Graph Theory and Optimisation

Made by Son of Anton

Graph theory (T.T 1 prep)

① Graph is also known as triple

where

$$\boxed{\{\text{Set}, \text{Set}, \text{Function}\}}$$

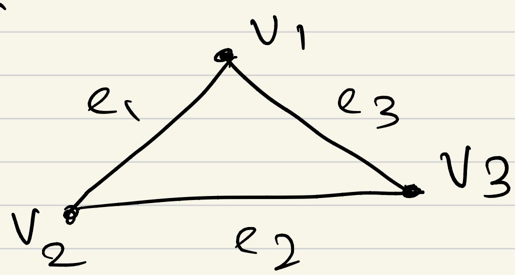
$\{V, E, I\}$

V = Set of Vertices

E = Set of edges

I = Incidence Function

Example

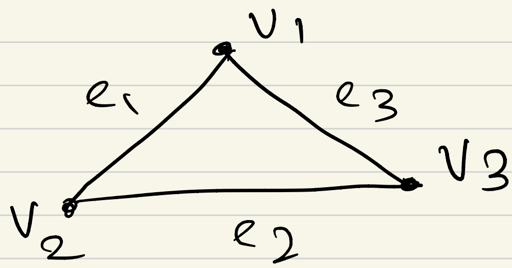


$V = \{v_1, v_2, v_3\}$

$E = \{e_1, e_2, e_3\}$

$$I = \left\{ \begin{array}{l} e_1(v_1\ v_2), \\ e_2(v_2\ \ v_3) \\ e_3(v_1\ \ v_3) \end{array} \right\}$$

→ lets take another escample of the Same graph



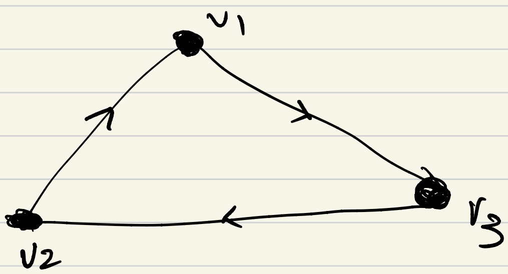
where we dont worry About derection of the graph then its called

① un-direction graph
② un-ordered graph

escample

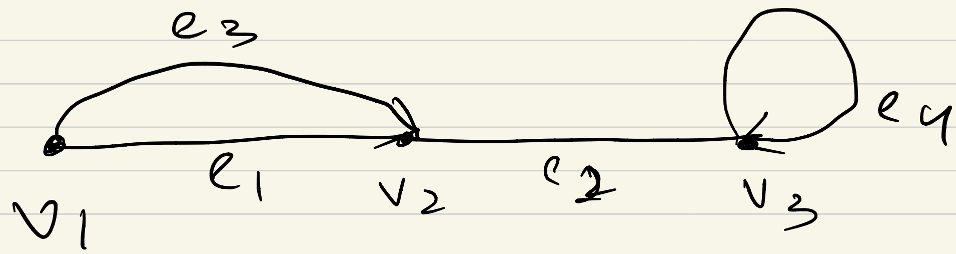
$$e_3(v_2 v_3) \cong e_3(v_3 v_2)$$

where as ordered graph would has the directions Specifying Some thing like this



Types of graph

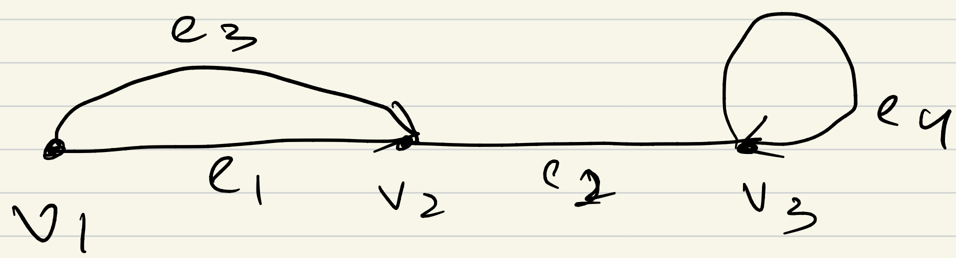
① Simple graph → no ||'el edges or Self loop



↑

This is not a Simple graph

② Multi graph → have ||'el edges and Self loop

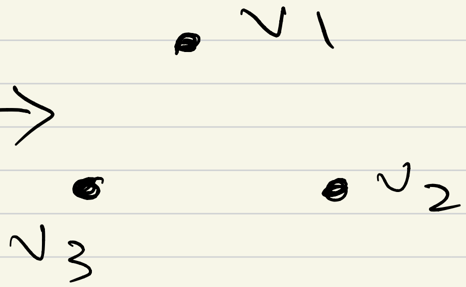


↑

This is a multi graph

③ Null graph

have Vertices and no edge



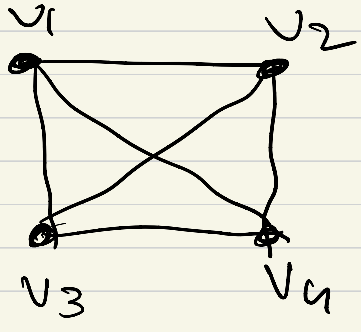
This is a null graph where we dont have edges

this are made of Isolated Vertices

have not edges

④ Complete graph

where every vertex has one edge with all the vertices present

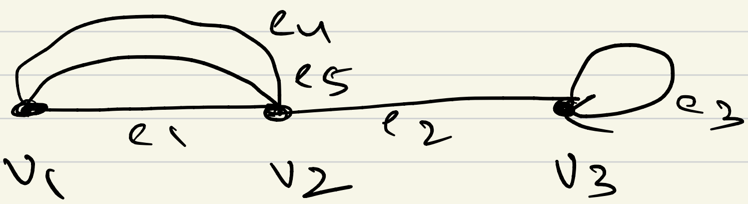


5 Regular graph

→ degree (vertices have them)

↓

The no of Edges that are Incident on that vertex.



$d(v_1) = 3$

$d(v_2) = 4$

$d(v_3) = 3$ (why? Self loop = 2 and Incoming = 1)

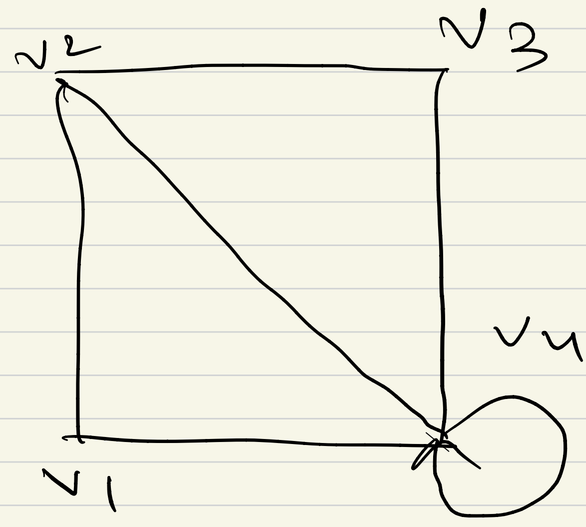
General Formula

$$\boxed{d(v) = n_e + 2n_L}$$

where, n_e = no of edges

n_L = no of loops

Example - 1



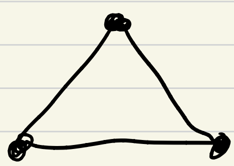
$d(v_1) = 2$, $d(v_2) = 3$, $d(v_3) = 2$

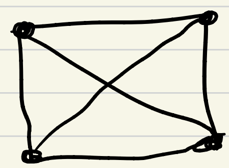
$d(v_4) = 5$

Now lets take about regular graph

→ Degree of all the vertices is the same.

Regular and complete graph



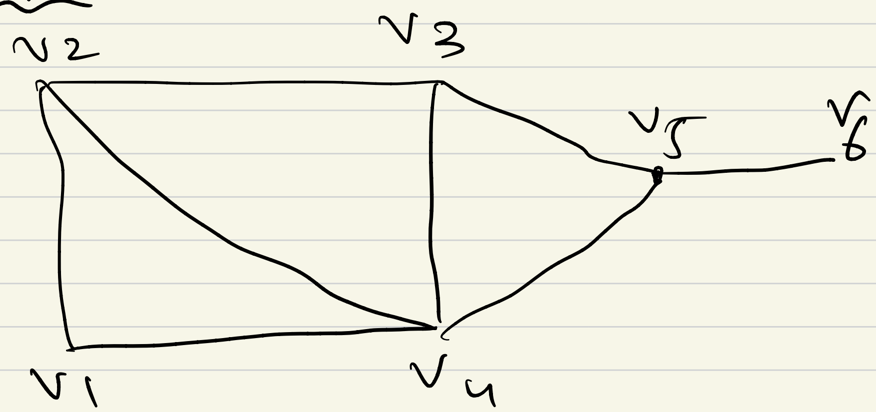


→ Both are regular and are complete or complete graphs are also regular.

Degree of a Vertex

→ No of edges on the vertex

Example - 1



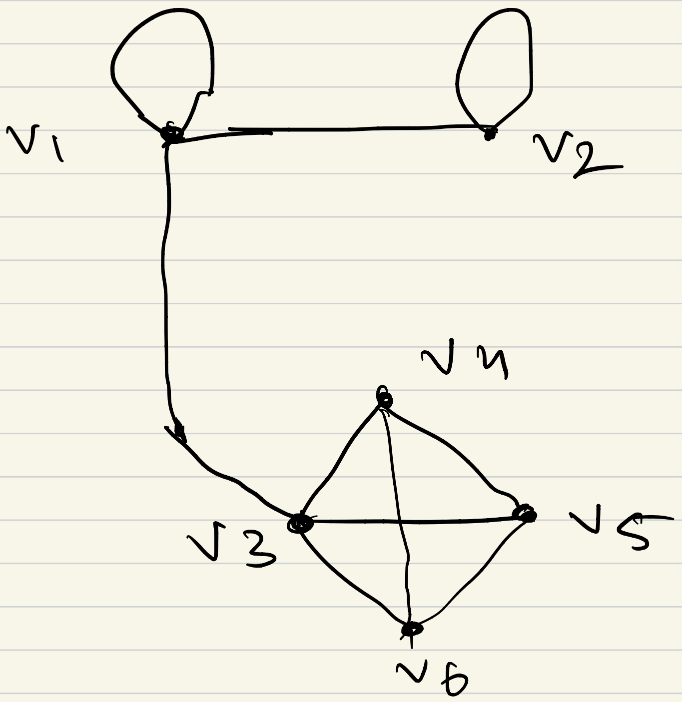
$d(v_1) = 2$, $d(v_2) = 3$

$d(v_3) = 3$, $d(v_4) = 4$, $d(v_5) = 3$

$d(v_6) = 1$ → Pendent vertex

Pendent vertex where degree is 1
one path to reach to that
vertex

③



$d(v_1) = 4$, $d(v_2) = 3$

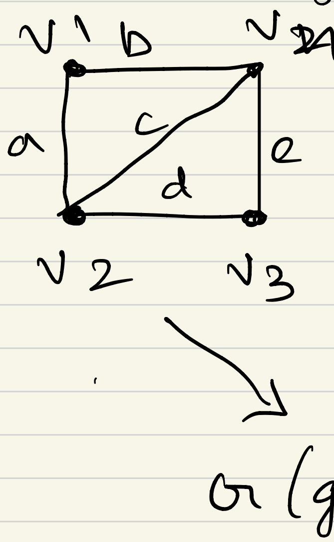
$d(v_3) = 4$, $d(v_4) = 3$,

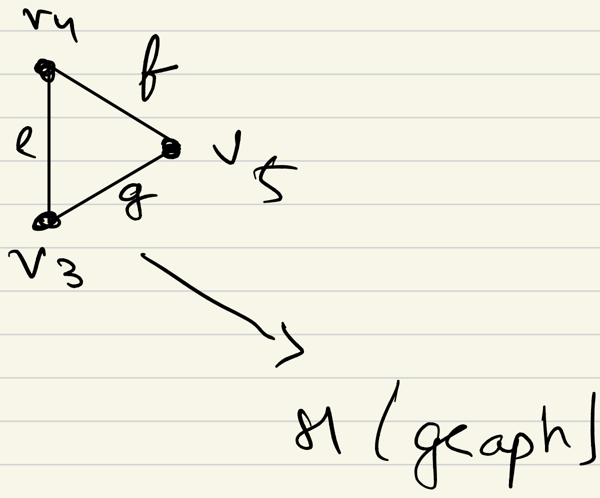
$d(v_5) = 3$, $d(v_6) = 3$

If the graph is regular then
We can define for graphs

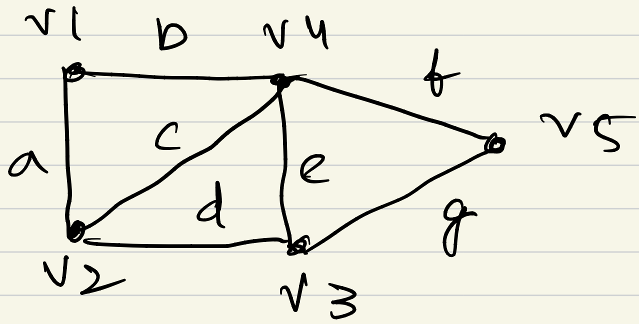
Operations on graph

① union graph

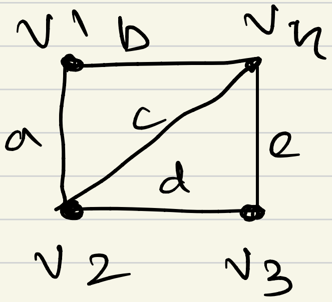


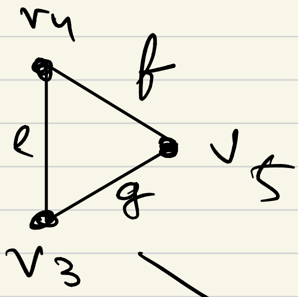


when they have to perform union then it looks like



② Intersection graph

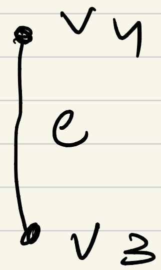




G (graph)

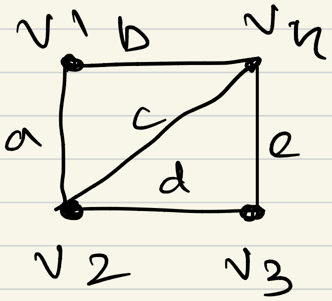
H (graph)

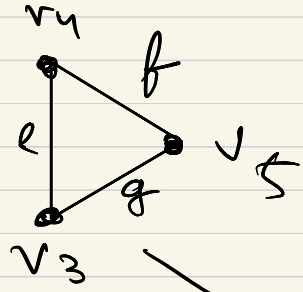
Everything that are commen



This is the result of Intersection of the graph.

3) Difference



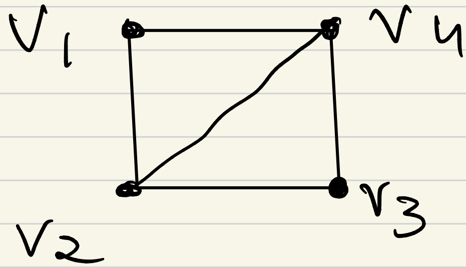


G (graph)

H (graph)

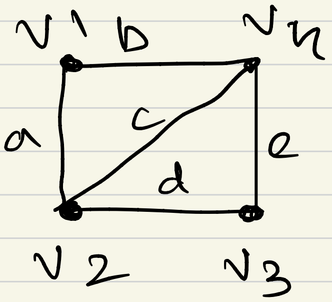
→ $\boxed{G - H}$

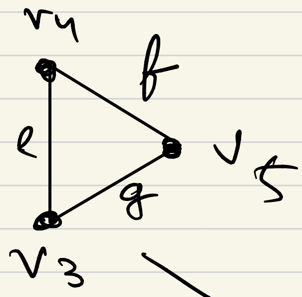
So, All the edges in G but not in H



(4) Sum of the graph (Ring-Sum)

$\boxed{G \oplus H}$





G (graph)

H (graph)

→ Edges in G

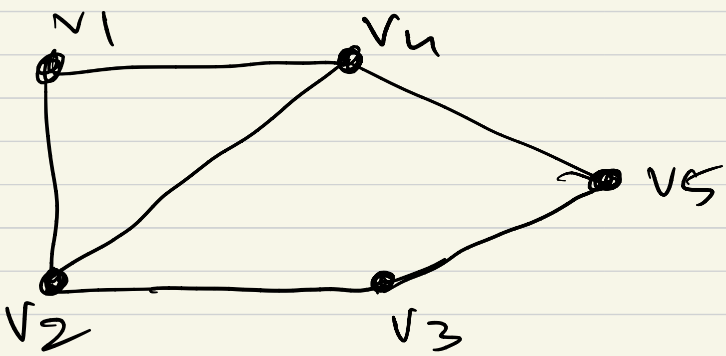
→ Edges in H

→ Not edges in both G and H

$$\boxed{G \oplus H = (G \cup H) - (G \cap H)}$$

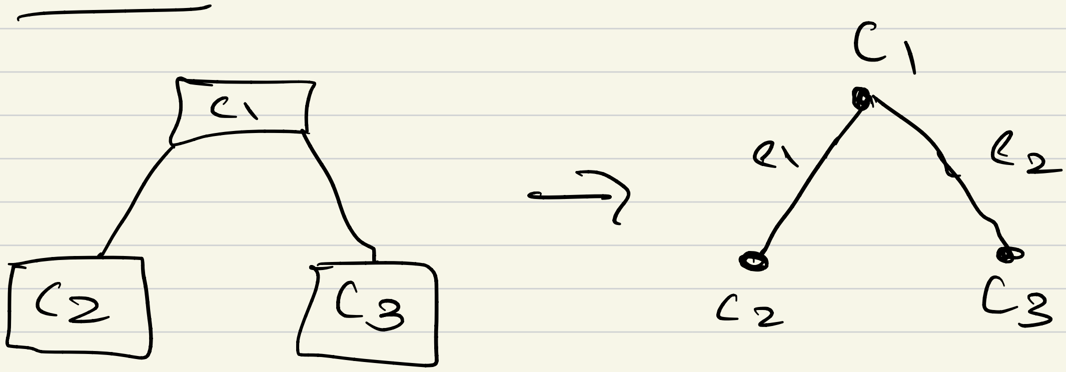
So this results in,

⇒



Representation of the graphs

① Pictorial



② Set Representation

$$V = \{v_1, v_2, v_3, v_4\}$$

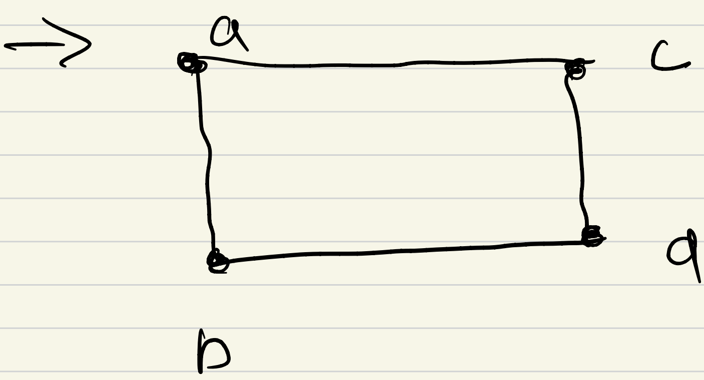
$$E = \{e_1, e_2, e_3, e_4\}$$

$$I = \{e_1(v_1\ v_5),\ e_2(v_2\ v_3),\ e_3(v_2\ v_3),\ e_4(v_3\ v_4)\}$$

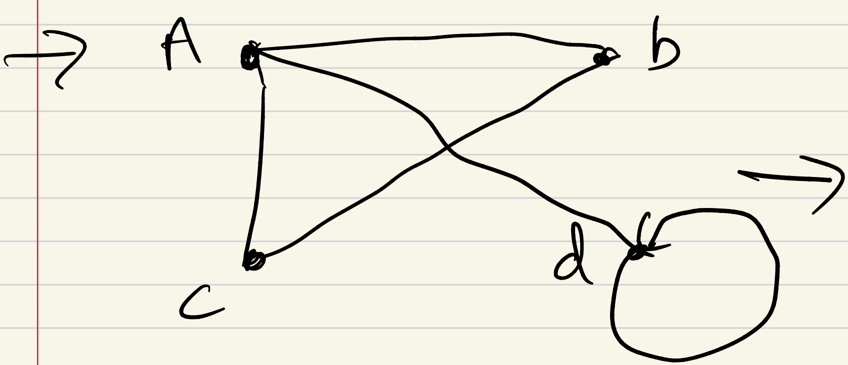
✲ Draw a graph here!!!

3 Adjacency Matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



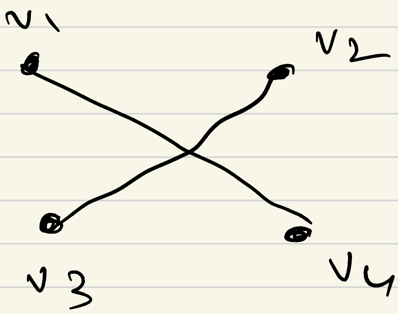
Solution



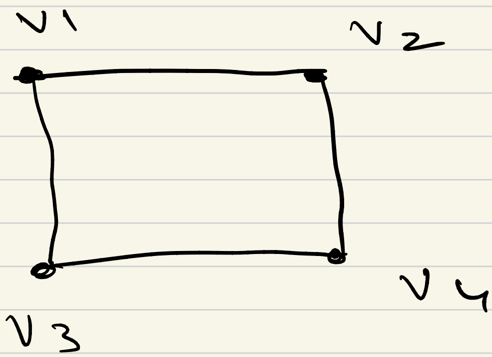
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Complement of a graph

→ G
Normal

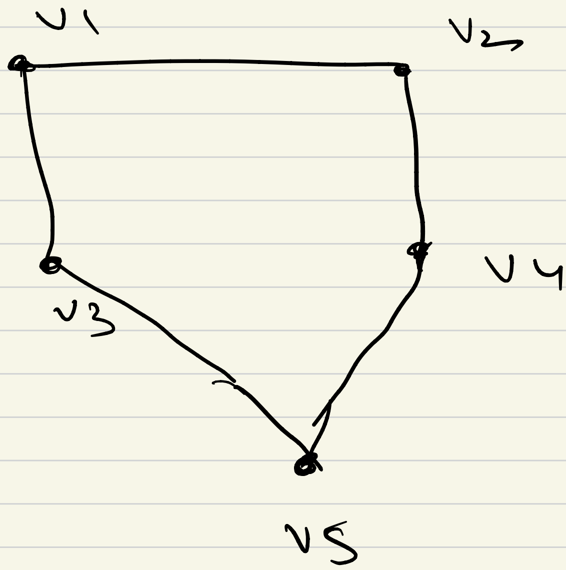


G_c → complement

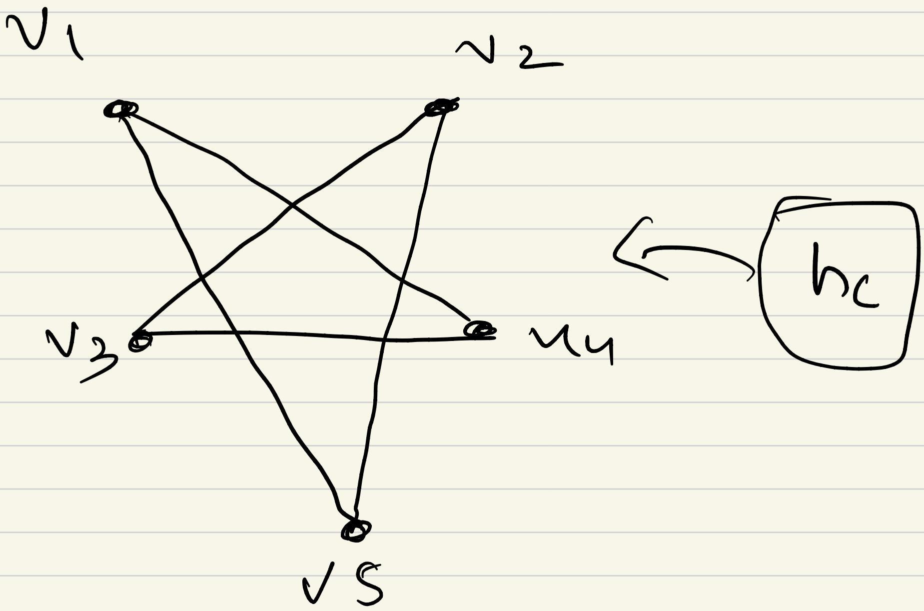


Basically fill out whatever was left off.

example - 2

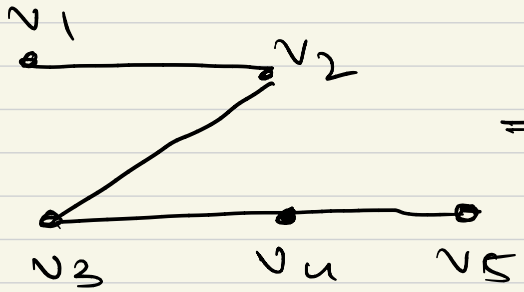


(n)



Also they are a Star, Haha lol !!!

Complement using Adjecency list



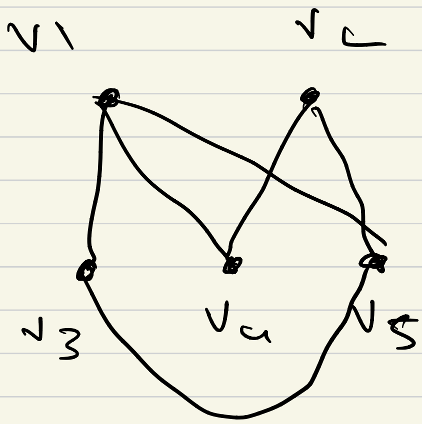
$\Rightarrow$

v	connected to
v_1	v_2
v_2	v_1, v_3
v_3	v_2, v_4
v_4	v_3, v_5
v_5	v_4

G_c

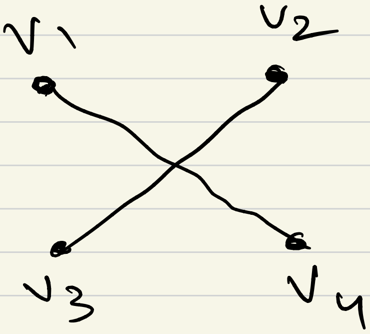
v	connected to
v_1	v_3, v_4, v_5
v_2	v_4, v_5
v_3	v_1, v_5
v_4	$v_1, v_2,$
v_5	v_1, v_2, v_3

$\Rightarrow$

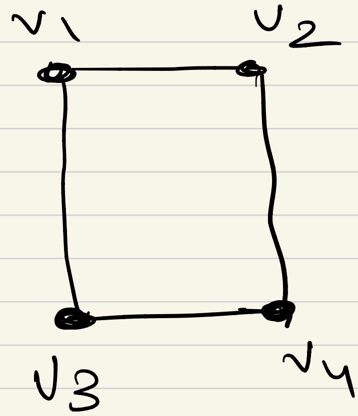


Union of complementary Graphs

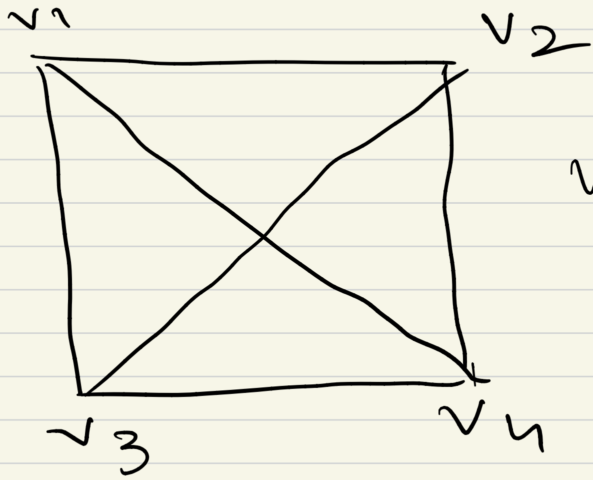
$H \Rightarrow$



$H_c \Rightarrow$



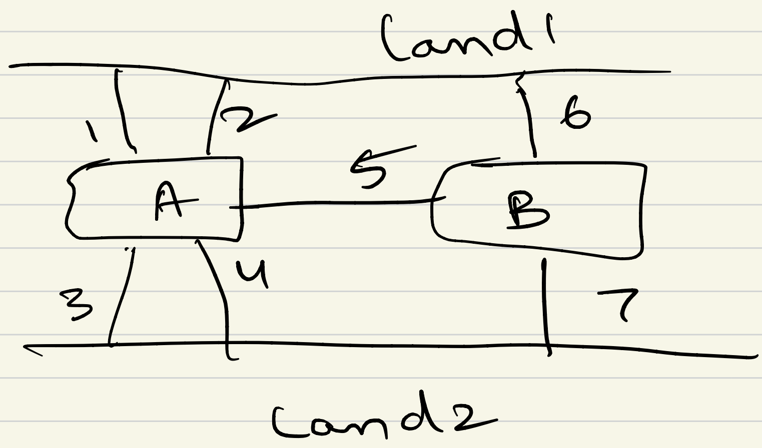
$\Rightarrow (H \cup H_c)$



We got a regular graph with degree - 3

So when you do union of complement and given graph you end up with regular graph.

Eular Graphs



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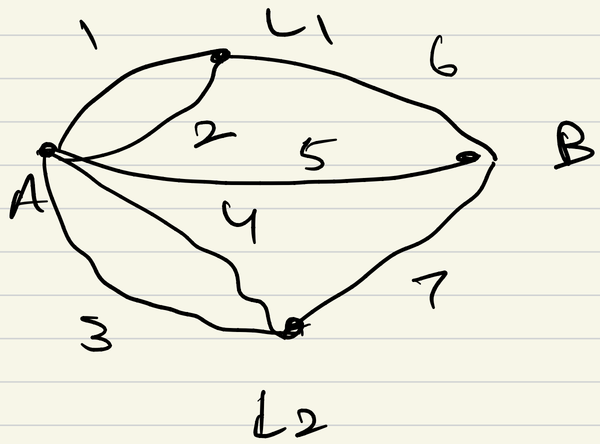
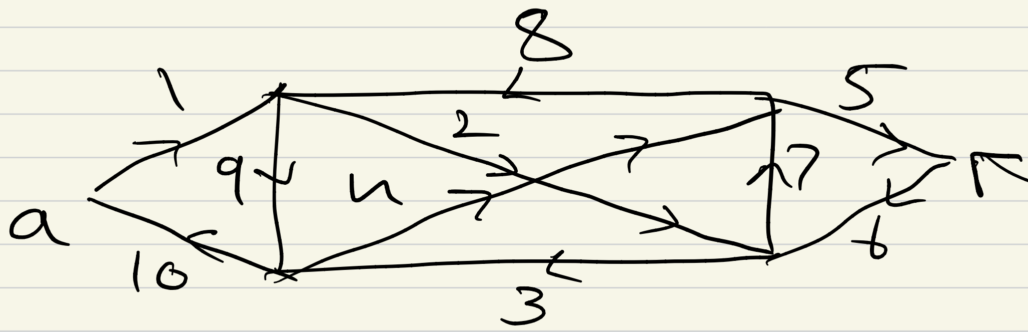


Eular path — Euler-Circuit — Start at one vertex, visit each edge only once and return to same vertex.

Eular path — Start at one Vertex visit each edge only once and go to other vertex.

Example - 1



$E_c \rightarrow$ It is present

$E_p \rightarrow$ No.

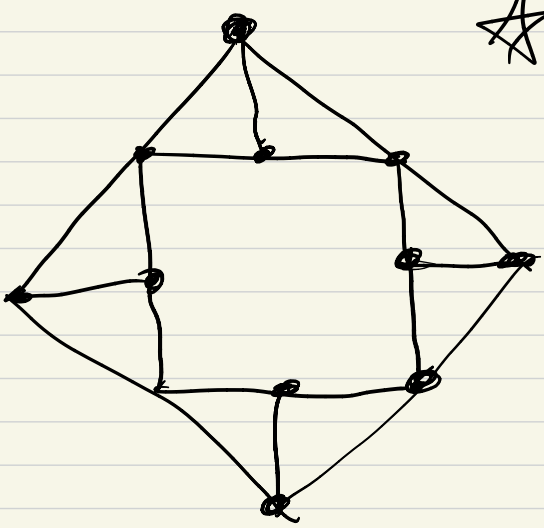
a → b → e → f → c → d → e → c → b → f → a

degree of All the vertices is even the it is a Euler circuit.

Euler path

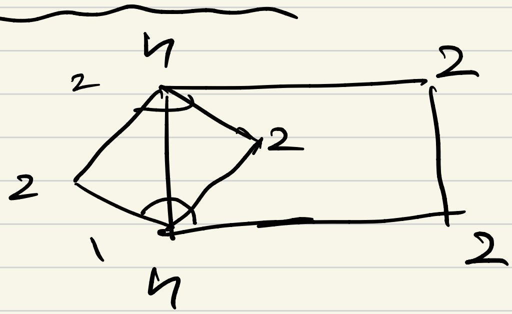
No of odd degree Vertices Should be ≤ 2 (0, 1, 2)

Example - 2



☆ → no path or circuit

Example - 3



→ n

has circuit has a path.

$n \leq 2$ odd vertices

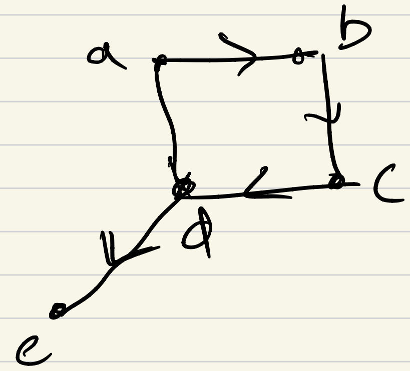
Hamiltonian Graph

→ Traverse all the vertices only once.

→ Transe any edge only once

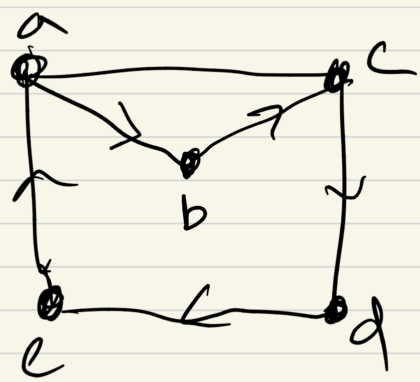
→ Don't need to traverse all the edges.

Example -1



Hamiltonion path

Example -2

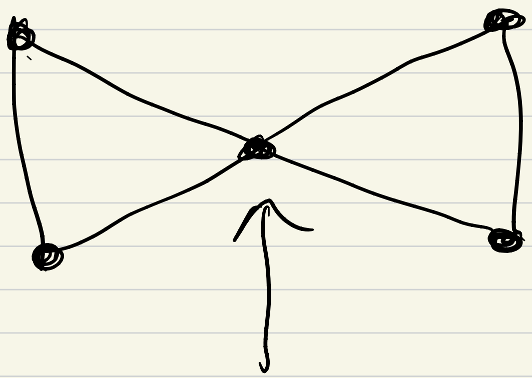


Hamiltonion cycle

Filters

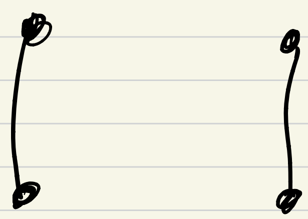
→ If you have a pendent vertex then no possibility of a cycle but might have a path.

Example



Euler graph
Ham graph } → only for connected graphs

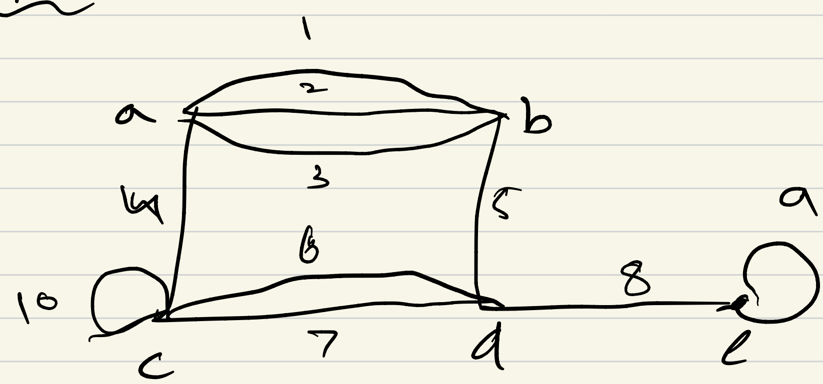
If you remove a vertex then you have to remove the edge as well



Handshaking theorem / lemma (Sum of degree)

Psedo graph → Regular Edge → Simple
→ ||el Edge → Multi
→ Loop → Psedo

Example



$d(a) = 4$, $d(b) = 4$

$d(c) = 5$, $d(d) = 4$

$d(e) = 3$

$\Rightarrow$ 5, 4, 4, 4, 3 → degree Sequence

Add Everything

$= 5 + 4 + 4 + 4 + 3 = 20$

No of Edges = 10 (In previous graph)

Formula

$$\boxed{\sum_{i=1}^{n} d(v_i) = 2E}$$

$$\Rightarrow \sum_{\text{even}} d(v_i) + \sum_{\text{odd}} d(v_j) = \sum_{i=1}^{n} d(v_i)$$

Even

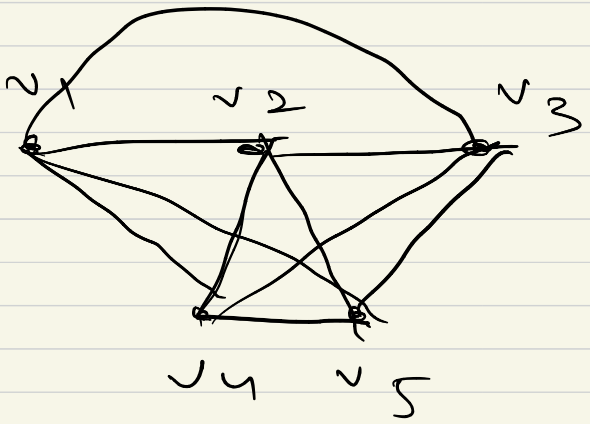
when you add Even no of odd # in degree no then it becomes Even

→ A complete graph is represented by the no of vertices

$K_n \rightarrow$ Complete graph

n = How many Vertices

K_5 graph



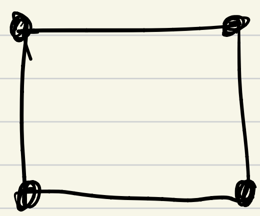
Regular graph

The degree of all Vertex is Equal

→ K-regular graph

$$d(v_i) = K$$

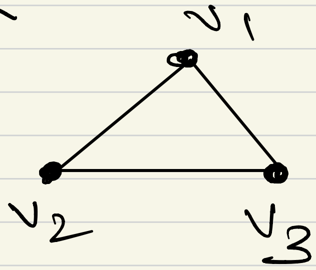
Example



$d(2)$

2 - regular graph with 4 Vertices

Example -3



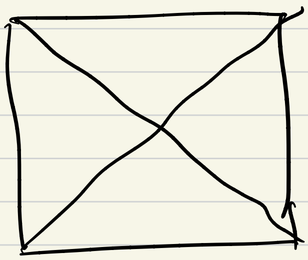
Complete graph

K_3 graph

as well as

2- Regular graph

Example -4



K_4 - graph

3 - Regular graph

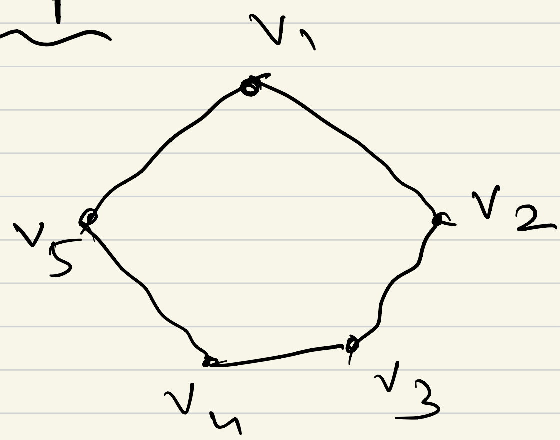
Cycle graph

C_N

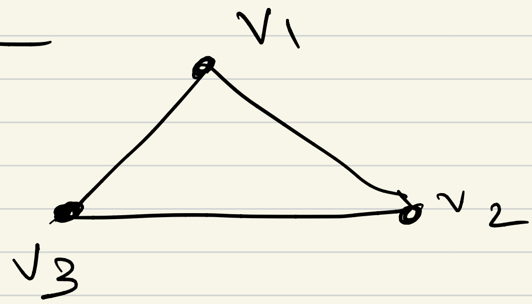
$C \rightarrow$ cyclic

$N \rightarrow$ no of vertices

C_5 graph



C_3 graph



Defination

A cyclic graph C_N has N vertices and N edges with the incidence

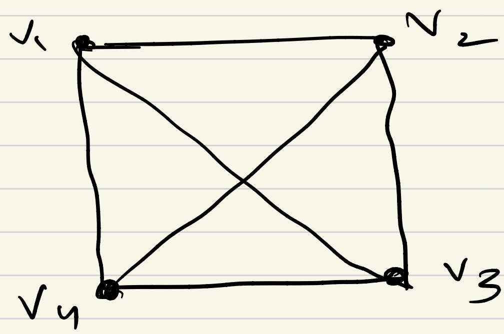
$E_1(v_1 v_2)$, $E_2(v_2 v_3)$, $E_3(v_3 v_4)$ ...... $E_{n-1}(v_{n-1}, v_N)$, $E_n(v_n, v_1)$

Total no Edges = Total no vertices

$d(v_i) = 2$ always //

→ Total no of Edges in Complete graph?

Ky graph



$d(v_i) = n-1$ (where n = no of total vertices)

$= 4-1$

$= 3$

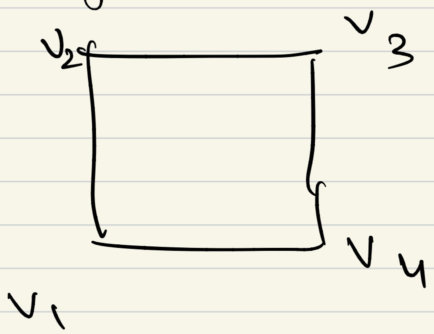
Total no of Edges in Complete graph is

$$\boxed{E(K_n) = \frac{n \times (n-1)}{2}} = \frac{4 \times (3)}{2} = \frac{\cancel{12}^{6}}{\cancel{2}_{1}}$$

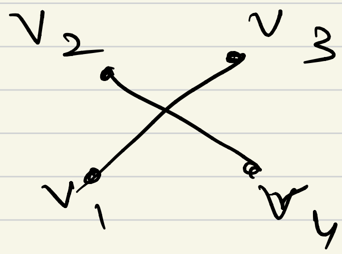
6 Edges for Ky graph

Edges in graph and its complement.

$G \Rightarrow$



$\bar{G} \Rightarrow$

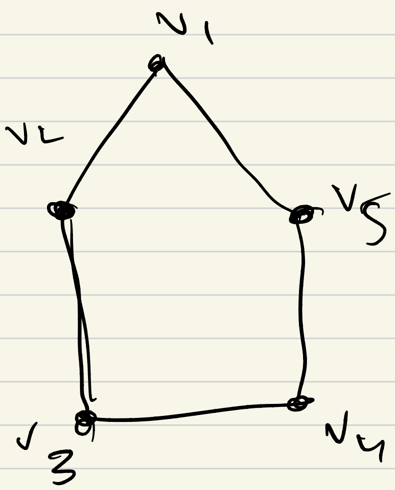


Any graph with its complement on union is a complete graph.

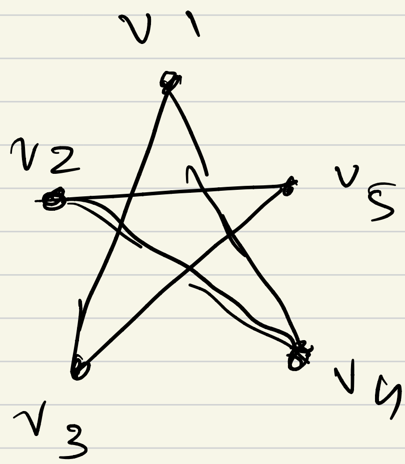
$$\boxed{G \cup \bar{G} = K_n}$$

$n = 5$

$G =$



$\bar{G} =$



Formula for the edges in complement graph

$$\boxed{E_G + E_{\bar{G}} = K_n}$$

Formula's (Summary)

① $\sum_{i=1}^{n} d(v_i) = 2E$ (sum of degrees)

② $\sum_{odd} d(v_i) + \sum_{even} d(v_j) = 2E$

③ In $K_n = \frac{n \times (n-1)}{2}$ [Total No of edgs in the complete graph]

③ $K_n = E_G + E_{\bar{G}}$

④ $r = E - V + 2$ // Euler theorm

Bipartite Graph

→ Set of Vertices in graph G Should be able to divide into two

$$\begin{array}{ccc} & V(G) & \\ \swarrow & & \searrow \\ V_1(G) & & V_2(G) \end{array}$$

Example

$$V(G) = \{1, 2, 3, 4, 5\}$$

$$V_1(G) = \{1, 2\}$$

$$V_2(G) = \{3, 4, 5\}$$

→ $V_1(G)$, $V_2(G)$ → i → non-empty

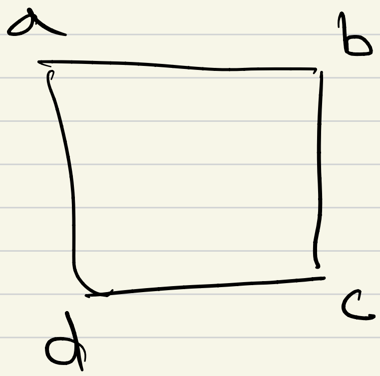
ii → no-disjoint

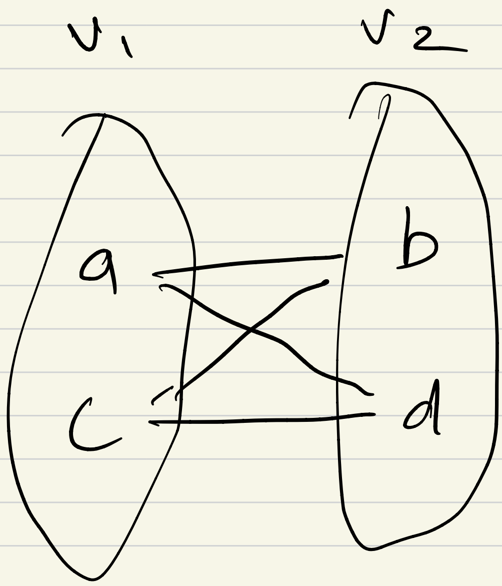
meaning no repeating elements

- Edges

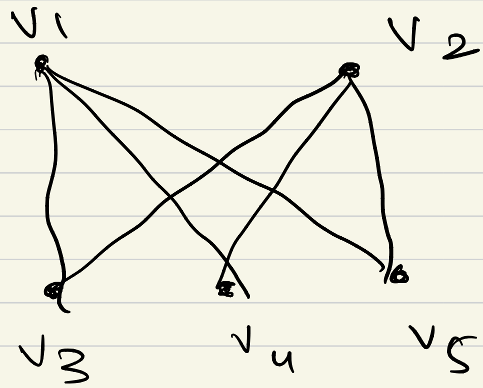
→ Each Edge Should connect one vertex from V_1 to one vertex from V_2

→ Should not have Same Edges from the Subset of the vertices





this is a bipartite graph

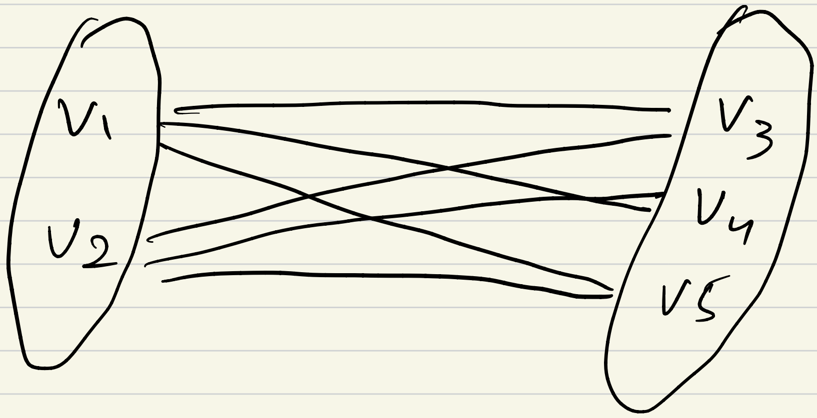


Set 1

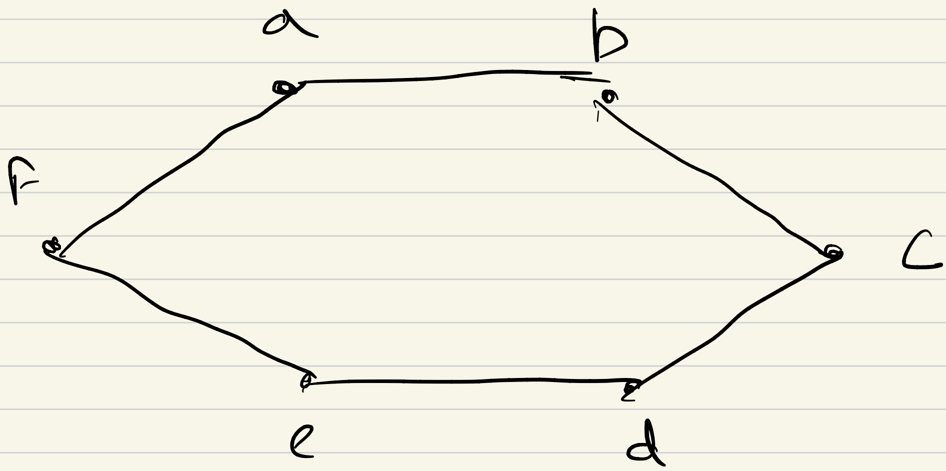
$v = \{v_1, v_2\}$

Set 2

$= \{v_3, v_4, v_5\}$



this is also a bipartite graph



Set 1

c
e
a

Not bipartite ?

Set 2

f
d
b

Complete graph cannot be bipartite graph

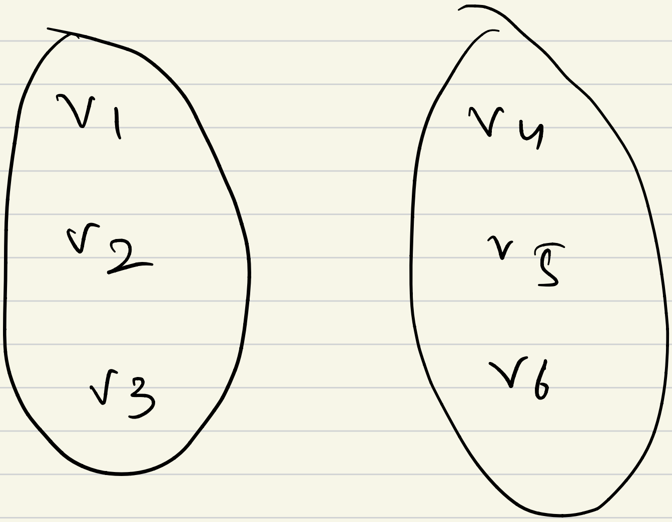
$\Rightarrow K_{MN} \rightarrow$ Bipartite graph

$M = \text{Set} \to 1 = 2$ $\quad K_{23}$

$N = \text{Set } 2 = 3$

$\Rightarrow$ complete bipartite graph

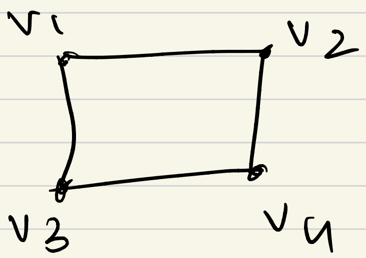
complete $K_{3,3}$



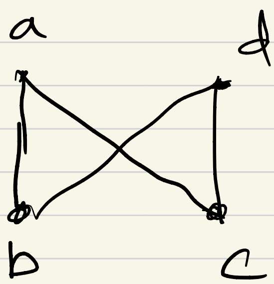
Every vertices should be connected to all the vertices in each set

Isomorphic Graph

Two graphs are Isomorphic if they are Same graph and they are drawn differently.



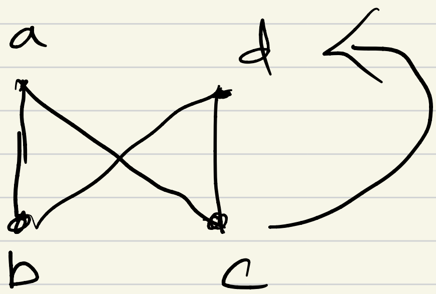
(h_1)



(h_2)

→ h_1 and h_2 are Isomorphic.

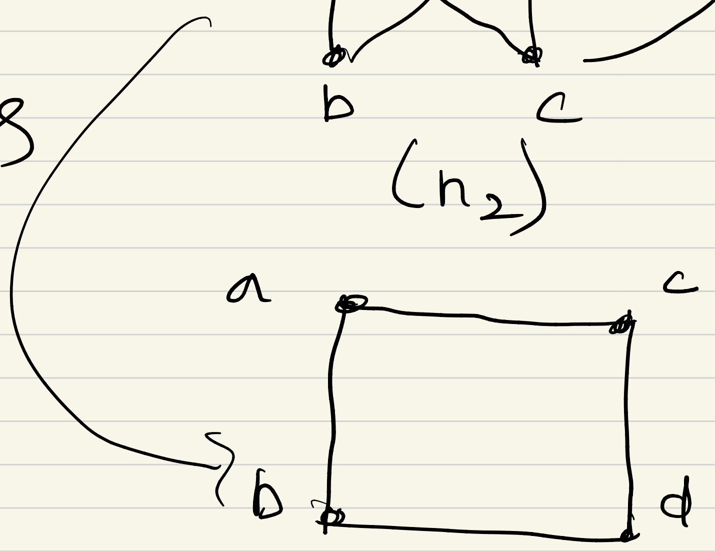
lets redraw h_2,,



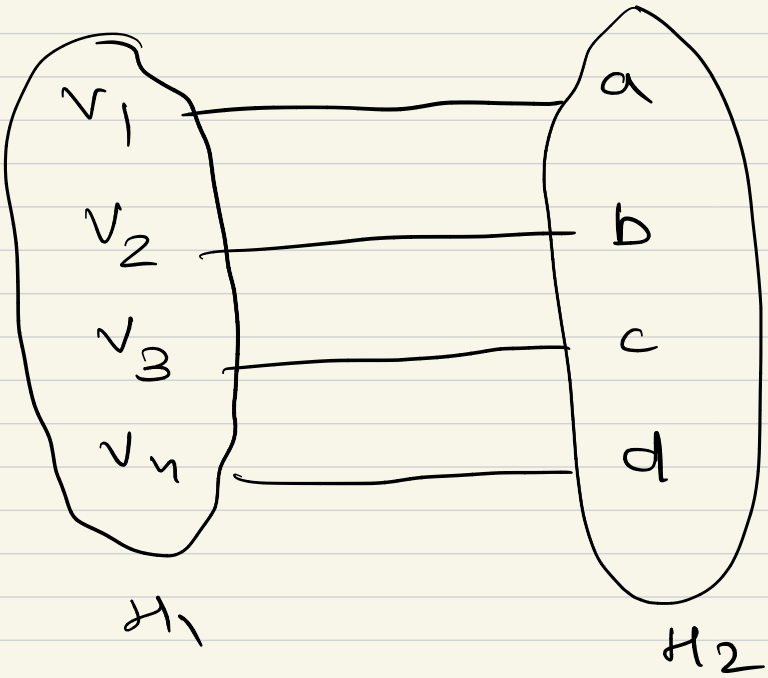
take c up

(h_2)

Becomes this

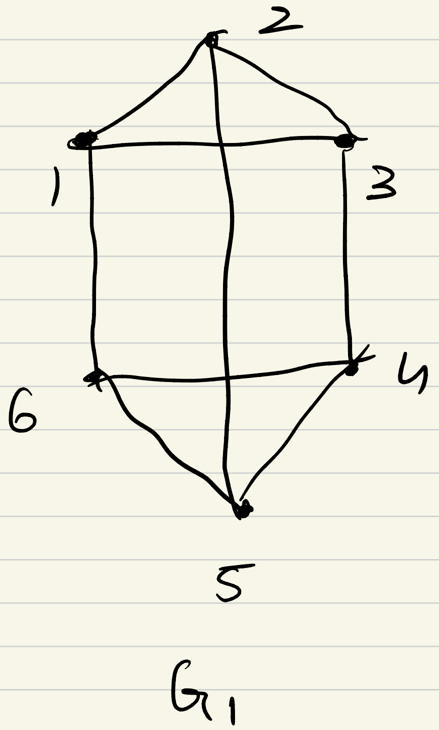


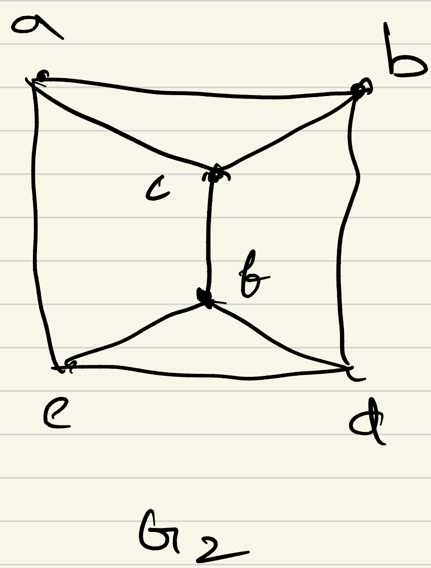
So

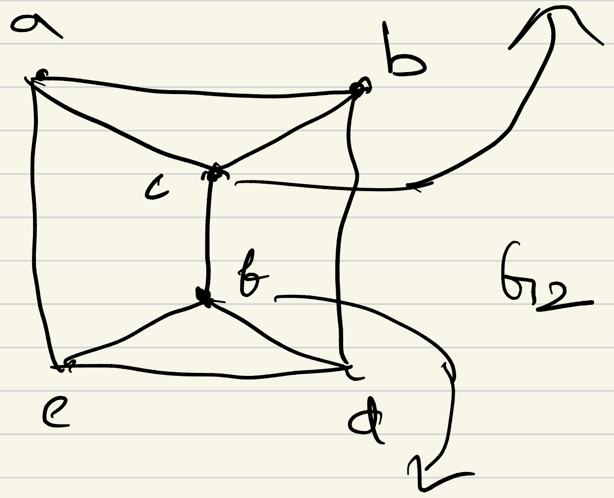


H_1 and H_2 are Isomorphic.

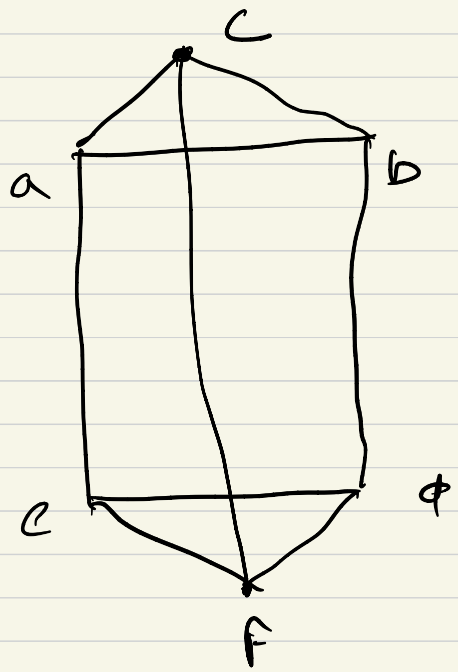
Example - 1





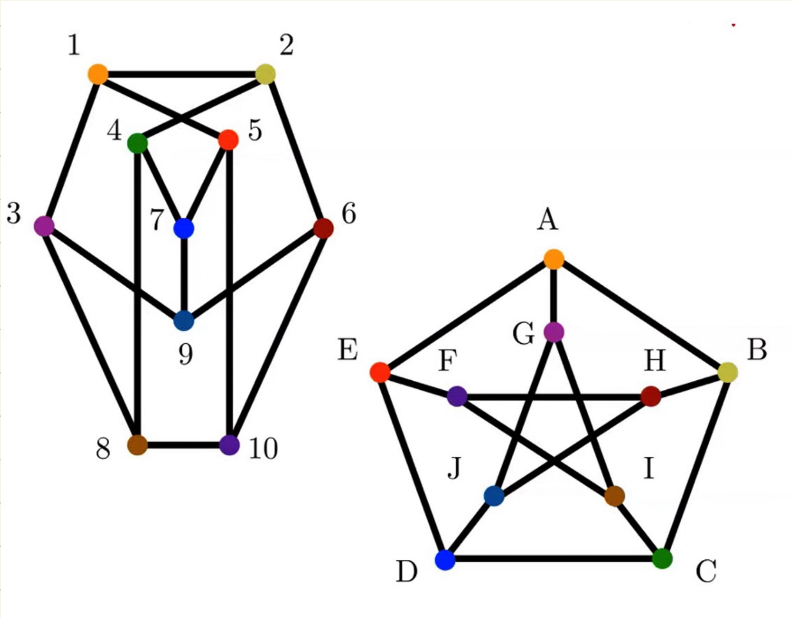


Now lets Just open it as envolop



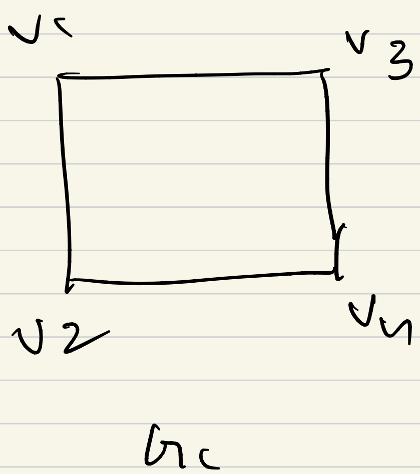
Now its Isomorphic G_1 and G_2

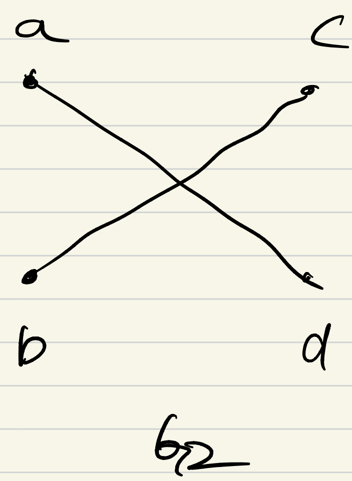
Example - 2



It is Isomorphic. (Map the coloured dots.)

Example - 3





This is not Isomorphic

Mathematical conclusion

$G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are Isomorphic if there is

→ one to one Mapping b/w V_1 & V_2

→ onto mapping b/w V_1 & V_2

→ Two Adj vertices in G_1 should be adj in G_2

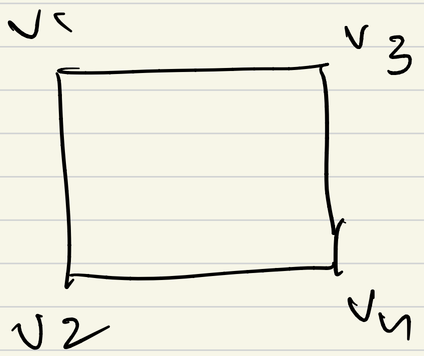
Logical Conclusion ☆☆☆

Two Isomorphic graphs exhibit the same behaviour for all the graph releated properties

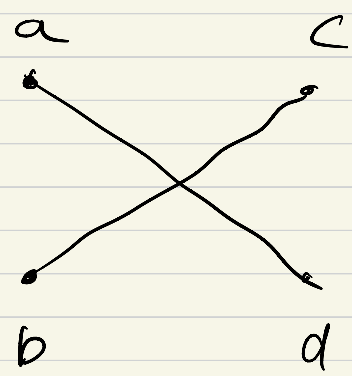
Same properties

1. Same vertices
2. 〃 edges
3. 〃 no of vertices with given degree
4. 〃 Sum of degrees
5. Both planer
6. Both cyclic
7. Both pendent
8. Both Isolated

Example - 4



G_1



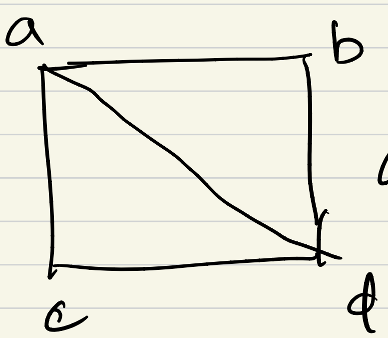
G_2

$G_1 \rightarrow$ 4 vertices with degree 2

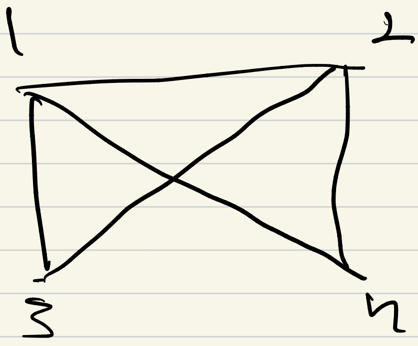
$G_2 \rightarrow$ 4 vertices " " 1

NOT Isomorphic !!!

Example - 5



G_1



G_2

degree,

$G_1 = \{2, 2, 3, 3\}$

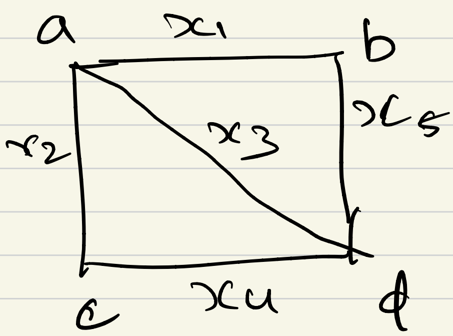
$G_2 = \{2, 2, 3, 3\}$

$a \to 1$

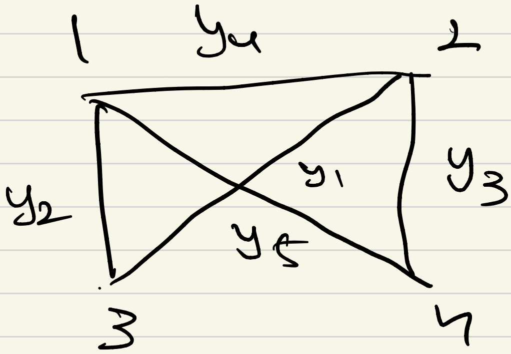
$b \to 2$

$c \to 3$

$d \to 4$



G_1



G_2

$x_1 \to y_1$

$x_2 \to y_2$

$x_3 \to y_5$

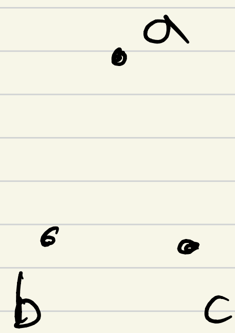
$x_4 \to y_4$

$x_5 \to y_5$

Isomorphic graph !!!

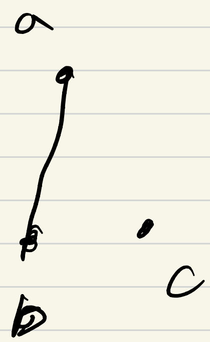
Isomorphism
- → two graphs Find If they are Isomorphic?
- → Draw all possiable Isomorphic graphs given Vertices and edges?

① Draw all possiable Isomorphic graph with 3 vertices and any no of edges.



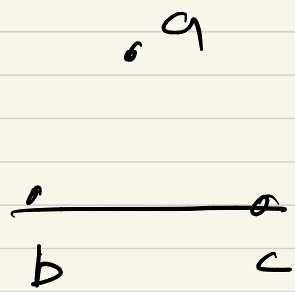
$E=0$

(P_1)



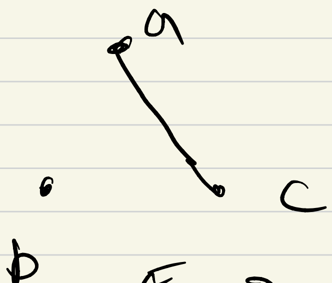
$E=1$

(P_2)



$E=1$

(P_3)

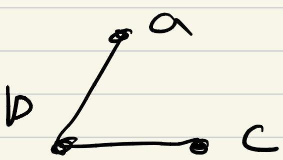


$E=2$

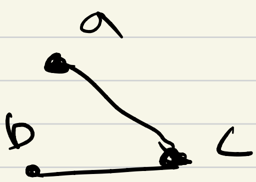
P_4

☆ P = possibility

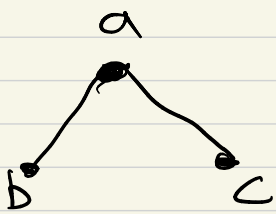
E_2



(P_5)

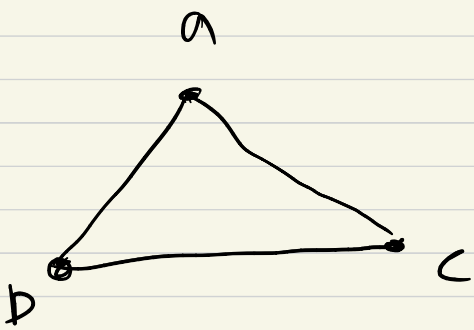


(P_6)



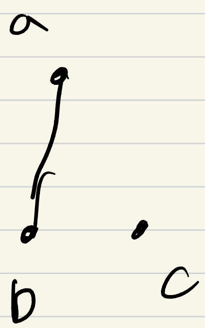
(P_7)

E_3

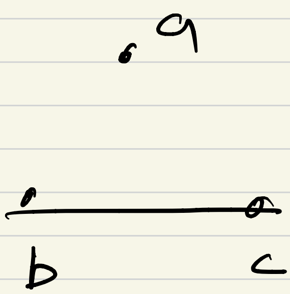


(P_8)

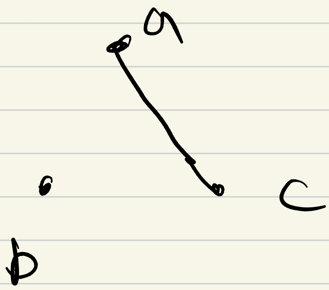
Now lets take $V = 3$ and $E = 1$



P_1



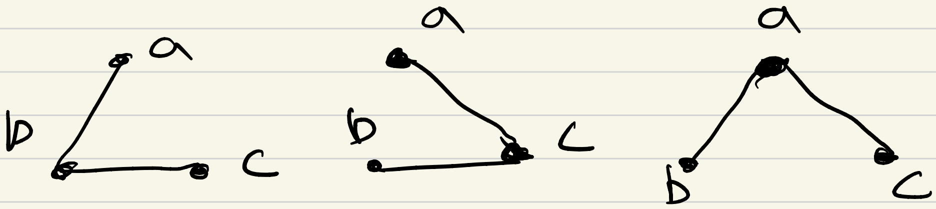
P_2



P_3

Properties

1. 3 vertices
2. 1 edge
3. 1 Isolated Vertex
4. 2 Vertices with degree 1
   1 Vertex with degree 0
5. Sum of degree = 2



(P_5) (P_6) (P_7)

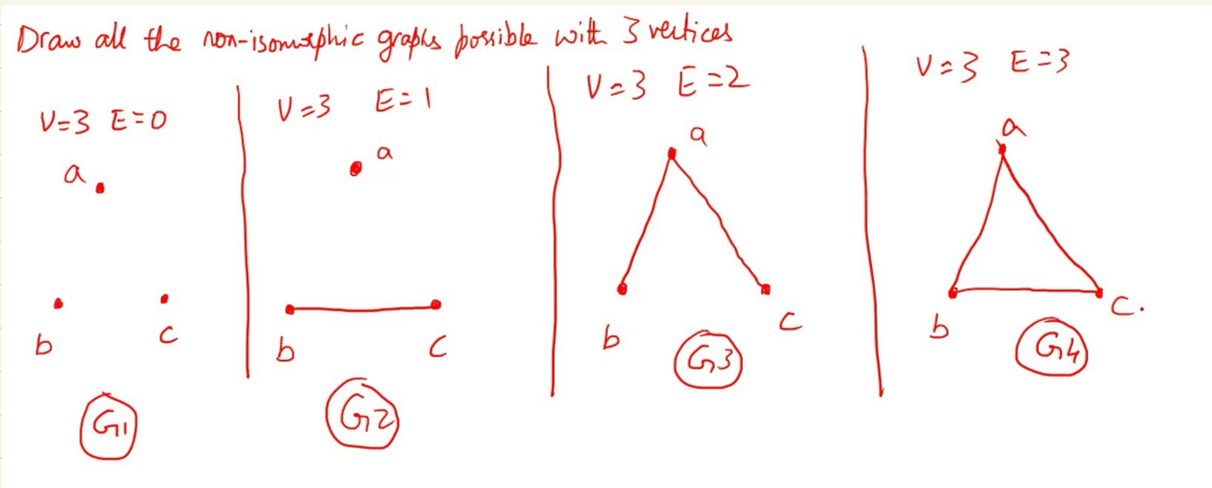
Properties

→ 3 vertices, 2 edges

→ 1 vertex with degree 2
2 vertices with degree 1

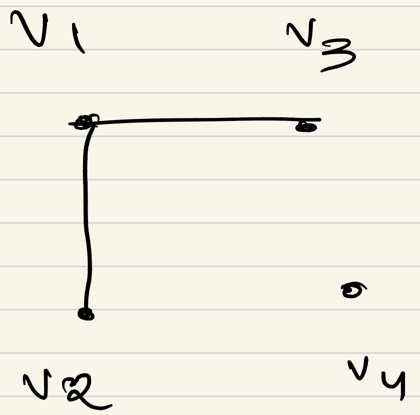
→ Sum of degree = 4

→ 1 vertex with degree 2 with 2 neighbours whose degree - 1

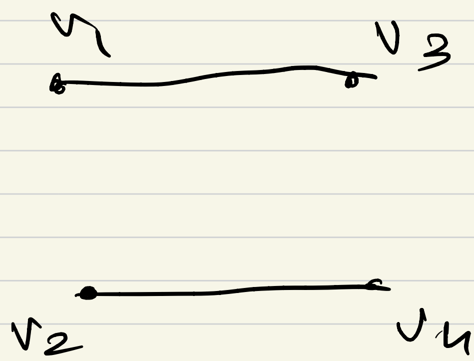


Only 4 non-isomorphic are possible with V=3

② Draw all possible non-Isomorphic graph with $V = 4$ and $E = 2$

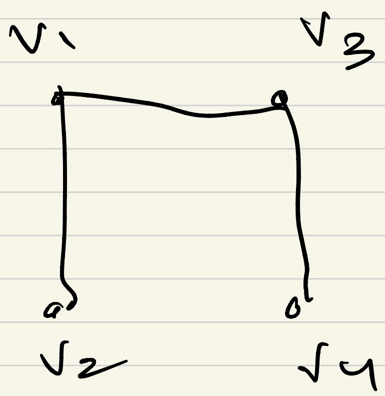


G_1

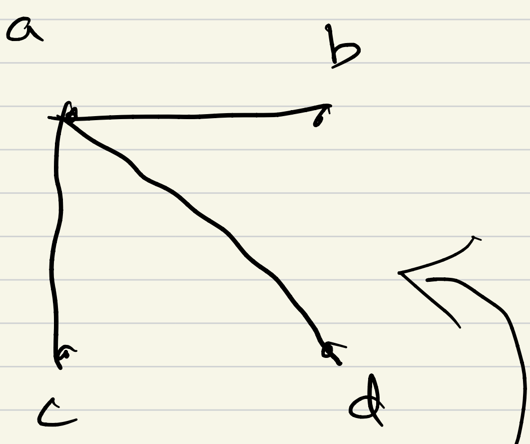


G_2

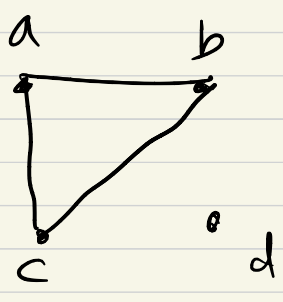
③ Draw all possible non - Isomorphic graphs with $V = 4$ and $E = 3$



G_1

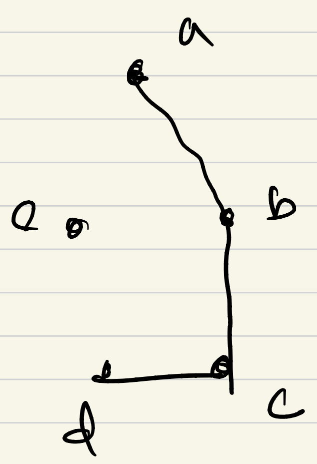


G_2

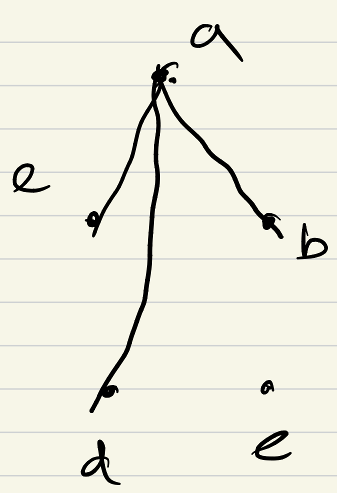


G_3

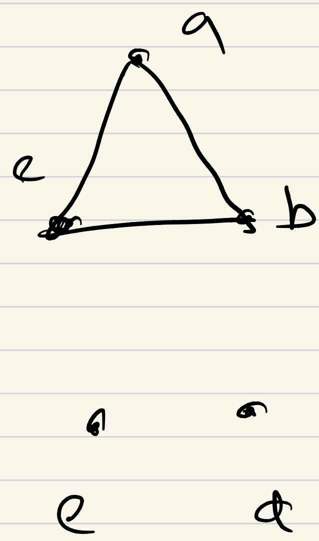
(4) with $V = 5$ and $E = 3$



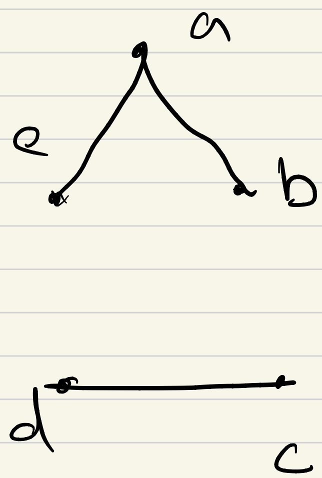
G_1



G_2



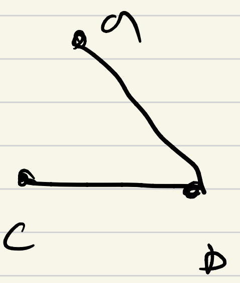
G_3

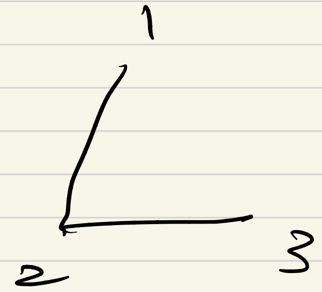


G_4

Given two graphs are Isomorphic or not?

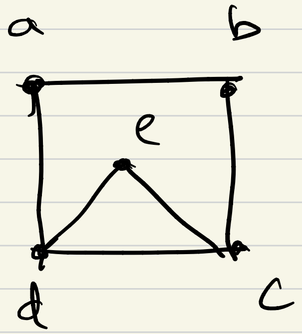
Example - 1

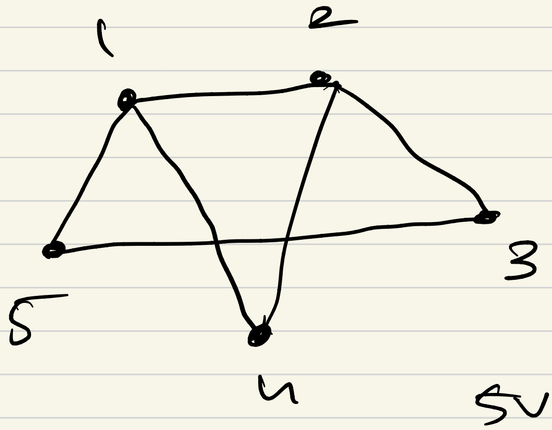




G_1 G_2

Example - 2





5V
6E

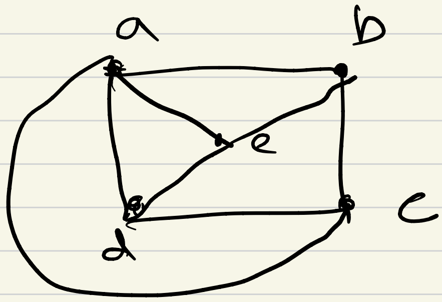
5V
6E

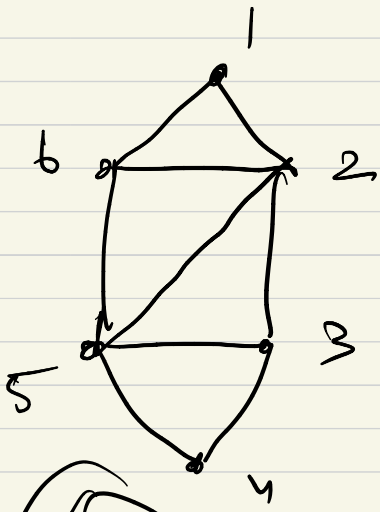
a ⟶ 5
b ⟶ 3
c ⟶ 2
d ⟶ 1
e ⟶ 4

Mapping

$\boxed{\text{Incedince should be preserved}}$

Example - 3



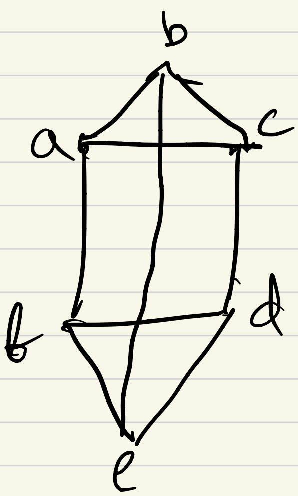


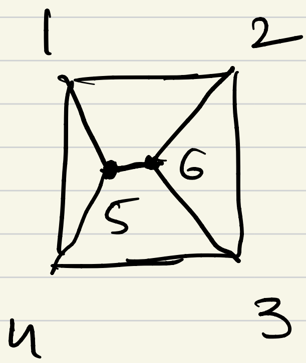
5 V

Not Isomorphic

6V

Example - 4





a — 1

b — 5

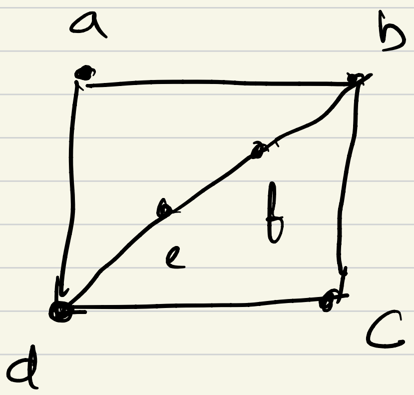
c — 4

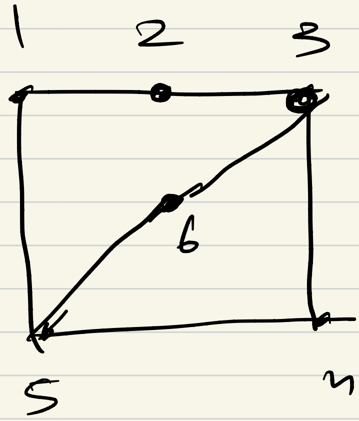
d — 2

e — 6

f — 3

Example - 5





$G_1 = 2, 3, 2, 3, 2, 2$

2v with d(3)

4v with d(2)

$G_2 = 2, 2, 3, 2, 3, 2$

2v with d(3)

4v with d(2)

a — 6

b — 3

c — 4

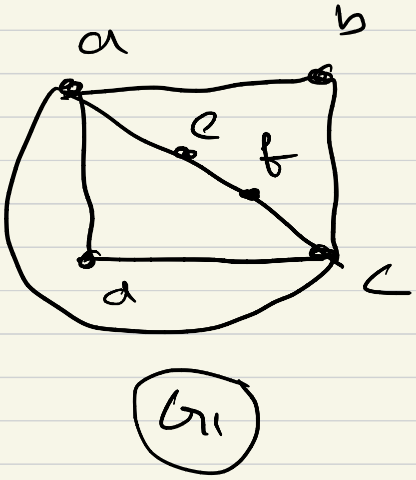
d — 5

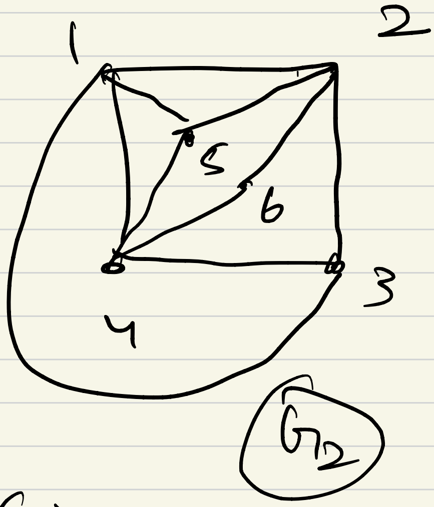
e — 1

f — 2

Isomorphic

Example - 6





$d(a) = 4$

$d(b) = 2$

$d(c) = 4$

$d(d) = 2$

$d(e) = 2$

$d(f) = 2$

$\text{sum}(d) = 16$

$d(1) = 4$

$d(2) = 4$

$d(3) = 3$

$d(4) = 4$

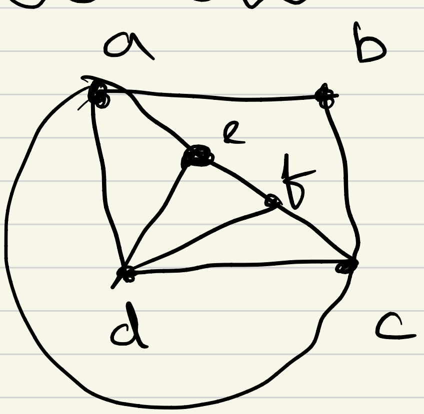
$d(5) = 3$

$d(6) = 2$

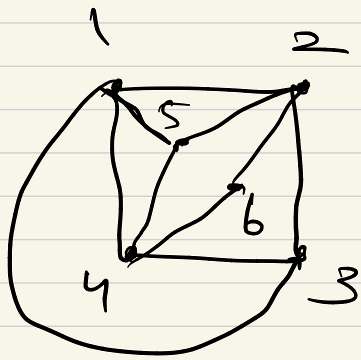
$\text{sum}(d) = 20$

Not Isomorphic //

Example - 7



G_1



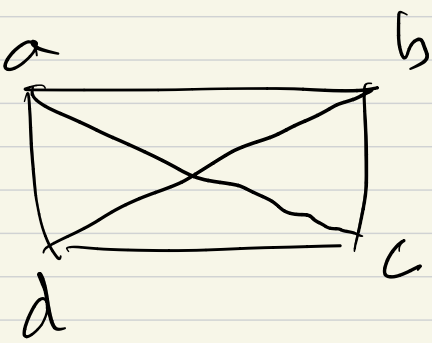
G_2

not Isomorphic !!!

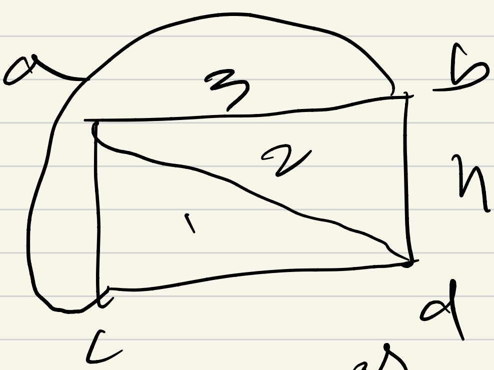
Planar graphs

Planar : where edges dont cross

non planar : where edges cross



→



non planar

Planar

no of Regions → 1 and 2, 3
and 4 outer

Therom

Complete graph with 5 vertices is a non-planar graph

Eular's Formula

let G be a connected Planar graph with 'e' - edges and 'v' - vertices. let 'r' be the no of regions in Planar

$$\boxed{r = e - v + 2}$$

r = regions

Q 1) $v = 20$ and 3 degree each

$\Sigma(v) = 2(E)$ → Sum of degrees

$\Sigma \deg(v) = 20 \times 3 = \frac{\cancel{60}\,30}{\cancel{2}} = 30$

$= 60 \Rightarrow r = 30 - 20 + 2$

$= 12$

$\boxed{r = 12}$

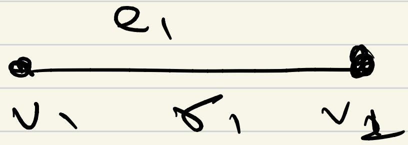
Eular Method by Induction

① Basic Step

② Induction Step [assume it would be true for k]

③ verify result by k+1

Q1



$$r_1 = e_1 - v_1 + 2$$

$$= 1 - \cancel{2} + \cancel{2}$$

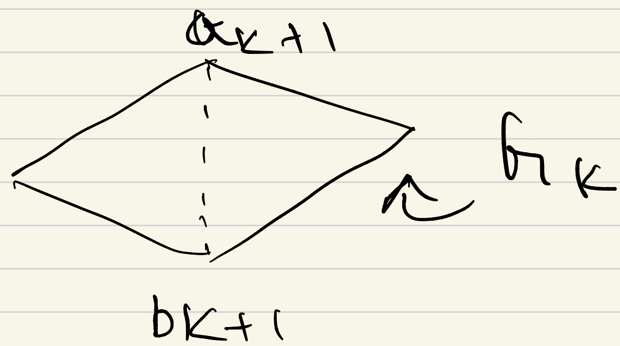
$\boxed{r_1 = 1}$ true.

Q2 $r_k = e_k - v_k + 2$ is true

Q3 verification

let (a_{k+1}, b_{k+1}) be the edge that is added to G_k $\longrightarrow$

case-1



$$r(k+1) = r_k + 1$$

$$e(k+1) = e_k + 1$$

$$\left.v_{k+1} = v_k\right] \because \text{both are inside the } G_k$$

$$r_{k+1} = e_{k+1} - v_{(k+1)} + 2$$

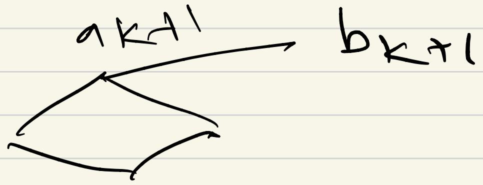
$$r_k \cancel{+1} = e_k \cancel{+1} - v_k + 2$$

$$\boxed{r_k = e_k - v_k + 2}$$

True for $k+1$

Inside the region

Case - 2



$r_{k+1} = r_k$

Out side the region

$e_{k+1} = e_{k+1}$

$v_{k+1} = v_{k+1}$

$$r_{k+1} = e_{k\cancel{+1}} - [v_{k\cancel{+1}}] + 2$$

$$\boxed{r_k = e_k - v_k + 2}$$

Hence proven

Formula's

① Sum of All degrees (Ring Sum)

$$\boxed{\star \quad \sum_{i=1}^{n} d(v_i) = 2E}$$

② Sum of degrees with odd and even degree of vertices

$$\boxed{\sum_{odd} d(v_i) + \sum_{even} d(v_j) = 2E}$$

③ Find edges in complete graph

$$\boxed{K_n = \frac{n*(n-1)}{2}}$$

To find the complete graph edges with its complement graph

$$\boxed{K_n = E_G + E_{\bar{G}}}$$

(4) Eular's theorm

$$\boxed{r = E - V + 2}$$

(5) To find the degree of an Complete graph

$$\boxed{K_n = n - 1}$$

(6) K- regular graph

where K = degree of the vertex

3 - regular graph with vertices - 4