Ordinary Differential Equations

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Preface

This preliminary note is written for the undergraduate course "Ordinary Differential Equations" that I lectured in the fall semester of 2023 at Department of Mathematical Sciences, Tsinghua University. It is to give a basic introduction to the theory of ordinary differential equations, covering fundamental topics on initial value problem, boundary value problem and analytic method, as well as applied topics on calculus of variations and numerical methods.

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