Chapter 2

Convex sets

2.1 Affine and convex sets

2.1.1 Lines and line segments

Suppose $x_1 \neq x_2$ are two points in $\mathbf{R}^n$. Points of the form

$$y = \theta x_1 + (1 - \theta)x_2,$$

where $\theta \in \mathbf{R}$, form the *line* passing through x_1 and x_2. The parameter value $\theta = 0$ corresponds to $y = x_2$, and the parameter value $\theta = 1$ corresponds to $y = x_1$. Values of the parameter θ between 0 and 1 correspond to the (closed) *line segment* between x_1 and x_2.

Expressing y in the form

$$y = x_2 + \theta(x_1 - x_2)$$

gives another interpretation: y is the sum of the *base point* x_2 (corresponding to $\theta = 0$) and the *direction* $x_1 - x_2$ (which points from x_2 to x_1) scaled by the parameter θ. Thus, θ gives the fraction of the way from x_2 to x_1 where y lies. As θ increases from 0 to 1, the point y moves from x_2 to x_1; for $\theta > 1$, the point y lies on the line beyond x_1. This is illustrated in figure 2.1.

2.1.2 Affine sets

A set $C \subseteq \mathbf{R}^n$ is *affine* if the line through any two distinct points in C lies in C, *i.e.*, if for any $x_1,\ x_2 \in C$ and $\theta \in \mathbf{R}$, we have $\theta x_1 + (1-\theta)x_2 \in C$. In other words, C contains the linear combination of any two points in C, provided the coefficients in the linear combination sum to one.

This idea can be generalized to more than two points. We refer to a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_1 + \cdots + \theta_k = 1$, as an *affine combination* of the points $x_1, \ldots, x_k$. Using induction from the definition of affine set (*i.e.*, that it contains every affine combination of two points in it), it can be shown that

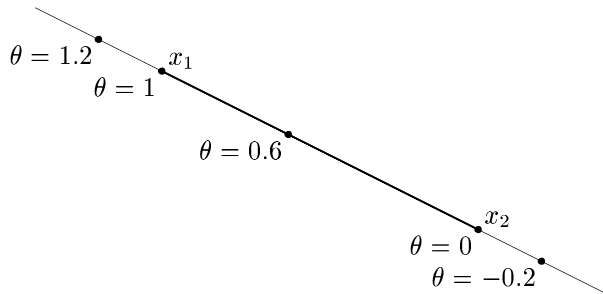


Figure 2.1 The line passing through x_1 and x_2 is described parametrically by $\theta x_1 + (1-\theta)x_2$, where θ varies over $\mathbf{R}$. The line segment between x_1 and x_2, which corresponds to θ between 0 and 1, is shown darker.

an affine set contains every affine combination of its points: If C is an affine set, $x_1, \ldots, x_k \in C$, and $\theta_1 + \cdots + \theta_k = 1$, then the point $\theta_1 x_1 + \cdots + \theta_k x_k$ also belongs to C.

If C is an affine set and $x_0 \in C$, then the set

$$V = C - x_0 = \{x - x_0 \mid x \in C\}$$

is a subspace, *i.e.*, closed under sums and scalar multiplication. To see this, suppose $v_1,\ v_2 \in V$ and $\alpha,\ \beta \in \mathbf{R}$. Then we have $v_1 + x_0 \in C$ and $v_2 + x_0 \in C$, and so

$$\alpha v_1 + \beta v_2 + x_0 = \alpha(v_1 + x_0) + \beta(v_2 + x_0) + (1 - \alpha - \beta)x_0 \in C,$$

since C is affine, and $\alpha + \beta + (1 - \alpha - \beta) = 1$. We conclude that $\alpha v_1 + \beta v_2 \in V$, since $\alpha v_1 + \beta v_2 + x_0 \in C$.

Thus, the affine set C can be expressed as

$$C = V + x_0 = \{v + x_0 \mid v \in V\},$$

i.e., as a subspace plus an offset. The subspace V associated with the affine set C does not depend on the choice of x_0, so x_0 can be chosen as any point in C. We define the *dimension* of an affine set C as the dimension of the subspace $V = C - x_0$, where x_0 is any element of C.

Example 2.1 *Solution set of linear equations.* The solution set of a system of linear equations, $C = \{x \mid Ax = b\}$, where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$, is an affine set. To show this, suppose $x_1,\ x_2 \in C$, *i.e.*, $Ax_1 = b$, $Ax_2 = b$. Then for any θ, we have

$$\begin{array}{rcl} A(\theta x_1 + (1-\theta)x_2) & = & \theta A x_1 + (1-\theta) A x_2 \\ & = & \theta b + (1-\theta) b \\ & = & b, \end{array}$$

which shows that the affine combination $\theta x_1 + (1-\theta)x_2$ is also in C. The subspace associated with the affine set C is the nullspace of A.

We also have a converse: every affine set can be expressed as the solution set of a system of linear equations.

The set of all affine combinations of points in some set $C \subseteq \mathbf{R}^n$ is called the *affine hull* of C, and denoted $\mathbf{aff}\, C$:

$$\mathbf{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \ldots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1\}.$$

The affine hull is the smallest affine set that contains C, in the following sense: if S is any affine set with $C \subseteq S$, then $\mathbf{aff}\, C \subseteq S$.

2.1.3 Affine dimension and relative interior

We define the *affine dimension* of a set C as the dimension of its affine hull. Affine dimension is useful in the context of convex analysis and optimization, but is not always consistent with other definitions of dimension. As an example consider the unit circle in $\mathbf{R}^2$, *i.e.*, $\{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 = 1\}$. Its affine hull is all of $\mathbf{R}^2$, so its affine dimension is two. By most definitions of dimension, however, the unit circle in $\mathbf{R}^2$ has dimension one.

If the affine dimension of a set $C \subseteq \mathbf{R}^n$ is less than n, then the set lies in the affine set $\mathbf{aff}\, C \neq \mathbf{R}^n$. We define the *relative interior* of the set C, denoted $\mathbf{relint}\, C$, as its interior relative to $\mathbf{aff}\, C$:

$$\mathbf{relint}\, C = \{x \in C \mid B(x, r) \cap \mathbf{aff}\, C \subseteq C \text{ for some } r > 0\},$$

where $B(x, r) = \{y \mid \|y - x\| \leq r\}$, the ball of radius r and center x in the norm $\|\cdot\|$. (Here $\|\cdot\|$ is any norm; all norms define the same relative interior.) We can then define the *relative boundary* of a set C as $\mathbf{cl}\, C \setminus \mathbf{relint}\, C$, where $\mathbf{cl}\, C$ is the closure of C.

Example 2.2 Consider a square in the (x_1, x_2)-plane in $\mathbf{R}^3$, defined as

$$C = \{x \in \mathbf{R}^3 \mid -1 \leq x_1 \leq 1,\ -1 \leq x_2 \leq 1,\ x_3 = 0\}.$$

Its affine hull is the (x_1, x_2)-plane, *i.e.*, $\mathbf{aff}\, C = \{x \in \mathbf{R}^3 \mid x_3 = 0\}$. The interior of C is empty, but the relative interior is

$$\mathbf{relint}\, C = \{x \in \mathbf{R}^3 \mid -1 < x_1 < 1,\ -1 < x_2 < 1,\ x_3 = 0\}.$$

Its boundary (in $\mathbf{R}^3$) is itself; its relative boundary is the wire-frame outline,

$$\{x \in \mathbf{R}^3 \mid \max\{|x_1|, |x_2|\} = 1,\ x_3 = 0\}.$$

2.1.4 Convex sets

A set C is *convex* if the line segment between any two points in C lies in C, *i.e.*, if for any $x_1,\ x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta x_1 + (1 - \theta) x_2 \in C.$$

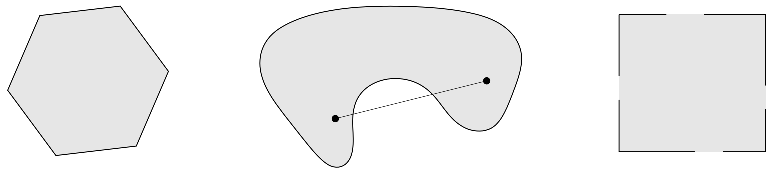


Figure 2.2 Some simple convex and nonconvex sets. *Left.* The hexagon, which includes its boundary (shown darker), is convex. *Middle.* The kidney shaped set is not convex, since the line segment between the two points in the set shown as dots is not contained in the set. *Right.* The square contains some boundary points but not others, and is not convex.

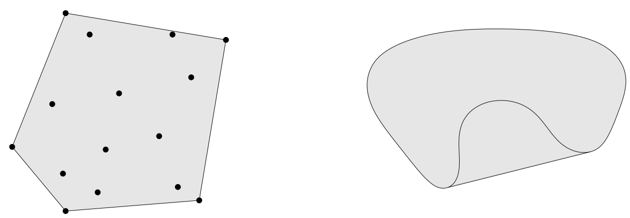


Figure 2.3 The convex hulls of two sets in $\mathbf{R}^2$. *Left.* The convex hull of a set of fifteen points (shown as dots) is the pentagon (shown shaded). *Right.* The convex hull of the kidney shaped set in figure 2.2 is the shaded set.

Roughly speaking, a set is convex if every point in the set can be seen by every other point, along an unobstructed straight path between them, where unobstructed means lying in the set. Every affine set is also convex, since it contains the entire line between any two distinct points in it, and therefore also the line segment between the points. Figure 2.2 illustrates some simple convex and nonconvex sets in $\mathbf{R}^2$.

We call a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_1 + \cdots + \theta_k = 1$ and $\theta_i \geq 0$, $i = 1, \ldots, k$, a *convex combination* of the points $x_1, \ldots, x_k$. As with affine sets, it can be shown that a set is convex if and only if it contains every convex combination of its points. A convex combination of points can be thought of as a *mixture* or *weighted average* of the points, with θ_i the fraction of x_i in the mixture.

The *convex hull* of a set C, denoted $\mathbf{conv}\, C$, is the set of all convex combinations of points in C:

$$\mathbf{conv}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \ldots, k,\ \theta_1 + \cdots + \theta_k = 1\}.$$

As the name suggests, the convex hull $\mathbf{conv}\, C$ is always convex. It is the smallest convex set that contains C: If B is any convex set that contains C, then $\mathbf{conv}\, C \subseteq B$. Figure 2.3 illustrates the definition of convex hull.

The idea of a convex combination can be generalized to include infinite sums, integrals, and, in the most general form, probability distributions. Suppose $\theta_1, \theta_2, \ldots$

satisfy

$$\theta_i \geq 0, \quad i = 1, 2, \ldots, \qquad \sum_{i=1}^{\infty} \theta_i = 1,$$

and $x_1, x_2, \ldots \in C$, where $C \subseteq \mathbf{R}^n$ is convex. Then

$$\sum_{i=1}^{\infty} \theta_i x_i \in C,$$

if the series converges. More generally, suppose $p : \mathbf{R}^n \to \mathbf{R}$ satisfies $p(x) \geq 0$ for all $x \in C$ and $\int_C p(x)\, dx = 1$, where $C \subseteq \mathbf{R}^n$ is convex. Then

$$\int_C p(x) x \, dx \in C,$$

if the integral exists.

In the most general form, suppose $C \subseteq \mathbf{R}^n$ is convex and x is a random vector with $x \in C$ with probability one. Then $\mathbf{E}\, x \in C$. Indeed, this form includes all the others as special cases. For example, suppose the random variable x only takes on the two values x_1 and x_2, with $\mathbf{prob}(x = x_1) = \theta$ and $\mathbf{prob}(x = x_2) = 1 - \theta$, where $0 \leq \theta \leq 1$. Then $\mathbf{E}\, x = \theta x_1 + (1 - \theta) x_2$, and we are back to a simple convex combination of two points.

2.1.5 Cones

A set C is called a *cone*, or *nonnegative homogeneous*, if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$. A set C is a *convex cone* if it is convex and a cone, which means that for any $x_1,\ x_2 \in C$ and $\theta_1,\ \theta_2 \geq 0$, we have

$$\theta_1 x_1 + \theta_2 x_2 \in C.$$

Points of this form can be described geometrically as forming the two-dimensional pie slice with apex 0 and edges passing through x_1 and x_2. (See figure 2.4.)

A point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1, \ldots, \theta_k \geq 0$ is called a *conic combination* (or a *nonnegative linear combination*) of $x_1, \ldots, x_k$. If x_i are in a convex cone C, then every conic combination of x_i is in C. Conversely, a set C is a convex cone if and only if it contains all conic combinations of its elements. Like convex (or affine) combinations, the idea of conic combination can be generalized to infinite sums and integrals.

The *conic hull* of a set C is the set of all conic combinations of points in C, *i.e.*,

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \ldots, k\},$$

which is also the smallest convex cone that contains C (see figure 2.5).

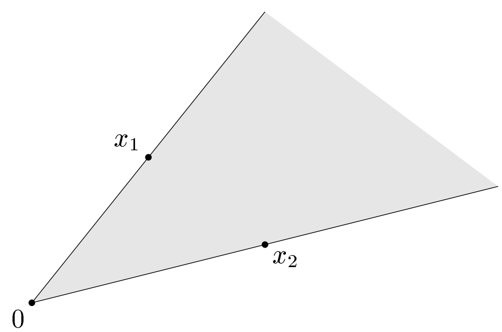


Figure 2.4 The pie slice shows all points of the form $\theta_1 x_1 + \theta_2 x_2$, where $\theta_1,\ \theta_2 \geq 0$. The apex of the slice (which corresponds to $\theta_1 = \theta_2 = 0$) is at 0; its edges (which correspond to $\theta_1 = 0$ or $\theta_2 = 0$) pass through the points x_1 and x_2.

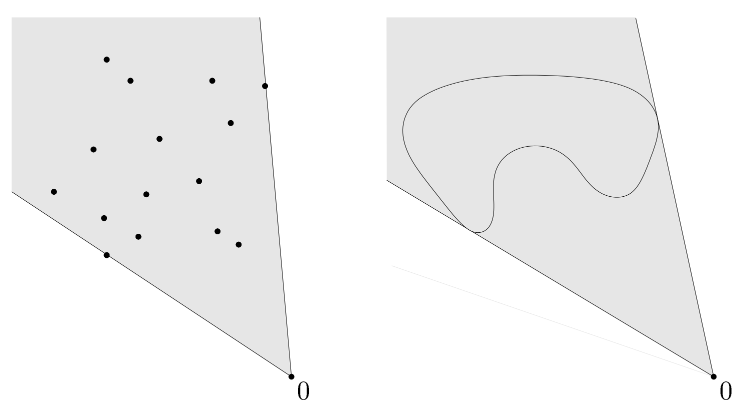


Figure 2.5 The conic hulls (shown shaded) of the two sets of figure 2.3.

2.2 Some important examples

In this section we describe some important examples of convex sets which we will encounter throughout the rest of the book. We start with some simple examples.

- The empty set $\emptyset$, any single point (*i.e.*, singleton) $\{x_0\}$, and the whole space $\mathbf{R}^n$ are affine (hence, convex) subsets of $\mathbf{R}^n$.
- Any line is affine. If it passes through zero, it is a subspace, hence also a convex cone.
- A line segment is convex, but not affine (unless it reduces to a point).
- A *ray*, which has the form $\{x_0 + \theta v \mid \theta \geq 0\}$, where $v \neq 0$, is convex, but not affine. It is a convex cone if its base x_0 is 0.
- Any subspace is affine, and a convex cone (hence convex).

2.2.1 Hyperplanes and halfspaces

A *hyperplane* is a set of the form

$$\{x \mid a^T x = b\},$$

where $a \in \mathbf{R}^n$, $a \neq 0$, and $b \in \mathbf{R}$. Analytically it is the solution set of a nontrivial linear equation among the components of x (and hence an affine set). Geometrically, the hyperplane $\{x \mid a^T x = b\}$ can be interpreted as the set of points with a constant inner product to a given vector a, or as a hyperplane with *normal vector* a; the constant $b \in \mathbf{R}$ determines the offset of the hyperplane from the origin. This geometric interpretation can be understood by expressing the hyperplane in the form

$$\{x \mid a^T(x - x_0) = 0\},$$

where x_0 is any point in the hyperplane (*i.e.*, any point that satisfies $a^T x_0 = b$). This representation can in turn be expressed as

$$\{x \mid a^T(x - x_0) = 0\} = x_0 + a^\perp,$$

where $a^\perp$ denotes the orthogonal complement of a, *i.e.*, the set of all vectors orthogonal to it:

$$a^\perp = \{v \mid a^T v = 0\}.$$

This shows that the hyperplane consists of an offset x_0, plus all vectors orthogonal to the (normal) vector a. These geometric interpretations are illustrated in figure 2.6.

A hyperplane divides $\mathbf{R}^n$ into two *halfspaces*. A (closed) halfspace is a set of the form

$$\{x \mid a^T x \leq b\}, \tag{2.1}$$

where $a \neq 0$, *i.e.*, the solution set of one (nontrivial) linear inequality. Halfspaces are convex, but not affine. This is illustrated in figure 2.7.

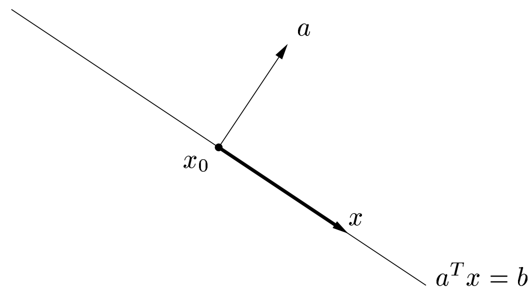


Figure 2.6 Hyperplane in $\mathbf{R}^2$, with normal vector a and a point x_0 in the hyperplane. For any point x in the hyperplane, $x - x_0$ (shown as the darker arrow) is orthogonal to a.

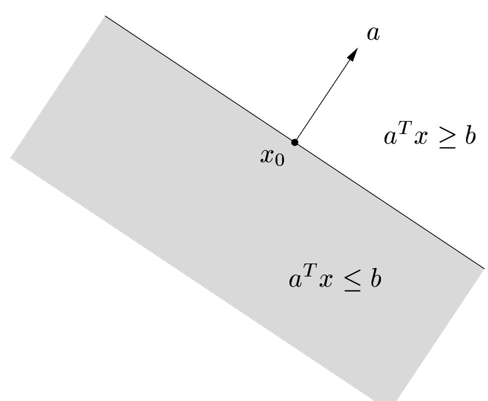


Figure 2.7 A hyperplane defined by $a^T x = b$ in $\mathbf{R}^2$ determines two halfspaces. The halfspace determined by $a^T x \geq b$ (not shaded) is the halfspace extending in the direction a. The halfspace determined by $a^T x \leq b$ (which is shown shaded) extends in the direction $-a$. The vector a is the outward normal of this halfspace.

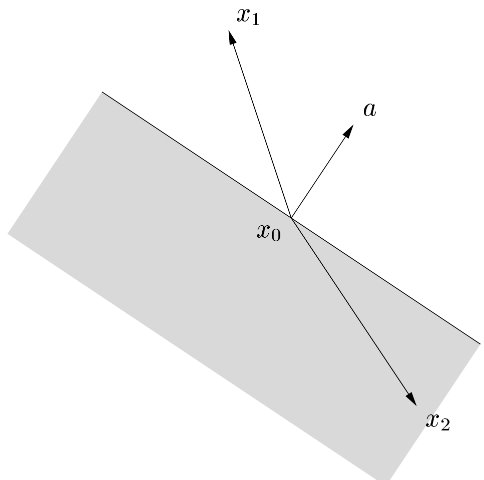


Figure 2.8 The shaded set is the halfspace determined by $a^T(x - x_0) \leq 0$. The vector $x_1 - x_0$ makes an acute angle with a, so x_1 is not in the halfspace. The vector $x_2 - x_0$ makes an obtuse angle with a, and so is in the halfspace.

The halfspace (2.1) can also be expressed as

$$\{x \mid a^T(x - x_0) \leq 0\}, \tag{2.2}$$

where x_0 is any point on the associated hyperplane, *i.e.*, satisfies $a^T x_0 = b$. The representation (2.2) suggests a simple geometric interpretation: the halfspace consists of x_0 plus any vector that makes an obtuse (or right) angle with the (outward normal) vector a. This is illustrated in figure 2.8.

The boundary of the halfspace (2.1) is the hyperplane $\{x \mid a^T x = b\}$. The set $\{x \mid a^T x < b\}$, which is the interior of the halfspace $\{x \mid a^T x \leq b\}$, is called an *open halfspace*.

2.2.2 Euclidean balls and ellipsoids

A *(Euclidean) ball* (or just ball) in $\mathbf{R}^n$ has the form

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x \mid (x - x_c)^T(x - x_c) \leq r^2\},$$

where $r > 0$, and $\|\cdot\|_2$ denotes the Euclidean norm, *i.e.*, $\|u\|_2 = (u^T u)^{1/2}$. The vector x_c is the *center* of the ball and the scalar r is its *radius*; $B(x_c, r)$ consists of all points within a distance r of the center x_c. Another common representation for the Euclidean ball is

$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \leq 1\}.$$

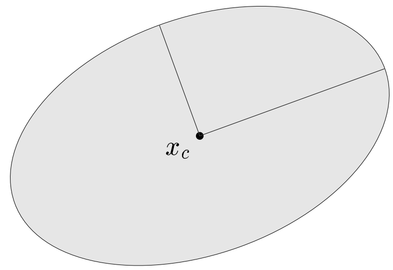


Figure 2.9 An ellipsoid in $\mathbf{R}^2$, shown shaded. The center x_c is shown as a dot, and the two semi-axes are shown as line segments.

A Euclidean ball is a convex set: if $\|x_1 - x_c\|_2 \le r$, $\|x_2 - x_c\|_2 \le r$, and $0 \le \theta \le 1$, then

$$\begin{array}{rcl} \|\theta x_1 + (1-\theta)x_2 - x_c\|_2 & = & \|\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)\|_2 \\ & \le & \theta\|x_1 - x_c\|_2 + (1-\theta)\|x_2 - x_c\|_2 \\ & \le & r. \end{array}$$

(Here we use the homogeneity property and triangle inequality for $\|\cdot\|_2$; see §A.1.2.)

A related family of convex sets is the *ellipsoids*, which have the form

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}, \tag{2.3}$$

where $P = P^T \succ 0$, *i.e.*, P is symmetric and positive definite. The vector $x_c \in \mathbf{R}^n$ is the *center* of the ellipsoid. The matrix P determines how far the ellipsoid extends in every direction from x_c; the lengths of the semi-axes of $\mathcal{E}$ are given by $\sqrt{\lambda_i}$, where λ_i are the eigenvalues of P. A ball is an ellipsoid with $P = r^2 I$. Figure 2.9 shows an ellipsoid in $\mathbf{R}^2$.

Another common representation of an ellipsoid is

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \le 1\}, \tag{2.4}$$

where A is square and nonsingular. In this representation we can assume without loss of generality that A is symmetric and positive definite. By taking $A = P^{1/2}$, this representation gives the ellipsoid defined in (2.3). When the matrix A in (2.4) is symmetric positive semidefinite but singular, the set in (2.4) is called a *degenerate ellipsoid*; its affine dimension is equal to the rank of A. Degenerate ellipsoids are also convex.

2.2.3 Norm balls and norm cones

Suppose $\|\cdot\|$ is any norm on $\mathbf{R}^n$ (see §A.1.2). From the general properties of norms it can be shown that a *norm ball* of radius r and center x_c, given by $\{x \mid \|x - x_c\| \le r\}$, is convex. The *norm cone* associated with the norm $\|\cdot\|$ is the set

$$C = \{(x,t) \mid \|x\| \le t\} \subseteq \mathbf{R}^{n+1}.$$

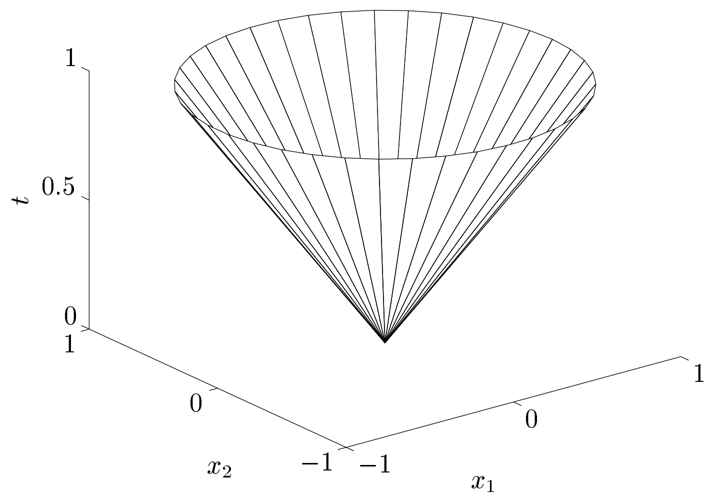


Figure 2.10 Boundary of second-order cone in $\mathbf{R}^3$, $\{(x_1, x_2, t) \mid (x_1^2+x_2^2)^{1/2} \leq t\}$.

It is (as the name suggests) a convex cone.

Example 2.3 *The second-order cone* is the norm cone for the Euclidean norm, *i.e.*,

$$\begin{aligned} C &= \{(x,t) \in \mathbf{R}^{n+1} \mid \|x\|_2 \leq t\} \\ &= \left\{ \left[\begin{array}{c} x \\ t \end{array}\right] \middle| \left[\begin{array}{c} x \\ t \end{array}\right]^T \left[\begin{array}{cc} I & 0 \\ 0 & -1 \end{array}\right] \left[\begin{array}{c} x \\ t \end{array}\right] \leq 0,\ t \geq 0 \right\}. \end{aligned}$$

The second-order cone is also known by several other names. It is called the *quadratic cone*, since it is defined by a quadratic inequality. It is also called the *Lorentz cone* or *ice-cream cone*. Figure 2.10 shows the second-order cone in $\mathbf{R}^3$.

2.2.4 Polyhedra

A *polyhedron* is defined as the solution set of a finite number of linear equalities and inequalities:

$$\mathcal{P} = \{x \mid a_j^T x \leq b_j,\ j = 1, \ldots, m,\ c_j^T x = d_j,\ j = 1, \ldots, p\}. \tag{2.5}$$

A polyhedron is thus the intersection of a finite number of halfspaces and hyperplanes. Affine sets (*e.g.*, subspaces, hyperplanes, lines), rays, line segments, and halfspaces are all polyhedra. It is easily shown that polyhedra are convex sets. A bounded polyhedron is sometimes called a *polytope*, but some authors use the opposite convention (*i.e.*, polytope for any set of the form (2.5), and polyhedron

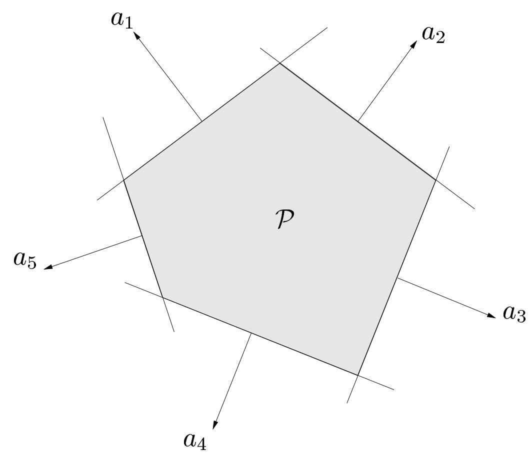


Figure 2.11 The polyhedron $\mathcal{P}$ (shown shaded) is the intersection of five halfspaces, with outward normal vectors $a_1, \ldots, a_5$.

when it is bounded). Figure 2.11 shows an example of a polyhedron defined as the intersection of five halfspaces.

It will be convenient to use the compact notation

$$\mathcal{P} = \{x \mid Ax \preceq b,\ Cx = d\} \tag{2.6}$$

for (2.5), where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \qquad C = \begin{bmatrix} c_1^T \\ \vdots \\ c_p^T \end{bmatrix},$$

and the symbol $\preceq$ denotes *vector inequality* or *componentwise inequality* in $\mathbf{R}^m$: $u \preceq v$ means $u_i \leq v_i$ for $i = 1, \ldots, m$.

Example 2.4 The *nonnegative orthant* is the set of points with nonnegative components, *i.e.*,

$$\mathbf{R}_+^n = \{x \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\} = \{x \in \mathbf{R}^n \mid x \succeq 0\}.$$

(Here $\mathbf{R}_+$ denotes the set of nonnegative numbers: $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x \geq 0\}$.) The nonnegative orthant is a polyhedron and a cone (and therefore called a *polyhedral cone*).

Simplexes

Simplexes are another important family of polyhedra. Suppose the $k+1$ points $v_0, \ldots, v_k \in \mathbf{R}^n$ are *affinely independent*, which means $v_1 - v_0, \ldots, v_k - v_0$ are linearly independent. The simplex determined by them is given by

$$C = \mathbf{conv}\{v_0, \ldots, v_k\} = \{\theta_0 v_0 + \cdots + \theta_k v_k \mid \theta \succeq 0,\ \mathbf{1}^T\theta = 1\}, \tag{2.7}$$

where $\mathbf{1}$ denotes the vector with all entries one. The affine dimension of this simplex is k, so it is sometimes referred to as a k-dimensional simplex in $\mathbf{R}^n$.

Example 2.5 *Some common simplexes.* A 1-dimensional simplex is a line segment; a 2-dimensional simplex is a triangle (including its interior); and a 3-dimensional simplex is a tetrahedron.

The *unit simplex* is the n-dimensional simplex determined by the zero vector and the unit vectors, *i.e.*, $0,\ e_1, \ldots, e_n \in \mathbf{R}^n$. It can be expressed as the set of vectors that satisfy

$$x \succeq 0, \qquad \mathbf{1}^T x \leq 1.$$

The *probability simplex* is the $(n-1)$-dimensional simplex determined by the unit vectors $e_1, \ldots, e_n \in \mathbf{R}^n$. It is the set of vectors that satisfy

$$x \succeq 0, \qquad \mathbf{1}^T x = 1.$$

Vectors in the probability simplex correspond to probability distributions on a set with n elements, with x_i interpreted as the probability of the ith element.

To describe the simplex (2.7) as a polyhedron, *i.e.*, in the form (2.6), we proceed as follows. By definition, $x \in C$ if and only if $x = \theta_0 v_0 + \theta_1 v_1 + \cdots + \theta_k v_k$ for some $\theta \succeq 0$ with $\mathbf{1}^T \theta = 1$. Equivalently, if we define $y = (\theta_1, \ldots, \theta_k)$ and

$$B = \left[\begin{array}{ccc} v_1 - v_0 & \cdots & v_k - v_0 \end{array}\right] \in \mathbf{R}^{n \times k},$$

we can say that $x \in C$ if and only if

$$x = v_0 + By \tag{2.8}$$

for some $y \succeq 0$ with $\mathbf{1}^T y \leq 1$. Now we note that affine independence of the points $v_0, \ldots, v_k$ implies that the matrix B has rank k. Therefore there exists a nonsingular matrix $A = (A_1, A_2) \in \mathbf{R}^{n \times n}$ such that

$$AB = \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] B = \left[\begin{array}{c} I \\ 0 \end{array}\right].$$

Multiplying (2.8) on the left with A, we obtain

$$A_1 x = A_1 v_0 + y, \qquad A_2 x = A_2 v_0.$$

From this we see that $x \in C$ if and only if $A_2 x = A_2 v_0$, and the vector $y = A_1 x - A_1 v_0$ satisfies $y \succeq 0$ and $\mathbf{1}^T y \leq 1$. In other words we have $x \in C$ if and only if

$$A_2 x = A_2 v_0, \qquad A_1 x \succeq A_1 v_0, \qquad \mathbf{1}^T A_1 x \leq 1 + \mathbf{1}^T A_1 v_0,$$

which is a set of linear equalities and inequalities in x, and so describes a polyhedron.

Convex hull description of polyhedra

The convex hull of the finite set $\{v_1, \ldots, v_k\}$ is

$$\mathbf{conv}\{v_1, \ldots, v_k\} = \{\theta_1 v_1 + \cdots + \theta_k v_k \mid \theta \succeq 0,\ \mathbf{1}^T\theta = 1\}.$$

This set is a polyhedron, and bounded, but (except in special cases, *e.g.*, a simplex) it is not simple to express it in the form (2.5), *i.e.*, by a set of linear equalities and inequalities.

A generalization of this convex hull description is

$$\{\theta_1 v_1 + \cdots + \theta_k v_k \mid \theta_1 + \cdots + \theta_m = 1,\ \theta_i \geq 0,\ i = 1, \ldots, k\}, \tag{2.9}$$

where $m \leq k$. Here we consider nonnegative linear combinations of v_i, but only the first m coefficients are required to sum to one. Alternatively, we can interpret (2.9) as the convex hull of the points $v_1, \ldots, v_m$, plus the conic hull of the points $v_{m+1}, \ldots, v_k$. The set (2.9) defines a polyhedron, and conversely, every polyhedron can be represented in this form (although we will not show this).

The question of how a polyhedron is represented is subtle, and has very important practical consequences. As a simple example consider the unit ball in the ℓ_∞-norm in $\mathbf{R}^n$,

$$C = \{x \mid |x_i| \leq 1,\ i = 1, \ldots, n\}.$$

The set C can be described in the form (2.5) with $2n$ linear inequalities $\pm e_i^T x \leq 1$, where e_i is the ith unit vector. To describe it in the convex hull form (2.9) requires at least 2^n points:

$$C = \mathbf{conv}\{v_1, \ldots, v_{2^n}\},$$

where $v_1, \ldots, v_{2^n}$ are the 2^n vectors all of whose components are 1 or -1. Thus the size of the two descriptions differs greatly, for large n.

2.2.5 The positive semidefinite cone

We use the notation $\mathbf{S}^n$ to denote the set of symmetric $n \times n$ matrices,

$$\mathbf{S}^n = \{X \in \mathbf{R}^{n\times n} \mid X = X^T\},$$

which is a vector space with dimension $n(n+1)/2$. We use the notation $\mathbf{S}^n_+$ to denote the set of symmetric positive semidefinite matrices:

$$\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\},$$

and the notation $\mathbf{S}^n_{++}$ to denote the set of symmetric positive definite matrices:

$$\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}.$$

(This notation is meant to be analogous to $\mathbf{R}_+$, which denotes the nonnegative reals, and $\mathbf{R}_{++}$, which denotes the positive reals.)

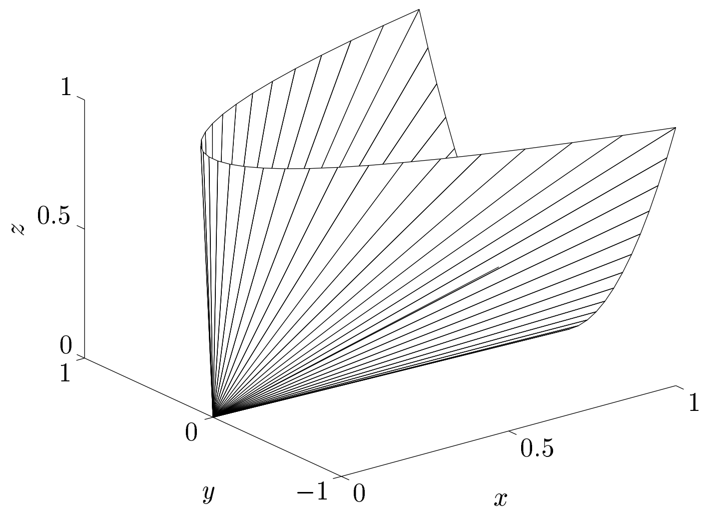


Figure 2.12 Boundary of positive semidefinite cone in $\mathbf{S}^2$.

The set $\mathbf{S}_+^n$ is a convex cone: if $\theta_1, \theta_2 \geq 0$ and $A,\ B \in \mathbf{S}_+^n$, then $\theta_1 A + \theta_2 B \in \mathbf{S}_+^n$. This can be seen directly from the definition of positive semidefiniteness: for any $x \in \mathbf{R}^n$, we have

$$x^T(\theta_1 A + \theta_2 B)x = \theta_1 x^T A x + \theta_2 x^T B x \geq 0,$$

if $A \succeq 0$, $B \succeq 0$ and $\theta_1, \theta_2 \geq 0$.

Example 2.6 *Positive semidefinite cone in* $\mathbf{S}^2$. We have

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2 \quad \Longleftrightarrow \quad x \geq 0, \quad z \geq 0, \quad xz \geq y^2.$$

The boundary of this cone is shown in figure 2.12, plotted in $\mathbf{R}^3$ as (x, y, z).

2.3 Operations that preserve convexity

In this section we describe some operations that preserve convexity of sets, or allow us to construct convex sets from others. These operations, together with the simple examples described in §2.2, form a calculus of convex sets that is useful for determining or establishing convexity of sets.

2.3.1 Intersection

Convexity is preserved under intersection: if S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex. This property extends to the intersection of an infinite number of sets: if S_α is convex for every $\alpha \in \mathcal{A}$, then $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex. (Subspaces, affine sets, and convex cones are also closed under arbitrary intersections.) As a simple example, a polyhedron is the intersection of halfspaces and hyperplanes (which are convex), and therefore is convex.

Example 2.7 The positive semidefinite cone $\mathbf{S}^n_+$ can be expressed as

$$\bigcap_{z \neq 0} \{X \in \mathbf{S}^n \mid z^T X z \geq 0\}.$$

For each $z \neq 0$, $z^T X z$ is a (not identically zero) linear function of X, so the sets

$$\{X \in \mathbf{S}^n \mid z^T X z \geq 0\}$$

are, in fact, halfspaces in $\mathbf{S}^n$. Thus the positive semidefinite cone is the intersection of an infinite number of halfspaces, and so is convex.

Example 2.8 We consider the set

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}, \tag{2.10}$$

where $p(t) = \sum_{k=1}^m x_k \cos kt$. The set S can be expressed as the intersection of an infinite number of *slabs*: $S = \bigcap_{|t| \leq \pi/3} S_t$, where

$$S_t = \{x \mid -1 \leq (\cos t, \ldots, \cos mt)^T x \leq 1\},$$

and so is convex. The definition and the set are illustrated in figures 2.13 and 2.14, for $m = 2$.

In the examples above we establish convexity of a set by expressing it as a (possibly infinite) intersection of halfspaces. We will see in §2.5.1 that a converse holds: *every* closed convex set S is a (usually infinite) intersection of halfspaces. In fact, a closed convex set S is the intersection of all halfspaces that contain it:

$$S = \bigcap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace}, \ S \subseteq \mathcal{H}\}.$$

2.3.2 Affine functions

Recall that a function $f : \mathbf{R}^n \to \mathbf{R}^m$ is *affine* if it is a sum of a linear function and a constant, *i.e.*, if it has the form $f(x) = Ax + b$, where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. Suppose $S \subseteq \mathbf{R}^n$ is convex and $f : \mathbf{R}^n \to \mathbf{R}^m$ is an affine function. Then the image of S under f,

$$f(S) = \{f(x) \mid x \in S\},$$

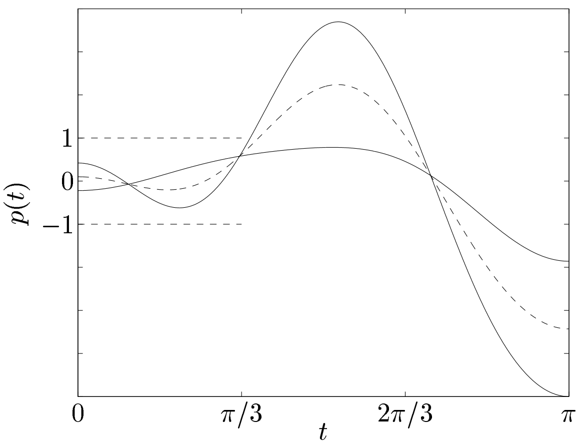


Figure 2.13 Three trigonometric polynomials associated with points in the set S defined in (2.10), for $m = 2$. The trigonometric polynomial plotted with dashed line type is the average of the other two.

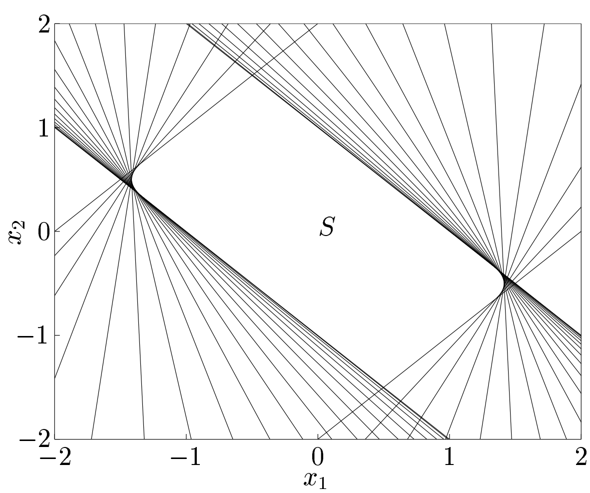


Figure 2.14 The set S defined in (2.10), for $m = 2$, is shown as the white area in the middle of the plot. The set is the intersection of an infinite number of slabs (20 of which are shown), hence convex.

is convex. Similarly, if $f : \mathbf{R}^k \to \mathbf{R}^n$ is an affine function, the *inverse image* of S under f,

$$f^{-1}(S) = \{x \mid f(x) \in S\},$$

is convex.

Two simple examples are *scaling* and *translation*. If $S \subseteq \mathbf{R}^n$ is convex, $\alpha \in \mathbf{R}$, and $a \in \mathbf{R}^n$, then the sets αS and $S + a$ are convex, where

$$\alpha S = \{\alpha x \mid x \in S\}, \qquad S + a = \{x + a \mid x \in S\}.$$

The *projection* of a convex set onto some of its coordinates is convex: if $S \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is convex, then

$$T = \{x_1 \in \mathbf{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbf{R}^n\}$$

is convex.

The *sum* of two sets is defined as

$$S_1 + S_2 = \{x + y \mid x \in S_1,\ y \in S_2\}.$$

If S_1 and S_2 are convex, then $S_1 + S_2$ is convex. To see this, if S_1 and S_2 are convex, then so is the direct or Cartesian product

$$S_1 \times S_2 = \{(x_1, x_2) \mid x_1 \in S_1,\ x_2 \in S_2\}.$$

The image of this set under the linear function $f(x_1, x_2) = x_1 + x_2$ is the sum $S_1 + S_2$.

We can also consider the *partial sum* of $S_1,\ S_2 \in \mathbf{R}^n \times \mathbf{R}^m$, defined as

$$S = \{(x, y_1 + y_2) \mid (x, y_1) \in S_1,\ (x, y_2) \in S_2\},$$

where $x \in \mathbf{R}^n$ and $y_i \in \mathbf{R}^m$. For $m = 0$, the partial sum gives the intersection of S_1 and S_2; for $n = 0$, it is set addition. Partial sums of convex sets are convex (see exercise 2.16).

Example 2.9 *Polyhedron.* The polyhedron $\{x \mid Ax \preceq b,\ Cx = d\}$ can be expressed as the inverse image of the Cartesian product of the nonnegative orthant and the origin under the affine function $f(x) = (b - Ax, d - Cx)$:

$$\{x \mid Ax \preceq b,\ Cx = d\} = \{x \mid f(x) \in \mathbf{R}_+^m \times \{0\}\}.$$

Example 2.10 *Solution set of linear matrix inequality.* The condition

$$A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B, \tag{2.11}$$

where $B,\ A_i \in \mathbf{S}^m$, is called a *linear matrix inequality* (LMI) in x (Note the similarity to an ordinary linear inequality,

$$a^T x = x_1 a_1 + \cdots + x_n a_n \le b,$$

with $b,\ a_i \in \mathbf{R}$.)

The solution set of a linear matrix inequality, $\{x \mid A(x) \preceq B\}$, is convex. Indeed, it is the inverse image of the positive semidefinite cone under the affine function $f : \mathbf{R}^n \to \mathbf{S}^m$ given by $f(x) = B - A(x)$.

Example 2.11 *Hyperbolic cone.* The set

$$\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\}$$

where $P \in \mathbf{S}^n_+$ and $c \in \mathbf{R}^n$, is convex, since it is the inverse image of the second-order cone,

$$\{(z,t) \mid z^T z \leq t^2,\ t \geq 0\},$$

under the affine function $f(x) = (P^{1/2}x, c^T x)$.

Example 2.12 *Ellipsoid.* The ellipsoid

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\},$$

where $P \in \mathbf{S}^n_{++}$, is the image of the unit Euclidean ball $\{u \mid \|u\|_2 \leq 1\}$ under the affine mapping $f(u) = P^{1/2}u + x_c$. (It is also the inverse image of the unit ball under the affine mapping $g(x) = P^{-1/2}(x - x_c)$.)

2.3.3 Linear-fractional and perspective functions

In this section we explore a class of functions, called *linear-fractional*, that is more general than affine but still preserves convexity.

The perspective function

We define the *perspective function* $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$, with domain $\mathbf{dom}\, P = \mathbf{R}^n \times \mathbf{R}_{++}$, as $P(z,t) = z/t$. (Here $\mathbf{R}_{++}$ denotes the set of positive numbers: $\mathbf{R}_{++} = \{x \in \mathbf{R} \mid x > 0\}$.) The perspective function scales or normalizes vectors so the last component is one, and then drops the last component.

Remark 2.1 We can interpret the perspective function as the action of a *pin-hole camera.* A pin-hole camera (in $\mathbf{R}^3$) consists of an opaque horizontal plane $x_3 = 0$, with a single pin-hole at the origin, through which light can pass, and a horizontal image plane $x_3 = -1$. An object at x, above the camera (*i.e.*, with $x_3 > 0$), forms an image at the point $-(x_1/x_3, x_2/x_3, 1)$ on the image plane. Dropping the last component of the image point (since it is always -1), the image of a point at x appears at $y = -(x_1/x_3, x_2/x_3) = -P(x)$ on the image plane. This is illustrated in figure 2.15.

If $C \subseteq \mathbf{dom}\, P$ is convex, then its image

$$P(C) = \{P(x) \mid x \in C\}$$

is convex. This result is certainly intuitive: a convex object, viewed through a pin-hole camera, yields a convex image. To establish this fact we show that line segments are mapped to line segments under the perspective function. (This too

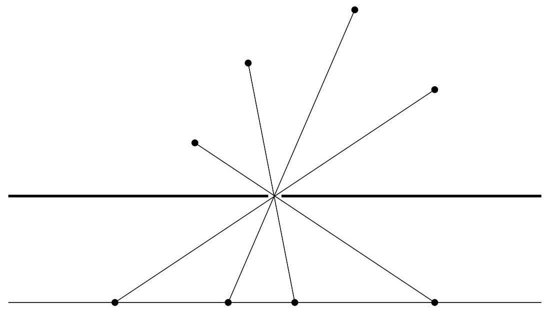


Figure 2.15 Pin-hole camera interpretation of perspective function. The dark horizontal line represents the plane $x_3 = 0$ in $\mathbf{R}^3$, which is opaque, except for a pin-hole at the origin. Objects or light sources above the plane appear on the image plane $x_3 = -1$, which is shown as the lighter horizontal line. The mapping of the position of a source to the position of its image is related to the perspective function.

makes sense: a line segment, viewed through a pin-hole camera, yields a line segment image.) Suppose that $x = (\tilde{x}, x_{n+1})$, $y = (\tilde{y}, y_{n+1}) \in \mathbf{R}^{n+1}$ with $x_{n+1} > 0$, $y_{n+1} > 0$. Then for $0 \le \theta \le 1$,

$$P(\theta x + (1-\theta)y) = \frac{\theta\tilde{x} + (1-\theta)\tilde{y}}{\theta x_{n+1} + (1-\theta)y_{n+1}} = \mu P(x) + (1-\mu)P(y),$$

where

$$\mu = \frac{\theta x_{n+1}}{\theta x_{n+1} + (1-\theta)y_{n+1}} \in [0,1].$$

This correspondence between θ and μ is monotonic: as θ varies between 0 and 1 (which sweeps out the line segment $[x,y]$), μ varies between 0 and 1 (which sweeps out the line segment $[P(x), P(y)]$). This shows that $P([x,y]) = [P(x), P(y)]$.

Now suppose C is convex with $C \subseteq \mathbf{dom}\, P$ (*i.e.*, $x_{n+1} > 0$ for all $x \in C$), and $x,\ y \in C$. To establish convexity of $P(C)$ we need to show that the line segment $[P(x), P(y)]$ is in $P(C)$. But this line segment is the image of the line segment $[x,y]$ under P, and so lies in $P(C)$.

The inverse image of a convex set under the perspective function is also convex: if $C \subseteq \mathbf{R}^n$ is convex, then

$$P^{-1}(C) = \{(x,t) \in \mathbf{R}^{n+1} \mid x/t \in C,\ t > 0\}$$

is convex. To show this, suppose $(x,t) \in P^{-1}(C)$, $(y,s) \in P^{-1}(C)$, and $0 \le \theta \le 1$. We need to show that

$$\theta(x,t) + (1-\theta)(y,s) \in P^{-1}(C),$$

i.e., that

$$\frac{\theta x + (1-\theta)y}{\theta t + (1-\theta)s} \in C$$

($\theta t + (1-\theta)s > 0$ is obvious). This follows from

$$\frac{\theta x + (1-\theta)y}{\theta t + (1-\theta)s} = \mu(x/t) + (1-\mu)(y/s),$$

where

$$\mu = \frac{\theta t}{\theta t + (1-\theta)s} \in [0,1].$$

Linear-fractional functions

A *linear-fractional function* is formed by composing the perspective function with an affine function. Suppose $g : \mathbf{R}^n \to \mathbf{R}^{m+1}$ is affine, *i.e.*,

$$g(x) = \left[\begin{array}{c} A \\ c^T \end{array}\right] x + \left[\begin{array}{c} b \\ d \end{array}\right], \tag{2.12}$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, and $d \in \mathbf{R}$. The function $f : \mathbf{R}^n \to \mathbf{R}^m$ given by $f = P \circ g$, *i.e.*,

$$f(x) = (Ax+b)/(c^T x + d), \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}, \tag{2.13}$$

is called a *linear-fractional* (or *projective*) function. If $c = 0$ and $d > 0$, the domain of f is $\mathbf{R}^n$, and f is an affine function. So we can think of affine and linear functions as special cases of linear-fractional functions.

Remark 2.2 *Projective interpretation.* It is often convenient to represent a linear-fractional function as a matrix

$$Q = \left[\begin{array}{cc} A & b \\ c^T & d \end{array}\right] \in \mathbf{R}^{(m+1)\times(n+1)} \tag{2.14}$$

that acts on (multiplies) points of form $(x,1)$, which yields $(Ax+b, c^T x + d)$. This result is then scaled or normalized so that its last component is one, which yields $(f(x), 1)$.

This representation can be interpreted geometrically by associating $\mathbf{R}^n$ with a set of rays in $\mathbf{R}^{n+1}$ as follows. With each point z in $\mathbf{R}^n$ we associate the (open) ray $\mathcal{P}(z) = \{t(z,1) \mid t > 0\}$ in $\mathbf{R}^{n+1}$. The last component of this ray takes on positive values. Conversely any ray in $\mathbf{R}^{n+1}$, with base at the origin and last component which takes on positive values, can be written as $\mathcal{P}(v) = \{t(v,1) \mid t \geq 0\}$ for some $v \in \mathbf{R}^n$. This (projective) correspondence $\mathcal{P}$ between $\mathbf{R}^n$ and the halfspace of rays with positive last component is one-to-one and onto.

The linear-fractional function (2.13) can be expressed as

$$f(x) = \mathcal{P}^{-1}(Q\mathcal{P}(x)).$$

Thus, we start with $x \in \mathbf{dom}\, f$, *i.e.*, $c^T x + d > 0$. We then form the ray $\mathcal{P}(x)$ in $\mathbf{R}^{n+1}$. The linear transformation with matrix Q acts on this ray to produce another ray $Q\mathcal{P}(x)$. Since $x \in \mathbf{dom}\, f$, the last component of this ray assumes positive values. Finally we take the inverse projective transformation to recover $f(x)$.

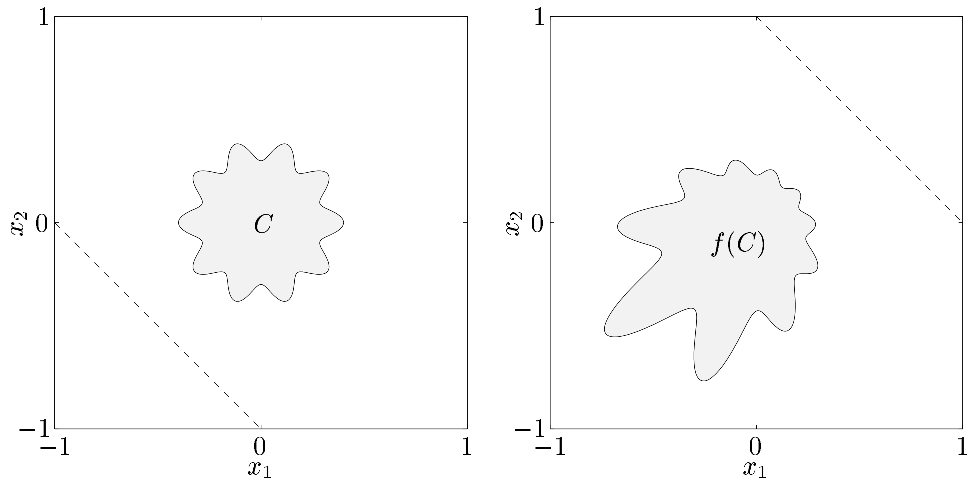


Figure 2.16 *Left.* A set $C \subseteq \mathbf{R}^2$. The dashed line shows the boundary of the domain of the linear-fractional function $f(x) = x/(x_1 + x_2 + 1)$ with $\mathbf{dom}\, f = \{(x_1, x_2) \mid x_1 + x_2 + 1 > 0\}$. *Right.* Image of C under f. The dashed line shows the boundary of the domain of f^{-1}.

Like the perspective function, linear-fractional functions preserve convexity. If C is convex and lies in the domain of f (*i.e.*, $c^T x + d > 0$ for $x \in C$), then its image $f(C)$ is convex. This follows immediately from results above: the image of C under the affine mapping (2.12) is convex, and the image of the resulting set under the perspective function P, which yields $f(C)$, is convex. Similarly, if $C \subseteq \mathbf{R}^m$ is convex, then the inverse image $f^{-1}(C)$ is convex.

Example 2.13 *Conditional probabilities.* Suppose u and v are random variables that take on values in $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively, and let p_{ij} denote $\mathbf{prob}(u = i, v = j)$. Then the conditional probability $f_{ij} = \mathbf{prob}(u = i | v = j)$ is given by

$$f_{ij} = \frac{p_{ij}}{\sum_{k=1}^{n} p_{kj}}.$$

Thus f is obtained by a linear-fractional mapping from p.

It follows that if C is a convex set of joint probabilities for (u, v), then the associated set of conditional probabilities of u given v is also convex.

Figure 2.16 shows a set $C \subseteq \mathbf{R}^2$, and its image under the linear-fractional function

$$f(x) = \frac{1}{x_1 + x_2 + 1} x, \qquad \mathbf{dom}\, f = \{(x_1, x_2) \mid x_1 + x_2 + 1 > 0\}.$$

2.4 Generalized inequalities

2.4.1 Proper cones and generalized inequalities

A cone $K \subseteq \mathbf{R}^n$ is called a *proper cone* if it satisfies the following:

- K is convex.
- K is closed.
- K is *solid*, which means it has nonempty interior.
- K is *pointed*, which means that it contains no line (or equivalently, $x \in K,\ -x \in K \implies x = 0$).

A proper cone K can be used to define a *generalized inequality*, which is a partial ordering on $\mathbf{R}^n$ that has many of the properties of the standard ordering on $\mathbf{R}$. We associate with the proper cone K the partial ordering on $\mathbf{R}^n$ defined by

$$x \preceq_K y \iff y - x \in K.$$

We also write $x \succeq_K y$ for $y \preceq_K x$. Similarly, we define an associated strict partial ordering by

$$x \prec_K y \iff y - x \in \mathbf{int}\, K,$$

and write $x \succ_K y$ for $y \prec_K x$. (To distinguish the generalized inequality $\preceq_K$ from the strict generalized inequality, we sometimes refer to $\preceq_K$ as the nonstrict generalized inequality.)

When $K = \mathbf{R}_+$, the partial ordering $\preceq_K$ is the usual ordering $\leq$ on $\mathbf{R}$, and the strict partial ordering $\prec_K$ is the same as the usual strict ordering $<$ on $\mathbf{R}$. So generalized inequalities include as a special case ordinary (nonstrict and strict) inequality in $\mathbf{R}$.

Example 2.14 *Nonnegative orthant and componentwise inequality.* The nonnegative orthant $K = \mathbf{R}^n_+$ is a proper cone. The associated generalized inequality $\preceq_K$ corresponds to componentwise inequality between vectors: $x \preceq_K y$ means than $x_i \leq y_i$, $i = 1, \ldots, n$. The associated strict inequality corresponds to componentwise strict inequality: $x \prec_K y$ means than $x_i < y_i$, $i = 1, \ldots, n$.

The nonstrict and strict partial orderings associated with the nonnegative orthant arise so frequently that we drop the subscript $\mathbf{R}^n_+$; it is understood when the symbol $\preceq$ or $\prec$ appears between vectors.

Example 2.15 *Positive semidefinite cone and matrix inequality.* The positive semidefinite cone $\mathbf{S}^n_+$ is a proper cone in $\mathbf{S}^n$. The associated generalized inequality $\preceq_K$ is the usual matrix inequality: $X \preceq_K Y$ means $Y - X$ is positive semidefinite. The interior of $\mathbf{S}^n_+$ (in $\mathbf{S}^n$) consists of the positive definite matrices, so the strict generalized inequality also agrees with the usual strict inequality between symmetric matrices: $X \prec_K Y$ means $Y - X$ is positive definite.

Here, too, the partial ordering arises so frequently that we drop the subscript: for symmetric matrices we write simply $X \preceq Y$ or $X \prec Y$. It is understood that the generalized inequalities are with respect to the positive semidefinite cone.

Example 2.16 *Cone of polynomials nonnegative on* $[0,1]$. Let K be defined as

$$K = \{c \in \mathbf{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\}, \tag{2.15}$$

i.e., K is the cone of (coefficients of) polynomials of degree $n-1$ that are nonnegative on the interval $[0,1]$. It can be shown that K is a proper cone; its interior is the set of coefficients of polynomials that are positive on the interval $[0,1]$.

Two vectors $c,\ d \in \mathbf{R}^n$ satisfy $c \preceq_K d$ if and only if

$$c_1 + c_2 t + \cdots + c_n t^{n-1} \leq d_1 + d_2 t + \cdots + d_n t^{n-1}$$

for all $t \in [0,1]$.

Properties of generalized inequalities

A generalized inequality $\preceq_K$ satisfies many properties, such as

- $\preceq_K$ *is preserved under addition*: if $x \preceq_K y$ and $u \preceq_K v$, then $x+u \preceq_K y+v$.
- $\preceq_K$ *is transitive*: if $x \preceq_K y$ and $y \preceq_K z$ then $x \preceq_K z$.
- $\preceq_K$ *is preserved under nonnegative scaling*: if $x \preceq_K y$ and $\alpha \geq 0$ then $\alpha x \preceq_K \alpha y$.
- $\preceq_K$ *is reflexive:* $x \preceq_K x$.
- $\preceq_K$ *is antisymmetric:* if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
- $\preceq_K$ *is preserved under limits:* if $x_i \preceq_K y_i$ for $i = 1,\ 2, \ldots$, $x_i \to x$ and $y_i \to y$ as $i \to \infty$, then $x \preceq_K y$.

The corresponding strict generalized inequality $\prec_K$ satisfies, for example,

- if $x \prec_K y$ then $x \preceq_K y$.
- if $x \prec_K y$ and $u \preceq_K v$ then $x + u \prec_K y + v$.
- if $x \prec_K y$ and $\alpha > 0$ then $\alpha x \prec_K \alpha y$.
- $x \not\prec_K x$.
- if $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

These properties are inherited from the definitions of $\preceq_K$ and $\prec_K$, and the properties of proper cones; see exercise 2.30.

2.4.2 Minimum and minimal elements

The notation of generalized inequality (*i.e.*, $\preceq_K$, $\prec_K$) is meant to suggest the analogy to ordinary inequality on $\mathbf{R}$ (*i.e.*, $\leq$, $<$). While many properties of ordinary inequality do hold for generalized inequalities, some important ones do not. The most obvious difference is that $\leq$ on $\mathbf{R}$ is a *linear ordering*: any two points are *comparable*, meaning either $x \leq y$ or $y \leq x$. This property does not hold for other generalized inequalities. One implication is that concepts like minimum and maximum are more complicated in the context of generalized inequalities. We briefly discuss this in this section.

We say that $x \in S$ is the *minimum* element of S (with respect to the generalized inequality $\preceq_K$) if for every $y \in S$ we have $x \preceq_K y$. We define the *maximum* element of a set S, with respect to a generalized inequality, in a similar way. If a set has a minimum (maximum) element, then it is unique. A related concept is *minimal element*. We say that $x \in S$ is a *minimal* element of S (with respect to the generalized inequality $\preceq_K$) if $y \in S$, $y \preceq_K x$ only if $y = x$. We define *maximal* element in a similar way. A set can have many different minimal (maximal) elements.

We can describe minimum and minimal elements using simple set notation. A point $x \in S$ is the minimum element of S if and only if

$$S \subseteq x + K.$$

Here $x + K$ denotes all the points that are comparable to x and greater than or equal to x (according to $\preceq_K$). A point $x \in S$ is a minimal element if and only if

$$(x - K) \cap S = \{x\}.$$

Here $x - K$ denotes all the points that are comparable to x and less than or equal to x (according to $\preceq_K$); the only point in common with S is x.

For $K = \mathbf{R}_+$, which induces the usual ordering on $\mathbf{R}$, the concepts of minimal and minimum are the same, and agree with the usual definition of the minimum element of a set.

Example 2.17 Consider the cone $\mathbf{R}^2_+$, which induces componentwise inequality in $\mathbf{R}^2$. Here we can give some simple geometric descriptions of minimal and minimum elements. The inequality $x \preceq y$ means y is above and to the right of x. To say that $x \in S$ is the minimum element of a set S means that all other points of S lie above and to the right. To say that x is a minimal element of a set S means that no other point of S lies to the left and below x. This is illustrated in figure 2.17.

Example 2.18 *Minimum and minimal elements of a set of symmetric matrices.* We associate with each $A \in \mathbf{S}^n_{++}$ an ellipsoid centered at the origin, given by

$$\mathcal{E}_A = \{x \mid x^T A^{-1} x \leq 1\}.$$

We have $A \preceq B$ if and only if $\mathcal{E}_A \subseteq \mathcal{E}_B$.

Let $v_1, \ldots, v_k \in \mathbf{R}^n$ be given and define

$$S = \{P \in \mathbf{S}^n_{++} \mid v_i^T P^{-1} v_i \leq 1,\ i = 1, \ldots, k\},$$

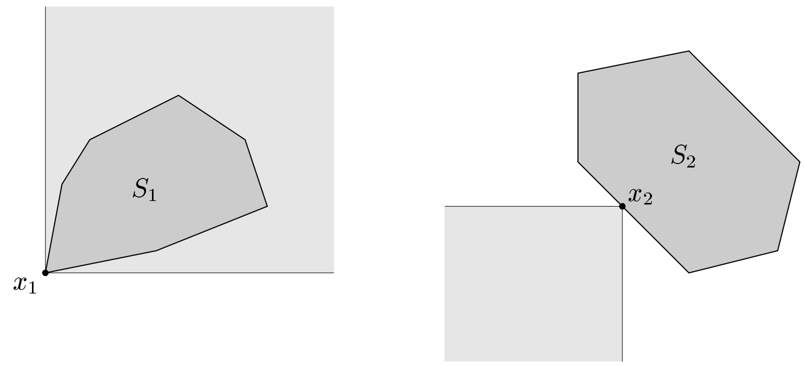


Figure 2.17 *Left.* The set S_1 has a minimum element x_1 with respect to componentwise inequality in $\mathbf{R}^2$. The set $x_1 + K$ is shaded lightly; x_1 is the minimum element of S_1 since $S_1 \subseteq x_1 + K$. *Right.* The point x_2 is a minimal point of S_2. The set $x_2 - K$ is shown lightly shaded. The point x_2 is minimal because $x_2 - K$ and S_2 intersect only at x_2.

which corresponds to the set of ellipsoids that contain the points $v_1, \ldots, v_k$. The set S does not have a minimum element: for any ellipsoid that contains the points $v_1, \ldots, v_k$ we can find another one that contains the points, and is not comparable to it. An ellipsoid is minimal if it contains the points, but no smaller ellipsoid does. Figure 2.18 shows an example in $\mathbf{R}^2$ with $k = 1$.

2.5 Separating and supporting hyperplanes

2.5.1 Separating hyperplane theorem

In this section we describe an idea that will be important later: the use of hyperplanes or affine functions to separate convex sets that do not intersect. The basic result is the *separating hyperplane theorem*: Suppose C and D are two convex sets that do not intersect, *i.e.*, $C \cap D = \emptyset$. Then there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$. In other words, the affine function $a^T x - b$ is nonpositive on C and nonnegative on D. The hyperplane $\{x \mid a^T x = b\}$ is called a *separating hyperplane* for the sets C and D, or is said to *separate* the sets C and D. This is illustrated in figure 2.19.

Proof of separating hyperplane theorem

Here we consider a special case, and leave the extension of the proof to the general case as an exercise (exercise 2.22). We assume that the (Euclidean) *distance* between C and D, defined as

$$\mathbf{dist}(C, D) = \inf\{\|u - v\|_2 \mid u \in C,\ v \in D\},$$

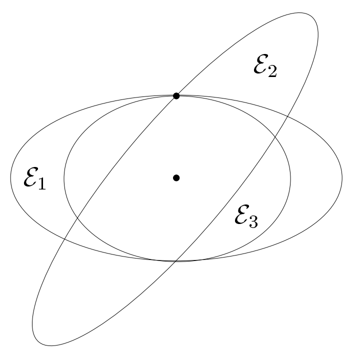


Figure 2.18 Three ellipsoids in $\mathbf{R}^2$, centered at the origin (shown as the lower dot), that contain the point shown as the upper dot. The ellipsoid $\mathcal{E}_1$ is not minimal, since there exist ellipsoids that contain the point, and are smaller (*e.g.*, $\mathcal{E}_3$). $\mathcal{E}_3$ is not minimal for the same reason. The ellipsoid $\mathcal{E}_2$ is minimal, since no other ellipsoid (centered at the origin) contains the point and is contained in $\mathcal{E}_2$.

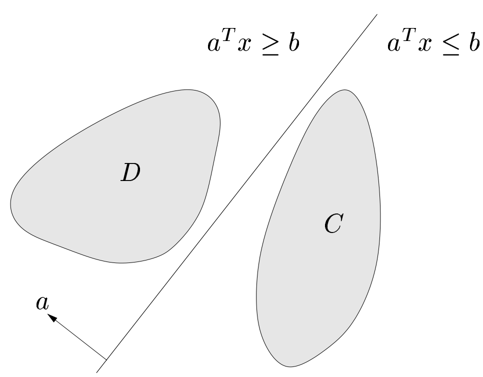


Figure 2.19 The hyperplane $\{x \mid a^T x = b\}$ separates the disjoint convex sets C and D. The affine function $a^T x - b$ is nonpositive on C and nonnegative on D.

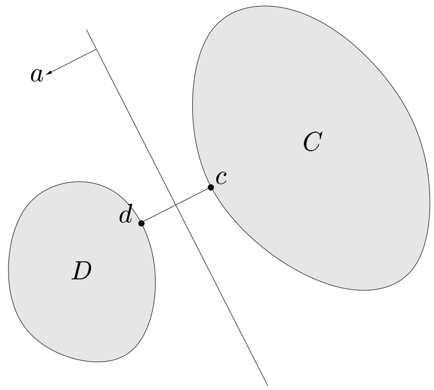


Figure 2.20 Construction of a separating hyperplane between two convex sets. The points $c \in C$ and $d \in D$ are the pair of points in the two sets that are closest to each other. The separating hyperplane is orthogonal to, and bisects, the line segment between c and d.

is positive, and that there exist points $c \in C$ and $d \in D$ that achieve the minimum distance, *i.e.*, $\|c - d\|_2 = \mathbf{dist}(C, D)$. (These conditions are satisfied, for example, when C and D are closed and one set is bounded.)

Define

$$a = d - c, \qquad b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}.$$

We will show that the affine function

$$f(x) = a^T x - b = (d - c)^T (x - (1/2)(d + c))$$

is nonpositive on C and nonnegative on D, *i.e.*, that the hyperplane $\{x \mid a^T x = b\}$ separates C and D. This hyperplane is perpendicular to the line segment between c and d, and passes through its midpoint, as shown in figure 2.20.

We first show that f is nonnegative on D. The proof that f is nonpositive on C is similar (or follows by swapping C and D and considering $-f$). Suppose there were a point $u \in D$ for which

$$f(u) = (d - c)^T (u - (1/2)(d + c)) < 0. \tag{2.16}$$

We can express $f(u)$ as

$$f(u) = (d - c)^T (u - d + (1/2)(d - c)) = (d - c)^T (u - d) + (1/2)\|d - c\|_2^2.$$

We see that (2.16) implies $(d - c)^T (u - d) < 0$. Now we observe that

$$\left. \frac{d}{dt} \|d + t(u - d) - c\|_2^2 \right|_{t=0} = 2(d - c)^T (u - d) < 0,$$

so for some small $t > 0$, with $t \le 1$, we have

$$\|d + t(u - d) - c\|_2 < \|d - c\|_2,$$

i.e., the point $d + t(u - d)$ is closer to c than d is. Since D is convex and contains d and u, we have $d + t(u - d) \in D$. But this is impossible, since d is assumed to be the point in D that is closest to C.

Example 2.19 *Separation of an affine and a convex set.* Suppose C is convex and D is affine, *i.e.*, $D = \{Fu + g \mid u \in \mathbf{R}^m\}$, where $F \in \mathbf{R}^{n \times m}$. Suppose C and D are disjoint, so by the separating hyperplane theorem there are $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.

Now $a^T x \geq b$ for all $x \in D$ means $a^T F u \geq b - a^T g$ for all $u \in \mathbf{R}^m$. But a linear function is bounded below on $\mathbf{R}^m$ only when it is zero, so we conclude $a^T F = 0$ (and hence, $b \leq a^T g$).

Thus we conclude that there exists $a \neq 0$ such that $F^T a = 0$ and $a^T x \leq a^T g$ for all $x \in C$.

Strict separation

The separating hyperplane we constructed above satisfies the stronger condition that $a^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$. This is called *strict separation* of the sets C and D. Simple examples show that in general, disjoint convex sets need not be strictly separable by a hyperplane (even when the sets are closed; see exercise 2.23). In many special cases, however, strict separation can be established.

Example 2.20 *Strict separation of a point and a closed convex set.* Let C be a closed convex set and $x_0 \notin C$. Then there exists a hyperplane that strictly separates x_0 from C.

To see this, note that the two sets C and $B(x_0, \epsilon)$ do not intersect for some $\epsilon > 0$. By the separating hyperplane theorem, there exist $a \neq 0$ and b such that $a^T x \leq b$ for $x \in C$ and $a^T x \geq b$ for $x \in B(x_0, \epsilon)$.

Using $B(x_0, \epsilon) = \{x_0 + u \mid \|u\|_2 \leq \epsilon\}$, the second condition can be expressed as

$$a^T (x_0 + u) \geq b \quad \text{for all} \quad \|u\|_2 \leq \epsilon.$$

The u that minimizes the lefthand side is $u = -\epsilon a / \|a\|_2$; using this value we have

$$a^T x_0 - \epsilon \|a\|_2 \geq b.$$

Therefore the affine function

$$f(x) = a^T x - b - \epsilon \|a\|_2 / 2$$

is negative on C and positive at x_0.

As an immediate consequence we can establish a fact that we already mentioned above: a closed convex set is the intersection of all halfspaces that contain it. Indeed, let C be closed and convex, and let S be the intersection of all halfspaces containing C. Obviously $x \in C \Rightarrow x \in S$. To show the converse, suppose there exists $x \in S$, $x \notin C$. By the strict separation result there exists a hyperplane that strictly separates x from C, *i.e.*, there is a halfspace containing C but not x. In other words, $x \notin S$.

Converse separating hyperplane theorems

The converse of the separating hyperplane theorem (*i.e.*, existence of a separating hyperplane implies that C and D do not intersect) is not true, unless one imposes additional constraints on C or D, even beyond convexity. As a simple counterexample, consider $C = D = \{0\} \subseteq \mathbf{R}$. Here the hyperplane $x = 0$ separates C and D.

By adding conditions on C and D various converse separation theorems can be derived. As a very simple example, suppose C and D are convex sets, with C open, and there exists an affine function f that is nonpositive on C and nonnegative on D. Then C and D are disjoint. (To see this we first note that f must be negative on C; for if f were zero at a point of C then f would take on positive values near the point, which is a contradiction. But then C and D must be disjoint since f is negative on C and nonnegative on D.) Putting this converse together with the separating hyperplane theorem, we have the following result: any two convex sets C and D, at least one of which is open, are disjoint if and only if there exists a separating hyperplane.

Example 2.21 *Theorem of alternatives for strict linear inequalities.* We derive the necessary and sufficient conditions for solvability of a system of strict linear inequalities

$$Ax \prec b. \tag{2.17}$$

These inequalities are infeasible if and only if the (convex) sets

$$C = \{b - Ax \mid x \in \mathbf{R}^n\}, \qquad D = \mathbf{R}^m_{++} = \{y \in \mathbf{R}^m \mid y \succ 0\}$$

do not intersect. The set D is open; C is an affine set. Hence by the result above, C and D are disjoint if and only if there exists a separating hyperplane, *i.e.*, a nonzero $\lambda \in \mathbf{R}^m$ and $\mu \in \mathbf{R}$ such that $\lambda^T y \leq \mu$ on C and $\lambda^T y \geq \mu$ on D.

Each of these conditions can be simplified. The first means $\lambda^T(b - Ax) \leq \mu$ for all x. This implies (as in example 2.19) that $A^T\lambda = 0$ and $\lambda^T b \leq \mu$. The second inequality means $\lambda^T y \geq \mu$ for all $y \succ 0$. This implies $\mu \leq 0$ and $\lambda \succeq 0$, $\lambda \neq 0$.

Putting it all together, we find that the set of strict inequalities (2.17) is infeasible if and only if there exists $\lambda \in \mathbf{R}^m$ such that

$$\lambda \neq 0, \qquad \lambda \succeq 0, \qquad A^T\lambda = 0, \qquad \lambda^T b \leq 0. \tag{2.18}$$

This is also a system of linear inequalities and linear equations in the variable $\lambda \in \mathbf{R}^m$. We say that (2.17) and (2.18) form a pair of *alternatives*: for any data A and b, exactly one of them is solvable.

2.5.2 Supporting hyperplanes

Suppose $C \subseteq \mathbf{R}^n$, and x_0 is a point in its boundary $\mathbf{bd}\, C$, *i.e.*,

$$x_0 \in \mathbf{bd}\, C = \mathbf{cl}\, C \setminus \mathbf{int}\, C.$$

If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$, then the hyperplane $\{x \mid a^T x = a^T x_0\}$ is called a *supporting hyperplane* to C at the point x_0. This is equivalent to saying

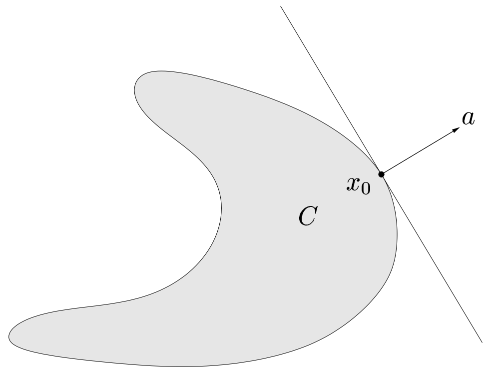


Figure 2.21 The hyperplane $\{x \mid a^T x = a^T x_0\}$ supports C at x_0.

that the point x_0 and the set C are separated by the hyperplane $\{x \mid a^T x = a^T x_0\}$. The geometric interpretation is that the hyperplane $\{x \mid a^T x = a^T x_0\}$ is tangent to C at x_0, and the halfspace $\{x \mid a^T x \leq a^T x_0\}$ contains C. This is illustrated in figure 2.21.

A basic result, called the *supporting hyperplane theorem*, states that for any nonempty convex set C, and any $x_0 \in \mathbf{bd}\, C$, there exists a supporting hyperplane to C at x_0. The supporting hyperplane theorem is readily proved from the separating hyperplane theorem. We distinguish two cases. If the interior of C is nonempty, the result follows immediately by applying the separating hyperplane theorem to the sets $\{x_0\}$ and $\mathbf{int}\, C$. If the interior of C is empty, then C must lie in an affine set of dimension less than n, and any hyperplane containing that affine set contains C and x_0, and is a (trivial) supporting hyperplane.

There is also a partial converse of the supporting hyperplane theorem: If a set is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary, then it is convex. (See exercise 2.27.)

2.6 Dual cones and generalized inequalities

2.6.1 Dual cones

Let K be a cone. The set

$$K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\} \tag{2.19}$$

is called the *dual cone* of K. As the name suggests, K^* is a cone, and is always convex, even when the original cone K is not (see exercise 2.31).

Geometrically, $y \in K^*$ if and only if $-y$ is the normal of a hyperplane that supports K at the origin. This is illustrated in figure 2.22.

Example 2.22 *Subspace.* The dual cone of a subspace $V \subseteq \mathbf{R}^n$ (which is a cone) is its orthogonal complement $V^\perp = \{y \mid y^T v = 0 \text{ for all } v \in V\}$.

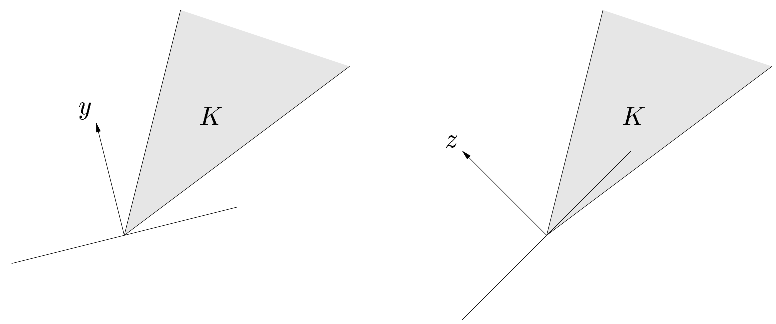


Figure 2.22 *Left.* The halfspace with inward normal y contains the cone K, so $y \in K^*$. *Right.* The halfspace with inward normal z does not contain K, so $z \notin K^*$.

Example 2.23 *Nonnegative orthant.* The cone $\mathbf{R}^n_+$ is its own dual:

$$y^T x \geq 0 \text{ for all } x \succeq 0 \iff y \succeq 0.$$

We call such a cone *self-dual*.

Example 2.24 *Positive semidefinite cone.* On the set of symmetric $n \times n$ matrices $\mathbf{S}^n$, we use the standard inner product $\mathbf{tr}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$ (see §A.1.1). The positive semidefinite cone $\mathbf{S}^n_+$ is self-dual, *i.e.*, for $X, Y \in \mathbf{S}^n$,

$$\mathbf{tr}(XY) \geq 0 \text{ for all } X \succeq 0 \iff Y \succeq 0.$$

We will establish this fact.

Suppose $Y \notin \mathbf{S}^n_+$. Then there exists $q \in \mathbf{R}^n$ with

$$q^T Y q = \mathbf{tr}(qq^T Y) < 0.$$

Hence the positive semidefinite matrix $X = qq^T$ satisfies $\mathbf{tr}(XY) < 0$; it follows that $Y \notin (\mathbf{S}^n_+)^*$.

Now suppose $X, Y \in \mathbf{S}^n_+$. We can express X in terms of its eigenvalue decomposition as $X = \sum_{i=1}^n \lambda_i q_i q_i^T$, where (the eigenvalues) $\lambda_i \geq 0$, $i = 1, \ldots, n$. Then we have

$$\mathbf{tr}(YX) = \mathbf{tr}\left(Y \sum_{i=1}^n \lambda_i q_i q_i^T \right) = \sum_{i=1}^n \lambda_i q_i^T Y q_i \geq 0.$$

This shows that $Y \in (\mathbf{S}^n_+)^*$.

Example 2.25 *Dual of a norm cone.* Let $\|\cdot\|$ be a norm on $\mathbf{R}^n$. The dual of the associated cone $K = \{(x,t) \in \mathbf{R}^{n+1} \mid \|x\| \leq t\}$ is the cone defined by the dual norm, *i.e.*,

$$K^* = \{(u,v) \in \mathbf{R}^{n+1} \mid \|u\|_* \leq v\},$$

where the dual norm is given by $\|u\|_* = \sup\{u^T x \mid \|x\| \leq 1\}$ (see (A.1.6)).

To prove the result we have to show that

$$x^T u + tv \geq 0 \text{ whenever } \|x\| \leq t \iff \|u\|_* \leq v. \tag{2.20}$$

Let us start by showing that the righthand condition on (u, v) implies the lefthand condition. Suppose $\|u\|_* \leq v$, and $\|x\| \leq t$ for some $t > 0$. (If $t = 0$, x must be zero, so obviously $u^T x + vt \geq 0$.) Applying the definition of the dual norm, and the fact that $\|-x/t\| \leq 1$, we have

$$u^T(-x/t) \leq \|u\|_* \leq v,$$

and therefore $u^T x + vt \geq 0$.

Next we show that the lefthand condition in (2.20) implies the righthand condition in (2.20). Suppose $\|u\|_* > v$, *i.e.*, that the righthand condition does not hold. Then by the definition of the dual norm, there exists an x with $\|x\| \leq 1$ and $x^T u > v$. Taking $t = 1$, we have

$$u^T(-x) + v < 0,$$

which contradicts the lefthand condition in (2.20).

Dual cones satisfy several properties, such as:

- K^* is closed and convex.
- $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.
- If K has nonempty interior, then K^* is pointed.
- If the closure of K is pointed then K^* has nonempty interior.
- K^{**} is the closure of the convex hull of K. (Hence if K is convex and closed, $K^{**} = K$.)

(See exercise 2.31.) These properties show that if K is a proper cone, then so is its dual K^*, and moreover, that $K^{**} = K$.

2.6.2 Dual generalized inequalities

Now suppose that the convex cone K is proper, so it induces a generalized inequality $\preceq_K$. Then its dual cone K^* is also proper, and therefore induces a generalized inequality. We refer to the generalized inequality $\preceq_{K^*}$ as the *dual* of the generalized inequality $\preceq_K$.

Some important properties relating a generalized inequality and its dual are:

- $x \preceq_K y$ if and only if $\lambda^T x \leq \lambda^T y$ for all $\lambda \succeq_{K^*} 0$.
- $x \prec_K y$ if and only if $\lambda^T x < \lambda^T y$ for all $\lambda \succeq_{K^*} 0$, $\lambda \neq 0$.

Since $K = K^{**}$, the dual generalized inequality associated with $\preceq_{K^*}$ is $\preceq_K$, so these properties hold if the generalized inequality and its dual are swapped. As a specific example, we have $\lambda \preceq_{K^*} \mu$ if and only if $\lambda^T x \leq \mu^T x$ for all $x \succeq_K 0$.

Example 2.26 *Theorem of alternatives for linear strict generalized inequalities.* Suppose $K \subseteq \mathbf{R}^m$ is a proper cone. Consider the strict generalized inequality

$$Ax \prec_K b, \tag{2.21}$$

where $x \in \mathbf{R}^n$.

We will derive a theorem of alternatives for this inequality. Suppose it is infeasible, *i.e.*, the affine set $\{b - Ax \mid x \in \mathbf{R}^n\}$ does not intersect the open convex set $\mathbf{int}\, K$. Then there is a separating hyperplane, *i.e.*, a nonzero $\lambda \in \mathbf{R}^m$ and $\mu \in \mathbf{R}$ such that $\lambda^T(b - Ax) \leq \mu$ for all x, and $\lambda^T y \geq \mu$ for all $y \in \mathbf{int}\, K$. The first condition implies $A^T\lambda = 0$ and $\lambda^T b \leq \mu$. The second condition implies $\lambda^T y \geq \mu$ for all $y \in K$, which can only happen if $\lambda \in K^*$ and $\mu \leq 0$.

Putting it all together we find that if (2.21) is infeasible, then there exists λ such that

$$\lambda \neq 0, \qquad \lambda \succeq_{K^*} 0, \qquad A^T\lambda = 0, \qquad \lambda^T b \leq 0. \tag{2.22}$$

Now we show the converse: if (2.22) holds, then the inequality system (2.21) cannot be feasible. Suppose that both inequality systems hold. Then we have $\lambda^T(b - Ax) > 0$, since $\lambda \neq 0$, $\lambda \succeq_{K^*} 0$, and $b - Ax \succ_K 0$. But using $A^T\lambda = 0$ we find that $\lambda^T(b - Ax) = \lambda^T b \leq 0$, which is a contradiction.

Thus, the inequality systems (2.21) and (2.22) are alternatives: for any data A, b, exactly one of them is feasible. (This generalizes the alternatives (2.17), (2.18) for the special case $K = \mathbf{R}^m_+$.)

2.6.3 Minimum and minimal elements via dual inequalities

We can use dual generalized inequalities to characterize minimum and minimal elements of a (possibly nonconvex) set $S \subseteq \mathbf{R}^m$ with respect to the generalized inequality induced by a proper cone K.

Dual characterization of minimum element

We first consider a characterization of the *minimum* element: x is the minimum element of S, with respect to the generalized inequality $\preceq_K$, if and only if for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over $z \in S$. Geometrically, this means that for any $\lambda \succ_{K^*} 0$, the hyperplane

$$\{z \mid \lambda^T(z - x) = 0\}$$

is a strict supporting hyperplane to S at x. (By strict supporting hyperplane, we mean that the hyperplane intersects S only at the point x.) Note that convexity of the set S is *not* required. This is illustrated in figure 2.23.

To show this result, suppose x is the minimum element of S, *i.e.*, $x \preceq_K z$ for all $z \in S$, and let $\lambda \succ_{K^*} 0$. Let $z \in S$, $z \neq x$. Since x is the minimum element of S, we have $z - x \succeq_K 0$. From $\lambda \succ_{K^*} 0$ and $z - x \succeq_K 0$, $z - x \neq 0$, we conclude $\lambda^T(z - x) > 0$. Since z is an arbitrary element of S, not equal to x, this shows that x is the unique minimizer of $\lambda^T z$ over $z \in S$. Conversely, suppose that for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over $z \in S$, but x is not the minimum

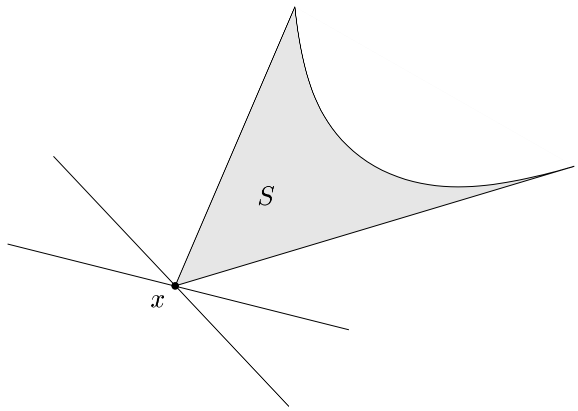


Figure 2.23 Dual characterization of minimum element. The point x is the minimum element of the set S with respect to $\mathbf{R}_+^2$. This is equivalent to: for every $\lambda \succ 0$, the hyperplane $\{z \mid \lambda^T(z-x)=0\}$ strictly supports S at x, *i.e.*, contains S on one side, and touches it only at x.

element of S. Then there exists $z \in S$ with $z \not\succeq_K x$. Since $z - x \not\succeq_K 0$, there exists $\tilde{\lambda} \succeq_{K^*} 0$ with $\tilde{\lambda}^T(z-x) < 0$. Hence $\lambda^T(z-x) < 0$ for $\lambda \succ_{K^*} 0$ in the neighborhood of $\tilde{\lambda}$. This contradicts the assumption that x is the unique minimizer of $\lambda^T z$ over S.

Dual characterization of minimal elements

We now turn to a similar characterization of *minimal elements*. Here there is a gap between the necessary and sufficient conditions. If $\lambda \succ_{K^*} 0$ and x minimizes $\lambda^T z$ over $z \in S$, then x is minimal. This is illustrated in figure 2.24.

To show this, suppose that $\lambda \succ_{K^*} 0$, and x minimizes $\lambda^T z$ over S, but x is not minimal, *i.e.*, there exists a $z \in S$, $z \neq x$, and $z \preceq_K x$. Then $\lambda^T(x-z) > 0$, which contradicts our assumption that x is the minimizer of $\lambda^T z$ over S.

The converse is in general false: a point x can be minimal in S, but not a minimizer of $\lambda^T z$ over $z \in S$, for any λ, as shown in figure 2.25. This figure suggests that convexity plays an important role in the converse, which is correct. Provided the set S is convex, we can say that for any minimal element x there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over $z \in S$.

To show this, suppose x is minimal, which means that $((x-K) \setminus \{x\}) \cap S = \emptyset$. Applying the separating hyperplane theorem to the convex sets $(x-K) \setminus \{x\}$ and S, we conclude that there is a $\lambda \neq 0$ and μ such that $\lambda^T(x-y) \le \mu$ for all $y \in K$, and $\lambda^T z \ge \mu$ for all $z \in S$. From the first inequality we conclude $\lambda \succeq_{K^*} 0$. Since $x \in S$ and $x \in x - K$, we have $\lambda^T x = \mu$, so the second inequality implies that μ is the minimum value of $\lambda^T z$ over S. Therefore, x is a minimizer of $\lambda^T z$ over S, where $\lambda \neq 0$, $\lambda \succeq_{K^*} 0$.

This converse theorem cannot be strengthened to $\lambda \succ_{K^*} 0$. Examples show that a point x can be a minimal point of a convex set S, but not a minimizer of

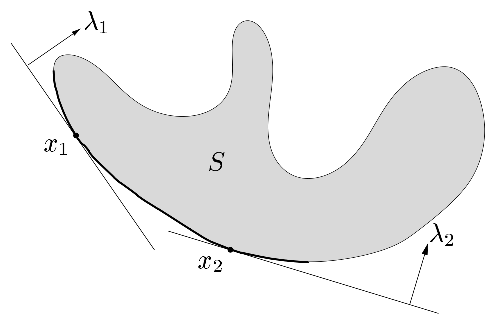


Figure 2.24 A set $S \subseteq \mathbf{R}^2$. Its set of minimal points, with respect to $\mathbf{R}^2_+$, is shown as the darker section of its (lower, left) boundary. The minimizer of $\lambda_1^T z$ over S is x_1, and is minimal since $\lambda_1 \succ 0$. The minimizer of $\lambda_2^T z$ over S is x_2, which is another minimal point of S, since $\lambda_2 \succ 0$.

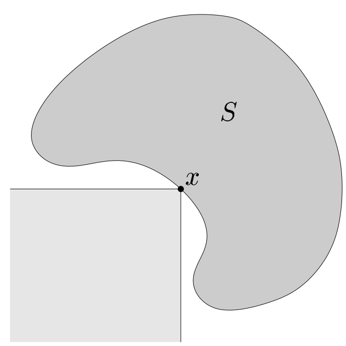


Figure 2.25 The point x is a minimal element of $S \subseteq \mathbf{R}^2$ with respect to $\mathbf{R}^2_+$. However there exists no λ for which x minimizes $\lambda^T z$ over $z \in S$.

Figure 2.26 *Left.* The point $x_1 \in S_1$ is minimal, but is not a minimizer of $\lambda^T z$ over S_1 for any $\lambda \succ 0$. (It does, however, minimize $\lambda^T z$ over $z \in S_1$ for $\lambda = (1, 0)$.) *Right.* The point $x_2 \in S_2$ is *not* minimal, but it does minimize $\lambda^T z$ over $z \in S_2$ for $\lambda = (0, 1) \succeq 0$.

$\lambda^T z$ over $z \in S$ for any $\lambda \succ_{K^*} 0$. (See figure 2.26, left.) Nor is it true that any minimizer of $\lambda^T z$ over $z \in S$, with $\lambda \succeq_{K^*} 0$, is minimal (see figure 2.26, right.)

Example 2.27 *Pareto optimal production frontier.* We consider a product which requires n resources (such as labor, electricity, natural gas, water) to manufacture. The product can be manufactured or produced in many ways. With each production method, we associate a *resource vector* $x \in \mathbf{R}^n$, where x_i denotes the amount of resource i consumed by the method to manufacture the product. We assume that $x_i \geq 0$ (*i.e.*, resources are consumed by the production methods) and that the resources are valuable (so using less of any resource is preferred).

The *production set* $P \subseteq \mathbf{R}^n$ is defined as the set of all resource vectors x that correspond to some production method.

Production methods with resource vectors that are minimal elements of P, with respect to componentwise inequality, are called *Pareto optimal* or *efficient*. The set of minimal elements of P is called the *efficient production frontier*.

We can give a simple interpretation of Pareto optimality. We say that one production method, with resource vector x, is *better* than another, with resource vector y, if $x_i \leq y_i$ for all i, and for some i, $x_i < y_i$. In other words, one production method is better than another if it uses no more of each resource than another method, and for at least one resource, actually uses less. This corresponds to $x \preceq y$, $x \neq y$. Then we can say: A production method is Pareto optimal or efficient if there is no better production method.

We can find Pareto optimal production methods (*i.e.*, minimal resource vectors) by minimizing

$$\lambda^T x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

over the set P of production vectors, using any λ that satisfies $\lambda \succ 0$.

Here the vector λ has a simple interpretation: λ_i is the *price* of resource i. By minimizing $\lambda^T x$ over P we are finding the overall cheapest production method (for the resource prices λ_i). As long as the prices are positive, the resulting production method is guaranteed to be efficient.

These ideas are illustrated in figure 2.27.

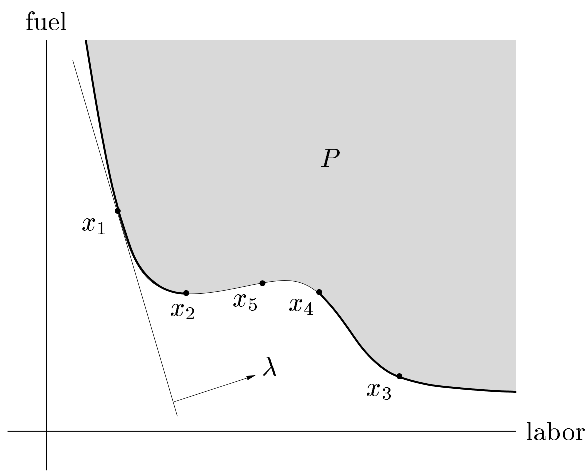


Figure 2.27 The production set P, for a product that requires labor and fuel to produce, is shown shaded. The two dark curves show the efficient production frontier. The points x_1, x_2 and x_3 are efficient. The points x_4 and x_5 are not (since in particular, x_2 corresponds to a production method that uses no more fuel, and less labor). The point x_1 is also the minimum cost production method for the price vector λ (which is positive). The point x_2 is efficient, but cannot be found by minimizing the total cost $\lambda^T x$ for any price vector $\lambda \succeq 0$.

Bibliography

Minkowski is generally credited with the first systematic study of convex sets, and the introduction of fundamental concepts such as supporting hyperplanes and the supporting hyperplane theorem, the Minkowski distance function (exercise 3.34), extreme points of a convex set, and many others.

Some well known early surveys are Bonnesen and Fenchel [BF48], Eggleston [Egg58], Klee [Kle63], and Valentine [Val64]. More recent books devoted to the geometry of convex sets include Lay [Lay82] and Webster [Web94]. Klee [Kle71], Fenchel [Fen83], Tikhomorov [Tik90], and Berger [Ber90] give very readable overviews of the history of convexity and its applications throughout mathematics.

Linear inequalities and polyhedral sets are studied extensively in connection with the linear programming problem, for which we give references at the end of chapter 4. Some landmark publications in the history of linear inequalities and linear programming are Motzkin [Mot33], von Neumann and Morgenstern [vNM53], Kantorovich [Kan60], Koopmans [Koo51], and Dantzig [Dan63]. Dantzig [Dan63, Chapter 2] includes an historical survey of linear inequalities, up to around 1963.

Generalized inequalities were introduced in nonlinear optimization during the 1960s (see Luenberger [Lue69, §8.2] and Isii [Isi64]), and are used extensively in cone programming (see the references in chapter 4). Bellman and Fan [BF63] is an early paper on sets of generalized linear inequalities (with respect to the positive semidefinite cone).

For extensions and a proof of the separating hyperplane theorem we refer the reader to Rockafellar [Roc70, part III], and Hiriart-Urruty and Lemaréchal [HUL93, volume 1, §III4]. Dantzig [Dan63, page 21] attributes the term *theorem of the alternative* to von Neumann and Morgenstern [vNM53, page 138]. For more references on theorems of alternatives, see chapter 5.

The terminology of example 2.27 (including Pareto optimality, efficient production, and the price interpretation of λ) is discussed in detail by Luenberger [Lue95].

Convex geometry plays a prominent role in the classical theory of moments (Krein and Nudelman [KN77], Karlin and Studden [KS66]). A famous example is the duality between the cone of nonnegative polynomials and the cone of power moments; see exercise 2.37.

Exercises

Definition of convexity

2.1 Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbf{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint.* Use induction on k.

2.2 Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

2.3 *Midpoint convexity.* A set C is *midpoint convex* if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. It can be proved that under mild conditions midpoint convexity implies convexity. As a simple case, prove that if C is closed and midpoint convex, then C is convex.

2.4 Show that the convex hull of a set S is the intersection of all convex sets that contain S. (The same method can be used to show that the conic, or affine, or linear hull of a set S is the intersection of all conic sets, or affine sets, or subspaces that contain S.)

Examples

2.5 What is the distance between two parallel hyperplanes $\{x \in \mathbf{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbf{R}^n \mid a^T x = b_2\}$?

2.6 *When does one halfspace contain another?* Give conditions under which

$$\{x \mid a^T x \leq b\} \subseteq \{x \mid \tilde{a}^T x \leq \tilde{b}\}$$

(where $a \neq 0$, $\tilde{a} \neq 0$). Also find the conditions under which the two halfspaces are equal.

2.7 *Voronoi description of halfspace.* Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, *i.e.*, $\{x \mid \|x-a\|_2 \leq \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

2.8 Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x \mid Ax \preceq b,\ Fx = g\}$.

(a) $S = \{y_1 a_1 + y_2 a_2 \mid -1 \leq y_1 \leq 1,\ -1 \leq y_2 \leq 1\}$, where $a_1, a_2 \in \mathbf{R}^n$.

(b) $S = \{x \in \mathbf{R}^n \mid x \succeq 0,\ \mathbf{1}^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, \ldots, a_n \in \mathbf{R}$ and $b_1, b_2 \in \mathbf{R}$.

(c) $S = \{x \in \mathbf{R}^n \mid x \succeq 0,\ x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$.

(d) $S = \{x \in \mathbf{R}^n \mid x \succeq 0,\ x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^n |y_i| = 1\}$.

2.9 *Voronoi sets and polyhedral decomposition.* Let $x_0, \ldots, x_K \in \mathbf{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, *i.e.*,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2,\ i = 1, \ldots, K\}.$$

V is called the *Voronoi region* around x_0 with respect to $x_1, \ldots, x_K$.

(a) Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$.

(b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \ldots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \ldots, x_K$.

(c) We can also consider the sets

$$V_k = \{x \in \mathbf{R}^n \mid \|x - x_k\|_2 \leq \|x - x_i\|_2,\ i \neq k\}.$$

The set V_k consists of points in $\mathbf{R}^n$ for which the closest point in the set $\{x_0, \ldots, x_K\}$ is x_k.

The sets $V_0, \ldots, V_K$ give a polyhedral decomposition of $\mathbf{R}^n$. More precisely, the sets V_k are polyhedra, $\bigcup_{k=0}^K V_k = \mathbf{R}^n$, and $\mathbf{int}\, V_i \cap \mathbf{int}\, V_j = \emptyset$ for $i \neq j$, *i.e.*, V_i and V_j intersect at most along a boundary.

Suppose that $P_1, \ldots, P_m$ are polyhedra such that $\bigcup_{i=1}^m P_i = \mathbf{R}^n$, and $\mathbf{int}\, P_i \cap \mathbf{int}\, P_j = \emptyset$ for $i \neq j$. Can this polyhedral decomposition of $\mathbf{R}^n$ be described as the Voronoi regions generated by an appropriate set of points?

2.10 *Solution set of a quadratic inequality.* Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbf{R}^n \mid x^T A x + b^T x + c \leq 0\},$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.

(a) Show that C is convex if $A \succeq 0$.

(b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbf{R}$.

Are the converses of these statements true?

2.11 *Hyperbolic sets.* Show that the *hyperbolic set* $\{x \in \mathbf{R}_+^2 \mid x_1 x_2 \geq 1\}$ is convex. As a generalization, show that $\{x \in \mathbf{R}_+^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex. *Hint.* If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$; see §3.1.9.

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

2.13 *Conic hull of outer products.* Consider the set of rank-k *outer products*, defined as $\{XX^T \mid X \in \mathbf{R}^{n \times k},\ \mathbf{rank}\, X = k\}$. Describe its conic hull in simple terms.

2.14 *Expanded and restricted sets.* Let $S \subseteq \mathbf{R}^n$, and let $\|\cdot\|$ be a norm on $\mathbf{R}^n$.

(a) For $a \geq 0$ we define S_a as $\{x \mid \mathbf{dist}(x, S) \leq a\}$, where $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$. We refer to S_a as S *expanded* or *extended* by a. Show that if S is convex, then S_a is convex.

(b) For $a \geq 0$ we define $S_{-a} = \{x \mid B(x, a) \subseteq S\}$, where $B(x, a)$ is the ball (in the norm $\|\cdot\|$), centered at x, with radius a. We refer to S_{-a} as S *shrunk* or *restricted* by a, since S_{-a} consists of all points that are at least a distance a from $\mathbf{R}^n \backslash S$. Show that if S is convex, then S_{-a} is convex.

2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbf{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

(a) $\alpha \leq \mathbf{E} f(x) \leq \beta$, where $\mathbf{E} f(x)$ is the expected value of $f(x)$, *i.e.*, $\mathbf{E} f(x) = \sum_{i=1}^n p_i f(a_i)$. (The function $f : \mathbf{R} \to \mathbf{R}$ is given.)

(b) $\mathbf{prob}(x > \alpha) \leq \beta$.

(c) $\mathbf{E}|x^3| \leq \alpha\, \mathbf{E}|x|$.

(d) $\mathbf{E}\, x^2 \leq \alpha$.

(e) $\mathbf{E}\, x^2 \geq \alpha$.

(f) $\mathbf{var}(x) \leq \alpha$, where $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}\, x)^2$ is the variance of x.

(g) $\mathbf{var}(x) \geq \alpha$.

(h) $\mathbf{quartile}(x) \geq \alpha$, where $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(i) $\mathbf{quartile}(x) \leq \alpha$.

Operations that preserve convexity

2.16 Show that if S_1 and S_2 are convex sets in $\mathbf{R}^{m \times n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbf{R}^m,\ y_1,\ y_2 \in \mathbf{R}^n, (x, y_1) \in S_1,\ (x, y_2) \in S_2\}.$$

2.17 *Image of polyhedral sets under perspective function.* In this problem we study the image of hyperplanes, halfspaces, and polyhedra under the perspective function $P(x, t) = x/t$, with $\mathbf{dom}\, P = \mathbf{R}^n \times \mathbf{R}_{++}$. For each of the following sets C, give a simple description of

$$P(C) = \{v/t \mid (v, t) \in C,\ t > 0\}.$$

(a) The polyhedron $C = \mathbf{conv}\{(v_1, t_1), \ldots, (v_K, t_K)\}$ where $v_i \in \mathbf{R}^n$ and $t_i > 0$.

(b) The hyperplane $C = \{(v, t) \mid f^T v + gt = h\}$ (with f and g not both zero).

(c) The halfspace $C = \{(v, t) \mid f^T v + gt \leq h\}$ (with f and g not both zero).

(d) The polyhedron $C = \{(v, t) \mid Fv + gt \preceq h\}$.

2.18 *Invertible linear-fractional functions.* Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be the linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}.$$

Suppose the matrix

$$Q = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$$

is nonsingular. Show that f is invertible and that f^{-1} is a linear-fractional mapping. Give an explicit expression for f^{-1} and its domain in terms of A, b, c, and d. *Hint.* It may be easier to express f^{-1} in terms of Q.

2.19 *Linear-fractional functions and convex sets.* Let $f : \mathbf{R}^m \to \mathbf{R}^n$ be the linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}.$$

In this problem we study the inverse image of a convex set C under f, *i.e.*,

$$f^{-1}(C) = \{x \in \mathbf{dom}\, f \mid f(x) \in C\}.$$

For each of the following sets $C \subseteq \mathbf{R}^n$, give a simple description of $f^{-1}(C)$.

(a) The halfspace $C = \{y \mid g^T y \leq h\}$ (with $g \neq 0$).

(b) The polyhedron $C = \{y \mid Gy \preceq h\}$.

(c) The ellipsoid $\{y \mid y^T P^{-1} y \leq 1\}$ (where $P \in \mathbf{S}^n_{++}$).

(d) The solution set of a linear matrix inequality, $C = \{y \mid y_1 A_1 + \cdots + y_n A_n \preceq B\}$, where $A_1, \ldots, A_n, B \in \mathbf{S}^p$.

Separation theorems and supporting hyperplanes

2.20 *Strictly positive solution of linear equations.* Suppose $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$. Show that there exists an x satisfying

$$x \succ 0, \qquad Ax = b$$

if and only if there exists no λ with

$$A^T\lambda \succeq 0, \qquad A^T\lambda \neq 0, \qquad b^T\lambda \leq 0.$$

Hint. First prove the following fact from linear algebra: $c^T x = d$ for all x satisfying $Ax = b$ if and only if there is a vector λ such that $c = A^T\lambda$, $d = b^T\lambda$.

2.21 *The set of separating hyperplanes.* Suppose that C and D are disjoint subsets of $\mathbf{R}^n$. Consider the set of $(a, b) \in \mathbf{R}^{n+1}$ for which $a^T x \leq b$ for all $x \in C$, and $a^T x \geq b$ for all $x \in D$. Show that this set is a convex cone (which is the singleton $\{0\}$ if there is no hyperplane that separates C and D).

2.22 Finish the proof of the separating hyperplane theorem in §2.5.1: Show that a separating hyperplane exists for two disjoint convex sets C and D. You can use the result proved in §2.5.1, *i.e.*, that a separating hyperplane exists when there exist points in the two sets whose distance is equal to the distance between the two sets.

Hint. If C and D are disjoint convex sets, then the set $\{x - y \mid x \in C,\ y \in D\}$ is convex and does not contain the origin.

2.23 Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

2.24 *Supporting hyperplanes.*

(a) Express the closed convex set $\{x \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

(b) Let $C = \{x \in \mathbf{R}^n \mid \|x\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbf{R}^n$, and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

2.25 *Inner and outer polyhedral approximations.* Let $C \subseteq \mathbf{R}^n$ be a closed convex set, and suppose that $x_1, \ldots, x_K$ are on the boundary of C. Suppose that for each i, $a_i^T(x - x_i) = 0$ defines a supporting hyperplane for C at x_i, *i.e.*, $C \subseteq \{x \mid a_i^T(x - x_i) \leq 0\}$. Consider the two polyhedra

$$P_{\text{inner}} = \mathbf{conv}\{x_1, \ldots, x_K\}, \qquad P_{\text{outer}} = \{x \mid a_i^T(x - x_i) \leq 0,\ i = 1, \ldots, K\}.$$

Show that $P_{\text{inner}} \subseteq C \subseteq P_{\text{outer}}$. Draw a picture illustrating this.

2.26 *Support function.* The support function of a set $C \subseteq \mathbf{R}^n$ is defined as

$$S_C(y) = \sup\{y^T x \mid x \in C\}.$$

(We allow $S_C(y)$ to take on the value $+\infty$.) Suppose that C and D are closed convex sets in $\mathbf{R}^n$. Show that $C = D$ if and only if their support functions are equal.

2.27 *Converse supporting hyperplane theorem.* Suppose the set C is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary. Show that C is convex.

Convex cones and generalized inequalities

2.28 *Positive semidefinite cone for* $n = 1,\ 2,\ 3$. Give an explicit description of the positive semidefinite cone $\mathbf{S}^n_+$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1,\ 2,\ 3$. To describe a general element of $\mathbf{S}^n$, for $n = 1,\ 2,\ 3$, use the notation

$$x_1, \qquad \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

2.29 *Cones in* $\mathbf{R}^2$. Suppose $K \subseteq \mathbf{R}^2$ is a closed convex cone.

(a) Give a simple description of K in terms of the polar coordinates of its elements ($x = r(\cos\phi, \sin\phi)$ with $r \geq 0$).

(b) Give a simple description of K^*, and draw a plot illustrating the relation between K and K^*.

(c) When is K pointed?

(d) When is K proper (hence, defines a generalized inequality)? Draw a plot illustrating what $x \preceq_K y$ means when K is proper.

2.30 *Properties of generalized inequalities.* Prove the properties of (nonstrict and strict) generalized inequalities listed in §2.4.1.

2.31 *Properties of dual cones.* Let K^* be the dual cone of a convex cone K, as defined in (2.19). Prove the following.

(a) K^* is indeed a convex cone.

(b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.

(c) K^* is closed.

(d) The interior of K^* is given by $\mathbf{int}\, K^* = \{y \mid y^T x > 0 \text{ for all } x \in K\}$.

(e) If K has nonempty interior then K^* is pointed.

(f) K^{**} is the closure of K. (Hence if K is closed, $K^{**} = K$.)

(g) If the closure of K is pointed then K^* has nonempty interior.

2.32 Find the dual cone of $\{Ax \mid x \succeq 0\}$, where $A \in \mathbf{R}^{m\times n}$.

2.33 *The monotone nonnegative cone.* We define the *monotone nonnegative cone* as

$$K_{\text{m+}} = \{x \in \mathbf{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

i.e., all nonnegative vectors with components sorted in nonincreasing order.

(a) Show that $K_{\text{m+}}$ is a proper cone.

(b) Find the dual cone $K^*_{\text{m+}}$. *Hint.* Use the identity

$$\begin{aligned} \sum_{i=1}^n x_i y_i &= (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \cdots \\ &\quad + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n). \end{aligned}$$

2.34 *The lexicographic cone and ordering.* The *lexicographic cone* is defined as

$$K_{\text{lex}} = \{0\} \cup \{x \in \mathbf{R}^n \mid x_1 = \cdots = x_k = 0,\ x_{k+1} > 0,\ \text{for some } k,\ 0 \leq k < n\},$$

i.e., all vectors whose first nonzero coefficient (if any) is positive.

(a) Verify that K_{lex} is a cone, but *not* a proper cone.

(b) We define the *lexicographic ordering* on $\mathbf{R}^n$ as follows: $x \leq_{\rm lex} y$ if and only if $y - x \in K_{\rm lex}$. (Since $K_{\rm lex}$ is not a proper cone, the lexicographic ordering is not a generalized inequality.) Show that the lexicographic ordering is a *linear ordering*: for any $x,\ y \in \mathbf{R}^n$, either $x \leq_{\rm lex} y$ or $y \leq_{\rm lex} x$. Therefore any set of vectors can be sorted with respect to the lexicographic cone, which yields the familiar sorting used in dictionaries.

(c) Find $K_{\rm lex}^*$.

2.35 *Copositive matrices.* A matrix $X \in \mathbf{S}^n$ is called *copositive* if $z^T X z \geq 0$ for all $z \succeq 0$. Verify that the set of copositive matrices is a proper cone. Find its dual cone.

2.36 *Euclidean distance matrices.* Let $x_1, \ldots, x_n \in \mathbf{R}^k$. The matrix $D \in \mathbf{S}^n$ defined by $D_{ij} = \|x_i - x_j\|_2^2$ is called a *Euclidean distance matrix.* It satisfies some obvious properties such as $D_{ij} = D_{ji}$, $D_{ii} = 0$, $D_{ij} \geq 0$, and (from the triangle inequality) $D_{ik}^{1/2} \leq D_{ij}^{1/2} + D_{jk}^{1/2}$. We now pose the question: When is a matrix $D \in \mathbf{S}^n$ a Euclidean distance matrix (for some points in $\mathbf{R}^k$, for some k)? A famous result answers this question: $D \in \mathbf{S}^n$ is a Euclidean distance matrix if and only if $D_{ii} = 0$ and $x^T D x \leq 0$ for all x with $\mathbf{1}^T x = 0$. (See §8.3.3.)

Show that the set of Euclidean distance matrices is a convex cone. Find the dual cone.

2.37 *Nonnegative polynomials and Hankel LMIs.* Let $K_{\rm pol}$ be the set of (coefficients of) nonnegative polynomials of degree $2k$ on $\mathbf{R}$:

$$K_{\rm pol} = \{x \in \mathbf{R}^{2k+1} \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k} \geq 0 \text{ for all } t \in \mathbf{R}\}.$$

(a) Show that $K_{\rm pol}$ is a proper cone.

(b) A basic result, known as the *Markov-Lucacs theorem*, states that a polynomial of degree $2k$ is nonnegative on $\mathbf{R}$ if and only if it can be expressed as the sum of squares of two polynomials of degree k or less. In other words, $x \in K_{\rm pol}$ if and only if the polynomial $p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k}$ can be expressed as

$$p(t) = r(t)^2 + s(t)^2$$

where r and s are polynomials of degree k.

Use the Markov-Lucacs theorem to show that

$$K_{\rm pol} = \left\{ x \in \mathbf{R}^{2k+1} \;\middle|\; x_i = \sum_{m+n=i+1} Y_{mn} \text{ for some } Y \in \mathbf{S}_+^{k+1} \right\}.$$

In other words, $p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k}$ is nonnegative if and only if there exists a matrix $Y \in \mathbf{S}_+^{k+1}$ such that

$$\begin{array}{rcl} x_1 & = & Y_{11} \\ x_2 & = & Y_{12} + Y_{21} \\ x_3 & = & Y_{13} + Y_{22} + Y_{31} \\ & \vdots & \\ x_{2k+1} & = & Y_{k+1,k+1}. \end{array}$$

(c) Show that $K_{\rm pol}^* = K_{\rm han}$ where

$$K_{\rm han} = \{z \in \mathbf{R}^{2k+1} \mid H(z) \succeq 0\}$$

and

$$H(z) = \begin{bmatrix} z_1 & z_2 & z_3 & \cdots & z_k & z_{k+1} \\ z_2 & z_3 & z_4 & \cdots & z_{k+1} & z_{k+2} \\ z_3 & z_4 & z_5 & \cdots & z_{k+2} & z_{k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z_k & z_{k+1} & z_{k+2} & \cdots & z_{2k-1} & z_{2k} \\ z_{k+1} & z_{k+2} & z_{k+3} & \cdots & z_{2k} & z_{2k+1} \end{bmatrix}.$$

(This is the *Hankel matrix* with coefficients $z_1, \dots, z_{2k+1}$.)

(d) Let $K_{\rm mom}$ be the conic hull of the set of all vectors of the form $(1, t, t^2, \dots, t^{2k})$, where $t \in \mathbf{R}$. Show that $y \in K_{\rm mom}$ if and only if $y_1 \geq 0$ and

$$y = y_1(1, \mathbf{E}\,u, \mathbf{E}\,u^2, \dots, \mathbf{E}\,u^{2k})$$

for some random variable u. In other words, the elements of $K_{\rm mom}$ are nonnegative multiples of the moment vectors of all possible distributions on $\mathbf{R}$. Show that $K_{\rm pol} = K_{\rm mom}^*$.

(e) Combining the results of (c) and (d), conclude that $K_{\rm han} = \mathbf{cl}\, K_{\rm mom}$.

As an example illustrating the relation between $K_{\rm mom}$ and $K_{\rm han}$, take $k = 2$ and $z = (1, 0, 0, 0, 1)$. Show that $z \in K_{\rm han}$, $z \notin K_{\rm mom}$. Find an explicit sequence of points in $K_{\rm mom}$ which converge to z.

2.38 [Roc70, pages 15, 61] *Convex cones constructed from sets.*

(a) The *barrier cone* of a set C is defined as the set of all vectors y such that $y^T x$ is bounded above over $x \in C$. In other words, a nonzero vector y is in the barrier cone if and only if it is the normal vector of a halfspace $\{x \mid y^T x \leq \alpha\}$ that contains C. Verify that the barrier cone is a convex cone (with no assumptions on C).

(b) The *recession cone* (also called *asymptotic cone*) of a set C is defined as the set of all vectors y such that for each $x \in C$, $x - ty \in C$ for all $t \geq 0$. Show that the recession cone of a convex set is a convex cone. Show that if C is nonempty, closed, and convex, then the recession cone of C is the dual of the barrier cone.

(c) The *normal cone* of a set C at a boundary point x_0 is the set of all vectors y such that $y^T(x - x_0) \leq 0$ for all $x \in C$ (*i.e.*, the set of vectors that define a supporting hyperplane to C at x_0). Show that the normal cone is a convex cone (with no assumptions on C). Give a simple description of the normal cone of a polyhedron $\{x \mid Ax \preceq b\}$ at a point in its boundary.

2.39 *Separation of cones.* Let K and $\tilde{K}$ be two convex cones whose interiors are nonempty and disjoint. Show that there is a nonzero y such that $y \in K^*$, $-y \in \tilde{K}^*$.

Chapter 3

Convex functions

3.1 Basic properties and examples

3.1.1 Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is *convex* if $\mathbf{dom}\, f$ is a convex set and if for all x, $y \in \mathbf{dom}\, f$, and θ with $0 \le \theta \le 1$, we have

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y). \tag{3.1}$$

Geometrically, this inequality means that the line segment between $(x, f(x))$ and $(y, f(y))$, which is the *chord* from x to y, lies above the graph of f (figure 3.1). A function f is *strictly convex* if strict inequality holds in (3.1) whenever $x \ne y$ and $0 < \theta < 1$. We say f is *concave* if $-f$ is convex, and *strictly concave* if $-f$ is strictly convex.

For an affine function we always have equality in (3.1), so all affine (and therefore also linear) functions are both convex and concave. Conversely, any function that is convex and concave is affine.

A function is convex if and only if it is convex when restricted to any line that intersects its domain. In other words f is convex if and only if for all $x \in \mathbf{dom}\, f$ and

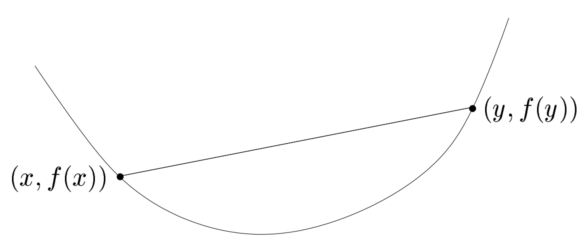


Figure 3.1 Graph of a convex function. The chord (*i.e.*, line segment) between any two points on the graph lies above the graph.

all v, the function $g(t) = f(x+tv)$ is convex (on its domain, $\{t \mid x+tv \in \mathbf{dom}\, f\}$). This property is very useful, since it allows us to check whether a function is convex by restricting it to a line.

The *analysis* of convex functions is a well developed field, which we will not pursue in any depth. One simple result, for example, is that a convex function is continuous on the relative interior of its domain; it can have discontinuities only on its relative boundary.

3.1.2 Extended-value extensions

It is often convenient to extend a convex function to all of $\mathbf{R}^n$ by defining its value to be ∞ outside its domain. If f is convex we define its *extended-value extension* $\tilde{f} : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbf{dom}\, f \\ \infty & x \notin \mathbf{dom}\, f. \end{cases}$$

The extension $\tilde{f}$ is defined on all $\mathbf{R}^n$, and takes values in $\mathbf{R} \cup \{\infty\}$. We can recover the domain of the original function f from the extension $\tilde{f}$ as $\mathbf{dom}\, f = \{x \mid \tilde{f}(x) < \infty\}$.

The extension can simplify notation, since we do not need to explicitly describe the domain, or add the qualifier 'for all $x \in \mathbf{dom}\, f$' every time we refer to $f(x)$. Consider, for example, the basic defining inequality (3.1). In terms of the extension $\tilde{f}$, we can express it as: for $0 < \theta < 1$,

$$\tilde{f}(\theta x + (1-\theta)y) \leq \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

for *any* x and y. (For $\theta = 0$ or $\theta = 1$ the inequality always holds.) Of course here we must interpret the inequality using extended arithmetic and ordering. For x and y both in $\mathbf{dom}\, f$, this inequality coincides with (3.1); if either is outside $\mathbf{dom}\, f$, then the righthand side is ∞, and the inequality therefore holds. As another example of this notational device, suppose f_1 and f_2 are two convex functions on $\mathbf{R}^n$. The pointwise sum $f = f_1 + f_2$ is the function with domain $\mathbf{dom}\, f = \mathbf{dom}\, f_1 \cap \mathbf{dom}\, f_2$, with $f(x) = f_1(x) + f_2(x)$ for any $x \in \mathbf{dom}\, f$. Using extended-value extensions we can simply say that for any x, $\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$. In this equation the domain of f has been automatically defined as $\mathbf{dom}\, f = \mathbf{dom}\, f_1 \cap \mathbf{dom}\, f_2$, since $\tilde{f}(x) = \infty$ whenever $x \notin \mathbf{dom}\, f_1$ or $x \notin \mathbf{dom}\, f_2$. In this example we are relying on extended arithmetic to automatically define the domain.

In this book we will use the same symbol to denote a convex function and its extension, whenever there is no harm from the ambiguity. This is the same as assuming that all convex functions are implicitly extended, *i.e.*, are defined as ∞ outside their domains.

Example 3.1 *Indicator function of a convex set.* Let $C \subseteq \mathbf{R}^n$ be a convex set, and consider the (convex) function I_C with domain C and $I_C(x) = 0$ for all $x \in C$. In other words, the function is identically zero on the set C. Its extended-value extension

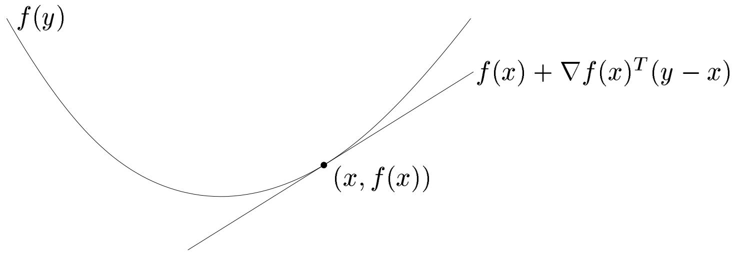


Figure 3.2 If f is convex and differentiable, then $f(x)+\nabla f(x)^T(y-x) \leq f(y)$ for all $x,\ y \in \mathbf{dom}\, f$.

is given by

$$\tilde{I}_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C. \end{cases}$$

The convex function $\tilde{I}_C$ is called the *indicator function* of the set C.

We can play several notational tricks with the indicator function $\tilde{I}_C$. For example the problem of minimizing a function f (defined on all of $\mathbf{R}^n$, say) on the set C is the same as minimizing the function $f + \tilde{I}_C$ over all of $\mathbf{R}^n$. Indeed, the function $f + \tilde{I}_C$ is (by our convention) f restricted to the set C.

In a similar way we can extend a concave function by defining it to be $-\infty$ outside its domain.

3.1.3 First-order conditions

Suppose f is differentiable (*i.e.*, its gradient ∇f exists at each point in $\mathbf{dom}\, f$, which is open). Then f is convex if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^T(y-x) \tag{3.2}$$

holds for all $x,\ y \in \mathbf{dom}\, f$. This inequality is illustrated in figure 3.2.

The affine function of y given by $f(x)+\nabla f(x)^T(y-x)$ is, of course, the first-order Taylor approximation of f near x. The inequality (3.2) states that for a convex function, the first-order Taylor approximation is in fact a *global underestimator* of the function. Conversely, if the first-order Taylor approximation of a function is always a global underestimator of the function, then the function is convex.

The inequality (3.2) shows that from *local information* about a convex function (*i.e.*, its value and derivative at a point) we can derive *global information* (*i.e.*, a global underestimator of it). This is perhaps the most important property of convex functions, and explains some of the remarkable properties of convex functions and convex optimization problems. As one simple example, the inequality (3.2) shows that if $\nabla f(x) = 0$, then for all $y \in \mathbf{dom}\, f$, $f(y) \geq f(x)$, *i.e.*, x is a global minimizer of the function f.

Strict convexity can also be characterized by a first-order condition: f is strictly convex if and only if $\mathbf{dom}\, f$ is convex and for $x,\ y \in \mathbf{dom}\, f$, $x \neq y$, we have

$$f(y) > f(x) + \nabla f(x)^T (y - x). \tag{3.3}$$

For concave functions we have the corresponding characterization: f is concave if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) \leq f(x) + \nabla f(x)^T (y - x)$$

for all $x,\ y \in \mathbf{dom}\, f$.

Proof of first-order convexity condition

To prove (3.2), we first consider the case $n = 1$: We show that a differentiable function $f : \mathbf{R} \to \mathbf{R}$ is convex if and only if

$$f(y) \geq f(x) + f'(x)(y - x) \tag{3.4}$$

for all x and y in $\mathbf{dom}\, f$.

Assume first that f is convex and $x,\ y \in \mathbf{dom}\, f$. Since $\mathbf{dom}\, f$ is convex (*i.e.*, an interval), we conclude that for all $0 < t \leq 1$, $x + t(y - x) \in \mathbf{dom}\, f$, and by convexity of f,

$$f(x + t(y - x)) \leq (1 - t)f(x) + tf(y).$$

If we divide both sides by t, we obtain

$$f(y) \geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t},$$

and taking the limit as $t \to 0$ yields (3.4).

To show sufficiency, assume the function satisfies (3.4) for all x and y in $\mathbf{dom}\, f$ (which is an interval). Choose any $x \neq y$, and $0 \leq \theta \leq 1$, and let $z = \theta x + (1 - \theta)y$. Applying (3.4) twice yields

$$f(x) \geq f(z) + f'(z)(x - z), \qquad f(y) \geq f(z) + f'(z)(y - z).$$

Multiplying the first inequality by θ, the second by $1 - \theta$, and adding them yields

$$\theta f(x) + (1 - \theta)f(y) \geq f(z),$$

which proves that f is convex.

Now we can prove the general case, with $f : \mathbf{R}^n \to \mathbf{R}$. Let $x,\ y \in \mathbf{R}^n$ and consider f restricted to the line passing through them, *i.e.*, the function defined by $g(t) = f(ty + (1 - t)x)$, so $g'(t) = \nabla f(ty + (1 - t)x)^T (y - x)$.

First assume f is convex, which implies g is convex, so by the argument above we have $g(1) \geq g(0) + g'(0)$, which means

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

Now assume that this inequality holds for any x and y, so if $ty + (1 - t)x \in \mathbf{dom}\, f$ and $\tilde{t}y + (1 - \tilde{t})x \in \mathbf{dom}\, f$, we have

$$f(ty + (1 - t)x) \geq f(\tilde{t}y + (1 - \tilde{t})x) + \nabla f(\tilde{t}y + (1 - \tilde{t})x)^T (y - x)(t - \tilde{t}),$$

i.e., $g(t) \geq g(\tilde{t}) + g'(\tilde{t})(t - \tilde{t})$. We have seen that this implies that g is convex.

3.1.4 Second-order conditions

We now assume that f is twice differentiable, that is, its *Hessian* or second derivative $\nabla^2 f$ exists at each point in $\mathbf{dom}\, f$, which is open. Then f is convex if and only if $\mathbf{dom}\, f$ is convex and its Hessian is positive semidefinite: for all $x \in \mathbf{dom}\, f$,

$$\nabla^2 f(x) \succeq 0.$$

For a function on $\mathbf{R}$, this reduces to the simple condition $f''(x) \geq 0$ (and $\mathbf{dom}\, f$ convex, *i.e.*, an interval), which means that the derivative is nondecreasing. The condition $\nabla^2 f(x) \succeq 0$ can be interpreted geometrically as the requirement that the graph of the function have positive (upward) curvature at x. We leave the proof of the second-order condition as an exercise (exercise 3.8).

Similarly, f is concave if and only if $\mathbf{dom}\, f$ is convex and $\nabla^2 f(x) \preceq 0$ for all $x \in \mathbf{dom}\, f$. Strict convexity can be partially characterized by second-order conditions. If $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex. The converse, however, is not true: for example, the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^4$ is strictly convex but has zero second derivative at $x = 0$.

Example 3.2 *Quadratic functions.* Consider the quadratic function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathbf{dom}\, f = \mathbf{R}^n$, given by

$$f(x) = (1/2)x^T Px + q^T x + r,$$

with $P \in \mathbf{S}^n$, $q \in \mathbf{R}^n$, and $r \in \mathbf{R}$. Since $\nabla^2 f(x) = P$ for all x, f is convex if and only if $P \succeq 0$ (and concave if and only if $P \preceq 0$).

For quadratic functions, strict convexity is easily characterized: f is strictly convex if and only if $P \succ 0$ (and strictly concave if and only if $P \prec 0$).

Remark 3.1 The separate requirement that $\mathbf{dom}\, f$ be convex cannot be dropped from the first- or second-order characterizations of convexity and concavity. For example, the function $f(x) = 1/x^2$, with $\mathbf{dom}\, f = \{x \in \mathbf{R} \mid x \neq 0\}$, satisfies $f''(x) > 0$ for all $x \in \mathbf{dom}\, f$, but is not a convex function.

3.1.5 Examples

We have already mentioned that all linear and affine functions are convex (and concave), and have described the convex and concave quadratic functions. In this section we give a few more examples of convex and concave functions. We start with some functions on $\mathbf{R}$, with variable x.

- *Exponential.* e^{ax} is convex on $\mathbf{R}$, for any $a \in \mathbf{R}$.
- *Powers.* x^a is convex on $\mathbf{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.
- *Powers of absolute value.* $|x|^p$, for $p \geq 1$, is convex on $\mathbf{R}$.
- *Logarithm.* $\log x$ is concave on $\mathbf{R}_{++}$.

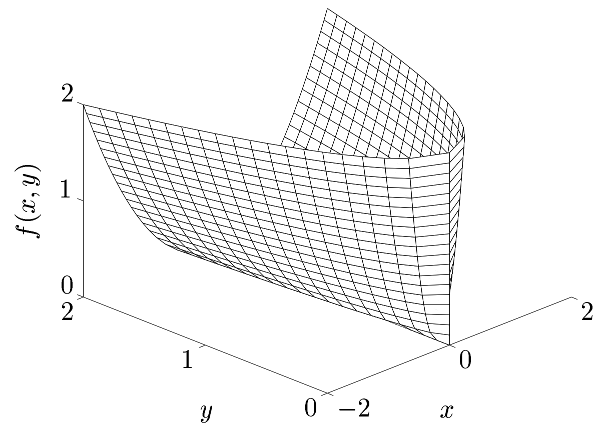


Figure 3.3 Graph of $f(x, y) = x^2/y$.

- *Negative entropy.* $x \log x$ (either on $\mathbf{R}_{++}$, or on $\mathbf{R}_+$, defined as 0 for $x = 0$) is convex.

Convexity or concavity of these examples can be shown by verifying the basic inequality (3.1), or by checking that the second derivative is nonnegative or nonpositive. For example, with $f(x) = x \log x$ we have

$$f'(x) = \log x + 1, \qquad f''(x) = 1/x,$$

so that $f''(x) > 0$ for $x > 0$. This shows that the negative entropy function is (strictly) convex.

We now give a few interesting examples of functions on $\mathbf{R}^n$.

- *Norms.* Every norm on $\mathbf{R}^n$ is convex.
- *Max function.* $f(x) = \max\{x_1, \ldots, x_n\}$ is convex on $\mathbf{R}^n$.
- *Quadratic-over-linear function.* The function $f(x, y) = x^2/y$, with

$$\mathbf{dom}\, f = \mathbf{R} \times \mathbf{R}_{++} = \{(x, y) \in \mathbf{R}^2 \mid y > 0\},$$

  is convex (figure 3.3).
- *Log-sum-exp.* The function $f(x) = \log(e^{x_1} + \cdots + e^{x_n})$ is convex on $\mathbf{R}^n$. This function can be interpreted as a differentiable (in fact, analytic) approximation of the max function, since

$$\max\{x_1, \ldots, x_n\} \le f(x) \le \max\{x_1, \ldots, x_n\} + \log n$$

  for all x. (The second inequality is tight when all components of x are equal.) Figure 3.4 shows f for $n = 2$.

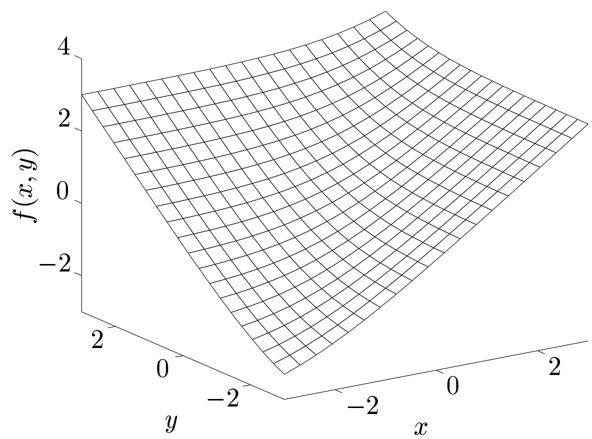


Figure 3.4 Graph of $f(x, y) = \log(e^x + e^y)$.

- *Geometric mean.* The geometric mean $f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{R}_{++}^n$.
- *Log-determinant.* The function $f(X) = \log\det X$ is concave on $\mathbf{dom}\, f = \mathbf{S}_{++}^n$.

Convexity (or concavity) of these examples can be verified in several ways, such as directly verifying the inequality (3.1), verifying that the Hessian is positive semidefinite, or restricting the function to an arbitrary line and verifying convexity of the resulting function of one variable.

Norms. If $f : \mathbf{R}^n \to \mathbf{R}$ is a norm, and $0 \leq \theta \leq 1$, then

$$f(\theta x + (1 - \theta)y) \leq f(\theta x) + f((1 - \theta)y) = \theta f(x) + (1 - \theta)f(y).$$

The inequality follows from the triangle inequality, and the equality follows from homogeneity of a norm.

Max function. The function $f(x) = \max_i x_i$ satisfies, for $0 \leq \theta \leq 1$,

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= \max_i(\theta x_i + (1 - \theta)y_i) \\ &\leq \theta \max_i x_i + (1 - \theta) \max_i y_i \\ &= \theta f(x) + (1 - \theta)f(y). \end{aligned}$$

Quadratic-over-linear function. To show that the quadratic-over-linear function $f(x, y) = x^2/y$ is convex, we note that (for $y > 0$),

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0.$$

Log-sum-exp. The Hessian of the log-sum-exp function is

$$\nabla^2 f(x) = \frac{1}{(\mathbf{1}^T z)^2} \left((\mathbf{1}^T z) \,\mathbf{diag}(z) - zz^T \right),$$

where $z = (e^{x_1}, \ldots, e^{x_n})$. To verify that $\nabla^2 f(x) \succeq 0$ we must show that for all v, $v^T \nabla^2 f(x) v \geq 0$, *i.e.*,

$$v^T \nabla^2 f(x) v = \frac{1}{(\mathbf{1}^T z)^2} \left(\left(\sum_{i=1}^n z_i \right) \left(\sum_{i=1}^n v_i^2 z_i \right) - \left(\sum_{i=1}^n v_i z_i \right)^2 \right) \geq 0.$$

But this follows from the Cauchy-Schwarz inequality $(a^T a)(b^T b) \geq (a^T b)^2$ applied to the vectors with components $a_i = v_i \sqrt{z_i}$, $b_i = \sqrt{z_i}$.

Geometric mean. In a similar way we can show that the geometric mean $f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{R}^n_{++}$. Its Hessian $\nabla^2 f(x)$ is given by

$$\frac{\partial^2 f(x)}{\partial x_k^2} = -(n-1) \frac{\left(\prod_{i=1}^n x_i\right)^{1/n}}{n^2 x_k^2}, \qquad \frac{\partial^2 f(x)}{\partial x_k \partial x_l} = \frac{\left(\prod_{i=1}^n x_i\right)^{1/n}}{n^2 x_k x_l} \quad \text{for } k \neq l,$$

and can be expressed as

$$\nabla^2 f(x) = -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2} \left(n \,\mathbf{diag}(1/x_1^2, \ldots, 1/x_n^2) - qq^T \right)$$

where $q_i = 1/x_i$. We must show that $\nabla^2 f(x) \preceq 0$, *i.e.*, that

$$v^T \nabla^2 f(x) v = -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2} \left(n \sum_{i=1}^n v_i^2 / x_i^2 - \left(\sum_{i=1}^n v_i / x_i \right)^2 \right) \leq 0$$

for all v. Again this follows from the Cauchy-Schwarz inequality $(a^T a)(b^T b) \geq (a^T b)^2$, applied to the vectors $a = \mathbf{1}$ and $b_i = v_i / x_i$.

Log-determinant. For the function $f(X) = \log\det X$, we can verify concavity by considering an arbitrary line, given by $X = Z + tV$, where Z, $V \in \mathbf{S}^n$. We define $g(t) = f(Z + tV)$, and restrict g to the interval of values of t for which $Z + tV \succ 0$. Without loss of generality, we can assume that $t = 0$ is inside this interval, *i.e.*, $Z \succ 0$. We have

$$\begin{aligned} g(t) &= \log\det(Z + tV) \\ &= \log\det(Z^{1/2}(I + tZ^{-1/2} V Z^{-1/2}) Z^{1/2}) \\ &= \sum_{i=1}^n \log(1 + t\lambda_i) + \log\det Z \end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $Z^{-1/2} V Z^{-1/2}$. Therefore we have

$$g'(t) = \sum_{i=1}^n \frac{\lambda_i}{1 + t\lambda_i}, \qquad g''(t) = -\sum_{i=1}^n \frac{\lambda_i^2}{(1 + t\lambda_i)^2}.$$

Since $g''(t) \leq 0$, we conclude that f is concave.

3.1.6 Sublevel sets

The α-*sublevel set* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}.$$

Sublevel sets of a convex function are convex, for any value of α. The proof is immediate from the definition of convexity: if $x,\ y \in C_\alpha$, then $f(x) \leq \alpha$ and $f(y) \leq \alpha$, and so $f(\theta x + (1-\theta)y) \leq \alpha$ for $0 \leq \theta \leq 1$, and hence $\theta x + (1-\theta)y \in C_\alpha$.

The converse is not true: a function can have all its sublevel sets convex, but not be a convex function. For example, $f(x) = -e^x$ is not convex on $\mathbf{R}$ (indeed, it is strictly concave) but all its sublevel sets are convex.

If f is concave, then its α-*superlevel set*, given by $\{x \in \mathbf{dom}\, f \mid f(x) \geq \alpha\}$, is a convex set. The sublevel set property is often a good way to establish convexity of a set, by expressing it as a sublevel set of a convex function, or as the superlevel set of a concave function.

Example 3.3 The geometric and arithmetic means of $x \in \mathbf{R}^n_+$ are, respectively,

$$G(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}, \qquad A(x) = \frac{1}{n}\sum_{i=1}^n x_i,$$

(where we take $0^{1/n} = 0$ in our definition of G). The arithmetic-geometric mean inequality states that $G(x) \leq A(x)$.

Suppose $0 \leq \alpha \leq 1$, and consider the set

$$\{x \in \mathbf{R}^n_+ \mid G(x) \geq \alpha A(x)\},$$

i.e., the set of vectors with geometric mean at least as large as a factor α times the arithmetic mean. This set is convex, since it is the 0-superlevel set of the function $G(x) - \alpha A(x)$, which is concave. In fact, the set is positively homogeneous, so it is a convex cone.

3.1.7 Epigraph

The graph of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$\{(x, f(x)) \mid x \in \mathbf{dom}\, f\},$$

which is a subset of $\mathbf{R}^{n+1}$. The *epigraph* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$\mathbf{epi}\, f = \{(x, t) \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\},$$

which is a subset of $\mathbf{R}^{n+1}$. ('Epi' means 'above' so epigraph means 'above the graph'.) The definition is illustrated in figure 3.5.

The link between convex sets and convex functions is via the epigraph: A function is convex if and only if its epigraph is a convex set. A function is concave if and only if its *hypograph*, defined as

$$\mathbf{hypo}\, f = \{(x, t) \mid t \leq f(x)\},$$

is a convex set.

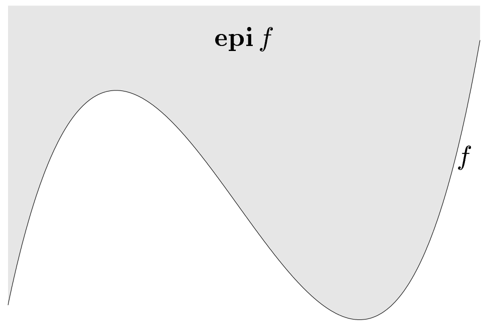


Figure 3.5 Epigraph of a function f, shown shaded. The lower boundary, shown darker, is the graph of f.

Example 3.4 *Matrix fractional function.* The function $f : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$, defined as

$$f(x, Y) = x^T Y^{-1} x$$

is convex on $\mathbf{dom}\, f = \mathbf{R}^n \times \mathbf{S}^n_{++}$. (This generalizes the quadratic-over-linear function $f(x, y) = x^2/y$, with $\mathbf{dom}\, f = \mathbf{R} \times \mathbf{R}_{++}$.)

One easy way to establish convexity of f is via its epigraph:

$$\begin{aligned} \mathbf{epi}\, f &= \{(x, Y, t) \mid Y \succ 0,\ x^T Y^{-1} x \leq t\} \\ &= \left\{ (x, Y, t) \;\middle|\; \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succeq 0,\ Y \succ 0 \right\}, \end{aligned}$$

using the Schur complement condition for positive semidefiniteness of a block matrix (see §A.5.5). The last condition is a linear matrix inequality in (x, Y, t), and therefore $\mathbf{epi}\, f$ is convex.

For the special case $n = 1$, the matrix fractional function reduces to the quadratic-over-linear function x^2/y, and the associated LMI representation is

$$\begin{bmatrix} y & x \\ x & t \end{bmatrix} \succeq 0, \qquad y > 0$$

(the graph of which is shown in figure 3.3).

Many results for convex functions can be proved (or interpreted) geometrically using epigraphs, and applying results for convex sets. As an example, consider the first-order condition for convexity:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x),$$

where f is convex and x, $y \in \mathbf{dom}\, f$. We can interpret this basic inequality geometrically in terms of $\mathbf{epi}\, f$. If $(y, t) \in \mathbf{epi}\, f$, then

$$t \geq f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

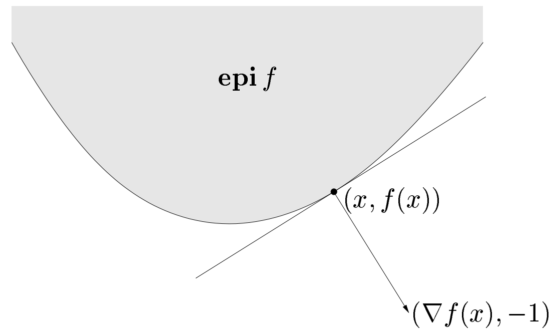


Figure 3.6 For a differentiable convex function f, the vector $(\nabla f(x), -1)$ defines a supporting hyperplane to the epigraph of f at x.

We can express this as:

$$(y, t) \in \mathbf{epi}\, f \implies \begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0.$$

This means that the hyperplane defined by $(\nabla f(x), -1)$ supports $\mathbf{epi}\, f$ at the boundary point $(x, f(x))$; see figure 3.6.

3.1.8 Jensen's inequality and extensions

The basic inequality (3.1), *i.e.*,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y),$$

is sometimes called *Jensen's inequality*. It is easily extended to convex combinations of more than two points: If f is convex, $x_1, \ldots, x_k \in \mathbf{dom}\, f$, and $\theta_1, \ldots, \theta_k \geq 0$ with $\theta_1 + \cdots + \theta_k = 1$, then

$$f(\theta_1 x_1 + \cdots + \theta_k x_k) \leq \theta_1 f(x_1) + \cdots + \theta_k f(x_k).$$

As in the case of convex sets, the inequality extends to infinite sums, integrals, and expected values. For example, if $p(x) \geq 0$ on $S \subseteq \mathbf{dom}\, f$, $\int_S p(x)\, dx = 1$, then

$$f\left(\int_S p(x) x\, dx \right) \leq \int_S f(x) p(x)\, dx,$$

provided the integrals exist. In the most general case we can take any probability measure with support in $\mathbf{dom}\, f$. If x is a random variable such that $x \in \mathbf{dom}\, f$ with probability one, and f is convex, then we have

$$f(\mathbf{E}\, x) \leq \mathbf{E}\, f(x), \tag{3.5}$$

provided the expectations exist. We can recover the basic inequality (3.1) from this general form, by taking the random variable x to have support $\{x_1, x_2\}$, with

$\mathbf{prob}(x = x_1) = \theta$, $\mathbf{prob}(x = x_2) = 1 - \theta$. Thus the inequality (3.5) characterizes convexity: If f is not convex, there is a random variable x, with $x \in \mathbf{dom}\, f$ with probability one, such that $f(\mathbf{E}\, x) > \mathbf{E}\, f(x)$.

All of these inequalities are now called *Jensen's inequality*, even though the inequality studied by Jensen was the very simple one

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

Remark 3.2 We can interpret (3.5) as follows. Suppose $x \in \mathbf{dom}\, f \subseteq \mathbf{R}^n$ and z is any zero mean random vector in $\mathbf{R}^n$. Then we have

$$\mathbf{E}\, f(x+z) \geq f(x).$$

Thus, randomization or dithering (*i.e.*, adding a zero mean random vector to the argument) cannot decrease the value of a convex function on average.

3.1.9 Inequalities

Many famous inequalities can be derived by applying Jensen's inequality to some appropriate convex function. (Indeed, convexity and Jensen's inequality can be made the foundation of a theory of inequalities.) As a simple example, consider the arithmetic-geometric mean inequality:

$$\sqrt{ab} \leq (a+b)/2 \tag{3.6}$$

for $a, b \geq 0$. The function $-\log x$ is convex; Jensen's inequality with $\theta = 1/2$ yields

$$-\log\left(\frac{a+b}{2}\right) \leq \frac{-\log a - \log b}{2}.$$

Taking the exponential of both sides yields (3.6).

As a less trivial example we prove Hölder's inequality: for $p > 1$, $1/p + 1/q = 1$, and $x,\ y \in \mathbf{R}^n$,

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n |y_i|^q\right)^{1/q}.$$

By convexity of $-\log x$, and Jensen's inequality with general θ, we obtain the more general arithmetic-geometric mean inequality

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b,$$

valid for $a,\ b \geq 0$ and $0 \leq \theta \leq 1$. Applying this with

$$a = \frac{|x_i|^p}{\sum_{j=1}^n |x_j|^p}, \qquad b = \frac{|y_i|^q}{\sum_{j=1}^n |y_j|^q}, \qquad \theta = 1/p,$$

yields

$$\left(\frac{|x_i|^p}{\sum_{j=1}^n |x_j|^p}\right)^{1/p} \left(\frac{|y_i|^q}{\sum_{j=1}^n |y_j|^q}\right)^{1/q} \leq \frac{|x_i|^p}{p\sum_{j=1}^n |x_j|^p} + \frac{|y_i|^q}{q\sum_{j=1}^n |y_j|^q}.$$

Summing over i then yields Hölder's inequality.

3.2 Operations that preserve convexity

In this section we describe some operations that preserve convexity or concavity of functions, or allow us to construct new convex and concave functions. We start with some simple operations such as addition, scaling, and pointwise supremum, and then describe some more sophisticated operations (some of which include the simple operations as special cases).

3.2.1 Nonnegative weighted sums

Evidently if f is a convex function and $\alpha \geq 0$, then the function αf is convex. If f_1 and f_2 are both convex functions, then so is their sum $f_1 + f_2$. Combining nonnegative scaling and addition, we see that the set of convex functions is itself a convex cone: a nonnegative weighted sum of convex functions,

$$f = w_1 f_1 + \cdots + w_m f_m,$$

is convex. Similarly, a nonnegative weighted sum of concave functions is concave. A nonnegative, nonzero weighted sum of strictly convex (concave) functions is strictly convex (concave).

These properties extend to infinite sums and integrals. For example if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, and $w(y) \geq 0$ for each $y \in \mathcal{A}$, then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y)\, dy$$

is convex in x (provided the integral exists).

The fact that convexity is preserved under nonnegative scaling and addition is easily verified directly, or can be seen in terms of the associated epigraphs. For example, if $w \geq 0$ and f is convex, we have

$$\mathbf{epi}(wf) = \begin{bmatrix} I & 0 \\ 0 & w \end{bmatrix} \mathbf{epi}\, f,$$

which is convex because the image of a convex set under a linear mapping is convex.

3.2.2 Composition with an affine mapping

Suppose $f : \mathbf{R}^n \to \mathbf{R}$, $A \in \mathbf{R}^{n \times m}$, and $b \in \mathbf{R}^n$. Define $g : \mathbf{R}^m \to \mathbf{R}$ by

$$g(x) = f(Ax + b),$$

with $\mathbf{dom}\, g = \{x \mid Ax + b \in \mathbf{dom}\, f\}$. Then if f is convex, so is g; if f is concave, so is g.

3.2.3 Pointwise maximum and supremum

If f_1 and f_2 are convex functions then their *pointwise maximum* f, defined by

$$f(x) = \max\{f_1(x), f_2(x)\},$$

with $\mathbf{dom}\, f = \mathbf{dom}\, f_1 \cap \mathbf{dom}\, f_2$, is also convex. This property is easily verified: if $0 \le \theta \le 1$ and $x,\ y \in \mathbf{dom}\, f$, then

$$\begin{array}{rcl} f(\theta x + (1-\theta)y) & = & \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\ & \le & \max\{\theta f_1(x) + (1-\theta)f_1(y), \theta f_2(x) + (1-\theta)f_2(y)\} \\ & \le & \theta \max\{f_1(x), f_2(x)\} + (1-\theta)\max\{f_1(y), f_2(y)\} \\ & = & \theta f(x) + (1-\theta)f(y), \end{array}$$

which establishes convexity of f. It is easily shown that if $f_1, \ldots, f_m$ are convex, then their pointwise maximum

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

is also convex.

Example 3.5 *Piecewise-linear functions.*The function

$$f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$$

defines a piecewise-linear (or really, affine) function (with L or fewer regions). It is convex since it is the pointwise maximum of affine functions.

The converse can also be shown: any piecewise-linear convex function with L or fewer regions can be expressed in this form. (See exercise 3.29.)

Example 3.6 *Sum of r largest components.* For $x \in \mathbf{R}^n$ we denote by $x_{[i]}$ the ith largest component of x, *i.e.*,

$$x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$$

are the components of x sorted in nonincreasing order. Then the function

$$f(x) = \sum_{i=1}^{r} x_{[i]},$$

i.e., the sum of the r largest elements of x, is a convex function. This can be seen by writing it as

$$f(x) = \sum_{i=1}^{r} x_{[i]} = \max\{x_{i_1} + \cdots + x_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\},$$

i.e., the maximum of all possible sums of r different components of x. Since it is the pointwise maximum of $n!/(r!(n-r)!)$ linear functions, it is convex.

As an extension it can be shown that the function $\sum_{i=1}^{r} w_i x_{[i]}$ is convex, provided $w_1 \ge w_2 \ge \cdots \ge w_r \ge 0$. (See exercise 3.19.)

The pointwise maximum property extends to the pointwise supremum over an infinite set of convex functions. If for each $y \in \mathcal{A}$, $f(x, y)$ is convex in x, then the function g, defined as

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y) \tag{3.7}$$

is convex in x. Here the domain of g is

$$\mathbf{dom}\, g = \{x \mid (x, y) \in \mathbf{dom}\, f \text{ for all } y \in \mathcal{A},\ \sup_{y \in \mathcal{A}} f(x, y) < \infty\}.$$

Similarly, the pointwise infimum of a set of concave functions is a concave function.

In terms of epigraphs, the pointwise supremum of functions corresponds to the intersection of epigraphs: with f, g, and $\mathcal{A}$ as defined in (3.7), we have

$$\mathbf{epi}\, g = \bigcap_{y \in \mathcal{A}} \mathbf{epi}\, f(\cdot, y).$$

Thus, the result follows from the fact that the intersection of a family of convex sets is convex.

Example 3.7 *Support function of a set.* Let $C \subseteq \mathbf{R}^n$, with $C \neq \emptyset$. The *support function* S_C associated with the set C is defined as

$$S_C(x) = \sup\{x^T y \mid y \in C\}$$

(and, naturally, $\mathbf{dom}\, S_C = \{x \mid \sup_{y \in C} x^T y < \infty\}$).

For each $y \in C$, $x^T y$ is a linear function of x, so S_C is the pointwise supremum of a family of linear functions, hence convex.

Example 3.8 *Distance to farthest point of a set.* Let $C \subseteq \mathbf{R}^n$. The distance (in any norm) to the farthest point of C,

$$f(x) = \sup_{y \in C} \|x - y\|,$$

is convex. To see this, note that for any y, the function $\|x - y\|$ is convex in x. Since f is the pointwise supremum of a family of convex functions (indexed by $y \in C$), it is a convex function of x.

Example 3.9 *Least-squares cost as a function of weights.* Let $a_1, \ldots, a_n \in \mathbf{R}^m$. In a weighted least-squares problem we minimize the objective function $\sum_{i=1}^n w_i (a_i^T x - b_i)^2$ over $x \in \mathbf{R}^m$. We refer to w_i as *weights*, and allow negative w_i (which opens the possibility that the objective function is unbounded below).

We define the (optimal) *weighted least-squares cost* as

$$g(w) = \inf_x \sum_{i=1}^n w_i (a_i^T x - b_i)^2,$$

with domain

$$\mathbf{dom}\, g = \left\{ w \;\middle|\; \inf_x \sum_{i=1}^n w_i (a_i^T x - b_i)^2 > -\infty \right\}.$$

Since g is the infimum of a family of linear functions of w (indexed by $x \in \mathbf{R}^m$), it is a concave function of w.

We can derive an explicit expression for g, at least on part of its domain. Let $W = \mathbf{diag}(w)$, the diagonal matrix with elements $w_1, \ldots, w_n$, and let $A \in \mathbf{R}^{n \times m}$ have rows a_i^T, so we have

$$g(w) = \inf_x (Ax - b)^T W (Ax - b) = \inf_x (x^T A^T W A x - 2b^T W A x + b^T W b).$$

From this we see that if $A^T W A \not\succeq 0$, the quadratic function is unbounded below in x, so $g(w) = -\infty$, *i.e.*, $w \notin \mathbf{dom}\, g$. We can give a simple expression for g when $A^T W A \succ 0$ (which defines a strict linear matrix inequality), by analytically minimizing the quadratic function:

$$\begin{aligned} g(w) &= b^T W b - b^T W A (A^T W A)^{-1} A^T W b \\ &= \sum_{i=1}^n w_i b_i^2 - \sum_{i=1}^n w_i^2 b_i^2 a_i^T \left(\sum_{j=1}^n w_j a_j a_j^T \right)^{-1} a_i. \end{aligned}$$

Concavity of g from this expression is not immediately obvious (but does follow, for example, from convexity of the matrix fractional function; see example 3.4).

Example 3.10 *Maximum eigenvalue of a symmetric matrix.* The function $f(X) = \lambda_{\max}(X)$, with $\mathbf{dom}\, f = \mathbf{S}^m$, is convex. To see this, we express f as

$$f(X) = \sup\{y^T X y \mid \|y\|_2 = 1\},$$

i.e., as the pointwise supremum of a family of linear functions of X (*i.e.*, $y^T X y$) indexed by $y \in \mathbf{R}^m$.

Example 3.11 *Norm of a matrix.* Consider $f(X) = \|X\|_2$ with $\mathbf{dom}\, f = \mathbf{R}^{p \times q}$, where $\|\cdot\|_2$ denotes the spectral norm or maximum singular value. Convexity of f follows from

$$f(X) = \sup\{u^T X v \mid \|u\|_2 = 1,\ \|v\|_2 = 1\},$$

which shows it is the pointwise supremum of a family of linear functions of X.

As a generalization suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbf{R}^p$ and $\mathbf{R}^q$, respectively. The induced norm of a matrix $X \in \mathbf{R}^{p \times q}$ is defined as

$$\|X\|_{a,b} = \sup_{v \neq 0} \frac{\|Xv\|_a}{\|v\|_b}.$$

(This reduces to the spectral norm when both norms are Euclidean.) The induced norm can be expressed as

$$\begin{aligned} \|X\|_{a,b} &= \sup\{\|Xv\|_a \mid \|v\|_b = 1\} \\ &= \sup\{u^T X v \mid \|u\|_{a*} = 1,\ \|v\|_b = 1\}, \end{aligned}$$

where $\|\cdot\|_{a*}$ is the dual norm of $\|\cdot\|_a$, and we use the fact that

$$\|z\|_a = \sup\{u^T z \mid \|u\|_{a*} = 1\}.$$

Since we have expressed $\|X\|_{a,b}$ as a supremum of linear functions of X, it is a convex function.

Representation as pointwise supremum of affine functions

The examples above illustrate a good method for establishing convexity of a function: by expressing it as the pointwise supremum of a family of affine functions. Except for a technical condition, a converse holds: almost every convex function can be expressed as the pointwise supremum of a family of affine functions. For example, if $f : \mathbf{R}^n \to \mathbf{R}$ is convex, with $\mathbf{dom}\, f = \mathbf{R}^n$, then we have

$$f(x) = \sup\{g(x) \mid g \text{ affine},\ g(z) \leq f(z) \text{ for all } z\}.$$

In other words, f is the pointwise supremum of the set of all affine global underestimators of it. We give the proof of this result below, and leave the case where $\mathbf{dom}\, f \neq \mathbf{R}^n$ as an exercise (exercise 3.28).

Suppose f is convex with $\mathbf{dom}\, f = \mathbf{R}^n$. The inequality

$$f(x) \geq \sup\{g(x) \mid g \text{ affine},\ g(z) \leq f(z) \text{ for all } z\}$$

is clear, since if g is any affine underestimator of f, we have $g(x) \leq f(x)$. To establish equality, we will show that for each $x \in \mathbf{R}^n$, there is an affine function g, which is a global underestimator of f, and satisfies $g(x) = f(x)$.

The epigraph of f is, of course, a convex set. Hence we can find a supporting hyperplane to it at $(x, f(x))$, *i.e.*, $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$ with $(a, b) \neq 0$ and

$$\left[\begin{array}{c} a \\ b \end{array}\right]^T \left[\begin{array}{c} x - z \\ f(x) - t \end{array}\right] \leq 0$$

for all $(z, t) \in \mathbf{epi}\, f$. This means that

$$a^T(x - z) + b(f(x) - f(z) - s) \leq 0 \tag{3.8}$$

for all $z \in \mathbf{dom}\, f = \mathbf{R}^n$ and all $s \geq 0$ (since $(z, t) \in \mathbf{epi}\, f$ means $t = f(z) + s$ for some $s \geq 0$). For the inequality (3.8) to hold for all $s \geq 0$, we must have $b \geq 0$. If $b = 0$, then the inequality (3.8) reduces to $a^T(x - z) \leq 0$ for all $z \in \mathbf{R}^n$, which implies $a = 0$ and contradicts $(a, b) \neq 0$. We conclude that $b > 0$, *i.e.*, that the supporting hyperplane is not vertical.

Using the fact that $b > 0$ we rewrite (3.8) for $s = 0$ as

$$g(z) = f(x) + (a/b)^T(x - z) \leq f(z)$$

for all z. The function g is an affine underestimator of f, and satisfies $g(x) = f(x)$.

3.2.4 Composition

In this section we examine conditions on $h : \mathbf{R}^k \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}^k$ that guarantee convexity or concavity of their composition $f = h \circ g : \mathbf{R}^n \to \mathbf{R}$, defined by

$$f(x) = h(g(x)), \qquad \mathbf{dom}\, f = \{x \in \mathbf{dom}\, g \mid g(x) \in \mathbf{dom}\, h\}.$$

Scalar composition

We first consider the case $k = 1$, so $h : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$. We can restrict ourselves to the case $n = 1$ (since convexity is determined by the behavior of a function on arbitrary lines that intersect its domain).

To discover the composition rules, we start by assuming that h and g are twice differentiable, with $\mathbf{dom}\, g = \mathbf{dom}\, h = \mathbf{R}$. In this case, convexity of f reduces to $f'' \geq 0$ (meaning, $f''(x) \geq 0$ for all $x \in \mathbf{R}$).

The second derivative of the composition function $f = h \circ g$ is given by

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x). \tag{3.9}$$

Now suppose, for example, that g is convex (so $g'' \geq 0$) and h is convex and nondecreasing (so $h'' \geq 0$ and $h' \geq 0$). It follows from (3.9) that $f'' \geq 0$, *i.e.*, f is convex. In a similar way, the expression (3.9) gives the results:

$$\begin{array}{l} f \text{ is convex if } h \text{ is convex and nondecreasing, and } g \text{ is convex,} \\ f \text{ is convex if } h \text{ is convex and nonincreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave and nondecreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave and nonincreasing, and } g \text{ is convex.} \end{array} \tag{3.10}$$

These statements are valid when the functions g and h are twice differentiable and have domains that are all of $\mathbf{R}$. It turns out that very similar composition rules hold in the general case $n > 1$, without assuming differentiability of h and g, or that $\mathbf{dom}\, g = \mathbf{R}^n$ and $\mathbf{dom}\, h = \mathbf{R}$:

$$\begin{array}{l} f \text{ is convex if } h \text{ is convex, } \tilde{h} \text{ is nondecreasing, and } g \text{ is convex,} \\ f \text{ is convex if } h \text{ is convex, } \tilde{h} \text{ is nonincreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave, } \tilde{h} \text{ is nondecreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave, } \tilde{h} \text{ is nonincreasing, and } g \text{ is convex.} \end{array} \tag{3.11}$$

Here $\tilde{h}$ denotes the extended-value extension of the function h, which assigns the value ∞ ($-\infty$) to points not in $\mathbf{dom}\, h$ for h convex (concave). The only difference between these results, and the results in (3.10), is that we require that the *extended-value extension* function $\tilde{h}$ be nonincreasing or nondecreasing, on all of $\mathbf{R}$.

To understand what this means, suppose h is convex, so $\tilde{h}$ takes on the value ∞ outside $\mathbf{dom}\, h$. To say that $\tilde{h}$ is nondecreasing means that for *any* $x, y \in \mathbf{R}$, with $x < y$, we have $\tilde{h}(x) \leq \tilde{h}(y)$. In particular, this means that if $y \in \mathbf{dom}\, h$, then $x \in \mathbf{dom}\, h$. In other words, the domain of h extends infinitely in the negative direction; it is either $\mathbf{R}$, or an interval of the form $(-\infty, a)$ or $(-\infty, a]$. In a similar way, to say that h is convex and $\tilde{h}$ is nonincreasing means that h is nonincreasing and $\mathbf{dom}\, h$ extends infinitely in the positive direction. This is illustrated in figure 3.7.

Example 3.12 Some simple examples will illustrate the conditions on h that appear in the composition theorems.

- The function $h(x) = \log x$, with $\mathbf{dom}\, h = \mathbf{R}_{++}$, is concave and satisfies $\tilde{h}$ nondecreasing.

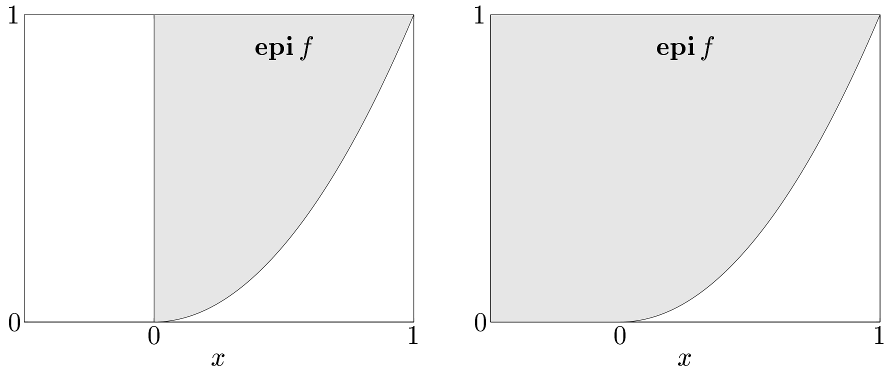


Figure 3.7 *Left.* The function x^2, with domain $\mathbf{R}_+$, is convex and nondecreasing on its domain, but its extended-value extension is *not* nondecreasing. *Right.* The function $\max\{x, 0\}^2$, with domain $\mathbf{R}$, is convex, and its extended-value extension is nondecreasing.

- The function $h(x) = x^{1/2}$, with $\mathbf{dom}\, h = \mathbf{R}_+$, is concave and satisfies the condition $\tilde{h}$ nondecreasing.
- The function $h(x) = x^{3/2}$, with $\mathbf{dom}\, h = \mathbf{R}_+$, is convex but *does not* satisfy the condition $\tilde{h}$ nondecreasing. For example, we have $\tilde{h}(-1) = \infty$, but $\tilde{h}(1) = 1$.
- The function $h(x) = x^{3/2}$ for $x \geq 0$, and $h(x) = 0$ for $x < 0$, with $\mathbf{dom}\, h = \mathbf{R}$, is convex and *does* satisfy the condition $\tilde{h}$ nondecreasing.

The composition results (3.11) can be proved directly, without assuming differentiability, or using the formula (3.9). As an example, we will prove the following composition theorem: if g is convex, h is convex, and $\tilde{h}$ is nondecreasing, then $f = h \circ g$ is convex. Assume that $x,\ y \in \mathbf{dom}\, f$, and $0 \leq \theta \leq 1$. Since $x,\ y \in \mathbf{dom}\, f$, we have that $x,\ y \in \mathbf{dom}\, g$ and $g(x),\ g(y) \in \mathbf{dom}\, h$. Since $\mathbf{dom}\, g$ is convex, we conclude that $\theta x + (1-\theta)y \in \mathbf{dom}\, g$, and from convexity of g, we have

$$g(\theta x + (1-\theta)y) \leq \theta g(x) + (1-\theta)g(y). \tag{3.12}$$

Since $g(x),\ g(y) \in \mathbf{dom}\, h$, we conclude that $\theta g(x) + (1-\theta)g(y) \in \mathbf{dom}\, h$, *i.e.*, the righthand side of (3.12) is in $\mathbf{dom}\, h$. Now we use the assumption that $\tilde{h}$ is nondecreasing, which means that its domain extends infinitely in the negative direction. Since the righthand side of (3.12) is in $\mathbf{dom}\, h$, we conclude that the lefthand side, *i.e.*, $g(\theta x + (1-\theta)y) \in \mathbf{dom}\, h$. This means that $\theta x + (1-\theta)y \in \mathbf{dom}\, f$. At this point, we have shown that $\mathbf{dom}\, f$ is convex.

Now using the fact that $\tilde{h}$ is nondecreasing and the inequality (3.12), we get

$$h(g(\theta x + (1-\theta)y)) \leq h(\theta g(x) + (1-\theta)g(y)). \tag{3.13}$$

From convexity of h, we have

$$h(\theta g(x) + (1-\theta)g(y)) \leq \theta h(g(x)) + (1-\theta)h(g(y)). \tag{3.14}$$

Putting (3.13) and (3.14) together, we have

$$h(g(\theta x + (1-\theta)y)) \le \theta h(g(x)) + (1-\theta)h(g(y)).$$

which proves the composition theorem.

Example 3.13 *Simple composition results.*

- If g is convex then $\exp g(x)$ is convex.
- If g is concave and positive, then $\log g(x)$ is concave.
- If g is concave and positive, then $1/g(x)$ is convex.
- If g is convex and nonnegative and $p \ge 1$, then $g(x)^p$ is convex.
- If g is convex then $-\log(-g(x))$ is convex on $\{x \mid g(x) < 0\}$.

Remark 3.3 The requirement that monotonicity hold for the extended-value extension $\tilde{h}$, and not just the function h, cannot be removed. For example, consider the function $g(x) = x^2$, with $\mathbf{dom}\, g = \mathbf{R}$, and $h(x) = 0$, with $\mathbf{dom}\, h = [1, 2]$. Here g is convex, and h is convex and nondecreasing. But the function $f = h \circ g$, given by

$$f(x) = 0, \qquad \mathbf{dom}\, f = [-\sqrt{2}, -1] \cup [1, \sqrt{2}],$$

is not convex, since its domain is not convex. Here, of course, the function $\tilde{h}$ is *not* nondecreasing.

Vector composition

We now turn to the more complicated case when $k \ge 1$. Suppose

$$f(x) = h(g(x)) = h(g_1(x), \ldots, g_k(x)),$$

with $h : \mathbf{R}^k \to \mathbf{R}$, $g_i : \mathbf{R}^n \to \mathbf{R}$. Again without loss of generality we can assume $n = 1$. As in the case $k = 1$, we start by assuming the functions are twice differentiable, with $\mathbf{dom}\, g = \mathbf{R}$ and $\mathbf{dom}\, h = \mathbf{R}^k$, in order to discover the composition rules. We have

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x), \tag{3.15}$$

which is the vector analog of (3.9). Again the issue is to determine conditions under which $f(x)'' \ge 0$ for all x (or $f(x)'' \le 0$ for all x for concavity). From (3.15) we can derive many rules, for example:

f is convex if h is convex, h is nondecreasing in each argument, and g_i are convex,

f is convex if h is convex, h is nonincreasing in each argument, and g_i are concave,

f is concave if h is concave, h is nondecreasing in each argument, and g_i are concave.

As in the scalar case, similar composition results hold in general, with $n > 1$, no assumption of differentiability of h or g, and general domains. For the general results, the monotonicity condition on h must hold for the extended-value extension $\tilde{h}$.

To understand the meaning of the condition that the extended-value extension $\tilde{h}$ be monotonic, we consider the case where $h : \mathbf{R}^k \to \mathbf{R}$ is convex, and $\tilde{h}$ nondecreasing, *i.e.*, whenever $u \preceq v$, we have $\tilde{h}(u) \leq \tilde{h}(v)$. This implies that if $v \in \mathbf{dom}\, h$, then so is u: the domain of h must extend infinitely in the $-\mathbf{R}^k_+$ directions. We can express this compactly as $\mathbf{dom}\, h - \mathbf{R}^k_+ = \mathbf{dom}\, h$.

Example 3.14 *Vector composition examples.*

- Let $h(z) = z_{[1]} + \cdots + z_{[r]}$, the sum of the r largest components of $z \in \mathbf{R}^k$. Then h is convex and nondecreasing in each argument. Suppose $g_1, \ldots, g_k$ are convex functions on $\mathbf{R}^n$. Then the composition function $f = h \circ g$, *i.e.*, the pointwise sum of the r largest g_i's, is convex.
- The function $h(z) = \log(\sum_{i=1}^k e^{z_i})$ is convex and nondecreasing in each argument, so $\log(\sum_{i=1}^k e^{g_i})$ is convex whenever g_i are.
- For $0 < p \leq 1$, the function $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$ on $\mathbf{R}^k_+$ is concave, and its extension (which has the value $-\infty$ for $z \not\succeq 0$) is nondecreasing in each component. So if g_i are concave and nonnegative, we conclude that $f(x) = (\sum_{i=1}^k g_i(x)^p)^{1/p}$ is concave.
- Suppose $p \geq 1$, and $g_1, \ldots, g_k$ are convex and nonnegative. Then the function $(\sum_{i=1}^k g_i(x)^p)^{1/p}$ is convex.

  To show this, we consider the function $h : \mathbf{R}^k \to \mathbf{R}$ defined as

  $$h(z) = \left(\sum_{i=1}^k \max\{z_i, 0\}^p \right)^{1/p},$$

  with $\mathbf{dom}\, h = \mathbf{R}^k$, so $h = \tilde{h}$. This function is convex, and nondecreasing, so we conclude $h(g(x))$ is a convex function of x. For $z \succeq 0$, we have $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$, so our conclusion is that $(\sum_{i=1}^k g_i(x)^p)^{1/p}$ is convex.
- The geometric mean $h(z) = (\prod_{i=1}^k z_i)^{1/k}$ on $\mathbf{R}^k_+$ is concave and its extension is nondecreasing in each argument. It follows that if $g_1, \ldots, g_k$ are nonnegative concave functions, then so is their geometric mean, $(\prod_{i=1}^k g_i)^{1/k}$.

3.2.5 Minimization

We have seen that the maximum or supremum of an arbitrary family of convex functions is convex. It turns out that some special forms of minimization also yield convex functions. If f is convex in (x, y), and C is a convex nonempty set, then the function

$$g(x) = \inf_{y \in C} f(x, y) \tag{3.16}$$

is convex in x, provided $g(x) > -\infty$ for some x (which implies $g(x) > -\infty$ for all x). The domain of g is the projection of $\mathbf{dom}\, f$ on its x-coordinates, *i.e.*,

$$\mathbf{dom}\, g = \{x \mid (x, y) \in \mathbf{dom}\, f \text{ for some } y \in C\}.$$

We prove this by verifying Jensen's inequality for $x_1,\ x_2 \in \mathbf{dom}\, g$. Let $\epsilon > 0$. Then there are $y_1,\ y_2 \in C$ such that $f(x_i, y_i) \le g(x_i) + \epsilon$ for $i = 1,\ 2$. Now let $\theta \in [0, 1]$. We have

$$\begin{aligned} g(\theta x_1 + (1-\theta)x_2) &= \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y) \\ &\le f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\ &\le \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2) \\ &\le \theta g(x_1) + (1-\theta) g(x_2) + \epsilon. \end{aligned}$$

Since this holds for any $\epsilon > 0$, we have

$$g(\theta x_1 + (1-\theta)x_2) \le \theta g(x_1) + (1-\theta) g(x_2).$$

The result can also be seen in terms of epigraphs. With f, g, and C defined as in (3.16), we have

$$\mathbf{epi}\, g = \{(x, t) \mid (x, y, t) \in \mathbf{epi}\, f \text{ for some } y \in C\}.$$

Thus $\mathbf{epi}\, g$ is convex, since it is the projection of a convex set on some of its components.

Example 3.15 *Schur complement.* Suppose the quadratic function

$$f(x, y) = x^T A x + 2x^T B y + y^T C y,$$

(where A and C are symmetric) is convex in (x, y), which means

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0.$$

We can express $g(x) = \inf_y f(x, y)$ as

$$g(x) = x^T (A - BC^\dagger B^T) x,$$

where $C^\dagger$ is the pseudo-inverse of C (see §A.5.4). By the minimization rule, g is convex, so we conclude that $A - BC^\dagger B^T \succeq 0$.

If C is invertible, *i.e.*, $C \succ 0$, then the matrix $A - BC^{-1}B^T$ is called the *Schur complement* of C in the matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

(see §A.5.5).

Example 3.16 *Distance to a set.* The distance of a point x to a set $S \subseteq \mathbf{R}^n$, in the norm $\|\cdot\|$, is defined as

$$\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|.$$

The function $\|x - y\|$ is convex in (x, y), so if the set S is convex, the distance function $\mathbf{dist}(x, S)$ is a convex function of x.

Example 3.17 Suppose h is convex. Then the function g defined as

$$g(x) = \inf\{h(y) \mid Ay = x\}$$

is convex. To see this, we define f by

$$f(x, y) = \begin{cases} h(y) & \text{if } Ay = x \\ \infty & \text{otherwise,} \end{cases}$$

which is convex in (x, y). Then g is the minimum of f over y, and hence is convex. (It is not hard to show directly that g is convex.)

3.2.6 Perspective of a function

If $f : \mathbf{R}^n \to \mathbf{R}$, then the *perspective* of f is the function $g : \mathbf{R}^{n+1} \to \mathbf{R}$ defined by

$$g(x, t) = tf(x/t),$$

with domain

$$\mathbf{dom}\, g = \{(x, t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}.$$

The perspective operation preserves convexity: If f is a convex function, then so is its perspective function g. Similarly, if f is concave, then so is g.

This can be proved several ways, for example, direct verification of the defining inequality (see exercise 3.33). We give a short proof here using epigraphs and the perspective mapping on $\mathbf{R}^{n+1}$ described in §2.3.3 (which will also explain the name 'perspective'). For $t > 0$ we have

$$\begin{aligned} (x, t, s) \in \mathbf{epi}\, g \quad &\Longleftrightarrow \quad tf(x/t) \le s \\ &\Longleftrightarrow \quad f(x/t) \le s/t \\ &\Longleftrightarrow \quad (x/t, s/t) \in \mathbf{epi}\, f. \end{aligned}$$

Therefore $\mathbf{epi}\, g$ is the inverse image of $\mathbf{epi}\, f$ under the perspective mapping that takes (u, v, w) to $(u, w)/v$. It follows (see §2.3.3) that $\mathbf{epi}\, g$ is convex, so the function g is convex.

Example 3.18 *Euclidean norm squared.* The perspective of the convex function $f(x) = x^T x$ on $\mathbf{R}^n$ is

$$g(x, t) = t(x/t)^T(x/t) = \frac{x^T x}{t},$$

which is convex in (x, t) for $t > 0$.

We can deduce convexity of g using several other methods. First, we can express g as the sum of the quadratic-over-linear functions x_i^2/t, which were shown to be convex in §3.1.5. We can also express g as a special case of the matrix fractional function $x^T(tI)^{-1}x$ (see example 3.4).

Example 3.19 *Negative logarithm.* Consider the convex function $f(x) = -\log x$ on $\mathbf{R}_{++}$. Its perspective is

$$g(x,t) = -t\log(x/t) = t\log(t/x) = t\log t - t\log x,$$

and is convex on $\mathbf{R}^2_{++}$. The function g is called the *relative entropy* of t and x. For $x = 1$, g reduces to the negative entropy function.

From convexity of g we can establish convexity or concavity of several interesting related functions. First, the relative entropy of two vectors $u,\ v \in \mathbf{R}^n_{++}$, defined as

$$\sum_{i=1}^n u_i \log(u_i/v_i),$$

is convex in (u,v), since it is a sum of relative entropies of $u_i,\ v_i$.

A closely related function is the *Kullback-Leibler divergence* between $u,\ v \in \mathbf{R}^n_{++}$, given by

$$D_{\mathrm{kl}}(u,v) = \sum_{i=1}^n \left(u_i \log(u_i/v_i) - u_i + v_i\right), \tag{3.17}$$

which is convex, since it is the relative entropy plus a linear function of (u,v). The Kullback-Leibler divergence satisfies $D_{\mathrm{kl}}(u,v) \geq 0$, and $D_{\mathrm{kl}}(u,v) = 0$ if and only if $u = v$, and so can be used as a measure of deviation between two positive vectors; see exercise 3.13. (Note that the relative entropy and the Kullback-Leibler divergence are the same when u and v are probability vectors, *i.e.*, satisfy $\mathbf{1}^T u = \mathbf{1}^T v = 1$.)

If we take $v_i = \mathbf{1}^T u$ in the relative entropy function, we obtain the concave (and homogeneous) function of $u \in \mathbf{R}^n_{++}$ given by

$$\sum_{i=1}^n u_i \log(\mathbf{1}^T u/u_i) = (\mathbf{1}^T u)\sum_{i=1}^n z_i \log(1/z_i),$$

where $z = u/(\mathbf{1}^T u)$, which is called the *normalized entropy* function. The vector $z = u/\mathbf{1}^T u$ is a normalized vector or probability distribution, since its components sum to one; the normalized entropy of u is $\mathbf{1}^T u$ times the entropy of this normalized distribution.

Example 3.20 Suppose $f : \mathbf{R}^m \to \mathbf{R}$ is convex, and $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, and $d \in \mathbf{R}$. We define

$$g(x) = (c^T x + d) f\left((Ax+b)/(c^T x + d)\right),$$

with

$$\mathbf{dom}\, g = \{x \mid c^T x + d > 0,\ (Ax+b)/(c^T x+d) \in \mathbf{dom}\, f\}.$$

Then g is convex.

3.3 The conjugate function

In this section we introduce an operation that will play an important role in later chapters.

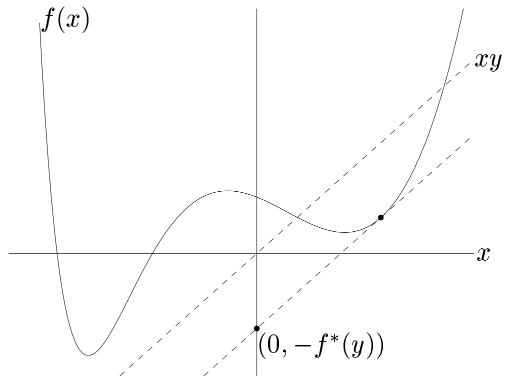


Figure 3.8 A function $f : \mathbf{R} \to \mathbf{R}$, and a value $y \in \mathbf{R}$. The conjugate function $f^*(y)$ is the maximum gap between the linear function yx and $f(x)$, as shown by the dashed line in the figure. If f is differentiable, this occurs at a point x where $f'(x) = y$.

3.3.1 Definition and examples

Let $f : \mathbf{R}^n \to \mathbf{R}$. The function $f^* : \mathbf{R}^n \to \mathbf{R}$, defined as

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} \left(y^T x - f(x) \right), \tag{3.18}$$

is called the *conjugate* of the function f. The domain of the conjugate function consists of $y \in \mathbf{R}^n$ for which the supremum is finite, *i.e.*, for which the difference $y^T x - f(x)$ is bounded above on $\mathbf{dom}\, f$. This definition is illustrated in figure 3.8.

We see immediately that f^* is a convex function, since it is the pointwise supremum of a family of convex (indeed, affine) functions of y. This is true whether or not f is convex. (Note that when f is convex, the subscript $x \in \mathbf{dom}\, f$ is not necessary since, by convention, $y^T x - f(x) = -\infty$ for $x \notin \mathbf{dom}\, f$.)

We start with some simple examples, and then describe some rules for conjugating functions. This allows us to derive an analytical expression for the conjugate of many common convex functions.

Example 3.21 We derive the conjugates of some convex functions on $\mathbf{R}$.

- *Affine function.* $f(x) = ax + b$. As a function of x, $yx - ax - b$ is bounded if and only if $y = a$, in which case it is constant. Therefore the domain of the conjugate function f^* is the singleton $\{a\}$, and $f^*(a) = -b$.
- *Negative logarithm.* $f(x) = -\log x$, with $\mathbf{dom}\, f = \mathbf{R}_{++}$. The function $xy + \log x$ is unbounded above if $y \geq 0$ and reaches its maximum at $x = -1/y$ otherwise. Therefore, $\mathbf{dom}\, f^* = \{y \mid y < 0\} = -\mathbf{R}_{++}$ and $f^*(y) = -\log(-y) - 1$ for $y < 0$.
- *Exponential.* $f(x) = e^x$. $xy - e^x$ is unbounded if $y < 0$. For $y > 0$, $xy - e^x$ reaches its maximum at $x = \log y$, so we have $f^*(y) = y \log y - y$. For $y = 0$,

$f^*(y) = \sup_x -e^x = 0$. In summary, $\mathbf{dom}\, f^* = \mathbf{R}_+$ and $f^*(y) = y \log y - y$ (with the interpretation $0 \log 0 = 0$).

- *Negative entropy.* $f(x) = x \log x$, with $\mathbf{dom}\, f = \mathbf{R}_+$ (and $f(0) = 0$). The function $xy - x \log x$ is bounded above on $\mathbf{R}_+$ for all y, hence $\mathbf{dom}\, f^* = \mathbf{R}$. It attains its maximum at $x = e^{y-1}$, and substituting we find $f^*(y) = e^{y-1}$.
- *Inverse.* $f(x) = 1/x$ on $\mathbf{R}_{++}$. For $y > 0$, $yx - 1/x$ is unbounded above. For $y = 0$ this function has supremum 0; for $y < 0$ the supremum is attained at $x = (-y)^{-1/2}$. Therefore we have $f^*(y) = -2(-y)^{1/2}$, with $\mathbf{dom}\, f^* = -\mathbf{R}_+$.

Example 3.22 *Strictly convex quadratic function.* Consider $f(x) = \frac{1}{2}x^TQx$, with $Q \in \mathbf{S}^n_{++}$. The function $y^Tx - \frac{1}{2}x^TQx$ is bounded above as a function of x for all y. It attains its maximum at $x = Q^{-1}y$, so

$$f^*(y) = \frac{1}{2}y^TQ^{-1}y.$$

Example 3.23 *Log-determinant.* We consider $f(X) = \log\det X^{-1}$ on $\mathbf{S}^n_{++}$. The conjugate function is defined as

$$f^*(Y) = \sup_{X \succ 0} \left(\mathbf{tr}(YX) + \log\det X\right),$$

since $\mathbf{tr}(YX)$ is the standard inner product on $\mathbf{S}^n$. We first show that $\mathbf{tr}(YX) + \log\det X$ is unbounded above unless $Y \prec 0$. If $Y \not\prec 0$, then Y has an eigenvector v, with $\|v\|_2 = 1$, and eigenvalue $\lambda \geq 0$. Taking $X = I + tvv^T$ we find that

$$\mathbf{tr}(YX) + \log\det X = \mathbf{tr}\, Y + t\lambda + \log\det(I + tvv^T) = \mathbf{tr}\, Y + t\lambda + \log(1+t),$$

which is unbounded above as $t \to \infty$.

Now consider the case $Y \prec 0$. We can find the maximizing X by setting the gradient with respect to X equal to zero:

$$\nabla_X \left(\mathbf{tr}(YX) + \log\det X\right) = Y + X^{-1} = 0$$

(see §A.4.1), which yields $X = -Y^{-1}$ (which is, indeed, positive definite). Therefore we have

$$f^*(Y) = \log\det(-Y)^{-1} - n,$$

with $\mathbf{dom}\, f^* = -\mathbf{S}^n_{++}$.

Example 3.24 *Indicator function.* Let I_S be the indicator function of a (not necessarily convex) set $S \subseteq \mathbf{R}^n$, *i.e.*, $I_S(x) = 0$ on $\mathbf{dom}\, I_S = S$. Its conjugate is

$$I_S^*(y) = \sup_{x \in S} y^Tx,$$

which is the support function of the set S.

Example 3.25 *Log-sum-exp function.* To derive the conjugate of the log-sum-exp function $f(x) = \log(\sum_{i=1}^n e^{x_i})$, we first determine the values of y for which the maximum over x of $y^T x - f(x)$ is attained. By setting the gradient with respect to x equal to zero, we obtain the condition

$$y_i = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, \quad i = 1, \ldots, n.$$

These equations are solvable for x if and only if $y \succ 0$ and $\mathbf{1}^T y = 1$. By substituting the expression for y_i into $y^T x - f(x)$ we obtain $f^*(y) = \sum_{i=1}^n y_i \log y_i$. This expression for f^* is still correct if some components of y are zero, as long as $y \succeq 0$ and $\mathbf{1}^T y = 1$, and we interpret $0 \log 0$ as 0.

In fact the domain of f^* is exactly given by $\mathbf{1}^T y = 1$, $y \succeq 0$. To show this, suppose that a component of y is negative, say, $y_k < 0$. Then we can show that $y^T x - f(x)$ is unbounded above by choosing $x_k = -t$, and $x_i = 0$, $i \neq k$, and letting t go to infinity.

If $y \succeq 0$ but $\mathbf{1}^T y \neq 1$, we choose $x = t\mathbf{1}$, so that

$$y^T x - f(x) = t\mathbf{1}^T y - t - \log n.$$

If $\mathbf{1}^T y > 1$, this grows unboundedly as $t \to \infty$; if $\mathbf{1}^T y < 1$, it grows unboundedly as $t \to -\infty$.

In summary,

$$f^*(y) = \begin{cases} \sum_{i=1}^n y_i \log y_i & \text{if } y \succeq 0 \text{ and } \mathbf{1}^T y = 1 \\ \infty & \text{otherwise.} \end{cases}$$

In other words, the conjugate of the log-sum-exp function is the negative entropy function, restricted to the probability simplex.

Example 3.26 *Norm.* Let $\|\cdot\|$ be a norm on $\mathbf{R}^n$, with dual norm $\|\cdot\|_*$. We will show that the conjugate of $f(x) = \|x\|$ is

$$f^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ \infty & \text{otherwise,} \end{cases}$$

i.e., the conjugate of a norm is the indicator function of the dual norm unit ball.

If $\|y\|_* > 1$, then by definition of the dual norm, there is a $z \in \mathbf{R}^n$ with $\|z\| \leq 1$ and $y^T z > 1$. Taking $x = tz$ and letting $t \to \infty$, we have

$$y^T x - \|x\| = t(y^T z - \|z\|) \to \infty,$$

which shows that $f^*(y) = \infty$. Conversely, if $\|y\|_* \leq 1$, then we have $y^T x \leq \|x\| \|y\|_*$ for all x, which implies for all x, $y^T x - \|x\| \leq 0$. Therefore $x = 0$ is the value that maximizes $y^T x - \|x\|$, with maximum value 0.

Example 3.27 *Norm squared.* Now consider the function $f(x) = (1/2)\|x\|^2$, where $\|\cdot\|$ is a norm, with dual norm $\|\cdot\|_*$. We will show that its conjugate is $f^*(y) = (1/2)\|y\|_*^2$. From $y^T x \leq \|y\|_* \|x\|$, we conclude

$$y^T x - (1/2)\|x\|^2 \leq \|y\|_* \|x\| - (1/2)\|x\|^2$$

for all x. The righthand side is a quadratic function of $\|x\|$, which has maximum value $(1/2)\|y\|_*^2$. Therefore for all x, we have

$$y^T x - (1/2)\|x\|^2 \leq (1/2)\|y\|_*^2,$$

which shows that $f^*(y) \leq (1/2)\|y\|_*^2$.

To show the other inequality, let x be any vector with $y^T x = \|y\|_* \|x\|$, scaled so that $\|x\| = \|y\|_*$. Then we have, for this x,

$$y^T x - (1/2)\|x\|^2 = (1/2)\|y\|_*^2,$$

which shows that $f^*(y) \geq (1/2)\|y\|_*^2$.

Example 3.28 *Revenue and profit functions.* We consider a business or enterprise that consumes n resources and produces a product that can be sold. We let $r = (r_1, \ldots, r_n)$ denote the vector of resource quantities consumed, and $S(r)$ denote the sales revenue derived from the product produced (as a function of the resources consumed). Now let p_i denote the price (per unit) of resource i, so the total amount paid for resources by the enterprise is $p^T r$. The profit derived by the firm is then $S(r) - p^T r$. Let us fix the prices of the resources, and ask what is the maximum profit that can be made, by wisely choosing the quantities of resources consumed. This maximum profit is given by

$$M(p) = \sup_r \left(S(r) - p^T r \right).$$

The function $M(p)$ gives the maximum profit attainable, as a function of the resource prices. In terms of conjugate functions, we can express M as

$$M(p) = (-S)^*(-p).$$

Thus the maximum profit (as a function of resource prices) is closely related to the conjugate of gross sales (as a function of resources consumed).

3.3.2 Basic properties

Fenchel's inequality

From the definition of conjugate function, we immediately obtain the inequality

$$f(x) + f^*(y) \geq x^T y$$

for all x, y. This is called *Fenchel's inequality* (or *Young's inequality* when f is differentiable).

For example with $f(x) = (1/2)x^T Q x$, where $Q \in \mathbf{S}_{++}^n$, we obtain the inequality

$$x^T y \leq (1/2)x^T Q x + (1/2)y^T Q^{-1} y.$$

Conjugate of the conjugate

The examples above, and the name 'conjugate', suggest that the conjugate of the conjugate of a convex function is the original function. This is the case provided a technical condition holds: if f is convex, and f is closed (*i.e.*, $\mathbf{epi}\, f$ is a closed set; see §A.3.3), then $f^{**} = f$. For example, if $\mathbf{dom}\, f = \mathbf{R}^n$, then we have $f^{**} = f$, *i.e.*, the conjugate of the conjugate of f is f again (see exercise 3.39).

Differentiable functions

The conjugate of a differentiable function f is also called the *Legendre transform* of f. (To distinguish the general definition from the differentiable case, the term *Fenchel conjugate* is sometimes used instead of conjugate.)

Suppose f is convex and differentiable, with $\mathbf{dom}\, f = \mathbf{R}^n$. Any maximizer x^* of $y^T x - f(x)$ satisfies $y = \nabla f(x^*)$, and conversely, if x^* satisfies $y = \nabla f(x^*)$, then x^* maximizes $y^T x - f(x)$. Therefore, if $y = \nabla f(x^*)$, we have

$$f^*(y) = x^{*T} \nabla f(x^*) - f(x^*).$$

This allows us to determine $f^*(y)$ for any y for which we can solve the gradient equation $y = \nabla f(z)$ for z.

We can express this another way. Let $z \in \mathbf{R}^n$ be arbitrary and define $y = \nabla f(z)$. Then we have

$$f^*(y) = z^T \nabla f(z) - f(z).$$

Scaling and composition with affine transformation

For $a > 0$ and $b \in \mathbf{R}$, the conjugate of $g(x) = af(x) + b$ is $g^*(y) = af^*(y/a) - b$.

Suppose $A \in \mathbf{R}^{n \times n}$ is nonsingular and $b \in \mathbf{R}^n$. Then the conjugate of $g(x) = f(Ax + b)$ is

$$g^*(y) = f^*(A^{-T} y) - b^T A^{-T} y,$$

with $\mathbf{dom}\, g^* = A^T \,\mathbf{dom}\, f^*$.

Sums of independent functions

If $f(u, v) = f_1(u) + f_2(v)$, where f_1 and f_2 are convex functions with conjugates f_1^* and f_2^*, respectively, then

$$f^*(w, z) = f_1^*(w) + f_2^*(z).$$

In other words, the conjugate of the sum of *independent* convex functions is the sum of the conjugates. ('Independent' means they are functions of different variables.)

3.4 Quasiconvex functions

3.4.1 Definition and examples

A function $f : \mathbf{R}^n \to \mathbf{R}$ is called *quasiconvex* (or *unimodal*) if its domain and all its sublevel sets

$$S_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\},$$

for $\alpha \in \mathbf{R}$, are convex. A function is *quasiconcave* if $-f$ is quasiconvex, *i.e.*, every superlevel set $\{x \mid f(x) \geq \alpha\}$ is convex. A function that is both quasiconvex and quasiconcave is called *quasilinear*. A function is quasilinear if its domain, and every level set $\{x \mid f(x) = \alpha\}$ is convex.

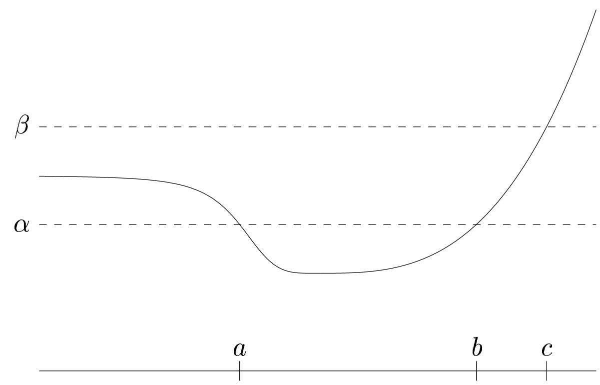


Figure 3.9 A quasiconvex function on $\mathbf{R}$. For each α, the α-sublevel set S_α is convex, *i.e.*, an interval. The sublevel set S_α is the interval $[a, b]$. The sublevel set S_β is the interval $(-\infty, c]$.

For a function on $\mathbf{R}$, quasiconvexity requires that each sublevel set be an interval (including, possibly, an infinite interval). An example of a quasiconvex function on $\mathbf{R}$ is shown in figure 3.9.

Convex functions have convex sublevel sets, and so are quasiconvex. But simple examples, such as the one shown in figure 3.9, show that the converse is not true.

Example 3.29 Some examples on $\mathbf{R}$:

- *Logarithm.* $\log x$ on $\mathbf{R}_{++}$ is quasiconvex (and quasiconcave, hence quasilinear).
- *Ceiling function.* $\operatorname{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$ is quasiconvex (and quasiconcave).

These examples show that quasiconvex functions can be concave, or discontinuous. We now give some examples on $\mathbf{R}^n$.

Example 3.30 *Length of a vector.* We define the *length* of $x \in \mathbf{R}^n$ as the largest index of a nonzero component, *i.e.*,

$$f(x) = \max\{i \mid x_i \neq 0\}.$$

(We define the length of the zero vector to be zero.) This function is quasiconvex on $\mathbf{R}^n$, since its sublevel sets are subspaces:

$$f(x) \leq \alpha \iff x_i = 0 \text{ for } i = \lfloor \alpha \rfloor + 1, \ldots, n.$$

Example 3.31 Consider $f : \mathbf{R}^2 \to \mathbf{R}$, with $\mathbf{dom}\, f = \mathbf{R}^2_+$ and $f(x_1, x_2) = x_1 x_2$. This function is neither convex nor concave since its Hessian

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is indefinite; it has one positive and one negative eigenvalue. The function f is quasiconcave, however, since the superlevel sets

$$\{x \in \mathbf{R}^2_+ \mid x_1 x_2 \geq \alpha\}$$

are convex sets for all α. (Note, however, that f is *not* quasiconcave on $\mathbf{R}^2$.)

Example 3.32 *Linear-fractional function.* The function

$$f(x) = \frac{a^T x + b}{c^T x + d},$$

with $\mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$, is quasiconvex, and quasiconcave, *i.e.*, quasilinear. Its α-sublevel set is

$$\begin{aligned} S_\alpha &= \{x \mid c^T x + d > 0,\ (a^T x + b)/(c^T x + d) \leq \alpha\} \\ &= \{x \mid c^T x + d > 0,\ a^T x + b \leq \alpha(c^T x + d)\}, \end{aligned}$$

which is convex, since it is the intersection of an open halfspace and a closed halfspace. (The same method can be used to show its superlevel sets are convex.)

Example 3.33 *Distance ratio function.* Suppose $a, b \in \mathbf{R}^n$, and define

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2},$$

i.e., the ratio of the Euclidean distance to a to the distance to b. Then f is quasiconvex on the halfspace $\{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$. To see this, we consider the α-sublevel set of f, with $\alpha \leq 1$ since $f(x) \leq 1$ on the halfspace $\{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$. This sublevel set is the set of points satisfying

$$\|x - a\|_2 \leq \alpha \|x - b\|_2.$$

Squaring both sides, and rearranging terms, we see that this equivalent to

$$(1 - \alpha^2) x^T x - 2(a - \alpha^2 b)^T x + a^T a - \alpha^2 b^T b \leq 0.$$

This describes a convex set (in fact a Euclidean ball) if $\alpha \leq 1$.

Example 3.34 *Internal rate of return.* Let $x = (x_0, x_1, \ldots, x_n)$ denote a cash flow sequence over n periods, where $x_i > 0$ means a payment to us in period i, and $x_i < 0$ means a payment by us in period i. We define the *present value* of a cash flow, with interest rate $r \geq 0$, to be

$$\mathrm{PV}(x, r) = \sum_{i=0}^{n} (1 + r)^{-i} x_i.$$

(The factor $(1 + r)^{-i}$ is a *discount factor* for a payment by or to us in period i.)

Now we consider cash flows for which $x_0 < 0$ and $x_0 + x_1 + \cdots + x_n > 0$. This means that we start with an investment of $|x_0|$ in period 0, and that the total of the remaining cash flow, $x_1 + \cdots + x_n$, (not taking any discount factors into account) exceeds our initial investment.

For such a cash flow, $\mathrm{PV}(x, 0) > 0$ and $\mathrm{PV}(x, r) \to x_0 < 0$ as $r \to \infty$, so it follows that for at least one $r \geq 0$, we have $\mathrm{PV}(x, r) = 0$. We define the *internal rate of return* of the cash flow as the smallest interest rate $r \geq 0$ for which the present value is zero:

$$\mathrm{IRR}(x) = \inf\{r \geq 0 \mid \mathrm{PV}(x, r) = 0\}.$$

Internal rate of return is a quasiconcave function of x (restricted to $x_0 < 0$, $x_1 + \cdots + x_n > 0$). To see this, we note that

$$\mathrm{IRR}(x) \geq R \iff \mathrm{PV}(x, r) \geq 0 \text{ for } 0 \leq r \leq R.$$

The lefthand side defines the R-superlevel set of IRR. The righthand side is the intersection of the sets $\{x \mid \mathrm{PV}(x, r) \geq 0\}$, indexed by r, over the range $0 \leq r \leq R$. For each r, $\mathrm{PV}(x, r) \geq 0$ defines a halfspace, so the righthand side defines a convex set.

3.4.2 Basic properties

The examples above show that quasiconvexity is a considerable generalization of convexity. Still, many of the properties of convex functions hold, or have analogs, for quasiconvex functions. For example, there is a variation on Jensen's inequality that characterizes quasiconvexity: A function f is quasiconvex if and only if $\mathbf{dom}\, f$ is convex and for any x, $y \in \mathbf{dom}\, f$ and $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta) y) \leq \max\{f(x), f(y)\}, \tag{3.19}$$

i.e., the value of the function on a segment does not exceed the maximum of its values at the endpoints. The inequality (3.19) is sometimes called Jensen's inequality for quasiconvex functions, and is illustrated in figure 3.10.

Example 3.35 *Cardinality of a nonnegative vector.* The *cardinality* or *size* of a vector $x \in \mathbf{R}^n$ is the number of nonzero components, and denoted $\mathbf{card}(x)$. The function $\mathbf{card}$ is quasiconcave on $\mathbf{R}^n_+$ (but not $\mathbf{R}^n$). This follows immediately from the modified Jensen inequality

$$\mathbf{card}(x + y) \geq \min\{\mathbf{card}(x), \mathbf{card}(y)\},$$

which holds for x, $y \succeq 0$.

Example 3.36 *Rank of positive semidefinite matrix.* The function $\mathbf{rank}\, X$ is quasiconcave on $\mathbf{S}^n_+$. This follows from the modified Jensen inequality (3.19),

$$\mathbf{rank}(X + Y) \geq \min\{\mathbf{rank}\, X, \mathbf{rank}\, Y\}$$

which holds for X, $Y \in \mathbf{S}^n_+$. (This can be considered an extension of the previous example, since $\mathbf{rank}(\mathbf{diag}(x)) = \mathbf{card}(x)$ for $x \succeq 0$.)

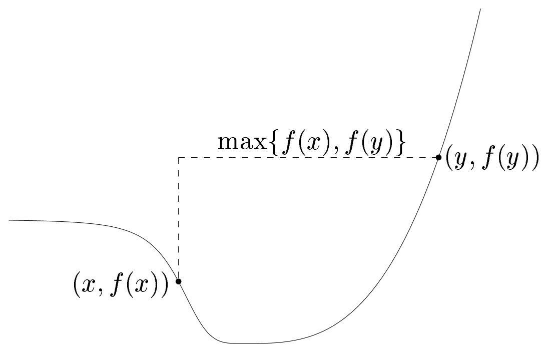


Figure 3.10 A quasiconvex function on $\mathbf{R}$. The value of f between x and y is no more than $\max\{f(x), f(y)\}$.

Like convexity, quasiconvexity is characterized by the behavior of a function f on lines: f is quasiconvex if and only if its restriction to any line intersecting its domain is quasiconvex. In particular, quasiconvexity of a function can be verified by restricting it to an arbitrary line, and then checking quasiconvexity of the resulting function on $\mathbf{R}$.

Quasiconvex functions on R

We can give a simple characterization of quasiconvex functions on $\mathbf{R}$. We consider continuous functions, since stating the conditions in the general case is cumbersome. A continuous function $f : \mathbf{R} \to \mathbf{R}$ is quasiconvex if and only if at least one of the following conditions holds:

- f is nondecreasing
- f is nonincreasing
- there is a point $c \in \mathbf{dom}\, f$ such that for $t \leq c$ (and $t \in \mathbf{dom}\, f$), f is nonincreasing, and for $t \geq c$ (and $t \in \mathbf{dom}\, f$), f is nondecreasing.

The point c can be chosen as any point which is a global minimizer of f. Figure (3.11) illustrates this.

3.4.3 Differentiable quasiconvex functions

First-order conditions

Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable. Then f is quasiconvex if and only if $\mathbf{dom}\, f$ is convex and for all x, $y \in \mathbf{dom}\, f$

$$f(y) \leq f(x) \Longrightarrow \nabla f(x)^T (y - x) \leq 0. \tag{3.20}$$

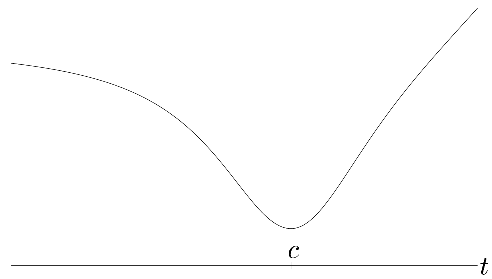


Figure 3.11 A quasiconvex function on $\mathbf{R}$. The function is nonincreasing for $t \leq c$ and nondecreasing for $t \geq c$.

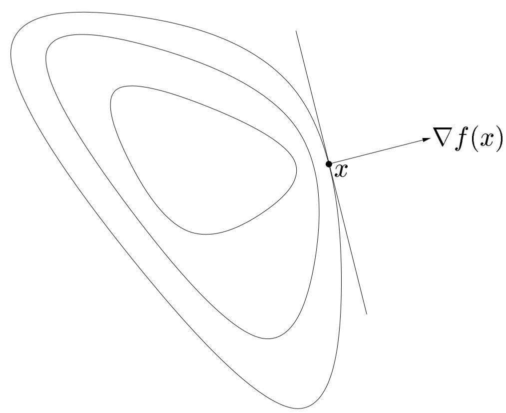


Figure 3.12 Three level curves of a quasiconvex function f are shown. The vector $\nabla f(x)$ defines a supporting hyperplane to the sublevel set $\{z \mid f(z) \leq f(x)\}$ at x.

This is the analog of inequality (3.2), for quasiconvex functions. We leave the proof as an exercise (exercise 3.43).

The condition (3.20) has a simple geometric interpretation when $\nabla f(x) \neq 0$. It states that $\nabla f(x)$ defines a supporting hyperplane to the sublevel set $\{y \mid f(y) \leq f(x)\}$, at the point x, as illustrated in figure 3.12.

While the first-order condition for convexity (3.2), and the first-order condition for quasiconvexity (3.20) are similar, there are some important differences. For example, if f is convex and $\nabla f(x) = 0$, then x is a global minimizer of f. But this statement is *false* for quasiconvex functions: it is possible that $\nabla f(x) = 0$, but x is not a global minimizer of f.

Second-order conditions

Now suppose f is twice differentiable. If f is quasiconvex, then for all $x \in \mathbf{dom}\, f$, and all $y \in \mathbf{R}^n$, we have

$$y^T \nabla f(x) = 0 \Longrightarrow y^T \nabla^2 f(x) y \geq 0. \tag{3.21}$$

For a quasiconvex function on $\mathbf{R}$, this reduces to the simple condition

$$f'(x) = 0 \Longrightarrow f''(x) \geq 0,$$

i.e., at any point with zero slope, the second derivative is nonnegative. For a quasiconvex function on $\mathbf{R}^n$, the interpretation of the condition (3.21) is a bit more complicated. As in the case $n = 1$, we conclude that whenever $\nabla f(x) = 0$, we must have $\nabla^2 f(x) \succeq 0$. When $\nabla f(x) \neq 0$, the condition (3.21) means that $\nabla^2 f(x)$ is positive semidefinite on the $(n-1)$-dimensional subspace $\nabla f(x)^\perp$. This implies that $\nabla^2 f(x)$ can have at most one negative eigenvalue.

As a (partial) converse, if f satisfies

$$y^T \nabla f(x) = 0 \Longrightarrow y^T \nabla^2 f(x) y > 0 \tag{3.22}$$

for all $x \in \mathbf{dom}\, f$ and all $y \in \mathbf{R}^n$, $y \neq 0$, then f is quasiconvex. This condition is the same as requiring $\nabla^2 f(x)$ to be positive definite for any point with $\nabla f(x) = 0$, and for all other points, requiring $\nabla^2 f(x)$ to be positive definite on the $(n-1)$-dimensional subspace $\nabla f(x)^\perp$.

Proof of second-order conditions for quasiconvexity

By restricting the function to an arbitrary line, it suffices to consider the case in which $f : \mathbf{R} \to \mathbf{R}$.

We first show that if $f : \mathbf{R} \to \mathbf{R}$ is quasiconvex on an interval (a, b), then it must satisfy (3.21), *i.e.*, if $f'(c) = 0$ with $c \in (a, b)$, then we must have $f''(c) \geq 0$. If $f'(c) = 0$ with $c \in (a, b)$, $f''(c) < 0$, then for small positive ϵ we have $f(c-\epsilon) < f(c)$ and $f(c+\epsilon) < f(c)$. It follows that the sublevel set $\{x \mid f(x) \leq f(c) - \epsilon\}$ is disconnected for small positive ϵ, and therefore not convex, which contradicts our assumption that f is quasiconvex.

Now we show that if the condition (3.22) holds, then f is quasiconvex. Assume that (3.22) holds, *i.e.*, for each $c \in (a, b)$ with $f'(c) = 0$, we have $f''(c) > 0$. This means that whenever the function f' crosses the value 0, it is strictly increasing. Therefore it can cross the value 0 at most once. If f' does not cross the value 0 at all, then f is either nonincreasing or nondecreasing on (a, b), and therefore quasiconvex. Otherwise it must cross the value 0 exactly once, say at $c \in (a, b)$. Since $f''(c) > 0$, it follows that $f'(t) \leq 0$ for $a < t \leq c$, and $f'(t) \geq 0$ for $c \leq t < b$. This shows that f is quasiconvex.

3.4.4 Operations that preserve quasiconvexity

Nonnegative weighted maximum

A nonnegative weighted maximum of quasiconvex functions, *i.e.*,

$$f = \max\{w_1 f_1, \ldots, w_m f_m\},$$

with $w_i \geq 0$ and f_i quasiconvex, is quasiconvex. The property extends to the general pointwise supremum

$$f(x) = \sup_{y \in C}(w(y)g(x,y))$$

where $w(y) \geq 0$ and $g(x,y)$ is quasiconvex in x for each y. This fact can be easily verified: $f(x) \leq \alpha$ if and only if

$$w(y)g(x,y) \leq \alpha \text{ for all } y \in C,$$

i.e., the α-sublevel set of f is the intersection of the α-sublevel sets of the functions $w(y)g(x,y)$ in the variable x.

Example 3.37 *Generalized eigenvalue.* The *maximum generalized eigenvalue* of a pair of symmetric matrices (X, Y), with $Y \succ 0$, is defined as

$$\lambda_{\max}(X,Y) = \sup_{u \neq 0} \frac{u^T X u}{u^T Y u} = \sup\{\lambda \mid \det(\lambda Y - X) = 0\}.$$

(See §A.5.3). This function is quasiconvex on $\mathbf{dom}\, f = \mathbf{S}^n \times \mathbf{S}^n_{++}$.

To see this we consider the expression

$$\lambda_{\max}(X,Y) = \sup_{u \neq 0} \frac{u^T X u}{u^T Y u}.$$

For each $u \neq 0$, the function $u^T X u / u^T Y u$ is linear-fractional in (X, Y), hence a quasiconvex function of (X, Y). We conclude that $\lambda_{\max}$ is quasiconvex, since it is the supremum of a family of quasiconvex functions.

Composition

If $g : \mathbf{R}^n \to \mathbf{R}$ is quasiconvex and $h : \mathbf{R} \to \mathbf{R}$ is nondecreasing, then $f = h \circ g$ is quasiconvex.

The composition of a quasiconvex function with an affine or linear-fractional transformation yields a quasiconvex function. If f is quasiconvex, then $g(x) = f(Ax + b)$ is quasiconvex, and $\tilde{g}(x) = f((Ax + b)/(c^T x + d))$ is quasiconvex on the set

$$\{x \mid c^T x + d > 0,\ (Ax + b)/(c^T x + d) \in \mathbf{dom}\, f\}.$$

Minimization

If $f(x,y)$ is quasiconvex jointly in x and y and C is a convex set, then the function

$$g(x) = \inf_{y \in C} f(x,y)$$

is quasiconvex.

To show this, we need to show that $\{x \mid g(x) \leq \alpha\}$ is convex, where $\alpha \in \mathbf{R}$ is arbitrary. From the definition of g, $g(x) \leq \alpha$ if and only if for any $\epsilon > 0$ there exists a $y \in C$ with $f(x, y) \leq \alpha + \epsilon$. Now let x_1 and x_2 be two points in the α-sublevel set of g. Then for any $\epsilon > 0$, there exists y_1, $y_2 \in C$ with

$$f(x_1, y_1) \leq \alpha + \epsilon, \qquad f(x_2, y_2) \leq \alpha + \epsilon,$$

and since f is quasiconvex in x and y, we also have

$$f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \leq \alpha + \epsilon,$$

for $0 \leq \theta \leq 1$. Hence $g(\theta x_1 + (1-\theta)x_2) \leq \alpha$, which proves that $\{x \mid g(x) \leq \alpha\}$ is convex.

3.4.5 Representation via family of convex functions

In the sequel, it will be convenient to represent the sublevel sets of a quasiconvex function f (which are convex) via inequalities of convex functions. We seek a family of convex functions $\phi_t : \mathbf{R}^n \to \mathbf{R}$, indexed by $t \in \mathbf{R}$, with

$$f(x) \leq t \iff \phi_t(x) \leq 0, \tag{3.23}$$

i.e., the t-sublevel set of the quasiconvex function f is the 0-sublevel set of the convex function ϕ_t. Evidently ϕ_t must satisfy the property that for all $x \in \mathbf{R}^n$, $\phi_t(x) \leq 0 \implies \phi_s(x) \leq 0$ for $s \geq t$. This is satisfied if for each x, $\phi_t(x)$ is a nonincreasing function of t, *i.e.*, $\phi_s(x) \leq \phi_t(x)$ whenever $s \geq t$.

To see that such a representation always exists, we can take

$$\phi_t(x) = \begin{cases} 0 & f(x) \leq t \\ \infty & \text{otherwise.} \end{cases}$$

i.e., ϕ_t is the indicator function of the t-sublevel of f. Obviously this representation is not unique; for example if the sublevel sets of f are closed, we can take

$$\phi_t(x) = \mathbf{dist}\left(x, \{z \mid f(z) \leq t\}\right).$$

We are usually interested in a family ϕ_t with nice properties, such as differentiability.

Example 3.38 *Convex over concave function.* Suppose p is a convex function, q is a concave function, with $p(x) \geq 0$ and $q(x) > 0$ on a convex set C. Then the function f defined by $f(x) = p(x)/q(x)$, on C, is quasiconvex.

Here we have

$$f(x) \leq t \iff p(x) - tq(x) \leq 0,$$

so we can take $\phi_t(x) = p(x) - tq(x)$ for $t \geq 0$. For each t, ϕ_t is convex and for each x, $\phi_t(x)$ is decreasing in t.

3.5 Log-concave and log-convex functions

3.5.1 Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is *logarithmically concave* or *log-concave* if $f(x) > 0$ for all $x \in \mathbf{dom}\, f$ and $\log f$ is concave. It is said to be *logarithmically convex* or *log-convex* if $\log f$ is convex. Thus f is log-convex if and only if $1/f$ is log-concave. It is convenient to allow f to take on the value zero, in which case we take $\log f(x) = -\infty$. In this case we say f is log-concave if the extended-value function $\log f$ is concave.

We can express log-concavity directly, without logarithms: a function $f : \mathbf{R}^n \to \mathbf{R}$, with convex domain and $f(x) > 0$ for all $x \in \mathbf{dom}\, f$, is log-concave if and only if for all $x,\ y \in \mathbf{dom}\, f$ and $0 \le \theta \le 1$, we have

$$f(\theta x + (1-\theta)y) \ge f(x)^\theta f(y)^{1-\theta}.$$

In particular, the value of a log-concave function at the average of two points is at least the *geometric mean* of the values at the two points.

From the composition rules we know that e^h is convex if h is convex, so a log-convex function is convex. Similarly, a nonnegative concave function is log-concave. It is also clear that a log-convex function is quasiconvex and a log-concave function is quasiconcave, since the logarithm is monotone increasing.

Example 3.39 *Some simple examples of log-concave and log-convex functions.*

- *Affine function.* $f(x) = a^T x + b$ is log-concave on $\{x \mid a^T x + b > 0\}$.
- *Powers.* $f(x) = x^a$, on $\mathbf{R}_{++}$, is log-convex for $a \le 0$, and log-concave for $a \ge 0$.
- *Exponentials.* $f(x) = e^{ax}$ is log-convex and log-concave.
- The cumulative distribution function of a Gaussian density,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

  is log-concave (see exercise 3.54).
- *Gamma function.* The Gamma function,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du,$$

  is log-convex for $x \ge 1$ (see exercise 3.52).
- *Determinant.* $\det X$ is log concave on $\mathbf{S}^n_{++}$.
- *Determinant over trace.* $\det X / \mathbf{tr}\, X$ is log concave on $\mathbf{S}^n_{++}$ (see exercise 3.49).

Example 3.40 *Log-concave density functions.* Many common probability density functions are log-concave. Two examples are the multivariate normal distribution,

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1} (x-\bar{x})}$$

(where $\bar{x} \in \mathbf{R}^n$ and $\Sigma \in \mathbf{S}^n_{++}$), and the exponential distribution on $\mathbf{R}^n_+$,

$$f(x) = \left(\prod_{i=1}^n \lambda_i\right) e^{-\lambda^T x}$$

(where $\lambda \succ 0$). Another example is the uniform distribution over a convex set C,

$$f(x) = \begin{cases} 1/\alpha & x \in C \\ 0 & x \notin C \end{cases}$$

where $\alpha = \mathbf{vol}(C)$ is the volume (Lebesgue measure) of C. In this case $\log f$ takes on the value $-\infty$ outside C, and $-\log\alpha$ on C, hence is concave.

As a more exotic example consider the Wishart distribution, defined as follows. Let $x_1, \ldots, x_p \in \mathbf{R}^n$ be independent Gaussian random vectors with zero mean and covariance $\Sigma \in \mathbf{S}^n$, with $p > n$. The random matrix $X = \sum_{i=1}^p x_i x_i^T$ has the Wishart density

$$f(X) = a\,(\det X)^{(p-n-1)/2}\, e^{-\frac{1}{2}\mathbf{tr}(\Sigma^{-1}X)},$$

with $\mathbf{dom}\, f = \mathbf{S}^n_{++}$, and a is a positive constant. The Wishart density is log-concave, since

$$\log f(X) = \log a + \frac{p-n-1}{2}\log\det X - \frac{1}{2}\mathbf{tr}(\Sigma^{-1}X),$$

which is a concave function of X.

3.5.2 Properties

Twice differentiable log-convex/concave functions

Suppose f is twice differentiable, with $\mathbf{dom}\, f$ convex, so

$$\nabla^2 \log f(x) = \frac{1}{f(x)}\nabla^2 f(x) - \frac{1}{f(x)^2}\nabla f(x)\nabla f(x)^T.$$

We conclude that f is log-convex if and only if for all $x \in \mathbf{dom}\, f$,

$$f(x)\nabla^2 f(x) \succeq \nabla f(x)\nabla f(x)^T,$$

and log-concave if and only if for all $x \in \mathbf{dom}\, f$,

$$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^T.$$

Multiplication, addition, and integration

Log-convexity and log-concavity are closed under multiplication and positive scaling. For example, if f and g are log-concave, then so is the pointwise product $h(x) = f(x)g(x)$, since $\log h(x) = \log f(x) + \log g(x)$, and $\log f(x)$ and $\log g(x)$ are concave functions of x.

Simple examples show that the sum of log-concave functions is not, in general, log-concave. Log-convexity, however, is preserved under sums. Let f and g be log-convex functions, *i.e.*, $F = \log f$ and $G = \log g$ are convex. From the composition rules for convex functions, it follows that

$$\log(\exp F + \exp G) = \log(f + g)$$

is convex. Therefore the sum of two log-convex functions is log-convex.

More generally, if $f(x,y)$ is log-convex in x for each $y \in C$ then

$$g(x) = \int_C f(x,y)\, dy$$

is log-convex.

Example 3.41 *Laplace transform of a nonnegative function and the moment and cumulant generating functions.* Suppose $p : \mathbf{R}^n \to \mathbf{R}$ satisfies $p(x) \geq 0$ for all x. The Laplace transform of p,

$$P(z) = \int p(x) e^{-z^T x}\, dx,$$

is log-convex on $\mathbf{R}^n$. (Here $\mathbf{dom}\, P$ is, naturally, $\{z \mid P(z) < \infty\}$.)

Now suppose p is a density, *i.e.*, satisfies $\int p(x)\, dx = 1$. The function $M(z) = P(-z)$ is called the *moment generating function* of the density. It gets its name from the fact that the moments of the density can be found from the derivatives of the moment generating function, evaluated at $z = 0$, *e.g.*,

$$\nabla M(0) = \mathbf{E}\, v, \qquad \nabla^2 M(0) = \mathbf{E}\, v v^T,$$

where v is a random variable with density p.

The function $\log M(z)$, which is convex, is called the *cumulant generating function* for p, since its derivatives give the cumulants of the density. For example, the first and second derivatives of the cumulant generating function, evaluated at zero, are the mean and covariance of the associated random variable:

$$\nabla \log M(0) = \mathbf{E}\, v, \qquad \nabla^2 \log M(0) = \mathbf{E}(v - \mathbf{E}\, v)(v - \mathbf{E}\, v)^T.$$

Integration of log-concave functions

In some special cases log-concavity is preserved by integration. If $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is log-concave, then

$$g(x) = \int f(x,y)\, dy$$

is a log-concave function of x (on $\mathbf{R}^n$). (The integration here is over $\mathbf{R}^m$.) A proof of this result is not simple; see the references.

This result has many important consequences, some of which we describe in the rest of this section. It implies, for example, that marginal distributions of log-concave probability densities are log-concave. It also implies that log-concavity is closed under convolution, *i.e.*, if f and g are log-concave on $\mathbf{R}^n$, then so is the convolution

$$(f * g)(x) = \int f(x-y) g(y)\, dy.$$

(To see this, note that $g(y)$ and $f(x-y)$ are log-concave in (x,y), hence the product $f(x-y)g(y)$ is; then the integration result applies.)

Suppose $C \subseteq \mathbf{R}^n$ is a convex set and w is a random vector in $\mathbf{R}^n$ with log-concave probability density p. Then the function

$$f(x) = \mathbf{prob}(x + w \in C)$$

is log-concave in x. To see this, express f as

$$f(x) = \int g(x+w)p(w)\, dw,$$

where g is defined as

$$g(u) = \begin{cases} 1 & u \in C \\ 0 & u \notin C, \end{cases}$$

(which is log-concave) and apply the integration result.

Example 3.42 The *cumulative distribution function* of a probability density function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$F(x) = \mathbf{prob}(w \preceq x) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f(z)\, dz_1 \cdots dz_n,$$

where w is a random variable with density f. If f is log-concave, then F is log-concave. We have already encountered a special case: the cumulative distribution function of a Gaussian random variable,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

is log-concave. (See example 3.39 and exercise 3.54.)

Example 3.43 *Yield function.* Let $x \in \mathbf{R}^n$ denote the nominal or target value of a set of parameters of a product that is manufactured. Variation in the manufacturing process causes the parameters of the product, when manufactured, to have the value $x + w$, where $w \in \mathbf{R}^n$ is a random vector that represents manufacturing variation, and is usually assumed to have zero mean. The *yield* of the manufacturing process, as a function of the nominal parameter values, is given by

$$Y(x) = \mathbf{prob}(x + w \in S),$$

where $S \subseteq \mathbf{R}^n$ denotes the set of acceptable parameter values for the product, *i.e.*, the product *specifications*.

If the density of the manufacturing error w is log-concave (for example, Gaussian) and the set S of product specifications is convex, then the yield function Y is log-concave. This implies that the α-*yield region*, defined as the set of nominal parameters for which the yield exceeds α, is convex. For example, the 95% yield region

$$\{x \mid Y(x) \geq 0.95\} = \{x \mid \log Y(x) \geq \log 0.95\}$$

is convex, since it is a superlevel set of the concave function $\log Y$.

Example 3.44 *Volume of polyhedron.* Let $A \in \mathbf{R}^{m \times n}$. Define

$$P_u = \{x \in \mathbf{R}^n \mid Ax \preceq u\}.$$

Then its volume $\mathbf{vol}\, P_u$ is a log-concave function of u.

To prove this, note that the function

$$\Psi(x, u) = \begin{cases} 1 & Ax \preceq u \\ 0 & \text{otherwise}, \end{cases}$$

is log-concave. By the integration result, we conclude that

$$\int \Psi(x, u)\, dx = \mathbf{vol}\, P_u$$

is log-concave.

3.6 Convexity with respect to generalized inequalities

We now consider generalizations of the notions of monotonicity and convexity, using generalized inequalities instead of the usual ordering on $\mathbf{R}$.

3.6.1 Monotonicity with respect to a generalized inequality

Suppose $K \subseteq \mathbf{R}^n$ is a proper cone with associated generalized inequality $\preceq_K$. A function $f : \mathbf{R}^n \to \mathbf{R}$ is called *K-nondecreasing* if

$$x \preceq_K y \Longrightarrow f(x) \leq f(y),$$

and *K-increasing* if

$$x \preceq_K y,\ x \neq y \Longrightarrow f(x) < f(y).$$

We define *K-nonincreasing* and *K-decreasing* functions in a similar way.

Example 3.45 *Monotone vector functions.* A function $f : \mathbf{R}^n \to \mathbf{R}$ is nondecreasing with respect to $\mathbf{R}^n_+$ if and only if

$$x_1 \leq y_1, \ldots, x_n \leq y_n \implies f(x) \leq f(y)$$

for all x, y. This is the same as saying that f, when restricted to any component x_i (*i.e.*, x_i is considered the variable while x_j for $j \neq i$ are fixed), is nondecreasing.

Example 3.46 *Matrix monotone functions.* A function $f : \mathbf{S}^n \to \mathbf{R}$ is called *matrix monotone* (increasing, decreasing) if it is monotone with respect to the positive semidefinite cone. Some examples of matrix monotone functions of the variable $X \in \mathbf{S}^n$:

- $\mathbf{tr}(WX)$, where $W \in \mathbf{S}^n$, is matrix nondecreasing if $W \succeq 0$, and matrix increasing if $W \succ 0$ (it is matrix nonincreasing if $W \preceq 0$, and matrix decreasing if $W \prec 0$).
- $\mathbf{tr}(X^{-1})$ is matrix decreasing on $\mathbf{S}^n_{++}$.
- $\det X$ is matrix increasing on $\mathbf{S}^n_+$.

Gradient conditions for monotonicity

Recall that a differentiable function $f : \mathbf{R} \to \mathbf{R}$, with convex (*i.e.*, interval) domain, is nondecreasing if and only if $f'(x) \geq 0$ for all $x \in \mathbf{dom}\, f$, and increasing if $f'(x) > 0$ for all $x \in \mathbf{dom}\, f$ (but the converse is not true). These conditions are readily extended to the case of monotonicity with respect to a generalized inequality. A differentiable function f, with convex domain, is K-nondecreasing if and only if

$$\nabla f(x) \succeq_{K^*} 0 \tag{3.24}$$

for all $x \in \mathbf{dom}\, f$. Note the difference with the simple scalar case: the gradient must be nonnegative in the *dual* inequality. For the strict case, we have the following: If

$$\nabla f(x) \succ_{K^*} 0 \tag{3.25}$$

for all $x \in \mathbf{dom}\, f$, then f is K-increasing. As in the scalar case, the converse is not true.

Let us prove these first-order conditions for monotonicity. First, assume that f satisfies (3.24) for all x, but is not K-nondecreasing, *i.e.*, there exist x, y with $x \preceq_K y$ and $f(y) < f(x)$. By differentiability of f there exists a $t \in [0, 1]$ with

$$\frac{d}{dt} f(x + t(y - x)) = \nabla f(x + t(y - x))^T (y - x) < 0.$$

Since $y - x \in K$ this means

$$\nabla f(x + t(y - x)) \notin K^*,$$

which contradicts our assumption that (3.24) is satisfied everywhere. In a similar way it can be shown that (3.25) implies f is K-increasing.

It is also straightforward to see that it is necessary that (3.24) hold everywhere. Assume (3.24) does not hold for $x = z$. By the definition of dual cone this means there exists a $v \in K$ with

$$\nabla f(z)^T v < 0.$$

Now consider $h(t) = f(z + tv)$ as a function of t. We have $h'(0) = \nabla f(z)^T v < 0$, and therefore there exists $t > 0$ with $h(t) = f(z + tv) < h(0) = f(z)$, which means f is not K-nondecreasing.

3.6.2 Convexity with respect to a generalized inequality

Suppose $K \subseteq \mathbf{R}^m$ is a proper cone with associated generalized inequality $\preceq_K$. We say $f : \mathbf{R}^n \to \mathbf{R}^m$ is *K-convex* if for all x, y, and $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta) y) \ \preceq_K \ \theta f(x) + (1 - \theta) f(y).$$

The function is *strictly K-convex* if

$$f(\theta x + (1-\theta)y) \prec_K \theta f(x) + (1-\theta)f(y)$$

for all $x \neq y$ and $0 < \theta < 1$. These definitions reduce to ordinary convexity and strict convexity when $m = 1$ (and $K = \mathbf{R}_+$).

Example 3.47 *Convexity with respect to componentwise inequality.* A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is convex with respect to componentwise inequality (*i.e.*, the generalized inequality induced by $\mathbf{R}^m_+$) if and only if for all x, y and $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \preceq \theta f(x) + (1-\theta)f(y),$$

i.e., each component f_i is a convex function. The function f is strictly convex with respect to componentwise inequality if and only if each component f_i is strictly convex.

Example 3.48 *Matrix convexity.* Suppose f is a symmetric matrix valued function, *i.e.*, $f : \mathbf{R}^n \to \mathbf{S}^m$. The function f is convex with respect to matrix inequality if

$$f(\theta x + (1-\theta)y) \preceq \theta f(x) + (1-\theta)f(y)$$

for any x and y, and for $\theta \in [0, 1]$. This is sometimes called *matrix convexity*. An equivalent definition is that the scalar function $z^T f(x) z$ is convex for all vectors z. (This is often a good way to prove matrix convexity). A matrix function is strictly matrix convex if

$$f(\theta x + (1-\theta)y) \prec \theta f(x) + (1-\theta)f(y)$$

when $x \neq y$ and $0 < \theta < 1$, or, equivalently, if $z^T f z$ is strictly convex for every $z \neq 0$.

Some examples:

- The function $f(X) = XX^T$ where $X \in \mathbf{R}^{n\times m}$ is matrix convex, since for fixed z the function $z^T XX^T z = \|X^T z\|_2^2$ is a convex quadratic function of (the components of) X. For the same reason, $f(X) = X^2$ is matrix convex on $\mathbf{S}^n$.
- The function X^p is matrix convex on $\mathbf{S}^n_{++}$ for $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and matrix concave for $0 \leq p \leq 1$.
- The function $f(X) = e^X$ is *not* matrix convex on $\mathbf{S}^n$, for $n \geq 2$.

Many of the results for convex functions have extensions to K-convex functions. As a simple example, a function is K-convex if and only if its restriction to any line in its domain is K-convex. In the rest of this section we list a few results for K-convexity that we will use later; more results are explored in the exercises.

Dual characterization of K-convexity

A function f is K-convex if and only if for every $w \succeq_{K^*} 0$, the (real-valued) function $w^T f$ is convex (in the ordinary sense); f is strictly K-convex if and only if for every nonzero $w \succeq_{K^*} 0$ the function $w^T f$ is strictly convex. (These follow directly from the definitions and properties of dual inequality.)

Differentiable K-convex functions

A differentiable function f is K-convex if and only if its domain is convex, and for all $x,\ y \in \mathbf{dom}\, f$,

$$f(y) \succeq_K f(x) + Df(x)(y - x).$$

(Here $Df(x) \in \mathbf{R}^{m\times n}$ is the derivative or Jacobian matrix of f at x; see §A.4.1.) The function f is strictly K-convex if and only if for all $x,\ y \in \mathbf{dom}\, f$ with $x \neq y$,

$$f(y) \succ_K f(x) + Df(x)(y - x).$$

Composition theorem

Many of the results on composition can be generalized to K-convexity. For example, if $g : \mathbf{R}^n \to \mathbf{R}^p$ is K-convex, $h : \mathbf{R}^p \to \mathbf{R}$ is convex, and $\tilde{h}$ (the extended-value extension of h) is K-nondecreasing, then $h \circ g$ is convex. This generalizes the fact that a nondecreasing convex function of a convex function is convex. The condition that $\tilde{h}$ be K-nondecreasing implies that $\mathbf{dom}\, h - K = \mathbf{dom}\, h$.

Example 3.49 The quadratic matrix function $g : \mathbf{R}^{m\times n} \to \mathbf{S}^n$ defined by

$$g(X) = X^T AX + B^T X + X^T B + C,$$

where $A \in \mathbf{S}^m$, $B \in \mathbf{R}^{m\times n}$, and $C \in \mathbf{S}^n$, is convex when $A \succeq 0$.

The function $h : \mathbf{S}^n \to \mathbf{R}$ defined by $h(Y) = -\log\det(-Y)$ is convex and increasing on $\mathbf{dom}\, h = -\mathbf{S}^n_{++}$.

By the composition theorem, we conclude that

$$f(X) = -\log\det(-(X^T AX + B^T X + X^T B + C))$$

is convex on

$$\mathbf{dom}\, f = \{X \in \mathbf{R}^{m\times n} \mid X^T AX + B^T X + X^T B + C \prec 0\}.$$

This generalizes the fact that

$$-\log(-(ax^2 + bx + c))$$

is convex on

$$\{x \in \mathbf{R} \mid ax^2 + bx + c < 0\},$$

provided $a \geq 0$.

Bibliography

The standard reference on convex analysis is Rockafellar [Roc70]. Other books on convex functions are Stoer and Witzgall [SW70], Roberts and Varberg [RV73], Van Tiel [vT84], Hiriart-Urruty and Lemaréchal [HUL93], Ekeland and Témam [ET99], Borwein and Lewis [BL00], Florenzano and Le Van [FL01], Barvinok [Bar02], and Bertsekas, Nedić, and Ozdaglar [Ber03]. Most nonlinear programming texts also include chapters on convex functions (see, for example, Mangasarian [Man94], Bazaraa, Sherali, and Shetty [BSS93], Bertsekas [Ber99], Polyak [Pol87], and Peressini, Sullivan, and Uhl [PSU88]).

Jensen's inequality appears in [Jen06]. A general study of inequalities, in which Jensen's inequality plays a central role, is presented by Hardy, Littlewood, and Pólya [HLP52], and Beckenbach and Bellman [BB65].

The term *perspective function* is from Hiriart-Urruty and Lemaréchal [HUL93, volume 1, page 100]. For the definitions in example 3.19 (relative entropy and Kullback-Leibler divergence), and the related exercise 3.13, see Cover and Thomas [CT91].

Some important early references on quasiconvex functions (as well as other extensions of convexity) are Nikaidô [Nik54], Mangasarian [Man94, chapter 9], Arrow and Enthoven [AE61], Ponstein [Pon67], and Luenberger [Lue68]. For a more comprehensive reference list, we refer to Bazaraa, Sherali, and Shetty [BSS93, page 126].

Prékopa [Pré80] gives a survey of log-concave functions. Log-convexity of the Laplace transform is mentioned in Barndorff-Nielsen [BN78, §7]. For a proof of the integration result of log-concave functions, see Prékopa [Pré71, Pré73].

Generalized inequalities are used extensively in the recent literature on cone programming, starting with Nesterov and Nemirovski [NN94, page 156]; see also Ben-Tal and Nemirovski [BTN01] and the references at the end of chapter 4. Convexity with respect to generalized inequalities also appears in the work of Luenberger [Lue69, §8.2] and Isii [Isi64]. Matrix monotonicity and matrix convexity are attributed to Löwner [Löw34], and are discussed in detail by Davis [Dav63], Roberts and Varberg [RV73, page 216] and Marshall and Olkin [MO79, §16E]. For the result on convexity and concavity of the function X^p in example 3.48, see Bondar [Bon94, theorem 16.1]. For a simple example that demonstrates that e^X is not matrix convex, see Marshall and Olkin [MO79, page 474].

Exercises

Definition of convexity

3.1 Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, and $a,\ b \in \mathbf{dom}\, f$ with $a < b$.

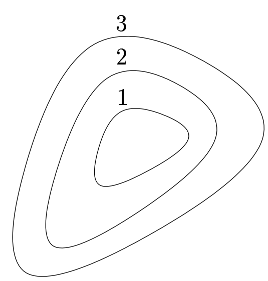
(a) Show that
$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$
for all $x \in [a, b]$.

(b) Show that
$$\frac{f(x) - f(a)}{x-a} \le \frac{f(b) - f(a)}{b-a} \le \frac{f(b) - f(x)}{b-x}$$
for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

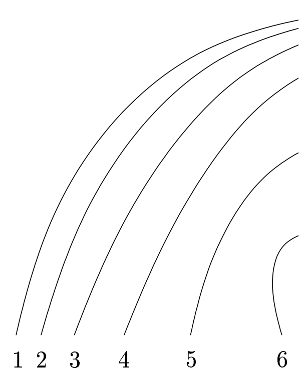
(c) Suppose f is differentiable. Use the result in (b) to show that
$$f'(a) \le \frac{f(b) - f(a)}{b-a} \le f'(b).$$
Note that these inequalities also follow from (3.2):
$$f(b) \ge f(a) + f'(a)(b-a), \qquad f(a) \ge f(b) + f'(b)(a-b).$$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(a) \ge 0$ and $f''(b) \ge 0$.

3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.



Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.



3.3 *Inverse of an increasing convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse, *i.e.*, the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?

3.4 [RV73, page 15] Show that a continuous function $f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if for every line segment, its average value on the segment is less than or equal to the average of its values at the endpoints of the segment: For every $x,\ y \in \mathbf{R}^n$,

$$\int_0^1 f(x + \lambda(y - x))\, d\lambda \leq \frac{f(x) + f(y)}{2}.$$

3.5 [RV73, page 22] *Running average of a convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, with $\mathbf{R}_+ \subseteq \mathbf{dom}\, f$. Show that its *running average* F, defined as

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt, \qquad \mathbf{dom}\, F = \mathbf{R}_{++},$$

is convex. You can assume f is differentiable.

3.6 *Functions and epigraphs.* When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

3.7 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex with $\mathbf{dom}\, f = \mathbf{R}^n$, and bounded above on $\mathbf{R}^n$. Show that f is constant.

3.8 *Second-order condition for convexity.* Prove that a twice differentiable function f is convex if and only if its domain is convex and $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbf{dom}\, f$. *Hint.* First consider the case $f : \mathbf{R} \to \mathbf{R}$. You can use the first-order condition for convexity (which was proved on page 70).

3.9 *Second-order conditions for convexity on an affine set.* Let $F \in \mathbf{R}^{n \times m}$, $\hat{x} \in \mathbf{R}^n$. The *restriction* of $f : \mathbf{R}^n \to \mathbf{R}$ to the affine set $\{Fz + \hat{x} \mid z \in \mathbf{R}^m\}$ is defined as the function $\tilde{f} : \mathbf{R}^m \to \mathbf{R}$ with

$$\tilde{f}(z) = f(Fz + \hat{x}), \qquad \mathbf{dom}\, \tilde{f} = \{z \mid Fz + \hat{x} \in \mathbf{dom}\, f\}.$$

Suppose f is twice differentiable with a convex domain.

(a) Show that $\tilde{f}$ is convex if and only if for all $z \in \mathbf{dom}\, \tilde{f}$

$$F^T \nabla^2 f(Fz + \hat{x}) F \succeq 0.$$

(b) Suppose $A \in \mathbf{R}^{p \times n}$ is a matrix whose nullspace is equal to the range of F, *i.e.*, $AF = 0$ and $\mathbf{rank}\, A = n - \mathbf{rank}\, F$. Show that $\tilde{f}$ is convex if and only if for all $z \in \mathbf{dom}\, \tilde{f}$ there exists a $\lambda \in \mathbf{R}$ such that

$$\nabla^2 f(Fz + \hat{x}) + \lambda A^T A \succeq 0.$$

Hint. Use the following result: If $B \in \mathbf{S}^n$ and $A \in \mathbf{R}^{p \times n}$, then $x^T B x \geq 0$ for all $x \in \mathcal{N}(A)$ if and only if there exists a λ such that $B + \lambda A^T A \succeq 0$.

3.10 *An extension of Jensen's inequality.* One interpretation of Jensen's inequality is that randomization or dithering hurts, *i.e.*, raises the average value of a convex function: For f convex and v a zero mean random variable, we have $\mathbf{E}\, f(x_0 + v) \geq f(x_0)$. This leads to the following conjecture. If f_0 is convex, then the larger the variance of v, the larger $\mathbf{E}\, f(x_0 + v)$.

(a) Give a counterexample that shows that this conjecture is false. Find zero mean random variables v and w, with $\mathbf{var}(v) > \mathbf{var}(w)$, a convex function f, and a point x_0, such that $\mathbf{E}\, f(x_0 + v) < \mathbf{E}\, f(x_0 + w)$.

(b) The conjecture is true when v and w are scaled versions of each other. Show that $\mathbf{E} f(x_0 + tv)$ is monotone increasing in $t \geq 0$, when f is convex and v is zero mean.

3.11 *Monotone mappings.* A function $\psi : \mathbf{R}^n \to \mathbf{R}^n$ is called *monotone* if for all $x,\ y \in \mathbf{dom}\,\psi$,

$$(\psi(x) - \psi(y))^T (x - y) \geq 0.$$

(Note that 'monotone' as defined here is not the same as the definition given in §3.6.1. Both definitions are widely used.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is a differentiable convex function. Show that its gradient ∇f is monotone. Is the converse true, *i.e.*, is every monotone mapping the gradient of a convex function?

3.12 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex, $g : \mathbf{R}^n \to \mathbf{R}$ is concave, $\mathbf{dom}\, f = \mathbf{dom}\, g = \mathbf{R}^n$, and for all x, $g(x) \leq f(x)$. Show that there exists an affine function h such that for all x, $g(x) \leq h(x) \leq f(x)$. In other words, if a concave function g is an underestimator of a convex function f, then we can fit an affine function between f and g.

3.13 *Kullback-Leibler divergence and the information inequality.* Let D_{kl} be the Kullback-Leibler divergence, as defined in (3.17). Prove the *information inequality*: $D_{\mathrm{kl}}(u, v) \geq 0$ for all $u,\ v \in \mathbf{R}^n_{++}$. Also show that $D_{\mathrm{kl}}(u, v) = 0$ if and only if $u = v$.

Hint. The Kullback-Leibler divergence can be expressed as

$$D_{\mathrm{kl}}(u, v) = f(u) - f(v) - \nabla f(v)^T (u - v),$$

where $f(v) = \sum_{i=1}^n v_i \log v_i$ is the negative entropy of v.

3.14 *Convex-concave functions and saddle-points.* We say the function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is *convex-concave* if $f(x, z)$ is a concave function of z, for each fixed x, and a convex function of x, for each fixed z. We also require its domain to have the product form $\mathbf{dom}\, f = A \times B$, where $A \subseteq \mathbf{R}^n$ and $B \subseteq \mathbf{R}^m$ are convex.

(a) Give a second-order condition for a twice differentiable function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ to be convex-concave, in terms of its Hessian $\nabla^2 f(x, z)$.

(b) Suppose that $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is convex-concave and differentiable, with $\nabla f(\tilde{x}, \tilde{z}) = 0$. Show that the *saddle-point property* holds: for all x, z, we have

$$f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z}).$$

Show that this implies that f satisfies the *strong max-min property*:

$$\sup_z \inf_x f(x, z) = \inf_x \sup_z f(x, z)$$

(and their common value is $f(\tilde{x}, \tilde{z})$).

(c) Now suppose that $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is differentiable, but not necessarily convex-concave, and the saddle-point property holds at $\tilde{x}, \tilde{z}$:

$$f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z})$$

for all x, z. Show that $\nabla f(\tilde{x}, \tilde{z}) = 0$.

Examples

3.15 *A family of concave utility functions.* For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\mathbf{dom}\, u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\, u_0 = \mathbf{R}_{++}$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$.

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (*i.e.*, the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation.*

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbf{R}^2_{++}$.

3.17 Suppose $p < 1$, $p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^n x_i^p \right)^{1/p}$$

with $\mathbf{dom}\, f = \mathbf{R}^n_{++}$ is concave. This includes as special cases $f(x) = (\sum_{i=1}^n x_i^{1/2})^2$ and the *harmonic mean* $f(x) = (\sum_{i=1}^n 1/x_i)^{-1}$. *Hint.* Adapt the proofs for the log-sum-exp function and the geometric mean in §3.1.5.

3.18 Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.

(a) $f(X) = \mathbf{tr}\left(X^{-1}\right)$ is convex on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

(b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

3.19 *Nonnegative weighted sums and integrals.*

(a) Show that $f(x) = \sum_{i=1}^r \alpha_i x_{[i]}$ is a convex function of x, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the ith largest component of x. (You can use the fact that $f(x) = \sum_{i=1}^k x_{[i]}$ is convex on $\mathbf{R}^n$.)

(b) Let $T(x, \omega)$ denote the trigonometric polynomial

$$T(x, \omega) = x_1 + x_2 \cos\omega + x_3 \cos 2\omega + \cdots + x_n \cos(n-1)\omega.$$

Show that the function

$$f(x) = -\int_0^{2\pi} \log T(x, \omega)\, d\omega$$

is convex on $\{x \in \mathbf{R}^n \mid T(x, \omega) > 0,\ 0 \leq \omega \leq 2\pi\}$.

3.20 *Composition with an affine function.* Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.

(a) $f(x) = \|Ax - b\|$, where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

(b) $f(x) = -\left(\det(A_0 + x_1 A_1 + \cdots + x_n A_n)\right)^{1/m}$, on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$, where $A_i \in \mathbf{S}^m$.

(c) $f(X) = \mathbf{tr}\left(A_0 + x_1 A_1 + \cdots + x_n A_n\right)^{-1}$, on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$, where $A_i \in \mathbf{S}^m$. (Use the fact that $\mathbf{tr}(X^{-1})$ is convex on $\mathbf{S}^m_{++}$; see exercise 3.18.)

3.21 *Pointwise maximum and supremum.* Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.

(a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$, where $A^{(i)} \in \mathbf{R}^{m\times n}$, $b^{(i)} \in \mathbf{R}^m$ and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

(b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbf{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (*i.e.*, $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

3.22 *Composition rules.* Show that the following functions are convex.

(a) $f(x) = -\log(-\log(\sum_{i=1}^m e^{a_i^T x + b_i}))$ on $\mathbf{dom}\, f = \{x \mid \sum_{i=1}^m e^{a_i^T x+b_i} < 1\}$. You can use the fact that $\log(\sum_{i=1}^n e^{y_i})$ is convex.

(b) $f(x,u,v) = -\sqrt{uv - x^T x}$ on $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^T x,\ u,\ v > 0\}$. Use the fact that $x^T x/u$ is convex in (x,u) for $u > 0$, and that $-\sqrt{x_1 x_2}$ is convex on $\mathbf{R}^2_{++}$.

(c) $f(x,u,v) = -\log(uv - x^T x)$ on $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^T x,\ u,\ v > 0\}$.

(d) $f(x,t) = -(t^p - \|x\|_p^p)^{1/p}$ where $p > 1$ and $\mathbf{dom}\, f = \{(x,t) \mid t \geq \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x,u) for $u > 0$ (see exercise 3.23), and that $-x^{1/p}y^{1-1/p}$ is convex on $\mathbf{R}^2_+$ (see exercise 3.16).

(e) $f(x,t) = -\log(t^p - \|x\|_p^p)$ where $p > 1$ and $\mathbf{dom}\, f = \{(x,t) \mid t > \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x,u) for $u > 0$ (see exercise 3.23).

3.23 *Perspective of a function.*

(a) Show that for $p > 1$,

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{\|x\|_p^p}{t^{p-1}}$$

is convex on $\{(x,t) \mid t > \|x\|_p\}$.

(b) Show that

$$f(x) = \frac{\|Ax + b\|_2^2}{c^T x + d}$$

is convex on $\{x \mid c^T x + d > 0\}$, where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$ and $d \in \mathbf{R}$.

3.24 *Some functions on the probability simplex.* Let x be a real-valued random variable which takes values in $\{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$. For each of the following functions of p (on the probability simplex $\{p \in \mathbf{R}^n_+ \mid \mathbf{1}^T p = 1\}$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

(a) $\mathbf{E}\, x$.

(b) $\mathbf{prob}(x \geq \alpha)$.

(c) $\mathbf{prob}(\alpha \leq x \leq \beta)$.

(d) $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.

(e) $\mathbf{var}\, x = \mathbf{E}(x - \mathbf{E}\, x)^2$.

(f) $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(g) The cardinality of the smallest set $\mathcal{A} \subseteq \{a_1, \ldots, a_n\}$ with probability $\geq 90\%$. (By cardinality we mean the number of elements in $\mathcal{A}$.)

(h) The minimum width interval that contains 90% of the probability, *i.e.*,

$$\inf\{\beta - \alpha \mid \mathbf{prob}(\alpha \leq x \leq \beta) \geq 0.9\}.$$

3.25 *Maximum probability distance between distributions.* Let $p,\ q \in \mathbf{R}^n$ represent two probability distributions on $\{1,\ldots,n\}$ (so $p,\ q \succeq 0$, $\mathbf{1}^T p = \mathbf{1}^T q = 1$). We define the *maximum probability distance* $d_{\rm mp}(p,q)$ between p and q as the maximum difference in probability assigned by p and q, over all events:

$$d_{\rm mp}(p,q) = \max\{|\,\mathbf{prob}(p,C) - \mathbf{prob}(q,C)| \mid C \subseteq \{1,\ldots,n\}\}.$$

Here $\mathbf{prob}(p,C)$ is the probability of C, under the distribution p, *i.e.*, $\mathbf{prob}(p,C) = \sum_{i\in C} p_i$.

Find a simple expression for $d_{\rm mp}$, involving $\|p-q\|_1 = \sum_{i=1}^n |p_i - q_i|$, and show that $d_{\rm mp}$ is a convex function on $\mathbf{R}^n \times \mathbf{R}^n$. (Its domain is $\{(p,q) \mid p,\ q \succeq 0,\ \mathbf{1}^T p = \mathbf{1}^T q = 1\}$, but it has a natural extension to all of $\mathbf{R}^n \times \mathbf{R}^n$.)

3.26 *More functions of eigenvalues.* Let $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ denote the eigenvalues of a matrix $X \in \mathbf{S}^n$. We have already seen several functions of the eigenvalues that are convex or concave functions of X.

- The maximum eigenvalue $\lambda_1(X)$ is convex (example 3.10). The minimum eigenvalue $\lambda_n(X)$ is concave.
- The sum of the eigenvalues (or trace), $\mathbf{tr}\, X = \lambda_1(X) + \cdots + \lambda_n(X)$, is linear.
- The sum of the inverses of the eigenvalues (or trace of the inverse), $\mathbf{tr}(X^{-1}) = \sum_{i=1}^n 1/\lambda_i(X)$, is convex on $\mathbf{S}^n_{++}$ (exercise 3.18).
- The geometric mean of the eigenvalues, $(\det X)^{1/n} = (\prod_{i=1}^n \lambda_i(X))^{1/n}$, and the logarithm of the product of the eigenvalues, $\log\det X = \sum_{i=1}^n \log \lambda_i(X)$, are concave on $X \in \mathbf{S}^n_{++}$ (exercise 3.18 and page 74).

In this problem we explore some more functions of eigenvalues, by exploiting variational characterizations.

(a) *Sum of k largest eigenvalues.* Show that $\sum_{i=1}^k \lambda_i(X)$ is convex on $\mathbf{S}^n$. *Hint.* [HJ85, page 191] Use the variational characterization

$$\sum_{i=1}^k \lambda_i(X) = \sup\{\mathbf{tr}(V^T X V) \mid V \in \mathbf{R}^{n\times k},\ V^T V = I\}.$$

(b) *Geometric mean of k smallest eigenvalues.* Show that $(\prod_{i=n-k+1}^n \lambda_i(X))^{1/k}$ is concave on $\mathbf{S}^n_{++}$. *Hint.* [MO79, page 513] For $X \succ 0$, we have

$$\left(\prod_{i=n-k+1}^n \lambda_i(X)\right)^{1/k} = \frac{1}{k}\inf\{\mathbf{tr}(V^T X V) \mid V \in \mathbf{R}^{n\times k},\ \det V^T V = 1\}.$$

(c) *Log of product of k smallest eigenvalues.* Show that $\sum_{i=n-k+1}^n \log \lambda_i(X)$ is concave on $\mathbf{S}^n_{++}$. *Hint.* [MO79, page 513] For $X \succ 0$,

$$\prod_{i=n-k+1}^n \lambda_i(X) = \inf\left\{ \prod_{i=1}^n (V^T X V)_{ii} \;\middle|\; V \in \mathbf{R}^{n\times k},\ V^T V = I \right\}.$$

3.27 *Diagonal elements of Cholesky factor.* Each $X \in \mathbf{S}^n_{++}$ has a unique *Cholesky factorization* $X = LL^T$, where L is lower triangular, with $L_{ii} > 0$. Show that L_{ii} is a concave function of X (with domain $\mathbf{S}^n_{++}$).

Hint. L_{ii} can be expressed as $L_{ii} = (w - z^T Y^{-1} z)^{1/2}$, where

$$\left[\begin{array}{cc} Y & z \\ z^T & w \end{array}\right]$$

is the leading $k \times k$ submatrix of X.

Operations that preserve convexity

3.28 *Expressing a convex function as the pointwise supremum of a family of affine functions.* In this problem we extend the result proved on page 83 to the case where $\mathbf{dom}\, f \neq \mathbf{R}^n$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function. Define $\tilde{f} : \mathbf{R}^n \to \mathbf{R}$ as the pointwise supremum of all affine functions that are global underestimators of f:

$$\tilde{f}(x) = \sup\{g(x) \mid g \text{ affine},\ g(z) \leq f(z) \text{ for all } z\}.$$

(a) Show that $f(x) = \tilde{f}(x)$ for $x \in \mathbf{int\,dom}\, f$.

(b) Show that $f = \tilde{f}$ if f is closed (*i.e.*, $\mathbf{epi}\, f$ is a closed set; see §A.3.3).

3.29 *Representation of piecewise-linear convex functions.* A function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathbf{dom}\, f = \mathbf{R}^n$, is called *piecewise-linear* if there exists a partition of $\mathbf{R}^n$ as

$$\mathbf{R}^n = X_1 \cup X_2 \cup \cdots \cup X_L,$$

where $\mathbf{int}\, X_i \neq \emptyset$ and $\mathbf{int}\, X_i \cap \mathbf{int}\, X_j = \emptyset$ for $i \neq j$, and a family of affine functions $a_1^T x + b_1, \ldots, a_L^T x + b_L$ such that $f(x) = a_i^T x + b_i$ for $x \in X_i$.
Show that this means that $f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$.

3.30 *Convex hull or envelope of a function.* The *convex hull* or *convex envelope* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$g(x) = \inf\{t \mid (x,t) \in \mathbf{conv\,epi}\, f\}.$$

Geometrically, the epigraph of g is the convex hull of the epigraph of f.
Show that g is the largest convex underestimator of f. In other words, show that if h is convex and satisfies $h(x) \leq f(x)$ for all x, then $h(x) \leq g(x)$ for all x.

3.31 [Roc70, page 35] *Largest homogeneous underestimator.* Let f be a convex function. Define the function g as

$$g(x) = \inf_{\alpha > 0} \frac{f(\alpha x)}{\alpha}.$$

(a) Show that g is homogeneous ($g(tx) = tg(x)$ for all $t \geq 0$).

(b) Show that g is the largest homogeneous underestimator of f: If h is homogeneous and $h(x) \leq f(x)$ for all x, then we have $h(x) \leq g(x)$ for all x.

(c) Show that g is convex.

3.32 *Products and ratios of convex functions.* In general the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbf{R}$. Prove the following.

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.

(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

3.33 *Direct proof of perspective theorem.* Give a direct proof that the perspective function g, as defined in §3.2.6, of a convex function f is convex: Show that $\mathbf{dom}\, g$ is a convex set, and that for $(x,t),\ (y,s) \in \mathbf{dom}\, g$, and $0 \leq \theta \leq 1$, we have

$$g(\theta x + (1-\theta)y, \theta t + (1-\theta)s) \leq \theta g(x,t) + (1-\theta) g(y,s).$$

3.34 *The Minkowski function.* The *Minkowski function* of a convex set C is defined as

$$M_C(x) = \inf\{t > 0 \mid t^{-1}x \in C\}.$$

(a) Draw a picture giving a geometric interpretation of how to find $M_C(x)$.

(b) Show that M_C is homogeneous, *i.e.*, $M_C(\alpha x) = \alpha M_C(x)$ for $\alpha \geq 0$.

(c) What is $\mathbf{dom}\, M_C$?

(d) Show that M_C is a convex function.

(e) Suppose C is also closed, symmetric (if $x \in C$ then $-x \in C$), and has nonempty interior. Show that M_C is a norm. What is the corresponding unit ball?

3.35 *Support function calculus.* Recall that the support function of a set $C \subseteq \mathbf{R}^n$ is defined as $S_C(y) = \sup\{y^T x \mid x \in C\}$. On page 81 we showed that S_C is a convex function.

(a) Show that $S_B = S_{\mathbf{conv}\, B}$.

(b) Show that $S_{A+B} = S_A + S_B$.

(c) Show that $S_{A\cup B} = \max\{S_A, S_B\}$.

(d) Let B be closed and convex. Show that $A \subseteq B$ if and only if $S_A(y) \leq S_B(y)$ for all y.

Conjugate functions

3.36 Derive the conjugates of the following functions.

(a) *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbf{R}^n$.

(b) *Sum of largest elements.* $f(x) = \sum_{i=1}^r x_{[i]}$ on $\mathbf{R}^n$.

(c) *Piecewise-linear function on* $\mathbf{R}$. $f(x) = \max_{i=1,\ldots,m}(a_i x + b_i)$ on $\mathbf{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \leq \cdots \leq a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.

(d) *Power function.* $f(x) = x^p$ on $\mathbf{R}_{++}$, where $p > 1$. Repeat for $p < 0$.

(e) *Geometric mean.* $f(x) = -(\prod x_i)^{1/n}$ on $\mathbf{R}^n_{++}$.

(f) *Negative generalized logarithm for second-order cone.* $f(x,t) = -\log(t^2 - x^T x)$ on $\{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 < t\}$.

3.37 Show that the conjugate of $f(X) = \mathbf{tr}(X^{-1})$ with $\mathbf{dom}\, f = \mathbf{S}^n_{++}$ is given by

$$f^*(Y) = -2\,\mathbf{tr}(-Y)^{1/2}, \qquad \mathbf{dom}\, f^* = -\mathbf{S}^n_+.$$

Hint. The gradient of f is $\nabla f(X) = -X^{-2}$.

3.38 *Young's inequality.* Let $f : \mathbf{R} \to \mathbf{R}$ be an increasing function, with $f(0) = 0$, and let g be its inverse. Define F and G as

$$F(x) = \int_0^x f(a)\, da, \qquad G(y) = \int_0^y g(a)\, da.$$

Show that F and G are conjugates. Give a simple graphical interpretation of Young's inequality,

$$xy \leq F(x) + G(y).$$

3.39 *Properties of conjugate functions.*

(a) *Conjugate of convex plus affine function.* Define $g(x) = f(x) + c^T x + d$, where f is convex. Express g^* in terms of f^* (and c, d).

(b) *Conjugate of perspective.* Express the conjugate of the perspective of a convex function f in terms of f^*.

(c) *Conjugate and minimization.* Let $f(x,z)$ be convex in (x,z) and define $g(x) = \inf_z f(x,z)$. Express the conjugate g^* in terms of f^*.
As an application, express the conjugate of $g(x) = \inf_z\{h(z) \mid Az+b = x\}$, where h is convex, in terms of h^*, A, and b.

(d) *Conjugate of conjugate.* Show that the conjugate of the conjugate of a closed convex function is itself: $f = f^{**}$ if f is closed and convex. (A function is closed if its epigraph is closed; see §A.3.3.) *Hint.* Show that f^{**} is the pointwise supremum of all affine global underestimators of f. Then apply the result of exercise 3.28.

3.40 *Gradient and Hessian of conjugate function.* Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex and twice continuously differentiable. Suppose $\bar{y}$ and $\bar{x}$ are related by $\bar{y} = \nabla f(\bar{x})$, and that $\nabla^2 f(\bar{x}) \succ 0$.

(a) Show that $\nabla f^*(\bar{y}) = \bar{x}$.

(b) Show that $\nabla^2 f^*(\bar{y}) = \nabla^2 f(\bar{x})^{-1}$.

3.41 *Domain of conjugate function.* Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is a twice differentiable convex function and $x \in \mathbf{dom}\, f$. Show that for small enough u we have

$$y = \nabla f(x) + \nabla^2 f(x) u \in \mathbf{dom}\, f^*,$$

i.e., $y^T x - f(x)$ is bounded above. It follows that $\dim(\mathbf{dom}\, f^*) \geq \mathbf{rank}\, \nabla^2 f(x)$.
Hint. Consider $\nabla f(x+tv)$, where t is small, and v is any vector in $\mathbf{R}^n$.

Quasiconvex functions

3.42 *Approximation width.* Let $f_0, \ldots, f_n : \mathbf{R} \to \mathbf{R}$ be given continuous functions. We consider the problem of approximating f_0 as a linear combination of $f_1, \ldots, f_n$. For $x \in \mathbf{R}^n$, we say that $f = x_1 f_1 + \cdots + x_n f_n$ approximates f_0 with tolerance $\epsilon > 0$ over the interval $[0,T]$ if $|f(t) - f_0(t)| \leq \epsilon$ for $0 \leq t \leq T$. Now we choose a fixed tolerance $\epsilon > 0$ and define the *approximation width* as the largest T such that f approximates f_0 over the interval $[0,T]$:

$$W(x) = \sup\{T \mid |x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \leq \epsilon \text{ for } 0 \leq t \leq T\}.$$

Show that W is quasiconcave.

3.43 *First-order condition for quasiconvexity.* Prove the first-order condition for quasiconvexity given in §3.4.3: A differentiable function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathbf{dom}\, f$ convex, is quasiconvex if and only if for all $x, y \in \mathbf{dom}\, f$,

$$f(y) \leq f(x) \Longrightarrow \nabla f(x)^T (y - x) \leq 0.$$

Hint. It suffices to prove the result for a function on $\mathbf{R}$; the general result follows by restriction to an arbitrary line.

3.44 *Second-order conditions for quasiconvexity.* In this problem we derive alternate representations of the second-order conditions for quasiconvexity given in §3.4.3. Prove the following.

(a) A point $x \in \mathbf{dom}\, f$ satisfies (3.21) if and only if there exists a σ such that

$$\nabla^2 f(x) + \sigma \nabla f(x) \nabla f(x)^T \succeq 0. \tag{3.26}$$

It satisfies (3.22) for all $y \neq 0$ if and only if there exists a σ such

$$\nabla^2 f(x) + \sigma \nabla f(x) \nabla f(x)^T \succ 0. \tag{3.27}$$

Hint. We can assume without loss of generality that $\nabla^2 f(x)$ is diagonal.

(b) A point $x \in \mathbf{dom}\, f$ satisfies (3.21) if and only if either $\nabla f(x) = 0$ and $\nabla^2 f(x) \succeq 0$, or $\nabla f(x) \neq 0$ and the matrix

$$H(x) = \begin{bmatrix} \nabla^2 f(x) & \nabla f(x) \\ \nabla f(x)^T & 0 \end{bmatrix}$$

has exactly one negative eigenvalue. It satisfies (3.22) for all $y \neq 0$ if and only if $H(x)$ has exactly one nonpositive eigenvalue.

Hint. You can use the result of part (a). The following result, which follows from the eigenvalue interlacing theorem in linear algebra, may also be useful: If $B \in \mathbf{S}^n$ and $a \in \mathbf{R}^n$, then

$$\lambda_n\left(\begin{bmatrix} B & a \\ a^T & 0 \end{bmatrix}\right) \geq \lambda_n(B).$$

3.45 Use the first and second-order conditions for quasiconvexity given in §3.4.3 to verify quasiconvexity of the function $f(x) = -x_1x_2$, with $\mathbf{dom}\, f = \mathbf{R}^2_{++}$.

3.46 *Quasilinear functions with domain* $\mathbf{R}^n$. A function on $\mathbf{R}$ that is quasilinear (*i.e.*, quasiconvex and quasiconcave) is monotone, *i.e.*, either nondecreasing or nonincreasing. In this problem we consider a generalization of this result to functions on $\mathbf{R}^n$.

Suppose the function $f : \mathbf{R}^n \to \mathbf{R}$ is quasilinear and continuous with $\mathbf{dom}\, f = \mathbf{R}^n$. Show that it can be expressed as $f(x) = g(a^Tx)$, where $g : \mathbf{R} \to \mathbf{R}$ is monotone and $a \in \mathbf{R}^n$. In other words, a quasilinear function with domain $\mathbf{R}^n$ must be a monotone function of a linear function. (The converse is also true.)

Log-concave and log-convex functions

3.47 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable, $\mathbf{dom}\, f$ is convex, and $f(x) > 0$ for all $x \in \mathbf{dom}\, f$. Show that f is log-concave if and only if for all $x, y \in \mathbf{dom}\, f$,

$$\frac{f(y)}{f(x)} \leq \exp\left(\frac{\nabla f(x)^T(y-x)}{f(x)}\right).$$

3.48 Show that if $f : \mathbf{R}^n \to \mathbf{R}$ is log-concave and $a \geq 0$, then the function $g = f - a$ is log-concave, where $\mathbf{dom}\, g = \{x \in \mathbf{dom}\, f \mid f(x) > a\}$.

3.49 Show that the following functions are log-concave.

(a) *Logistic function:* $f(x) = e^x/(1+e^x)$ with $\mathbf{dom}\, f = \mathbf{R}$.

(b) *Harmonic mean:*

$$f(x) = \frac{1}{1/x_1 + \cdots + 1/x_n}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}.$$

(c) *Product over sum:*

$$f(x) = \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}.$$

(d) *Determinant over trace:*

$$f(X) = \frac{\det X}{\mathbf{tr}\, X}, \qquad \mathbf{dom}\, f = \mathbf{S}^n_{++}.$$

3.50 *Coefficients of a polynomial as a function of the roots.* Show that the coefficients of a polynomial with real negative roots are log-concave functions of the roots. In other words, the functions $a_i : \mathbf{R}^n \to \mathbf{R}$, defined by the identity

$$s^n + a_1(\lambda)s^{n-1} + \cdots + a_{n-1}(\lambda)s + a_n(\lambda) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n),$$

are log-concave on $-\mathbf{R}^n_{++}$.
Hint. The function

$$S_k(x) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

with $\mathbf{dom}\, S_k \in \mathbf{R}^n_+$ and $1 \le k \le n$, is called the kth elementary symmetric function on $\mathbf{R}^n$. It can be shown that $S_k^{1/k}$ is concave (see [ML57]).

3.51 [BL00, page 41] Let p be a polynomial on $\mathbf{R}$, with all its roots real. Show that it is log-concave on any interval on which it is positive.

3.52 [MO79, §3.E.2] *Log-convexity of moment functions.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is nonnegative with $\mathbf{R}_+ \subseteq \mathbf{dom}\, f$. For $x \ge 0$ define

$$\phi(x) = \int_0^\infty u^x f(u)\, du.$$

Show that ϕ is a log-convex function. (If x is a positive integer, and f is a probability density function, then $\phi(x)$ is the xth moment of the distribution.)
Use this to show that the Gamma function,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du,$$

is log-convex for $x \ge 1$.

3.53 Suppose x and y are independent random vectors in $\mathbf{R}^n$, with log-concave probability density functions f and g, respectively. Show that the probability density function of the sum $z = x + y$ is log-concave.

3.54 *Log-concavity of Gaussian cumulative distribution function.* The cumulative distribution function of a Gaussian random variable,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt,$$

is log-concave. This follows from the general result that the convolution of two log-concave functions is log-concave. In this problem we guide you through a simple self-contained proof that f is log-concave. Recall that f is log-concave if and only if $f''(x)f(x) \le f'(x)^2$ for all x.

(a) Verify that $f''(x)f(x) \le f'(x)^2$ for $x \ge 0$. That leaves us the hard part, which is to show the inequality for $x < 0$.

(b) Verify that for any t and x we have $t^2/2 \ge -x^2/2 + xt$.

(c) Using part (b) show that $e^{-t^2/2} \le e^{x^2/2 - xt}$. Conclude that

$$\int_{-\infty}^x e^{-t^2/2}\, dt \le e^{x^2/2} \int_{-\infty}^x e^{-xt}\, dt.$$

(d) Use part (c) to verify that $f''(x)f(x) \le f'(x)^2$ for $x \le 0$.

3.55 *Log-concavity of the cumulative distribution function of a log-concave probability density.* In this problem we extend the result of exercise 3.54. Let $g(t) = \exp(-h(t))$ be a differentiable log-concave probability density function, and let

$$f(x) = \int_{-\infty}^{x} g(t)\, dt = \int_{-\infty}^{x} e^{-h(t)}\, dt$$

be its cumulative distribution. We will show that f is log-concave, *i.e.*, it satisfies $f''(x)f(x) \le (f'(x))^2$ for all x.

(a) Express the derivatives of f in terms of the function h. Verify that $f''(x)f(x) \le (f'(x))^2$ if $h'(x) \ge 0$.

(b) Assume that $h'(x) < 0$. Use the inequality

$$h(t) \ge h(x) + h'(x)(t - x)$$

(which follows from convexity of h), to show that

$$\int_{-\infty}^{x} e^{-h(t)}\, dt \le \frac{e^{-h(x)}}{-h'(x)}.$$

Use this inequality to verify that $f''(x)f(x) \le (f'(x))^2$ if $h'(x) \ge 0$.

3.56 *More log-concave densities.* Show that the following densities are log-concave.

(a) [MO79, page 493] The *gamma density*, defined by

$$f(x) = \frac{\alpha^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\alpha x},$$

with $\mathbf{dom}\, f = \mathbf{R}_+$. The parameters λ and α satisfy $\lambda \ge 1$, $\alpha > 0$.

(b) [MO79, page 306] The *Dirichlet density*

$$f(x) = \frac{\Gamma(\mathbf{1}^T\lambda)}{\Gamma(\lambda_1)\cdots\Gamma(\lambda_{n+1})} x_1^{\lambda_1 - 1} \cdots x_n^{\lambda_n - 1} \left(1 - \sum_{i=1}^n x_i\right)^{\lambda_{n+1}-1}$$

with $\mathbf{dom}\, f = \{x \in \mathbf{R}^n_{++} \mid \mathbf{1}^T x < 1\}$. The parameter λ satisfies $\lambda \succeq \mathbf{1}$.

Convexity with respect to a generalized inequality

3.57 Show that the function $f(X) = X^{-1}$ is matrix convex on $\mathbf{S}^n_{++}$.

3.58 *Schur complement.* Suppose $X \in \mathbf{S}^n$ partitioned as

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where $A \in \mathbf{S}^k$. The *Schur complement* of X (with respect to A) is $S = C - B^T A^{-1} B$ (see §A.5.5). Show that the Schur complement, viewed as function from $\mathbf{S}^n$ into $\mathbf{S}^{n-k}$, is matrix concave on $\mathbf{S}^n_{++}$.

3.59 *Second-order conditions for K-convexity.* Let $K \subseteq \mathbf{R}^m$ be a proper convex cone, with associated generalized inequality $\preceq_K$. Show that a twice differentiable function $f : \mathbf{R}^n \to \mathbf{R}^m$, with convex domain, is K-convex if and only if for all $x \in \mathbf{dom}\, f$ and all $y \in \mathbf{R}^n$,

$$\sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} y_i y_j \succeq_K 0,$$

i.e., the second derivative is a K-nonnegative bilinear form. (Here $\partial^2 f / \partial x_i \partial x_j \in \mathbf{R}^m$, with components $\partial^2 f_k / \partial x_i \partial x_j$, for $k = 1, \dots, m$; see §A.4.1.)

3.60 *Sublevel sets and epigraph of K-convex functions.* Let $K \subseteq \mathbf{R}^m$ be a proper convex cone with associated generalized inequality $\preceq_K$, and let $f : \mathbf{R}^n \to \mathbf{R}^m$. For $\alpha \in \mathbf{R}^m$, the α-sublevel set of f (with respect to $\preceq_K$) is defined as

$$C_\alpha = \{x \in \mathbf{R}^n \mid f(x) \preceq_K \alpha\}.$$

The epigraph of f, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K f = \{(x, t) \in \mathbf{R}^{n+m} \mid f(x) \preceq_K t\}.$$

Show the following:

(a) If f is K-convex, then its sublevel sets C_α are convex for all α.

(b) f is K-convex if and only if $\mathbf{epi}_K f$ is a convex set.