



## Lecture 12: Whitehead Theorem



Whitehead Theorem says the homotopy type of a CW complex is "determined" by its homotopy groups.



## Definition

We define the category **TopP** of topological pairs where an object

$$(X, A)$$

is a topological space  $X$  with a subspace  $A$ , and morphisms  $(X, A) \rightarrow (Y, B)$  are continuous maps

$$f: X \rightarrow Y \text{ such that } f(A) \subset B.$$

A homotopy between two maps  $f_1, f_2: (X, A) \rightarrow (Y, B)$  is a homotopy  $F: X \times I \rightarrow Y$  between  $f_0, f_1$  such that

$$F|_{X \times t}(A) \subset B \text{ for any } t \in I.$$



The quotient category of  $\underline{\text{TopP}}$  by homotopy of maps is denoted by  $\underline{\text{hTopP}}$ . The pointed versions are defined similarly and denoted by  $\underline{\text{TopP}}_*$  and  $\underline{\text{hTopP}}_*$ . Morphisms in  $\underline{\text{hTopP}}$  and  $\underline{\text{hTopP}}_*$  are denoted by

$$[(X, A), (Y, B)], \quad [(X, A), (Y, B)]_0.$$

When we work with the convenient category  $\underline{\mathcal{T}}$ , we have similar notions of  $\underline{\mathcal{TP}}$  for a pair of spaces,  $\underline{\text{hTP}}$  for the quotient homotopy category, and  $\underline{\mathcal{TP}}_*$ ,  $\underline{\text{hTP}}_*$  for the pointed cases.



## Theorem

Let  $f: (X, A) \rightarrow (Y, B)$  in  $\underline{h\mathcal{TP}}_*$ . Let  $\bar{f} = f|_A$ . Then the sequence

$$(X, A) \rightarrow (Y, B) \rightarrow (C_f, C_{\bar{f}}) \rightarrow \Sigma(X, A) \rightarrow \Sigma(Y, B) \rightarrow \Sigma(C_f, C_{\bar{f}}) \rightarrow \Sigma^2(X, A) \rightarrow \dots$$

is co-exact in  $\underline{h\mathcal{TP}}_*$ .

This generalizes the co-exact Puppe sequence to the pair case.



## Definition

Let  $(X, A) \in \mathcal{TP}_\star$ . We define the **relative homotopy group** by

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]_0.$$

We will also write  $\pi_n(X, A; x_0)$  when we specify the base point.

Note that

$$(D^n, S^{n-1}) \simeq \Sigma^{n-1}(D^1, S^0), \quad n \geq 2.$$

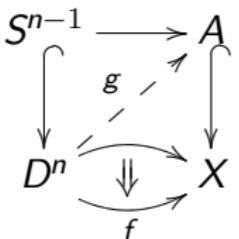
$\pi_n(X, A)$  is a group for  $n \geq 2$  due to the adjunct pair  $(\Sigma, \Omega)$ .



## Lemma

$f: (D^n, S^{n-1}) \rightarrow (X, A)$  is zero in  $\pi_n(X, A)$  if and only if  $f$  is homotopic rel  $S^{n-1}$  to a map whose image lies in  $A$ .

This lemma can be illustrated by the following diagram



Here  $g$  maps  $D^n$  to  $A$  and  $g \simeq f$  rel  $S^{n-1}$ .



# Proof

Assume  $[f]_0 = 0$  in  $\pi_n(X, A)$ . Then we can find a homotopy

$$F: D^n \times I \rightarrow X \quad \text{s.t.} \quad F(-, 0) = x_0, \quad F(-, t) \in A, \quad F(-, 1) = f(-).$$

Let us view the restriction of  $F$  to  $S^{n-1} \times I \cup D^n \times \{0\}$  as defining a map (via a natural homeomorphism)

$$g: (D^n, S^{n-1}) \rightarrow (X, A).$$

Then  $F$  can be viewed as a homotopy  $g \simeq f$  rel  $S^{n-1}$  as required.

Conversely, assume there exists  $g: (D^n, S^{n-1}) \rightarrow (X, A)$  such that  $g \simeq f$  rel  $S^{n-1}$ . Let

$$F: D^n \times I \rightarrow D^n$$

be a homotopy from the identity to the trivial map. Then

$$F \circ g: D^n \times I \rightarrow X$$

shows that  $[g]_0 = 0$ , hence  $[f]_0 = 0$  as well.



## Theorem

Let  $A \subset X$  in  $\mathcal{T}_*$ . Then there is a long exact sequence

$$\cdots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \rightarrow \pi_0(X)$$

Here the boundary map  $\partial$  sends  $\varphi \in [(D^n, S^{n-1}), (X, A)]_0$  to its restriction to  $S^{n-1}$ .



# Proof

Consider

$$f: (S^0, \{0\}) \rightarrow (S^0, S^0).$$

Let  $\bar{f} = f|_{\{0\}} : \{0\} \rightarrow S^0$ . It is easy to see that

$$(C_f, C_{\bar{f}}) \simeq (D^1, S^0).$$

Since  $\Sigma^n(S^0) = S^n$ ,  $\Sigma(D^n, S^{n-1}) = (D^{n+1}, S^n)$ , the co-exact sequence

$$(S^0, \{0\}) \rightarrow (S^0, S^0) \rightarrow (D^1, S^0) \rightarrow (S^1, \{0\}) \rightarrow (S^1, S^1) \rightarrow (D^2, S^1) \rightarrow (S^2, \{0\}) \rightarrow \dots$$

implies the exact sequence

$$\dots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \dots \rightarrow \pi_0(X)$$





## Definition

A pair  $(X, A)$  is called **n-connected** ( $n \geq 0$ ) if  $\pi_0(A) \rightarrow \pi_0(X)$  is surjective and

$$\pi_k(X, A; x_0) = 0 \quad \forall 1 \leq k \leq n, x_0 \in A.$$



From the long exact sequence

$$\cdots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \rightarrow \pi_0(X)$$

we see that  $(X, A)$  is  $n$ -connected if and only if for any  $x_0 \in A$

$$\begin{cases} \pi_r(A, x_0) \rightarrow \pi_r(X, x_0) \text{ is bijective for } r < n \\ \pi_n(A, x_0) \rightarrow \pi_n(X, x_0) \text{ is surjective} \end{cases}$$



## Definition

A map  $f: X \rightarrow Y$  is called an  **$n$ -equivalence** ( $n \geq 0$ ) if for any  $x_0 \in X$

$$\begin{cases} f_* : \pi_r(X, x_0) \rightarrow \pi_r(Y, f(x_0)) \text{ is bijective for } r < n \\ f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0)) \text{ is surjective} \end{cases}$$

$f$  is called **weak homotopy equivalence** or  **$\infty$ -equivalence** if  $f$  is  $n$ -equivalence for any  $n \geq 0$ .



## Example

For any  $n \geq 0$ , the pair  $(D^{n+1}, S^n)$  is  $n$ -connected.



## CW Complex



## Lemma

Let  $X$  be obtained from  $A$  by attaching  $n$ -cells. Let  $(Y, B)$  be a pair such that

$$\begin{cases} \pi_n(Y, B; b) = 0, \forall b \in B & \text{if } n \geq 1 \\ \pi_0(B) \rightarrow \pi_0(Y) \text{ is surjective} & \text{if } n = 0. \end{cases}$$

Then any map from  $(X, A) \rightarrow (Y, B)$  is homotopic rel  $A$  to a map from  $X$  to  $B$ .

## Proof.

This follows from the universal property of push-out

$$\begin{array}{ccccc} \coprod S^{n-1} & \longrightarrow & A & \longrightarrow & B \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ \coprod D^n & \longrightarrow & X & \xrightarrow{\quad} & Y \end{array}$$

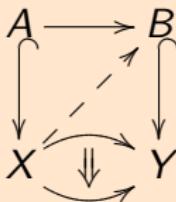


## Theorem

Let  $(X, A)$  be a relative CW complex with relative dimension  $\leq n$ .

Let  $(Y, B)$  be  $n$ -connected ( $0 \leq n \leq \infty$ ). Then any map from

$(X, A)$  to  $(Y, B)$  is homotopic relative to  $A$  to a map from  $X$  to  $B$ .



## Proof.

Apply the previous Lemma to

$$A \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

and observe that all embeddings are cofibrations. □



## Proposition

Let  $f: X \rightarrow Y$  be a weak homotopy equivalence,  $P$  be a CW complex. Then

$$f_* : [P, X] \rightarrow [P, Y]$$

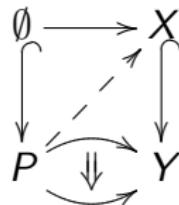
is a bijection.



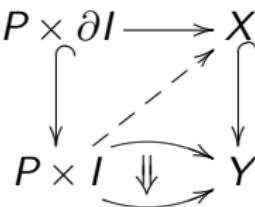
# Proof

We can assume  $f$  is an embedding and  $(Y, X)$  is  $\infty$ -connected.  
 Otherwise replace  $Y$  by  $M_f$ .

Surjectivity is illustrated by the diagram (applying previous Theorem to the pair  $(P, \emptyset)$ )



Injectivity is illustrated by the diagram (observing  $P \times I, P \times \partial I$  are CW complexes)





## Theorem (Whitehead Theorem)

A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.



# Proof

Let  $f: X \rightarrow Y$  be a weak homotopy equivalence between CW complexes. Apply previous Prop to  $P = X, Y$ , we find bijections

$$f_* : [X, X] \rightarrow [X, Y], \quad f_* : [Y, X] \rightarrow [Y, Y].$$

Let  $g \in [Y, X]$  such that  $f_*[g] = 1_Y$ . Then  $f \circ g \simeq 1_Y$ .

On the other hand,

$$f_*[g \circ f] = [f \circ g \circ f] \simeq [f \circ 1] = [f] = f_*[1_X].$$

We conclude  $[g \circ f] = 1_X$ . Therefore  $f$  is a homotopy equivalence. The reverse direction is obvious. □