Electromagnetism and Geometry

电磁与几何

(preliminary draft updated August 2023)

李思 *(Si Li)*

Tsinghua University

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You can also contact me at sili@mail.tsinghua.edu.cn. The draft will be updated on my homepage: https://sili-math.github.io/. Thank you.

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Preface

In April 2021, Qiuzhen College (求真书院) was newly established at Tsinghua University under the leadership of Professor Shing-Tung Yau. It homes the distinguished elite mathematics program in China starting in 2021: the "Yau Mathematical Sciences Leaders Program" (丘成桐数学科学领军人才培养计划). This program puts strong emphasis on basic sciences related to mathematics in a broad sense. Though majored in mathematics, students in this program are required to study fundamental theoretical physics such as classical mechanics, electromagnetism, quantum mechanics, and statistical mechanics, in order to understand global perspectives of theoretical sciences. It is an exciting challenge both for students and for instructors.

This preliminary note is written for the course "Electrodynamics" that I lectured at Qiuzhen College in the spring semester of 2023. The lecture note consists of two parts. The first part is to explain key physics ingredients of electromagnetism, such as Maxwell's equations, electrostatics, magnetostatics, electromagnetic waves, radiation, scattering, etc. The second part is of geometric nature, which explains Maxwell theory as U(1)-gauge theory in terms of fiber bundle theory, as well as its consistency with special relativity. We emphasize on different faces of concrete examples in order to understand the bridge between physics and mathematics. This note is in succession to [10] in this series, and assumes basic knowledge on differential forms and vector fields. Readers can consult [10] for preliminary geometric backgrounds.

I greatly appreciate the "*Notes Taker Program*" at Qiuzhen College, which has triggered and supported the production of this note. A preliminary version of this note was carefully typed by Yang Peng (杨鹏) all the way along the course, and I am extremely grateful to his great job for *Notes Taker*. I would also like to thank Zhou Jiawei (周嘉伟)、Dingxu Zhihan (丁徐祉晗)、Liu Jiuhe (刘九和) and Quan Hanwen (权瀚文) for their help on proofreading of the first version of this note.

@ Jingzhai (静斋)

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This is a drawing by my daughter expressing herself with her brother. I found it interesting as it illustrates precisely the essence of electromagnetism on the coupling of electric and magnetic fields as well as the topological nature of Maxwell's equations.