Lecture 5: Limit and colimit

Many constructions in algebraic topology are described by their universal properties. There are two important ways to define new objects of such types, called the limit and colimit, which are dual to each other. In this lecture, we briefly discuss these two notions.

Let $\mathcal{I}$ be a small category (i.e. objects form a set). Let $\mathcal{C}$ be a category. Recall that we have a functor category

$$\mathrm{Fun}(\mathcal{I}, \mathcal{C})$$

where objects are functors and morphisms are natural transformations.

Definition

We define the diagonal (or constant) functor

$$\Delta : \mathcal{C} \to \mathsf{Fun}(\mathcal{I}, \mathcal{C})$$

which assigns $X \in \mathcal{C}$ to the functor $\Delta(X) : \mathcal{I} \to \mathcal{C}$ that sends all objects in $\mathcal{I}$ to X and all morphisms to 1_X.

Limits and colimits amount to understand adjoints of this functor.

Diagram category

Definition

Let $\mathcal{I}$ be a diagram, with objects and arrows. We can define a category still denoted by $\mathcal{I}$

- $\mathrm{Obj}(\mathcal{I})$ = vertices (or objects) in the diagram $\mathcal{I}$
- morphisms are composites of all given arrows as well as additional "identity arrows" that compose like identity maps.

Example

The following diagram

$$\begin{tikzcd} \bullet \arrow[r] & \circ \arrow[d] \\ & \star \end{tikzcd}$$

defines a category with three objects $\bullet, \circ, \star$. There is only one morphism $\bullet \to \circ$, one morphism $\circ \to \star$, and one morphism $\bullet \to \star$ by the composite of the previous two morphisms.

Given $A \in \mathcal{C}$, the constant functor $\Delta(A) : \mathcal{I} \to \mathcal{C}$ is represented by

$$\begin{tikzcd} A \arrow[r, "1_A"] & A \arrow[d, "1_A"] \\ & A \end{tikzcd}$$

Example

The following diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \circ \arrow[d] \\ & \star \end{tikzcd}$$

defines a category with three objects $\bullet, \circ, \star$. There is only one morphism from $\bullet$ to $\circ$, one morphism from $\circ$ to $\star$. There are two morphisms from $\bullet$ to $\star$, one of them is the composite of the previous two, and the other one is represented by $\bullet \to \star$.

Example

The following diagram

$$\bullet \rightleftarrows \circ$$

defines a category with two objects $\bullet, \circ$. Morphisms from $\bullet$ to $\bullet$ contains $1_\bullet$, the composite of $\bullet \to \circ$ and $\circ \to \bullet$ and so on.

Given a diagram $\mathcal{I}$, a functor $F : \mathcal{I} \to \mathcal{C}$ is determined by assigning vertices and arrows the corresponding objects and morphisms in $\mathcal{C}$. For example, the following data

$$X \xrightarrow{f} Y \xleftarrow{g} Z , \qquad X, Y, Z \in \mathcal{C}$$

defines a functor from $\bullet \to \circ \leftarrow \star$ to $\mathcal{C}$. Such a data will be also called a $\mathcal{I}$-shaped diagram in $\mathcal{C}$.

Limit

Definition (Limit)

Let $F\colon \mathcal{I} \to \mathcal{C}$. A limit for F is an object P in $\mathcal{C}$ together with a natural transformation

$$\tau : \Delta(P) \Rightarrow F$$

such that for every object Q of $\mathcal{C}$ and every natural transformation $\eta : \Delta(Q) \Rightarrow F$, there exists a unique map $f\colon Q \to P$ such that $\tau \circ \Delta(f) = \eta$, i.e., the following diagram is commutative.

$$\begin{tikzcd}
\Delta(Q) \arrow[r, Rightarrow, dashed, "\exists! \Delta(f)"] \arrow[dr, Rightarrow, "\eta"'] & \Delta(P) \arrow[d, Rightarrow, "\tau"] \\
& F
\end{tikzcd}$$

For example, consider the following $\mathcal{I}$-shaped diagram in $\mathcal{C}$ which represents a functor $F : \mathcal{I} \to \mathcal{C}$

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Its limit is an object $P \in \mathcal{C}$ that fits into the commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Moreover for any other object Q fitting into the same commutative diagram, there exists a unique $f\colon Q \to A$ to making the following diagram commutative

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}$$

Proposition

Let $F : \mathcal{I} \to \mathcal{C}$ and P_1, P_2 be two limits of F with natural transformations $\tau_i : \Delta(A_i) \Rightarrow F$. Then there exists a unique isomorphism $P_1 \to P_2$ in $\mathcal{C}$ making the following commutative

$$\begin{tikzcd} \Delta(P_1) \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "\tau_1"'] & & \Delta(P_2) \arrow[dl, Rightarrow, "\tau_2"] \\ & F & \end{tikzcd}$$

This proposition follows from the universal property. It implies that if the limit of F exists, then it is unique up to isomorphism.

Definition

We denote the limit of $F : \mathcal{I} \to \mathcal{C}$ by $\lim F$ (if exists).

Theorem

Let $\mathcal{C}$ be a category. Then the following are equivalent

(1) Every $F : \mathcal{I} \to \mathcal{C}$ has a limit

(2) The constant functor $\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ has a right adjoint.

$$\Delta : \mathcal{C} \rightleftarrows \mathrm{Fun}(\mathcal{I}, \mathcal{C}) : \mathbf{lim}$$

In this case, the right adjoint of the constant functor is the limit.

The universal property of the limit gives the adjunction

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(\Delta(X), F) = \mathrm{Hom}_{\mathcal{C}}(X, \mathbf{lim}\, F).$$

Pullback

Example

The limit of the following diagram $X \to Y \leftarrow Z$ gives

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

which is called the pullback.

In **Set**, the pull-back exists and is given by the subset of $X \times Y$

$$P = \{(x, y) \in X \times Y | f(x) = g(y)\} \subset X \times Y.$$

Tower and inverse limit

Example

Consider the following tower-shaped diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r] & X_1
\end{tikzcd}
$$

The limit of tower diagram is also called the inverse limit of the tower and written as $\varprojlim X_i$.

$$
\begin{tikzcd}
 & & Q \arrow[d, dashed, "\exists!"] \arrow[dll] \arrow[drrr] & & & \\
 & & \varprojlim X_i \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\cdots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r] & X_1
\end{tikzcd}
$$

Theorem

Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be adjoint functors. Assume the limit of $F : \mathcal{I} \to \mathcal{D}$ exists. Then the limit of $R \circ F : \mathcal{I} \to \mathcal{C}$ also exists and is given by

$$\mathbf{lim}(R \circ F) = R(\mathbf{lim}\, F).$$

In other words, right adjoint functors preserve limit.

Proof

Let $A \in \mathcal{C}$. Assume we have a natural transformation

$$\tau : \Delta(A) \Rightarrow R \circ F.$$

By adjunction, this is equivalent to a natural transformation

$$\Delta(L(A)) \Rightarrow F.$$

By the universal property of limit, there exists a unique map $L(A) \to \lim F$ factorizing $\Delta(L(A)) \Rightarrow F$

$$\Delta(L(A)) \Rightarrow \lim F \Rightarrow F.$$

By adjunction again, this is equivalent to natural transformations

$$\Delta(A) \Rightarrow R(\lim F) \Rightarrow R \circ F.$$

This implies that $R(\lim F)$ is the limit of $R \circ F$. $\square$

Remark

A functor is called continuous if it preserves all limits. This theorem says if a functor has a left adjoint, then it is continuous.

Under certain conditions, the reverse is also true (Adjoint Functor Theorem).

Corollary

The forgetful functor $\mathrm{Forget} : \underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ preserves limit.

Proof.

$\mathrm{Forget} : \underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ has a left adjoint

$$\text{Discrete} : \underline{\mathbf{Set}} \rightleftarrows \underline{\mathbf{Top}} : \mathrm{Forget}$$

where Discrete associates a set X with discrete topology. $\square$

Example

Consider the following diagram in **Top**

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

We would like to understand its pull-back P in **Top**. By the previous Corollary, the underlying set for P (if exists) is

$$\mathrm{Forget}(P) = \{(x, y) \in X \times Y | f(x) = g(y)\} \subset X \times Y.$$

It is not hard to see that if we assign P is the subspace topology of the topological product $X \times Y$, then P is indeed the pull-back in **Top**. In particular, pull-back exists in **Top**.

Colimit

The notion of colimit is dual to limit.

Definition (Colimit)

Let $F : \mathcal{I} \to \mathcal{C}$. A colimit for F is an object P in $\mathcal{C}$ together with a natural transformation

$$\tau : F \Rightarrow \Delta(P)$$

such that for every object Q of $\mathcal{C}$ and every natural transformation $\eta : F \Rightarrow \Delta(Q)$, there exists a unique map $f : P \to Q$ such that $\Delta(f) \circ \tau = \eta$. In other words, the following diagram is commutative

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\tau"] \arrow[dr, Rightarrow, "\eta"'] & \Delta(P) \arrow[d, Rightarrow, dashed, "\exists!\Delta(f)"] \\ & \Delta(Q) \end{tikzcd}$$

The colimit, if exists, is unique up to isomorphism, and will be denoted by **colim** F.

The following theorems are dual to the limit case as well and can be proved dually.

Theorem

Let $\mathcal{C}$ be a category. Then the following are equivalent

(1) Every $F : \mathcal{I} \to \mathcal{C}$ has a limit

(2) The constant functor $\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ has a left adjoint.

$$\mathbf{colim} : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \rightleftarrows \mathcal{C} : \Delta$$

In this case, the left adjoint of the constant functor is the colimit.

Theorem

Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be adjoint functors. Assume the colimit of $F : \mathcal{I} \to \mathcal{C}$ exists. Then the colimit of $L \circ F : \mathcal{I} \to \mathcal{D}$ also exists and is given by

$$\mathbf{colim}(L \circ F) = L(\mathbf{colim}\, F).$$

In other words, left adjoint functors preserve colimit.

Remark

A functor is called co-continuous if it preserves all colimits.
This says if a functor has a right adjoint, then it is co-continuous.

Under certain conditions, the reverse is also true (Adjoint Functor Theorem).

Corollary

The forgetful functor $\mathrm{Forget} : \underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ preserves colimit.

Proof.

$\mathrm{Forget} : \underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ has a right adjoint

$$\mathrm{Forget} : \underline{\mathbf{Top}} \rightleftarrows \underline{\mathbf{Set}} : \mathrm{Triv}$$

where Triv associates a set X with trivial topology (only open subsets are $\emptyset$ and X). □

Pushout

Example

The colimit of the following diagram $X \leftarrow Y \rightarrow Z$ gives

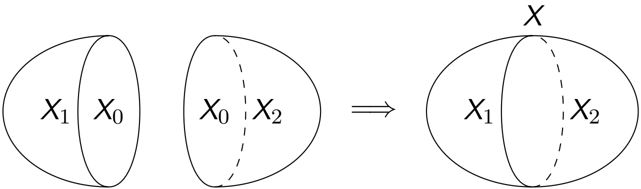
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

This colimit is called the pushout. The universal property is

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists!"] & \\ & & Q \end{tikzcd}$$

Here are some examples.

- Let $j_1 : X_0 \to X_1, j_2 : X_0 \to X_2$ in **Top**. Their pushout is the quotient of the disjoint union $X_1 \coprod X_2$ by identifying $j_1(y) \sim j_2(y), y \in X_0$. It glues X_1, X_2 along X_0 using j_1, j_2.



- Let $\rho_1 : H \to G_1, \rho_2 : H \to G_2$ be two morphisms in **Group**, then their pushout is

$$(G_1 * G_2)/N$$

where $G_1 * G_2$ is the free product and N is the normal subgroup generated by $\rho_1(h)\rho_2^{-1}(h), h \in H$.

Telescope and direct limit

Example

Consider the following telescope-shaped diagram

$$
\begin{tikzcd}
X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r] & \cdots
\end{tikzcd}
$$

The limit of telescope diagram is also called the direct limit of the telescope and written as $\varinjlim X_i$.

$$
\begin{tikzcd}
X_1 \arrow[r] \arrow[drrr] \arrow[ddrrr] & X_2 \arrow[r] \arrow[drr] & \cdots \arrow[r] \arrow[dr] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[dl] & \cdots \arrow[dll] \arrow[ddll] \\
 & & & \varinjlim X_i \arrow[d, dashed, "\exists!"] & & \\
 & & & Q & &
\end{tikzcd}
$$

Product and coproduct

Definition

Let $\mathcal{C}$ be a category, $\{A_\alpha\}_{\alpha\in I}$ be a set of objects in $\mathcal{C}$. Their product is an object A in $\mathcal{C}$ together with $\pi_\alpha : A \to A_\alpha$ satisfying the following universal property: for any X in $\mathcal{C}$ and $f_\alpha : X \to A_\alpha$, there exists a unique morphism $f\colon X \to A$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_\alpha"'] & A \arrow[d, "\pi_\alpha"] \\ & A_\alpha \end{tikzcd}$$

For product of two objects, we have the following diagram

$$
\begin{tikzcd}
 & A_{\times} A_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A_1 & & A_2 \\
 & X \arrow[ul, "f_1"] \arrow[ur, "u_2"'] \arrow[uu, dashed, "\exists ! f"'] &
\end{tikzcd}
$$

The product is a limit. In fact, let us equip the index set I with the category structure such that it has only identity morphisms. Then the data $\{A_\alpha\}_{\alpha \in I}$ is the same as a functor $F: I \to \mathcal{C}$. Their product is precisely $\lim F$. We denote it by

$$\prod_{\alpha \in I} A_\alpha.$$

A useful consequence is that the product is preserved under right adjoint functors (like forgetful functors).

Example

- Let $S_\alpha \in$ **Set**. $\prod\limits_\alpha S_\alpha = \{(s_\alpha) | s_\alpha \in S_\alpha\}$ is the Cartesian product.
- Let $X_\alpha \in$ **Top**. Then $\prod\limits_\alpha X_\alpha$ is the Cartesian product with induced product topology. Namely, we have $X \xrightarrow{f} \prod\limits_\alpha X_\alpha$ is continuous if and only if $\{X \xrightarrow{f_\alpha} X_\alpha\}$ are continuous for any α.
- Let $G_\alpha \in$ **Group**. Then $\prod\limits_\alpha G_\alpha$ is the Cartesian product with induced group structure, i.e.

$$\prod_\alpha G_\alpha = \{(g_\alpha) \mid g_\alpha \in G_\alpha\}$$

with $(g_\alpha) \cdot (g'_\alpha) = (g_\alpha \cdot g'_\alpha)$.

Definition

Let $\mathcal{C}$ be a category, $\{A_\alpha\}_{\alpha\in I}$ be a set of objects in $\mathcal{C}$. Their coproduct is an object A in $\mathcal{C}$ together with $i_\alpha : A_\alpha \to A$ satisfying the following universal property: for any X in $\mathcal{C}$ and $f_\alpha : A_\alpha \to X$, there exists a unique morphism $f\colon A \to X$ such that the following diagram commutes

$$\begin{tikzcd} X & & A \arrow[ll, dashed, "\exists! f"'] \\ & & A_\alpha \arrow[ull, "f_\alpha"] \arrow[u, "\pi_\alpha"'] \end{tikzcd}$$

The coproduct is a colimit. As in the discussion of product, the data $\{A_\alpha\}_{\alpha\in I}$ defines a functor $F: I \to \mathcal{C}$. Their coproduct is precisely $\mathbf{colim}\, F$, which is unique up to isomorphism if it exists. We denote it by

$$\coprod_{\alpha\in I} A_\alpha.$$

A useful consequence is that the coproduct is preserved under left adjoint functors (like free constructions).

Example

- Let $S_\alpha \in$ **Set**. $\coprod\limits_\alpha S_\alpha = \{(s_\alpha) | s_\alpha \in S_\alpha\}$ is the disjoint union of sets.
- Let $X_\alpha \in$ **Top**. Then $\coprod\limits_\alpha X_\alpha$ is the disjoint union of topological spaces. Clearly, continuous maps $\{X_\alpha \xrightarrow{f_\alpha} Y\}$ uniquely extends to $\coprod\limits_\alpha X_\alpha \to Y$.

- Let $G_\alpha \in$ **Group**. Then $\coprod_\alpha G_\alpha$ is the free product of groups.

$$\coprod_\alpha G_\alpha : = \{\text{word of finite length: } x_1 x_2 \cdots x_n \mid x_i \in G_{\alpha_i}\}/\sim,$$

where

$$x_1 \cdots x_i x_{i+1} \cdots x_n \sim x_1 \cdots (x_i \cdot x_{i+1}) \cdots x_n$$

if $x_i, x_{i+1} \in G_\alpha$ and $(x_i \cdot x_{i+1})$ is the group production in G_α. The group structure in $\coprod_\alpha G_\alpha$ is

$$(x_1 \cdots x_n) \cdot (y_1 \cdots y_m) : = x_1 \cdots x_n y_1 \cdots y_m.$$

Given group homomorphisms $G_\alpha \xrightarrow{f_\alpha} H$, it uniquely determines

$$f\colon \coprod_\alpha G_\alpha \to H, \quad x_q \cdots x_n \qquad \mapsto f_{\alpha_1}(x_1) \cdots f_{\alpha_n}(x_n).$$

When there are only finitely many G_α, we will also write

$$\coprod_\alpha G_\alpha =: G_1 \star G_2 \star \cdots \star G_n.$$

Wedge and smash product

Definition

We define the category $\mathbf{Top}_*$ of pointed topological space where

- an object (X, x_0) is a topological space X with a based point $x_0 \in X$
- morphisms are based continuous maps that map based point to based point.

Given a space X, we can define a pointed space X_+ by adding

$$X_+ = X \coprod \star, \quad \text{with basepoint } \star .$$

This defines a functor

$$()_+ : \mathbf{Top} \to \mathbf{Top}_* .$$

On the other hand, we have a forgetful functor

$$\mathrm{Forget} : \mathbf{Top}_* \to \mathbf{Top} .$$

They form an adjoint pair

$$()_+ : \mathbf{Top} \rightleftarrows \mathbf{Top}_* : \mathrm{Forget}$$

This implies that the limit in $\mathbf{Top}_*$ will be the same as the limit in $\mathbf{Top}$. In particular, the product of pointed spaces $\{(X_i, x_i)\}$ in $\mathbf{Top}_*$ is the topological product

$$\prod_i X_i, \quad \text{with base point } \{x_i\}.$$

In $\mathbf{Top}_*$, the coproduct of two pointed spaces X, Y is the wedge product $\vee$. Specifically,

$$X \vee Y = X \coprod Y / \sim$$

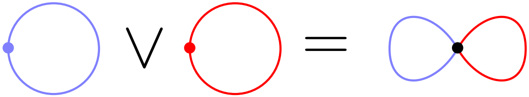
is the quotient of the disjoint union of X and Y by identifying the base points $x_0 \in X$ and $y_0 \in Y$. The identified based point is the new based point of $X \vee Y$. In general, we have

$$\bigvee_{i \in I} X_i = \coprod_{i \in I} X_i / \sim$$

where $\sim$ again identifies all based points in X_i's. In other words, $\bigvee$ is the joining of spaces at a single point.

Example

The Figure-8 can be identified with $S^1 \vee S^1$.



In $\mathbf{Top}_*$, there is another operation, called smash product $\wedge$, which will have adjunction property and play an important role in homotopy theory. Specifically,

$$X \wedge Y = X \times Y/\sim$$

is the quotient of the product space $X \times Y$ under the identifications $(x, y_0) \sim (x_0, y)$ for all $x \in X, y \in Y$. The identified point is the new based point of $X \wedge Y$. Note that we can write it as the quotient

$$X \wedge Y = X \times Y/X \vee Y.$$

Example

There is a natural homeomorphism

$$S^1 \wedge S^n \cong S^{n+1}.$$

This implies that $S^n \wedge S^m \cong S^{n+m}$.

Complete and cocomplete

Definition

A category $\mathcal{C}$ is called complete (cocomplete) if for any $F \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ with $\mathcal{I}$ a small category, the limit $\mathbf{lim}\, F$ ($\mathbf{colim}\, F$) exists.

Example

Set, **Group**, **Ab**, **Vect**, **Top** are complete and cocomplete.

In **Set**, the limit of $F: I \to \mathbf{Set}$ is given by

$$\lim F = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} F(i) \,\middle|\, x_j = F(f)(x_i) \text{ for any } i \xrightarrow{f} j \right\} \subset \prod_{i\in I} F(i)$$

which is a subset of $\prod\limits_{i\in I} F(i)$. The colimit is given by

$$\operatorname{colim} F = \coprod_{i\in I} F(i) \Big/ \left\{ x_i \sim F(f)(x_i) \text{ for any } i \xrightarrow{f} j,\ x_i \in F(i) \right\}$$

which is a quotient of $\coprod_{i\in I} F(i)$.

Let us consider $\mathbf{Top}$. Since the forgetful functor

$$\mathrm{Forget} : \mathbf{Top} \to \mathbf{Set}$$

has both a left adjoint and a right adjoint, it preserves both limits and colimits. Given $F : I \to \mathbf{Top}$, its limit $\mathbf{lim}\, F$ has the same underlying set as that in $\mathbf{Set}$ above, but equipped with the induced topology from product and subspace.

Similarly, the colimit $\mathbf{colim}\, F$ is the quotient of disjoint unions of $F(i)$ with the induced quotient topology.