Application period. The challenged investigator.

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1 Model

$$p(x, \mathbf{d}|\boldsymbol{\theta}) = p(\mathbf{d})p(x|\mathbf{d}, \boldsymbol{\theta})$$

Let's define indexes of face points as:

$$Face(\mathbf{d}) = \{(ij) : d_h \le i < d_h + h, d_w \le i < d_w + w\}$$

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$$p(x|\mathbf{d}, \boldsymbol{\theta}) = \prod_{(ij) \in Face(\mathbf{d})} \mathcal{N}(x_{ij}|f_{i-d_h, j-d_w}, \sigma^2) \prod_{(ij) \notin Face(\mathbf{d})} \mathcal{N}(x_{ij}|b_{ij}, \sigma^2)$$

$$p(d_h = i, d_w = j) = w_{ij}$$

Then latents variables are $\mathbf{D}$, parameters of model are $\boldsymbol{\theta} = \{\mathbf{W}, \mathbf{F}, \mathbf{B}, \sigma^2\}$.

2 EM-algorithm

2.1 E-step

$$p(\mathbf{D}|\mathbf{X}, \boldsymbol{\theta}) = \prod_{n=1}^{N} p(\mathbf{d}_n|x_n, \boldsymbol{\theta}) = \prod_{n=1}^{N} \frac{p(x_n, \mathbf{d}_n|\boldsymbol{\theta})}{p(x_n|\boldsymbol{\theta})} = \prod_{n=1}^{N} \frac{p(\mathbf{d_n})p(x_n|\mathbf{d_n}, \boldsymbol{\theta})}{\sum_{\mathbf{d}} p(\mathbf{d})p(x_n|\mathbf{d}, \boldsymbol{\theta})}$$

$$\gamma_{ij}^n = p(d_h^n = i, d_w^n = j|x_n, \boldsymbol{\theta})$$

2.2 Lower bound $\mathcal{L}(q, \boldsymbol{\theta})$

$$\mathcal{L}(q, \boldsymbol{\theta}) = \mathbb{E}[\log p(\mathbf{X}, \mathbf{D}|\boldsymbol{\theta})] + \mathbb{H}[q] =$$

$$= -\frac{NWH\log 2\pi}{2} - \frac{NWH}{2}\log\sigma^2 + \sum_{n=1}^{N}\sum_{ij}\gamma_{ij}^n \log w_{ij} -$$

$$-\frac{1}{2\sigma^2}\sum_{n=1}^{N}\sum_{ij}\gamma_{ij}^n\Big[\sum_{(kl)\notin Face(ij)}(x_{kl}^n - b_{kl})^2 + \sum_{(kl)\in Face(ij)}(x_{kl}^n - f_{k-i,l-j})^2\Big] -$$

$$-\sum_{n=1}^{N}\sum_{ij}\gamma_{ij}^n\log\gamma_{ij}^n$$

2.3 M-step

For $\mathbf{W}$:

$$\mathbb{E}[\log p(\mathbf{X}, \mathbf{D}|\boldsymbol{\theta})] = \sum_{n=1}^{N}\sum_{ij}\gamma_{ij}^n\log w_{ij} + const$$

$$\frac{\partial}{\partial w_{kl}}\Big[\sum_{n=1}^{N}\sum_{ij}\gamma_{ij}^n\log w_{ij} + \lambda(\sum_{ij} w_{ij} - 1) + const\Big] = \sum_{n=1}^{N}\frac{\gamma_{kl}^n}{w_{kl}} + \lambda = 0$$

$$\mathbf{W} = \frac{\sum\limits_{n=1}^{N}\boldsymbol{\Gamma}_n}{\sum\limits_{n=1}^{N}\sum\limits_{ij}\gamma_{ij}^n} = \frac{\sum\limits_{n=1}^{N}\boldsymbol{\Gamma}_n}{N}$$

For $\mathbf{F}$:

$$\mathbb{E}[\log p(\mathbf{X}, \mathbf{D}|\boldsymbol{\theta})] = -\sum_{ij}\sum_{n=1}^{N}\gamma_{ij}^n\sum_{(kl)\in Face(ij)}\frac{(x_{kl}^n - f_{k-i,l-j})^2}{2\sigma^2} + const$$

$$f_{ij} = \frac{\sum\limits_{kl}\sum\limits_{n=1}^{N}\gamma_{kl}^n x_{k+i,l+j}^n}{\sum\limits_{kl}\sum\limits_{n=1}^{N}\gamma_{kl}^n} = \frac{\sum\limits_{kl}\sum\limits_{n=1}^{N}\gamma_{kl}^n x_{k+i,l+j}^n}{N}$$

For $\mathbf{B}$:

$$\mathbb{E}[\log p(\mathbf{X}, \mathbf{D}|\boldsymbol{\theta})] = -\sum_{ij} \sum_{n=1}^{N} \gamma_{ij}^n \sum_{(kl) \notin Face(ij)} \frac{(x_{kl}^n - b_{kl})^2}{2\sigma^2} + const$$

$$b_{ij} = \frac{\sum\limits_{(ij) \notin Face(kl)} \sum\limits_{n=1}^{N} \gamma_{kl}^n x_{ij}^n}{\sum\limits_{(ij) \notin Face(kl)} \sum\limits_{n=1}^{N} \gamma_{kl}^n}$$

For σ^2:

$$\mathbb{E}[\log p(\mathbf{X}, \mathbf{D}|\boldsymbol{\theta})] = -\frac{NWH}{2} \log \sigma^2 -$$

$$-\frac{1}{2\sigma^2} \sum_{ij} \sum_{n=1}^{N} \gamma_{ij}^n \Big[\sum_{(kl) \in Face(ij)} (x_{kl}^n - f_{k-i,l-j})^2 + \sum_{(kl) \notin Face(ij)} (x_{kl}^n - b_{kl})^2 \Big] + const$$

$$\sigma^2 = \frac{1}{NHW} \sum_{ij} \sum_{n=1}^{N} \gamma_{ij}^n \Big[\sum_{(kl) \in Face(ij)} (x_{kl}^n - f_{k-i,l-j})^2 + \sum_{(kl) \notin Face(ij)} (x_{kl}^n - b_{kl})^2 \Big]$$

3 Hard EM-algorithm

$$q(d_n) = \delta(d - d_{MP})$$

3.1 E-step

$$d_n^* = \arg\max_{i,j} \Big[\log w_{ij} - \frac{1}{2\sigma^2} \sum_{(kl) \notin Face(ij)} (x_{kl}^n - b_{kl})^2 - \frac{1}{2\sigma^2} \sum_{(kl) \in Face(ij)} (x_{kl}^n - f_{k-i,l-j})^2 \Big]$$

3.2 Lower bound $\mathcal{L}(q, \boldsymbol{\theta})$

$$\mathcal{L}(q, \boldsymbol{\theta}) = \mathbb{E}[\log p(\mathbf{X}, \mathbf{D}|\boldsymbol{\theta})] - \mathbb{H}[q] =$$

$$= -\frac{NWH \log 2\pi}{2} - \frac{NWH}{2} \log \sigma^2 + \sum_{n=1}^{N} \log w_{d_h^n, d_w^n} -$$

$$-\frac{1}{2\sigma^2} \sum_{n=1}^{N} \Big[\sum_{(kl) \notin Face(d_h^n, d_w^n)} (x_{kl}^n - b_{kl})^2 + \sum_{(kl) \in Face(d_h^n, d_w^n)} (x_{kl}^n - f_{k-i,l-j})^2 \Big]$$

3.3 M-step

For $\mathbf{F}$:

$$f_{ij} = \frac{\sum_{n=1}^{N} x^n_{d^n_h+i, d^n_w+j}}{N}$$

For $\mathbf{B}$:

$$b_{ij} = \frac{\sum_{n=1}^{N} [(ij) \notin Face(d^n_h, d^n_w)] x^n_{ij}}{\sum_{n=1}^{N} [(ij) \notin Face(d^n_h, d^n_w)]}$$

For σ^2:

$$\sigma^2 = \frac{1}{NHW} \sum_{n=1}^{N} \Big[\sum_{(kl) \in Face(d^n_h, d^n_w)} (x^n_{kl} - f_{k-i, l-j})^2 + \sum_{(kl) \notin Face(d^n_h, d^n_w)} (x^n_{kl} - b_{kl})^2 \Big]$$