Introduction to Statistical Time Series
WAYNE A. FULLER

CHAPTER 7

The Periodogram, Estimated Spectrum

In this chapter we shall investigate estimators of the spectral density of time series with absolutely summable covariance function. The spectral density of a time series was defined in Section 4.1 as the Fourier transform of the covariance function. We also noted in Section 4.2 that the variances of the random variables defined by the Fourier coefficients of the original time series are, approximately, multiples of the spectral density. These two results suggest methods of estimating the spectral density.

The study of the Fourier coefficients was popular among economists in the 1920s and 1930s, and the student of economics may be interested in the discussion of Davis (1941) and the studies cited by Tintner (1952). Also see Granger and Hatanaka (1964) and Nold (1972). Recent applications are more common in the engineering literature. See, for example, Marple (1987).

7.1. THE PERIODOGRAM

Given a finite realization from a time series, we can represent the n observations by the trigonometric polynomial

$$X_t = \frac{a_0}{2} + \sum_{k=1}^{m} (a_k \cos \omega_k t + b_k \sin \omega_k t), \tag{7.1.1}$$

where

$$\omega_k = \frac{2\pi k}{n}, \qquad k = 0, 1, 2, \ldots, m,$$

$$a_k = \frac{2 \sum_{t=1}^{n} X_t \cos \omega_k t}{n}, \qquad k = 0, 1, 2, \ldots, m,$$

$$b_k = \frac{2 \sum_{t=1}^{n} X_t \sin \omega_k t}{n}, \qquad k = 1, 2, \ldots, m,$$

and we have assumed n odd and equal to $2m + 1$. As before, we recognize the Fourier coefficients as regression coefficients. We can use the standard regression analysis to partition the total sum of squares for the n observations. The sum of squares removed by the regression of X_t on $\cos \omega_k t$ is the regression coefficient multiplied by the sum of cross products; that is, the sum of squares due to a_k is

$$\frac{n}{2} a_k^2 = \frac{2}{n} \left(\sum_{t=1}^{n} X_t \cos \omega_k t \right)^2, \qquad k = 1, 2, \ldots, m, \tag{7.1.2}$$

and the sum of squares removed by $\cos \omega_k t$ and $\sin \omega_k t$ is

$$\frac{n}{2} (a_k^2 + b_k^2) = \frac{2}{n} \left[\left(\sum_{t=1}^{n} X_t \cos \omega_k t \right)^2 + \left(\sum_{t=1}^{n} X_t \sin \omega_k t \right)^2 \right]. \tag{7.1.3}$$

We might call the quantity in (7.1.3) the sum of squares associated with frequency ω_k. Thus the total sum of squares for the $n = 2m + 1$ observations may be partitioned into $m + 1$ components. One component is associated with the mean. Each of the remaining m components is the sum of the two squares associated with the m nonzero frequencies. The partition is displayed in Table 7.1.1.

Should the number of observations be even and denoted by $2m$, there is only one regression variable associated with the mth frequency: $\cos \pi t$. Then the sum of squares for the mth frequency has one degree of freedom and is given by $n^{-1}(\sum_{t=1}^{n} X_t \cos \pi t)^2 = \frac{1}{4} n a_m^2$.

One might divide all of the sums of squares of Table 7.1.1 by the degrees of

Table 7.1.1. Analysis of Variance Table for a Sample of Size $n = 2m + 1$

Source	Degrees of Freedom	Sum of Squares
Mean	1	$n\bar{x}_n^2 = \frac{1}{4} n a_0^2$
Frequency $\omega_1 = 2\pi/n$	2	$(n/2)(a_1^2 + b_1^2)$
Frequency $\omega_2 = 4\pi/n$	2	$(n/2)(a_2^2 + b_2^2)$
⋮	⋮	⋮
Frequency $\omega_m = 2\pi m/n$	2	$(n/2)(a_m^2 + b_m^2)$
Total	n	$\sum_{t=1}^{n} X_t^2$

freedom and consider the mean squares. However, it is more common to investigate the sums of squares, multiplying the sums of squares with one degree of freedom by two. The function of frequency given by these normalized sums of squares is called the *periodogram*. Thus, the periodogram is defined by

$$I_n(\omega_k) = \frac{n}{2}(a_k^2 + b_k^2), \qquad k = 1, 2, \ldots, m, \tag{7.1.4}$$

where m is the smallest integer greater than or equal to $(n-1)/2$.

Most computer programs designed to compute the periodogram for large data sets use an algorithm based on the fast Fourier transform. See Cooley, Lewis, and Welch (1967) for references on the fast Fourier transform. Singleton (1969) gives a Fortran program for the transform, and Bloomfield (1976, p. 61) discusses the procedure.

If $\{X_t\}$ is a sequence of normal independent $(0, \sigma^2)$ random variables, then the a_k and b_k, being linear combinations of the X_t, will be normally distributed. Since the sine and cosine functions are orthogonal, the a_k and b_k are independent. In this case those entries in Table 7.1.1 with two degrees of freedom divided by σ^2 are distributed as independent chi-squares with two degrees of freedom.

The periodogram may also be defined in terms of the original observations as

$$I_n(\omega_k) = \frac{2}{n}\left[\left(\sum_{t=1}^{n} X_t \cos \omega_k t\right)^2 + \left(\sum_{t=1}^{n} X_t \sin \omega_k t\right)^2\right], \qquad k = 0, 1, \ldots, m. \tag{7.1.5}$$

Note that if we define the complex coefficients c_k by

$$c_k = \frac{1}{n}\sum_{t=1}^{n} X_t e^{\epsilon\omega_k t},$$

then

$$I_n(\omega_k) = 2nc_k c_k^* = \frac{2}{n}\left|\sum_{t=1}^{n} X_t e^{\epsilon\omega_k t}\right|^2. \tag{7.1.6}$$

Thus, the periodogram ordinate at ω_k is a multiple of the squared norm of the complex Fourier coefficient of the time series associated with the frequency ω_k.

The periodogram is also expressible as a multiple of the Fourier transform of the estimated covariance function. If $\omega_k \neq 0$, we can write

$$\sum_{t=1}^{n} X_t \cos \omega_k t = \sum_{t=1}^{n} (X_t - \mu) \cos \omega_k t,$$

$$\sum_{t=1}^{n} X_t \sin \omega_k t = \sum_{t=1}^{n} (X_t - \mu) \sin \omega_k t,$$

where $\mu = E\{X_t\}$. Therefore,

$$
\begin{aligned}
I_n(\omega_k) &= \frac{2}{n}\left[\left\{\sum_{t=1}^{n}(X_t-\mu)\cos\omega_k t\right\}^2 + \left\{\sum_{t=1}^{n}(X_t-\mu)\sin\omega_k t\right\}^2\right] \\
&= \frac{2}{n}\left[\sum_{t=1}^{n}\sum_{j=1}^{n}(X_t-\mu)(X_j-\mu)\cos\omega_k t\cos\omega_k j\right. \\
&\qquad \left. + \sum_{t=1}^{n}\sum_{j=1}^{n}(X_t-\mu)(X_j-\mu)\sin\omega_k t\sin\omega_k j\right] \\
&= \frac{2}{n}\left[\sum_{t=1}^{n}\sum_{j=1}^{n}(X_t-\mu)(X_j-\mu)\cos\omega_k(t-j)\right]
\end{aligned}
$$

for $k = 1, 2, \ldots, m$. In this double sum there are n combinations with $t-j=0$, $n-1$ combinations with $t-j=1$, and so forth. Therefore, by letting $p = t-j$, we obtain the following result.

Result 7.1.1. The kth periodogram ordinate is given by

$$
I_n(\omega_k) = \begin{cases} 2n\bar{x}_n^2, & k=0, \\ 4\pi\hat{f}(\omega_k), & k=1,2,\ldots,m, \end{cases} \tag{7.1.7}
$$

where

$$
\hat{f}(\omega) = \frac{1}{2\pi}\sum_{p=-\infty}^{\infty}\frac{n-|p|}{n}\tilde{\gamma}(p)\cos\omega p \qquad (-\pi \le \omega \le \pi), \tag{7.1.8}
$$

$$
\tilde{\gamma}(-p) = \tilde{\gamma}(p) = \begin{cases} \dfrac{1}{n-p}\displaystyle\sum_{j=1}^{n-p}(X_j-\mu)(X_{j+p}-\mu), & 0 \le p \le n-1, \\ 0, & p > n-1. \end{cases} \tag{7.1.9}
$$

The coefficients a_k and b_k $(k \ne 0)$ that are computed using $X_t - \bar{x}_n$ are identical to those computed using X_t. Therefore, by substituting $X_t - \bar{x}_n$ for X_t in (7.1.5), we can also write

$$
I_n(\omega_k) = 2\sum_{p=-\infty}^{\infty}\hat{\gamma}(p)\cos\omega_k p, \tag{7.1.10}
$$

where $k \ne 0$ and

$$
\hat{\gamma}(h) = \hat{\gamma}(-h) = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{t=1}^{n-h}(X_t-\bar{x}_n)(X_{t+h}-\bar{x}_n), & 0 \le h \le n-1, \\ 0, & h > n-1. \end{cases}
$$

The function $\hat{f}(\omega)$ of equation (7.1.8) is a continuous periodic function of ω defined for all ω. The periodogram has been defined only for the discrete set of points $\omega_k = 2\pi k/n$, $k = 0, 1, 2, \ldots, m$. In investigating the limiting properties of the periodogram, it is convenient to have a function defined for all $\omega \in [0, \pi]$. To this end we introduce the function

$$K(n, \omega) = k \qquad \text{for} \quad \frac{\pi(2k-1)}{n} < \omega \le \frac{\pi(2k+1)}{n}, \quad k = 0, \pm 1, \pm 2, \ldots,$$

and take

$$I_n(\omega) = I_n(\omega_{K(n,\omega)}) .$$

Thus, $I_n(\omega)$ for $\omega \in [0, \pi]$ is a step function that takes the value $I_n(\omega_k)$ on the interval $(\pi\{2k-1\}/n, \pi\{2k+1\}/n)$.

The word periodogram is used with considerable flexibility in the literature. Despite the apparent conflict in terms, our definition of the periodogram as a function of frequency rather than of period is a common one. We have chosen to define the periodogram for the discrete frequencies $\omega_k = 2\pi k/n$, $k = 0, 1, 2, \ldots, m$, and to extend it to all ω as a step function. An alternative definition of the periodogram is $4\pi\hat{f}(\omega)$, where $\hat{f}(\omega)$ is given in (7.1.8), in which case one automatically has a function for all ω. Our definition will prove convenient in obtaining the limiting properties of estimators of the spectrum.

The distributional properties of the periodogram ordinates are easily obtained when the time series is normal white noise. To establish the properties of the periodogram for other time series, we first obtain the limiting value of the expectation of $I_n(\omega)$ for a time series with absolutely summable covariance function.

Theorem 7.1.1. Let X_t be a stationary time series with $E\{X_t\} = \mu$ and absolutely summable covariance function. Then

$$\lim_{n\to\infty} E\{I_n(\omega)\} = 4\pi f(\omega), \qquad \omega \ne 0,$$
$$\lim_{n\to\infty} E\{I_n(0) - 2n\mu^2\} = 4\pi f(0), \qquad \omega = 0.$$

Proof. Since $E\{\tilde{\gamma}(p)\} = \gamma(p)$, it follows from (7.1.7) that

$$E\{I_n(0)\} = 2 \sum_{h=-(n-1)}^{n-1} \frac{n-|h|}{n} \gamma(h) + 2n\mu^2 ,$$
$$E\{I_n(\omega_k)\} = 2 \sum_{h=-(n-1)}^{n-1} \frac{n-|h|}{n} \gamma(h) \cos \omega_k h , \qquad k = 1, 2, \ldots, m .$$

Now

$$\left| \sum_{h=-n}^{n} \frac{|h|}{n} \gamma(h) \cos \omega_k h \right| \le \sum_{h=-n}^{n} \frac{|h|}{n} |\gamma(h)| ,$$

and the latter sum goes to zero as $n \to \infty$ by Lemma 3.1.4. The sequence $g_n(\omega) = 2\sum_{h=-(n-1)}^{n-1} \gamma(h)\cos\omega h$ converges uniformly to $4\pi f(\omega)$ by Corollary 3.1.8, and $\omega_{K(n,\omega)}$ converges to ω by construction. ▲

The normalized coefficients $2^{-1/2}n^{1/2}a_k$ and $2^{-1/2}n^{1/2}b_k$ are the random variables obtained by applying to the observations the transformation discussed in Section 4.2. Therefore, for a time series with absolutely summable covariance function, these random variables are in the limit uncorrelated and have variance given by a multiple of the spectral density evaluated at the associated frequency.

The importance of this result is difficult to overemphasize. For a wide class of time series we are able to transform an observed set of n observations into a set of n statistics that are nearly uncorrelated. All except the first, the sample mean, have zero expected value. The variance of these random variables is, approximately, a simple function of the spectral density.

Theorem 7.1.2. Let X_t be a time series defined by

$$X_t = \sum_{j=0}^{\infty} \alpha_j e_{t-j},$$

where $\{\alpha_j\}$ is absolutely summable and the e_t are independent identically distributed $(0, \sigma^2)$ random variables. Let $f_X(\omega)$ be positive for all ω. Then, for ω and λ in $(0, \pi)$ and $\omega \neq \lambda$, the sequences $[2\pi f(\omega)]^{-1}I_n(\omega)$ and $[2\pi f(\lambda)]^{-1}I_n(\lambda)$ converge in distribution to independent chi-square random variables, each with two degrees of freedom.

Proof. Consider

$$\begin{aligned} 2^{-1/2}n^{1/2}a_{K(n,\omega)} &= 2^{1/2}n^{-1/2}\sum_{t=1}^{n} \cos\omega_{K(n,\omega)}t\, X_t \\ &= 2^{1/2}n^{-1/2}\sum_{t=1}^{n} \cos\omega_{K(n,\omega)}t \left(\sum_{j=0}^{r} \alpha_j e_{t-j} + \sum_{j=r+1}^{\infty} \alpha_j e_{t-j}\right), \end{aligned}$$

where

$$\lim_{r\to\infty} \operatorname{Var}\left\{2^{1/2}n^{-1/2}\sum_{t=1}^{n} \cos\omega_{K(n,\omega)}t \sum_{j=r+1}^{\infty} \alpha_j e_{t-j}\right\} = 0$$

uniformly in n. Fixing r, we have

$$2^{1/2}n^{-1/2}\sum_{t=1}^{n} \cos\omega_{K(n,\omega)}t \sum_{j=0}^{r} \alpha_j e_{t-j} = n^{-1/2}\sum_{t=1}^{n} \delta_{n\omega t}e_t + R_n,$$

where

$$\delta_{n\omega t} = 2^{1/2} \sum_{j=0}^{r} \alpha_j \cos \omega_{K(n,\omega)}(t+j),$$

$$R_n = 2^{1/2} n^{-1/2} \left[\sum_{j=0}^{r-1} \sum_{s=j+1}^{r} \alpha_s e_{-j} \cos \omega_{K(n,\omega)}(s-j) - \sum_{j=0}^{r-1} \sum_{s=j+1}^{r} \alpha_s e_{n-j} \cos \omega_{K(n,\omega)}(n+s-j) \right],$$

and R_n converges in probability to zero. As $\delta_{n\omega t}$ is uniformly bounded by (say) M, we have for $\epsilon > 0$

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} \int_{|\delta_{n\omega t}e| > \epsilon n^{1/2}} \delta_{n\omega t}^2 e^2 \, dF(e) \le \lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} \int_{|e| > \epsilon M^{-1} n^{1/2}} M^2 e^2 \, dF(e) = 0,$$

and $n^{-1/2} \sum_{t=1}^{n} \delta_{n\omega t} e_t$ converges in distribution to a normal random variable by the Lindeberg condition. The asymptotic normality of $2^{-1/2} n^{1/2} a_{K(n,\omega)}$ follows by Lemma 6.3.1. The same arguments hold for a linear combination of $n^{1/2} a_{K(n,\omega)}$, $n^{1/2} b_{K(n,\omega)}$, $n^{1/2} a_{K(n,\lambda)}$, $n^{1/2} b_{K(n,\lambda)}$, so that $2^{-1/2} n^{1/2} [a_{K(n,\omega)}, b_{K(n,\omega)}, a_{K(n,\lambda)}, b_{K(n,\lambda)}]$, $\omega \ne \lambda$, converges in distribution to a multivariate normal random variable. The covariance matrix is given by Theorem 4.2.1 and is

$$\operatorname{diag}\{2\pi f(\omega), 2\pi f(\omega), 2\pi f(\lambda), 2\pi f(\lambda)\}.$$

Hence, by Theorem 5.2.4, the limiting distribution of $I_n(\omega)/2\pi f(\omega)$ and $I_n(\lambda)/2\pi f(\lambda)$ is that of two independent chi-square random variables with two degrees of freedom. ▲

Thus, for many nonnormal processes, we may treat the periodogram ordinates as multiples of chi-square random variables. If the original time series is a sequence of independent $(0, \sigma^2)$ random variables, then the periodogram ordinates all have the same expected value. However, for a time series with a nonzero autocorrelation structure, the ordinates will have different expected values. These facts have been used in constructing tests based on the periodogram.

Perhaps it is most natural to use the periodogram to search for "cycles" or "periodicities" in the data. For example, let us hypothesize that a time series is well represented by

$$X_t = \mu + A \cos \omega t + B \sin \omega t + e_t, \tag{7.1.11}$$

where the e_t are normal independent $(0, \sigma^2)$ random variables and A and B are fixed.

First assume that ω is known and of the form $2\pi k/n$, where k is an integer. To test the hypothesis that $A = B = 0$ against the alternative $A \neq 0$ or $B \neq 0$, we can use

$$F^2_{2m-2} = \frac{(2m-2)(a_k^2 + b_k^2)}{2 \sum_{\substack{j=1 \\ j \neq k}}^{m} (a_j^2 + b_j^2)},$$

where F^2_{2m-2} has the F-distribution with 2 and $2m - 2$ degrees of freedom. Note that the sum of squares for the mean is not included in the denominator, since we postulated a general mean μ. If ω cannot be expressed as $2\pi k/n$, where k is an integer, then the regression associated with (7.1.11) can be computed and the usual regression test constructed.

We sometimes believe a time series contains a periodic component, but are unwilling to postulate the periodic function to be a perfect sine wave. For example, we may feel that a monthly time series contains a seasonal component, a large portion of which is because of a high value for December. We know that any periodic function defined on the integers, with integral period H, can be represented by

$$\frac{a_0}{2} + \sum_{k=1}^{L[H]} \left(a_k \cos \frac{2\pi k}{H} t + b_k \sin \frac{2\pi k}{H} t \right),$$

where $L[H]$ is the largest integer less than or equal to $H/2$. For the monthly time series one might postulate

$$Y_t = \mu + \sum_{k=1}^{6} \left(A_k \cos \frac{2\pi k}{12} t + B_k \sin \frac{2\pi k}{12} t \right) + e_t .$$

To test the hypothesis of no seasonal effect (i.e., all A_k and B_k equal zero), we form Snedecor's F as the ratio of the mean square for the six seasonal frequencies to the mean square for the remaining frequencies. Note that these tests assume e_t to be a sequence of normal independent $(0, \sigma^2)$ random variables.

The periodogram has also been used to search for "hidden periodicities." In the above examples we postulated the frequency or frequencies of interest and hence, under the null hypothesis, the ratio of the mean squares has the F-distribution. However, we might postulate the null model

$$X_t = \mu + e_t$$

and the alternative model

$$X_t = \mu + A \cos \omega t + B \sin \omega t + e_t ,$$

where ω is unknown.

In such a case one might search out the largest periodogram ordinate and ask if this ordinate can reasonably be considered the largest in a random sample of size m selected from a distribution function that is a multiple of a chi-square with two degrees of freedom. A statistic that can be used to test the hypothesis is

$$\xi = \left(\frac{1}{m}\sum_{k=1}^{m} I_n(\omega_k)\right)^{-1} I_n(L),$$

where $I_n(L)$ is the largest periodogram ordinate in a sample of m periodogram ordinates each with two degrees of freedom. Fisher (1929) demonstrated that, for $g > 0$,

$$P\{m^{-1}\xi > g\} = \sum_{j=1}^{k} (-1)^{j-1}\binom{m}{j}(1 - jg)^{m-1},$$

where ξ is constructed from the periodogram of a sequence of normal independent (μ, σ^2) random variables and k is the largest integer less than g^{-1}. A table of the distribution of ξ is given by Davis (1941). Wilks (1962, p. 529) contains a derivation of the distribution. In Table 7.1.2 we give the 1, 5, and 10 percentage points for the distribution.

Under the null hypothesis that a time series is normal white noise, the periodogram ordinates are multiples of independent chi-squares, each with two degrees of freedom. Hence, any number of other "goodness of fit" tests are available to test the hypothesis of independence. Bartlett (1966, p. 318) suggested a test based on the normalized cumulative periodogram

$$C_k = \text{cum}\{I_n(\omega_k)\} = \left[\sum_{j=1}^{m} I_n(\omega_j)\right]^{-1} \sum_{j=1}^{k} I_n(\omega_j). \tag{7.1.12}$$

The normalized cumulative periodogram for $k = 1, 2, \ldots, m-1$ has the same distribution function as that of an ordered sample of size $m-1$ selected from the uniform (0, 1) distribution. Therefore, if we plot the normalized periodogram as a sample distribution function and apply the Kolmogorov–Smirnov test of the hypothesis that it is a sample distribution function for a sample of $m-1$ selected from a uniform (0, 1) distribution, we have a test of the hypothesis that the original time series is white noise. This testing procedure has been discussed by Durbin (1967, 1969).

Example 7.1.1. In Section 9.2 a grafted quadratic trend is fitted to United States wheat yields from 1908 to 1971. The periodogram computed for the deviations from the trend is given in Table 7.1.3. As we are working with deviations from regression, the distributions of the test statistics are only approximately those discussed above. Durbin (1969) has demonstrated that the critical points for the Kolmogorov–Smirnov-type test statistic using deviations from regression differ from those for an unaltered time series by a quantity that is $O(n^{-1})$.

Table 7.1.2. Percentage Points for the Ratio of Largest Periodogram Ordinate to the Average

Number of Ordinates	Probability of a Larger Value		
	0.10	0.05	0.01
2	1.900	1.950	1.990
3	2.452	2.613	2.827
4	2.830	3.072	3.457
5	3.120	3.419	3.943
6	3.354	3.697	4.331
7	3.552	3.928	4.651
8	3.722	4.125	4.921
9	3.872	4.297	5.154
10	4.005	4.450	5.358
15	4.511	5.019	6.103
20	4.862	5.408	6.594
25	5.130	5.701	6.955
30	5.346	5.935	7.237
40	5.681	6.295	7.663
50	5.937	6.567	7.977
60	6.144	6.785	8.225
70	6.317	6.967	8.428
80	6.465	7.122	8.601
90	6.595	7.258	8.750
100	6.711	7.378	8.882
150	7.151	7.832	9.372
200	7.458	8.147	9.707
250	7.694	8.389	9.960
300	7.886	8.584	10.164
350	8.047	8.748	10.334
400	8.186	8.889	10.480
500	8.418	9.123	10.721
600	8.606	9.313	10.916
700	8.764	9.473	11.079
800	8.901	9.612	11.220
900	9.022	9.733	11.344
1000	9.130	9.842	11.454

To test the hypothesis that the largest ordinate is the largest in a random sample of 31 estimates, we form the ratio of the fourth ordinate to the average of ordinates 1 to 31:

Table 7.1.3. Periodogram of Deviations of United States Wheat Yields from Trend

k	Period in Years	Periodogram Ordinate
1	64.0	0.54
2	32.0	5.15
3	21.3	4.74
4	16.0	53.50
5	12.8	9.86
6	10.7	3.02
7	9.1	3.20
8	8.0	2.62
9	7.1	3.56
10	6.4	2.75
11	5.8	7.99
12	5.3	5.89
13	4.9	2.08
14	4.6	2.61
15	4.3	3.44
16	4.0	1.47
17	3.8	1.44
18	3.6	0.12
19	3.4	8.77
20	3.2	1.03
21	3.0	4.00
22	2.9	0.41
23	2.8	0.10
24	2.7	17.03
25	2.6	5.73
26	2.5	0.15
27	2.4	1.90
28	2.3	3.53
29	2.2	1.91
30	2.1	6.64
31	2.1	1.96
32	2.0	12.50

$$\xi = \frac{53.50}{5.391} = 9.92 .$$

From Table 7.1.2 we see that the 1% point for this ratio is about 7.28. Thus, the null hypothesis is rejected at this level. While we may be somewhat reluctant to accept the existence of a perfect sine cycle of length 16 years on the basis of 64 observations, it is unlikely that the deviation from trend is a white noise time series.

The cumulative periodogram for the wheat yield data is displayed in Figure

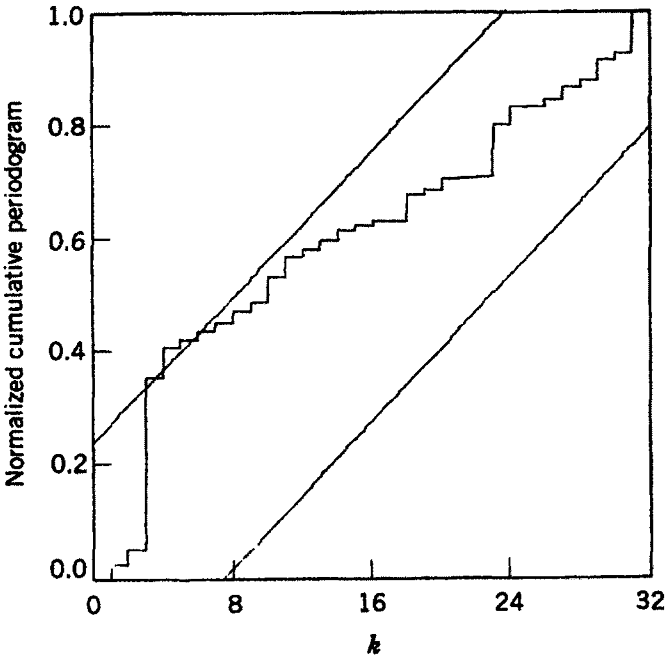

Figure 7.1.1. Normalized cumulative periodogram for deviations of wheat yield from trend.

7.1.1. We have followed the common practice of plotting the cumulative periodogram against k. The value of the cumulative periodogram for k is plotted on the interval $(k-1, k]$. The upper and lower 5% bounds for the Kolmogorov–Smirnov test have been drawn in the figure. The lines are $k/31 + 0.245$ and $k/31 - 0.245$, where 0.245 is the 5% point of the Kolmogorov–Smirnov statistic for sample size 31. The tables of Birnbaum (1952) indicate that for $m - 1 > 30$, the 95% point for the Kolmogorov–Smirnov statistic is approximately $1.36(m - 1)^{-1/2}$ and the 99% point is approximately $1.63(m-1)^{-1/2}$. In constructing the cumulative periodogram we included all 32 ordinates, even though the last ordinate has only one degree of freedom. Since the normalized cumulative periodogram passes above the upper 5% line, the data reject the hypothesis of independence, primarily because of the large ordinate at $k = 4$. ▲▲

7.2. SMOOTHING, ESTIMATING THE SPECTRUM

It is clear from the development of the distributional properties of the periodogram that increasing the sample size has little effect on the behavior of the estimated ordinate for a particular frequency. In fact, if the time series is normal (0, 1) white noise, the distribution of the periodogram ordinate is a two-degree-of-freedom chi-square independent of sample size. The number of periodogram ordinates increases as the sample size increases, but the efficiency of the estimator for a particular frequency remains unchanged.

If the spectral density is a continuous function of ω, it is natural to consider an average of local values of the periodogram to obtain a better estimate of the spectral density. Before treating such estimators, we present an additional result on the covariance properties of the periodogram.

In investigating local averages of the periodogram we find it convenient to make a somewhat stronger assumption about the rate at which the covariance function approaches zero. We shall assume that the covariances are such that

$$\sum_{h=-n}^{n} |h|\,|\gamma(h)| = O(n^{1/2}).$$

This is a fairly modest assumption and would be satisfied, for example, by any stationary finite autoregressive process. A sufficient condition for

$$\sum_{h=-n}^{n} |h|\,|\gamma(h)| = O(n^{1/2})$$

for a time series X_t is that

$$X_t = \sum_{j=0}^{\infty} \alpha_j e_{t-j},$$

where

$$\sum_{j=0}^{\infty} j^{1/2}|\alpha_j| < \infty$$

and $\{e_t\}$ is a sequence of uncorrelated $(0, \sigma^2)$ random variables. This is because

$$\begin{aligned} n^{-1/2} \sum_{h=0}^{n} h|\gamma(h)| &\le n^{-1/2} \sum_{h=0}^{n} \sum_{j=0}^{\infty} h|\alpha_j \alpha_{j+h}|\sigma^2 \\ &\le \sum_{h=0}^{n} \sum_{j=0}^{\infty} h^{1/2}|\alpha_j|\,|\alpha_{j+h}|\sigma^2 \\ &\le \sum_{j=0}^{\infty} \sum_{h=0}^{\infty} |\alpha_j|(j+h)^{1/2}|\alpha_{j+h}|\sigma^2 \\ &\le \sigma^2 \sum_{j=0}^{\infty} |\alpha_j| \sum_{s=0}^{\infty} s^{1/2}|\alpha_s|. \end{aligned}$$

Before giving a theorem on the covariance properties of the periodogram, we present two lemmas useful in the proof.

Lemma 7.2.1. For integer p, $0 \le p < n$, and real L,

$$\left| \sum_{j=1}^{n-p} \cos \omega_k (L+j) \right| \le \min\{p, n-p\}$$

and

$$\left| \sum_{j=1}^{n-p} \sin \omega_k (L+j) \right| \le \min\{p, n-p\},$$

where $\omega_k = 2\pi k/n$, $k = 1, 2, \ldots, n-1$.

Proof. Now

$$\begin{aligned}\sum_{j=1}^{n} \cos \omega_k (L+j) &= \sum_{j=1}^{n} (\cos \omega_k L \cos \omega_k j - \sin \omega_k L \sin \omega_k j) \\ &= (\cos \omega_k L) \sum_{j=1}^{n} \cos \omega_k j - (\sin \omega_k L) \sum_{j=1}^{n} \sin \omega_k j \\ &= 0 \qquad \text{for} \quad \omega_k \ne 0,\end{aligned}$$

where the zero sum follows from the arguments of Theorem 3.1.1. Now $|\cos \theta| \le 1$ for all θ, and the result follows. The analogous argument holds for the sum of sines. ▲

Lemma 7.2.2. For real L

$$\left| \sum_{t=1}^{n} \sum_{j=1}^{n} \gamma(t-j) \cos \omega_k (L+j) \right| \le \sum_{p=-(n-1)}^{n-1} |p| |\gamma(p)|,$$

$$\left| \sum_{t=1}^{n} \sum_{j=1}^{n} \gamma(t-j) \sin \omega_k (L+j) \right| \le \sum_{p=-(n-1)}^{n-1} |p| |\gamma(p)|,$$

where $\omega_k = 2\pi k/n$, $k = 1, 2, \ldots, n-1$.

Proof. We have

$$\begin{aligned}\left| \sum_{t=1}^{n} \sum_{j=1}^{n} \gamma(t-j) \cos \omega_k (L+j) \right| &= \left| \sum_{p=0}^{n-1} \sum_{j=1}^{n-p} \gamma(p) \cos \omega_k (L+j) \right. \\ &\qquad \left. + \sum_{p=-(n-1)}^{-1} \sum_{j=-p+1}^{n} \gamma(p) \cos \omega_k (L+j) \right| \\ &\le \sum_{p=-(n-1)}^{(n-1)} |p| |\gamma(p)|,\end{aligned}$$

where we have used Lemma 7.2.1. The sine result follows in a completely analogous manner. ▲

Theorem 7.2.1. Let the time series X_t be defined by

$$X_t = \sum_{j=0}^{\infty} \alpha_j e_{t-j},$$

where the e_t are independent $(0, \sigma^2)$ random variables with fourth moment $\eta\sigma^4$ and

$$\sum_{j=1}^{\infty} j^{1/2}|\alpha_j| < \infty.$$

Then

$$\operatorname{Cov}\{I_n(\omega_j), I_n(\omega_k)\} = \begin{cases} 2(4\pi)^2 f^2(0) + o(1), & \omega_j = \omega_k = 0, \\ (4\pi)^2 f^2(\omega_k) + o(1), & \omega_j = \omega_k, \omega_k \neq 0, \pi, \\ O(n^{-1}), & \omega_j \neq \omega_k. \end{cases}$$

Furthermore, for the sequence composed only of even-sized samples,

$$\operatorname{Var}\{I_n(\pi)\} = 2(4\pi)^2 f^2(\pi) + o(1).$$

Proof. By the definition (7.1.6) of the periodogram, we have

$$I_n(\omega_k) = \frac{2}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} X_t X_s e^{\iota\omega_k(t-s)}$$

and

$$\begin{aligned}
&E\{I_n(\omega_k)I_n(\omega_j)\} - E\{I_n(\omega_k)\}E\{I_n(\omega_j)\} \\
&\quad = E\left\{\frac{4}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} X_t X_s X_u X_v e^{\iota\omega_k(t-s)} e^{\iota\omega_j(u-v)}\right\} \\
&\qquad - 4 \sum_{h=-(n-1)}^{n-1} \frac{n-|h|}{n} \gamma(h) e^{-\iota\omega_k h} \sum_{q=-(n-1)}^{n-1} \frac{n-|q|}{n} \gamma(q) e^{-\iota\omega_j q} \\
&\quad = \frac{4(\eta-3)\sigma^4}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \sum_{r=-\infty}^{\infty} \alpha_r \alpha_{r+t-s} \alpha_{r+t-u} \alpha_{r+t-v} e^{\iota\omega_k(t-s)} e^{\iota\omega_j(u-v)} \\
&\qquad + \frac{4}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \gamma(t-u)\gamma(s-v) e^{\iota\omega_k(t-u)} e^{\iota\omega_j(s-v)} e^{\iota(\omega_j+\omega_k)(u-s)} \\
&\qquad + \frac{4}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \gamma(t-v)\gamma(u-s) e^{\iota\omega_k(t-v)} e^{\iota\omega_j(u-s)} e^{\iota(\omega_j-\omega_k)(s-v)},
\end{aligned} \tag{7.2.1}$$

where we have used (6.2.5) and $\alpha_r = 0$, $r < 0$. Now,

$$\frac{4}{n^2}\left|\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}\sum_{r=-\infty}^{\infty}\alpha_r\alpha_{r+t-s}\alpha_{r+t-u}\alpha_{r+t-v}e^{\iota\omega_k(t-s)}e^{\iota\omega_j(u-v)}\right|$$
$$\leq \frac{4}{n}\sum_{j=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}|\alpha_j||\alpha_{j+p}||\alpha_{j+p+q}||\alpha_{j+p+q+h}|$$
$$= O(n^{-1}),$$

by the absolute summability of α_j.

If $\omega_k = \omega_j = 0$, or if n is even and $\omega_k = \omega_j = \pi$, the second term of (7.2.1) is

$$\frac{4}{n^2}\left[\sum_{h=-(n-1)}^{n-1}(n-|h|)\gamma(h)\right]^2.$$

For $w_k = \omega_j \neq 0$, π the second term of (7.2.1) is

$$\frac{4}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}\gamma(t-u)\gamma(s-v)e^{\iota\omega_k[(t-u+2u)-(s-v+2v)]}$$
$$\leq\left[2\sum_{p=-(n-1)}^{n-1}\frac{|p|}{n}|\gamma(p)|\right]^2 = O(n^{-1}),$$

where the inequality follows from Lemma 7.2.2. For $\omega_k = \omega_j$ the third term reduces to

$$\left[2\sum_{p=-(n-1)}^{n-1}\frac{n-|p|}{n}\gamma(p)e^{\iota\omega_k p}\right]^2.$$

By Lemma 7.2.2, for $\omega_k \neq \omega_j$,

$$\left|\sum_{t=1}^{n}\sum_{u=1}^{n}\gamma(t-u)e^{\iota\omega_k(t-u)}e^{\iota(\omega_j+\omega_k)u}\right| \leq \sum_{p=-(n-1)}^{n-1}|p||\gamma(p)|$$

and the absolute value of the second term shown above is less than $[2n^{-1}\sum_{p=-(n-1)}^{n-1}|p||\gamma(p)|]^2$, which, by assumption, is $O(n^{-1})$. By a similar argument, the third term of (7.2.1) is $O(n^{-1})$ when $\omega_k \neq \omega_j$. Thus, if $\omega_k = \omega_j = \omega \neq 0$, π,

$$\lim_{n\to\infty}\frac{4}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}\gamma(t-v)\gamma(u-s)e^{\iota\omega_{K(n,\omega)}(t-v)}e^{\iota\omega_{K(n,\omega)}(u-s)}$$
$$=\lim_{n\to\infty}\left[2\sum_{h=-(n-1)}^{n-1}\frac{n-|h|}{n}\gamma(h)e^{-\iota\omega h}\right]^2 = (4\pi)^2 f^2(\omega).$$

If $\omega_j = \omega_k = 0$, then

$$\lim_{n\to\infty} \frac{4}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \gamma(t-u)\gamma(s-v)$$

$$+ \lim_{n\to\infty} \frac{4}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \gamma(t-v)\gamma(u-s)$$

$$= 2 \lim_{n\to\infty} \left[2 \sum_{h=-(n-1)}^{n-1} \frac{n-|h|}{n} \gamma(h) \right]^2 = 2(4\pi)^2 f^2(0),$$

and the stated results follow. ▲

Corollary 7.2.1. Let X_t be the time series defined in Theorem 7.2.1. In addition, assume the e_t are NI$(0, \sigma^2)$ random variables and that

$$\sum_{j=1}^{\infty} j|\alpha_j| < \infty .$$

Then

$$\operatorname{Cov}\{I_n(\omega_j), I_n(\omega_k)\} = O(n^{-2}), \qquad \omega_j \neq \omega_k .$$

Proof. Under normality, the first term of (7.2.1) is zero because $\eta = 3$ for the normal distribution. Under the assumption that $\sum_{j=1}^{\infty} j|\alpha_j| < \infty$, we have

$$\sum_{-\infty}^{\infty} |h|\,|\gamma(h)| < \infty ,$$

and the remaining two terms of (7.2.1) are $O(n^{-2})$. See Exercise 7.9. ▲

Since the covariances between periodogram ordinates are small, the variance of the periodogram estimator of the spectral density at a particular frequency can be reduced by averaging adjacent periodogram ordinates. The simplest such estimator is defined by

$$\bar{f}(\omega_k) = \frac{1}{2d+1} \sum_{j=-d}^{d} \hat{f}(\omega_{k+j}) .$$

where

$$\hat{f}(\omega_k) = \frac{1}{4\pi} I_n(\omega_k) .$$

In general, we consider the linear function of the periodogram ordinates

$$\bar{f}(\omega_k) = \sum_{j=-d}^{d} W(j) \hat{f}(\omega_{k+j}) , \tag{7.2.2}$$

where $\Sigma_{j=-d}^{d} W(j) = 1$. The weight function $W(j)$ is typically symmetric about zero with a maximum at zero.

We may extend $\bar{f}(\omega_k)$ to all ω by defining $\bar{f}(\omega) = \bar{f}(\omega_{K(n,\omega)})$ where the function $K(n, \omega)$ was introduced in Section 7.1. In practice the values of $\bar{f}(\omega_k)$ are often connected by lines to obtain a continuous function of ω.

Theorem 7.2.2. Let the time series X_t be defined by

$$X_t = \sum_{j=0}^{\infty} \alpha_j e_{t-j} ,$$

where the e_t are independent $(0, \sigma^2)$ random variables with fourth moment $\eta\sigma^4$ and

$$\sum_{j=1}^{\infty} j^{1/2}|\alpha_j| < \infty .$$

Let d_n be an increasing sequence of positive integers satisfying

$$\lim_{n\to\infty} d_n = \infty ,$$

$$\lim_{n\to\infty} \frac{d_n}{n} = 0 .$$

Let the weight function $W_n(j)$, $j = 0, \pm 1, \pm 2, \ldots, \pm d_n$, satisfy

$$\sum_{j=-d_n}^{d_n} W_n(j) = 1 ,$$

$$W_n(j) = W_n(-j) ,$$

$$\lim_{n\to\infty} \sum_{j=-d_n}^{d_n} W_n^2(j) = 0 .$$

Then $\bar{f}(\omega_{K(n,\omega)})$ defined by (7.2.2) satisfies

$$\lim_{n\to\infty} E\{\bar{f}(\omega_{K(n,\omega)})\} = f(\omega) ,$$

$$\lim_{n\to\infty} \left(\sum_{j=-d_n}^{d_n} W_n^2(j) \right)^{-1} \operatorname{Var}\{\bar{f}(\omega_{K(n,\omega)})\} = \begin{cases} f^2(\omega) , & \omega \neq 0, \pi, \\ 2f^2(\omega) , & \omega = 0, \pi . \end{cases}$$

Proof. Now

$$E\{\bar{f}(\omega_{K(n,\omega)}) = \frac{1}{2\pi} \sum_{j=-d_n}^{d_n} W_n(j) \sum_{h=-(n-1)}^{n-1} \frac{n-|h|}{n} \gamma(h) e^{-\iota h \omega_{K(n,\omega+2\pi j/n)}}$$

and

$$\lim_{n\to\infty} E\{\bar{f}(\omega_{K(n,\omega)})\} = \lim_{n\to\infty} \sum_{j=-d_n}^{d_n} W_n(j) f(\omega_{K(n,\omega+2\pi j/n)}) .$$

Since $f(\omega)$ is uniformly continuous, given $\epsilon > 0$, there exists an N such that for $n > N$, $|f(\delta) - f(\omega)| < \epsilon$ for δ in the interval $[\omega_{K(n,\omega-2\pi d_n/n)}, \omega_{K(n,\omega+2\pi d_n/n)}]$. Therefore,

$$\lim_{n\to\infty} E\{\bar{f}(\omega)\} = f(\omega) .$$

By Theorem 7.2.1, for $2\pi d_n/n \leq \omega \leq \pi - 2\pi d_n/n$,

$$\operatorname{Var}\{\bar{f}(\omega_{K(n,\omega)})\} = \sum_{j=-d_n}^{d_n} W_n^2(j) f^2(\omega_{K(n,\omega+2\pi j/n)}) + o\left(\sum_{j=-d_n}^{d_n} W_n^2(j)\right),$$

and, by the argument used for the expectation,

$$\lim_{n\to\infty} \left(\sum_{j=-d_n}^{d_n} W_n^2(j)\right)^{-1} \operatorname{Var}\{\bar{f}(\omega)\} = f^2(\omega) .$$

If $\omega = 0$ or π, only $d_n + 1$ (or d_n) estimates $\hat{f}(\omega_k)$ are averaged, since, for example, $\hat{f}(2\pi/n) = \hat{f}(-2\pi/n)$. Therefore,

$$\operatorname{Var}\{\bar{f}(0)\} = W_n^2(0) f^2(0) + 4\sum_{j=1}^{d_n} W_n^2(j) f^2(\omega_{K(n,\omega+2\pi j/n)}) + o\left(\sum_{j=-d_n}^{d_n} W_n^2(j)\right),$$

and the result follows. ▲

Corollary 7.2.2. Let the assumptions of Theorem 7.2.2 be satisfied with $W_n(j) = (2d_n + 1)^{-1}$. Then

$$\operatorname{Var}\{\bar{f}(\omega_{K(n,\omega)})\} = \begin{cases} \dfrac{1}{2d_n + 1} f^2(\omega) + o(d_n^{-1}), & \omega \neq 0, \pi, \\ \dfrac{2}{2d_n + 1} f^2(\omega) + o(d_n^{-1}), & \omega = 0, \pi . \end{cases}$$

Perhaps it is worthwhile to pause and summarize our results for the periodogram estimators. First, the periodogram ordinates $I_n(\omega_j)$ are the sums of squares associated with sine and cosine regression variables for the frequency ω_j. For a wide class of time series the $I_n(\omega_j)$ are approximately independently distributed as $[2\pi f(\omega_j)]\chi_2^2$, that is, as a multiple of a chi-square with two degrees of freedom (Theorems 7.1.1 and 7.2.1).

If $f(\omega)$ is a continuous function of ω, then, for large n, adjacent periodogram ordinates have approximately the same mean and variance. Therefore, an average of $2d+1$ adjacent ordinates has approximately the same mean and a variance $(2d+1)^{-1}$ times that of the original ordinates. It is possible to construct a sequence of estimators based on realizations of increasing size wherein the number of adjacent ordinates being averaged increases (at a slower rate than n) so that the average, when divided by 4π, is a consistent estimator of $f(\omega)$ (Theorem 7.2.2). The consistency result is less than fully appealing, since it does not tell us how many terms to include in the average for any particular time series. Some general conclusions are possible. For most time series the average of the periodogram ordinates will be a biased estimator of $4\pi f(\omega)$. For the largest portion of the range of most functions, this bias will increase as the number of terms being averaged increases. On the other hand, we can expect the variance of the average to decline as additional terms are added. (See Exercise 7.11.) Therefore, the mean square error of our average as an estimator of the spectral density will decline as long as the increase in the squared bias is less than the decrease in the variance. The white noise time series furnishes the limiting case. Since the spectral density is a constant function, the best procedure is to include all ordinates in the average [i.e., to use $\hat{\gamma}(0)$ to estimate $2\pi f(\omega)$ for all ω]. For a time series of known structure we could determine the optimum number of terms to include in the weight function. However, if we possess that degree of knowledge, the estimation problem is no longer of interest. The practitioner, as he works with data, will develop certain rules of thumb for particular kinds of data. For data with unknown structure it would seem advisable to construct several averages of varying length before reaching conclusions about the nature of the spectral density.

The approximate distributional properties of the smoothed periodogram can be used to construct a confidence interval for the estimated spectral density. Under the conditions of Theorem 7.2.2, the $I_n(\omega_k)$ are approximately distributed as independent chi-squares, and therefore $\bar{f}(\omega)$ is approximately distributed as a linear combination of chi-square random variables. One common approximation to such a distribution is a chi-square distribution with degrees of freedom determined by the variance of the distribution.

Result 7.2.1. Let X_t satisfy the assumptions of Theorem 7.2.2 and let $f(\omega) > 0$. Then, for $\pi d_n/n < \omega < \pi(1 - d_n/n)$, $f^{-1}(\omega)\bar{f}(\omega)$ is approximately distributed as a chi-square random variable divided by its degrees of freedom ν, where

$$\nu = 2\left[\sum_{j=-d_n}^{d_n} W_n^2(j)\right]^{-1}.$$

An approximate $1-\alpha$ level confidence interval for $f(\omega)$ can be constructed as

$$\frac{\nu\bar{f}(\omega)}{\chi^2_{\nu,\alpha/2}} \le f(\omega) \le \frac{\nu\bar{f}(\omega)}{\chi^2_{\nu,1-(\alpha/2)}}, \tag{7.2.3}$$

where $\chi^2_{\nu,\alpha/2}$ is the $\alpha/2$ tabular value for the chi-square distribution with ν degrees of freedom. (Point exceeded with probability $\alpha/2$.)

Since the variance of $\bar{f}(\omega)$ is a multiple of $[f(\omega)]^2$, the logarithm of $\bar{f}(\omega)$ for time series with $f(\omega)$ strictly positive will have approximately constant variance,

$$\operatorname{Var}\{\log \bar{f}(\omega)\} \doteq [f(\omega)]^{-2}\operatorname{Var}\{\bar{f}(\omega)\} \doteq \frac{2}{\nu}.$$

Therefore, $\log \bar{f}(\omega)$ is often plotted as a function of ω. Approximate confidence intervals for $\log f(\omega)$ are given by

$$\log \bar{f}(\omega) + \log\left(\frac{\nu}{\chi^2_{\nu,\alpha/2}}\right) \le \log f(\omega) \le \log \bar{f}(\omega) + \log\left(\frac{\nu}{\chi^2_{\nu,1-(\alpha/2)}}\right). \tag{7.2.4}$$

Example 7.2.1. As an illustration of spectral estimation, we consider an artificially generated time series. Table 7.2.1 contains 100 observations for the time series

$$X_t = 0.7\,X_{t-1} + e_t$$

where the e_t are computer generated normal independent (0, 1) random variables. The periodogram for this sample is given in Figure 7.2.1. We have connected the ordinates with straight lines. The approximate expected value of the periodogram ordinates is $4\pi f(\omega) = 2(1.49 - 1.4\cos\omega)^{-1}$ and has also been plotted in the figure. Both the average value and the variance are much larger for ordinates associated with the smaller frequencies.

We have labeled the frequency axis in radians. Thus, the fastest frequency we can observe is π, which is 0.50 cycles per time unit. This corresponds to a cycle with a period of two time units. In applications there are natural time units such as hours, months, or years, and one may choose to label the axis in terms of the frequency in these units.

In Figure 7.2.2 we display the smoothed periodogram where the smoothed value at ω_k is

$$\bar{I}(\omega_k) = \frac{1}{5}\sum_{j=-2}^{2} I_n(\omega_{k-j}).$$

The smoothed periodogram roughly assumes the shape of the spectral density. Because the standard error of the smoothed estimator for $2 < k < 48$ is about 0.45 of the true value, there is considerable variability in the plot. The smoothing

Table 7.2.1. One Hundred Observations from a First Order Autoregressive Time Series with $\rho = 0.7$

	First 25	Second 25	Third 25	Fourth 25
1	0.874	−0.613	−0.366	−0.955
2	0.850	0.110	−1.420	−0.948
3	2.345	0.113	−0.183	0.046
4	2.501	−0.308	−0.044	0.091
5	1.657	0.723	−0.391	0.254
6	1.649	−0.257	−0.095	2.750
7	2.498	1.051	−0.971	1.673
8	1.330	0.803	0.371	2.286
9	1.307	0.116	−1.622	1.220
10	3.404	−1.454	−2.941	−0.256
11	2.445	0.296	−2.814	0.252
12	2.805	1.501	−1.784	0.325
13	1.639	0.880	−2.471	−0.338
14	1.240	−0.672	−3.508	0.378
15	1.116	0.436	−2.979	0.127
16	0.448	0.930	−0.779	−2.006
17	0.377	1.168	0.869	−2.380
18	−0.488	1.999	1.786	−2.024
19	−0.960	1.376	0.123	−1.085
20	−0.579	1.613	0.093	1.037
21	−1.674	2.030	−0.731	−0.467
22	−0.366	0.616	−1.253	−0.794
23	−0.922	0.667	−2.213	−0.493
24	−1.174	0.707	−0.252	−0.157
25	−1.685	1.029	0.403	0.659

introduces a correlation between observations less than $2d + 1$ units apart. In one sense this was the objective of the smoothing, since it produces a plot that more nearly approximates that of the spectral density. On the other hand, if the estimated spectral density is above the true density, it is apt to remain so for some distance.

To compute smoothed values near zero and π, the periodic nature of the function $f(\omega)$ is used. In most applications the mean is estimated and therefore $I_n(0)$ is not computed. We follow this practice in our example. To compute the smoothed value at zero we set $I_n(\omega_0) = I_n(\omega_1)$ and compute

$$\bar{I}(\omega_0) = \tfrac{1}{5}\,[I_{100}(\omega_{-2}) + I_{100}(\omega_{-1}) + I_{100}(\omega_0) + I_{100}(\omega_1) + I_{100}(\omega_2)]\,,$$

which, by the even periodic property of $f(\omega)$, is given by

$$\bar{I}(\omega_0) = \tfrac{1}{5}[I_{100}(\omega_0) + 2I_{100}(\omega_1) + 2I_{100}(\omega_2)]\,.$$

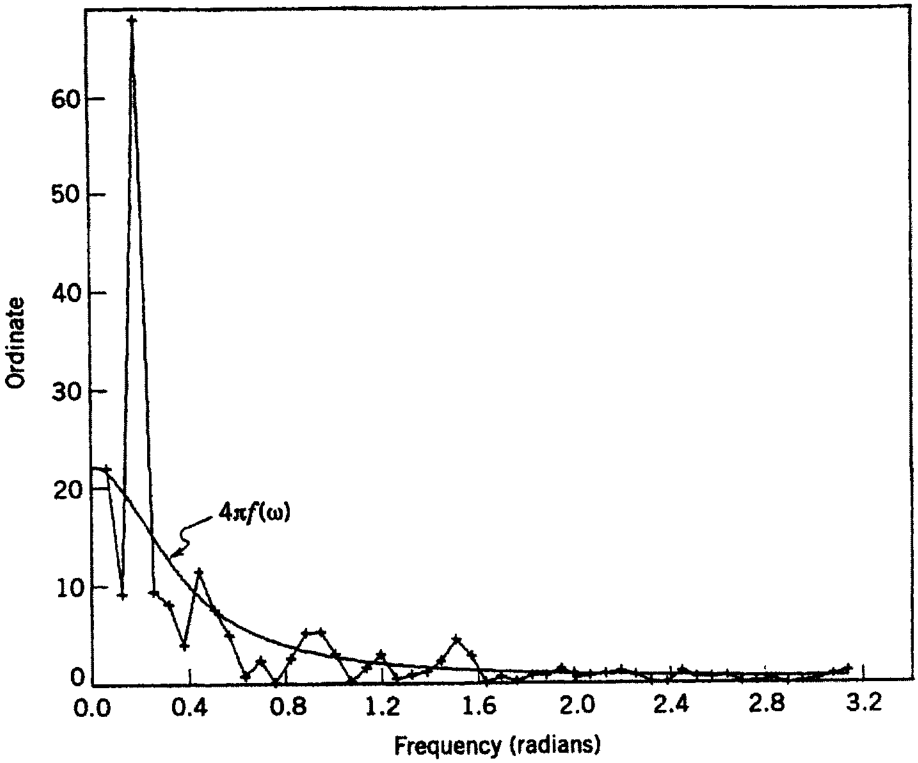

Figure 7.2.1. Periodogram computed from the 100 autoregressive observations of Table 7.2.1 compared with $4\pi f(\omega)$.

In our example, replacing $I_{100}(\omega_0)$ by $I_{100}(\omega_1)$,

$$\bar{I}(\omega_0) = \tfrac{1}{5}[3(22.004) + 2(9.230)] = 16.894 .$$

Similarly

$$\begin{aligned}\bar{I}(\omega_2) &= \tfrac{1}{5}[I_{100}(\omega_0) + I_{100}(\omega_1) + I_{100}(\omega_2) + I_{100}(\omega_3) + I_{100}(\omega_4)] \\ &= \tfrac{1}{5}[2(22.004) + (9.230) + (67.776) + (9.360)] = 26.075 .\end{aligned}$$

As the sample size is even, there is a one-degree-of-freedom periodogram ordinate for π. The smoothed estimate at π is

$$\begin{aligned}\bar{I}(\omega_{50}) &= \tfrac{1}{5}[I_{100}(\omega_{50}) + 2I_{100}(\omega_{49}) + 2I_{100}(\omega_{48})] \\ &= \tfrac{1}{5}[(1.166) + 2(0.870) + 2(0.440)] = 0.757 .\end{aligned}$$

In Figure 7.2.3 we plot the logarithm of the average of 11 periodogram ordinates ($d = 5$). The 95% confidence intervals are also plotted in Figure 7.2.3.

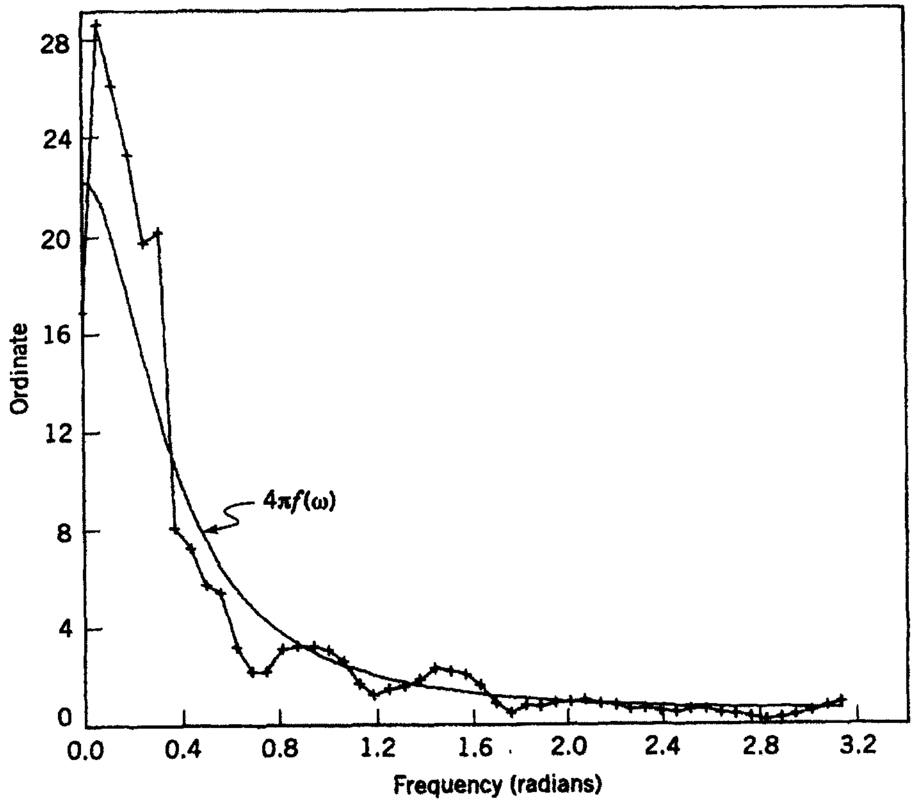

Figure 7.2.2. Smoothed periodogram ($d = 2$) computed from 100 autoregressive observations of Table 7.2.1 compared with $4\pi f(\omega)$.

They were constructed using (7.2.4), so that the upper bound is

$$\log \bar{I}(\omega) + \log\left(\frac{22}{10.98}\right)$$

and the lower bound is

$$\log \bar{I}(\omega) + \log\left(\frac{22}{36.78}\right).$$

This interval is appropriate for $6 \leq k \leq 44$ and adequate for $k = 45$. Confidence intervals for other values of k could be constructed using the variance of the estimator. For example, the smoothed value at zero is

$$\bar{I}(0) = \frac{1}{11}\left[3I_{100}(\omega_1) + 2\sum_{j=2}^{5} I_{100}(\omega_j)\right],$$

and the variance is approximately $\frac{25}{121}$ $[4\pi f(0)]^2 \doteq 0.21[4\pi f(0)]^2$. Since the

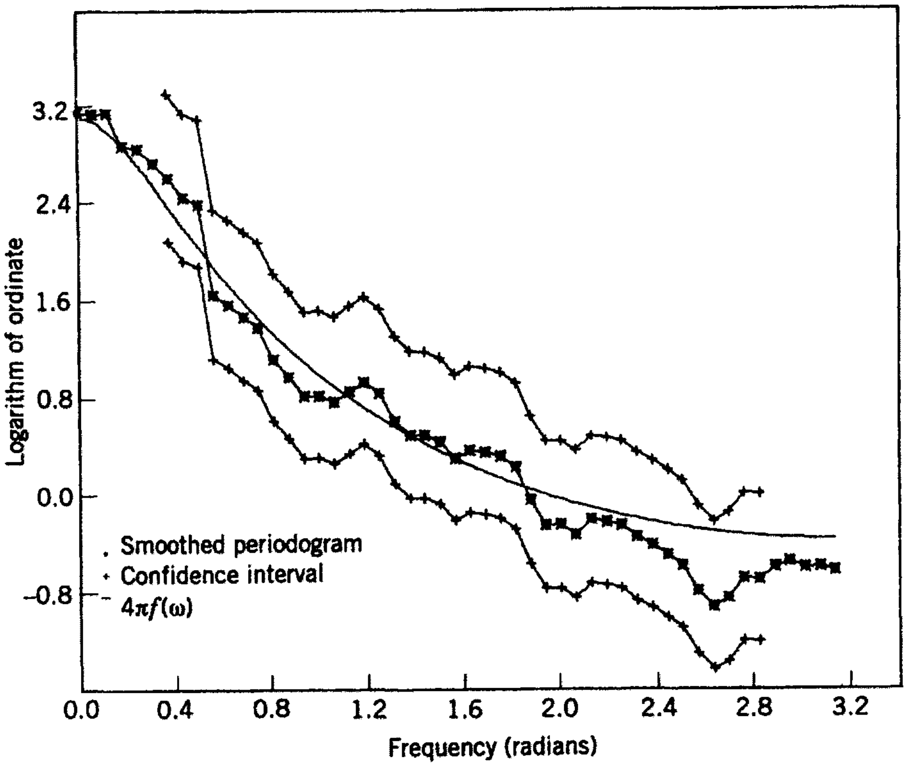

Figure 7.2.3. Logarithm of smoothed periodogram ($d = 5$) and confidence interval for logarithm of $4\pi f(\omega)$ computed from the 100 autoregressive observations of Table 7.2.1.

variance of a chi-square with 10 degrees of freedom divided by its degrees of freedom is 0.20, we can establish a confidence interval for $4\pi f(0)$ using the critical points for a chi-square with 10 degrees of freedom. A similar approach can be used for $1 \leq k \leq 5$ and $46 \leq k \leq 50$.

Example 7.2.2. As a second example of spectral estimation, we consider the United States monthly unemployment rate from October 1949 to September 1974, Figure 7.2.4 is a plot of the logarithm of the smoothed periodogram using the rectangular weights and $d = 5$. Also included in the figure are lines defining the 95% confidence interval. This plot displays characteristics typical of many economic time series. The smoothed periodogram is high at small frequencies, indicating a large positive autocorrelation for observations close together in time. Second, there are peaks at the seasonal frequencies π, $5\pi/6$, $2\pi/3$, $\pi/2$, $\pi/3$, $\pi/6$, indicating the presence of a seasonal component in the time series. The periodogram has been smoothed using the simple average, and the flatness of the seasonal peaks indicates that they are being dominated by the center frequencies. That is, the shape of the peak is roughly that of the weights being used in the smoothing. Table 7.2.2 contains the ordinates at and near the seasonal frequencies.

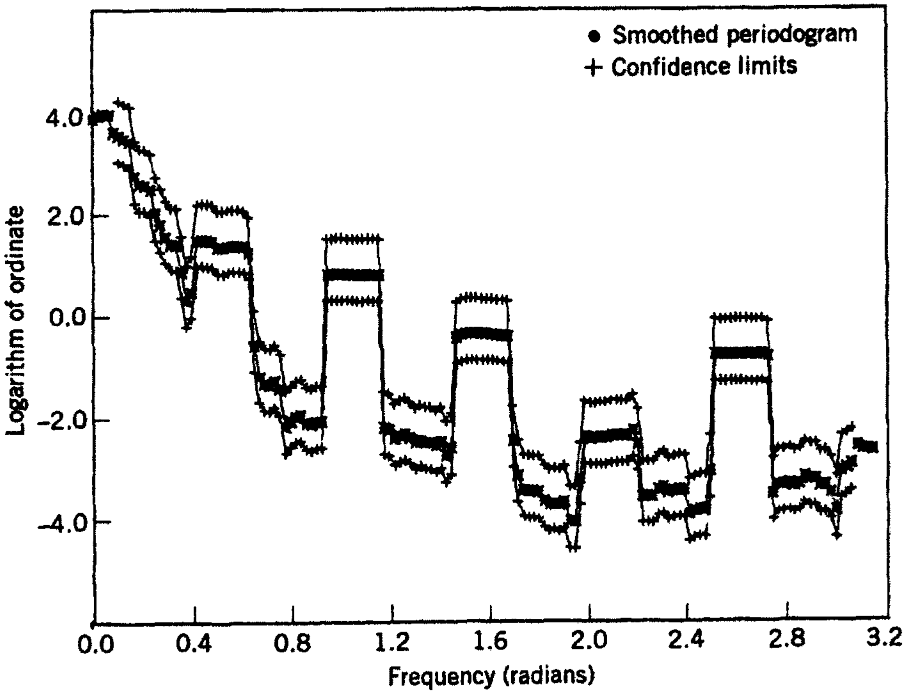

Figure 7.2.4. Logarithm of smoothed periodogram for monthly United States unemployment rate for October 1949 through September 1974 ($n = 300$, $d = 5$).

With the exception of π, the ordinates at the seasonal frequencies are much larger than the other ordinates. Also, it seems that the ordinates adjacent to the seasonal frequencies are larger than those farther away. This suggests that the "seasonality" in the time series is not perfectly periodic. That is, more than the six seasonal frequencies are required to completely explain the peaks in the estimated spectral density. ▲▲

7.3. OTHER ESTIMATORS OF THE SPECTRUM

The smoothed periodogram is a weighted average of the Fourier transform of the sample autocovariance. An alternative method of obtaining an estimated spectral density is to apply weights to the estimated covariance function and then transform the "smoothed" covariance function. The impetus for this procedure came from a desire to reduce the computational costs of computing covariances for realizations with very large numbers of observations. Thus, the weight function has traditionally been chosen to be nonzero for the first few autocovariances and zero otherwise. The development of computer routines using the fast Fourier transform reduced the cost of computing finite Fourier transforms and reduced the emphasis on the use of the weighted autocovariance estimator of the spectrum.

Table 7.2.2. Periodogram Ordinates Near Seasonal Frequencies

Frequency (Radians)	Ordinate	Frequency (Radians)	Ordinate
0.461	0.127	2.032	0.025
0.482	0.607	2.052	0.021
0.503	5.328	2.073	0.279
0.523	31.584	2.094	0.575
0.545	2.904	2.115	0.012
0.565	0.557	2.136	0.011
0.586	0.069	2.157	0.008
0.984	0.053	2.555	0.028
1.005	0.223	2.576	0.010
1.026	0.092	2.597	0.286
1.047	23.347	2.618	4.556
1.068	0.310	2.639	0.038
1.089	0.253	2.660	0.017
1.110	0.027	2.681	0.022
1.508	0.191	3.079	0.008
1.529	0.041	3.100	0.003
1.550	0.177	3.120	0.294
1.571	6.402	3.142	0.064
1.592	0.478		
1.613	0.142		
1.634	0.012		

Let $w(x)$ be a bounded even continuous function satisfying

$$\begin{aligned} w(0) &= 1\,, \\ w(x) &= 0\,, \qquad |x| > 1\,, \\ |w(x)| &\le 1 \qquad \text{for all } x\,. \end{aligned} \tag{7.3.1}$$

Then a weighted estimator of the spectral density is

$$\bar{f}(\omega) = \frac{1}{2\pi} \sum_{h=-g_n}^{g_n} w\left(\frac{h}{g_n}\right) \hat{\gamma}(h) e^{-\iota\omega h}\,, \tag{7.3.2}$$

where $g_n \le n$ is the chosen point of truncation and

$$\hat{\gamma}(h) = \hat{\gamma}(-h) = \frac{1}{h} \sum_{t=1}^{n-h} (X_t - \bar{x}_n)(X_{t+h} - \bar{x}_n)\,, \qquad h \ge 0\,.$$

From Section 3.4 we know that the transform of a convolution is the product of the transforms. Conversely, the transform of a product is the convolution of the transforms. In the current context we define the following function (or transform):

$$\hat{f}(\omega) = \frac{1}{2\pi} \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) e^{-\iota\omega h}$$
$$= \frac{1}{2\pi} \left[\hat{\gamma}(0) + 2 \sum_{h=1}^{n-1} \hat{\gamma}(h) \cos \omega h \right].$$

The function $\hat{f}(\omega)$ is a continuous function of ω and is an unweighted estimator of the continuous spectral density. By the uniqueness of Fourier transforms, we can write

$$\hat{\gamma}(h) = \int_{-\pi}^{\pi} \hat{f}(\omega) e^{\iota\omega h}\, d\omega\,, \qquad h = 0, \pm 1, \pm 2, \ldots, \pm(n-1)\,.$$

We define the transform of $w(x)$ similarly as

$$W(\omega) = \frac{1}{2\pi} \sum_{h=-(n-1)}^{n-1} w\left(\frac{h}{g_n}\right) e^{-\iota\omega h}\,,$$

where $W(\omega)$ is also a continuous function. It follows that

$$w\left(\frac{h}{g_n}\right) = \int_{-\pi}^{\pi} W(\omega) e^{\iota\omega h}\, d\omega\,, \qquad h = 0, \pm 1, \pm 2, \ldots, \pm(n-1)\,.$$

Note that $w(0) = 1$ means that $\int_{-\pi}^{\pi} W(s)\, ds = 1$. Then, by Exercise 3.14,

$$\bar{f}(\omega) = \frac{1}{2\pi} \sum_{h=-(n-1)}^{n-1} w\left(\frac{h}{g_n}\right) \hat{\gamma}(h) e^{-\iota\omega h}$$
$$= \int_{-\pi}^{\pi} W(s) \hat{f}(\omega - s)\, ds\,.$$

One should remember that both $W(s)$ and $\hat{f}(s)$ are even periodic functions. Thus, the estimated spectrum obtained from the weighted estimated covariances $w(h/g_n)\hat{\gamma}(h)$ is a weighted average (convolution) of the spectrum estimated from the original covariances, where the weight function is the transform of weights applied to the covariances.

The function $W(\omega)$ is called the *kernel* or *spectral window*. The weight function $w(x)$ is often called the *lag window*.

Theorem 7.3.1. Let g_n be a sequence of positive integers such that

$$\lim_{n\to\infty} g_n = \infty\,.$$
$$\lim_{n\to\infty} n^{-1} g_n = 0\,;$$

and let X_t be a time series defined by

$$X_t = \sum_{j=-\infty}^{\infty} \alpha_j e_{t-j},$$

where the e_t are independent $(0, \sigma^2)$ random variables with fourth moment $\eta\sigma^4$, and $\{\alpha_j\}$ is absolutely summable. Let

$$\tilde{\gamma}(h) = \tilde{\gamma}(-h) = \frac{1}{n-h} \sum_{t=1}^{n-h} X_t X_{t+h}, \qquad h \geq 0.$$

and

$$\tilde{f}(\omega) = \frac{1}{2\pi} \sum_{h=-g_n}^{g_n} w\left(\frac{h}{g_n}\right)\left(\frac{n-|h|}{n}\right) \tilde{\gamma}(h) e^{-\iota\omega h},$$

where $w(x)$ is a bounded even continuous function satisfying the conditions (7.2.5). Then

$$E\{\tilde{f}(\omega)\} = \frac{1}{2\pi} \sum_{h=-g_n}^{g_n} \frac{n-|h|}{n} w\left(\frac{h}{g_n}\right) \gamma(h) e^{-\iota\omega h}$$

and

$$\lim_{n\to\infty} \frac{n}{g_n} \operatorname{Cov}\{\tilde{f}(\omega), \tilde{f}(\lambda)\} = \begin{cases} 2f^2(\omega) \int_{-1}^{1} w^2(x)\, dx, & \omega = \lambda = 0, \pi, \\ f^2(\omega) \int_{-1}^{1} w^2(x)\, dx, & \omega = \lambda \neq 0, \pi, \\ 0, & \omega \neq \lambda. \end{cases}$$

Proof. See, for example, Anderson (1971, Chapter 9) or Hannan (1970, Chapter 5). ▲

The choice of a truncation point g_n for a particular sample of n observations is not determined by the asymptotic theory of Theorem 7.2.3. As in our discussion of the smoothed periodogram, the variance generally increases as g_n increases, but the bias in the estimator will typically decrease as g_n increases. It is possible to determine the order of the bias as a function of the properties of the weight function $w(x)$ and the speed with which $\gamma(h)$ approaches zero [see Parzen (1961)], but this still does not solve the problem for a given sample and unknown covariance structure.

Approximate confidence limits for the spectral density can be constructed in the same manner as that used for the smoothed periodogram estimator.

Result 7.3.1. Let X_t satisfy the assumptions of Theorem 7.3.1, let $w(x)$ satisfy

(7.3.1), and let $f(\omega) > 0$. Then for $\pi\nu/2n < \omega < \pi - \pi\nu/2n$, $f^{-1}(\omega)\tilde{f}(\omega)$ is approximately distributed as a chi-square random variable divided by its degrees of freedom ν, where

$$\nu = \frac{2n}{\int_{-1}^{1} w^2(x)\, dx} .$$

Considerable research has been conducted on the weights $w(x)$ to use in estimating the spectrum. One of the simplest windows is obtained by truncating the sequence of autocovariances at g_n. This procedure is equivalent to applying the window

$$w(x) = \begin{cases} 1, & |x| \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{7.3.3}$$

The function $w(x)$ is sometimes called a *truncated* or *rectangular* window. While $w(x)$ does not meet the conditions of Theorem 7.3.1, it can be shown that the conclusion holds. The spectral window for the function (7.3.3),

$$W(s) = \frac{1}{2\pi} \frac{\sin(g_n + \frac{1}{2})s}{\sin(s/2)} ,$$

takes on negative values, and it is possible that the weighted average $\tilde{f}(\omega)$ will be negative for some ω. This is generally considered an undesirable attribute, since $f(\omega) \ge 0$ for all ω.

Bartlett (1950) suggested splitting an observed time series of n observations into p groups of M observations each. The periodogram is then computed for each group, and the estimator for the ordinate associated with a particular frequency is taken to be the average of the p estimators; that is,

$$4\pi\tilde{f}(\omega_k) = \frac{1}{p} \sum_{s=0}^{p-1} I_{Ms}(\omega_k) ,$$

where $I_{Ms}(\omega_k)$ is the estimator for the ordinate at frequency ω_k obtained from the sth subsample.

Barlett's estimator is closely related to the estimator with lag window

$$w(x) = \begin{cases} 1 - |x|, & |x| \le 1, \\ 0 & \text{otherwise}. \end{cases}$$

This window has been called *modified Bartlett* or *triangular*. Setting $g_n = M$, the spectral window is given by

$$\begin{aligned} W_B(\omega) &= \frac{1}{2\pi} \sum_{h=-M}^{M} \frac{M - |h|}{M} \cos \omega h \\ &= \frac{1}{2\pi M} \sum_{h=0}^{M-1} \sum_{j=-h}^{h} \cos j\omega . \end{aligned}$$

Using Lemma 3.1.2,

$$W_B(\omega) = \frac{\sin^2 (M/2)\omega}{2\pi M \sin^2(\omega/2)} .$$

To evaluate the variance of the modified Bartlett estimator, we have

$$\sum_{h=-M}^{M} w^2\left(\frac{h}{M}\right) = \sum_{h=-M}^{M} \left(\frac{M-|h|}{M}\right)^2 \doteq \tfrac{2}{3} M .$$

Thus, for the modified Bartlett estimator with covariances truncated at M, the variance of the estimated spectrum is approximately

$$\operatorname{Var}\{\tilde{f}(\omega)\} \doteq \frac{2M}{3n} f^2(\omega) .$$

Blackman and Tukey (1959) suggested the weight function

$$w(x) = \begin{cases} 1 - 2a + 2a \cos \pi x , & |x| \le 1 , \\ 0 & \text{otherwise} . \end{cases}$$

The use of the window with $a = 0.23$ they called "hamming", and the use of the window with $a = 0.25$, "hanning." Parzen (1961) suggested the weight function

$$w(x) = \begin{cases} 1 - 6x^2 + 6|x|^3 , & |x| \le \frac{1}{2} , \\ 2(1 - |x|)^3 , & \frac{1}{2} \le |x| \le 1 , \\ 0 & \text{otherwise} . \end{cases}$$

This kernel will always produce nonnegative estimators. Brillinger (1975, Chapter 3) contains a discussion of these and other kernels.

Another method of estimating the spectral density is to estimate an autoregressive moving average model for the time series and then use the spectral density defined by that model as the estimator of the spectral density. Because of its simplicity, the pure autoregressive model is often used. The smoothness of the estimated spectral density is a function of the order of the model fit. A number of criteria for determining the order have been suggested. See Akaike (1969a), Parzen (1974, 1977), Marple (1987), Newton (1988), and the references cited in Section 8.4.

The Burg method of fitting the autoregressive process, discussed in Section 8.2.2, is often used in estimating it. The method has the advantage that the roots of the autoregressive process are always less than one and the computations are such that it is relatively easy to consider autoregressive models of different orders.

7.4. MULTIVARIATE SPECTRAL ESTIMATES

We now investigate estimators of the spectral parameters for vector time series. Let n observations be available on the bivariate time series $\mathbf{Y}_t = (Y_{1t}, Y_{2t})'$. The Fourier coefficients a_{1k}, b_{1k}, a_{2k}, and b_{2k} can be computed for the two time series by the formulas following (7.1.1). Having studied estimators of $f_{11}(\omega)$ and $f_{22}(\omega)$ in the preceding sections, it remains only to investigate estimators of $f_{12}(\omega)$ and of the associated quantities such as the phase spectrum and the squared coherency.

Recalling the transformation introduced in Section 4.2, which yields the normalized Fourier coefficients, we shall investigate the joint distributional properties of

$$
\begin{aligned}
2^{-1}n^{1/2}(a_{1k} + ib_{1k}) &= n^{-1/2}\sum_{t=1}^{n} Y_{1t}e^{i\omega_k t}, \\
2^{-1}n^{1/2}(a_{2k} + ib_{2k}) &= n^{-1/2}\sum_{t=1}^{n} Y_{2t}e^{i\omega_k t},
\end{aligned}
\tag{7.4.1}
$$

where $\omega_k = 2\pi k/n$, $k = 0, 1, \ldots, n-1$. Define a transformation matrix $\mathbf{H}$ by

$$
\mathbf{H} = \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix}, \tag{7.4.2}
$$

where $\mathbf{G}$ is an $n \times n$ matrix with rows given by

$$
g_{k.} = n^{-1/2}[1, e^{i2\pi k/n}, e^{i2\pi 2k/n}, \ldots, e^{i2\pi(n-1)k/n}], \qquad k = 0, 1, \ldots, n-1 .
$$

The matrix $\mathbf{G}$ was introduced in (4.2.6), and is the matrix that will diagonalize a circular matrix. Let

$$
\mathbf{V} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix} \tag{7.4.3}
$$

be the $2n \times 2n$ covariance matrix of $\mathbf{y} = (\mathbf{y}_1', \mathbf{y}_2')'$, where

$$
\begin{aligned}
\mathbf{y}_1' &= (Y_{11}, Y_{12}, \ldots, Y_{1n}), \\
\mathbf{y}_2' &= (Y_{21}, Y_{22}, \ldots, Y_{2n}),
\end{aligned}
$$

and

$$
\mathbf{V}_{12} = \begin{pmatrix}
\gamma_{12}(0) & \gamma_{12}(1) & \gamma_{12}(2) & \cdots & \gamma_{12}(n-1) \\
\gamma_{12}(-1) & \gamma_{12}(0) & \gamma_{12}(1) & \cdots & \gamma_{12}(n-2) \\
\gamma_{12}(-2) & \gamma_{12}(-1) & \gamma_{12}(0) & \cdots & \gamma_{12}(n-3) \\
\vdots & \vdots & \vdots & & \vdots \\
\gamma_{12}(-n+1) & \gamma_{12}(-n+2) & \gamma_{12}(-n+3) & \cdots & \gamma_{12}(0)
\end{pmatrix} . \tag{7.4.4}
$$

In Theorem 4.2.1, for time series with absolutely summable covariance function, we demonstrated that the elements of $\mathbf{G}\mathbf{V}_{ii}\mathbf{G}^*$ converge to the elements of the diagonal matrix $2\pi\mathbf{D}_{ii}$, where the elements of $\mathbf{D}_{ii}$ are $f_{ii}(\omega)$ evaluated at $\omega_k = 2\pi k/n$, $k = 0, 1, 2, \ldots, n-1$.

It remains to investigate the behavior of $\mathbf{GV}_{12}\mathbf{G}^*$. To this end define the circular matrix

$$\mathbf{V}_{12c} = \begin{pmatrix} \gamma_{12}(0) & \gamma_{12}(1) & \gamma_{12}(2) & \cdots & \gamma_{12}(-1) \\ \gamma_{12}(-1) & \gamma_{12}(0) & \gamma_{12}(1) & \cdots & \gamma_{12}(-2) \\ \gamma_{12}(-2) & \gamma_{12}(-1) & \gamma_{12}(0) & \cdots & \gamma_{12}(-3) \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_{12}(1) & \gamma_{12}(2) & \gamma_{12}(3) & \cdots & \gamma_{12}(0) \end{pmatrix}. \qquad (7.4.5)$$

Then $\mathbf{GV}_{12c}\mathbf{G}^*$ is a diagonal matrix with elements

$$\sum_{h=0}^{M} \gamma_{12}(h)e^{-\iota 2\pi kh/n} + \sum_{h=-M}^{-1} \gamma_{12}(h)e^{-\iota 2\pi k(n+h)/n}$$
$$= \sum_{h=-M}^{M} \gamma_{12}(h)e^{-\iota 2\pi kh/n}, \qquad k = 0, 1, \ldots, n-1, \quad (7.4.6)$$

where we have assumed n is odd and set $M = (n-1)/2$. If n is even, the sum is from $-M+1$ to M, where $M = n/2$. If $\gamma_{12}(h)$ is absolutely summable, we obtain the following theorem.

Theorem 7.4.1. Let $\mathbf{Y}_t$ be a stationary bivariate time series with absolutely summable autocovariance function. Let $\mathbf{V}$ of (7.4.3) be the covariance matrix for n observations. Then, given $\epsilon > 0$, there exists an N such that for $n > N$, every element of the matrix

$$\mathbf{HVH}^* - 2\pi\mathbf{D}$$

is less than ϵ in magnitude, where

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

$$D_{ij} = \operatorname{diag}\{f_{ij}(\omega_0), f_{ij}(\omega_1), \ldots, f_{ij}(\omega_{n-1})\}, \qquad i, j = 1, 2,$$

and $\omega_k = 2\pi k/n$, $k = 0, 1, \ldots, n-1$.

Proof. The result for $\mathbf{D}_{11}$ and $\mathbf{D}_{22}$ follows by Theorem 4.2.1. The result for $\mathbf{D}_{12}$ is obtained by arguments completely analogous to those of Section 4.2, by showing that the elements of $\mathbf{GV}_{12}\mathbf{G}^* - \mathbf{GV}_{12c}\mathbf{G}^*$ converge to zero as n increases. The details are reserved for the reader. ▲

If we make a stronger assumption about the autocovariance function we obtain the stronger result parallel to Corollary 4.2.1.

Corollary 7.4.1.1. Let $\mathbf{Y}_t$ be a stationary bivariate time series with an autocovariance function that satisfies

$$\sum_{h=-\infty}^{\infty} |h|\,|\gamma_{ij}(h)| < L < \infty\,, \qquad i, j = 1, 2\,.$$

Let $\mathbf{V}$ be as defined in (7.4.3), $\mathbf{H}$ as defined in (7.4.2), and $\mathbf{D}$ as defined in Theorem 7.4.1. Then every element of the matrix $\mathbf{HVH}^* - 2\pi\mathbf{D}$ is less than $3L/n$.

Proof. The proof parallels that of Corollary 4.2.1 and is reserved for the reader. ▲

By Theorem 7.4.1, the complex coefficients $2^{-1}n^{1/2}(a_{1k} + \imath b_{1k})$ and $2^{-1}n^{1/2}(a_{2j} + \imath b_{2j})$ are nearly uncorrelated in large samples if $j \neq k$. Since

$$a_{ik} - \imath b_{ik} = a_{i,n-k} + \imath b_{i,n-k}\,, \qquad i = 1, 2, \quad k = 1, 2, \ldots, n-1\,,$$

it follows that

$$\begin{aligned} &\lim_{n\to\infty} E\{a_{1k}a_{2k} - b_{1k}b_{2k}\} = 0\,, \\ &\lim_{n\to\infty} E\{b_{1k}a_{2k} + b_{2k}a_{1k}\} = 0\,. \end{aligned} \tag{7.4.7}$$

That is, the covariance between a_{1k} and a_{2k} is approximately equal to that between b_{1k} and b_{2k}, while the covariance between b_{1k} and a_{2k} is approximately the negative of the covariance between b_{2k} and a_{1k}.

We define the cross periodogram by

$$I_{12n}(\omega_k) = \begin{cases} \dfrac{n}{2}\,[a_{1k}a_{2k} + b_{1k}b_{2k} - \imath(a_{1k}b_{2k} - b_{1k}a_{2k})]\,, & \omega_k \neq 0,\ \pi\,, \\ 2na_{1k}a_{2k}\,, & \omega_k = 0,\ \pi\,. \end{cases} \tag{7.4.8}$$

To obtain a function defined at all ω, we recall the function

$$K(n, \omega) = k \qquad \text{for} \quad \frac{\pi(2k-1)}{n} < \omega \leq \frac{\pi(2k+1)}{n} \tag{7.4.9}$$

and take $I_{12n}(\omega) = I_{12n}(\omega_{K(n,\omega)})$.

Corollary 7.4.1.2. Let $\mathbf{Y}_t$ be a stationary bivariate time series with absolutely summable covariance function. Then

$$\lim_{n\to\infty} E\{I_{12n}(\omega)\} = 4\pi f_{12}(\omega)\,.$$

Proof. The result is an immediate consequence of Theorem 7.4.1. ▲

We now obtain some distributional results for spectral estimates. To simplify the presentation, we assume that $\mathbf{Y}_t$ is a normal time series.

Theorem 7.4.2. Let $\mathbf{Y}_t$ be a bivariate normal time series with covariance function that satisfies

$$\sum_{h=-\infty}^{\infty} |h|\,|\gamma_{ij}(h)| < L < \infty\,, \qquad i, j = 1, 2\,.$$

Then $\mathbf{r}_k = 2^{-1/2}n^{1/2}(a_{1k}, b_{1k}, a_{2k}, b_{2k})'$ is distributed as a multivariate normal random variable with zero mean and covariance matrix

$$E\{\mathbf{r}_k\mathbf{r}_k'\} = 2\pi\begin{pmatrix} f_{11}(\omega_k) & 0 & c_{12}(\omega_k) & q_{12}(\omega_k) \\ 0 & f_{11}(\omega_k) & -q_{12}(\omega_k) & c_{12}(\omega_k) \\ c_{12}(\omega_k) & -q_{12}(\omega_k) & f_{22}(\omega_k) & 0 \\ q_{12}(\omega_k) & c_{12}(\omega_k) & 0 & f_{22}(\omega_k) \end{pmatrix} + O(n^{-1})$$

for $\omega_k \neq 0,\ \pi$, where $f_{12}(\omega_k) = c_{12}(\omega_k) - \imath q_{12}(\omega_k)$. Also

$$E\{\mathbf{r}_k\mathbf{r}_j'\} = O(n^{-1})\,, \qquad j \neq k\,.$$

It follows that

$$E\left\{\frac{n}{2}(a_{1k}a_{2k} + b_{1k}b_{2k})\right\} = 4\pi c_{12}(\omega_k) + O(n^{-1})\,,$$

$$E\left\{\frac{n}{2}(a_{1k}b_{2k} - a_{2k}b_{1k})\right\} = 4\pi q_{12}(\omega_k) + O(n^{-1})\,,$$

$$E\{I_{12n}(\omega_k)\} = 4\pi f_{12}(\omega_k) + O(n^{-1})\,,$$

and for $\omega_k \neq 0,\ \pi$,

$$Cov\{I_{12n}(\omega_k), I_{12n}(\omega_j)\} = \begin{cases} (4\pi)^2 f_{11}(\omega_k) f_{22}(\omega_k) + O(n^{-1})\,, & j = k\,, \\ O(n^{-2})\,, & j \neq k\,. \end{cases}$$

Proof. Since the a_{ik} and b_{ik} are linear combinations of normal random variables, the normality is immediate. The moment properties follow from the moments of the normal distribution and from Corollary 7.4.1.1. ▲

We may construct smoothed estimators of the cross spectral density in the same manner that we constructed smoothed estimators in Section 7.2. Let

$$\bar{f}_{12}(\omega_k) = \sum_{j=-d}^{d} W_n(j)\hat{f}_{12}(\omega_{k+j})$$
$$= \frac{1}{4\pi}\bar{I}_{12}(\omega_k)\,, \qquad (7.4.10)$$

where

$$\hat{f}_{12}(\omega_k) = \frac{1}{4\pi} I_{12n}(\omega_k),$$

$$\bar{I}_{12}(\omega_k) = \sum_{j=-d}^{d} W_n(j) I_{12n}(\omega_{k+j}),$$

and $W_n(j)$, $j = 0, \pm 1, \pm 2, \ldots, \pm d$, is a weight function.

Theorem 7.4.3. Let $\mathbf{Y}_t$ be a bivariate normal time series with covariance function that satisfies $\sum_{h=-\infty}^{\infty} |h|\,|\gamma_{ij}(h)| < L < \infty$, $i, j = 1, 2$. Let d_n be a sequence of positive integers satisfying

$$\lim_{n\to\infty} d_n = \infty,$$

$$\lim_{n\to\infty} \frac{d_n}{n} = 0,$$

and let $W_n(j)$, $j = 0, \pm 1, \pm 2, \ldots, \pm d_n$, satisfy

$$\sum_{j=-d_n}^{d_n} W_n(j) = 1,$$

$$W_n(j) = W_n(-j),$$

$$\lim_{n\to\infty} \sum_{j=-d_n}^{d_n} W_n^2(j) = 0.$$

Then

$$\lim_{n\to\infty} E\{\bar{f}_{12}(\omega_{K(n,\omega)})\} = f_{12}(\omega),$$

$$\lim_{n\to\infty} \left(\sum_{j=-d_n}^{d_n} W_n^2(j) \right)^{-1} \operatorname{Var}\{\bar{f}_{12}(\omega_{K(n,\omega)})\} = \begin{cases} f_{11}(\omega) f_{22}(\omega), & \omega \neq 0, \pi, \\ 2 f_{11}(\omega) f_{22}(\omega), & \omega = 0, \pi, \end{cases}$$

where $\bar{f}_{12}(\omega_{K(n,\omega)})$ is defined by (7.4.10) and $K(n, \omega)$ is defined in (7.4.9).

Proof. Reserved for the reader. ▲

The properties of the other multivariate spectral estimators follow from Theorem 7.4.2 and Theorem 7.4.3. Table 7.4.1 will be used to illustrate the computations and emphasize the relationships to normal regression theory. Notice that the number of entries in a column of Table 7.4.1 is $2(2d + 1)$. We set

Table 7.4.1. Statistics Used in Computation of Cross Spectra at $\omega_k = 2\pi k/n$, $d = 2$

Fourier Coefficients for Y_2	Fourier Coefficients for Y_1	Signed Fourier Coefficients for Y_1
$a_{2,k-2}$	$a_{1,k-2}$	$-b_{1,k-2}$
$b_{2,k-2}$	$b_{1,k-2}$	$a_{1,k-2}$
$a_{2,k-1}$	$a_{1,k-1}$	$-b_{1,k-1}$
$b_{2,k-1}$	$b_{1,k-1}$	$a_{1,k-1}$
a_{2k}	a_{1k}	$-b_{1k}$
b_{2k}	b_{1k}	a_{1k}
$a_{2,k+1}$	$a_{1,k+1}$	$-b_{1,k+1}$
$b_{2,k+1}$	$b_{1,k+1}$	$a_{1,k+1}$
$a_{2,k+2}$	$a_{1,k+2}$	$-b_{1,k+2}$
$b_{2,k+2}$	$b_{1,k+2}$	$a_{1,k+2}$

$W_n(j) = (2d+1)^{-1}$ for $j = 0, \pm 1, \pm 2, \ldots, \pm d$. Then the cospectrum is estimated by

$$\bar{c}_{12}(\omega_k) = \frac{n}{8\pi(2d+1)} \sum_{j=-d}^{d} (a_{1,k-j}a_{2,k-j} + b_{1,k-j}b_{2,k-j}), \qquad (7.4.11)$$

which is the mean of the cross products of the first two columns of Table 7.4.1 multiplied by $n/4\pi$. The quadrature spectrum is estimated by

$$\bar{q}_{12}(\omega_k) = \frac{n}{8\pi(2d+1)} \sum_{j=-d}^{d} (a_{1,k-j}b_{2,k-j} - a_{2,k-j}b_{1,k-j}), \qquad (7.4.12)$$

which is the mean of the cross products of the first and third column of Table 7.4.1 multiplied by $n/4\pi$.

The estimator of the squared coherency for a bivariate time series computed from the smoothed periodogram estimator of $\mathbf{f}(\omega)$,

$$\bar{\mathbf{f}}(\omega_k) = \frac{1}{2d+1} \sum_{j=-d}^{d} \begin{pmatrix} \hat{f}_{11}(\omega_{k-j}) & \hat{f}_{12}(\omega_{k-j}) \\ \hat{f}_{21}(\omega_{k-j}) & \hat{f}_{22}(\omega_{k-j}) \end{pmatrix},$$

is given by

$$\overline{\mathcal{K}}^2_{12}(\omega_k) = \frac{|\bar{f}_{12}(\omega_k)|^2}{\bar{f}_{11}(\omega_k)\bar{f}_{22}(\omega_k)} = \frac{[\bar{c}_{12}(\omega_k)]^2 + [\bar{q}_{12}(\omega_k)]^2}{\bar{f}_{11}(\omega_k)\bar{f}_{22}(\omega_k)}. \qquad (7.4.13)$$

This quantity is recognizable as the multiple correlation coefficient of normal regression theory obtained by regressing the first column of Table 7.4.1 on the second and third columns. By construction, the second and third columns are orthogonal.

The estimation of the squared coherency generalizes immediately to higher dimensions. If there is a second explanatory variable, the Fourier coefficients of this variable are added to Table 7.4.1 in the same form as the columns for Y_1. Then the multiple squared coherency is the multiple correlation coefficient associated with the regression of the column for Y_2 on the four columns for the two explanatory variables.

An estimator of the error spectrum or residual spectrum of Y_2 after Y_1 is

$$\bar{f}_{ZZ}(\omega_k) = \bar{f}_{22}(\omega_k)[1 - \overline{\mathcal{K}}^2_{12}(\omega_k)] \frac{2d+1}{2d} . \tag{7.4.14}$$

This is the residual mean square for the regression of the first column of Table 7.4.1 on the second and third multiplied by $n/4\pi$. Many authors define the estimator of the error spectrum without the factor $(2d+1)/2d$. We include it to make the analogy to multiple regression complete. It also serves to remind us that $\overline{\mathcal{K}}^2_{12}(\omega_k)$ is identically one if computed for $d = 0$. A test of the hypothesis that $\mathcal{K}^2_{12}(\omega_k) = 0$ is given by the statistic

$$F^2_{4d} = \frac{4d\overline{\mathcal{K}}^2_{12}(\omega_k)}{2[1 - \overline{\mathcal{K}}^2_{12}(\omega_k)]} , \tag{7.4.15}$$

which is approximately distributed as Snedecor's F with 2 and $4d$ $(d > 0)$ degrees of freedom under the null hypothesis. This is the test of the hypothesis that the regression coefficients associated with columns two and three of Table 7.4.1 are zero.

If $\mathcal{K}^2_{12}(\omega_k) \neq 0$, the distribution of $\overline{\mathcal{K}}^2_{12}(\omega_k)$ is approximately that of the multiple correlation coefficient [see, for example, Anderson (1984, p. 134). Tables and graphs useful in constructing confidence intervals for $\mathcal{K}^2_{12}(\omega_k)$ have been given by Amos and Koopmans (1963). For many degrees of freedom and $\mathcal{K}^2_{12}(\omega_k) \neq 0$, $\overline{\mathcal{K}}^2_{12}(\omega_k)$ is approximately normally distributed with variance

$$\operatorname{Var}\{\overline{\mathcal{K}}^2_{12}(\omega_k)\} \doteq \frac{4\mathcal{K}^2_{12}(\omega_k)[1 - \mathcal{K}^2_{12}(\omega_k)]^2}{4d+2} . \tag{7.4.16}$$

The estimated phase spectrum is

$$\bar{\varphi}_{12}(\omega_k) = \tan^{-1}[-\bar{q}_{12}(\omega_k)/\bar{c}_{12}(\omega_k)] , \tag{7.4.17}$$

where it is understood that $\bar{\varphi}_{12}(\omega_k]$ is the angle in $(-\pi, \pi]$ between the positive half of the $c_{12}(\omega_k)$ axis and the ray from the origin through $(\bar{c}_{12}(\omega_k), -\bar{q}_{12}(\omega_k))$. The sample distribution of this quantity depends in a critical manner on the true coherency between the two time series. If $\mathcal{K}^2_{12}(\omega_k) = 0$, then, conditional on $\bar{f}_{11}(\omega_k)$, the variables $\bar{c}_{12}(\omega_k)/\bar{f}_{11}(\omega_k)$ and $\bar{q}_{12}(\omega_k)/\bar{f}_{11}(\omega_k)$ are approximately distributed as independent normal $(0, f_{ZZ}(\omega_k)[2(2d+1)\bar{f}_{11}(\omega_k)]^{-1})$ random variables. This is because $\bar{c}_{12}(\omega_k)/\bar{f}_{11}(\omega_k)$ and $\bar{q}_{12}(\omega_k)/\bar{f}_{11}(\omega_k)$ are the regression coefficients obtained by regressing column one of Table 7.4.1 on columns two and three. It is well known that the ratio of two independent normal random variables with zero mean and common variance has the Cauchy distribution, and that the arc tangent of the ratio has a uniform distribution. Therefore, if $\mathcal{K}_{12}^2(\omega_k) = 0$, the principal value of $\bar{\varphi}_{12}(\omega_k)$ will be approximately uniformly distributed on the interval $(-\pi/2, \pi/2)$.

If $\mathcal{K}_{12}^2(\omega_k) \neq 0$, then $\bar{\varphi}_{12}(\omega)$ will converge in distribution to a normal random variable. While approximate confidence limits could be established on the basis of the limiting distribution, it seems preferable to set confidence limits using the normality of $\bar{c}_{12}(\omega_k)/\bar{f}_{11}(\omega_k)$ and $\bar{q}_{12}(\omega_k)/\bar{f}_{11}(\omega_k)$.

Fieller's method [see Fieller (1954)] can be used to construct a confidence interval for $\varphi_{12}(\omega)$. This procedure follows from the fact that the statement $-q_{12}(\omega)/c_{12}(\omega) = R_{12}(\omega)$ is equivalent to the statement $c_{12}(\omega)R_{12}(\omega) + q_{12}(\omega) = 0$. Therefore, the method of setting confidence intervals for the sum $c_{12}(\omega)R_{12}(\omega) + q_{12}(\omega)$ can be used to determine a confidence interval for $R_{12}(\omega) = \tan \varphi_{12}(\omega)$ and hence for $\varphi_{12}(\omega)$. The $(1-\alpha)$-level confidence interval for the principal value of $\varphi_{12}(\omega)$ is the set of $\varphi_{12}(\omega)$ in $[-\pi/2, \pi/2]$ such that

$$\sin^2[\varphi_{12}(\omega) - \bar{\varphi}_{12}(\omega)] \leq t_\alpha^2[\bar{c}_{12}^2(\omega) + \bar{q}_{12}^2(\omega)]^{-1}\hat{\mathrm{V}}\mathrm{ar}\{\bar{c}_{12}(\omega)\}, \qquad (7.4.18)$$

where

$$\hat{\mathrm{V}}\mathrm{ar}\{\bar{c}_{12}(\omega)\} = (4d+2)^{-1}\bar{f}_{11}(\omega)\bar{f}_{ZZ}(\omega),$$

t_α is such that $P\{|t| > t_\alpha\} = \alpha$, and t is distributed as Student's t with 4d degrees of freedom.

To obtain a confidence interval for $\varphi_{12}(\omega)$ in the interval $(-\pi, \pi]$ it is necessary to modify Fieller's method. We suggest the following procedure to establish an approximate $(1-\alpha)$-level confidence interval. Let $F_{4d}^2(\alpha)$ denote the $\alpha\%$ point of the F-distribution with 2 and $4d$ degrees of freedom. The possibilities for the interval fall into two categories.

1. $\bar{c}_{12}^2(\omega) + \bar{q}_{12}^2(\omega) \leq 2F_{4d}^2(\alpha)\hat{\mathrm{V}}\mathrm{ar}\{\bar{c}_{12}(\omega)\}$. The confidence interval for $\varphi_{12}(\omega)$ is $(-\pi, \pi]$.
2. $\bar{c}_{12}^2(\omega) + \bar{q}_{12}^2(\omega) > 2F_{4d}^2(\alpha)\,\hat{\mathrm{V}}\mathrm{ar}\{\bar{c}_{12}(\omega)\}$. The confidence interval for $\varphi_{12}(\omega)$ is $[\bar{\varphi}_{12}(\omega) - \delta, \bar{\varphi}_{12}(\omega) + \delta]$, where

$$\begin{aligned}\delta &= \sin^{-1}[t_\alpha^2\{\bar{c}_{12}^2(\omega) + \bar{q}_{12}^2(\omega)\}^{-1}\,\hat{\mathrm{V}}\mathrm{ar}\{\bar{c}_{12}(\omega)\}]^{1/2}\\ &= \sin^{-1}\left[t_\alpha^2\frac{1-\overline{\mathcal{K}}_{12}^2(\omega)}{4d\overline{\mathcal{K}}_{12}^2(\omega)}\right]^{1/2}.\end{aligned}$$

Note that the criterion for category 1 is satisfied when the F-statistic of (7.4.15) is less than $F_{4d}^2(\alpha)$. Assuming $\bar{c}_{12}(\omega)$ and $\bar{q}_{12}(\omega)$ to be normally distributed, it can be proven that the outlined procedure furnishes a confidence interval with probability at least $1-\alpha$ of covering the true $\varphi(\omega)$. If the true coherency is zero, the interval will have length 2π with probability $1-\alpha$.

Recall that the cross amplitude spectrum is

$$A_{12}(\omega) = [c_{12}^2(\omega) + q_{12}^2(\omega)]^{1/2} = |f_{12}(\omega)|$$

and the gain of X_{2t} over X_{1t} is

$$\psi_{12}(\omega) = \frac{A_{12}(\omega)}{f_{11}(\omega)}.$$

Estimator of these quantities are

$$\bar{A}_{12}(\omega) = [\bar{c}_{12}^2(\omega) + \bar{q}_{12}^2(\omega)]^{1/2}, \tag{7.4.19}$$

$$\bar{\psi}_{12}(\omega) = [\bar{f}_{11}(\omega)]^{-1}\bar{A}_{12}(\omega). \tag{7.4.20}$$

It is possible to establish approximate confidence intervals for these quantities using the approximate normality of $\bar{c}_{12}(\omega)/\bar{f}_{11}(\omega)$ and $\bar{q}_{12}(\omega)/\bar{f}_{11}(\omega)$. As a consequence of this normality,

$$\frac{(2d+1)\{[\bar{c}_{12}(\omega) - c_{12}(\omega)]^2 + [\bar{q}_{12}(\omega) - q_{12}(\omega)]^2\}}{\bar{f}_{11}(\omega)\bar{f}_{ZZ}(\omega)} \tag{7.4.21}$$

has, approximately, the F-distribution with 2 and $4d$ degrees of freedom. Therefore, those $c_{12}(\omega)$ and $q_{12}(\omega)$ for which (7.4.21) is less than the α percentage tabular value of the F-distribution form a $(1-\alpha)$-level confidence region. Let

$$\begin{aligned} A_U(\omega) &= \bar{A}_{12}(\omega) + [(2d+1)^{-1}\bar{f}_{11}(\omega)\bar{f}_{ZZ}(\omega)F_{4d}^2(\alpha)]^{1/2}, \\ A_L(\omega) &= \max\{0, \bar{A}_{12}(\omega) - [(2d+1)^{-1}\bar{f}_{11}(\omega)\bar{f}_{ZZ}(\omega)F_{4d}^2(\alpha)]^{1/2}\}. \end{aligned} \tag{7.4.22}$$

Assuming normal $\bar{c}_{12}(\omega)$ and $\bar{q}_{12}(\omega)$, we conclude that $[A_L(\omega), A_U(\omega)]$ is a confidence interval for $A_{12}(\omega)$ of at least level $1-\alpha$. The confidence interval for gain is that of $A_{12}(\omega)$ divided by $\bar{f}_{11}(\omega)$.

Example 7.4.1. We use the data on the sediment in the Des Moines River discussed in Section 6.4 to illustrate some of the cross spectral computations. Table 7.4.2 contains the Fourier coefficients for the first 11 frequencies for the 205 observations. For $d=5$, these are the statistics used to estimate $f(\omega_6)$, where $\omega_6 = 0.0293$ cycles per day.

Using the rectangular weight function, we have

$$\bar{I}_{22,205}(0.0293) = \frac{1}{11}\sum_{j=-5}^{5} \tfrac{205}{2}(a_{2,6+j}^2 + b_{2,6+j}^2) = 4.0320,$$

$$\bar{I}_{11,205}(0.0293) = \frac{1}{11}\sum_{j=-5}^{5} \tfrac{205}{2}(a_{1,6+j}^2 + b_{1,6+j}^2) = 7.2999,$$

$$\bar{I}_{12,205}(0.0293) = \frac{1}{11}\sum_{j=-5}^{5} \tfrac{205}{2}(a_{1,6+j}a_{2,6+j} + b_{1,6+j}b_{2,6+j})$$

Table 7.4.2. Statistics Used in Computing Smoothed Estimates of Cross Spectrum for Sediment in the Des Moines River at Boone and Saylorville for Frequency of 0.0293 Cycles per Day, $d = 5$

			Coefficients		
Index (k)	Frequency (Cycles per Day)	Period (Days)	Saylorville (a_{2k}, b_{2k})	Boone (a_{1k}, b_{1k})	Boone $(-b_{1k}, a_{1k})$
1	0.0049	205.00	−0.194	−0.103	−0.504
			0.305	0.504	−0.103
2	0.0098	102.50	0.175	0.432	−0.196
			0.161	0.196	0.432
3	0.0146	68.33	0.011	0.047	0.039
			0.013	−0.039	0.047
4	0.0195	51.25	0.058	0.055	−0.088
			0.147	0.088	0.055
5	0.0244	41.00	0.067	0.202	−0.116
			0.042	0.116	0.202
6	0.0293	34.17	0.021	0.133	−0.072
			0.061	0.072	0.133
7	0.0341	29.29	0.322	0.234	0.012
			0.059	−0.012	0.234
8	0.0390	25.63	0.065	−0.017	0.019
			−0.067	−0.019	−0.017
9	0.0439	22.78	−0.053	−0.013	−0.098
			0.019	0.098	−0.013
10	0.0488	20.50	0.281	0.332	0.053
			0.037	−0.053	0.332
11	0.0537	18.64	0.081	0.152	−0.026
			−0.062	0.026	0.152

$$- \mathscr{i}\frac{205}{22} \sum_{j=-5}^{5} (b_{2,6+j}a_{1,6+j} - a_{2,6+j}b_{1,6+j})$$
$$= 4.6768 - 1.4313\mathscr{i} .$$

It follows that

$$\bar{f}_{22}(0.0293) = 0.3209 ,$$
$$\bar{f}_{11}(0.0293) = 0.5809 ,$$
$$\bar{f}_{12}(0.0293) = 0.3722 - 0.1138\mathscr{i} .$$

The error spectrum for Saylorville is estimated by the residual sum of squares obtained from the regression of the Saylorville column on the Boone columns multiplied by $205[2(10)(4\pi)]^{-1}$. We have

$$\begin{aligned}\bar{f}_{ZZ}(0.0293) &= \frac{205}{80\pi}[0.4327 - 0.6407(0.5019) - (-0.1960)(-0.1536)] \\ &= 0.0661 .\end{aligned}$$

The squared coherency is

$$\bar{\mathcal{K}}_{12}^2(0.0293) = \frac{|\bar{f}_{12}(0.0293)|^2}{\bar{f}_{11}(0.0293)\bar{f}_{22}(0.0293)} = 0.8128 .$$

The F-statistic to test the hypothesis of zero coherency is

$$F_{20}^2 = \frac{20\bar{\mathcal{K}}_{12}^2(0.0293)}{2[1 - \bar{\mathcal{K}}_{12}^2(0.0293)]} = 43.42 .$$

This is well beyond the 1% tabular value of 5.85 for 2 and 20 degrees of freedom, and it is clear that the two time series are not independent. The estimate of the phase spectrum is

$$\bar{\varphi}_{12}(0.0293) = \tan^{-1}[-1.4313/4.6768] = -0.2970 \text{ radians} .$$

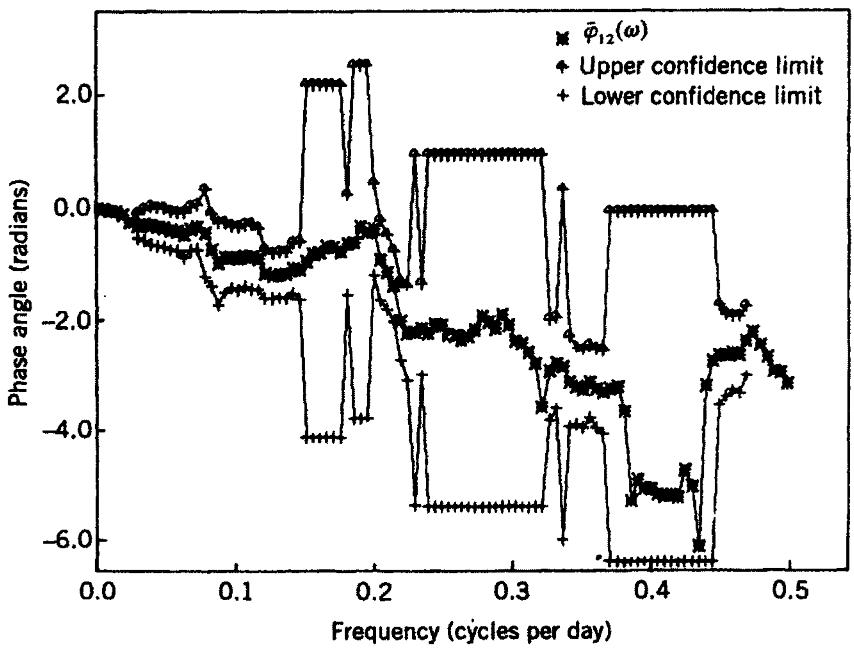

Figure 7.4.1. Estimated phase spectrum for Boone–Saylorville sediment data ($d = 5$ and rectangular weight function).

Let us establish a 95% confidence interval for $\varphi_{12}(0.0293)$. Because the F-test rejects the hypothesis of zero coherency at the 5% level, the criterion of category 2 is satisfied. Consequently the confidence interval for $\varphi_{12}(\omega_6)$ is $[-0.5228, -0.0712]$, where

$$\delta = \sin^{-1}\left[(2.086)^2 \frac{0.1872}{20(0.8128)}\right]^{1/2} = 0.2258 .$$

The estimated gain of X_{2t} over X_{1t} is

$$\bar{\psi}_{12}(\omega) = \frac{|\bar{f}_{12}(0.0293)|}{\bar{f}_{11}(0.0293)} = \frac{0.3892}{0.5809} = 0.6700 .$$

A 95% confidence interval for $\psi_{12}(\omega_6)$ is given by $[\psi_L, \psi_U]$, where

$$\psi_L = 0.6700 - [(11)^{-1}(0.5809)^{-1}(0.0661)(3.49)]^{1/2} = 0.4800 ,$$
$$\psi_U = 0.6700 + [(11)^{-1}(0.5809)^{-1}(0.0661)(3.49)]^{1/2} = 0.8600 .$$

Figure 7.4.1 is a plot of $\bar{\varphi}_{12}(\omega)$ and the confidence interval for $\varphi_{12}(\omega)$ ($d = 5$ and the rectangular weight function) for the Boone–Saylorville sediment data. Recall that $q_{12}(\omega)$ is an odd function of ω with $q_{12}(0) = q_{12}(\pi) = 0$. Therefore, we

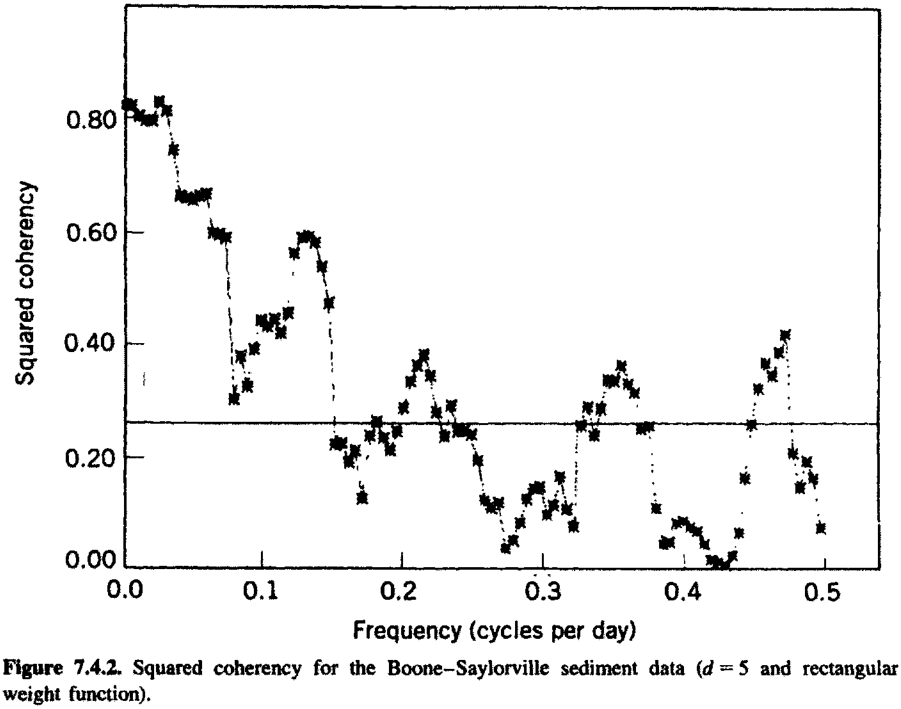

Figure 7.4.2. Squared coherency for the Boone–Saylorville sediment data ($d = 5$ and rectangular weight function).

define the imaginary part of $I_{12n}(\omega_{-k})$ to be the negative of the imaginary part of $I_{12n}(\omega_k)$. Likewise, the imaginary part of $I_{12n}(\pi + \lambda)$ is set equal to the negative of the imaginary part of $I_{12n}(\pi - \lambda)$. If $I_{12n}(0)$ or $I_{12n}(\pi)$ are computed, the imaginary part is zero. As a result, $\bar{q}_{12}(0) = 0$, and $\bar{q}_{12}(\pi) = 0$ if n is even. It follows that the estimated phase at zero is 0 if $\bar{c}_{12}(0) > 0$ and is π if $\bar{c}_{12}(0) < 0$. In plotting the estimated phase a continuous function of ω is desirable. Therefore, in creating Figure 7.4.1, that angle in the set

$$\{\bar{\varphi}_{12}(\omega_k) + j\pi: j = 0, \pm 2, \pm 4, \ldots\}$$

that differed least from the angle previously chosen for ω_{k-1}, $k = 2, 3, \ldots, 102$, was plotted.

The general downward slope of $\bar{\varphi}_{12}(\omega)$ is associated with the fact that the readings at Saylorville lag behind those at Boone. If the relationship were a perfect one period lag, $\bar{\varphi}_{12}(\omega)$ would be estimating a straight line with a negative slope of one radian per radian. The estimated function seems to differ enough from such a straight line to suggest a more complicated lag relationship.

Figure 7.4.2 contains a plot of squared coherency for the sediment data. The 5%

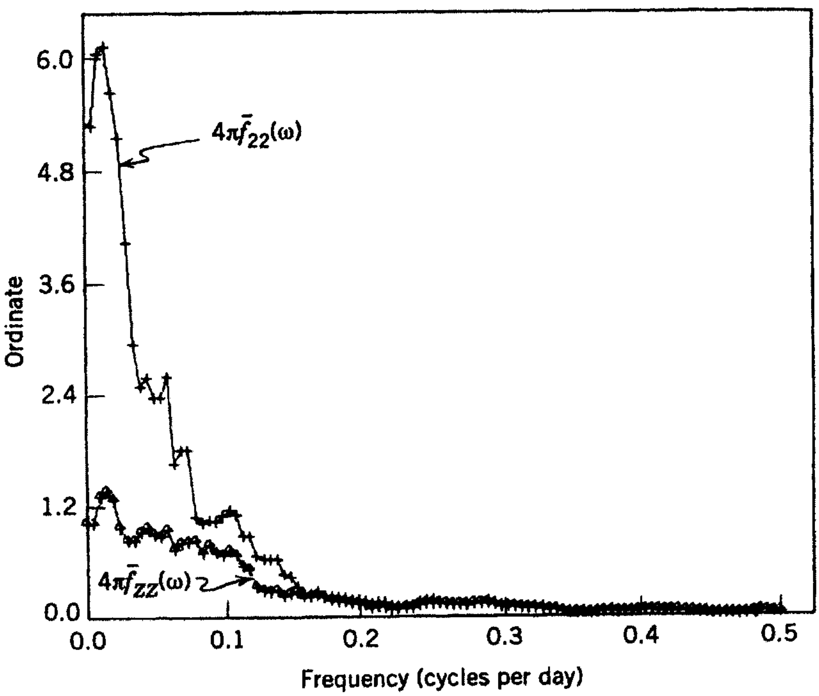

Figure 7.4.3. Plot of $4\pi\bar{f}_{22}(\omega)$ and $4\pi\bar{f}_{zz}(\omega)$ for Saylorville sediment ($d = 5$ and rectangular weight function).

point for the F-distribution with 2 and 20 degrees of freedom is 3.49. On the basis of (7.4.15), any $\bar{\mathscr{K}}_{12}^2(\omega)$ greater than 0.259 would be judged significant at that level. A line has been drawn at this height in the figure.

Similar information is contained in Figure 7.4.3, where the smoothed periodogram for Saylorville and $4\pi\bar{f}_{ZZ}(\omega)$ are plotted on the same graph. The estimated error spectrum lies well below the spectrum of the original time series for low frequencies, but the two nearly coincide for high frequencies. The estimated error spectrum is clearly not that of white noise, since it is considerably higher at low frequencies than at high. One might be led to consider a first or second order autoregressive process as a model for the error.

Figure 7.4.4 is a plot of the estimated gain of Saylorville over Boone. The 95% confidence interval plotted on the graph was computed using the limits (7.4.22) divided by $\bar{f}_{11}(\omega)$. Note that the lower confidence bound for gain is zero whenever the squared coherency falls below 0.259.

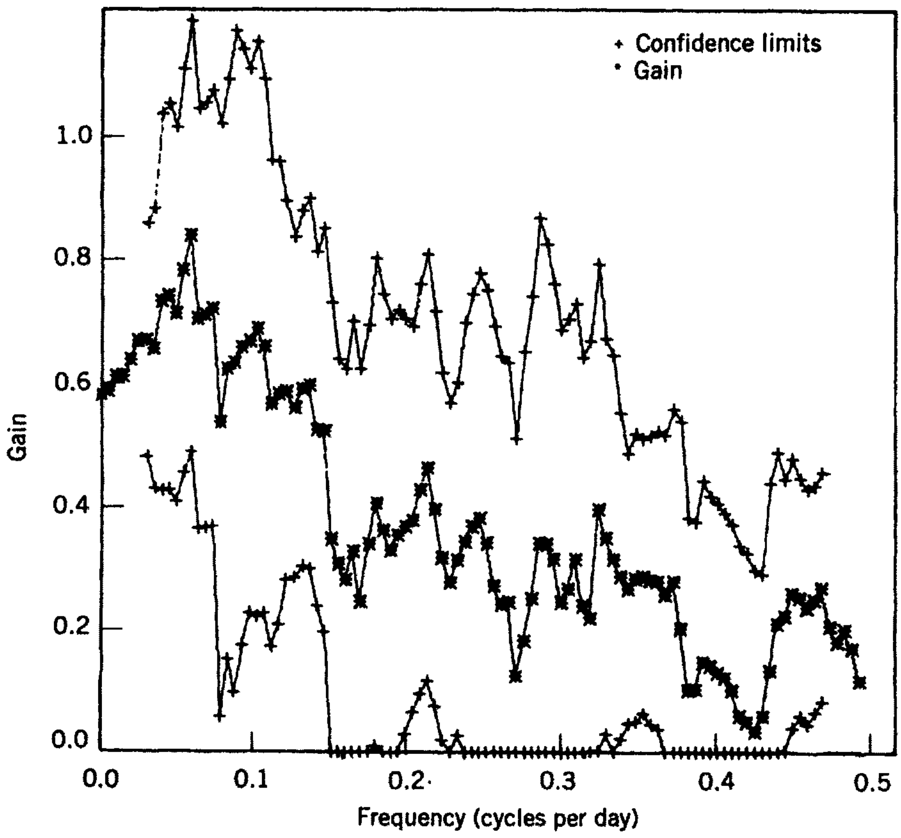

Figure 7.4.4. Gain of Saylorville over Boone ($d = 5$ and rectangular weight function).

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EXERCISES

1. Compute the periodogram for the data of Table 6.4.1. Calculate the smoothed periodogram using 2, 4 and 8 for d and rectangular weights. Plot the smoothed periodograms, and observe the differences in smoothness and in the height and width of the peak at zero. Compute a 95% confidence interval for $4\pi f(\omega)$ using the smoothed periodogram with $d = 4$. Plot the logarithm of the smoothed periodogram and the confidence interval for $\log 4\pi f(\omega)$.

2. Given in the accompanying table is the quarterly gasoline consumption in California from 1960 to 1973 in millions of gallons.

	Quarter			
Year	I	II	III	IV
1960	1335	1443	1529	1447
1961	1363	1501	1576	1495
1962	1464	1450	1611	1612
1963	1516	1660	1738	1652
1964	1639	1754	1839	1736
1965	1699	1812	1901	1821
1966	1763	1937	2001	1894
1967	1829	1966	2068	1983
1968	1939	2099	2201	2081
1969	2008	2232	2299	2204
1970	2152	2313	2393	2278
1971	2191	2402	2450	2387
1972	2391	2549	2602	2529
1973	2454	2647	2689	2549

Source. U.S. Dept. of Transportation (1975), *Review and Analysis of Gasoline Consumption in the United States from 1960 to the Present*, and U.S. Dept. of Transportation, *News*, various issues.

Using these data:

(a) Compute the periodogram.

(b) Obtain the smoothed periodogram by computing the centered moving average

$$\bar{I}_n(\omega_k) = \frac{1}{5} \sum_{j=-2}^{2} I_n(\omega_{k-j}) .$$

(c) Fit the regression model

$$Y_t = \alpha + \beta t + Z_t .$$

Repeat parts a and b for the regression residuals $\hat{Z}_t$.

(d) Compute the smoothed periodogram for $\hat{Z}_t$ of part c using the symmetric weight function $W_n(j)$ where

$$W_n(j) = \begin{cases} 0.3 , & j = 0 , \\ 0.2 , & j = 1 , \\ 0.1 , & j = 2 , \\ 0.05 , & j = 3 , \\ 0 , & |j| \geq 4 . \end{cases}$$

(e) Fit the regression model

$$Y_t = \sum_{j=1}^{4} \alpha_j \delta_{tj} + \beta t + u_t ,$$

where

$$\delta_{tj} = \begin{cases} 1 , & j\text{th quarter,} \\ 0 & \text{otherwise} . \end{cases}$$

Repeat parts a and b for the residuals from the fitted regression. Compute and plot a 95% confidence interval for the estimated spectral density.

3. Let the time series X_t be defined by

$$X_t = e_t + 0.6 e_{t-1} ,$$

where $\{e_t\}$ is a sequence of normal independent (0, 1) random variables. Given a sample of 10,000 observations from such a time series, what is the approximate joint distribution of the periodogram ordinates associated with $\omega_{2500} = 2\pi(2500)/10{,}000$ and $\omega_{1250} = 2\pi(1250)/10{,}000$?

4. Prove the sine result of Lemma 7.2.2.

5. Prove that the covariance function of a stationary finite autoregressive process

satisfies

$$\sum_{h=-n}^{n} |h|\,|\gamma(h)| = O(1).$$

6. Use the moment properties of the normal distribution to demonstrate the portion of Theorem 7.4.2 that states that

$$\operatorname{Cov}\{I_{12n}(\omega_k), I_{12n}(\omega_j)\} = \begin{cases} (4\pi)^2 f_{11}(\omega_k) f_{22}(\omega_k) + O(n^{-1}), & j = k, \\ O(n^{-2}), & j \neq k. \end{cases}$$

7. Let X_{1t} denote the United States quarterly unemployment rate of Table 6.4.1, and let X_{2t} denote the weekly gross hours per production worker given in Exercise 13 of Chapter 6. Compute the periodogram quantities $I_{11n}(\omega_k)$, $I_{12n}(\omega_k)$, and $I_{22n}(\omega_k)$. Compute the smoothed estimates using $d = 5$ and the rectangular weight function. Compute and plot $\bar{\mathscr{K}}^2_{12}(\omega_k)$, $\bar{\varphi}_{12}(\omega_k)$. Obtain and plot a confidence interval for $\varphi_{12}(\omega_k)$ and for the gain of X_{2t} over X_{1t}. Treating hours per week as the dependent variable, plot $4\pi \bar{f}_{ZZ}(\omega)$.

8. Show that

$$I_{XYn}(\omega_k) = 4\pi \hat{f}_{XY}(\omega_k), \qquad k = 1, 2, \ldots, m,$$

where

$$\hat{f}_{XY}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \hat{\gamma}_{XY}(h) e^{-\iota\omega h},$$

$$\hat{\gamma}_{XY}(h) = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{t=1}^{n-h} (X_t - \bar{x}_n)(Y_{t+h} - \bar{y}_n), & h = 0, 1, \ldots, n-1, \\ \dfrac{1}{n} \displaystyle\sum_{t=-h+1}^{n} (X_t - \bar{x}_n)(Y_{t+h} - \bar{y}_n), & h = -1, -2, \ldots, -(n-1), \\ 0 & \text{otherwise}. \end{cases}$$

9. Let X_t be a time series defined by

$$X_t = \sum_{j=0}^{\infty} \alpha_j e_{t-j},$$

where $\{e_t\}$ is a sequence of independent $(0, \sigma^2)$ random variables with fourth moment $\eta\sigma^4$ and

$$\sum_{j=0}^{\infty} j|\alpha_j| < \infty.$$

(a) Show that

$$\sum_{h=-\infty}^{\infty} |h|\,|\gamma(h)| < \infty$$

for such a time series.

(b) Let d_n, $W_n(j)$, and $\bar{f}(\omega)$ be as defined in Theorem 7.2.2. Show that

$$E\{\bar{f}(\omega)\} = f(\omega) + O(n^{-1}d_n)\,.$$

10. Given $X_1, X_2, \ldots, X_n$, let $\hat{\gamma}_X(h)$ be defined by (7.1.10). Consider the augmented observations

$$Y_t = \begin{cases} X_t - \bar{x}_n\,, & 1 \le t \le n\,, \\ 0\,, & n+1 \le t \le 2n-1\,. \end{cases}$$

(a) Show that the periodogram of Y_t can be written

$$I_{Y,2n-1}(\omega_j) = (2n-1)^{-1}2n \sum_{h=-n+1}^{n-1} \hat{\gamma}_X(h)e^{\epsilon\omega_j h}\,,$$

where $\omega_j = (2n-1)^{-1}2\pi j$.

(b) Show that

$$\hat{\gamma}_X(j) = (2n)^{-1}(2n-1)^{1/2}C_j\,,$$

where $\nu_j = (2n-1)^{-1}2\pi j$ and

$$C_j = (2n-1)^{-1/2} \sum_{r=0}^{2n-2} I_{Y,2n-1}(\omega_r)e^{-\epsilon\nu_j r}\,.$$

11. Suppose $X_1, X_2, \ldots, X_{p+1}$ are uncorrelated random variables. Show that

$$\operatorname{Var}\left\{(p+1)^{-1}\sum_{i=1}^{p+1} X_i\right\} \le \operatorname{Var}\left\{p^{-1}\sum_{i=1}^{p} X_i\right\}$$

unless

$$\operatorname{Var}\{X_{p+1}\} > (2p+1)\operatorname{Var}\left\{p^{-1}\sum_{i=1}^{p} X_i\right\}.$$