



Lecture 4: Covering and fibration II



G-principal covering



Definition

Let G be a discrete group. A continuous action $G \times X \rightarrow X$ is called **properly discontinuous** if for any $x \in X$, there exists an open neighborhood U of x such that

$$g(U) \cap U = \emptyset, \quad \forall g \neq 1 \in G.$$

We define the **orbit space**

$$X/G = X/\sim$$

where $x \sim g(x)$ for any $x \in X, g \in G$.



Proposition

Assume G acts properly discontinuously on X , then the quotient map $X \rightarrow X/G$ is a covering with fiber G .



Definition

A left (right) G -principal covering is a covering $p : E \rightarrow B$ with a left (right) properly discontinuous G -action on E over B

$$\begin{array}{ccc} E & \xrightarrow{g} & E, \quad \forall g \in G \\ & \searrow p \quad \swarrow p & \\ & B & \end{array}$$

such that the induced map $E/G \rightarrow B$ is a homeomorphism.



Example

$\exp: \mathbb{R}^1 \rightarrow S^1$ is a \mathbb{Z} -principal covering for the action
 $n: t \rightarrow t + n, \forall n \in \mathbb{Z}$.

Example

$S^n \rightarrow \mathbb{RP}^n \simeq S^n / \mathbb{Z}_2$ is a \mathbb{Z}_2 -principal covering.



Proposition

Let $p : E \rightarrow B$ be a G -principal covering. Then **transport commutes with G -action**, i.e.,

$$T_{[\gamma]} \circ g = g \circ T_{[\gamma]}, \quad \forall g \in G, \gamma \text{ a path in } B.$$



Theorem

Let $p : E \rightarrow B$ be a G -principal covering, E path connected, $e \in E, b = p(e)$. Then we have an exact sequence of groups

$$1 \rightarrow \pi_1(E, e) \rightarrow \pi_1(B, b) \rightarrow G \rightarrow 1.$$

In other words, $\pi_1(E, e)$ is a normal subgroup of $\pi_1(B, b)$ and

$$G = \pi_1(B, b)/\pi_1(E, e).$$



This can be illustrated by

$$\begin{array}{ccc} & \text{Stab}_e(\pi_1(B, b) \times G) & \\ \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\ \pi_1(B, b) & & G \end{array}$$

pr_1 is an isomorphism and pr_2 is an epimorphism with

$$\ker(\text{pr}_2) = \text{Stab}_e(\pi_1(B, b)) = \pi_1(E, e).$$



Example

Apply this Corollary to the covering $\exp: \mathbb{R}^1 \rightarrow S^1$, we find a group isomorphism (degree map)

$$\deg: \pi_1(S^1) \rightarrow \mathbb{Z}.$$

Example

As we will see, S^n is simply connected if $n > 1$. It follows that

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2, \quad n > 1.$$



Applications



Definition

$i: A \subset X$ be a subspace. A continuous map $r: X \rightarrow A$ is called a **retraction** if $r \circ i = 1_A$. It is called a **deformation retraction** if furthermore we have a homotopy $i \circ r \simeq 1_X$ rel A . We say A is a (deformation) retract of X if such a (deformation) retraction exists.



Proposition

If $i : A \subset X$ is a retract, then $r_* : \pi_1(A) \rightarrow \pi_1(X)$ is injective.

Corollary

Let D^2 be the unit disk in \mathbb{R}^2 . Then its boundary S^1 is not a retract of D^2 .



Theorem (Brouwer fixed point Theorem)

Let $f: D^2 \rightarrow D^2$. Then there exists $x \in D^2$ such that $f(x) = x$.



Theorem (Fundamental Theorem of Algebra)

Let $f(x) = x^n + c_1x^{n-1} + \cdots + c_n$ be a polynomial with $c_i \in \mathbb{C}$, $n > 0$. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.



Proposition (Antipode)

Let $f: S^1 \rightarrow S^1$ be an antipode-preserving map, i.e. $f(-x) = f(x)$.
Then $\deg(f)$ is odd. In particular, f is NOT null homotopic.



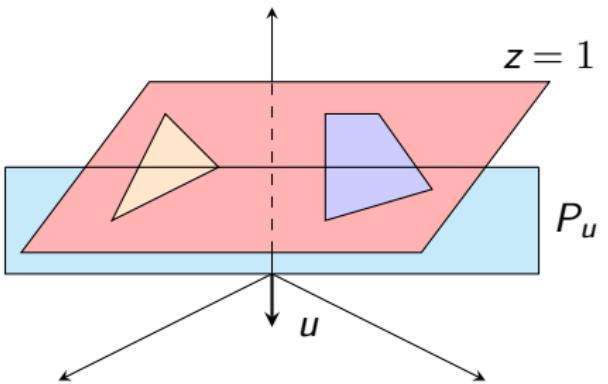
Theorem (Borsuk-Ulam)

Let $f: S^2 \rightarrow \mathbb{R}^2$. Then there exists $x \in S^2$ such that $f(x) = f(-x)$.



Corollary (Ham Sandwich Theorem)

Let A_1, A_2 be two bounded regions of positive areas in \mathbb{R}^2 . Then there exists a line which cuts each A_i into half of equal areas.





Classification of coverings



Definition

The **universal cover** of B is a covering map $p : E \rightarrow B$ with E simply connected.

The universal cover is unique (if exists) up to homeomorphism. This follows from the lifting criterion and the unique lifting property of covering maps.



Definition

A space is **semi-locally simply connected** if for any $x_0 \in X$, there is a neighbourhood U_0 such that the image of the map $i_*: \pi_1(U_0, x_0) \rightarrow \pi_1(X, x_0)$ is trivial.

We recall the following theorem from point-set topology.

Theorem (Existence of the universal cover)

Assume B is path connected and locally path connected. Then universal cover of B exists if and only if B is semi-locally simply connected space.



Definition

We define the category $\text{Cov}(B)$ of coverings of B where

- ▶ an object is a covering map $p : E \rightarrow B$
- ▶ a morphism between two coverings $p_1 : E_1 \rightarrow B$ and $p_2 : E_2 \rightarrow B$ is a map $f : E_1 \rightarrow E_2$ such that the following diagram is commutative

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array}$$

Definition

Let B be connected. We define $\text{Cov}_0(B) \subset \text{Cov}(B)$ to be the subcategory whose objects consist of connected coverings of B .



Proposition

Let B be connected and locally path connected. Then any morphism in $\text{Cov}_0(B)$ is a covering map.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array}$$

In other words, if B is connected and p_1, p_2 are coverings, then f is also a covering.



Definition

We define the category $G\text{-}\underline{\text{Set}}$ where

- ▶ an object is a set S with G -action
- ▶ morphisms are G -equivariant set maps, i.e. $f: S_1 \rightarrow S_2$ such that $f \circ g = g \circ f$, for any $g \in G$.

Given a covering $p: E \rightarrow B$, $b \in B$, the transport functor implies

$$p^{-1}(b) \in \pi_1(B, b)\text{-}\underline{\text{Set}}.$$



Lemma

Let B be path connected. Then $\pi_1(B, b)$ acts transitively on $p^{-1}(b)$ if and only if E is path connected.



Proposition

Assume B is path connected and locally path connected. Let $p_1, p_2 \in \text{Cov}(B)$. Then there is a set isomorphism

$$\text{Hom}_{\text{Cov}(B)}(p_1, p_2) \simeq \text{Hom}_{\pi_1(B, b)\text{-}\underline{\text{Set}}}(p_1^{-1}(b), p_2^{-1}(b))$$

for any $b \in B$.



Theorem

Assume B is path connected, locally path connected and semi-locally simply connected. $b \in B$. Then there exists an equivalence of categories

$$\text{Cov}(B) \simeq \pi_1(B, b) \text{-}\underline{\text{Set}}.$$



The equivalence is realized by the following functors

$$\text{Cov}(B) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \pi_1\text{-}\underline{\text{Set}}.$$

- ▶ Let $p : E \rightarrow B$ be a covering, we define

$$F(p) := p^{-1}(b).$$

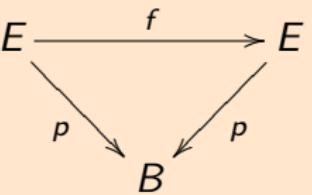
- ▶ Let $S \in \pi_1\text{-}\underline{\text{Set}}$, we define

$$G(S) := \widetilde{B} \times_{\pi_1} S = \widetilde{B} \times S / \sim.$$



Definition

Let B be path connected and $p : E \rightarrow B$ be a connected covering. A **deck transformation** (or **covering transformation**) of p is a homeomorphism $f : E \rightarrow E$ such that $p \circ f = p$.



Let $\text{Aut}(p)$ denote the group of deck transformation.



Note that $\text{Aut}(p)$ acts freely on E by the Uniqueness of Lifting.

Proposition

Let B be path connected and $p : E \rightarrow B$ be a connected covering. Then $\text{Aut}(p)$ acts properly discontinuous on E .



Theorem

Assume B is path connected, locally path connected. Let $p : E \rightarrow B$ be a connected covering, $e \in E, b = p(e) \in B$

$$G = \pi_1(B, b), H = \pi_1(E, e).$$

Then

$$\text{Aut}(p) \simeq N_G(H)/H$$

where

$$N_G(H) := \{r \in G \mid rHr^{-1} = H\}$$

is the normalizer of H in G .

This theorem is a direct consequence of the following computation

$$\text{Aut}(p) \simeq \text{Hom}_{G\text{-}\underline{\text{Set}}}(G/H, G/H) = N_G(H)/H.$$



Example

For the universal cover $p : \tilde{B} \rightarrow B$, this implies that

$$\text{Aut}(p) = \pi_1(B, b).$$

Therefore p is a $\pi_1(B, b)$ -principal covering.



Definition

We define the orbit category $\text{Orb}(G)$

- ▶ objects consist of (left) coset G/H , where H is a subgroup of G
- ▶ morphisms are G -equivariant maps: $G/H_1 \rightarrow G/H_2$.

$\text{Orb}(G)$ is a full subcategory of $G\text{-}\underline{\text{Set}}$ consisting of single orbits.

Remark

G/H_1 and G/H_2 are isomorphic in $\text{Orb}(G)$ if and only if H_1 and H_2 are conjugate subgroups of G .



If we restrict to connected coverings, we find an equivalence

$$\text{Cov}_0(B) \simeq \text{Orb}(\pi_1(B, b)).$$

$$\begin{array}{ccc} \pi_1(B, b) & \longrightarrow & \tilde{\pi}_1(B, b)/H \\ & \searrow & \swarrow \\ & 1 & \end{array} \iff \begin{array}{ccc} \tilde{B} & \xrightarrow{f} & \tilde{B}/H \\ & \searrow & \swarrow \\ & B & \end{array}$$

The universal cover $\tilde{B} \rightarrow B$ corresponds to the orbit $\pi_1(B, b)$. For the orbit $\pi_1(B, b)/H$, it corresponds to

$$E = \tilde{B}/H \rightarrow B.$$



A more intrinsic formulation is as follows. Given a covering $p: E \rightarrow B$, we obtain a transport functor

$$T_p: \Pi_1(B) \rightarrow \underline{\mathbf{Set}}.$$

Given a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array}$$

we find a natural transformation

$$\tau: T_{p_1} \Rightarrow T_{p_2}, \quad \tau = \{f: p_1^{-1}(b) \rightarrow p_2^{-1}(b) \mid b \in B\}.$$



The above structure can be summarized by a functor

$$T : \text{Cov}(B) \rightarrow \text{Fun}(\Pi_1(B), \underline{\text{Set}}).$$

Theorem

Assume B is path connected, locally path connected and semi-locally simply connected. Then

$$T : \text{Cov}(B) \rightarrow \text{Fun}(\Pi_1(B), \underline{\text{Set}})$$

is an equivalence of categories.