



Lecture 3: Covering and fibration



The goal of this lecture is to develop basic techniques to compute examples of fundamental groups through geometric covering. In particular, we will prove

$$\pi_1(S^1) = \mathbb{Z}.$$

Similar method applies to many other examples.



Fiber bundle and covering



Definition

Let $p : E \rightarrow B$ be in **Top**. A **trivialization** of p over an open set $U \subset B$ is a homeomorphism $\varphi : p^{-1}(U) \rightarrow U \times F$ over U , i.e. , the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & \searrow & \swarrow \\ & U & \end{array}$$

p is called **locally trivial** if there exists an open cover \mathcal{U} of B such that p has a trivialization over each open $U \in \mathcal{U}$. Such p is called a **fiber bundle**, F is called the **fiber** and B is called the **base**.



We denote it by

$$F \rightarrow E \rightarrow B$$

If we can find a trivialization of p over the whole B , then E is homeomorphic to $F \times B$

$$E \cong F \times B$$

and we say p is a **trivial fiber bundle**.



Example

The projection map

$$\mathbb{R}^{m+n} \rightarrow \mathbb{R}^n, \quad (x_1, \dots, x_n, \dots, x_{n+m}) \rightarrow (x_1, \dots, x_n)$$

is a trivial fiber bundle with fiber \mathbb{R}^m .

Example

A real vector bundle of rank n over a manifold is a fiber bundle with fiber \mathbb{R}^n .



Example

We identify S^{2n+1} as the unite sphere in \mathbb{C}^{n+1}

$$S^{2n+1} = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + |z_1|^2 + \dots + |z_n|^2 = 1\}.$$

There is a natural free S^1 -action on S^{2n+1} given by

$$e^{i\theta} : (z_0, \dots, z_n) \rightarrow (e^{i\theta} z_0, \dots, e^{i\theta} z_n), \quad e^{i\theta} \in S^1.$$

The orbit space is the n -dim complex projective space \mathbb{CP}^n

$$S^{2n+1}/S^1 \cong \mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*.$$

Then the projection $S^{2n+1} \rightarrow \mathbb{CP}^n$ is a nontrivial fiber bundle with fiber S^1 . The case when $n = 1$ gives the Hopf fibration

$$S^1 \rightarrow S^3 \rightarrow S^2.$$



Definition

A **covering (space)** (F -covering) is a locally trivial map $p : E \rightarrow B$ with **discrete fiber** F . A covering map which is a trivial fiber bundle is also called a **trivial covering**. If the fiber F has n points, we also call it a **n -fold covering**.

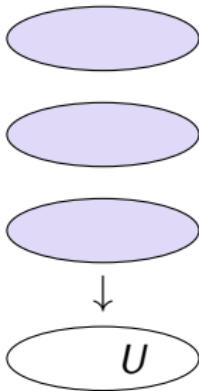


图: Local picture for a 3-fold covering



Example

The map $\exp : \mathbb{R}^1 \rightarrow S^1$, $t \mapsto e^{2\pi i t}$ is a \mathbb{Z} -covering.

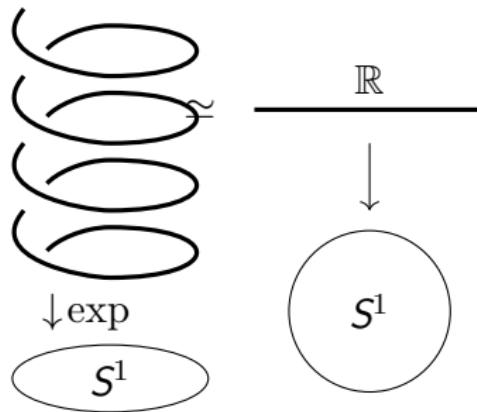


图: The \mathbb{Z} -covering of S^1

If $U = S^1 - \{-1\}$, then

$$\exp^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} \left(n - \frac{1}{2}, n + \frac{1}{2} \right).$$



Example

Denote by \mathbb{RP}^n the real projective space of dimension n , i.e.

$$\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\} / (\underline{x} \sim t\underline{x}), \quad \forall t \in \mathbb{R} - \{0\}, \underline{x} \in \mathbb{R}^{n+1} - \{0\}.$$

Let S^n be the n -sphere. Then there is a natural double cover

$$S^n \rightarrow \mathbb{RP}^n.$$



Example

The map $S^1 \rightarrow S^1$, $e^{2\pi i \theta} \mapsto e^{2\pi i n \theta}$ is $|n|$ -fold covering, $n \in \mathbb{Z} - \{0\}$.

Example

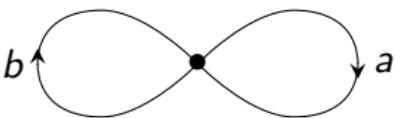
The map $\mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto z^n$, is not a covering (why?). But

- ▶ the map $\mathbb{C}^* \rightarrow \mathbb{C}^*$, $z \mapsto z^n$, is a $|n|$ -covering, where $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $n \in \mathbb{Z} - \{0\}$.
- ▶ the map $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$, $z \mapsto e^{2\pi i z}$ is a \mathbb{Z} -covering.

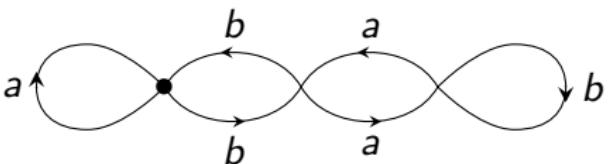
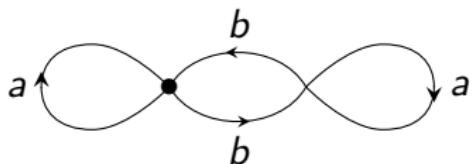


Example

The Figure-8



has two coverings as follows (the left is a 2-fold (or double) covering and the right is a 3-fold covering).





The 4-regular tree is a covering which is simply connected.

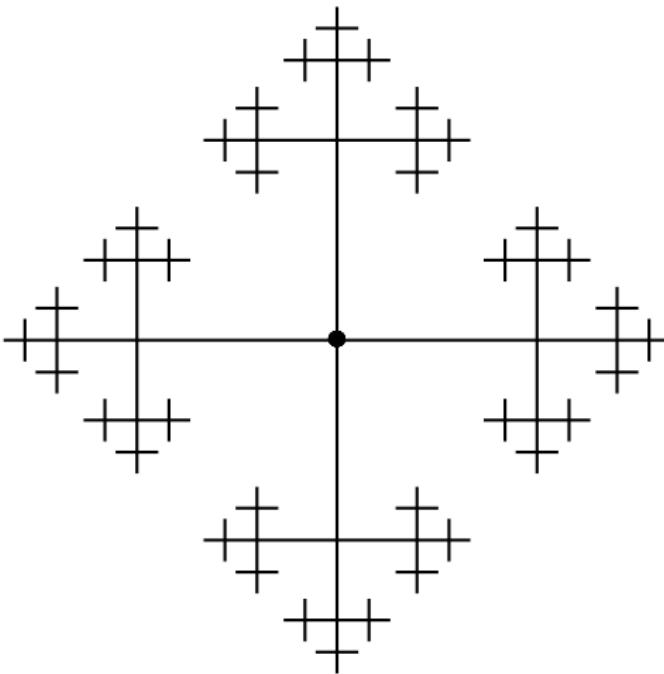


图: 4-regular tree



Example

Denote by $S_{g,b}$ the genus g surface with b boundary components.

- The surface $S_{4,0}$ admits a 7-fold covering from $S_{22,0}$.
- In general, $S_{g,b}$ admits a m -fold covering from $S_{mg-m+1,mb}$.

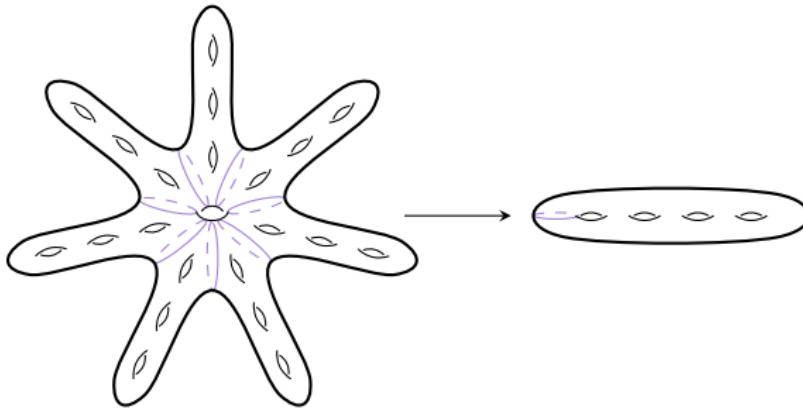


图: A 7-fold covering



Definition

Let $p: E \rightarrow B, f: X \rightarrow B$. A **lifting** of f along p is a map $F: X \rightarrow E$ such that $p \circ F = f$

$$\begin{array}{ccc} & E & \\ F \swarrow & \downarrow p & \\ X & \xrightarrow{f} & B \end{array}$$



Theorem (Uniqueness of lifting)

Let $p : E \rightarrow B$ be a covering. Let $F_0, F_1 : X \rightarrow E$ be two liftings of f . Suppose X is connected and F_0, F_1 agree somewhere. Then

$$F_0 = F_1.$$



We first state a simple lemma before proving the theorem.

Lemma

Let $p : E \rightarrow B$ be a covering. Let

$$D = \{(x, x) \in E \times E \mid x \in E\}$$

$$Z = \{(x, y) \in E \times E \mid p(x) = p(y)\}.$$

Then $D \subset Z$ is both open and closed.

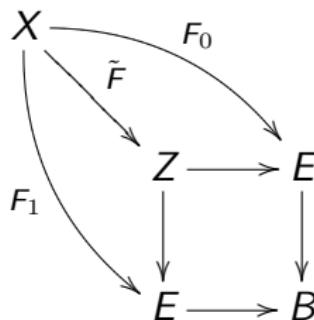


Proof of Theorem

Let D, Z be defined in Lemma. Consider the map

$\tilde{F} = (F_0, F_1) : X \rightarrow Z \subset E \times E$. By assumption, we have

$\tilde{F}(X) \cap D \neq \emptyset$. Moreover, Lemma implies that $\tilde{F}^{-1}(D)$ is both open and closed. Since X is connected, we find $\tilde{F}^{-1}(D) = X$ which is equivalent to $F_0 = F_1$.





Hurwitz fibration



Definition

A map $p : E \rightarrow B$ is said to have the **homotopy lifting property** (HLP) with respect to X if for any maps $\tilde{f} : X \rightarrow E$ and $F : X \times I \rightarrow B$ such that $p \circ \tilde{f} = F|_{X \times 0}$, there exists a lifting \tilde{F} of F along p such that $\tilde{F}|_{X \times 0} = \tilde{f}$.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{f}} & E \\ \downarrow & \exists \tilde{F} \nearrow & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$



Definition

A map $p : E \rightarrow B$ is called a **fibration** (or **Hurwitz fibration**) if p has HLP for any space.

Theorem

A covering map is a fibration.



Proof

Let $p : E \rightarrow B$, $f : X \rightarrow B$, $\tilde{f} : X \rightarrow E$, $F : X \times I \rightarrow B$ be the data as in the definition of HLP. We only need to show the existence of \tilde{F}_x for some neighbourhood N_x of any given point $x \in X$.

$$\begin{array}{ccc}
 & E & \\
 & \swarrow \tilde{F}_x & \downarrow p \\
 N_x \times I & \xrightarrow{F} & B
 \end{array}$$

In fact, for any two such neighbourhoods N_x and N_y with $N_x \cap N_y \neq \emptyset$, we have $\tilde{F}_x|_{N_0}$ and $\tilde{F}_y|_{N_0}$ agree at some point by $\tilde{f}|_{N_0}$ and hence agree everywhere in $N_x \cap N_y$ by the uniqueness of lifting. Thus $\{\tilde{F}_x \mid x \in X\}$ glue to give the required lifting \tilde{F} .



Next, we proceed to prove the existence. Since I is compact, given $x \in X$ we can find a neighbourhood N_x and a partition

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that p has a trivialization over open sets

$$U_i \supset F(N_x \times [t_i, t_{i+1}]).$$

Now we construct the lifting \tilde{F}_x on $N_x \times [t_0, t_k]$, for $1 \leq k \leq m$, by induction on k .



- ▶ For $k = 1$, the lifting \tilde{F}_x on $N_x \times [t_0, t_1]$ to one of the sheets of $p^{-1}(U_1)$ is determined by $\tilde{f}|_{N_x \times \{t_0\}}$:

$$\begin{array}{ccc}
 & p^{-1}(U_1) & = \bigsqcup_{\alpha} (\widetilde{U_1})^{\alpha} \\
 & \downarrow p & \\
 N_x \times [t_0, t_1] & \xrightarrow{F} & U_1 \\
 \nearrow \tilde{F}_x & \nearrow & \\
 \end{array}$$

- ▶ Assume that we have constructed \tilde{F}_x on $N_x \times [t_0, t_k]$ for some k . Now, the lifting of \tilde{F}_x on $N_x \times [t_k, t_{k+1}]$ to one of the sheets of $p^{-1}(U_k)$ is determined by $\tilde{f}|_{N_x \times \{t_k\}}$, which can be glued to the lifting on $N_x \times [t_0, t_k]$ by the uniqueness of lifting again. This finish the inductive step.

We obtain a lifting \tilde{F}_x of F on $N_X \times I$ as required. □

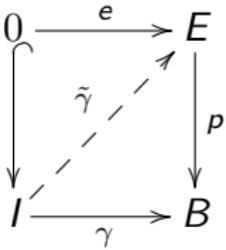


Corollary

Let $p : E \rightarrow B$ be a fibration. Then for any path $\gamma : I \rightarrow B$ and $e \in E$ such that $p(e) = \gamma(0)$, there exists a unique path $\tilde{\gamma} : I \rightarrow E$ which lifts γ and $\tilde{\gamma}(0) = e$.

Proof.

Apply HLP to $X = \text{pt}$.





Corollary

Let $p: E \rightarrow B$ be a covering. Then $\Pi_1(E) \rightarrow \Pi_1(B)$ is a faithful functor. In particular, the map $\pi_1(E, e) \rightarrow \pi_1(B, p(e))$ is injective.

Proof.

Let $\tilde{\gamma}_i: I \rightarrow E$ be two paths and $[\tilde{\gamma}_i] \in \text{Hom}_{\Pi_1(E)}(e_1, e_2)$. Let $\gamma_i = p \circ \tilde{\gamma}_i$. Suppose $[\gamma_1] = [\gamma_2]$ and we need to show that $[\tilde{\gamma}_1] = [\tilde{\gamma}_2]$. Let $F: \gamma_1 \simeq \gamma_2$ be a homotopy. Consider the following commutative diagram with the lifting \tilde{F} by HLP

$$\begin{array}{ccc}
 I \times \{0\} & \xrightarrow{\tilde{\gamma}_1} & E \\
 \downarrow & \exists \tilde{F} \nearrow & \downarrow p \\
 I \times I & \xrightarrow{F} & B
 \end{array}$$

Then the uniqueness of lifting implies $\tilde{F}|_{I \times \{1\}} = \tilde{\gamma}_2$. Thus, $\tilde{\gamma}_1 \simeq \tilde{\gamma}_2$.





Transport functor



Let $p : E \rightarrow B$ be a covering. Let $\gamma : I \rightarrow B$ be a path in B from b_1 to b_2 . It defines a map

$$T_\gamma : p^{-1}(b_1) \rightarrow p^{-1}(b_2)$$

$$e_1 \rightarrow \tilde{\gamma}(1)$$

where $\tilde{\gamma}$ is a lift of γ with initial condition $\tilde{\gamma}(0) = e_1$.

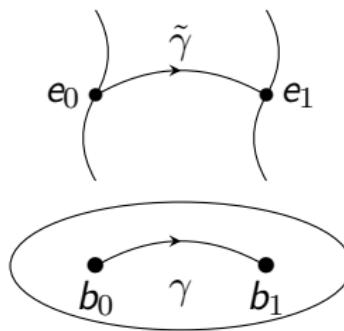


图: The transportation



Assume $[\gamma_1] = [\gamma_2]$ in B . HLP implies that $T_{\gamma_1} = T_{\gamma_2}$. We find a well-defined map:

$$\begin{aligned} T : \text{Hom}_{\Pi_1(B)}(b_1, b_2) &\rightarrow \text{Hom}_{\underline{\text{Set}}}(p^{-1}(b_1), p^{-1}(b_2)) \\ [\gamma] &\mapsto T_{[\gamma]} \end{aligned}$$

Definition

The following data

$$\begin{aligned} T : \Pi_1(B) &\rightarrow \underline{\text{Set}} \\ b &\mapsto p^{-1}(b) \\ [\gamma] &\mapsto T_{[\gamma]}. \end{aligned}$$

defines a functor, called the **transport functor**. In particular, we have a well-defined map

$$\pi_1(B, b) = \text{Aut}_{\Pi_1(B)}(b) \rightarrow \text{Aut}_{\underline{\text{Set}}}(p^{-1}(b)).$$



Example

Consider the covering map

$$\mathbb{Z} \rightarrow \mathbb{R}^1 \xrightarrow{\exp} S^1.$$

Consider the following path representing an element of $\pi_1(S^1)$

$$\gamma_n : I \rightarrow S^1, \quad t \rightarrow \exp(nt) = e^{2\pi i nt}, \quad n \in \mathbb{Z}.$$

Start with any point $m \in \mathbb{Z}$ in the fiber, γ_n lifts to a map to \mathbb{R}^1

$$\tilde{\gamma}_n : I \rightarrow \mathbb{R}^1, \quad t \rightarrow m + nt.$$

We find $T_{[\gamma_n]}(m) = \tilde{\gamma}(1) = m + n$. Therefore $T_{[\gamma_n]} \in \text{Aut}_{\text{Set}}(\mathbb{Z})$ is

$$T_{[\gamma_n]} : \mathbb{Z} \rightarrow \mathbb{Z}, \quad m \mapsto m + n.$$



Proposition

Let $p : E \rightarrow B$ be a covering, E be path connected. Let $e \in E, b = p(e) \in B$. Then the action of $\pi_1(B, b)$ on $p^{-1}(b)$ is transitive, whose stabilizer at e is $\pi_1(E, e)$. In other words,

$$p^{-1}(b) \simeq \pi_1(B, b) / \pi_1(E, e)$$

as a coset space, i.e. we have the following **short exact sequence**

$$1 \rightarrow \pi_1(E, e) \rightarrow \pi_1(B, b) \xrightarrow{\partial_e} p^{-1}(b) \rightarrow 1.$$
$$[\gamma] \mapsto T_\gamma(e)$$



Proof

For any point $e' \in p^{-1}(b)$, let $\tilde{\gamma}: e \rightarrow e'$ be a path in E and $\gamma = p \circ \tilde{\gamma}$. Then $e' = \partial_e(\gamma)$. This shows the surjectivity of ∂_e .

HLP implies that $p_*: \pi_1(E, e) \rightarrow \pi_1(B, b)$ is injective and we can view $\pi_1(E, e)$ as a subgroup of $\pi_1(B, b)$. By definition, for $\tilde{\gamma} \in \pi_1(E, e)$, we have $\partial_e([p \circ \tilde{\gamma}]) = \tilde{\gamma}(1) = e$, i.e. $\pi_1(E, e) \subset \text{stab}_e(\pi_1(B, b))$. On the other hand, if $T_\gamma(e) = e$, then the lift $\tilde{\gamma}$ of γ is a loop, i.e. $\tilde{\gamma} \in \pi_1(E, e)$. Therefore, $\pi_1(E, e) \supset \text{stab}_e(\pi_1(B, b))$. This implies

$$\pi_1(E, e) = \text{stab}_e(\pi_1(B, b)).$$





Example

Consider the covering map

$$\mathbb{Z} \rightarrow \mathbb{R}^1 \xrightarrow{\exp} S^1.$$

Apply the previous proposition, we find an identification (as sets)

$$\deg : \pi_1(S^1) \simeq \mathbb{Z}.$$

This is called the **degree map**. An example of degree n map is

$$S^1 \rightarrow S^1, \quad e^{i\theta} \rightarrow e^{in\theta}.$$



The element $\gamma_n \in \pi_1(S^1)$ with $\deg(\gamma_n) = n$ acts on the fiber \mathbb{Z} as

$$\begin{aligned}T_{\gamma_n} : \mathbb{Z} &\rightarrow \mathbb{Z} \\a &\rightarrow a + n.\end{aligned}$$

It is easy to see that

$$T_{\gamma_n} \circ T_{\gamma_m} = T_{\gamma_{n+m}}.$$

This implies that the degree map

$$\deg : \pi_1(S^1) \rightarrow \mathbb{Z}$$

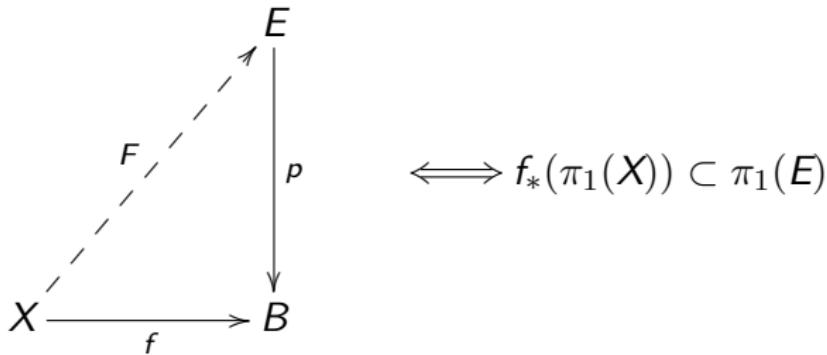
is a group isomorphism. Therefore $\pi_1(S^1) = \mathbb{Z}$.



Theorem (Lifting Criterion)

Let $p: E \rightarrow B$ be a covering. Let $f: X \rightarrow B$ where X is path connected and locally path connected. Let $e_0 \in E, x_0 \in X$ such that $f(x_0) = p(e_0)$. Then there exists a lift F of f with $F(x_0) = e_0$ if and only if

$$f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0)).$$





Proof

If such F exists, then

$$f_*(\pi_1(X, x_0)) = p_*\left(F_*(\pi_1(X, x_0))\right) \subset p_*(\pi_1(E, e_0)).$$

Conversely, let

$$\tilde{E} = \{(x, e) \in X \times E \mid f(x) = p(e)\} \subset X \times E$$

and consider the following commutative diagram

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ \tilde{p} \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B. \end{array}$$



The projection \tilde{p} is also a covering. We have an induced commutative diagram of functors

$$\begin{array}{ccc} \Pi_1(X) & \longrightarrow & \Pi_1(B) \\ & \searrow T & \downarrow T \\ & & \underline{\text{Set}} \end{array}$$

which induces natural group homomorphisms

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(B, b_0) \rightarrow \text{Aut}(\tilde{p}^{-1}(x_0)) = \text{Aut}(p^{-1}(b_0)).$$

Here $b_0 = f(x_0) = p(e)$. Let $\tilde{e}_0 = (x_0, e_0) \in \tilde{E}$. The condition $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$ says that $\pi_1(X, x_0)$ stabilizes \tilde{e}_0 . By the previous Proposition, this implies an isomorphism

$$\tilde{p}_* : \pi_1(\tilde{E}, \tilde{e}_0) \simeq \pi_1(X, x_0).$$



Since X is locally path connected, \tilde{E} is also locally path connected. Then path connected components and connected components of \tilde{E} coincide. Let \tilde{X} be the (path) connected component of \tilde{E} containing \tilde{e} , then $\pi_1(\tilde{E}, \tilde{e}) \simeq \pi_1(X, x_0)$ implies that $\tilde{p} : \tilde{X} \rightarrow X$ is a covering with fiber a single point, hence a homeomorphism. Its inverse defines a continuous map $X \rightarrow \tilde{E}$ whose composition with $\tilde{E} \rightarrow E$ gives F .

$$\begin{array}{ccc}
 \tilde{E} & \longrightarrow & E \\
 \tilde{p} \downarrow & \nearrow & \downarrow p \\
 X & \xrightarrow{f} & B.
 \end{array}$$

