§7. Topological Quantum Mechanics-I

Last time: Deformation Quantization

$$\left(C^\infty(X), \{-,-\}_{\omega^{-1}}\right) \longmapsto \left(C^\infty(X)[[\hbar]], *\right)$$

w/. canonical trace map

$$\mathrm{Tr}: C^\infty(X)[[\hbar]] \longmapsto \mathbb{R}((\hbar))$$

Algebraic Index Thm:

$$\mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar} \hat{A}(X)$$

Today: Prove this using the effective renormalization method in a topological quantum mechanical model.

We follow the presentation in

- Gui - L - Xu CMP 2021
- Grady - Li - L AIM 2017

· Local model

Let's consider the standard phase space

$$(V, \omega)$$

$V \simeq \mathbb{R}^{2n}$ w/. coordinate

$$(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n}) = (q^1, \cdots, q^n, p_1, \cdots, p_n)$$

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$$

Let S^1_{dR} be the space w/.

topology $= S^1$ structure sheaf $= \Omega^\bullet_{S^1}$

$$\mathcal{O}(S^1_{dR}) = \Omega^\bullet_{S^1} \text{ dg ring}, \quad d = \text{de Rham diff}$$

Consider the space of maps

$$\varphi : S^1_{dR} \longmapsto V$$

Such φ can be identified w/. an element in $\Omega^\bullet_{S^1} \otimes V$

Explicitly, let θ be coordinate on S^1 $(\theta \sim \theta+1)$

$$\varphi = \{ \mathbb{P}_i(\theta), \mathbb{Q}^i(\theta) \}_{i=1,\dots,n}, \quad \mathbb{P}_i, \mathbb{Q}^i \in \Omega^\bullet_{S^1}$$

We will write in form component as

$$\mathbb{P}_i(\theta) = P_i(\theta) + \eta_i(\theta) d\theta, \quad \mathbb{Q}_i(\theta) = q^i(\theta) + \xi^i(\theta)\, d\theta$$

So the space of fields is

$$\mathcal{E} = \Omega^\bullet_{S^1} \otimes V \ni \varphi$$

Then $\left(\Omega^\bullet_{S^1} \otimes V, d, \int_{S^1} \langle -,- \rangle_\omega\right)$ is (-1)-symplectic

The action is the free one

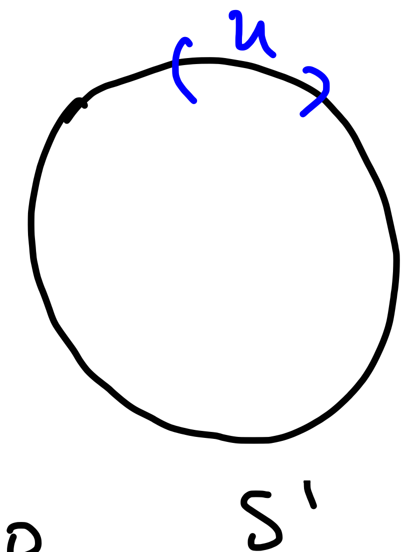
$$S[\varphi] := \int_{S^1} \langle \varphi, d\varphi \rangle_\omega$$

$$= \sum_i \int_{S^1} \mathbb{P}_i d\mathbb{Q}^i = \sum_i \int_{S^1} P_i(\theta) dq^i(\theta)$$

RK:
- This is the 1st order formalism of TQM
- This is 1d CS theory via AKSZ construction

RK:

• Local observable on $\mathcal{U}$ (classical)



$$= \mathcal{O}(\Omega^\bullet_{\mathcal{U}} \otimes V)$$

Since $\mathcal{U}$ is contractible, $(\Omega^\bullet_{\mathcal{U}}, d) \simeq \mathbb{R}$

$$\simeq \mathcal{O}(V)$$

• global observable on S^1 (classical)

$$= \mathcal{O}(\Omega^\bullet_{S^1} \otimes V)$$

$$\simeq \mathcal{O}(H^\bullet(S^1) \otimes V) = \mathcal{O}(V \oplus V\,dt)$$

$$\simeq \Omega^{-\bullet}_V$$

local $\to$ global

$\Rightarrow$ Hochschild-Kostant-Rosenberg (HKR)

As we will see, TQM gives an explicit deformed HKR

when $\mathcal{O}(V) \overset{\text{deform}}{\rightsquigarrow} \mathcal{W}(V)$ Weyl algebra

- Propagator

Let us choose the standard flat metric on S^1.

Let $d^* =$ adjoint of d. The Laplacian is

$$[d, d^*] = -\left(\frac{d}{d\theta}\right)^2$$

Let $\Pi = \omega^{-1} = \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} = \frac{1}{2}\sum_i \left(\frac{\partial}{\partial p_i}\otimes\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i}\otimes\frac{\partial}{\partial p_i}\right)$

$\Pi \in \Lambda^2 V$ is the Poisson Kernel.

Let

$$h_t(\theta_1, \theta_2) = \frac{1}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} e^{-\frac{(\theta_1-\theta_2+n)^2}{4t}}$$

be the standard heat kernel on S^1. Then the regularized propagator is

$$P_\varepsilon^L = \underbrace{\int_\varepsilon^L \partial_{\theta_1} h_t(\theta_1,\theta_2)\, dt}_{\in\, C^\infty(S^1\times S^1)} \otimes \underbrace{\Pi}_{\in\, V\otimes V} \quad \in \mathcal{E}\otimes\mathcal{E}$$

Let us denote

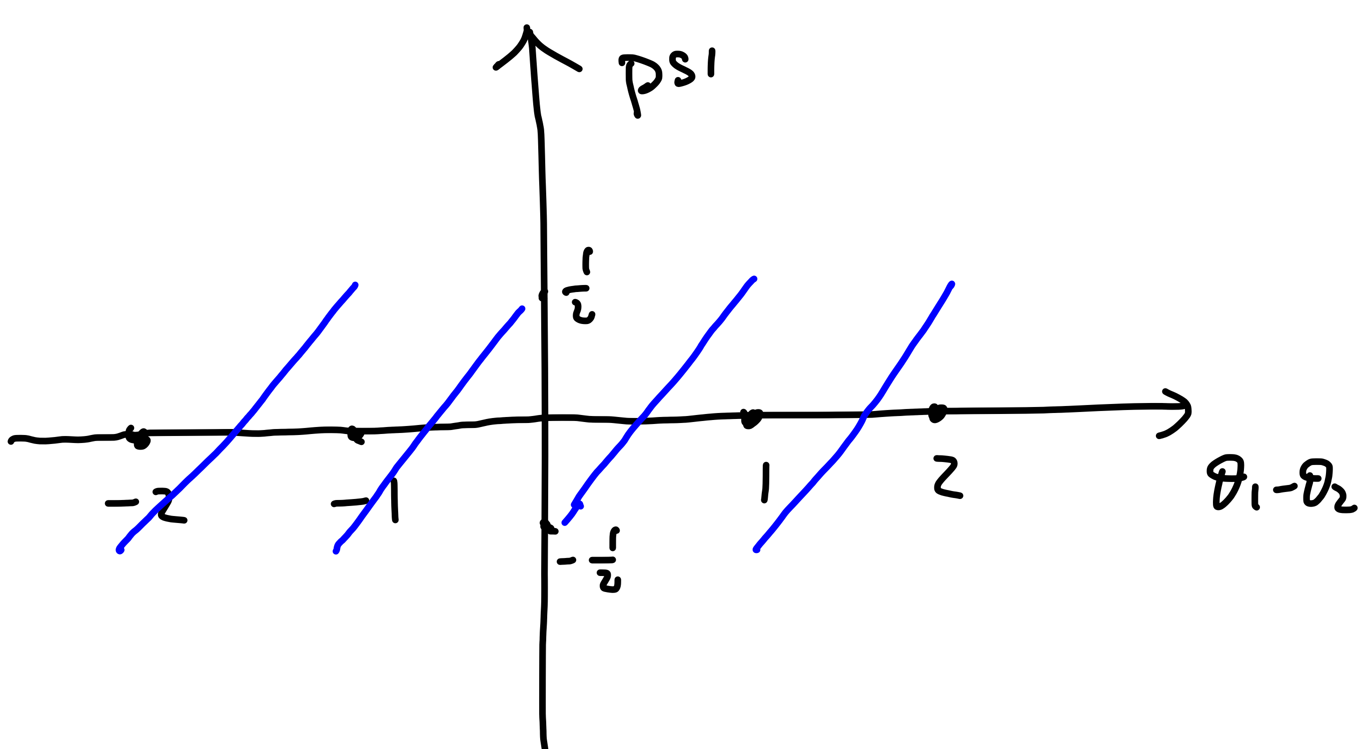
$$P^{S^1}(\theta_1,\theta_2) = \int_0^\infty \partial_{\theta_1} h_t(\theta_1,\theta_2)\,dt$$

Then the full propagator is given by

$$P_0^\infty = P^{S^1} \otimes \mathbb{1}$$

Prop: $P^{S^1}(\theta_1,\theta_2)$ is the following periodic function of $\theta_1-\theta_2 \in \mathbb{R}/\mathbb{Z}$ where

$$P^{S^1}(\theta_1,\theta_2) = \theta_1-\theta_2-\frac{1}{2} \quad \text{if } 0<\theta_1-\theta_2<1$$



In particular, P^{S^1} is not smooth function on $S^1\times S^1$ (as expected), but it is bounded.

- Correlation map

Let us denote

$$\mathcal{W}_{2n} = (\mathbb{R}[[p_i, q^i]]((\hbar)) \ *)$$

and $\mathcal{W}_{2n}^{+} = (\mathbb{R}[[p_i, q^i]][[\hbar]], *)$

Moyal-Weyl product.

We can identify

$$\mathcal{W}_{2n}^{+} \simeq (\hat{\mathcal{O}}(V)[[\hbar]], *)$$

as (formal) functions on V (deformation quantization)

Given $f_0, f_1, \cdots, f_m \in \mathcal{W}_{2n}$, define

$$\Theta_{f_0, f_1, \cdots, f_m} \in \mathcal{O}(\mathcal{E})((\hbar)) \text{ by}$$

$$\Theta_{f_0, f_1, \cdots, f_m}[\varphi] \qquad \varphi \in \Omega^{\cdot}_{S^1} \otimes V$$

$$:= \int_{0<\theta_1<\theta_2<\cdots<\theta_m<1} d\theta_1 d\theta_2 \cdots d\theta_m \ f_0^{(0)}(\varphi(\theta_0)) f_1^{(1)}(\varphi(\theta_1)) \cdots f_m^{(1)}(\varphi(\theta_m))$$

Here $f(\varphi(\theta)) = f(\mathbb{P}_i(\theta), \mathbb{Q}^i(\theta)) \in \Omega^\cdot_{S^1}$

and we decompose as

$$f(\varphi(\theta)) = f^{(0)}(\varphi(\theta)) + f^{(1)}(\varphi(\theta))\,d\theta$$

RK. $f^{(1)}(\varphi)$ is the topological descent of $f^{(0)}(\varphi)$ (in the sense of Witten)

Now let's apply the homotopy RG flow:

$$e^{\hbar P_0^\infty}\left(\mathcal{O}_{f_0, f_1, \cdots, f_m}\right)$$

Since P_0^∞ is bounded, the above is convergent and well-defined! (UV finite property)

As we have discussed, at $L=\infty$, we can view is as defining a function on zero modes or

$$\mathbb{H} = H^\cdot(\Omega^\cdot_{S^1} \otimes V, d) = H^\cdot(S^1) \otimes V$$

We have $\mathcal{O}(\mathbb{H}) = \widehat{\Omega}^{-\bullet}_{2n}$ forms on V

Def'n We define the following correlation map

$$\langle \cdots \rangle_{free} : \mathcal{W}_{2n} \otimes \cdots \otimes \mathcal{W}_{2n} \longmapsto \hat{\Omega}^{-\bullet}_{2n}((\hbar))$$

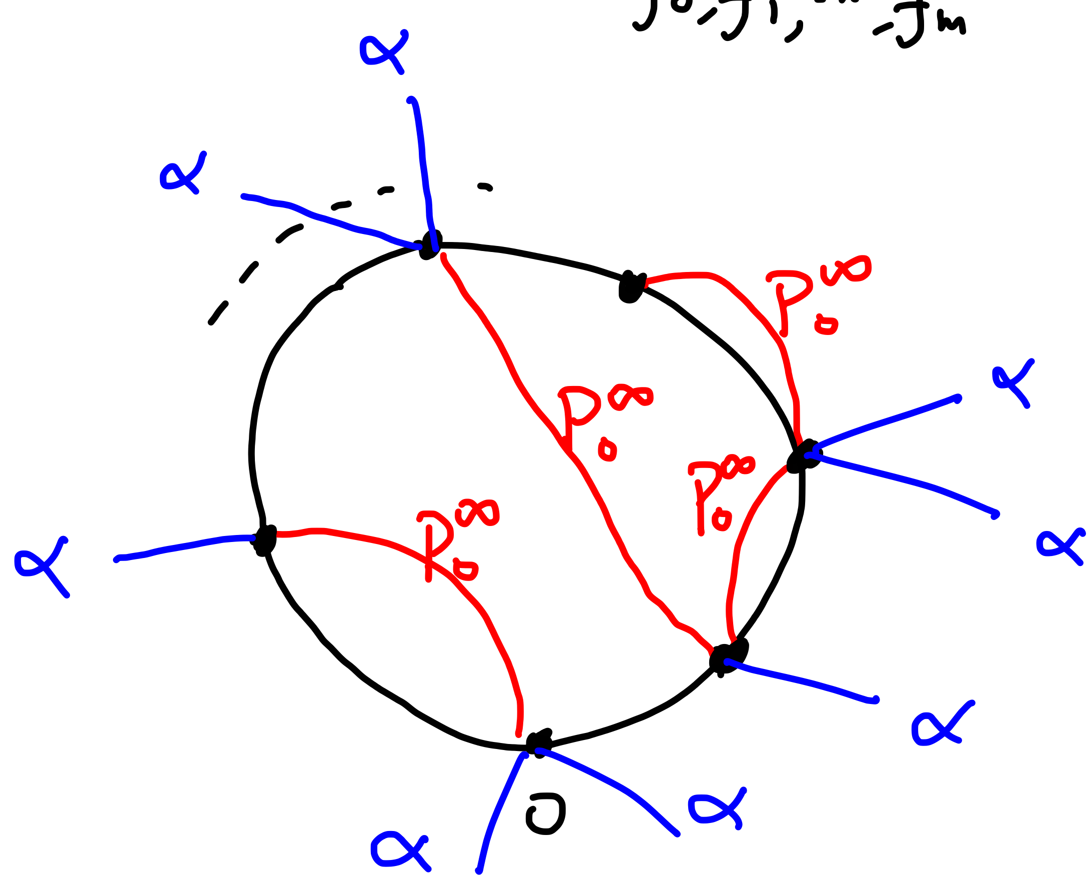
by

$$\langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{free} := e^{\hbar P_0^\infty} \left(\mathcal{O}_{f_0, f_1, \cdots, f_m} \right) \Big|_{\mathbb{H}}$$

In the path integral perspective, this is

$$\langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{free} (\alpha) \qquad \alpha \in \mathbb{H} = H^\bullet(S^1) \otimes V$$

$$= \int_{\mathrm{Im} d^* \subset \mathcal{E}} [D\varphi]\, e^{-S[\varphi+\alpha]/\hbar}\, \mathcal{O}_{f_0, f_1, \cdots, f_m}[\varphi+\alpha]$$



· (Cyclic) Hochschild Complex Reviewed

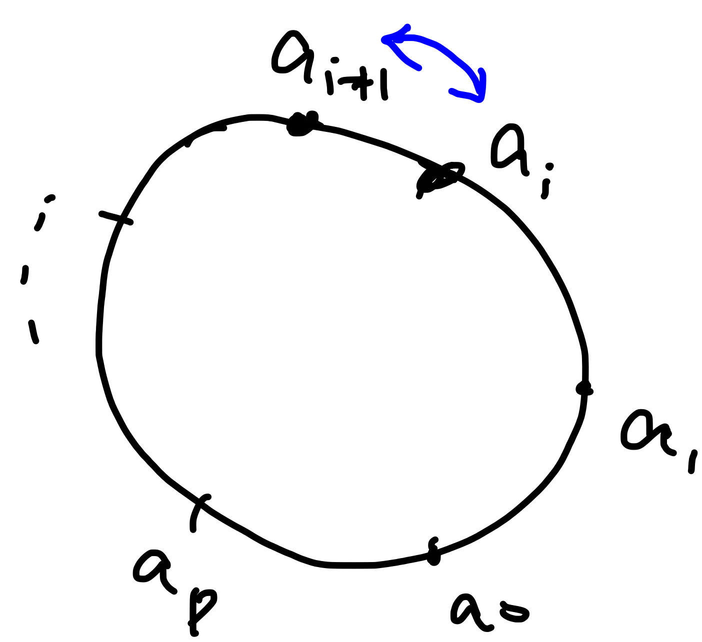
Let A be a unital associative algebra. $\bar{A} = A/\mathbb{C}\cdot 1$

Let
$$C_{-p}(A) := A \otimes \bar{A}^{\otimes p} \quad \text{cyclic } p\text{-chains}$$

Define the Hochschild differential

$$b : C_{-p}(A) \longmapsto C_{-p+1}(A) \qquad p \geq 1$$

by
$$b(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p$$



Then associativity implies

$$\boxed{b \circ b = 0}$$

$\Rightarrow \left(C_{-\bullet}(A), b\right)$

is the Hochschild Chain Complex

We can also define Connes operator

$$B: C_{-p}(A) \mapsto C_{-p-1}(A) \quad \text{by}$$

$$B(a_0\otimes\cdots\otimes a_p) = 1\otimes a_0\otimes\cdots\otimes a_p$$
$$+\sum_{i=1}^{p}(-1)^{pi}\, 1\otimes a_i\otimes\cdots\otimes a_p\otimes a_0\otimes\cdots\otimes a_{i-1}$$

We have the following relations

$$b^2=0 \qquad B^2=0 \qquad [b,B]=bB+Bb=0$$

Let u be a formal variable of $\deg = 2$. Then

$$(b+uB)^2=0$$

This defines a complex

$$CC_{-\bullet}^{per}(A) = \left(C_{-\bullet}(A)[u,u^{-1}],\ b+uB\right)$$

called periodic cyclic complex.

· Correlation map continued.

It is not hard to see via type reason by

$$\langle \cdots \rangle_{free} : C_{-p}(\mathcal{W}_{2n}) \longmapsto \hat{\Omega}^{-p}_{2n}((\hbar))$$

i.e. $\langle f_0 \otimes f_1 \otimes \cdots \otimes f_p \rangle_{free}$ is a p-form

Recall that $\hat{\Omega}^{-\bullet}_{2n}$ is equipped w/. a BV operator

$$\Delta = \mathcal{L}_{\omega^{-1}} = \mathcal{L}_{\pi}$$

<u>Prop</u>: $\langle b(-) \rangle_{free} = \hbar \Delta \langle \cdots \rangle_{free}$

$$\langle B(-) \rangle_{free} = d_{2n} \langle \cdots \rangle_{free}$$

Here $d_{2n} : \hat{\Omega}^{-\bullet}_{2n} \mapsto \hat{\Omega}^{-(\bullet+1)}_{2n}$ is the de Rham differential

In other words, the correlation map

$$\langle \cdots \rangle_{free} : C_{-\bullet}(\mathcal{W}_{2n}) \longmapsto \hat{\Omega}^{-\bullet}_{2n}((\hbar))$$

interfwines b with $\hbar\Delta$

B with d_{2n}

We can combine the above two equations

$$\langle \cdots \rangle_{free} : CC^{per}_{-\bullet}(\mathcal{W}_{2n}) \longmapsto \hat{\Omega}^{-\bullet}_{2n}((\hbar))[u,u^{-1}]$$

$$b + uB \qquad\qquad \hbar\Delta + u d_{2n}$$

- BV integral on zero modes

We can define a BV integration map on the BV algebra $(\hat{\Omega}^{-\bullet}_{2n}, \Delta)$ which is only non-zero on top forms $\hat{\Omega}^{-2n}_{2n}$ and send

$$\beta \in \hat{\Omega}^{-2n}_{2n} \longmapsto \frac{\hbar^n}{n!} \iota_\pi^n \beta \Big|_{p=q=0}$$

This is the Berezin integral over the purely fermionic super-Lagrangian

We can extend this BV integration to an S^1-equivariant version by

$$\int_{BV}: \quad \widehat{\Omega}^{-\bullet}_{2n}[u,u^{-1}] \longmapsto \mathbb{R}((\hbar))[u,u^{-1}]$$

$$\beta \longmapsto \left(u^n e^{\hbar \iota_\pi / u} \beta \right) \Big|_{p=q=0}$$

Then it has the following property

$$\int_{BV} (\hbar \Delta + u d_{2n})(-) = 0$$

Rk. For $\beta \in \widehat{\Omega}^{-\bullet}_{2n}$, the equivariant limit

$$\lim_{u \to 0} \int_{BV} \beta = \frac{\hbar^n}{n!} \iota_\pi \beta \Big|_{p=q=0}$$

goes back to the previous BV integration.

Combining the above maps, we define

$$\mathrm{Tr} := \int_{BV} \circ \langle \cdots \rangle_{free} : CC_{-\bullet}^{per}(\mathcal{W}_{2n}) \longmapsto \mathbb{R}((\hbar))[u, u^{-1}]$$

which satisfies the following equation

$$\mathrm{Tr}\left((b + uB)(-)\right) = 0$$

Therefore Tr descends to periodic cyclic homology.

This is essentially the Feigin-Felder-Shoikhet formula.

- A graded version

We can generalize slightly by considering

V = graded vector space

w/. $\deg = 0$ symplectic pairing ω

We still have the canonical quantization

$$\left(\widehat{\mathcal{O}}(V)[[\hbar]], \ * \right)$$

and similarly can define BV algebra of forms

$$\left(\widehat{\Omega}_V^{-\bullet}, \ \Delta = \mathcal{L}_{\omega^{-1}} \right)$$

The same trace map gives

$$\langle \cdots \rangle_{free} : \underset{b}{C_{-\bullet}\left(\widehat{\mathcal{O}}(V)[[\hbar]] \right)} \longmapsto \underset{\hbar\Delta}{\widehat{\Omega}_V^{-\bullet}((\hbar))}$$

Given $\gamma \in \widehat{\mathcal{O}}(V)$, $\deg(\gamma) = 1$, it defines an action

$$I_\gamma = \int_{S^1} \gamma(\varphi) \quad \forall \varphi \in \Omega^\bullet(S^1) \otimes V$$

Let's treat I_γ as an interaction and consider

$$\underbrace{\frac{1}{2} \int_{S^1} \langle \varphi, d\varphi \rangle}_{\text{free part}} + \underbrace{\int_{S^1} \gamma(\varphi)}_{= I_\gamma}$$

Then we run the RG flow to get

$$e^{\frac{1}{\hbar} I_\gamma[\infty]} := e^{\hbar P_0^\infty} e^{\frac{1}{\hbar} I_\gamma}$$

which is well-defined since P_0^∞ is bounded.

Let's now analyze the QME. By construction

$$e^{\frac{1}{\hbar} I_\gamma[\infty]} = \langle 1 \otimes e^{\gamma/\hbar} \rangle_{free}$$

Assume $\gamma * \gamma = \frac{1}{2}[\gamma, \gamma]_* = 0$. Then

$$\hbar \triangle e^{\frac{1}{\hbar} I_\gamma[\infty]} = \langle b(1 \otimes e^{\gamma/\hbar}) \rangle_{free} = 0$$

Prop [Grady-Li-L] If $[\gamma, \gamma]_* = 0$. Then the local interaction $I_\gamma = \int_{S^1} \gamma(\varphi)$ defines a family of sol'n of effective QME $I_\gamma[L]$ at scale $L > 0$ by

$$e^{\frac{1}{\hbar} I_\gamma[L]} := \lim_{\varepsilon \to 0} e^{\hbar P_\varepsilon^L} e^{\frac{1}{\hbar} I_\gamma}.$$