Lecture 1: Categories and Functors

Topology studies properties (such as shapes) of spaces preserved under continuous deformations: stretching, twisting, bending...

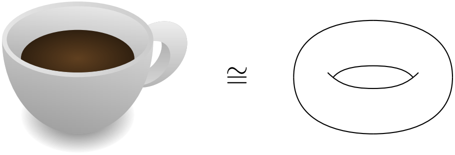


图: Cup vs Torus

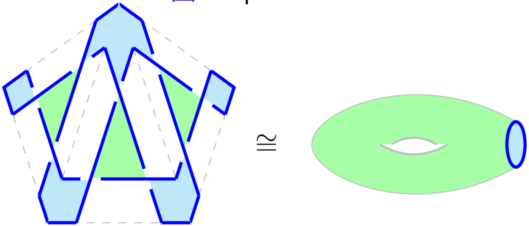
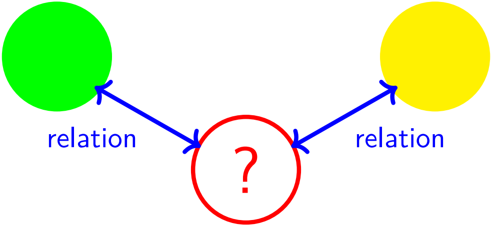


图: Torus via Seifert surface of a knot

Our perspective is categorical as it is the modern (mathematical) language and it helps to reveal analogies between areas in different branches of mathematics.

Roughly speaking, the philosophy of category says that we should study an object by understanding how it is related to the other objects. Most of the time such relation is more important than and actually defines the object itself.



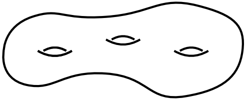
The central object in this lecture is

$$\underline{\mathbf{Top}} := \text{the category of topological spaces.}$$

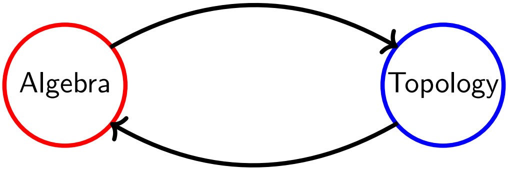
In algebraic topology, we will study functors from $\underline{\mathbf{Top}}$ to certain categories, such as vector spaces, (abelian) groups, rings, etc.:

$$\begin{array}{ccc} \text{“continuous”} & & \text{“discrete”} \\ \underline{\mathbf{Top}} = \{\text{topological spaces}\} & \xrightarrow{\text{functors}} & \{\text{algebraic objects}\}, \end{array}$$

e.g.

 $\longmapsto$ genus$=$ #holes$=3$.

Traditionally, we use (discrete) algebraic tools to study (continuous) topology.



In modern development, topological spaces are essentially used to construct and study many novel and deep algebraic structures (e.g. homotopy algebras, factorization algebras, etc.).

In category theory, we will encounter many presentations in terms of diagrams. Roughly speaking, a diagram is a collection of 'objects' denoted by $A, B, C, X, Y, \cdots$, and 'arrows' between them denoted by $f, g, \cdots$, as in the examples

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "g_1"'] & Y \arrow[d, "f_2"] \\ Z \arrow[r, "g_2"] & W \end{tikzcd}$$

We will alway have an operation $\circ$ to compose arrows.

The diagram is called commutative if all the composite paths between two objects ultimately compose to give the same arrow. For the above examples, they are commutative if

$$h = g \circ f \qquad f_2 \circ f_1 = g_2 \circ g_1.$$

Definition

A category $\mathcal{C}$ consists of

1. A class of objects: $\mathrm{Obj}(\mathcal{C})$
2. A set of morphisms: $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for each $A, B \in \mathrm{Obj}(\mathcal{C})$.

$$A \xrightarrow{f} B \quad \text{or} \quad f\colon A \to B, \qquad f \in \mathrm{Hom}_{\mathcal{C}}(A, B).$$

3. A composition operation $\circ$ between morphisms

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

These are subject to the following axioms:

1. Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ holds, and will be denoted by $h \circ g \circ f$ without ambiguity.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & B \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

2. Identity: for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists the identity element $1_A \in \mathrm{Hom}(A, A)$ such that the following diagrams are commutative

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "f"'] & A \arrow[d, "f"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "1_B"] \\ & B \end{tikzcd}$$

Definition

A morphism $f\colon A \to B$ is called an isomorphism (or invertible) if there exists $g\colon B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$, i.e. we have the following commutative diagram

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[loop right, "1_B"] \end{tikzcd}$$

Two objects A, B are called isomorphic if there exists an isomorphism $f : A \to B$.

Definition

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ (denoted by $\mathcal{C}' \subset \mathcal{C}$) is a category such that

1. $\mathrm{Obj}(\mathcal{C}') \subset \mathrm{Obj}(\mathcal{C})$
2. $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B), \quad \forall A, B \in \mathrm{Obj}(\mathcal{C}')$
3. compositions in $\mathcal{C}'$ coincide with that in $\mathcal{C}$ under the above inclusion.

Definition

$\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$ if

- $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B), \quad \forall A, B \in \mathrm{Obj}(\mathcal{C}')$.

Example

- $\mathcal{C} = \mathbf{Set}$, the category of sets:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{set}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{set map } A \to B\}.$$

- $\mathcal{C} = \mathbf{Vect}_k$, the category of vector spaces over a field k:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{k-vector space}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{k-linear } A \to B\}.$$

  $\mathbf{Vect}_k$ is a subcategory of $\mathbf{Set}$, but not a full subcategory.

Example

- $\mathcal{C} = \mathbf{Group}$, the category of groups:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{group}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{group homo } A \to B\}.$$

- It has a full subcategory

$$\mathbf{Ab}, \text{ the category of abelian groups.}$$

Example

- $\mathcal{C} = \mathbf{Ring}$, the category of rings:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{ring}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{ring homo } A \to B\}.$$

  $\mathbf{Ring}$ is a subcategory of $\mathbf{Ab}$, but not a full subcategory.
- $\mathbf{Ring}$ has a full subcategory

$$\mathbf{CRing}, \text{ the category of commutative rings.}$$

Example

The main object of our interest is

$$\boxed{\underline{\mathbf{Top}}: \ = \text{the category of topological spaces}}$$

- whose objects are topological spaces and
- whose morphisms $f: X \to Y$ are continuous maps.

Example

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We can construct a new category $\mathcal{C} \times \mathcal{D}$, called the product of $\mathcal{C}$ and $\mathcal{D}$, as follows.

- An object of $\mathcal{C} \times \mathcal{D}$ is a pair

$$(X, Y), \quad X \in \mathcal{C}, Y \in D.$$

- A morphism $(f, g) : (X_1, Y_1) \to (X_2, Y_2)$ is pair of

$$f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2), g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2).$$

- Compositions are componentwise.

Let $\mathcal{C}$ be a category. Let $\simeq$ be an equivalence relation defined on each $\mathrm{Hom}_{\mathcal{C}}(A, B), A, B \in \mathrm{Obj}(\mathcal{C})$ and compatible with the composition in the following sense

$$f_1 \simeq f_2, \quad g_1 \simeq g_2 \Longrightarrow g_1 \circ f_1 \simeq g_2 \circ f_2.$$

The compatibility can be represented by the following diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\simeq"] & B \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\simeq"] & C \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[r, bend left, "g_1 \circ f_1"] \arrow[r, bend right, "g_2 \circ f_2"'] \arrow[r, phantom, "\simeq"] & C \end{tikzcd}$$

We say $\simeq$ defines an equivalence relation on $\mathcal{C}$.

Definition

Let $\simeq$ be an equivalence relation on $\mathcal{C}$. The quotient category

$$\mathcal{C}' = \mathcal{C}/ \simeq$$

is defined by

- $\mathrm{Obj}(\mathcal{C}') = \mathrm{Obj}(\mathcal{C}')$
- $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/ \simeq, \quad \forall A, B \in \mathrm{Obj}(\mathcal{C}')$.

Let $I = [0, 1]$.

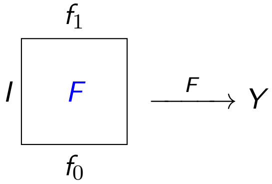
Definition

Two morphisms $f_0, f_1 : X \to Y$ in **Top** are said to be homotopic, denoted by $f_0 \simeq f_1$, if

$$\exists F : X \times I \to Y \text{ such that } F|_{X\times 0} = f_0 \text{ and } F|_{X\times 1} = f_1.$$

We will also write $F : f_0 \simeq f_1$ or $f_0 \overset{F}{\simeq} f_1$ to specify the homotopy F.

This can be illustrated as



Let $f\colon X \to Y$ be a morphism in **Top**. We define its homotopy class

$$[f] : \{g \in \mathrm{Hom}(X, Y) \mid g \simeq f\}.$$

We denote

$$[X, Y] : = \mathrm{Hom}(X, Y)/ \simeq .$$

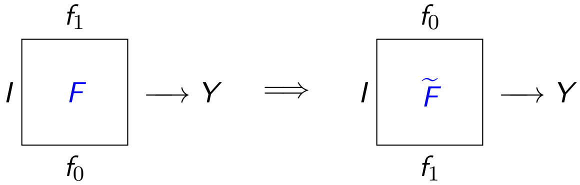
Theorem

Homotopy defines an equivalence relation on **Top**.

Proof:

We first check that $\simeq$ defines an equivalence relation on morphisms.

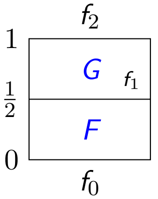
- Reflexivity: Take F such that $F|_{X\times t} = f$ for any $t \in I$.
- Symmetry: Assume we have a homotopy $F: f_0 \simeq f_1$. Then reversing I as



i.e. taking $\widetilde{F}(x,t) = F(x, 1-t): X \times I \to Y$, gives $f_1 \simeq f_0$.

Proof:

- Transitivity: Assume we have two homotopies $F : f_0 \simeq f_1$ and $G : f_1 \simeq f_2$, then putting them together gives $\widetilde{F}$: $f_0 \simeq f_2$ as

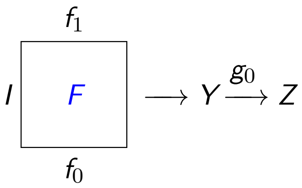


$$\xrightarrow{\widetilde{F}} Y \quad \widetilde{F}(x,t) = \begin{cases} G(x, 2t-1), & 1/2 \le t \le 1; \\ F(x, 2t), & 0 \le t \le 1/2. \end{cases}$$

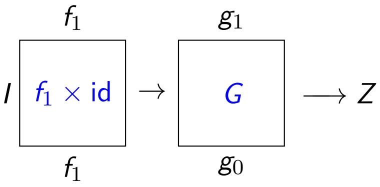
We next check $\simeq$ is compatible with compositions.

Proof:

Let $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$. Assume $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$.



$\Longrightarrow g_0 \circ f_0 \simeq g_0 \circ f_1$



$\Longrightarrow g_0 \circ f_1 \simeq g_1 \circ f_1$

Bt transitivity, $g_0 \circ f_0 \simeq g_0 \circ f_1 \simeq g_1 \circ f_1$. $\square$

We denote the quotient category of $\mathbf{Top}$ under homotopy relation

$$\mathbf{hTop} = \mathbf{Top} / \simeq$$

with morphisms $\mathrm{Hom}_{\mathbf{hTop}}(X, Y) = [X, Y]$.

Definition

Two topological spaces X, Y are said to have the same homotopy type (or homotopy equivalent) if they are isomorphic in $\mathbf{hTop}$.

Example

$\mathbb{R}$ and $\mathbb{R}^2$ are homotopic equivalent, but not homeomorphic. In other words, they are isomorphic in $\mathbf{hTop}$, but not isomorphic in $\mathbf{Top}$. As we will see, $\mathbb{R}^1$ and S^1 are not homotopic equivalent.

Definition

Let $\mathcal{C}, \mathcal{D}$ be two categories. A covariant functor (resp. contravariant functor) $F : \mathcal{C} \to \mathcal{D}$ consists of

1. $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, $A \to F(A)$
2. $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)), \forall A, B \in \mathrm{Obj}(\mathcal{C})$.

$$A \xrightarrow{f} B \Longrightarrow F(A) \xrightarrow{F(f)} F(B)$$

(resp. $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)), \quad \forall A, B \in \mathrm{Obj}(\mathcal{C})$,

$$A \xleftarrow{f} B \Longrightarrow F(B) \xrightarrow{F(f)} F(A). \quad)$$

satisfying

1. $F(g \circ f) = F(g) \circ F(f)$ (resp. $F(g \circ f) = F(f) \circ F(g)$) for any composable morphisms f, g

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \Longrightarrow \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g)\circ F(f)"'] & F(B) \arrow[d, "F(g)"] \\ & F(C) \end{tikzcd}$$

(resp. reversing all arrows in diagram on the right).

2. $F(1_A) = 1_{F(A)}, \quad \forall A \in \mathrm{Obj}(\mathcal{C})$.

Definition

F is called faithful (or full) if $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective (or surjective) $\forall A, B \in \mathrm{Obj}(\mathcal{C})$.

Example

The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ maps

$$1_{\mathcal{C}}(A) = A, \quad 1_{\mathcal{C}}(f) = f$$

for any object A and morphism f.

Example

$\forall X \in \mathrm{Obj}(\mathcal{C})$,

$$\begin{aligned} \mathrm{Hom}(X, -) : \mathcal{C} &\to \mathbf{Set}, \\ A &\mapsto \mathrm{Hom}(X, A) \end{aligned}$$

defines a covariant functor and

$$\begin{aligned} \mathrm{Hom}(-, X) : \mathcal{C} &\to \mathbf{Set}, \\ A &\mapsto \mathrm{Hom}(A, X) \end{aligned}$$

defines a contravariant functor.
Functors of these two types are called representable (by X).

Example

The forgetful functor

$$Forget : \underline{\mathbf{Group}} \to \underline{\mathbf{Set}}$$

(mapping a group to its set of group elements) is representable by the free group with one generator (i.e. $\mathbb{Z}$).

$$G = \mathrm{Hom}_{\underline{\mathbf{Group}}}(\mathbb{Z}, G) \in \underline{\mathbf{Set}} .$$

Example

Let G be an abelian group. Given $X \in$ **Top**, we will study its n-th cohomology $\mathrm{H}^n(X; G)$ with coefficient in G. It defines a contravariant functor

$$\mathrm{H}^n(-; G) : \mathbf{hTop} \to \mathbf{Set}, \quad X \to \mathrm{H}^n(X; G).$$

We will see that this functor is representable by the Eilenberg-Maclane space $K(G, n)$ if we work with the subcategory of CW-complexes.

Example

We define a contravariant functor

$$\mathrm{Fun} : \underline{\mathbf{Top}} \to \underline{\mathbf{Ring}}, \quad X \to \mathrm{Fun}(X) = \mathrm{Hom}_{\underline{\mathbf{Top}}}(X, \mathbb{R}).$$

$\mathrm{Fun}(X)$ are continuous real functions on X. A classical result of Gelfand-Kolmogoroff says that two compact Hausdorff spaces X, Y are homeomorphic (i.e. equivalent in $\underline{\mathbf{Top}}$) if and only if $\mathrm{Fun}(X)$ and $\mathrm{Fun}(Y)$ are ring isomorphism (i.e. equivalent in $\underline{\mathbf{Ring}}$).

Proposition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Suppose $f \colon A \to B$ is an isomorphism in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$.

Definition

Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\tau : F \Rightarrow G$ consists of morphisms

$$\tau = \{\tau_A : F(A) \to G(A) | \forall A \in \mathrm{Obj}(\mathcal{C})\}$$

such that the diagram commutes for any $A, B \in \mathcal{C}$ (here $f: A \to B$ if F, G are covariant and $f: B \to A$ if F, G are contravariant)

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\tau_A"'] & F(B) \arrow[d, "\tau_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

τ is called a natural isomorphism if τ_A is an isomorphism for any $A \in \mathrm{Obj}(\mathcal{C})$ and we write $F \simeq G$.

Example

We consider the following two functors

$$\mathrm{GL}_n, (-)^\times : \underline{\mathbf{CRing}} \to \underline{\mathbf{Group}}.$$

Given a commutative ring $R \in \underline{\mathbf{CRing}}$, $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in R, and $R^\times$ is the multiplicative group of invertible elements of R.

The determinant defines a natural transformation

$$\det : \mathrm{GL}_n \to (-)^\times$$

where $\det_R : \mathrm{GL}_n(R) \to R^\times$ is the determinant of the matrix.

Example

- Let $F : \mathcal{C} \to \mathbf{Set}$ be a contravariant functor. Then any $\varphi \in F(A)$ induces a natural transformation

$$\mathrm{Hom}(-, A) \Rightarrow F$$

by assigning $f \in \mathrm{Hom}(B, A)$ to $F(f)(\varphi) \in F(B)$.

- Let $G : \mathcal{C} \to \mathbf{Set}$ be a covariant functor. Then any $\varphi \in G(A)$ induces a natural transformation

$$\mathrm{Hom}(A, -) \Rightarrow G$$

by assigning $f \in \mathrm{Hom}(A, B)$ to $G(f)(\varphi) \in G(B)$.

Definition

Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors and $\tau_1 : F \Rightarrow G, \tau_2 : G \Rightarrow H$ be two natural transformations. The composition $\tau_2 \circ \tau_1$ is a natural transformation from F to H defined by

$$(\tau_2 \circ \tau_1)_A : F(A) \xrightarrow{\tau_1} G(A) \xrightarrow{\tau_2} H(A), \quad \forall A \in \mathrm{Obj}(\mathcal{C}).$$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left=60, "F"{name=F}] \arrow[rr, "G"{name=G, description}] \arrow[rr, bend right=60, "H"'{name=H}] & & D \arrow[from=F, to=G, Rightarrow, "\tau_1"] \arrow[from=G, to=H, Rightarrow, "\tau_2"] \end{tikzcd} \quad \text{compose} \quad \begin{tikzcd} \mathcal{C} \arrow[rr, bend left=60, "F"{name=F}] \arrow[rr, bend right=60, "H"'{name=H}] & & D \arrow[from=F, to=H, Rightarrow, "\tau_2 \circ \tau_1"] \end{tikzcd}$$

Definition

Two categories $\mathcal{C}, \mathcal{D}$ are called isomorphic if

$$\exists F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$$

such that

$$F \circ G = 1_{\mathcal{D}}, G \circ F = 1_{\mathcal{C}}.$$

They are called equivalent if $\exists F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that

$$F \circ G \simeq 1_{\mathcal{D}}, G \circ F \simeq 1_{\mathcal{C}}.$$

We say $F : \mathcal{C} \rightrightarrows \mathcal{D}$ gives an isomorphism/equivalence of categories.

In applications, isomorphism is a too strong condition to impose for most interesting functors. Equivalence is more realistic and equally good essentially. The following proposition will be very useful.

Proposition

Let $F : \mathcal{C} \Rightarrow \mathcal{D}$ be an equivalence of categories. Then F is fully faithful.

Definition

Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ be a category. We define the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

- Objects: functors from $\mathcal{C}$ to $\mathcal{D}$

$$F : \mathcal{C} \to \mathcal{D}.$$

- Morphisms: natural transformations between two functors (which is indeed a set since $\mathcal{C}$ is small).

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd}$$

Theorem (Yoneda Lemma)

Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. Denote the two functors

$$h_A = \mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \to \mathbf{Set}, \quad h^A = \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}.$$

1. Contravariant version: Let $F : \mathcal{C} \to \mathbf{Set}$ be a contravariant functor. Then there is an isomorphism of sets

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathbf{Set})}(h_A, F) \cong F(A).$$

   This isomorphism is functorial in A.

2. Covariant version: Let $G : \mathcal{C} \to \mathbf{Set}$ be a covariant functor. Then there is an isomorphism of sets

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathbf{Set})}\left(h^A, G\right) \cong G(A).$$

   This isomorphism is functorial in A.

The precise meaning of functoriality in A is that we have isomorphisms of functors $\mathcal{C} \to \mathbf{Set}$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}\left(h_{(-)}, F\right) \cong F(-), \quad \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}\left(h^{(-)}, G\right) \cong G(-).$$

Adjunction

Let $\mathcal{C}, \mathcal{D}$ be two categories, and let

$$L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C}$$

be two (covariant) functors. The rules $(A \in \mathcal{C}, D \in \mathcal{D})$

$$(A, B) \to \mathrm{Hom}_{\mathcal{D}}(L(A), B), \quad (A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, R(B))$$

define two functors

$$\mathrm{Hom}_{\mathcal{D}}(L(-), -), \mathrm{Hom}_{\mathcal{C}}(-, R(-)) : \quad \mathcal{C} \times \mathcal{D} \to \underline{\mathbf{Set}}\,.$$

We say L and R are adjoint to one another (more precisely, L is the left adjoint, R is the right adjoint), if there is a natural isomorphism

$$\tau : \mathrm{Hom}_{\mathcal{D}}(L(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, R(-));$$

that is, for each $A \in \mathcal{C}$, $B \in \mathcal{D}$, we have a set isomorphism

$$\tau_{A,B} : \mathrm{Hom}_{\mathcal{D}}(L(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R(B))$$

and this isomorphism is functorial both in A and in B. We sometimes write adjoint functors as

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R.$$

Example (Free vs Forget)

Let X be a set, and $F(X) = \bigoplus_{x \in X} \mathbb{Z}$ denote the free abelian group generated by X. This defines a functor

$$F : \underline{\mathbf{Set}} \to \underline{\mathbf{Ab}}, \quad X \to F(X).$$

Forgetting the group structure defines a functor (such functor is often called a forgetful functor)

$$G : \underline{\mathbf{Ab}} \to \underline{\mathbf{Set}}, \quad A \to A.$$

These two functors are adjoint to each other

$$F : \underline{\mathbf{Ab}} \rightleftarrows \underline{\mathbf{Set}} : G.$$

In fact, many "free constructions" in mathematics are left adjoint to certain forgetful functors.