



Lecture 6: Seifert-van Kampen Theorem



Seifert-van Kampen Theorem gives a very effective method to compute fundamental groups of a space from its building blocks.



Theorem (Seifert-van Kampen Theorem, Groupoid version)

Let $X = U \cup V$ where $U, V \subset X$ are open. Then the diagram

$$\begin{array}{ccc} \Pi(U \cap V) & \longrightarrow & \Pi(U) \\ \downarrow & & \downarrow \\ \Pi(V) & \longrightarrow & \Pi(X) \end{array}$$

is a pushout in the category [Groupoid](#).



Proof

Let \mathcal{C} be a groupoid fitting into the commutative diagram

$$\begin{array}{ccc} \Pi(U \cap V) & \longrightarrow & \Pi(U) \\ \downarrow & & \downarrow \\ \Pi(V) & \longrightarrow & \mathcal{C} \end{array}$$

and we need to show that

$$\begin{array}{ccccc} \Pi(U \cap V) & \longrightarrow & \Pi(U) & & \\ \downarrow & & \downarrow & & \\ \Pi(V) & \longrightarrow & \Pi(X) & \searrow \exists! F & \\ & \searrow & & & \mathcal{C} \end{array}$$



Uniqueness: Let $\gamma: I \rightarrow X$ be a path in X with $x_t = \gamma(t)$. We subdivide I (by its compactness) into

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $\gamma_i := \gamma(t_{i-1}, t_i)$ lies entirely in U or V . Then

$$F([\gamma]) = F([\gamma_m]) \cdots F([\gamma_1])$$

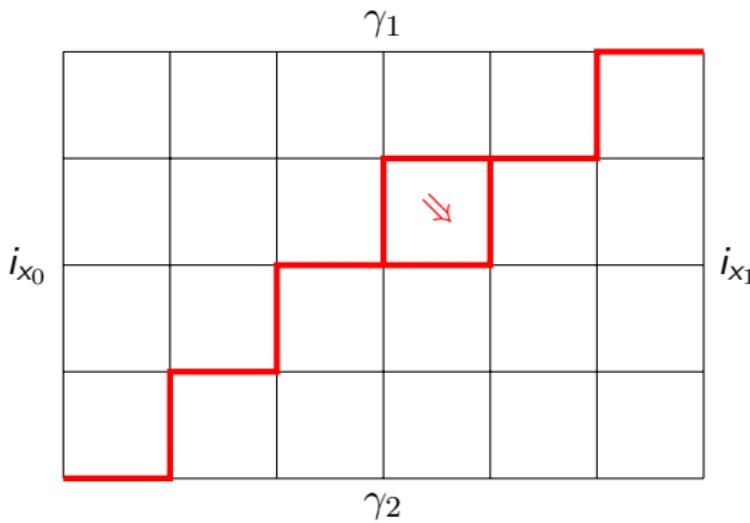
is determined uniquely in \mathcal{C} as each term is.

Existence: Given a path γ , we can define $F([\gamma])$ using a subdivision of γ , where the result does not depend on the choice of the subdivision (by the definition of pushout). We need to show that this is well-defined on homotopy class.



This follows from a refined double subdivision of $I \times I$, as shown in the picture below. Each square represents a homotopy lying entirely in either U or V and combining them together gives the required homotopy.

$$F(\gamma_1) \simeq F(\gamma_1 \star i_{x_0}) \simeq F(i_{x_1} \star \gamma_2) \simeq F(\gamma_2)$$





Theorem (Seifert-van Kampen Theorem, Group version)

Let $X = U \cup V$ where $U, V \subset X$ are open and $U, V, U \cap V$ are path connected. Let $x_0 \in U \cap V$. Then the following diagram

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \longrightarrow & \pi_1(U, x_0) \\ \downarrow & & \downarrow \\ \pi(V, x_0) & \longrightarrow & \pi(X, x_0) \end{array}$$

is a pushout in the category [Group](#).



Proof

Denote by \underline{G} the groupoid with one object for a group G .

For each $x \in X$, we fix a choice of

$$[\gamma_x] \in \text{Hom}(x_0, x)$$

such that γ_x lies entirely in U when $x \in U$ and γ_x lies entirely in V when $x \in V$. Note that γ_x lies entirely in $U \cap V$ when $x \in U \cap V$. This can be achieved since $U, V, U \cap V$ are all path connected.



Consider the following functors

$$\Pi_1(U) \rightarrow \underline{\pi_1(U, x_0)}$$

$$\Pi_1(V) \rightarrow \underline{\pi_1(V, x_0)}$$

$$\Pi_1(U \cap V) \rightarrow \underline{\pi_1(U \cap V, x_0)}$$

$$\gamma \mapsto \gamma_{x_2}^{-1} \star \gamma \star \gamma_{x_1}, \quad \gamma \in \text{Hom}(x_1, x_2).$$

These functors are all retracts in Groupoid, in other words,

$$\underline{\pi_1(U, x_0)} \rightarrow \Pi_1(U) \rightarrow \underline{\pi_1(U, x_0)}$$

$$\underline{\pi_1(V, x_0)} \rightarrow \Pi_1(V) \rightarrow \underline{\pi_1(V, x_0)}$$

$$\underline{\pi_1(U \cap V, x_0)} \rightarrow \Pi_1(U \cap V) \rightarrow \underline{\pi_1(U \cap V, x_0)}$$

are all identity functors.



Suppose there is a group G that fits into the following commutative diagram:

$$\begin{array}{ccccc} \Pi_1(U) & \xleftarrow{\quad} & \Pi_1(U \cap V) & \xrightarrow{\quad} & \Pi_1(V) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\pi_1(U, x_0)} & \xleftarrow{\quad} & \underline{\pi_1(U \cap V, x_0)} & \xrightarrow{\quad} & \underline{\pi_1(V, x_0)} \\ & \searrow & \downarrow & \swarrow & \\ & & G & & \end{array}$$



By Seifert-van Kampen Theorem (Groupoid version),

$$\begin{array}{ccccc} \Pi_1(U) & \xleftarrow{\quad} & \Pi_1(U \cap V) & \xrightarrow{\quad} & \Pi_1(V) \\ \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\ \underline{\pi_1(U, x_0)} & & \Pi_1(X) & & \underline{\pi_1(V, x_0)} \\ & & \downarrow & & \\ & & \exists! F & & \\ & & \downarrow & & \\ & & \underline{G} & & \end{array}$$



Thus, we obtain a morphism

$$\underline{\pi_1(X, x_0)} \hookrightarrow \underline{\Pi_1(X)} \xrightarrow{F} \underline{G}$$

which fits into a commutative diagram

$$\begin{array}{ccc}
 \underline{\pi_1(U \cap V, x_0)} & \longrightarrow & \underline{\pi_1(U, x_0)} \\
 \downarrow & & \downarrow \\
 \underline{\pi(V, x_0)} & \longrightarrow & \underline{\pi(X, x_0)} \\
 & \searrow & \swarrow \\
 & & \underline{G}
 \end{array}$$

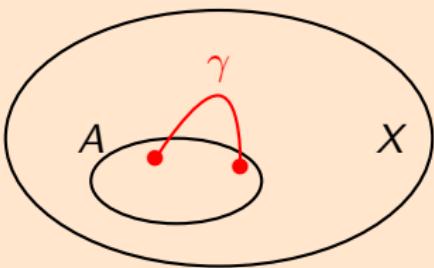
This leads to the required diagram in the category Group. □



We also have the relative version.

Definition

Let $A \subset X$, we define $\Pi_1(X, A)$ be the full subcategory of $\Pi_1(X)$ consists of objects in A .



For instance, when $A = \{x_0\}$, we have

$$\Pi_1(X, x_0) = \pi_1(X, x_0).$$



Theorem

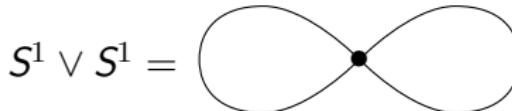
Let $X = U \cup V$, U, V be open and $A \subset X$ intersects each path connected components of $U, V, U \cap V$. Then we have a pushout

$$\begin{array}{ccc} \Pi_1(U \cap V, A) & \longrightarrow & \Pi_1(U, A) \\ \downarrow & & \downarrow \\ \Pi_1(V, A) & \longrightarrow & \Pi_1(X, A). \end{array}$$



Example

For the Figure-8, which is $S^1 \vee S^1$.



It can be decomposed into U, V as follows



Since U, V are homotopic to S^1 , and $U \cap V$ is homotopic to a point, Seifert-van Kampen Theorem implies

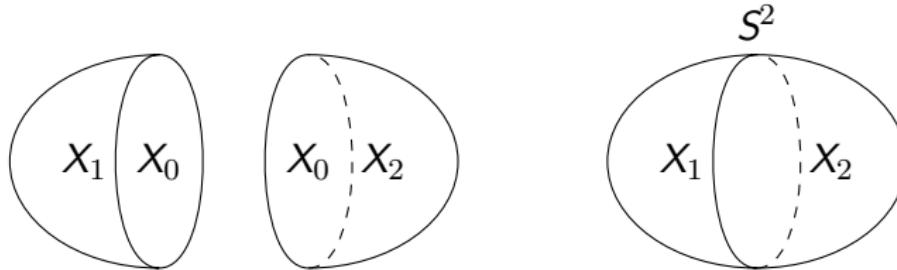
$$\pi_1(S^1 \vee S^1) = \pi_1(S^1) \star \pi_1(S^1) = \mathbb{Z} \star \mathbb{Z}.$$

In general, we have $\pi_1(\bigvee_{i=1}^n S^1) = \underbrace{\mathbb{Z} \star \cdots \star \mathbb{Z}}_n$.



Example

Consider the 2-sphere $S^2 = D_1 \cup D_2$ where D_i are open disks and $D_0 = D_1 \cap D_2$ is an annulus. D_i is an open neighbourhood of X_i .



Since $\pi_1(D_1) = \pi_1(D_2) = 1$, $\pi_1(D_0) = \pi_1(S^1) = \mathbb{Z}$, we deduce that

$$\pi_1(S^2) = (1 \star 1)/\mathbb{Z} = 1.$$

Similar argument shows that

$$\pi_1(S^n) = 1, \quad n \geq 2.$$



Example

Let us identify $X = S^1$ with the unit circle in \mathbb{R}^2 . Consider

$$U = \{(x, y) \in S^1 \mid y > -1/2\}, \quad V = \{(x, y) \in S^1 \mid y < 1/2\}$$

and $A = \{(\pm 1, 0)\}$. Then we obtain a pushout

$$\begin{array}{ccc} \Pi_1(U \cap V, A) & \longrightarrow & \Pi_1(U, A) \\ \downarrow & & \downarrow \\ \Pi_1(V, A) & \longrightarrow & \Pi_1(S^1, A). \end{array}$$



This implies that the groupoid $\Pi_1(S^1, A)$ contains two objects $A_1 = (1, 0)$, $A_2 = (-1, 0)$ with morphisms

$$\text{Hom}_{\Pi_1(S^1, A)}(A_1, A_1) = \{(\gamma_- \gamma_+)^n\}_{n \in \mathbb{Z}}$$

$$\text{Hom}_{\Pi_1(S^1, A)}(A_1, A_2) = \{(\gamma_+ \gamma_-)^n \gamma_+\}_{n \in \mathbb{Z}}$$

$$\text{Hom}_{\Pi_1(S^1, A)}(A_2, A_1) = \{(\gamma_- \gamma_+)^n \gamma_-\}_{n \in \mathbb{Z}}$$

$$\text{Hom}_{\Pi_1(S^1, A)}(A_2, A_2) = \{(\gamma_+ \gamma_-)^n\}_{n \in \mathbb{Z}}.$$

Here γ_+ represent the semi-circle from $(1, 0)$ to $(-1, 0)$ anti-clockwise, and γ_- represent the semi-circle from $(-1, 0)$ to $(1, 0)$ anti-clockwise.

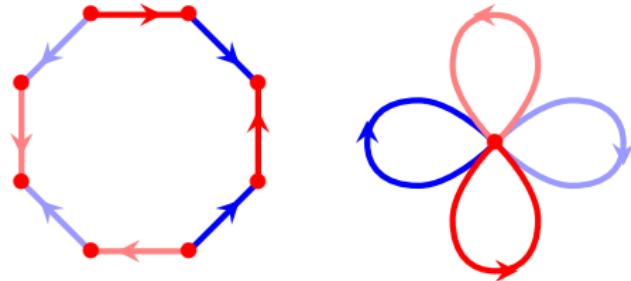


Example

Consider the closed orientable surface Σ_g of genus g , which admits a polygon presentation

$$P = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Here is a figure for $g = 2$.





The edges of the polygon form $V_{2g} = \bigvee_{i=1}^{2g} S^1$.

Let U be the interior of the polygon and V be a small open neighbourhood of V_{2g} . $U \cap V$ is an annulus, which is homotopy to S_1 with generator P as above. Thus

$$\begin{aligned}\pi_1(\Sigma_g) &= \left(\coprod_{i=1}^{2g} \mathbb{Z} \right) * 0/\mathbb{Z} \\ &= \langle a_i, b_i \mid i = 1, \dots, g \rangle / (a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}).\end{aligned}$$



Example

Using the polygon presentation $P = a^2$, we can similarly compute

$$\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$$



The Jordan Curve Theorem



We give an application of Seifert-van Kampen Theorem to prove the Jordan Curve Theorem. This is an example which sounds totally obvious intuitively, but turns out to be very difficult to prove rigorously.



Definition

A simple closed curve is a subset of \mathbb{R}^2 (or S^2) which is homeomorphic to the circle S^1 .

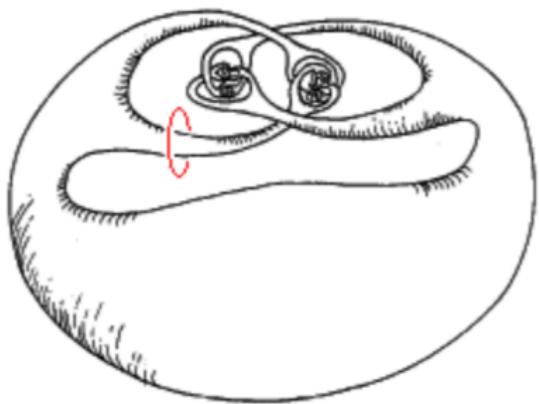
Theorem (The Jordan Curve Theorem)

Let C be a simple closed curve in the sphere S^2 . Then the complement of C has exactly two connected components.



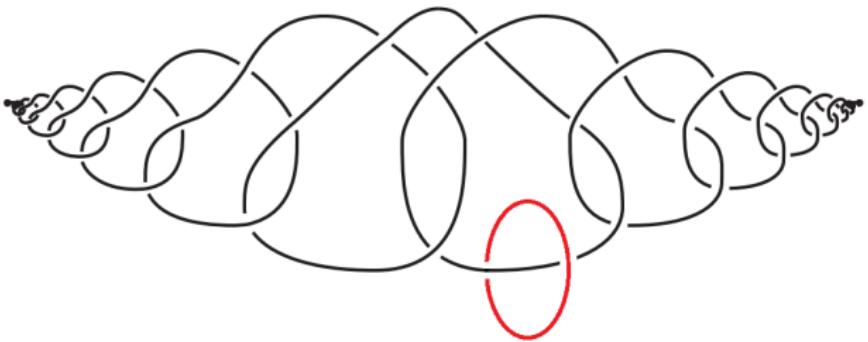


Alexander's Horned Sphere





Wild arc in \mathbb{R}^3





Proof

We sketch a proof here. Since S^2 is locally path connected, we would not distinguish connected and path connected here.

By an **arc**, we mean a subset of S^2 which is homeomorphic to the interval I . We first show

if A is an arc in S^2 , then $S^2 \setminus A$ is connected.



In fact, assume there are two points $\{a, b\}$ which are disconnected in $S^2 \setminus A$. Let us subdivide $A = A_1 \cup A_2$ into two intervals where $A_1 = [0, 1/2]$, $A_2 = [1/2, 1]$ using the homeomorphism $A = [0, 1]$.

We argue that a, b are disconnected in either $S^2 \setminus A_1$ or $S^2 \setminus A_2$.

Let us choose a set D which contains one point from each connected component of $S^2 \setminus A$ and such that $\{a, b\} \subset D$.

Apply Seifert-van Kampen Theorem to

$$V_1 = S^2 \setminus A_1, \quad V_2 = S^2 \setminus A_2, \quad V_1 \cap V_2 = S^2 \setminus A.$$



We obtain a pushout in Groupoid

$$\begin{array}{ccc}
 \Pi_1(V_1 \cap V_2, D) & \longrightarrow & \Pi_1(V_2, D) \\
 \downarrow & & \downarrow \\
 \Pi_1(V_1, D) & \longrightarrow & \Pi_1(Y, D).
 \end{array}$$

Here $Y = V_1 \cup V_2$ is the complement of a point in S^2 . If $\{a, b\}$ are connected in both V_1 and V_2 , then the pushout implies that there exists a nontrivial morphism $a \rightarrow a$ in $\Pi_1(Y, D)$ by a composition

$$a \xrightarrow{\text{in } V_1} b \xrightarrow{\text{in } V_1 \cap V_2} b \xrightarrow{\text{in } V_2} a$$

But this can not be true since Y is contractible. So let us assume a, b are disconnected in $V_1 = S^2 \setminus A_1$. Run the above process replacing A by A_1 , and keep doing this, we end up with contradiction in the limit. This proves our claim above for the arc.



Secondly, we show

the complement of C in S^2 is disconnected.

Otherwise, assume $S^2 \setminus C$ is connected. Let us divide $C = A_1 \cup A_2$ into two intervals A_1, A_2 which intersect at two endpoints $\{a, b\}$.

$$U_1 = S^2 \setminus A_1, U_2 = S^2 \setminus A_2, U_1 \cap U_2 = S^2 \setminus C, X = U_1 \cup U_2 = S^2 \setminus \{a, b\}.$$

Since $U_1, U_2, U_1 \cap U_2$ are all connected, Seifert-van Kampen Theorem leads to a pushout in Group

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2) & \longrightarrow & \pi_1(U_2) \\ \downarrow & & \downarrow \\ \pi_1(U_1) & \longrightarrow & \pi_1(X). \end{array}$$

Observe $\pi_1(X) = \mathbb{Z}$. We show both $\pi_1(U_i) \rightarrow \pi_1(X)$ are trivial. This would lead to a contradiction.



Let us identify $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and assume $a = 0, b = \infty$, so A_1 is parametrized by a path α from 0 to ∞ . Let γ be an arbitrary loop in U_1 , we need to show γ becomes trivial in X . Let $R > 0$ be sufficient large such that γ is contained in the ball of radius R centered at the origin in \mathbb{R}^2 . Consider the homotopy

$$F(t, s) = \gamma(t) - \alpha(s), \quad \gamma_s := F(-, s).$$

We have $\gamma_0 = \gamma$. Assume $\alpha(t_0) > R$, then γ_{t_0} lies inside the ball of radius R centered at $\alpha(t_0)$, which is contractible in X . This implies γ is trivial in X . The same argument applies to A_2 .



Finally, we show

the complement of C in S^2 has exactly two connected components.

Let $C = A_1 \cup A_2$ and U_1, U_2 as in the previous step. We have a pushout in Groupoid

$$\begin{array}{ccc} \Pi_1(U_1 \cap U_2) & \longrightarrow & \Pi_1(U_2) \\ \downarrow & & \downarrow \\ \Pi_1(U_1) & \longrightarrow & \Pi_1(X). \end{array}$$



Assume $S^2 \setminus C$ has at least three components, so we can choose p_1, p_2, p_3 from three different connected components. Since U_1, U_2 are connected, and p_i 's are disconnected in $U_1 \cap U_2$, the following two compositions

$$p_1 \xrightarrow{\text{in } U_1} p_2 \xrightarrow{\text{in } U_1 \cap U_2} p_2 \xrightarrow{\text{in } U_2} p_1, \quad p_1 \xrightarrow{\text{in } U_1} p_3 \xrightarrow{\text{in } U_1 \cap U_2} p_3 \xrightarrow{\text{in } U_2} p_1$$

give two free generators in $\pi_1(X, p_1)$.

But $\pi_1(X, p_1) = \mathbb{Z}$. This is a contradiction. □