Lecture 12: Whitehead Theorem

Whitehead Theorem says the homotopy type of a CW complex is "determined" by its homotopy groups.

Definition

We define the category **TopP** of topological pairs where an object

$$(X, A)$$

is a topological space X with a subspace A, and morphisms $(X, A) \to (Y, B)$ are continuous maps

$$f : X \to Y \quad \text{such that} \quad f(A) \subset B.$$

A homotopy between two maps $f_1, f_2 : (X, A) \to (Y, B)$ is a homotopy $F : X \times I \to Y$ between f_0, f_1 such that

$$F|_{X \times t}(A) \subset B \quad \text{for any} \quad t \in I.$$

The quotient category of $\mathbf{TopP}$ by homotopy of maps is denoted by $\mathbf{hTopP}$. The pointed versions are defined similarly and denoted by $\mathbf{TopP}_*$ and $\mathbf{hTopP}_*$. Morphisms in $\mathbf{hTopP}$ and $\mathbf{hTopP}_*$ are denoted by

$$[(X,A),(Y,B)], \qquad [(X,A),(Y,B)]_0.$$

When we work with the convenient category $\mathscr{T}$, we have similar notions of $\mathscr{T}\mathbf{P}$ for a pair of spaces, $\mathrm{h}\mathscr{T}\mathbf{P}$ for the quotient homotopy category, and $\mathscr{T}\mathbf{P}_\star, \mathrm{h}\mathscr{T}\mathbf{P}_\star$ for the pointed cases.

Theorem

Let $f\colon (X,A) \to (Y,B)$ in $\underline{\mathrm{h}\mathscr{T}\mathbf{P}_\star}$. Let $\bar{f} = f|_A$. Then the sequence

$$(X,A)\to(Y,B)\to(C_f,C_{\bar{f}})\to\Sigma(X,A)\to\Sigma(Y,B)\to\Sigma(C_f,C_{\bar{f}})\to\Sigma^2(X,A)\to\cdots$$

is co-exact in $\underline{\mathrm{h}\mathscr{T}\mathbf{P}_\star}$.

This generalizes the co-exact Puppe sequence to the pair case.

Definition

Let $(X, A) \in \underline{\mathscr{T}\mathbf{P}_\star}$. We define the relative homotopy group by

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]_0.$$

We will also write $\pi_n(X, A; x_0)$ when we specify the base point.

Note that

$$(D^n, S^{n-1}) \simeq \Sigma^{n-1}(D^1, S^0), \quad n \geq 2.$$

$\pi_n(X, A)$ is a group for $n \geq 2$ due to the adjunct pair (Σ, Ω).

Lemma

$f\colon (D^n, S^{n-1}) \to (X, A)$ is zero in $\pi_n(X, A)$ if and only if f is homotopic rel S^{n-1} to a map whose image lies in A.

This lemma can be illustrated by the following diagram

$$
\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\
D^n \arrow[ur, dashed, "g"] \arrow[r, bend left] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow"] & X
\end{tikzcd}
$$

Here g maps D^n to A and $g \simeq f$ rel S^{n-1}.

Proof

Assume $[f]_0 = 0$ in $\pi_n(X, A)$. Then we can find a homotopy

$F : D^n \times I \to X$ s.t. $F(-, 0) = x_0$, $F(-, t) \in A$, $F(-, 1) = f(-)$.

Let us view the restriction of F to $S^{n-1} \times I \cup D^n \times \{0\}$ as defining a map (via a natural homeomorphism)

$$g : (D^n, S^{n-1}) \to (X, A).$$

Then F can be viewed as a homotopy $g \simeq f$ rel S^{n-1} as required. Conversely, assume there exists $g : (D^n, S^{n-1}) \to (X, A)$ such that $g \simeq f$ rel S^{n-1}. Let

$$F : D^n \times I \to D^n$$

be a homotopy from the identity to the trivial map. Then

$$F \circ g : D^n \times I \to X$$

shows that $[g]_0 = 0$, hence $[f]_0 = 0$ as well.

Theorem

Let $A \subset X$ in $\mathscr{T}_\star$. Then there is a long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

Here the boundary map ∂ sends $\varphi \in [(D^n, S^{n-1}), (X, A)]_0$ to its restriction to S^{n-1}.

Proof

Consider

$$f\colon (S^0,\{0\})\to (S^0,S^0).$$

Let $\bar{f}=f|_{\{0\}}:\{0\}\to S^0$. It is easy to see that

$$(C_f,C_{\bar{f}})\simeq (D^1,S^0).$$

Since $\Sigma^n(S^0)=S^n,\Sigma(D^n,S^{n-1})=(D^{n+1},S^n)$, the co-exact sequence

$$(S^0,\{0\})\to(S^0,S^0)\to(D^1,S^0)\to(S^1,\{0\})\to(S^1,S^1)\to(D^2,S^1)\to(S^2,\{0\})\to\cdots$$

implies the exact sequence

$$\cdots\to\pi_n(A)\xrightarrow{i_*}\pi_n(X)\xrightarrow{j_*}\pi_n(X,A)\xrightarrow{\partial}\pi_{n-1}(A)\cdots\to\pi_0(X)$$

$\square$

Definition

A pair (X, A) is called n-connected $(n \geq 0)$ if $\pi_0(A) \to \pi_0(X)$ is surjective and

$$\pi_k(X, A; x_0) = 0 \quad \forall 1 \leq k \leq n, x_0 \in A.$$

From the long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

we see that (X, A) is n-connected if and only if for any $x_0 \in A$

$$\begin{cases} \pi_r(A, x_0) \to \pi_r(X, x_0) \text{ is bijective for } r < n \\ \pi_n(A, x_0) \to \pi_n(X, x_0) \text{ is surjective} \end{cases}$$

Definition

A map $f: X \to Y$ is called an n-equivalence ($n \geq 0$) if for any $x_0 \in X$

$$\begin{cases} f_* : \pi_r(X, x_0) \to \pi_r(Y, f(x_0)) \text{ is bijective for } r < n \\ f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \text{ is surjective} \end{cases}$$

f is called weak homotopy equivalence or ∞-equivalence if f is n-equivalence for any $n \geq 0$.

Example

For any $n \geq 0$, the pair (D^{n+1}, S^n) is n-connected.

CW Complex

Lemma

Let X be obtained from A by attaching n-cells. Let (Y, B) be a pair such that

$$\begin{cases} \pi_n(Y, B; b) = 0, \forall b \in B & \text{if} \quad n \geq 1 \\ \pi_0(B) \to \pi_0(Y) \text{ is surjective} & \text{if} \quad n = 0. \end{cases}$$

Then any map from $(X, A) \to (Y, B)$ is homotopic rel A to a map from X to B.

Proof.

This follows from the universal property of push-out

$$\begin{tikzcd}
\coprod S^{n-1} \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\
\coprod D^n \arrow[r] \arrow[urr, dashed] & X \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

Theorem

Let (X, A) be a relative CW complex with relative dimension $\leq n$. Let (Y, B) be n-connected $(0 \leq n \leq \infty)$. Then any map from (X, A) to (Y, B) is homotopic relative to A to a map from X to B.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\
X \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

Proof.

Apply the previous Lemma to

$$A \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

and observe that all embeddings are cofibrations. $\square$

Proposition

Let $f\colon X \to Y$ be a weak homotopy equivalence, P be a CW complex. Then

$$f_* : [P, X] \to [P, Y]$$

is a bijection.

Proof

We can assume f is an embedding and (Y, X) is ∞-connected. Otherwise replace Y by M_f.

Surjectivity is illustrated the diagram (applying previous Theorem to the pair $(P, \emptyset)$)

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ P \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

Injectivity is illustrated by the diagram (observing $P \times I, P \times \partial I$ are CW complexes)

$$\begin{tikzcd} P \times \partial I \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ P \times I \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

Theorem (Whitehead Theorem)

A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.

Proof

Let $f : X \to Y$ be a weak homotopy equivalence between CW complexes. Apply previous Prop to $P = X, Y$, we find bijections

$$f_* : [X, X] \to [X, Y], \quad f_* : [Y, X] \to [Y, Y].$$

Let $g \in [Y, X]$ such that $f_*[g] = 1_Y$. Then $f \circ g \simeq 1_Y$.

On the other hand,

$$f_*[g \circ f] = [f \circ g \circ f] \simeq [f \circ 1] = [f] = f_*[1_X].$$

We conclude $[g \circ f] = 1_X$. Therefore f is a homotopy equivalence. The reverse direction is obvious. $\square$