Benders Decomposition

$\mathcal{R}$: the set of real numbers.

$\mathcal{R}_+$: the set of nonnegative real numbers.

$\mathcal{R}_+^n$: the set of n-dimensional nonnegative real vectors.

$\mathcal{Z}$: the set of integers.

$\mathcal{Z}_+$: the set of nonnegative integers.

$\mathcal{Z}_+^n$: the set of n-dimensional nonnegative integer vectors.

$\mathbf{c} \in \mathcal{R}^m$, $\mathbf{d} \in \mathcal{R}^n$, $\mathbf{f} \in \mathcal{R}^p$, $\mathbf{g} \in \mathcal{R}^q$, $\mathbf{A} \in \mathcal{R}^{p\times m}$, $\mathbf{B} \in \mathcal{R}^{p\times n}$, $\mathbf{E} \in \mathcal{R}^{q\times m}$.

The original mixed integer linear programming model is formulated as follows.

$$
\begin{aligned}
\text{(P1)} \qquad z_{\text{P1}} = \min\, & \mathbf{c}'\mathbf{x} + \mathbf{d}'\mathbf{y}, && (1)\\
\text{s.t.} \qquad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{f}, && (2)\\
& \mathbf{E}\mathbf{x} \geq \mathbf{g}, && (3)\\
& \mathbf{x} \in \mathcal{Z}_+^m, && (4)\\
& \mathbf{y} \in \mathcal{R}_+^n. && (5)
\end{aligned}
$$

Given the value of $\mathbf{x}$, the optimal objective value associated with variable $\mathbf{y}$ can be described as follows.

$$
\begin{aligned}
\text{(P2(}\mathbf{x}\text{))} \qquad z_{\text{P2}(\mathbf{x})} = \min\, & \mathbf{d}'\mathbf{y}, && (6)\\
\text{s.t.} \qquad & \mathbf{B}\mathbf{y} \geq \mathbf{f} - \mathbf{A}\mathbf{x}, && (7)\\
& \mathbf{y} \in \mathcal{R}_+^n, && (8)
\end{aligned}
$$

where $z_{\text{P2}(\mathbf{x})} = +\infty$ if model P2($\mathbf{x}$) is infeasible.

The dual of model P2($\mathbf{x}$) is as follows.

$$
\begin{aligned}
\text{(D2(}\mathbf{x}\text{))} \qquad z_{\text{D2}(\mathbf{x})} = \max\, & \boldsymbol{\pi}'(\mathbf{f} - \mathbf{A}\mathbf{x}), && (9)\\
\text{s.t.} \qquad & \boldsymbol{\pi}'\mathbf{B} \leq \mathbf{d}', && (10)\\
& \boldsymbol{\pi} \in \mathcal{R}_+^p. && (11)
\end{aligned}
$$

Note that the feasible set $\Pi = \{\boldsymbol{\pi} \in \mathcal{R}_+^p : \boldsymbol{\pi}'\mathbf{B} \leq \mathbf{d}'\}$ is independent of $\mathbf{x}$. If Π is empty, then D2($\mathbf{x}$) is infeasible for all $\mathbf{x} \in \mathcal{Z}_+^m$. Because the infeasibility of D2($\mathbf{x}$) implies that either (i) P2($\mathbf{x}$) is infeasible, or (ii) $z_{\text{P2}(\mathbf{x})} = -\infty$, we conclude that the original problem P1 is either infeasible or unbounded.

In the following, we assume that the original problem P1 is feasible with a finite optimal objective value, i.e., Π is nonempty. Furthermore, denote $\{\boldsymbol{\pi}_i : i \in \mathcal{I}\}$ as the set of extreme points of Π, and $\{\boldsymbol{\pi}_j : j \in \mathcal{J}\}$ as a complete set of extreme rays of Π. Then we have

- $\mathrm{P2}(\mathbf{x})$ is feasible if and only if $\boldsymbol{\pi}_j^{'}(\mathbf{f}-\mathbf{A}\mathbf{x}) \leq 0,\ \forall\ j \in \mathcal{J}$.
- If $\mathrm{P2}(\mathbf{x})$ is feasible with finite optimal objective value, then $z_{\mathrm{P2}(\mathbf{x})} = z_{\mathrm{D2}(\mathbf{x})} = \max_{i\in\mathcal{I}} \boldsymbol{\pi}_i^{'}(\mathbf{f}-\mathbf{A}\mathbf{x})$.

Therefore,

$$
\begin{aligned}
z_{\mathrm{P1}} = \min\left\{\mathbf{c}'\mathbf{x} + z_{\mathrm{P2}(\mathbf{x})} : \mathbf{E}\mathbf{x} \geq \mathbf{g}, \mathbf{x} \in \mathcal{Z}_+^m\right\} = \min \mathbf{c}'\mathbf{x} + \eta, && (12)\\
\text{s.t.} \qquad \mathbf{E}\mathbf{x} \geq \mathbf{g}, && (13)\\
\eta \geq \boldsymbol{\pi}_i^{'}(\mathbf{f}-\mathbf{A}\mathbf{x}), \qquad \forall\ i \in \mathcal{I}, && (14)\\
\boldsymbol{\pi}_j^{'}(\mathbf{f}-\mathbf{A}\mathbf{x}) \leq 0, \qquad \forall\ j \in \mathcal{J}, && (15)\\
\mathbf{x} \in \mathcal{Z}_+^m. && (16)
\end{aligned}
$$