, the "product" (binary operation) depends on the location holomorphically, leading to $\infty$-ly many binary operations.

**Definition 10.1.** A **vertex algebra** is a collection of data:

- (space of states) a $\mathbb{Z}$-graded superspace $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$,
- (vacuum) a vector $|0\rangle \in \mathcal{V}_0$,
- (translation operator) an even linear map $T : \mathcal{V} \to \mathcal{V}$,
- (state-field correspondence) an even linear operation (vertex operation)

$$Y(-, z) : \mathcal{V} \to \operatorname{End} \mathcal{V}[[z, z^{-1}]], \quad A \mapsto Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$$

  such that $Y(A, z)B \in \mathcal{V}((z))$ for any $A, B \in \mathcal{V}$.

The data are required to satisfy the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = 1_{d_{\mathcal{V}}}$, i.e. for any $A \in \mathcal{V}$,

$$Y(A, z)|0\rangle \in \mathcal{V}[[z]] \quad \text{and} \quad \lim_{z \to 0} Y(A, z)|0\rangle = A,$$

- (translation axiom) $T|0\rangle = 0$, i.e. for any $A \in \mathcal{V}$,

$$[T, Y(A, z)] = \partial_z Y(A, z),$$

- (locality axiom) all $\{Y(A,z)\}_{a\in\mathcal{V}}$ are mutually local.

  Roughly speaking, mutual locality implies for any $A, B \in \mathcal{V}$, we can expand as

$$Y(A,z)Y(B,w) = \sum_{n\in\mathbb{Z}} \frac{Y(A_{(n)}\cdot B, w)}{(z-w)^{n+1}}.$$

This is called the **operator product expansion (OPE)**. $\{A_{(n)}\cdot B\}$ from the expansion coefficient can be viewed as defining an infinite tower of products. For simplicity, we will write

$$A(z) \equiv Y(A,z) \quad \text{for } A \in \mathcal{V}.$$

Then the OPE can be written as

$$A(z)B(w) = \sum_{n\in\mathbb{Z}} \frac{A_{(n)}\cdot B(w)}{(z-w)^{n+1}}.$$

We also write, whenever only the *singular* parts matter,

$$A(z)B(w) \sim \sum_{n\geq 0} \frac{A_{(n)}\cdot B(w)}{(z-w)^{n+1}}.$$

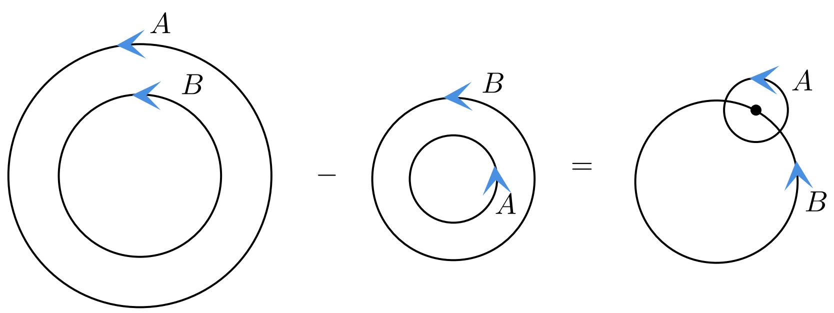
Given a vertex algebra, we can define its **modes Lie algebra**

$$\oint \mathcal{V} := \mathrm{Span}_{\mathbb{C}} \left\{ \oint dz\ z^k A(z) = A_{(k)} \right\}_{A\in\mathcal{V},\ k\in\mathbb{Z}}.$$

The Lie bracket of contour integrals is determined by the OPE,

$$\left[ \oint dz\ z^m A(z),\ \oint dw\ w^n B(w) \right] = \oint dw\ w^n \oint_w dz\ z^m \sum_{j\in\mathbb{Z}} \frac{A_{(j)}\cdot B(w)}{(z-w)^{j+1}},$$

where only the singular part matters in the integration. The Lie bracket is represented diagrammatically as follows.




**Example 10.2** ($\beta\gamma$-system)**.** *The $\beta\gamma$-system is generated by two* bosonic *fields $\beta(z), \gamma(z)$ with the contractions*

$$\beta(z)\gamma(w) \sim \frac{\hbar}{z-w} \sim -\gamma(z)\beta(w).$$

*The vertex algebra $\mathcal{V}$ is identified with the differential ring*

$$\mathcal{V} = :\mathbb{C}[[\partial^i\beta, \partial^i\gamma]]:\ [[\hbar]],$$

*where* $: \; :$ *is the* normal ordering operator*. The general OPE is obtained via* ***Wick contractions****. For example,*

$$
\begin{aligned}
:\beta(z)\gamma(z): \, :\beta(w)\gamma(w): \; &= \underbrace{\frac{\hbar}{z-w} \; :\gamma(z)\beta(w): \; - \frac{\hbar}{z-w} \; :\beta(z)\gamma(w):}_{\text{1 contraction}} - \underbrace{\left(\frac{\hbar}{z-w}\right)^2}_{\text{2 contractions}} \\
&= \sum_{k\geq 0} \frac{\hbar}{z-w} \frac{(z-w)^k}{k!} \; :\partial^k\gamma(w)\beta(w) - \partial^k\beta(w)\gamma(w): \; - \frac{\hbar^2}{(z-w)^2}.
\end{aligned}
$$

**Example 10.3** ($bc$-system)**.** *The $bc$-system is generated by two* fermionic *fields $b(z), c(z)$ with*

$$
b(z)c(w) \sim \frac{\hbar}{z-w} \sim c(z)b(w).
$$

*The vertex algebra $\mathcal{V}$ is identified with the differential ring*

$$
\mathcal{V} = :\mathbb{C}[[\partial^i b, \partial^i c]]: \; [[\hbar]].
$$

*The general OPE is generated in the similar way as the $\beta\gamma$-system (but we need to take care of the signs).*

More generally, we can define a general $\beta\gamma - bc$ system by considering a $\mathbb{Z}_2$-graded space

$$
h = h_0 \oplus h_1
$$

with an even symplectic pairing

$$
\langle -, - \rangle : \bigwedge^2 h \to \mathbb{C}.
$$

Let $\{a_i\}$ be a basis of $h$, then we can define a vertex algebra $\mathcal{V}_h$ by

$$
\mathcal{V}_h = :\mathbb{C}[[\partial^k a_i]]: \; [[\hbar]].
$$

The OPE is generated by

$$
a_i(z)a_j(w) \sim \frac{\hbar}{z-w} \langle a_i, a_j \rangle .
$$

In particular, $h_0$ represents the copies of $\beta\gamma$-system; $h_1$ represents the copies of $bc$-system.

## 10.2 Chiral deformation of $\beta\gamma - bc$ systems

We consider the following data:

- an elliptic curve $E$ (topologically a torus $T^2$) with linear coordinate $z$ such that $z \sim z+1 \sim z+\tau$,
- a graded symplectic space $h = h_0 \oplus h_1$ with an even symplectic pairing $\langle -, - \rangle$.

This defines a field theory in BV formalism where the space of fields is

$$
\mathcal{E} = \Omega^{0,\bullet}(E) \otimes h
$$

with $(-1)$-symplectic pair by

$$
\omega(\varphi_1, \varphi_2) = \int_E dz \, \langle \varphi_1, \varphi_2 \rangle , \quad \varphi_i \in \mathcal{E}.
$$

Note that $\omega$ has deg $= -1$ since we need exactly one $\overline{dz}$ from $\varphi_1, \varphi_2$ to be integrated.

The free theory is given by

$$\frac{1}{2}\int_E dz \left\langle \varphi, \bar{\partial}\varphi \right\rangle, \quad \varphi \in \mathcal{E}.$$

The local quantum observables form exactly $\beta\gamma - bc$ system. The propagator is given by the **Szegö kernel**

$$\bar{\partial}^{-1} \sim \frac{1}{z-w} + \text{regular}.$$

We would like to consider a general interacting theory by turning on **chiral deformations** of the form

$$\int \mathcal{L}\left(\varphi, \partial_z \varphi, \partial_z^2 \varphi, \cdots\right)$$

which involves only *holomorphic* derivatives. This is related precisely to the vertex algebra

$$\mathcal{V}_{h^\vee} = \mathbb{C}[[\partial^i h^\vee]][[\hbar]]$$

as follows. Define a map

$$I : \mathcal{V}_{h^\vee} \to \mathscr{O}_{loc}(\mathcal{E}), \quad \gamma \mapsto I_\gamma.$$

Explicitly, if $\gamma = \sum \partial^{k_1} a_1 \cdots \partial^{k_m} a_m$, then

$$I_\gamma(\varphi) = i \int_E dz \sum \pm \partial_z^{k_1} a_1(\varphi) \cdots \partial_z^{k_m} a_m(\varphi).$$

Here $a_i \in h^\vee$ and $a_i(\varphi) \in \omega^{0,\bullet}(E)$.

**Theorem 10.4** ([29]). *For any $\gamma \in \mathcal{V}_{h^\vee}$, the chiral deformed theory*

$$\frac{1}{2}\int_E dz \left\langle \varphi, \bar{\partial}\varphi \right\rangle + I_\gamma(\varphi)$$

*is **UV finite** in the sense that the limit*

$$e^{\frac{1}{\hbar} I_\gamma[L]} = \lim_{\varepsilon \to 0} e^{\hbar \partial_{P_\varepsilon^L}} e^{\frac{1}{\hbar} I_\gamma} \quad \textit{exists.}$$

*Remark* 10.5. The proof to the UV finiteness theorem is a bit technical. The reason is different from topological QM, where we saw that the propagator is bounded (although not continuous). Here the graph integral is NOT absolute convergent. See the next section for a geometric interpretation [34] of this fact.

Once we have a well-defined $I_\gamma[L]$ described above, we can formulate the **effective QME**

$$\bar{\partial} I_\gamma[L] + \hbar \Delta_L I_\gamma[L] + \frac{1}{2}\left\{I_\gamma[L], I_\gamma[L]\right\}_L = 0$$

and ask for the condition of $\gamma$ to satisfy the equation. It turns out that the answer is very simple.

**Theorem 10.6** ([29]). *Consider $\gamma \in \mathcal{V}_{h^\vee}$ and the effective functional $I_\gamma[L]$ defined above via the UV finiteness. Then $I_\gamma[L]$ satisfies the effective QME*

$$\bar{\partial} I_\gamma[L] + \hbar \Delta_L I_\gamma[L] + \frac{1}{2}\left\{I_\gamma[L], I_\gamma[L]\right\}_L = 0$$

*if and only if*

$$\left[\oint \gamma, \oint \gamma\right] = 0 \ \in \oint \mathcal{V}.$$

*Remark* 10.7. The local quantum observable of the chiral deformed theory is the vertex algebra $H^\bullet(\mathcal{V}_{h^\vee}, [\oint \gamma, -])$. So $[\oint \gamma, -]$ plays the role of BRST reduction. Reversing this reasoning, vertex algebras coming from the BRST reduction of free field realizations can be realized via the model of chiral deformations above.

The above theorem can be glued for a *chiral $\sigma$-model*

$$\varphi : E \to X$$

which produces a bundle $\mathcal{V}(X) \to X$ of chiral vertex algebras on $X$. Then the solution of effective QME asks for a flat connection on $\mathcal{V}(X)$ of the form

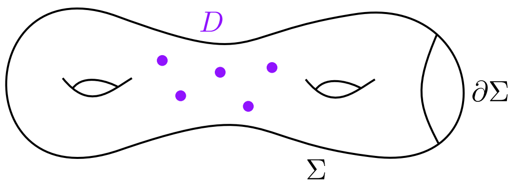
$$D = d + \frac{1}{\hbar}\left[\oint \gamma, -\right], \text{ such that } D^2 = 0.$$

Here $\gamma \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \gamma$ is fiberwise chiral mode operator. This can be viewed as the *chiral analogue of Fedosov connection.*

## 10.3 Regularized integral and UV finiteness

The propagator $\bar{\partial}^{-1}$ is given by the Szegö kernel which exhibits holomorphic poles $\frac{1}{z-w}$ along the diagonal. In general, the Feynman diagram involves $\int_{\Sigma^n} \Omega$, where $\Omega$ exhibits holomorphic poles of arbitrary order when $z_i \to z_j$. It turns out that such looking divergent integral has an intrinsic *regularization* via its conformal structure.

For simplicity, we start by considering such an integral $\int_\Sigma \omega$. Here $\Sigma$ is a Riemann surface, possibly with boundary $\partial\Sigma$, $\omega$ is a 2-from on $\Sigma$ with meromorphic poles of arbitrary order along a finite set $D \subset \Sigma$, such that $D \cap \partial\Sigma = \varnothing$.




Let $p \in D$ and $z$ be a local coordinate centered at $p$. Then locally $\omega$ can be written as

$$\omega = \frac{\eta}{z^n}$$

where $\eta$ is smooth, and $n \in \mathbb{Z}$. Since the pole order can be arbitrarily large, the naive $\int_\Sigma \omega$ is divergent in general. One homological way out of this divergence problem [34] is as follows. We can decompose $\omega$ into

$$\omega = \alpha + \partial\beta,$$

where $\alpha$ is a 2-form with at most *logarithmic pole* along $D$, $\beta$ is a $(0,1)$-form with *arbitrary order of poles* along $D$, and $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham differential. Such a decomposition *exists* and is *not unique*.

**Definition 10.8** ([34]). Define the **regularized integral**

$$⨍_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial\Sigma} \beta$$

as a recipe to integrate the singular form $\omega$ on $\Sigma$. It has the following properties

- it does NOT depend on the choice of $\alpha, \beta$,
- $\fint_\Sigma$ is invariant under conformal transformations,
- $\fint_\Sigma \partial(-) = \int_{\partial\Sigma}(-)$,
- $\fint_\Sigma \bar\partial(-) = \mathrm{Res}(-)$.

The regularized integral extends the usual integral for smooth forms, i.e., the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}^2(\Sigma) \arrow[rr, hook] \arrow[dr, "\int_\Sigma"'] & & \mathcal{A}^2(\Sigma, \star D) \arrow[dl, "\fint_\Sigma"] \\ & \mathbb{C} & \end{tikzcd}$$

We can use this to define integrals on configuration space of $\Sigma$

$$\mathrm{Conf}_n(\Sigma) = \Sigma^n - \Delta = \{\, (p_1, \cdots, p_n) \in \Sigma^n |\, p_i \neq p_j, \forall i \neq j\}$$

and define

$$\fint_{\Sigma^n} : \mathcal{A}^{2n}(\Sigma^n, \star\Delta) \to \mathbb{C}$$

by

$$\fint_{\Sigma^n} (-) = \fint_{\Sigma_1} \fint_{\Sigma_2} \cdots \fint_{\Sigma_n} (-).$$

It does NOT depend on the choice of the ordering of the factors in $\Sigma^n$; Fubini-type theorem holds. This gives an intrinsically regularized meaning for $\fint_{\Sigma^n} \Omega$, where $\Omega$ is the Feynman diagram integrand. This explains why the theory is UV finite.

## 10.4 Homological Aspects of BV quantization

Roughly speaking, BV quantization in QFT leads to

- factorization algebra Obs of observables [11],
- a chain complex $(C_\bullet(\mathrm{Obs}), d)$ via algebraic structure of Obs,
- a BV algebra $(\mathcal{A}, \Delta)$ describing the zero modes (at $L = \infty$) with a BV integration map
$$\int_{BV} : \mathcal{A} \to \mathbb{C},$$
- a $\mathbb{C}[[\hbar]]$-linear map
$$\langle - \rangle : C_\bullet(\mathrm{Obs}) \to \mathcal{A}((\hbar))$$
which is the HRG flow from $L = 0$ to $L = \infty$ satisfying the QME $(d + \hbar\Delta)\langle - \rangle = 0$; the QME says that $\langle - \rangle$ is a chain map intertwining $d$ and $\hbar\Delta$,
- partition function leading to the index theorem
$$\mathrm{Index} = \int_{BV} \langle 1 \rangle .$$

Recall in the example of TQM, we have the following data.

- The factorization algebra is the Weyl algebra: $\mathrm{Obs} = \mathcal{W}_{2n}$.
- The factorization complex is the Hochschild chain complex $(C_\bullet(\mathrm{Obs}), d)$.
- BV algebra on zero modes: $(\mathcal{A}, \Delta) = \left(\Omega^\bullet(\mathbb{R}^{2n}), \mathcal{L}_{\omega^{-1}}\right)$.

- Free correlation map

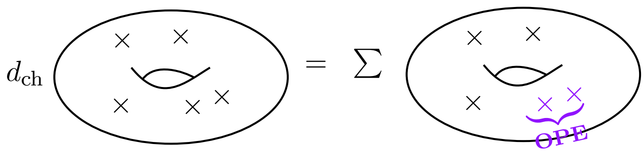
$$\langle - \rangle : C_\bullet(\mathcal{W}_{2n}) \to \Omega^\bullet(\mathbb{R}^{2n})((\hbar)), \quad b \mapsto \hbar \mathcal{L}_{\omega^{-1}}.$$

- Index $= \int_{BV} \langle 1 \rangle = \left[ e^{\omega_\hbar/\hbar} \widehat{A} \right]$.

In the example of a 2d chiral QFT, we will have a similar story [23].

## 10.5 Chiral Chain Complex

Intuitively, chiral chain complex can be viewed as a 2d chiral analogue of Hochschild chain complex.




- [41] studies the space of genus 1 conformal blocks (i.e. the 0th elliptic chiral homology).
- [5] explores the chiral homology for general algebraic curves.

**The construction of Beilinson-Drinfeld.** Given the following data:

- a category of right $\mathcal{D}$-modules $\mathcal{M}(X)$ on $X = \Sigma$,
- a category of right $\mathcal{D}$-modules $\mathcal{M}(X^S)$ on $X^S$, such that each element $M \in \mathcal{M}(X^S)$ is a collection that assigns every finite index set $I \in S$ a right $\mathcal{D}$-module $\mathcal{M}_{X^I}$ on the product $X^I$ satisfying certain compatibility conditions,
- there is an exact fully faithful embedding

$$\Delta_\star^{(S)} : \mathcal{M}(X) \hookrightarrow \mathcal{M}(X^S)$$

  via the diagonal map $\Delta^{(I)} : X \hookrightarrow X^I$,
- $\mathcal{M}(X^S)$ carries a (chiral) tensor structure $\otimes^{\text{ch}}$,

then a chiral algebra $\mathcal{A}$ is a **Lie algebraic object** via $\Delta_\star^{(S)}$.

*Remark* 10.9. The chiral algebra $\mathcal{A}$ collects all "normal ordering operators."

We consider the Chevalley-Eilenberg (CE) complex

$$(C(\mathcal{A}), d_{\text{CE}}) = \left( \bigoplus_{\bullet > 0} \text{Sym}^\bullet_{\otimes^{\text{ch}}} \left( \Delta_\star^{(S)} \mathcal{A}[1] \right), d_{\text{CE}} \right).$$

The chiral homology for this complex is

$$C^{\text{ch}}(X, \mathcal{A}) = R\Gamma_{DR}(X^S, C(\mathcal{A})).$$

We will focus on $\beta\gamma - bc$ system, where the vertex operator algebra (VOA) $\mathcal{V}^{\beta\gamma - bc}$ is the chiral algebra $\mathcal{A}^{\beta\gamma - bc}$.

**Theorem 10.10** ([23])**.** *Let $E$ be an elliptic curve. Then the HRG flow gives a map*

$$\langle - \rangle_{2d} : C^{\text{ch}}(E, \mathcal{A}^{\beta\gamma - bc}) \to \mathcal{A}((\hbar))$$

*satisfying the QME $(d_{\text{ch}} + \hbar\Delta) \langle - \rangle_{2d} = 0$. Roughly speaking, $\langle - \rangle$ is defined by*

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{E^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$$

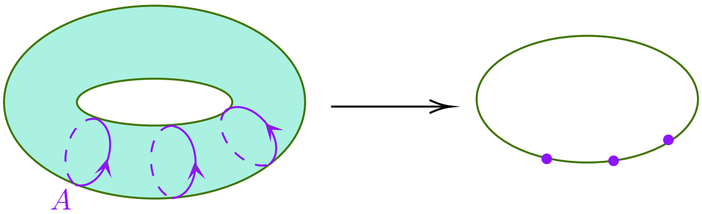
*where $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is a local correlator given by the Feynman diagram, and $\fint_{E^n}$ is the regularized integral.*

### 10.6 2d $\to$ 1d reduction

We summarize our discussion as follows.

| 1d TQM | 2d chiral QFT |
|---|---|
| Associative algebra | Vertex/chiral algebra |
| Hochschild homology | Chiral homology |
| QME $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$ | QME $(\hbar\Delta + d_{\text{ch}})\langle - \rangle_{2d} = 0$ |
| $\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} = \int_{\text{Conf}_n(S^1)}$ | $\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} = ⨍_{\Sigma^n}$ |

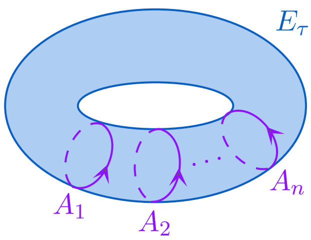
In physics, the partition functions/correlation functions on elliptic curves are described by QM on $S^1$.




$$\langle - \rangle_{2d} \longrightarrow \operatorname{Tr}_{\mathcal{H}} \langle \cdots \rangle$$

Now we can define 2d correlation function using *regularized integral* $⨍_E$. In 1d, operators are described by $A$-cycle $\oint_A$. These two integrals are not exactly the same, but related to each other by *holomorphic anomaly*.

**Theorem 10.11** ([34]). *Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$ which is holomorphic away from diagonals. Let $A_1, \cdots, A_n$ be $n$ disjoint $A$-cycles on the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$.*




*Then the regularized integral*

$$⨍_{E_\tau^n} \left( \prod_{i=1}^n \frac{d^2 z_i}{\operatorname{Im}\tau} \right) \Phi(z_1, \cdots, z_n; \tau)$$

*lies in $\mathscr{O}_{\mathbb{H}}\left[\frac{1}{\operatorname{Im}\tau}\right]$. Moreover, we have*

$$\lim_{\bar\tau \to \infty} ⨍_{E_\tau^n} \left( \prod_{i=1}^n \frac{d^2 z_i}{\operatorname{Im}\tau} \right) \Phi = \frac{1}{n!} \sum_{\sigma \in S_n} \oint_{A_{\sigma(1)}} dz_1 \cdots \oint_{A_{\sigma(n)}} dz_n \, \Phi,$$

*where $S_n$ is the permutation group on $n$. Essentially this means*

$$⨍_{E^n} \xrightarrow{\lim_{\bar\tau\to\infty}} \textit{averaged} \oint_A,$$

*where $⨍_{E^n}$ is almost holomorphic modular, whereas the averaged $\int_A$ is quasi-modular.*

The anti-holomorphic dependence has a precise description.

**Theorem 10.12** ([35]). *Let $\Phi$ be an almost-elliptic function. Then one has*

$$\partial_Y \fint_{E_\tau^n} \Phi = \fint_{E_\tau^n} \partial_Y \Phi - \sum_{i<j} \fint_{E_\tau^{n-1}} \mathrm{Res}_{z_i=z_j}\left((z_i - z_j)\Phi\right).$$

*Here $Y = \frac{1}{\mathrm{Im}\,\tau}$. This gives the* ***holomorphic anomaly equation****.*

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