



Lecture 9: Exact Puppe sequence



Definition

A sequence of maps of sets with base points (i.e. in $\underline{\text{Set}}_*$)

$$(A, a_0) \xrightarrow{f} (B, b_0) \xrightarrow{g} (C, c_0)$$

is said to be **exact** at B if $\text{im}(f) = \ker(g)$ where

$$\text{im}(f) = f(A), \quad \ker(g) = g^{-1}(c_0).$$

A sequence

$$\cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots$$

is called an **exact sequence** if it is exact at every A_i .



Example

Let $H \triangleleft G$ be a normal subgroup. There is a short exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$$

in Group. Here we view Group as a subcategory of Set_{*} where a group is based at its identity element.



Definition

A sequence of maps in $\underline{h\mathcal{T}_\star}$

$$\cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots$$

is called exact if for any $Y \in \underline{h\mathcal{T}_\star}$, the following sequence of pointed sets is exact

$$\cdots \rightarrow [Y, X_{n+1}]_0 \rightarrow [Y, X_n]_0 \rightarrow [Y, X_{n-1}]_0 \rightarrow \cdots$$



Definition

Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a map in \mathcal{T}_\star . We define its **homotopy fiber** F_f in \mathcal{T}_\star by the pull-back diagram

$$\begin{array}{ccc} F_f & \longrightarrow & P_{y_0} Y \\ \pi \downarrow & & \downarrow p_1 \\ X & \xrightarrow{f} & Y \end{array}$$

$$F_f = \{(x, \gamma) \in X \times PY \mid \gamma(0) = y_0, \gamma(1) = f(x)\}$$



Recall that $p_1: P_{y_0} Y \rightarrow Y$ is a fibration, thus

Lemma

$\pi: F_f \rightarrow X$ is a fibration.

Note that F_f is precisely the fiber of $P_f \rightarrow Y$ over y_0 :

$$\begin{array}{ccc} F_f & \longrightarrow & P_f \\ \pi \downarrow & & \downarrow p_0 \\ y_0 & \hookrightarrow & Y. \end{array}$$

So this is the same as our definition before. We will put extra emphasize on the role of based point in this section.



Lemma

If $f: X \rightarrow Y$ is a fibration, then $f^{-1}(y_0)$ is homotopy equivalent to its homotopy fiber F_f .

$$\begin{array}{ccc}
 X & \xrightarrow{\iota} & P_f \\
 & \searrow f & \swarrow p_0 \\
 & Y &
 \end{array}$$

For arbitrary map $f: X \rightarrow Y$, we still have a canonical map

$$j: f^{-1}(y_0) \rightarrow F_f$$

which may not be a homotopy equivalence. The homotopy fiber can be viewed as a good replacement of fiber in homotopy category that behaves nicely for fibrations.



Lemma

The sequence

$$F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$

is exact at X in $\underline{h\mathcal{T}}_*$.

Proof: Let y_0 be the base point of Y . We first observe that $f \circ \pi$ factors through $P_{y_0} Y$ which is contractible. Therefore $f \circ \pi$ is null homotopy. Let $Z \in \underline{h\mathcal{T}}_*$. Consider

$$[Z, F_f]_0 \xrightarrow{\pi_*} [Z, X]_0 \xrightarrow{f_*} [Z, Y]_0.$$

Since $f \circ \pi$ is null homotopic, we have $\text{im } \pi_* \subset \ker f_*$.



Let $g: Z \rightarrow X$, $[g]_0 \in \ker f_*$. Assume $f \circ g$ is based homotopic to the trivial map by

$$G: Z \times I \rightarrow Y.$$

Since $G|_{Z \times \{0\}} = y_0$, it can be regarded as a map

$$G: Z \rightarrow P_{y_0} Y$$

that fits into the following diagram

$$\begin{array}{ccccc}
 & Z & & & \\
 & \searrow & \downarrow G & & \\
 & & F_f & \rightarrow & P_{y_0} Y \\
 & \swarrow g & \downarrow \pi & & \downarrow p_1 \\
 X & \xrightarrow{f} & Y & &
 \end{array}$$

Therefore the pair (G, g) factors through F_f . So $[g]_0 \in \text{im } \pi_{*}$. □ ↻



Notice that the fiber of F_f over x_0 is precisely ΩY

$$\begin{array}{ccc} \Omega Y & \longrightarrow & F_f \\ \downarrow & & \downarrow \pi \\ x_0 & \hookrightarrow & X. \end{array}$$

We find the following sequence of pointed maps

$$\Omega X \xrightarrow{\Omega f} \Omega Y \rightarrow F_f \xrightarrow{\pi} X \xrightarrow{f} Y.$$



Lemma

The sequence $\Omega X \xrightarrow{\Omega f} \Omega Y \rightarrow F_f \xrightarrow{\pi} X \xrightarrow{f} Y$ is exact in $\underline{h\mathcal{T}}_*$.

Proof: We will construct the following commutative diagram in $\underline{h\mathcal{T}}_*$ with all vertical arrows homotopy equivalences

$$\begin{array}{ccccccc}
 \Omega X & \xrightarrow{\Omega f} & \Omega Y & \longrightarrow & F_f & \xrightarrow{\pi} & X & \xrightarrow{f} & Y \\
 \downarrow j' & & \downarrow j & & \downarrow & & \downarrow & & \downarrow \\
 F_{\pi'} & \xrightarrow{\pi''} & F_{\pi} & \xrightarrow{\pi'} & F_f & \xrightarrow{\pi} & X & \xrightarrow{f} & Y
 \end{array}$$



F_π is the homotopy fiber of $\pi : F_f \rightarrow X$, given by the pull-back

$$\begin{array}{ccc} F_\pi & \longrightarrow & P_{x_0} X \\ \downarrow & & \downarrow p_1 \\ F_f & \xrightarrow{\pi} & X \end{array}$$

or explicitly

$$F_\pi = \{([\gamma], [\beta]) \in P_{y_0} Y \times P_{x_0} X \mid f(\beta(1)) = \gamma(1)\}.$$

Since $F_f \xrightarrow{\pi} X$ is a fibration with fiber ΩY , the map

$$j : \Omega Y \rightarrow F_\pi, \quad j([\beta]) = ([1_{x_0}], [\beta])$$

is the natural map of fiber into homotopy fiber which is a homotopy equivalence. This gives the second square.



Similarly, the fiber of the fibration $F_\pi \rightarrow F_f$ is ΩX . We find the following commutative diagram

$$\begin{array}{ccc} \Omega X & & \\ j' \downarrow & \searrow k & \\ F_{\pi'} & \xrightarrow{\pi''} & F_\pi \end{array}$$

Here $F_{\pi'}$ is the homotopy fiber of π' and j' is a homotopy equivalence as before. Consider the diagram

$$\begin{array}{ccc} \Omega X & \xrightarrow{\Omega f} & \Omega Y \\ & \searrow k & \downarrow j \\ & & F_\pi. \end{array}$$

This diagram is NOT commutative in \mathcal{T}_* .



However, $j \circ \Omega f$ is homotopic to k , so this diagram commutes in $h\mathcal{T}_*$. To see this, let us explicitly write

$$k([\gamma]) = ([\gamma], [1_{y_0}]), \quad (j \circ \Omega f)(\gamma) = ([1_{x_0}], [f(\gamma)]).$$

They are homotopic via

$$F([\gamma], t) = \left([\gamma|_{[0,t]}], f[\gamma|_{[t,1-t]}] \right).$$

So the first square commutes in $h\mathcal{T}_*$. The lemma follows. □



Lemma

Let $X_1 \rightarrow X_2 \rightarrow X_3$ be exact in $\underline{h\mathcal{T}}_*$, then so is

$$\Omega X_1 \rightarrow \Omega X_2 \rightarrow \Omega X_3.$$

Proof.

For any Y , apply $[Y, -]_0$ to the exact sequence $X_1 \rightarrow X_2 \rightarrow X_3$ and use the fact that Ω is right adjoint to the suspension Σ , i.e.

$$[\Sigma Y, X_i]_0 = [Y, \Omega X_i]_0,$$

we obtain an exact sequence. This implies the lemma. □



Theorem (Exact Puppe Sequence)

Let $f: X \rightarrow Y$ in $\underline{\mathcal{T}}_*$. Then the following sequence is exact in $\underline{h\mathcal{T}}_*$

$$\cdots \rightarrow \Omega^2 Y \rightarrow \Omega F_f \rightarrow \Omega X \rightarrow \Omega Y \rightarrow F_f \rightarrow X \rightarrow Y.$$



Theorem

Let $p : E \rightarrow B$ be a map in $\underline{\mathcal{T}}_*$. Assume p is a fibration whose fiber over the base point is F . Then we have an exact sequence of homotopy groups

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \cdots \rightarrow \pi_0(E) \rightarrow \pi_0(B).$$



Proof: Since p is a fibration, F is homotopy equivalent to F_p .
Observe that

$$[S^0, \Omega^n X]_0 = [\Sigma^n S_0, X]_0 = [S^n, X] = \pi_n(X).$$

The theorem follows by applying $[S^0, -]_0$ to the Puppe Sequence associated to $p : E \rightarrow B$. □



Example

Consider the universal cover $\exp : \mathbb{R}^1 \rightarrow S^1$. The associated long exact sequence implies

$$\pi_n(S^1) = 0, \quad \forall n > 1.$$



Proposition

If $i < n$, then $\pi_i(S^n) = 0$.

Proof.

Let $f: S^i \rightarrow S^n$. We need the following fact: any continuous map from a compact smooth manifold X to S^n can be uniformly approximated by a smooth map. Furthermore, two smooth maps are continuously homotopic, then they are smoothly homotopic.

Thus, we can assume that f is homotopic to a smooth map f' .

Then f' is not surjective (for dimension reason). Thus,

$f': S^i \rightarrow (S^n - \{\text{pt}\}) \simeq \mathbb{R}^n \simeq \{\text{pt}\}$ is null homotopic. □



Example

Consider the Hopf fibration $S^3 \rightarrow S^2$ with fiber S^1 . The associated long exact sequence of homotopy groups implies

$$\pi_2(S^2) \simeq \mathbb{Z}, \quad \pi_n(S^3) \simeq \pi_n(S^2) \text{ for } n \geq 3.$$