Lecture 4: Covering and fibration II

G-principal covering

Definition

Let G be a discrete group. A continuous action $G \times X \to X$ is called properly discontinuous if for any $x \in X$, there exists an open neighborhood U of x such that

$$g(U) \cap U = \emptyset, \quad \forall g \neq 1 \in G.$$

We define the orbit space

$$X/G = X/\sim$$

where $x \sim g(x)$ for any $x \in X, g \in G$.

Proposition

Assume G acts properly discontinuously on X, then the quotient map $X \to X/G$ is a covering with fiber G.

Proof.

For any $x \in X$, let U be the neighbourhood satisfying

$$g(U) \cap U = \emptyset, \forall g \neq 1 \in G.$$

Then

$$p^{-1}(p(U)) = \bigsqcup_{g \in G} gU$$

is a disjoint union of open sets. Thus, p is locally trivial with discrete fiber G, hence a covering. $\square$

Definition

A left (right) G-principal covering is a covering $p : E \to B$ with a left (right) properly discontinuous G-action on E over B

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}, \quad \forall g \in G$$

such that the induced map $E/G \to B$ is a homeomorphism.

Example

$\exp\colon \mathbb{R}^1 \to S^1$ is a $\mathbb{Z}$-principal covering for the action $n : t \to t + n, \forall n \in \mathbb{Z}$.

Example

$S^n \to \mathbb{RP}^n \simeq S^n/\mathbb{Z}_2$ is a $\mathbb{Z}_2$-principal covering.

Proposition

Let $p : E \to B$ be a G-principal covering. Then transport commutes with G-action, i.e.,

$$T_{[\gamma]} \circ g = g \circ T_{[\gamma]}, \quad \forall g \in G, \gamma \text{ a path in } B.$$

Proof.

Let $\gamma : b_0 \to b_1$ and $e_0 \in p^{-1}(b_0)$. Then $\tilde{\gamma} : e_0 \to e_1 = T_{[\gamma]}(e_0)$ for some $e_1 \in p^{-1}(b_1)$. If we apply the transformation g to the path $\tilde{\gamma}$, we find another lift of γ but with endpoints $g(e_0)$ and $g(e_1)$. Therefore

$$T_{[\gamma]}(g(e_0)) = g(e_1).$$

It follows that $T_{[\gamma]}(g(e_0)) = g(e_1) = g(T_{[\gamma]}(e_0))$. $\square$

Theorem

Let $p : E \to B$ be a G-principal covering, E path connected, $e \in E, b = p(e)$. Then we have an exact sequence of groups

$$1 \to \pi_1(E, e) \to \pi_1(B, b) \to G \to 1.$$

In other words, $\pi_1(E, e)$ is a normal subgroup of $\pi_1(B, b)$ and

$$G = \pi_1(B, b)/\pi_1(E, e).$$

Proof

Let $F = p^{-1}(b)$. The previous proposition implies $\pi_1(B, b)$-action and G-action on F commute. It induces a $\pi_1(B, b) \times G$ action on F. Consider its stabilizer at e and two projections

$$\begin{tikzcd} & \mathrm{Stab}_e(\pi_1(B, b) \times G) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \pi_1(B, b) & & G \end{tikzcd}$$

pr_1 is an isomorphism and pr_2 is an epimorphism with $\ker(\mathrm{pr}_2) = \mathrm{Stab}_e(\pi_1(B, b)) = \pi_1(E, e)$. $\square$

Corollary

If $E \to B$ is a G-covering and E is a simply connected, then

$$\pi_1(B) = G.$$

Example

Apply this Corollary to the covering $\exp\colon \mathbb{R}^1 \to S^1$, we find a group isomorphism (degree map)

$$\deg : \pi_1(S^1) \to \mathbb{Z}.$$

Example

As we will see, S^n is simply connected if $n > 1$. It follows that

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2, \quad n > 1.$$

Applications

Definition

$i : A \subset X$ be a subspace. A continuous map $r : X \to A$ is called a retraction if $r \circ i = 1_A$. It is called a deformation retraction if furthermore we have a homotopy $i \circ r \simeq 1_X \text{ rel } A$. We say A is a (deformation) retract of X if such a (deformation) retraction exists.

Proposition

If $i : A \subset X$ is a retract, then $r_* : \pi_1(A) \to \pi_1(X)$ is injective.

Corollary

Let D^2 be the unit disk in $\mathbb{R}^2$. Then its boundary S^1 is not a retract of D^2.

Proof.

Since D^2 is contractible, we have $\pi_1(D^2) = 1$. But $\pi_1(S^1) = \mathbb{Z}$. Then the corollary follows from the proposition above. $\square$

Theorem (Brouwer fixed point Theorem)

Let $f: D^2 \to D^2$. Then there exists $x \in D^2$ such that $f(x) = x$.

Proof.

Assume f has no fixed point. Let l_x be the ray starting from $f(x)$ pointing toward x. Then

$$D^2 \to S^1, \quad x \to l_x \cap \partial D^2$$

is a retraction of $\partial D^2 = S^1 \subset D^2$. Contradiction. □

Theorem (Fundamental Theorem of Algebra)

Let $f(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ be a polynomial with $c_i \in \mathbb{C}, n > 0$. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.

Proof.

Assume f has no root in $\mathbb{C}$. Define a homotopy of maps

$$F : S^1 \times I \to S^1, \quad F(e^{i\theta}, t) = \frac{f(\tan(\frac{\pi t}{2}) e^{i\theta})}{\left| f(\tan(\frac{\pi t}{2}) e^{i\theta}) \right|}.$$

On one hand, $\deg(F|_{S^1 \times 0}) = 0$. On the other hand, $\deg(F|_{S^1 \times 1}) = n$. But they are homotopic hence representing the same element in $\pi_1(S^1)$. Contradiction. $\square$

Proposition (Antipode)

Let $f\colon S^1 \to S^1$ be an antipode-preserving map, i.e. $f(-x) = f(-x)$. Then $\deg(f)$ is odd. In particular, f is NOT null homotopic.

Proof.

Let $\sigma : S^1 \to S^1$ be the antipode map, with $\sigma(x) = -x$. Then $\deg(\sigma) = -1$. Let

$$F\colon \mathbb{R}^1 \to \mathbb{R}^1$$
$$F(x+1) = F(x) + \deg(f),$$

be a lifting of f. Since f is antipode-preserving, $F(x+1/2) = F(x) + m$ for $m \in \mathbb{Z} + 1/2$. So $F(x+1) = F(x) + 2m$ which implies $\deg(f) = 2m$ is odd. $\square$

Theorem (Borsuk-Ulam)

Let $f\colon S^2 \to \mathbb{R}^2$. Then there exists $x \in S^2$ such that $f(x) = f(-x)$.

Proof.

Assume $f(x) \neq f(-x), \forall x \in S^2$. Define

$$\rho : S^2 \to S^1, \quad \rho(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

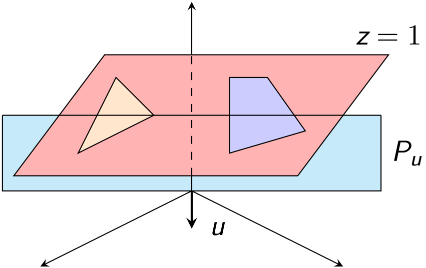
Let D^2 be the upper hemi-sphere of S^2. It defines a homotopy between constant map and $\rho|_{\partial D^2} : S^1 \to S^1$, hence $\deg(\rho|_{\partial D^2}) = 0$. On the other hand, $\rho|_{\partial D^2}$ is antipode-preserving:

$$\rho|_{\partial D^2}(-x) = -\rho|_{\partial D^2}(x).$$

Hence $\deg(\rho|_{\partial D^2})$ is odd. Contradiction. $\square$

Corollary (Ham Sandwich Theorem)

Let A_1, A_2 be two bounded regions of positive areas in $\mathbb{R}^2$. Then there exists a line which cuts each A_i into half of equal areas.



Proof

Let $A_1, A_2 \subset \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$.
Given $u \in S^2$, let P_u be the plane passing the origin and perpendicular to the unit vector u. Let $A_i(u) = \{p \in A_i | p \cdot u \leq 0\}$. Define the map

$$f : S^2 \to \mathbb{R}^2, \quad f_i(u) = \text{Area}(A_i(u)).$$

By Borsuk-Ulam, $\exists u$ such that $f(u) = f(-u)$. The intersection $\mathbb{R}^2 \times \{1\} \cap P_u$ gives the required line since

$$f(u) = f(-u) \Longleftrightarrow f_i(u) = \frac{1}{2}(A_i).$$

□

Classification of coverings

Definition

The universal cover of B is a covering map $p : E \to B$ with E simply connected.

The universal cover is unique (if exists) up to homeomorphism. This follows from the lifting criterion and the unique lifting property of covering maps.

Definition

A space is semi-locally simply connected if for any $x_0 \in X$, there is a neighbourhood U_0 such that the image of the map $i_*\colon \pi_1(U_0, x_0) \to \pi_1(X, x_0)$ is trivial.

We recall the following theorem from point-set topology.

Theorem (Existence of the universal cover)

Assume B is path connected and locally path connected. Then universal cover of B exists if and only if B is semi-locally simply connected space.

Definition

We define the category $\text{Cov}(B)$ of coverings of B where

- an object is a covering map $p: E \to B$
- a morphism between two coverings $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ is a map $f: E_1 \to E_2$ such that the following diagram is commutative

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Definition

Let B be connected. We define $\text{Cov}_0(B) \subset \text{Cov}(B)$ to be the subcategory whose objects consist of connected coverings of B.

Proposition

Let B be connected and locally path connected. Then any morphism in $\mathrm{Cov}_0(B)$ is a covering map.

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

In other words, if B is connected and p_1, p_2 are coverings, then f is also a covering.

Definition

We define the category G-**Set** where

- an object is a set S with G-action
- morphisms are G-equivariant set maps, i.e. $f\colon S_1 \to S_2$ such that $f \circ g = g \circ f$, for any $g \in G$.

Given a covering $p : E \to B$, $b \in B$, the transport functor implies

$$p^{-1}(b) \in \pi_1(B, b)\text{-}\mathbf{Set}.$$

Lemma

Let B be path connected. Then $\pi_1(B, b)$ acts transitively on $p^{-1}(b)$ if and only if E is path connected.

Proof.

We prove the "only if" part.

Let us fix a point $e_0 \in p^{-1}(b)$. Assume $\pi_1(B, b)$ acts transitively on $p^{-1}(b)$. This implies that any point in $p^{-1}(b)$ is connected to e_0 by a path. Given any point $e \in E$, let γ be a path in B that connects $p(e)$ to b. The transport functor $T_{[\gamma]}$ gives a path connecting e to some point in $p^{-1}(b)$. This further implies that e is path connected to e_0. This proves the "only if" part.

$\square$

Proposition

Assume B is path connected and locally path connected. Let $p_1, p_2 \in \mathrm{Cov}(B)$. Then there is a set isomorphism

$$\mathrm{Hom}_{\mathrm{Cov}(B)}(p_1, p_2) \simeq \mathrm{Hom}_{\pi_1(B,b)\text{-}\underline{\mathbf{Set}}}(p_1^{-1}(b), p_2^{-1}(b))$$

for any $b \in B$.

Proof

Let $f \in \mathrm{Hom}_{\mathsf{Cov}(B)}(p_1, p_2)$

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

It induces a map by restricting f to the fiber $p^{-1}(b)$

$$f_b \colon p_1^{-1}(b) \to p_2^{-1}(b).$$

f_b is $\pi_1(B, b)$-equivariant (why?). Thus we obtain a map

$$\Phi \colon \mathrm{Hom}_{\mathsf{Cov}(B)}(p_1, p_2) \to \mathrm{Hom}_{\pi_1(B,b)\text{-}\mathbf{Set}}(p_1^{-1}(b), p_2^{-1}(b))$$
$$f \to f_b$$

The injectivity of Φ comes from the uniqueness of the lifting.

To prove surjectivity, we can assume E_1 is path connected, and $\pi_1(B, b)$ acts transitively on $p_1^{-1}(b)$ (why?). Given

$$f_b \colon p_1^{-1}(b) \to p_2^{-1}(b),$$

let us fix two points $e_i \in p_i^{-1}(b)$ such that $f(e_1) = e_2$. The $\pi_1(B, b)$-equivariance of f_b gives rise to the homomorphism

$$\begin{array}{lcl} \mathrm{Stab}_{e_1}(\pi_1(B, b)) & \longrightarrow & \mathrm{Stab}_{e_2}(\pi_1(B, b)) \\ = \pi_1(E_1, e_1) & & = \pi_1(E_2, e_2). \end{array}$$

By Lifting Criterion, we obtain $f \colon E_1 \to E_2$ as required. $\square$

Theorem

Assume B is path connected, locally path connected and semi-locally simply connected. $b \in B$. Then there exists an equivalence of categories

$$\boxed{\mathrm{Cov}(B) \simeq \pi_1(B, b)\text{-}\mathbf{Set}.}$$

Proof

Let us denote $\pi_1 = \pi_1(B, b)$. Let $\tilde{p} : \widetilde{B} \to B$ be a fixed universal cover of B and $\tilde{b} \in \pi^{-1}(b)$ chosen.

We will define the following functors

$$\mathrm{Cov}(B) \underset{G}{\overset{F}{\rightleftarrows}} \pi_1\text{-}\mathbf{Set}.$$

Let $p : E \to B$ be a covering, we define

$$F(p) := p^{-1}(b).$$

Let $S \in \pi_1$ -**Set**, we define

$$G(S) := \widetilde{B} \times_{\pi_1} S = \widetilde{B} \times S / \sim,$$

where $(e \cdot g, s) \sim (e, g \cdot s)$ for any $e \in \widetilde{B}, s \in S, g \in \pi_1$. Note that here $e \cdot g$ represents the (right) π_1-action on $\widetilde{B}$.

We next construct natural isomorphisms

$$F \circ G \overset{\eta}{\simeq} 1, \quad G \circ F \overset{\tau}{\simeq} 1.$$

η is the natural isomorphism

$$\eta_S \in \mathrm{Hom}_{\pi_1\text{-}\mathbf{Set}}(F \circ G(S), S), \quad \eta_S(e, s) = g \cdot s \quad \text{if } e = \tilde{b} \cdot g.$$

τ is the natural isomorphism

$$\tau_p \in \mathrm{Hom}_{\mathsf{Cov}(B)}(G \circ F(p), p) \simeq \mathrm{Hom}_{\pi_1\text{-}\mathbf{Set}}(p^{-1}(b), p^{-1}(b))$$

$$G \circ F(p) : \widetilde{B} \times_{\pi_1} p^{-1}(b) \to B$$

determined by the identity in $\mathrm{Hom}_{\pi_1\text{-}\mathbf{Set}}(p^{-1}(b), p^{-1}(b))$. $\square$

Definition

Let B be path connected and $p: E \to B$ be a connected covering. A deck transformation (or covering transformation) of p is a homeomorphism $f: E \to E$ such that $p \circ f = p$.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Let $\mathrm{Aut}(p)$ denote the group of deck transformation.

Note that $\mathrm{Aut}(p)$ acts freely on E by the Uniqueness of Lifting.

Proposition

Let B be path connected and $p : E \to B$ be a connected covering. Then $\mathrm{Aut}(p)$ acts properly discontinuous on E.

Theorem

Assume B is path connected, locally path connected. Let $p : E \to B$ be a connected covering, $e \in E, b = p(e) \in B$

$$G = \pi_1(B, b), H = \pi_1(E, e).$$

Then

$$\mathrm{Aut}(p) \simeq N_G(H)/H$$

where

$$N_G(H) := \{r \in G \mid rHr^{-1} = H\}$$

is the normalizer of H in G.

Proof.

By the above proposition,

$$\mathrm{Aut}(p) \simeq \mathrm{Hom}_{G\text{-}\underline{\mathbf{Set}}}(G/H, G/H) = N_G(H)/H.$$

$\square$

Example

For the universal cover $p : \tilde{B} \to B$, this implies that

$$\mathrm{Aut}(p) = \pi_1(B, b).$$

Therefore p is a $\pi_1(B, b)$-principal covering.

Definition

We define the orbit category $\mathrm{Orb}(G)$

- objects consist of (left) coset G/H, where H is a subgroup of G
- morphisms are G-equivariant maps: $G/H_1 \to G/H_2$.

$\mathrm{Orb}(G)$ is a full subcategory of G-**Set** consisting of single orbits.

Remark

G/H_1 and G/H_2 are isomorphic in $\mathrm{Orb}(G)$ if and only if H_1 and H_2 are conjugate subgroups of G.

If we restrict to connected coverings, we find an equivalence

$$\mathrm{Cov}_0(B) \simeq \mathrm{Orb}(\pi_1(B,b)).$$

$$\begin{tikzcd} \pi_1(B,b) \arrow[rr] \arrow[dr] & & \tilde{\pi}_1(B,b)/H \arrow[dl] \\ & 1 & \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} \widetilde{B} \arrow[rr, "f"] \arrow[dr] & & \widetilde{B}/H \arrow[dl] \\ & B & \end{tikzcd}$$

The universal cover $\widetilde{B} \to B$ corresponds to the orbit $\pi_1(B,b)$. For the orbit $\pi_1(B,b)/H$, it corresponds to

$$E = \widetilde{B}/H \to B.$$

A more intrinsic formulation is as follows. Given a covering $p : E \to B$, we obtain a transport functor

$$T_p : \Pi_1(B) \to \underline{\mathbf{Set}}\,.$$

Given a commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

we find a natural transformation

$$\tau : T_{p_1} \Longrightarrow T_{p_2}, \quad \tau = \{f : p_1^{-1}(b) \to p_2^{-1}(b) | b \in B\}.$$

The above structure can be summarized by a functor

$$\boxed{T : \mathrm{Cov}(B) \to \mathrm{Fun}(\Pi_1(B), \mathbf{Set})}.$$

Theorem

Assume B is path connected, locally path connected and semi-locally simply connected. Then

$$T : \mathrm{Cov}(B) \to \mathrm{Fun}(\Pi_1(B), \mathbf{Set})$$

is an equivalence of categories.