

Classical Mechanics and Geometry

经典力学与几何

(preliminary draft updated July 2023)



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You can also contact me at sili@mail.tsinghua.edu.cn. The draft will be updated on my homepage: <https://sili-math.github.io/>. Thank you.

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Contents

Preface	5
Chapter 1 Lagrangian Mechanics	6
1.1 Principle of Least Action	6
1.1.1 Newtonian Mechanics	6
1.1.2 Action Functional	7
1.1.3 Principle of Least Time	8
1.1.4 Principle of Minimum Energy	9
1.2 Euler-Lagrange Equation	10
1.2.1 Calculus of Variations	10
1.2.2 Comparison with Calculus	12
1.2.3 Examples	13
1.2.4 Plane Motion with Central Force	19
1.3 Noether's Theorem	21
1.3.1 Infinitesimal Symmetry	21
1.3.2 Noether's Theorem	22
1.3.3 Energy, Momentum, and Angular Momentum	24
1.4 Kepler Problem	26
1.4.1 Harmonic Oscillator	27
1.4.2 The Inverse Square Law	28
1.4.3 Laplace-Runge-Lenz Vector	32
1.5 Rigid Body	34
1.5.1 Angular Velocity	35
1.5.2 Inertia Tensor	37
1.5.3 Euler's Equation	40
1.5.4 Free Tops	41
1.5.5 Euler's Equation in Lax Form	43
Chapter 2 Hamiltonian Mechanics	46
2.1 Hamilton's Equations	46
2.1.1 Hamilton's Equations	46
2.1.2 Legendre Transform	49



2.2	Poisson Bracket	51
2.2.1	Phase Space and Poisson Bracket	51
2.2.2	Constant of Motion	53
2.3	Liouville's Theorem	55
2.3.1	Phase Flow and Liouville's Theorem	55
2.3.2	Liouville's Equation	58
2.3.3	Poincaré's Recurrence Theorem	58
2.4	Canonical Transformation	59
2.4.1	Time-independent Canonical Transformation	59
2.4.2	Infinitesimal Canonical Transformation	62
2.4.3	Time-dependent Canonical Transformation	63
2.5	Hamilton-Jacobi Equation	66
2.5.1	Extremal Action and Hamilton-Jacobi Equation	66
2.5.2	Canonical Transformation via Hamilton-Jacobi	72
2.6	Geometric Optics	73
2.6.1	Eikonal Equation	73
2.6.2	Wavefront	75
2.6.3	Maxwell Fisheye	76
Chapter 3 Interlude of Symplectic Geometry		79
3.1	Vector Field and Differential Form	79
3.1.1	Vector Field	79
3.1.2	Differential Form	85
3.1.3	Lie Derivative	89
3.1.4	Stoke's Theorem	91
3.2	Cartan Formula and Poincaré Lemma	96
3.2.1	Cartan Formula	96
3.2.2	Poincaré Lemma	99
3.3	Symplectic Form	101
3.3.1	Symplectic Vector Space	101
3.3.2	Symplectic Form	103
3.3.3	Darboux Theorem	107
3.4	Geometry of Canonical Transformations	108
3.4.1	Canonical Transformation Revisited	108
3.4.2	Poincaré's Integral Invariant	110
3.5	Symplectic Manifold	111
3.5.1	Smooth Manifold	111
3.5.2	Symplectic Manifold	119
3.5.3	Lagrangian Submanifold	120
3.6	Moment Map	120



3.6.1	Lie Group and Lie Algebra	121
3.6.2	Moment Map	124
3.6.3	Symplectic Reduction	128
Chapter 4 Integrable System		132
4.1	Liouville Integrability	132
4.1.1	Liouville Integrability and Liouville Tori	132
4.1.2	Liouville-Arnol'd Theorem	134
4.2	Integrability of Kepler Problem	137
4.2.1	Complete Integrability	137
4.2.2	Action-Angle Variables	138
4.3	Hamilton-Jacobi v.s. Liouville Integrability	142
4.3.1	Local Complete Solution	142
4.3.2	Integrability via Hamilton-Jacobi Theory	144
4.4	Toda Lattice	148
4.4.1	Toda Equations in Lax Form	148
4.4.2	Integrability of Toda Lattice	150
4.5	Calogero-Moser System	152
4.5.1	Calogero-Moser Space via Symplectic Reduction	152
4.5.2	Integrability of Calogero-Moser System	155
Bibliography		158



Preface

In April 2021, Qiu Zhen College (求真书院) was newly established at Tsinghua University under the leadership of Professor Shing-Tung Yau. It homes the distinguished elite mathematics program in China starting in 2021: the “Yau Mathematical Sciences Leaders Program” (丘成桐数学科学领军人才培养计划). This program puts strong emphasis on basic sciences related to mathematics in a broad sense. Though majored in mathematics, students in this program are required to study fundamental theoretical physics such as classical mechanics, electromagnetism, quantum mechanics, and statistical mechanics, in order to understand global perspectives of theoretical sciences. It is an exciting challenge both for students and for instructors.

This preliminary note is written for the course “Classical Mechanics” that I lectured at Qiu Zhen College in the fall semester of 2022. It is to explain key physics ingredients of Lagrangian and Hamiltonian mechanics, as well as their connections with modern geometric development. We put heavy emphasis on different faces of concrete examples in order to understand the bridge between mathematics and physics. Examples such as Toda lattice and Calogero-Moser System are still active research topics nowadays in areas of integrable system, representation theory and mathematical physics. A large part of this note relies on the beautiful books of “Mechanics” by Landau-Lifshitz, and “Mathematical Methods of Classical Mechanics” by Arnol’d, which themselves show different faces of this classical subject. Other useful resources that we consulted are listed at the end of this note.

I would like to thank 杨鹏 and 王进一, who have done amazing jobs of teaching assistant for this course. An early version of this note was typed by 杨鹏, including all those beautiful figures that are better arts than my blackboard drawings. I want to thank 丁徐祉晗 and 刘九和 for their help on careful proofreading of this note, as well as their important roles of being excellent students for the whole semester. I want to thank my colleague 周杰, the collaboration and discussion with whom in this year have kept my brain fresh during the preparation of this note. Special thank goes to 程子钰 from office of Teaching Affairs at Qiu Zhen College, whose tremendous help has saved me alive from heavy administrative service to finish this note.

静斋

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Chapter 1 Lagrangian Mechanics

1.1 Principle of Least Action

1.1.1 Newtonian Mechanics

Recall **Newton's Second Law**

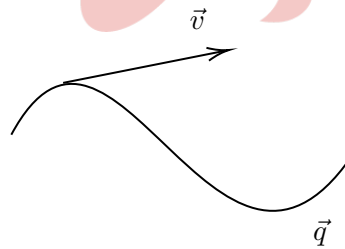
$$\vec{F} = m\vec{a}.$$

Consider a particle moving in \mathbb{R}^n at time $t \in \mathbb{R}$ with position $\vec{q}(t) \in \mathbb{R}^n$. Let us describe this motion as a map

$$\begin{aligned}\vec{q}: \mathbb{R} &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \vec{q}(t) = (q_1(t), \dots, q_n(t))\end{aligned}$$

We have

- **velocity:** $\vec{v} = \dot{\vec{q}} = \frac{d\vec{q}}{dt}$
- **acceleration:** $\vec{a} = \dot{\vec{v}} = \frac{d^2\vec{q}}{dt^2}$



Assume the force \vec{F} depends only on the position. Then

$$m\ddot{\vec{q}} = \vec{F}(\vec{q}(t)),$$

which has a unique solution if we fix initial value $(\vec{q}(t_0), \dot{\vec{q}}(t_0))$ at some time t_0 .

Definition 1.1.1. Define the **kinetic energy** of the motion

$$K = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\frac{d\vec{q}}{dt}\right)^2 = \frac{1}{2}\sum_i m\left(\frac{dq_i}{dt}\right)^2.$$