Push-Relabel Extra Work

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We will let $d(x, y)$ be the length of the shortest path from x to y in the residual graph if such a path exists, ∞ otherwise. We need to prove that :
$h(u) < |V| \Rightarrow h(u) \leq d(u, t)$
$h(u) \geq |V| \Rightarrow h(u) - |V| \leq d(u, s)$

Lemma

For every edge (u, v) in the residual graph, we have $h(v) \geq h(u) - 1$.

Proof

We will use induction to prove the claim. Initially, the source has height $|V|$ and every other node height 0. The only edges including s in the residual graph are of the type (u, s) for which we have $h(s) = |V| \geq -1 = h(u) - 1$, because every node coming out of s is saturated at the very beginning of the algorithm. For edges between all other nodes u and v, we have $h(u) = 0 \geq -1 = h(v) - 1$.

By doing a push operation for an edge (u, v), we need to have $h(u) = h(v)+1$. After pushing, the edge changes orientation in the residual graph, thus becoming (v, u) and the property remains true: $h(u) = h(v) + 1 \geq h(v) - 1$.

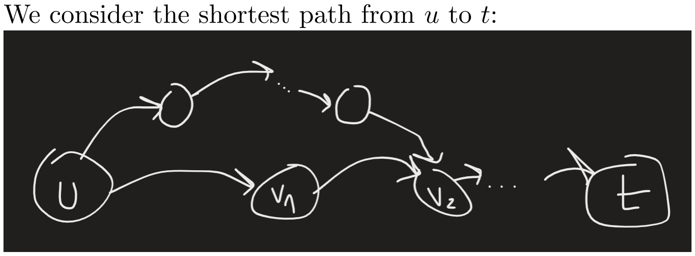
By doing a relabel operation, we will let $h(u) = \min_{\exists(u,v)} 1 + h(v)$. Consequently, $h(u) \leq h(v) + 1 \Rightarrow h(v) \geq h(u) - 1$ for every relevant edge (u, v).

Therefore, the lemma will hold during every step of the algorithm.

Solution

In fact, the claims are always true regardless of $h(u)$. If a path from u to t or from u to s respectively doesn't exist in the residual graph, we are done.

We consider the shortest path from u to t:



From the previous lemma, we get $h(i) \leq h(j)+1$. Applying this successively, we get $h(u) \leq h(v_1) + 1 \leq h(v_2) + 2 \leq ... \leq h(t) + d(u, t) = 0 + d(u, t) = d(u, t)$. A longer path between the nodes u and t does not affect the answer, it merely provides a weaker upper bound.

Similarly, by looking at the shortest path from u to s we get $h(u) \leq h(v_1) + 1 \leq h(v_2) + 2 \leq ... \leq h(s) + d(u, s) = |V| + d(u, s)$, which yields the desired $h(u) - |V| \leq d(u, s)$.