



Lecture 2: Fundamental Groupoid



Path connected components



Definition

Let $X \in \underline{\text{Top}}$.

- ▶ A map $\gamma : I \rightarrow X$ is called a **path** from $\gamma(0)$ to $\gamma(1)$.
- ▶ We denote γ^{-1} be the path from $\gamma(1)$ to $\gamma(0)$ defined by $\gamma^{-1}(t) = \gamma(1 - t)$
- ▶ We denote $i_{x_0} : I \rightarrow X$ be the constant map to $x_0 \in X$.

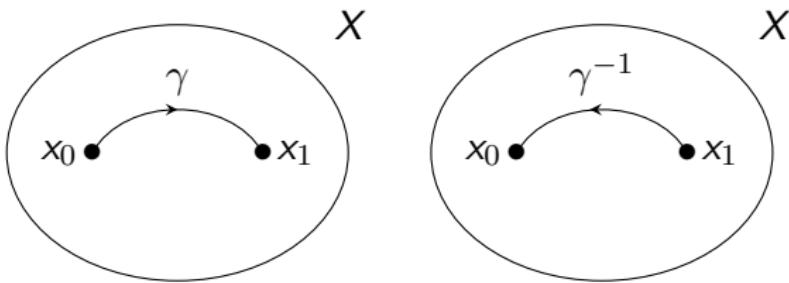


图: A path γ in a topological space X and its inverse



Let us introduce an equivalence relation on X by

$$x_0 \sim x_1 \iff \exists \text{ a path from } x_0 \text{ to } x_1.$$

We denote the quotient space

$$\pi_0(X) = X / \sim$$

which is the set of **path connected components** of X .

Theorem

$\pi_0: \underline{\text{hTop}} \rightarrow \underline{\text{Set}}$ defines a covariant functor.

Corollary

If X, Y are homotopy equivalent, then $\pi_0(X) \simeq \pi_0(Y)$.



Path category / fundamental groupoid



Definition

Let $\gamma : I \rightarrow X$ be a path. We define the **path class** of γ by

$$[\gamma] = \{\tilde{\gamma} : I \rightarrow X \mid \gamma \simeq \tilde{\gamma} \text{ rel } \partial I = \{0, 1\}\}.$$

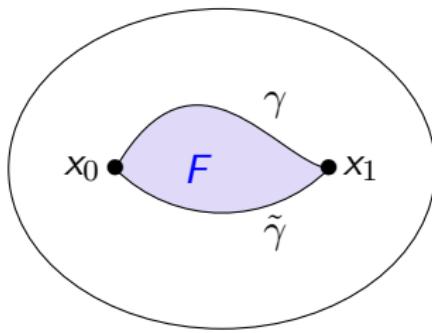


图: In a path class, $F: \gamma \simeq \tilde{\gamma}$ rel ∂I

$[\gamma]$ is the class of all paths that can be continuously deformed to γ while fixing the endpoints.



Definition

Let $\gamma_1, \gamma_2 : I \rightarrow X$ such that $\gamma_1(1) = \gamma_2(0)$. We define the **composite path**

$$\gamma_2 * \gamma_1 : I \rightarrow X$$

by

$$\gamma_2 * \gamma_1(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t - 1) & 1/2 \leq t \leq 1, \end{cases}$$

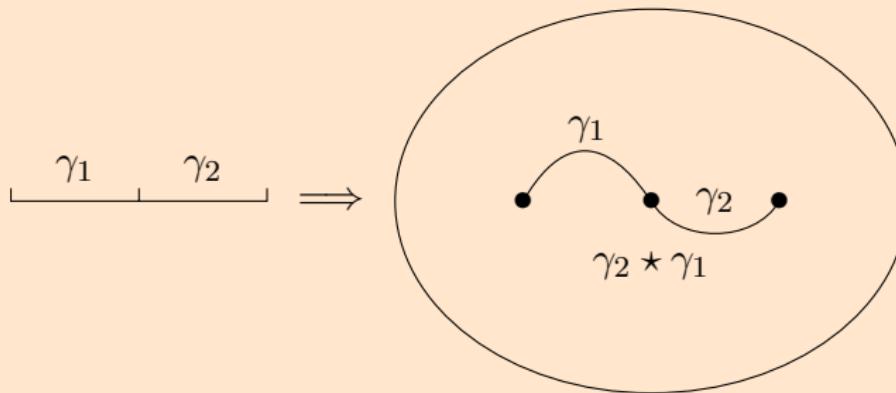


图: Composition of paths



Proposition

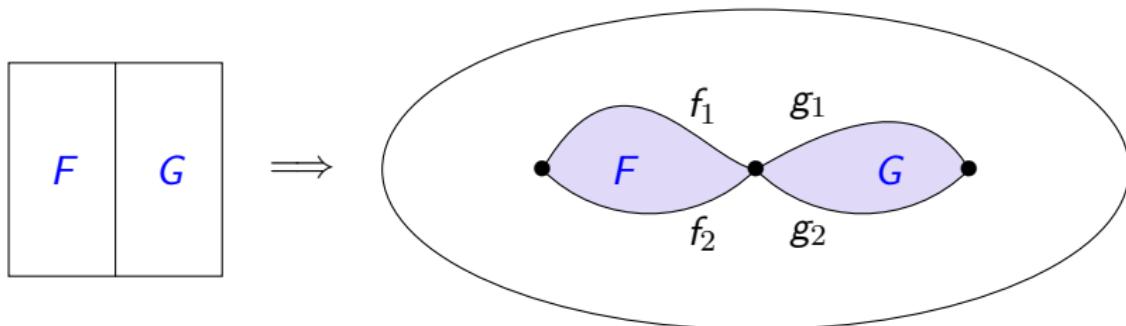
Let f_1, f_2, g_1, g_2 be paths, such that $f_i(1) = g_i(0)$, $[f_1] = [f_2]$, $[g_1] = [g_2]$. Then

$$[g_1 \star f_1] = [g_2 \star f_2].$$

Therefore \star is well-defined for path classes.

Proof.

We illustrate the proof as follows





Proposition (Associativity)

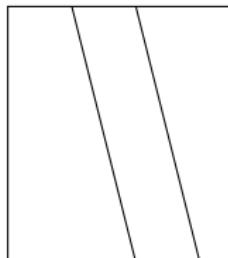
Let $f, g, h: I \rightarrow X$ with $f(1) = g(0)$ and $g(1) = h(0)$. Then

$$([h] \star [g]) \star [f] = [h] \star ([g] \star [f]).$$

Proof.

We illustrate the proof as follows

$$[h] \star ([g] \star [f]) = \begin{array}{ccc} f & g & h \end{array}$$



$$\xrightarrow{F} X$$

$$([h] \star [g]) \star [f] = \begin{array}{ccc} f & g & h \end{array}$$



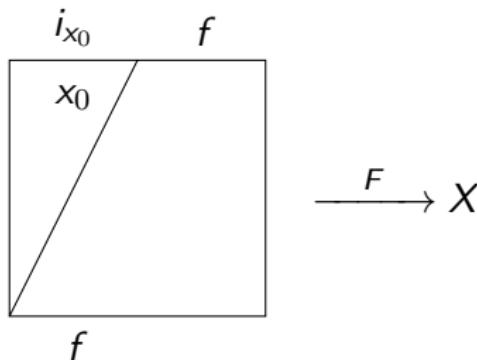
Proposition

Let $f: I \rightarrow X$ with endpoints $f(0) = x_0$ and $f(1) = x_1$. Then

$$[f] \star [i_{x_0}] = [f] = [i_{x_1}] \star [f].$$

Proof.

We only show the first equality, which follows from the figure





Definition

Let $X \in \underline{\text{Top}}$. We define a category $\Pi_1(X)$ as follows:

- ▶ $\text{Obj}(\Pi_1(X)) = X$.
- ▶ $\text{Hom}_{\Pi_1(X)}(x_0, x_1) = \text{path classes from } x_0 \text{ to } x_1$.
- ▶ $1_{x_0} = i_{x_0}$.

The propositions above imply $\Pi_1(X)$ is a well-defined category. $\Pi_1(X)$ is called the **path category** or **fundamental groupoid** of X .



Groupoid



Definition

A category where all morphisms are isomorphisms is called a **groupoid**. All groupoids form a category **Groupoid**.

Example

A group G can be regarded as a groupoid \underline{G} with

- ▶ $\text{Obj}(\underline{G}) = \{\star\}$ consists of a single object.
- ▶ $\text{Hom}_{\underline{G}}(\star, \star) = G$ and composition is group multiplication.

Thus we have a fully faithful functor **Group** \rightarrow **Groupoid**.



Theorem

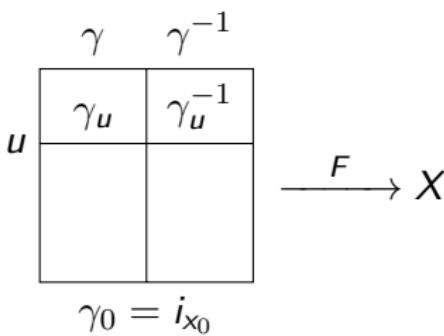
Let $\gamma: I \rightarrow X$ with endpoints $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then

$$[\gamma] \star [\gamma^{-1}] = [1_{x_1}], \quad \text{and} \quad [\gamma^{-1}] \star [\gamma] = [1_{x_0}].$$

Therefore \$\Pi_1(X)\$ is a groupoid.

Proof.

Let $\gamma_u: I \rightarrow X$ such that $\gamma_u(t) = \gamma(tu)$. The following figure gives the homotopy $\gamma^{-1} \star \gamma \simeq 1_{x_0}$:





Definition

Let \mathcal{C} be a groupoid. Let $A \in \text{Obj}(\mathcal{C})$, we define its **automorphism group** by

$$\text{Aut}_{\mathcal{C}}(A) := \text{Hom}_{\mathcal{C}}(A, A).$$

Note that this indeed forms a group.

$$\text{Aut}_{\mathcal{C}}(A) = \left\{ \subset A \right\}$$



For any $f: A \rightarrow B$, it induces a group isomorphism

$$\begin{aligned}\text{Ad}_f: \text{Aut}_{\mathcal{C}}(A) &\rightarrow \text{Aut}_{\mathcal{C}}(B) \\ g &\mapsto f \circ g \circ f^{-1}.\end{aligned}$$

Here is a figure to illustrate

$$\text{Ad}_f: \text{maps } g \subset A \text{ to } g \subset A \xrightleftharpoons[f^{-1}]{f}$$



This naturally defines a functor

$\mathcal{C} \rightarrow \underline{\text{Group}}$ by assigning $A \mapsto \text{Aut}_{\mathcal{C}}(A)$, $f \mapsto \text{Ad}_f$.

Specialize this to topological spaces, we find a functor

$\Pi_1(X) \rightarrow \underline{\text{Group}}$.

Definition

Let $x_0 \in X$, the group

$$\pi_1(X, x_0) := \text{Aut}_{\Pi_1(X)}(x_0)$$

is called the fundamental group of the pointed space (X, x_0) .



Theorem

Let X be path connected. Then for $x_0, x_1 \in X$, we have a group isomorphism

$$\pi_1(X, x_0) \simeq \pi_1(X, x_1).$$

Proof.

Consider the functor $\Pi_1(X) \rightarrow \text{Group}$ described above. Since X is path connected and $\Pi_1(X)$ is a groupoid, any two points x_0 and x_1 are isomorphic in $\Pi_1(X)$. Since functors preserves isomorphism, we conclude $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. □

In the path connected case, we will simply denote by $\pi_1(X)$ the **fundamental group** without mentioning the reference point.



Let $f: X \rightarrow Y$ be a continuous map. It defines a functor

$\Pi_1(f): \Pi_1(X) \rightarrow \Pi_1(Y)$ by assigning $x \mapsto f(x)$, $[\gamma] \mapsto [f \circ \gamma]$.

Proposition

Π_1 defines a functor

$$\boxed{\Pi_1 : \underline{\text{Top}} \rightarrow \underline{\text{Groupoid}}},$$

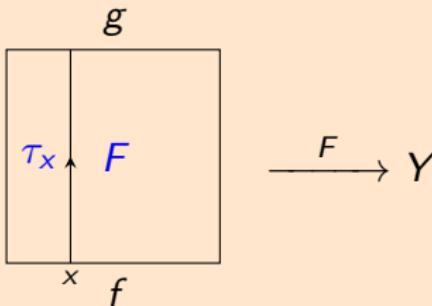
that sends X to $\Pi_1(X)$.



Proposition

Let $f, g: X \rightarrow Y$ be maps which are homotopic by $F: X \times I \rightarrow Y$.
Let us define path classes

$$\tau_x = [F|_{x \times I}] \in \text{Hom}_{\Pi_1(Y)}(f(x), g(x)),$$



Then τ defines a **natural transformation**

$$\tau_F: \Pi_1(f) \Rightarrow \Pi_1(g).$$



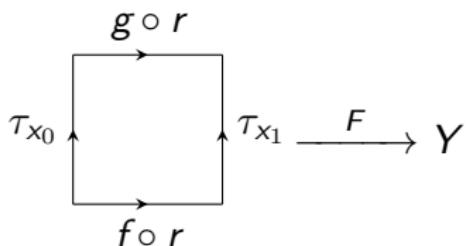
Proof:

Let $r: I \rightarrow X$ with $r(t) = x_t$. We only need to show that the following diagram is commutative at the level of path classes:

$$\begin{array}{ccc} f(x_0) = & \Pi_1(f)(x_0) \xrightarrow{f \circ r} \Pi_1(f)(x_1) & = f(x_1) \\ & \downarrow \tau_{x_0} & \downarrow \tau_{x_1} \\ g(x_0) = & \Pi_1(g)(x_0) \xrightarrow{g \circ r} \Pi_1(g)(x_1) & = g(x_1) \end{array}$$



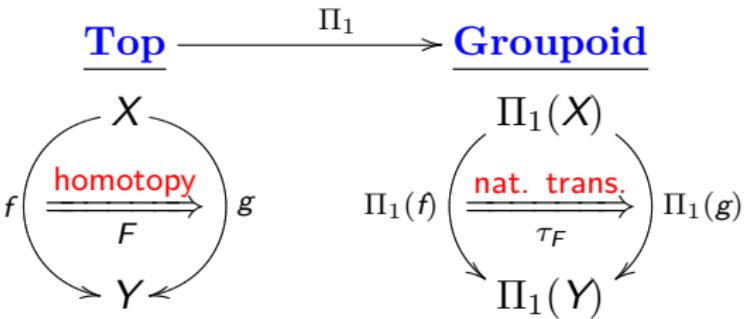
The composition $F \circ (r \times I)$ gives the following diagram:



which implies that $[g \circ r] \star [\tau_{x_0}] = [\tau_{x_1}] \star [f \circ r]$ as required. □



This proposition can be pictured by the following diagram





The following theorem is a formal consequence of the above proposition

Theorem

Let $f: X \rightarrow Y$ be a homotopy equivalence. Then

$$\Pi_1(f) : \Pi_1(X) \rightarrow \Pi_1(Y)$$

is an equivalence of categories. In particular, it induces a group isomorphism

$$\pi_1(X, x_0) \simeq \pi_1(Y, f(x_0)),$$



Proof.

Let $g: Y \rightarrow X$ represent the inverse of f in \mathbf{hTop} . Applying Π_1 to $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$, we find $\Pi_1(f) \circ \overline{\Pi_1(g)}$ and $\Pi_1(g) \circ \Pi_1(f)$ are naturally equivalent to identity functors. Thus the first statement follows.

The second statement follows from the fact that equivalence functors are fully faithful. □



Proposition

Let $X, Y \in \underline{\text{Top}}$. Then we have an isomorphism of categories

$$\Pi_1(X \times Y) \cong \Pi_1(X) \times \Pi_1(Y).$$

In particular, for any $x_0 \in X, y_0 \in Y$, we have a group isomorphism

$$\pi_1(X \times Y, x_0 \times y_0) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0).$$



Example

For a point $X = \text{pt}$, $\pi_1(\text{pt}) = 0$ is trivial. It is not hard to see that \mathbb{R}^n is homotopy equivalent to a point. It follows that

$$\pi_1(\mathbb{R}^n) = 0 \quad n \geq 0.$$



Example

As we will see,

$$\pi_1(S^1) = \mathbb{Z}, \quad \text{and} \quad \pi_1(S^n) = 0, \forall n > 1.$$

Let $T^n = (S^1)^n$ be the n -dim torus. Then

$$\pi_1(T^n) = \mathbb{Z}^n.$$



Example (Braid groups)

Artin's braid group Br_n of n strings has the finite presentation:

$$\text{Br}_n = \langle b_1, \dots, b_{n-1} \mid \begin{array}{l} b_i b_j b_i = b_j b_i b_j \quad \forall |j-i| = 1, \\ b_j b_i = b_i b_j \quad \forall |j-i| > 1 \end{array} \rangle.$$

Braid groups can be realized as fundamental groups.



The n^{th} (ordered) configuration space of X is

$$\text{Conf}_n(X) := \{\underline{x} = (x_1, \dots, x_n) \in X^n \mid x_i \neq x_j, \forall i \neq j\}.$$

It carries a natural action of the permutation group S_n

$$\begin{aligned} S_n \times \text{Conf}_n(X) &\longrightarrow \text{Conf}_n(X) \\ (\sigma, \underline{x}) &\longmapsto \sigma(\underline{x}) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}). \end{aligned}$$

The unordered configuration space of X is the orbit space :

$$\text{conf}_n(X) = \text{Conf}_n(X)/S_n.$$

A classical result says

$$\text{Br}_n \cong \pi_1(\text{conf}_n(\mathbb{R}^2)) \cong \pi_1(\text{conf}_n(D^2)).$$



Fix n distinct points Z_1, \dots, Z_n in \mathbb{R}^2 . A geometric braid is an n -tuple $\Psi = (\psi_1, \dots, \psi_n)$ of paths

$$\psi_i: [0, 1] \rightarrow \mathbb{R}^2$$

such that

- ▶ $\psi_i(0) = Z_i$;
- ▶ $\psi_i(1) = Z_{\nu(i)}$ for some permutation ν of $\{1, \dots, n\}$;
- ▶ $\{\psi_1(t), \dots, \psi_n(t)\}$ are distinct points in \mathbb{R}^2 , for $0 < t < 1$.



The product of geometric braids follows the same way of products of paths (in the fundamental group setting). All braids on \mathbb{R}^2 with the product above form the braid group.

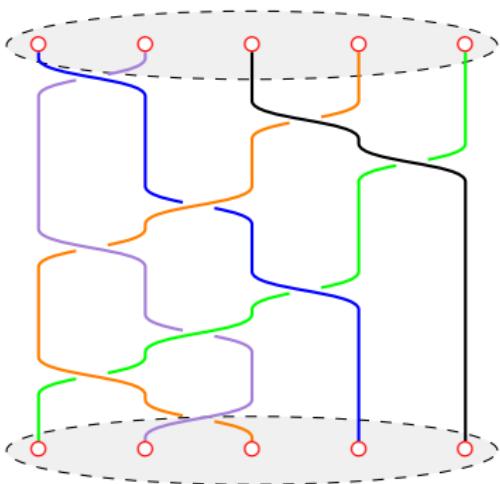


图: Classical braids