# Classical Mechanics and Geometry

# 经典力学与几何

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You can also contact me at sili@mail.tsinghua.edu.cn. The draft will be updated on my homepage: https://sili-math.github.io/. Thank you.

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# Preface

In April 2021, Qiuzhen College (求真书院) was newly established at Tsinghua University under the leadership of Professor Shing-Tung Yau. It homes the distinguished elite mathematics program in China starting in 2021: the "Yau Mathematical Sciences Leaders Program" (丘成桐数学科学领军人才培养计划). This program puts strong emphasis on basic sciences related to mathematics in a broad sense. Though majored in mathematics, students in this program are required to study fundamental theoretical physics such as classical mechanics, electromagnetism, quantum mechanics, and statistical mechanics, in order to understand global perspectives of theoretical sciences. It is an exciting challenge both for students and for instructors.

This preliminary note is written for the course "Classical Mechanics" that I lectured at Qiuzhen College in the fall semester of 2022. It is to explain key physics ingredients of Lagrangian and Hamiltonian mechanics, as well as their connections with modern geometric development. We put heavy emphasis on different faces of concrete examples in order to understand the bridge between mathematics and physics. Examples such at Toda lattice and Calogero-Moser System are still active research topics nowadays in areas of integrable system, representation theory and mathematical physics. A large part of this note relies on the beautiful books of "Mechanics" by Landau-Lifshitz, and "Mathematical Methods of Classical Mechanics" by Arnol'd, which themselves show different faces of this classical subject. Other useful resources that we consulted are listed at the end of this note.

I would like to thank 杨鹏 and 王进一, who have done amazing jobs of teaching assistant for this course. An early version of this note was typed by 杨鹏, including all those beautiful figures that are better arts than my blackboard drawings. I want to thank 丁徐祉晗 and 刘九和 for their help on careful proofreading of this note, as well as their important roles of being excellent students for the whole semester. I want to thank my colleague 周杰, the collaboration and discussion with whom in this year have kept my brain fresh during the preparation of this note. Special thank goes to 程子钰 from office of Teaching Affairs at Qiuzhen College, whose tremendous help has saved me alive from heavy administrative service to finish this note.

静斋

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# Chapter 1 Lagrangian Mechanics

## 1.1 Principle of Least Action

### 1.1.1 Newtonian Mechanics

Recall **Newton's Second Law**

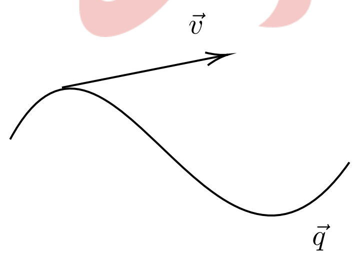
$$\vec{F} = m\vec{a}.$$

Consider a particle moving in $\mathbb{R}^n$ at time $t \in \mathbb{R}$ with position $\vec{q}(t) \in \mathbb{R}^n$. Let us describe this motion as a map

$$\begin{aligned} \vec{q} : \mathbb{R} &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \vec{q}(t) = (q_1(t), \cdots, q_n(t)) \end{aligned}$$

We have

- **velocity**: $\vec{v} = \dot{\vec{q}} = \frac{d\vec{q}}{dt}$
- **acceleration**: $\vec{a} = \dot{\vec{v}} = \frac{d^2\vec{q}}{dt^2}$




Assume the force $\vec{F}$ depends only on the position. Then

$$m\ddot{\vec{q}} = \vec{F}(q(t)),$$

which has a unique solution if we fix initial value $(\vec{q}(t_0), \dot{\vec{q}}(t_0))$ at some time $t_0$.

**Definition 1.1.1.** Define the **kinetic energy** of the motion

$$K = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\frac{d\vec{q}}{dt}\right)^2 = \frac{1}{2}\sum_i m\left(\frac{dq_i}{dt}\right)^2.$$