CHAPTER 3

Introduction to Fourier Analysis

3.1. SYSTEMS OF ORTHOGONAL FUNCTIONS—FOURIER COEFFICIENTS

In many areas of applied mathematics it is convenient to approximate a function by a linear combination of elementary functions. The reader is familiar with the Weierstrass theorem, which states that any continuous function on a compact set may be approximated by a polynomial. Likewise, it may be convenient to construct a set of vectors called a *basis* such that all other vectors may be expressed as linear combinations of the elements of the basis. Often the basis vectors are constructed to be orthogonal, that is, the sum of the products of the elements of any pair is zero. The system that is of particular interest to us in the analysis of time series is the system of trigonometric polynomials.

Assume that we have a function defined on a finite number of points, N. We shall investigate the properties of the set of functions $\{\cos(2\pi mt/N), \sin(2\pi mt/N)$: $t = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, L[N]\}$, where $L[N]$ is the largest integer less than or equal to $N/2$. The reader should observe that the points have been indexed by t running from zero to $N-1$, while m runs from zero to $L[N]$. Note that for $m = 0$ the cosine is identically equal to 1. The sine function is identically zero for $m = 0$ and for $m = N/2$ if N is even. Therefore, it is to be understood that these sine functions are not included in a discussion of the collection of functions. The collection of interest will always contain exactly N functions, none of which is identically zero. We shall demonstrate that these functions, defined on the integers $0, 1, \ldots, N-1$, are orthogonal, and we shall derive the sum of squares for each function. This constitutes a proof that the N functions defined on the integers furnish an orthogonal basis for the N-dimensional vector space.

Theorem 3.1.1. Given that m and r are contained in the set $\{0, 1, 2, \ldots, L[N]\}$, then

$$\sum_{t=0}^{N-1} \cos\frac{2\pi m}{N}t \cos\frac{2\pi r}{N}t = \begin{cases} N, & m = r = 0 \quad\text{or}\quad \dfrac{N}{2}, \\ \dfrac{N}{2}, & m = r \neq 0 \quad\text{or}\quad \dfrac{N}{2}, \\ 0, & m \neq r; \end{cases}$$

$$\sum_{t=0}^{N-1} \sin \frac{2\pi m}{N} t \cos \frac{2\pi r}{N} t = 0\,, \qquad \forall m, r\,;$$

$$\sum_{t=0}^{N-1} \sin \frac{2\pi m}{N} t \sin \frac{2\pi r}{N} t = \begin{cases} \frac{N}{2}\,, & m = r \neq 0 \quad \text{or} \quad \frac{N}{2}\,, \\ 0\,, & m \neq r\,. \end{cases}$$

Proof. Consider first the sum of the products of two cosine functions, and let

$$\begin{aligned} S(m, r) &= \sum_{t=0}^{N-1} \cos \frac{2\pi m}{N} t \cos \frac{2\pi r}{N} t \\ &= \frac{1}{2} \sum_{t=0}^{N-1} \left[\cos \frac{2\pi t}{N} (m + r) + \cos \frac{2\pi t}{N} (m - r) \right]. \end{aligned} \tag{3.1.1}$$

For $m = r = 0$ or $m = r = N/2$ (N even) the cosine terms on the right-hand side of (3.1.1) are always equal to one, and we have

$$S(m, r) = \frac{1}{2} \sum_{t=0}^{N-1} (1 + 1) = N\,.$$

For $m = r$, but not equal to zero and not equal to $N/2$ if N is even, the sum (3.1.1) reduces to

$$S(m, r) = \frac{1}{2} \sum_{t=0}^{N-1} \cos \frac{2\pi t}{N} 2m + \frac{1}{2} N\,,$$

where the summation of cosines is given by

$$\sum_{t=0}^{N-1} \cos \frac{2\pi t}{N} 2m = \frac{1}{2} \sum_{t=0}^{N-1} [e^{\acute{e}(2m)2\pi t/N} + e^{-\acute{e}(2m)2\pi t/N}]\,.$$

The two sums are geometric series whose rates are $\exp\{\acute{e}(2m)2\pi/N\}$ and $\exp\{-\acute{e}(2m)2\pi/N\}$, respectively, and the first term is one in both cases. Now $\acute{e}(2m)2\pi/N$ is not an integer multiple of 2π; therefore the rates are not unity. Applying the well-known formula $\Sigma_{t=0}^{N-1} \lambda^t = (1 - \lambda^N)/(1 - \lambda)$, the partial sum is

$$\frac{1 - (e^{\acute{e}(2m)2\pi/N})^N}{1 - e^{\acute{e}(2m)2\pi/N}} + \frac{1 - (e^{-\acute{e}(2m)2\pi/N})^N}{1 - e^{-\acute{e}(2m)2\pi/N}}\,.$$

Since $\exp\{\acute{e}(2m)2\pi\} = \exp\{\acute{e}2\pi\} = 1$, the numerators of the partial sums are 0, and $S(m, r)$ reduces to $N/2$. For $m \neq r$, we have

$$\begin{aligned} S(m, r) &= \frac{1}{2} \sum_{t=0}^{N-1} \left[\cos \frac{2\pi t}{N}(m+r) + \cos \frac{2\pi t}{N}(m-r)\right] \\ &= \frac{1}{4} \sum_{t=0}^{N-1} [e^{i(2\pi t/N)(m+r)} + e^{-i(2\pi t/N)(m+r)} \\ &\quad + e^{i(2\pi t/N)(m-r)} + e^{-i(2\pi t/N)(m-r)}] \\ &= 0 . \end{aligned}$$

The sum of products of a sine and cosine function is given by

$$\begin{aligned} &\sum_{t=0}^{N-1} \sin \frac{2\pi m}{N} t \cos \frac{2\pi r}{N} t \\ &\qquad = \frac{1}{2} \sum_{t=0}^{N-1} \left[\sin(m+r) \frac{2\pi t}{N} + \sin(m-r) \frac{2\pi t}{N}\right] = 0 \qquad \forall m, r, \end{aligned}$$

where the proof follows the same pattern as that for the product of cosines with $m = r \neq 0$. We leave the details to the reader. ▲

Having demonstrated that the N functions form an orthogonal basis, it follows that any function $f(t)$ defined on N integers can be represented by

$$f(t) = \sum_{m=0}^{L[N]} (a_m \cos \omega_m t + b_m \sin \omega_m t) , \qquad t = 0, 1, \ldots, N-1 , \tag{3.1.2}$$

where

$$\omega_m = \frac{2\pi m}{N} , \qquad m = 0, 1, 2, \ldots, L[N]:$$

$$a_m = \begin{cases} \dfrac{2 \sum_{t=0}^{N-1} f(t) \cos \omega_m t}{N} , & m = 1, 2, \ldots, L[N-1] , \\ \dfrac{\sum_{t=0}^{N-1} f(t) \cos \omega_m t}{N} , & m = 0 , \text{ and } m = \dfrac{N}{2} \text{ if } N \text{ is even:} \end{cases}$$

$$b_m = \frac{2 \sum_{t=0}^{N-1} f(t) \sin \omega_m t}{N} , \qquad m = 1, 2, \ldots, L[N-1] .$$

The a_m and b_m are called *Fourier coefficients*. One way to obtain the representation (3.1.2) is to find the a_m and b_m such that

$$\sum_{t=0}^{N-1} \left\{ f(t) - \sum_{m=0}^{L[N]} (a_m \cos \omega_m t + b_m \sin \omega_m t) \right\}^2 \tag{3.1.3}$$

is a minimum. Differentiating with respect to the a_j and b_j and setting the derivatives equal to zero, we obtain

$$\sum_{t=0}^{N-1} \left\{ f(t) - \sum_{m=0}^{L[N]} (a_m \cos \omega_m t + b_m \sin \omega_m t) \right\} \cos \omega_j t = 0 ,$$

$$j = 0, 1, 2, \ldots, L[N] ,$$

$$\sum_{t=0}^{N-1} \left\{ f(t) - \sum_{m=0}^{L[N]} (a_m \cos \omega_m t + b_m \sin \omega_m t) \right\} \sin \omega_j t = 0 ,$$

$$j = 1, 2, \ldots, L[N-1] .$$

By the results of Theorem 3.1.1, these equations reduce to

$$\sum_{t=0}^{N-1} f(t) \cos \omega_m t = a_m \sum_{t=0}^{N-1} \cos^2 \omega_m t , \qquad m = 0, 1, \ldots, L[N] ,$$

$$\sum_{t=0}^{N-1} f(t) \sin \omega_m t = b_m \sum_{t=0}^{N-1} \sin^2 \omega_m t , \qquad m = 1, \ldots, L[N-1] ,$$

and we have the coefficients of (3.1.2). Thus we see that the coefficients are the regression coefficients obtained by regressing the vector $f(t)$ on the vectors $\cos \omega_m t$ and $\sin \omega_m t$. By the orthogonality of the functions, the multiple regression coefficients are the simple regression coefficients.

Because of the different sum of squares for $\cos 0t$ and $\cos \pi t$, the a's are sometimes all defined with a common divisor, $N/2$, and the first and last terms of the series modified accordingly. Often N is restricted to be odd to simplify the discussion. Specifically, for N odd, we have

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{(N-1)/2} (a_m \cos \omega_m t + b_m \sin \omega_m t) , \qquad t = 0, 1, \ldots, N-1 , \tag{3.1.4}$$

where

$$a_m = \frac{2 \sum_{t=0}^{N-1} f(t) \cos \omega_m t}{N} , \qquad m = 0, 1, \ldots, \frac{N-1}{2} ,$$

$$b_m = \frac{2 \sum_{t=0}^{N-1} f(t) \sin \omega_m t}{N} , \qquad m = 1, 2, \ldots \frac{N-1}{2} .$$

The reader will have no difficulty in identifying the definitions being used. If the leading term is given as $a_0/2$, the definition (3.1.4) is being used; if not, (3.1.2) is being used.

The preceding material demonstrates that any finite vector can be represented as a linear combination of vectors defined by the sine and cosine functions. We now turn to the investigation of similar representations for functions defined on the real line.

We first consider functions defined on a finite interval of the real line. Since the interval is finite, we may code the end points in any convenient manner. When dealing with trigonometric functions, it will be most convenient to treat intervals whose length is a multiple of 2π. We shall most frequently take the interval to be $[-\pi, \pi]$.

Definition 3.1.1. An infinite system of square integrable functions $\{\varphi_j\}_{j=0}^{\infty}$ is orthogonal on $[a, b]$ if

$$\int_a^b \varphi_j(x)\varphi_m(x)\,dx = 0\,, \qquad j \neq m\,, \quad j, m = 0, 1, \ldots,$$

and

$$\int_a^b |\varphi_j(x)|^2\,dx \neq 0\,, \qquad j = 0, 1, \ldots.$$

The following theorem states that the system of trigonometric functions is orthogonal on the interval $[-\pi, \pi]$.

Theorem 3.1.2. Given that m and j are nonnegative integers, then

$$\int_{-\pi}^{\pi} \sin mx \cos jx\,dx = 0 \qquad \forall m,\ j;$$

$$\int_{-\pi}^{\pi} \sin mx \sin jx\,dx = \begin{cases} 0\,, & m \neq j\,, \\ \pi\,, & m = j \neq 0\,, \\ 0\,, & m = j = 0\,; \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos jx\,dx = \begin{cases} 2\pi\,, & m = j = 0\,, \\ \pi\,, & m = j \neq 0\,, \\ 0\,, & m \neq j\,. \end{cases}$$

Proof. Reserved for the reader. ▲

The sum

$$S_n(x) = \sum_{k=0}^{n} (a_k \cos \lambda_k x + b_k \sin \lambda_k x), \tag{3.1.5}$$

$$\lambda_k = \frac{2\pi k}{T}, \qquad k = 0, 1, \ldots, n,$$

is called a *trigonometric polynomial* of order (or degree) n and period T. We shall take $T = 2\pi$ (i.e., $\lambda_k = k$) in the sequel unless stated otherwise. If we let n increase without bound, we have the infinite trigonometric series

$$S_\infty(x) = \sum_{k=0}^{\infty} (a_k \cos \lambda_k x + b_k \sin \lambda_k x). \tag{3.1.6}$$

By Theorem 3.1.2 we know that the sine and cosine functions are orthogonal on the interval $[-\pi, \pi]$, or on any interval of length 2π. We shall investigate the ability of a trigonometric polynomial to approximate a function defined on this interval.

In the finite theory of least squares we are given a vector $\mathbf{y}$ of dimension n, which we desire to approximate by a linear combination of the vectors $\mathbf{x}_k$, $k = 1, 2, \ldots, p$. The coefficients of the predicting equation

$$\hat{Y}_j = \sum_{k=1}^{p} b_k x_{kj}$$

are obtained by minimizing

$$\sum_{j=1}^{n} \left(Y_j - \sum_{k=1}^{p} b_k x_{kj} \right)^2$$

with respect to the b_k.

The analogous procedure for a function $f(x)$ defined on the interval $[-\pi, \pi]$ is to minimize the integral

$$\int_{-\pi}^{\pi} \left[f(x) - \sum_{k=0}^{n} (a_k \cos \lambda_k x + b_k \sin \lambda_k x) \right]^2 dx. \tag{3.1.7}$$

Of course, $[f(x) - \sum_{k=0}^{n} (a_k \cos \lambda_k x + b_k \sin \lambda_k x)]^2$ must be integrable. In this section we shall say that a function $g(x)$ is integrable over $[a, b]$ if it is continuous, or if it has a finite number of discontinuities [at which $g(x)$ can be either bounded or unbounded], provided the improper Riemann integral exists. More general definitions could be used. Our treatment in the sequel follows closely that of Tolstov (1962), and we use his definitions.

The reader may verify by expansion of the square and differentiation of the resultant products that the coefficients that minimize (3.1.7) are

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos \lambda_k x\, f(x)\, dx\,, \qquad k = 0, 1, 2, \ldots, \tag{3.1.8}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin \lambda_k x\, f(x)\, dx\,, \qquad k = 1, 2, \ldots, \tag{3.1.9}$$

where the first term is written as $a_0/2$. In this form the approximating sum is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos \lambda_k x + b_k \sin \lambda_k x)\,. \tag{3.1.10}$$

When T, the length of the interval, is not equal to 2π, λ_k is set equal to $2\pi k/T$, and we obtain the formulas

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} \cos \lambda_k x\, f(x)\, dx\,, \qquad k = 0, 1, 2, \ldots,$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} \sin \lambda_k x\, f(x)\, dx\,, \qquad k = 1, 2, \ldots,$$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos \lambda_k x + b_k \sin \lambda_k x)\,. \tag{3.1.11}$$

Observe that $\pi[(a_0^2/2) + \sum_{k=1}^{n} (a_k^2 + b_k^2)]$ is completely analogous to the sum of squares due to regression on n orthogonal independent variables of finite least squares theory. We repeat below the theorem analogous to the statement in finite least squares that the multiple correlation coefficient is less than or equal to one.

Theorem 3.1.3 (Bessel's inequality). Let a_k, b_k, and $S_n(x)$ be defined by (3.1.8), (3.1.9), and (3.1.10), respectively. If $f(x)$ defined on $[-\pi, \pi]$ is square integrable, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx \geq \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\,. \tag{3.1.12}$$

Proof. Since the square is always nonnegative,

$$0 \leq \int_{-\pi}^{\pi} |f(x) - S_n(x)|^2\, dx = \int_{-\pi}^{\pi} [f^2(x) - 2f(x)S_n(x) + S_n^2(x)]\, dx$$

$$= \int_{-\pi}^{\pi} f^2(x)\, dx - 2\int_{-\pi}^{\pi} f(x)S_n(x)\, dx + \int_{-\pi}^{\pi} S_n^2(x)\, dx\,.$$

By the definitions of a_k and b_k and the orthogonality of the sine and cosine functions, we have

$$\int_{-\pi}^{\pi} |f(x) - S_n(x)|^2 \, dx = \int_{-\pi}^{\pi} f^2(x) \, dx - 2\left[\frac{a_0^2}{2}\pi + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2)\right]$$
$$+ \left[\frac{a_0^2}{2}\pi + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2)\right]$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x) \, dx \geqslant \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \, . \qquad \blacktriangle$$

Definition 3.1.2. A system of functions $\{\varphi_j(x)\}_{j=0}^{\infty}$ is *complete* if there does not exist a function $f(x)$ such that

$$\int_a^b |f(x)| \, dx \neq 0$$

and

$$\int_a^b f(x)\varphi_j(x) \, dx = 0 \, , \qquad j = 0, 1, \ldots \, .$$

We shall give only a few of the theorems of Fourier analysis. The following two theorems constitute a proof of completeness of the trigonometric functions. Lebesgue's proofs are given in Rogosinski (1959).

Theorem 3.1.4. If $f(x)$ is continuous on the interval $[-\pi, \pi]$, then all the Fourier coefficients are zero if and only if

$$f(x) \equiv 0 \, .$$

Proof. Omitted. ▲

Theorem 3.1.5. If $f(x)$ defined on the interval $[-\pi, \pi]$ is integrable and if all the Fourier coefficients $\{a_k, b_k \colon k = 0, 1, 2, \ldots\}$ are zero, then

$$\int_{-\pi}^{\pi} |f(x)| \, dx = 0 \, .$$

Proof. Omitted. ▲

By Theorem 3.1.4 the Fourier coefficients of a continuous function are unique. Theorem 3.1.5 generalizes the result to absolutely integrable functions. Neither furnishes information on the nature of the convergence of the sequences $\{a_k\}$ and $\{b_k\}$.

Theorem 3.1.6 (Parseval's theorem) answers the question of convergence for square integrable functions.

Theorem 3.1.6. If $f(x)$ defined on $[-\pi, \pi]$ is square integrable, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right|^2 dx = 0$$

or, equivalently,

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx .$$

Proof. By Bessel's inequality, the partial sum of squares of the Fourier coefficients converges for any square integrable function. (The sum of squares is monotone increasing and bounded.) Therefore, the function

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) ,$$

where a_k and b_k are the Fourier coefficients of a square integrable function $f(x)$, is square integrable on $[-\pi, \pi]$. Now $D(x) = f(x) - g(x)$ is square integrable and all Fourier coefficients of $D(x)$ are zero; hence, by Theorem 3.1.5,

$$\int_{-\pi}^{\pi} |f(x) - g(x)| \, dx = \int_{-\pi}^{\pi} |f(x) - g(x)|^2 \, dx = 0 . \qquad \blacktriangle$$

Thus, any square integrable function defined on an interval can be approximated in mean square by a trigonometric polynomial.

As one might suspect, the convergence of the sequence of functions $S_n(x)$ at a point x_0 requires additional conditions. We shall closely follow Tolstov's (1962) text in presenting a proof of the pointwise convergence of $S_n(x_0)$ to $f(x_0)$. The following definitions are needed.

Definition 3.1.3. The *left limit* of $f(x)$ is given by

$$\lim_{\substack{x\to x_0 \\ x<x_0}} f(x) = f(x_0^-)$$

provided the limit exists and is finite. The limit of $f(x)$ as $x \to x_0$, $x > x_0$, is called the *right limit* of $f(x)$ and is denoted by $f(x_0^+)$, provided it exists and is finite.

Definition 3.1.4. The *right derivative* of $f(x)$ at $x = x_0$ is defined by

$$f'(x_0^+) = \lim_{\substack{h\to 0 \\ h>0}} \frac{f(x_0 + h) - f(x_0^+)}{h} ,$$

and the *left derivative* by

$$f'(x_0^-) = \lim_{\substack{h \to 0 \\ h>0}} \frac{f(x_0 - h) - f(x_0^-)}{h},$$

provided the limits exist and are finite.

Definition 3.1.5. The function $f(x)$ has a *jump discontinuity* at the point x_0 if $f(x_0^+) \neq f(x_0^-)$.

From the definitions it is clear that the jump $f(x_0^+) - f(x_0^-)$ is finite.

Definition 3.1.6. A function $f(x)$ is *smooth* on the interval $[c, d]$ if it has a continuous derivative on the interval $[c, d]$.

Definition 3.1.7. A function $f(x)$ is *piecewise smooth* on the interval $[c, d]$ if

(i) $f(x)$ and $f'(x)$ are both continuous, or
(ii) $f(x)$ and $f'(x)$ have a finite number of jump discontinuities.

Bessel's inequality guarantees that the Fourier coefficients of any square integrable function go to zero as the frequency increases; that is,

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 0.$$

The following lemma is a proof of the same result for a different class of functions.

Lemma 3.1.1. If $f(x)$ is a piecewise smooth function on $[c, d]$, then

$$\lim_{m\to\infty} \int_c^d f(x)\cos mx\, dx = \lim_{m\to\infty} \int_c^d f(x)\sin mx\, dx = 0.$$

Proof. Integrating by parts gives

$$\int_c^d f(x)\cos mx\, dx = \frac{1}{m}\left\{[f(x)\sin mx]_c^d - \int_c^d f'(x)\sin mx\, dx\right\}.$$

Since both $f(x)$ and $f'(x)$ contain at most a finite number of jump discontinuities on the interval $[c, d]$, they are both bounded. Therefore, the quantity contained within the curly braces is bounded, and the result follows immediately. ▲

Using the fact that any absolutely integrable function can be approximated in the mean by a piecewise smooth function, it is possible to extend this result to any absolutely integrable function. We state this generalization without proof.

Lemma 3.1.1A. The Fourier coefficients a_m and b_m of an absolutely integrable function defined on a finite interval approach zero as $m \to \infty$.

Lemmas 3.1.2 and 3.1.3 are presented as preliminaries to the proof of Theorem 3.1.7.

Lemma 3.1.2. Define

$$A_n(u) = \frac{1}{2} + \sum_{j=1}^{n} \cos ju .$$

Then

$$A_n(u) = \frac{\sin (n + \frac{1}{2})u}{2 \sin(u/2)} \tag{3.1.13}$$

and

$$\frac{1}{\pi} \int_0^{\pi} A_n(u)\, du = \frac{1}{\pi} \int_{-\pi}^{0} A_n(u)\, du = \frac{1}{2} . \tag{3.1.14}$$

Proof. Multiplying both sides of the definition of $A_n(u)$ by $2\sin(u/2)$, we have

$$\begin{aligned} 2A_n(u)\sin(u/2) &= \sin(u/2) + 2 \sum_{j=1}^{n} \cos ju \sin(u/2) \\ &= \sin(u/2) + \sum_{j=1}^{n} [\sin (j + \tfrac{1}{2})u - \sin (j - \tfrac{1}{2})u] \\ &= \sin (n + \tfrac{1}{2})u , \end{aligned}$$

and the result (3.1.13) follows. Since the integral of $\cos ju$, $j = 1, 2, \ldots,$ is zero on an interval of length 2π,

$$\int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{j=1}^{n} \cos ju \right] du = \pi ,$$

and hence

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin (n + \frac{1}{2})u}{2 \sin(u/2)}\, du = 1 . \tag{3.1.15}$$

The result (3.1.14) follows, since $A_n(u)$ is an even function. ▲

Lemma 3.1.3. The partial sum

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

may be expressed in the form

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin (n+\frac{1}{2})u}{2\sin(u/2)} \, du \,, \tag{3.1.16}$$

where a_k and b_k are the Fourier coefficients of a periodic function $f(x)$ of period 2π defined in (3.1.8) and (3.1.9).

Proof. Substituting (3.1.8) and (3.1.9) into (3.1.10), we have

$$\begin{aligned} S_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt \\ &\quad + \frac{1}{\pi} \sum_{k=1}^{n} \left[\left\{ \int_{-\pi}^{\pi} f(t) \cos kt \, dt \right\} \cos kx + \left\{ \int_{-\pi}^{\pi} f(t) \sin kt \, dt \right\} \sin kx \right] \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \cos k(t-x) \right] dt \,. \end{aligned}$$

Using (3.1.13) of Lemma 3.1.2, we have

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin[(n+\frac{1}{2})(t-x)]}{2\sin[\frac{1}{2}(t-x)]} \, dt \,.$$

Setting $u = t - x$ and noting that the integral of a periodic function of period 2π over the interval $[-\pi - x, \pi - x]$ is the same as the integral over $[-\pi, \pi]$, we have

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin (n+\frac{1}{2})u}{2\sin(u/2)} \, du \,. \qquad \blacktriangle$$

Equation (3.1.16) is called the *integral formula* for the partial sum of a Fourier series.

Theorem 3.1.7. Let $f(x)$ be an absolutely integrable function of period 2π. Then:

(i) At a point of continuity where $f(x)$ has a right derivative and a left derivative,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \,.$$

(ii) At every point of discontinuity where $f(x)$ has a right and a left derivative,

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \frac{f(x^+) + f(x^-)}{2}.$$

Proof. Consider first a point of continuity. We wish to show that the difference

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin(n+\frac{1}{2})u}{2\sin(u/2)}\, du - f(x) \tag{3.1.17}$$

equals zero. Multiplying (3.1.15) of Lemma 3.1.2 by $f(x)$, we have

$$f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{\sin(n+\frac{1}{2})u}{2\sin(u/2)}\, du. \tag{3.1.18}$$

Therefore, the difference (3.1.17) can be written as

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+u) - f(x)] \frac{\sin(n+\frac{1}{2})u}{2\sin(u/2)}\, du$$
$$= \lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{f(x+u)-f(x)}{u}\right)\left(\frac{u}{2\sin(u/2)}\right) \sin(n+\tfrac{1}{2})u\, du.$$

Now, $u^{-1}[f(x+u) - f(x)]$ is absolutely integrable, since the existence of the left and right derivatives means that the ratio is bounded as u approaches zero. Also, $[2\sin(u/2)]^{-1}u$ is bounded. Therefore,

$$\frac{f(x+u)-f(x)}{u}\left(\frac{u}{2\sin(u/2)}\right) \overset{\text{(say)}}{=} g(u)$$

is absolutely integrable. Hence, by Lemma 3.1.1A,

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} g(u) \sin(n+\tfrac{1}{2})u\, du = 0,$$

giving us the desired result for a point of continuity.

For a point of jump discontinuity we must prove

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin(n+\frac{1}{2})u}{2\sin(u/2)}\, du = \frac{f(x^+) + f(x^-)}{2}.$$

Using (3.1.14) of Lemma 3.1.2, we have

$$\frac{f(x^+)}{2} = \frac{1}{\pi} \int_0^{\pi} f(x^+) \frac{\sin(n+\frac{1}{2})u}{2\sin(u/2)}\, du$$

and

$$\frac{f(x^-)}{2} = \frac{1}{\pi}\int_{-\pi}^{0} f(x^-)\frac{\sin(n+\frac{1}{2})u}{2\sin(u/2)}\,du\,.$$

The same arguments on boundedness and integrability that were used for a point of continuity can be used to show that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x+u)-f(x^+)}{u}\,\frac{u}{2\sin(u/2)}\sin(n+\tfrac{1}{2})u\,du \\ = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{0}\frac{f(x+u)-f(x^-)}{u}\,\frac{u}{2\sin(u/2)}\sin(n+\tfrac{1}{2})u\,du \\ = 0\,. \end{aligned}$$

▲

Theorem 3.1.8. Let $f(x)$ be a continuous periodic function of period 2π with derivative $f'(x)$ that is square integrable. Then the Fourier series of $f(x)$ converges to $f(x)$ absolutely and uniformly.

Proof. Under the assumptions we may integrate by parts to obtain

$$\begin{aligned} a_h &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos hx\,dx \\ &= \frac{1}{\pi h}[f(x)\sin hx]_{-\pi}^{\pi} - \frac{1}{\pi h}\int_{-\pi}^{\pi} f'(x)\sin hx\,dx \\ &= -\frac{1}{\pi h}\int_{-\pi}^{\pi} f'(x)\sin hx\,dx\,. \end{aligned}$$

Therefore, the Fourier coefficients of the function are directly related to the Fourier coefficients of the derivative by $a_h = -b'_h/h$, $b_h = a'_h/h$, where

$$\begin{aligned} a'_h &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos hx\,dx\,, \\ b'_h &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin hx\,dx\,. \end{aligned}$$

The Fourier coefficients of the derivative are well defined by the assumption that $f'(x)$ is square integrable. Furthermore, by Theorem 3.1.6 (Parseval's theorem), the series

$$\sum_{h=1}^{\infty}(|a'_h|^2 + |b'_h|^2)$$

converges. Since

$$\left(|a_h'| - \frac{1}{h}\right)^2 = |a_h'|^2 - \frac{2}{h}|a_h'| + \frac{1}{h^2} \geq 0$$

we have

$$\frac{1}{h}|a_h'| + \frac{1}{h}|b_h'| \leq \tfrac{1}{2}(|a_h'|^2 + |b_h'|^2) + \frac{1}{h^2}.$$

It follows that $\Sigma_{h=1}^{\infty}(|a_h| + |b_h|)$ converges. Now,

$$\begin{aligned}|a_h \cos hx + b_h \sin hx| &\leq |a_h \cos hx| + |b_h \sin hx| \\ &\leq |a_h| + |b_h|,\end{aligned}$$

and therefore, by the Weierstrass M-test, the trigonometric series

$$\frac{a_0}{2} + \sum_{h=1}^{\infty}(a_h \cos hx + b_h \sin hx)$$

converges to $f(x)$ absolutely and uniformly. ▲

As a simple consequence of the proof of Theorem 3.1.8, we have the following important result.

Corollary 3.1.8. If

$$\sum_{h=1}^{\infty}(|a_h| + |b_h|)$$

converges, then the associated trigonometric series

$$\frac{a_0}{2} + \sum_{h=1}^{\infty}(a_h \cos hx + b_h \sin hx)$$

converges absolutely and uniformly to a continuous periodic function of period 2π of which it is the Fourier series.

We are now in a position to prove a portion of Theorem 1.4.4 of Chapter 1. In the proof we require a result that will be used at later points, and therefore we state it as a lemma.

Lemma 3.1.4. (Kronecker's lemma) If the sequence $\{a_j\}$ is such that

$$\lim_{n\to\infty}\sum_{j=0}^{n}|a_j| = A < \infty,$$

then

$$\lim_{n\to\infty}\sum_{j=0}^{n}\frac{j}{n}|a_j| = 0 .$$

Proof. By assumption, given $\epsilon > 0$, there exists an N such that

$$\sum_{j=N+1}^{\infty}|a_j| < \epsilon .$$

Therefore, for $n > N$, we have

$$\sum_{j=0}^{n}\frac{j}{n}|a_j| < \frac{1}{n}\sum_{j=0}^{N} j|a_j| + \epsilon .$$

Clearly, for fixed N,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{N} j|a_j| = 0 ,$$

and since ϵ was arbitrary, the result follows. ▲

Theorem 3.1.9. Let the correlation function $\rho(h)$ of a stationary time series be absolutely summable. Then there exists a continuous function $f(\omega)$ such that:

(i) $\rho(h) = \int_{-\pi}^{\pi} f(\omega)\cos \omega h \, d\omega$.
(ii) $f(\omega) \geq 0$.
(iii) $\int_{-\pi}^{\pi} f(\omega)\, d\omega = 1$.
(iv) $f(\omega)$ is an even function.

Proof. By Corollary 3.1.8

$$g(\omega) = \frac{1}{2} + \sum_{h=1}^{\infty} \rho(h)\cos h\omega$$

is a well-defined continuous function. Now, by the positive semidefinite property of the correlation function,

$$\sum_{m=1}^{n}\sum_{q=1}^{n} \rho(m-q)\cos m\omega \cos q\omega \geq 0$$

and

$$\sum_{m=1}^{n}\sum_{q=1}^{n} \rho(m-q)\sin m\omega \sin q\omega \geq 0 .$$

Hence,

$$\sum_{m=1}^{n}\sum_{q=1}^{n}\rho(m-q)[\cos m\omega\cos q\omega+\sin m\omega\sin q\omega]$$

$$=\sum_{m=1}^{n}\sum_{q=1}^{n}\rho(m-q)\cos(m-q)\omega\geq 0\,.$$

Letting $m-q=h$, we have

$$\sum_{h=-(n-1)}^{n-1}\left(\frac{n-|h|}{n}\right)\rho(h)\cos h\omega\geq 0\,.$$

Now, $\rho(h)\cos h\omega$ is absolutely summable, and hence, by Lemma 3.1.4,

$$\lim_{n\to\infty}\sum_{h=-(n-1)}^{n-1}\frac{|h|}{n}\rho(h)\cos h\omega=0\,.$$

Therefore,

$$\lim_{n\to\infty}\sum_{h=-(n-1)}^{n-1}\frac{n-|h|}{n}\rho(h)\cos h\omega=\sum_{h=-\infty}^{\infty}\rho(h)\cos h\omega=2g(\omega)\geq 0\,.$$

Having shown that $g(\omega)$ satisfies conditions (i) and (ii), we need only multiply $g(\omega)$ by a constant to meet condition (iii). Since

$$\frac{1}{\pi}\int_{-\pi}^{\pi}g(\omega)\,d\omega=1\,,$$

the appropriate constant is π^{-1}, and we define $f(\omega)$ by

$$f(\omega)=\frac{1}{\pi}\left[\frac{1}{2}+\sum_{h=1}^{\infty}\rho(h)\cos h\omega\right]$$

$$=\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\rho(h)\cos h\omega\,.$$

The function $f(\omega)$ is an even function, since it is the uniform limit of a sum of even functions (cosines). ▲

In Theorem 3.1.8 the square integrability of the derivative was used to demonstrate the convergence of the Fourier series. In fact, the Fourier series of a continuous function not meeting such restrictions need not converge. However, Cesàro's method may be used to recover any continuous periodic function from its

Fourier series. Given the sequence $\{S_j\}_{j=1}^{\infty}$, the sequence $\{C_n\}$ defined by

$$C_n = \frac{1}{n}\sum_{j=1}^{n} S_j$$

is called the sequence of arithmetic means of $\{S_j\}$. If the sequence $\{C_n\}$ is convergent, we say the sequence $\{S_j\}$ is *Cesàro summable*. If the original sequence was convergent, then $\{C_n\}$ converges.

Lemma 3.1.5. If the sequence $\{S_j\}$ converges to s, then the sequence $\{C_n\}$ converges to s.

Proof. By hypothesis, given $\epsilon > 0$, we may choose an N such that $|S_j - s| < \frac{1}{2}\epsilon$ for all $j > N$. For $n > N$, we have

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=1}^{n} S_j - s\right| &\le \frac{1}{n}\sum_{j=1}^{N}|S_j - s| + \frac{1}{n}\sum_{j=N+1}^{n}|S_j - s| \\ &\le \frac{1}{n}\sum_{j=1}^{N}|S_j - s| + \tfrac{1}{2}\epsilon\,.\end{aligned}$$

Since we can choose an n large enough so that the first term is less than $\frac{1}{2}\epsilon$, the result follows. ▲

Theorem 3.1.10. Let $f(\omega)$ be a continuous function of period 2π. Then the Fourier series of $f(\omega)$ is uniformly summable to $f(\omega)$ by the method of Cesàro.

Proof. The Cesàro sum is

$$\begin{aligned}C_n(\omega) &= \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}(a_k\cos k\omega + b_k\sin k\omega) \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{\pi}\int_{-\pi}^{\pi} f(\omega+u)\frac{\sin(j+\frac{1}{2})u}{2\sin(u/2)}\,du \\ &= \frac{1}{n\pi}\int_{-\pi}^{\pi}\frac{f(\omega+u)}{2\sin^2(u/2)}\sum_{j=0}^{n-1}\sin(u/2)\sin(j+\tfrac{1}{2})u\,du\,,\end{aligned}$$

where we have used Lemma 3.1.3. Now,

$$\begin{aligned}\sum_{j=0}^{n-1}\sin(u/2)\sin(j+\tfrac{1}{2})u &= \tfrac{1}{2}\sum_{j=0}^{n-1}[\cos ju - \cos(j+1)u] \\ &= \tfrac{1}{2}(1-\cos nu) \\ &= \sin^2(nu/2)\,. \qquad (3.1.19)\end{aligned}$$

Therefore,

$$C_n(\omega) - f(\omega) = \frac{1}{n\pi}\int_{-\pi}^{0} [f(\omega+u) - f(\omega)]\frac{\sin^2(nu/2)}{2\sin^2(u/2)}\,du$$
$$+ \frac{1}{n\pi}\int_{0}^{\pi} [f(\omega+u) - f(\omega)]\frac{\sin^2(nu/2)}{2\sin^2(u/2)}\,du\,, \quad (3.1.20)$$

where we have used

$$\frac{1}{\pi n}\int_{-\pi}^{\pi} \frac{\sin^2(nu/2)}{2\sin^2(u/2)}\,du = 1\,.$$

Since $f(\omega)$ is uniformly continuous on $[-\pi, \pi]$, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(\omega+u) - f(\omega)| < \frac{\epsilon}{2}$$

for all $|u| < \delta$ and all $\omega \in [-\pi, \pi]$. We write the second integral of (3.1.20) as

$$\frac{1}{\pi n}\int_{0}^{\delta} [f(\omega+u) - f(\omega)]\frac{\sin^2(nu/2)}{2\sin^2(u/2)}\,du$$
$$+ \frac{1}{\pi n}\int_{\delta}^{\pi} [f(\omega+u) - f(\omega)]\frac{\sin^2(nu/2)}{2\sin^2(u/2)}\,du$$
$$\leqslant \frac{\epsilon}{2\pi n}\int_{0}^{\delta} \frac{\sin^2(nu/2)}{2\sin^2(u/2)}\,du$$
$$+ \frac{1}{2\pi n\sin^2(\delta/2)}\int_{0}^{\pi} |f(\omega+u) - f(\omega)|\,du$$
$$\leqslant \frac{\epsilon}{4} + \frac{M}{n\sin^2(\delta/2)}\,.$$

where M is the maximum of $|f(\omega)|$ on $[-\pi, \pi]$. A similar argument holds for the first integral of (3.1.20). Therefore, there exists an N such that (3.1.20) is less than ϵ for all $n > N$ and all $\omega \in [-\pi, \pi]$. ▲

3.2. COMPLEX REPRESENTATION OF TRIGONOMETRIC SERIES

We can represent the trigonometric series in a complex form that is somewhat more compact and that will prove useful in certain applications. Since

$$\cos\theta = \frac{\cos\theta + \imath\sin\theta + \cos\theta - \imath\sin\theta}{2} = \frac{e^{\imath\theta} + e^{-\imath\theta}}{2}$$

and

$$\sin\theta = \frac{\cos\theta + \imath\sin\theta - (\cos\theta - \imath\sin\theta)}{2\imath} = \frac{(-\imath)(e^{\imath\theta} - e^{-\imath\theta})}{2},$$

we have

$$\begin{aligned} a_k\cos kx + b_k\sin kx &= a_k\left(\frac{e^{\imath kx} + e^{-\imath kx}}{2}\right) - b_k\imath\left(\frac{e^{\imath kx} - e^{-\imath kx}}{2}\right) \\ &= \left(\frac{a_k - \imath b_k}{2}\right)e^{\imath kx} + \left(\frac{a_k + \imath b_k}{2}\right)e^{-\imath kx}. \end{aligned}$$

Thus we can write the approximating sum for a function $f(x)$, defined on $[-\pi, \pi]$, as

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx) = \sum_{k=-n}^{n} c_k e^{\imath kx},$$

where

$$c_k = \frac{a_k - \imath b_k}{2}, \quad c_{-k} = c_k^* = \frac{a_k + \imath b_k}{2}, \qquad k = 0, 1, 2, \ldots.$$

The coefficients c_k are given by the integrals

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\imath kx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos kx - \imath\sin kx]\,dx \\ &= \tfrac{1}{2}(a_k - \imath b_k). \end{aligned} \tag{3.2.1}$$

Consider now a function X_t defined on N integers. Let us identify the integers by $\{-m, -(m-1), \ldots, -1, 0, 1, \ldots, m\}$ if N is odd and by $\{-(m-1), -(m-2), \ldots, -1, 0, 1, \ldots, m\}$ if N is even. Taking N to be even, the c_k are given by

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{t=-(m-1)}^{m} X_t e^{-\imath 2\pi kt/N} \\ &= \frac{1}{N}\sum_{t=-(m-1)}^{m} X_t\left(\cos\frac{2\pi kt}{N} - \imath\sin\frac{2\pi kt}{N}\right), \\ k &= -(m-1), \ldots, 0, \ldots, m, \end{aligned} \tag{3.2.2}$$

and

$$X_t = \sum_{k=-(m-1)}^{m} c_k e^{\iota 2\pi kt/N} .$$

Note that c_0 and c_m do not require separate definitions. The complex form thus makes manipulation somewhat easier, since the correct divisor need not be specified by frequency.

If N is odd and X_t is an even function (i.e., $X_t = X_{-t}$), then the coefficients c_k are real. Conversely, if X_t is an odd function, the coefficients c_k of equation (3.2.1) are pure imaginary.

3.3. FOURIER TRANSFORM—FUNCTIONS DEFINED ON THE REAL LINE

The results of Theorems 3.1.8 and 3.1.9 represent a special case of a more general result known as the *Fourier integral theorem*. By Theorem 3.1.8 the sequence of Fourier coefficients for a continuous periodic function with square integrable derivative can be used to construct a sequence of functions that converges to the original function. This result can be stated in a very compact form by substituting the definition of a_k and b_k into the statement of Theorem 3.1.7 to obtain

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega) \cos k\omega \cos kx \, d\omega \right. \\ &\qquad \left. + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega) \sin k\omega \sin kx \, d\omega \right] \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega) \cos k(\omega - x) \, d\omega \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-\iota kx} \int_{-\pi}^{\pi} f(\omega) e^{\iota k\omega} \, d\omega . \end{aligned} \tag{3.3.1}$$

Since the reciprocal relationships are well defined, we could also write, using the notation of Theorem 3.1.9,

$$\rho(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\iota k\omega} \sum_{h=-\infty}^{\infty} \rho(h) e^{-\iota h\omega} \, d\omega .$$

We say that $\rho(k)$ and $f(x)$ form a *transform pair*. We shall associate the constant and the negative exponential with one transform and call this the *Fourier transform* or *spectral density*. The terms *spectrum* or *spectral function* are also used. Thus, in Theorem 3.1.9, the function $f(\omega)$ defined by

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho(h) e^{-\iota\omega h}$$

is the Fourier transform of $\rho(h)$, or the spectral density associated with $\rho(h)$.

The transform in (3.3.1) with positive exponent and no constant we call the *inverse transform* or the *characteristic function*. Thus the correlation function

$$\rho(h) = \int_{-\pi}^{\pi} f(\omega) e^{\iota h\omega}\, d\omega$$

is the inverse transform, or characteristic function, of $f(\omega)$. We have mentioned several times that this is the statistical characteristic function if $f(\omega)$ is a probability density.

These definitions are merely to aid us in remembering the transform to which we have attached the constant $1/2\pi$. We trust that the reader will not be disturbed to find that our definitional placement of the constant may differ from that of authors in other fields.

Theorem 3.1.8 was for a periodic function. However, there exists a considerable body of theory applicable to integrable functions defined on the real line. For an integrable function $f(x)$ defined on the real line we formally define the Fourier transform by

$$c(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-\iota ux}\, dx\,, \tag{3.3.2}$$

where $u \in (-\infty, \infty)$. The Fourier integral theorem states that if the function $f(x)$ meets certain regularity conditions, the inverse transform of the Fourier transform is again the function. We state one version without proof. [See Tolstov (1962, p. 188) for a proof.]

Theorem 3.3.1. Let $f(x)$ be an absolutely integrable continuous function defined on the real line with a right and a left derivative at every point. Then, for all $x \in (-\infty, \infty)$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\iota ux} \int_{-\infty}^{\infty} f(t) e^{-\iota ut}\, dt\, du\,. \tag{3.3.3}$$

Table 3.3.1 contains a summary of the Fourier transforms of the different types of functions we have considered. We have presented theorems for more general functions than the continuous functions described in the table. Likewise, the finite transform obviously holds for a function defined on an odd number of integers.

For the first, third, and fourth entries in the column headed "Inverse Transform" we have first listed the domain of the original function. However, the inverse transform, being a multiple of the transform of the transform, is defined for the values indicated in parentheses.

Table 3.3.1. Summary of Fourier transforms

Type of Function	Domain of Function	Fourier Transform	Inverse Transform
Finite sequence X_t	$-(n-1), \ldots, 0, \ldots, n$	$c_k = \frac{1}{2n} \sum_{t=-(n-1)}^{n} X_t e^{-i\pi kt/n}$ $k = -(n-1), \ldots, 0, \ldots, n$	$X_t = \sum_{k=-(n-1)}^{n} c_k e^{i\pi kt/n}$ $t = -(n-1), \ldots, 0, \ldots, n$ (can be extended to $t = 0, \pm 1, \pm 2, \ldots$)
Infinite sequence $\gamma(h)$ absolutely summable	$(0, \pm 1, \pm 2, \ldots)$	$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\omega h}$ $\omega \in (-\pi, \pi)$ $f(\omega)$ periodic of period 2π	$\gamma(h) = \int_{-\pi}^{\pi} f(\omega) e^{i\omega h}\, d\omega$ $h = 0, \pm 1, \pm 2, \ldots$
Continuous piecewise smooth $f(x)$	$[-\pi, \pi]$	$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$ $k = 0, \pm 1, \pm 2, \ldots$	$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ $x \in [-\pi, \pi]$ (can be extended as a periodic function on the real line)
Continuous periodic piecewise smooth $f(x) = f(x+T)$	$(-T/2, T/2)$ (can be extended to the real line)	$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i2\pi kx/T}\, dx$ $k = 0, \pm 1, \pm 2, \ldots$	$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i2\pi kx/T}$ $x \in (-T/2, T/2)$ (can be extended to real line)
Continuous square integrable $f(x)$	$(-\infty, \infty)$	$c(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iux}\, dx$ $u \in (-\infty, \infty)$	$\bar{f}(x) = \int_{-\infty}^{\infty} c(u) e^{iux}\, du$ $x \in (-\infty, \infty)$

Given a continuous function defined on an interval, it may be convenient to record the function at a finite number of equally spaced points and find the Fourier transform of the N points. Time series are often created by reading "continuous" functions of time at fixed intervals. For example, river level and temperature have the appearance of continuous functions of time, but we may record for analysis the readings at only a few times during a day or season.

Let $g(x)$ be a continuous function with square integrable derivative defined on the interval $[-T, T]$. We evaluate the function at the $2m$ points

$$x_t = \left(\frac{t}{m}\right)T\,, \qquad t = -(m-1), -(m-2), \ldots, m-1, m\,.$$

The complex form of the Fourier coefficients for the vector $g(x_t)$ is given by

$$c_k^{(m)} = \frac{1}{2m} \sum_{t=-(m-1)}^{m} g(x_t)e^{-\iota\pi tk/m}\,, \qquad k = -(m-1), -(m-2), \ldots, m\,,$$

and $g(x_t)$ is expressible as

$$g(x_t) = \sum_{k=-(m-1)}^{m} c_k^{(m)} e^{\iota\pi kt/m}\,, \qquad t = -(m-1), -(m-2), \ldots, m\,. \quad (3.3.4)$$

The superscript (m) on the coefficients is to remind us that they are based on $2m$ equally spaced values of $g(x)$.

If we compute the Fourier coefficients for the function using the integral form (3.2.1), we have

$$c_k = \frac{1}{2T}\int_{-T}^{T} e^{-\iota(k\pi/T)x} g(x)\,dx\,, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

and the original function evaluated at the points x_t, $t = -(m-1)$, $-(m-2)$, $\ldots, m$, is given by

$$\begin{aligned} g(x_t) &= \sum_{k=-\infty}^{\infty} c_k e^{\iota(k\pi/T)Tt/m} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{\iota\pi kt/m}\,. \end{aligned}$$

By the periodicity of the complex exponential we have

$$g(x_t) = \sum_{k=-(m-1)}^{m} e^{\iota\pi kt/m}[c_k + (c_{k-2m} + c_{k+2m}) + (c_{k-4m} + c_{k+4m}) + \cdots]\,. \quad (3.3.5)$$

Table 3.3.2. Computed Coefficients for Function (3.3.6) Observed at 10 Points

Period	Frequency	Cosine Coefficient $c_k^{(5)} + c_{-k}^{(5)}$	Contributing Alias Frequencies
—	0	0.5	10/20
20	1/20	0.0	—
10	2/20	0.0	—
20/3	3/20	1.6	13/20, 23/20
5	4/20	0.0	—
4	5/20	0.3	15/20

Equating the coefficients of $e^{\iota\pi kt/m}$ in (3.3.4) and (3.3.5), we have

$$c_k^{(m)} = c_k + \sum_{s=1}^{\infty} (c_{k-2ms} + c_{k+2ms}) .$$

It follows that the coefficient for the kth frequency of the function defined at $2m$ points is the sum of the coefficients of the continuous function at the k, $k+2m$, $k-2m$, $k+4m$, $k-4m, \dots$ frequencies. The frequencies $k \pm 2m$, $k \pm 4m, \dots$ are called the *aliases* of the kth frequency.

In our representation the distance between two points is T/m. The cosine of period $2T/m$ or frequency $m/2T$ is the function of highest frequency that will be used in the Fourier transform. The frequency $m/2T$ is called the *Nyquist* frequency. The aliases of an observed frequency are frequencies obtained by adding or subtracting integer multiples of twice the Nyquist frequency.

To illustrate these ideas, let $g(x)$ defined on $[-10, 10]$ be given by

$$\begin{aligned} g(x) &= 1.0 \cos 2\pi \tfrac{3}{20} x + 0.5 \cos \pi x \\ &\quad + 0.4 \cos 2\pi \tfrac{13}{20} x + 0.3 \cos 2\pi \tfrac{15}{20} x \\ &\quad + 0.2 \cos 2\pi \tfrac{23}{20} x . \end{aligned} \tag{3.3.6}$$

If the function is observed at the 10 points $-8, -6, \dots, 8, 10$ and the $\{c_k^{(5)}$: $k = -4, -3, \dots, 4, 5\}$ are computed by

$$c_k^{(5)} = \frac{1}{10} \sum_{t=-4}^{5} g\left(\frac{t}{5}\, 10\right) e^{-\iota\pi tk/5} ,$$

we will obtain the coefficients given in Table 3.3.2.

3.4. FOURIER TRANSFORM OF A CONVOLUTION

Fourier transform theory is particularly useful for certain function-of-function problems. We consider one such problem that occurs in statistics and statistical time series.

For absolutely integrable functions $f(x)$ and $g(x)$ defined on the real line, the function

$$\varphi(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

is called the *convolution* of $f(x)$ and $g(x)$. Since $f(x)$ and $g(x)$ are absolutely integrable, $\varphi(x)$ is absolutely integrable.

We have the following theorem on the Fourier transform of a convolution.

Theorem 3.4.1. If the functions $f(x)$ and $g(x)$ are absolutely integrable, then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is given by

$$\begin{aligned} c_\varphi(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x)e^{-i\omega x}\,dx \\ &= 2\pi c_f(\omega)c_g(\omega)\,, \end{aligned}$$

where

$$c_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

and

$$c_g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-i\omega x}\,dx$$

are the Fourier transforms of $f(x)$ and $g(x)$.

Proof. The integrals defining $c_\varphi(\omega)$, $c_f(\omega)$, and $c_g(\omega)$ are absolutely convergent. Hence,

$$\begin{aligned} c_\varphi(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x)e^{-i\omega x}\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)e^{-i\omega x}\,dy\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)e^{-i\omega(x-y)}g(y)e^{-i\omega y}\,dy\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)e^{-i\omega z}\,dz\int_{-\infty}^{\infty} g(y)e^{-i\omega y}\,dy \\ &= 2\pi c_f(\omega)c_g(\omega)\,, \end{aligned} \tag{3.4.1}$$

where the absolute integrability has enabled us to interchange the order of integration. ▲

We discussed the convolution of two absolutely summable sequences in Section 2.2 and demonstrated that the convolution was absolutely summable.

Corollary 3.4.1.1. Given that $\{a_j\}$ and $\{b_j\}$ are absolutely summable, the Fourier transform of

$$d_m = \sum_{j=-\infty}^{\infty} a_{m-j} b_j$$

is

$$f_d(\omega) = 2\pi f_a(\omega) f_b(\omega),$$

where

$$f_a(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} a_j e^{-\iota\omega j}$$

and

$$f_b(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} b_j e^{-\iota\omega j}.$$

Proof. The Fourier transform of $\{d_m\}$ is

$$\begin{aligned} f_d(\omega) &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} d_m e^{-\iota\omega m} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{m-j} b_j e^{-\iota\omega m} \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{m-j} b_j e^{-\iota\omega(m-j)} e^{-\iota\omega j} \\ &= 2\pi f_a(\omega) f_b(\omega). \end{aligned}$$ ▲

We may paraphrase these results as follows: the spectral density (Fourier transform) of a convolution is the product of the spectral densities (Fourier transforms) multiplied by 2π. That the converse is also true is clear from the proofs. We give a direct statement and proof for the product of absolutely summable sequences.

Corollary 3.4.1.2. Let $\{a_m\}$ and $\{b_m\}$ be absolutely summable, and define d_m by

$$d_m = a_m b_m.$$

Then

$$f_d(\omega) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} d_m e^{-\iota\omega m} = \int_{-\pi}^{\pi} f_a(u) f_b(\omega - u)\, du$$
$$= \int_{-\pi}^{\pi} f_a(\omega - u) f_b(u)\, du\,,$$

where $f_a(\omega)$ and $f_b(\omega)$ are defined in Corollary 3.4.1.1.

Proof. We have

$$\begin{aligned} f_d(\omega) &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} d_m e^{-\iota\omega m} \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} a_m b_m e^{-\iota\omega m} \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} f_a(x) e^{\iota m x}\, dx \right) b_m e^{-\iota\omega m} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_a(x) \sum_{m=-\infty}^{\infty} b_m e^{-\iota m(\omega - x)}\, dx \\ &= \int_{-\pi}^{\pi} f_a(x) f_b(\omega - x)\, dx\,. \end{aligned}$$

▲

REFERENCES

Jenkins and Watts (1968), Lighthill (1970), Nerlove (1964), Rogosinski (1959), Tolstov (1962), Zygmund (1959).

EXERCISES

1. Give the 12 orthogonal sine and cosine functions that furnish an orthogonal basis for the 12-dimensional vector space. Find the linear combinations of these vectors that yield the vectors $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$.

2. Expand the following functions defined on $[-\pi, \pi]$ in Fourier series:
   (a) $f(\omega) = \cos a\omega$, where a is not an integer.
   (b) $f(\omega) = \sin a\omega$, where a is not an integer.
   (c) $f(\omega) = e^{a|\omega|}$, where $a \neq 0$.
   (d) $f(\omega) = \begin{cases} 0\,, & -\pi \leq \omega \leq 0 \\ \sin \omega\,, & 0 < \omega < \pi\,. \end{cases}$

3. Define $g(x)$ by

$$g(x) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ 1, & 0 < x < \pi. \end{cases}$$

(a) Find the Fourier coefficients for $g(x)$.
(b) What is the maximum value for

$$S_3(x) = \frac{1}{2} + \sum_{k=1}^{3} (a_k \cos kx + b_k \sin kx)?$$

Where does this maximum occur? What is the maximum value of $S_4(x)$? $S_5(x)$? The fact that the approximating function always overestimates the true function near the point of discontinuity is called *Gibbs' phenomenon.*

4. Prove Theorem 3.1.2.

5. Let $f(x)$ be the periodic function defined on the real line by

$$f(x) = \begin{cases} b^{-1}, & 2\pi j - b < x < 2\pi j + b, \\ 0 & \text{otherwise}, \end{cases}$$

where $j = 0, \pm 1, \pm 2, \ldots$ and $0 < b < \pi$. Find the Fourier transform of $f(x)$.

6. Let

$$f(x) = \begin{cases} 1, & -b \leq x < b, \\ 0 & \text{otherwise}, \end{cases}$$

where b is a positive number and $f(x)$ is defined on the real line. Find the Fourier transform of $f(x)$. Show that the limit of this transform at zero is infinity as $b \to \infty$. Show that as $b \to \infty$ the transform is bounded except at zero.

7. Let

$$\delta_n(x) = \begin{cases} 2\pi n, & -\dfrac{1}{2n} \leq x < \dfrac{1}{2n}, \\ 0 & \text{otherwise}, \end{cases}$$

where n is a positive integer and $\delta_n(x)$ is defined on the real line. Find the Fourier transform of $\delta_n(x)$. Show that, as $n \to \infty$, the transform tends to the constant function of unit height.

8. Let the *generalized function* $\delta(x)$ represent a sequence of functions $\{\delta_n(x)\}_{n=1}^{\infty}$

such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)\,dx = f(0)\,,$$

where $f(x)$ is a continuous absolutely integrable function defined on the real line. Then $\delta(x)$ is called *Dirac's delta function*. Show that the sequence of functions $\{\delta_n(x)\}$ of Exercise 7 defines a generalized function.

9. Let $g_n(x) = (n/\pi)^{1/2}e^{-nx^2}$ for $x \in (-\infty, \infty)$. Show that the sequence $\{g_n(x)\}_{n=1}^{\infty}$ yields a Dirac delta function as defined in Exercise 8.

10. Let $f(x)$ defined for $x \in (-\infty, \infty)$ have the Fourier transform $c(u)$. Show that the Fourier transform of $f(ax + b)$ is $|a|^{-1}e^{\iota bu/a}c(u/a)$, $a \neq 0$.

11. Assume that price of a commodity is recorded on the last day of each month for a period of 144 months. The finite Fourier coefficients for the data are computed using the formulas following (3.1.2). Which coefficients will be affected if there is a weekly periodicity in prices that is perfectly represented by a sine wave of period 7 days? Assume that there are 30.437 days in a month. Which coefficients will be affected if the weekly periodicity in prices can be represented by the sum of two sine waves, one of period 7 days and one of period $3\frac{1}{2}$ days? See Granger and Hatanaka (1964).

12. Let $f(x)$ and $g(x)$ be absolutely integrable functions, and define

$$\varphi(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy\,.$$

Show that $\varphi(x)$ satisfies

$$\int_{-\infty}^{\infty}|\varphi(x)|\,dx \le \left(\int_{-\infty}^{\infty}|f(x)|\,dx\right)\left(\int_{-\infty}^{\infty}|g(y)|\,dy\right).$$

13. Let $f(x)$ and $g(x)$ be continuous absolutely integrable functions defined on the real line. State and prove the result analogous to Corollary 3.4.1.2 for $\psi(x) = f(x)g(x)$.

14. Give a direct proof of Corollary 3.4.1.2 for finite transforms. That is, for the two functions $\gamma(h)$ and $w(h)$ defined on the $2n-1$ integers $h = 0, \pm 1,$

$\pm 2, \ldots, \pm(n-1)$, show that

$$g(\omega_s) = \frac{1}{2n-1} \sum_{h=-(n-1)}^{n-1} w(h)\gamma(h)e^{-\iota\omega_s h}$$

$$= \sum_{k=-(n-1)}^{n-1} W(\omega_k)f(\omega_{s-k}),$$

where

$$\omega_k = \frac{2\pi k}{2n-1}, \qquad k = 0, \pm 1, \pm 2, \ldots, \pm(n-1),$$

$$W(\omega_k) = \frac{1}{2n-1} \sum_{h=-(n-1)}^{n-1} w(h)e^{-\iota\omega_k h},$$

$$f(\omega_k) = \frac{1}{2n-1} \sum_{h=-(n-1)}^{n-1} \gamma(h)e^{-\iota\omega_k h}.$$

15. Let $f(\omega)$ be a nonnegative even continuous periodic function of period 2π. Show that

$$c(h) = \int_{-\pi}^{\pi} f(\omega)e^{-\iota\omega h}\, d\omega, \qquad h = 0, \pm 1, \pm 2, \ldots,$$

is an even positive semidefinite function.