



Lecture 12: Whitehead Theorem



Whitehead Theorem says the homotopy type of a CW complex is "determined" by its homotopy groups.



Definition

We define the category **TopP** of topological pairs where an object

$$(X, A)$$

is a topological space X with a subspace A , and morphisms $(X, A) \rightarrow (Y, B)$ are continuous maps

$$f: X \rightarrow Y \text{ such that } f(A) \subset B.$$

A homotopy between two maps $f_1, f_2: (X, A) \rightarrow (Y, B)$ is a homotopy $F: X \times I \rightarrow Y$ between f_0, f_1 such that

$$F|_{X \times t}(A) \subset B \text{ for any } t \in I.$$



The quotient category of $\underline{\text{TopP}}$ by homotopy of maps is denoted by $\underline{\text{hTopP}}$. The pointed versions are defined similarly and denoted by $\underline{\text{TopP}}_*$ and $\underline{\text{hTopP}}_*$. Morphisms in $\underline{\text{hTopP}}$ and $\underline{\text{hTopP}}_*$ are denoted by

$$[(X, A), (Y, B)], \quad [(X, A), (Y, B)]_0.$$

When we work with the convenient category $\underline{\mathcal{T}}$, we have similar notions of $\underline{\mathcal{TP}}$ for a pair of spaces, $\underline{\text{hTP}}$ for the quotient homotopy category, and $\underline{\mathcal{TP}}_*$, $\underline{\text{hTP}}_*$ for the pointed cases.



Theorem

Let $f: (X, A) \rightarrow (Y, B)$ in $\underline{h\mathcal{TP}}_*$. Let $\bar{f} = f|_A$. Then the sequence

$$(X, A) \rightarrow (Y, B) \rightarrow (C_f, C_{\bar{f}}) \rightarrow \Sigma(X, A) \rightarrow \Sigma(Y, B) \rightarrow \Sigma(C_f, C_{\bar{f}}) \rightarrow \Sigma^2(X, A) \rightarrow \dots$$

is co-exact in $\underline{h\mathcal{TP}}_*$.

This generalizes the co-exact Puppe sequence to the pair case.



Definition

Let $(X, A) \in \mathcal{TP}_\star$. We define the **relative homotopy group** by

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]_0.$$

We will also write $\pi_n(X, A; x_0)$ when we specify the base point.

Note that

$$(D^n, S^{n-1}) \simeq \Sigma^{n-1}(D^1, S^0), \quad n \geq 2.$$

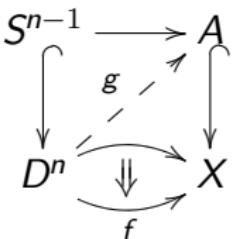
$\pi_n(X, A)$ is a group for $n \geq 2$ due to the adjunct pair (Σ, Ω) .



Lemma

$f: (D^n, S^{n-1}) \rightarrow (X, A)$ is zero in $\pi_n(X, A)$ if and only if f is homotopic rel S^{n-1} to a map whose image lies in A .

This lemma can be illustrated by the following diagram



Here g maps D^n to A and $g \simeq f$ rel S^{n-1} .



Proof

Assume $[f]_0 = 0$ in $\pi_n(X, A)$. Then we can find a homotopy

$$F: D^n \times I \rightarrow X \quad \text{s.t.} \quad F(-, 0) = x_0, \quad F(-, t) \in A, \quad F(-, 1) = f(-).$$

Let us view the restriction of F to $S^{n-1} \times I \cup D^n \times \{0\}$ as defining a map (via a natural homeomorphism)

$$g: (D^n, S^{n-1}) \rightarrow (X, A).$$

Then F can be viewed as a homotopy $g \simeq f$ rel S^{n-1} as required.

Conversely, assume there exists $g: (D^n, S^{n-1}) \rightarrow (X, A)$ such that $g \simeq f$ rel S^{n-1} . Let

$$F: D^n \times I \rightarrow D^n$$

be a homotopy from the identity to the trivial map. Then

$$F \circ g: D^n \times I \rightarrow X$$

shows that $[g]_0 = 0$, hence $[f]_0 = 0$ as well.



Theorem

Let $A \subset X$ in \mathcal{T}_* . Then there is a long exact sequence

$$\cdots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \rightarrow \pi_0(X)$$

Here the boundary map ∂ sends $\varphi \in [(D^n, S^{n-1}), (X, A)]_0$ to its restriction to S^{n-1} .



Proof

Consider

$$f: (S^0, \{0\}) \rightarrow (S^0, S^0).$$

Let $\bar{f} = f|_{\{0\}} : \{0\} \rightarrow S^0$. It is easy to see that

$$(C_f, C_{\bar{f}}) \simeq (D^1, S^0).$$

Since $\Sigma^n(S^0) = S^n$, $\Sigma(D^n, S^{n-1}) = (D^{n+1}, S^n)$, the co-exact sequence

$$(S^0, \{0\}) \rightarrow (S^0, S^0) \rightarrow (D^1, S^0) \rightarrow (S^1, \{0\}) \rightarrow (S^1, S^1) \rightarrow (D^2, S^1) \rightarrow (S^2, \{0\}) \rightarrow \dots$$

implies the exact sequence

$$\dots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \dots \rightarrow \pi_0(X)$$





Definition

A pair (X, A) is called **n-connected** ($n \geq 0$) if $\pi_0(A) \rightarrow \pi_0(X)$ is surjective and

$$\pi_k(X, A; x_0) = 0 \quad \forall 1 \leq k \leq n, x_0 \in A.$$



From the long exact sequence

$$\cdots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \rightarrow \pi_0(X)$$

we see that (X, A) is n -connected if and only if for any $x_0 \in A$

$$\begin{cases} \pi_r(A, x_0) \rightarrow \pi_r(X, x_0) \text{ is bijective for } r < n \\ \pi_n(A, x_0) \rightarrow \pi_n(X, x_0) \text{ is surjective} \end{cases}$$



Definition

A map $f: X \rightarrow Y$ is called an **n -equivalence** ($n \geq 0$) if for any $x_0 \in X$

$$\begin{cases} f_* : \pi_r(X, x_0) \rightarrow \pi_r(Y, f(x_0)) \text{ is bijective for } r < n \\ f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0)) \text{ is surjective} \end{cases}$$

f is called **weak homotopy equivalence** or **∞ -equivalence** if f is n -equivalence for any $n \geq 0$.



Example

For any $n \geq 0$, the pair (D^{n+1}, S^n) is n -connected.



CW Complex



Lemma

Let X be obtained from A by attaching n -cells. Let (Y, B) be a pair such that

$$\begin{cases} \pi_n(Y, B; b) = 0, \forall b \in B & \text{if } n \geq 1 \\ \pi_0(B) \rightarrow \pi_0(Y) \text{ is surjective} & \text{if } n = 0. \end{cases}$$

Then any map from $(X, A) \rightarrow (Y, B)$ is homotopic rel A to a map from X to B .

Proof.

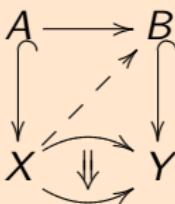
This follows from the universal property of push-out

$$\begin{array}{ccccc} \coprod S^{n-1} & \longrightarrow & A & \longrightarrow & B \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ \coprod D^n & \longrightarrow & X & \xrightarrow{\quad} & Y \end{array}$$



Theorem

Let (X, A) be a relative CW complex with relative dimension $\leq n$.
 Let (Y, B) be n -connected ($0 \leq n \leq \infty$). Then any map from (X, A) to (Y, B) is homotopic relative to A to a map from X to Y .



Proof.

Apply the previous Lemma to

$$A \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

and observe that all embeddings are cofibrations. □



Proposition

Let $f: X \rightarrow Y$ be a weak homotopy equivalence, P be a CW complex. Then

$$f_* : [P, X] \rightarrow [P, Y]$$

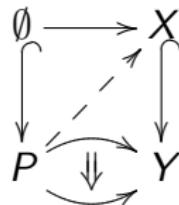
is a bijection.



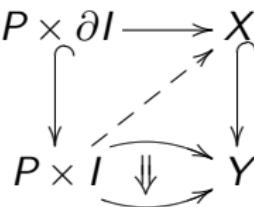
Proof

We can assume f is an embedding and (Y, X) is ∞ -connected.
 Otherwise replace Y by M_f .

Surjectivity is illustrated by the diagram (applying Theorem 0.45 to the pair (P, \emptyset))



Injectivity is illustrated by the diagram (observing $P \times I, P \times \partial I$ are CW complexes)





Theorem (Whitehead Theorem)

A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.



Proof

Let $f: X \rightarrow Y$ be a weak homotopy equivalence between CW complexes. Apply previous Prop to $P = X, Y$, we find bijections

$$f_* : [X, X] \rightarrow [X, Y], \quad f_* : [Y, X] \rightarrow [Y, Y].$$

Let $g \in [Y, X]$ such that $f_*[g] = 1_Y$. Then $f \circ g \simeq 1_Y$.

On the other hand,

$$f_*[g \circ f] = [f \circ g \circ f] \simeq [f \circ 1] = [f] = f_*[1_X].$$

We conclude $[g \circ f] = 1_X$. Therefore f is a homotopy equivalence. The reverse direction is obvious. □