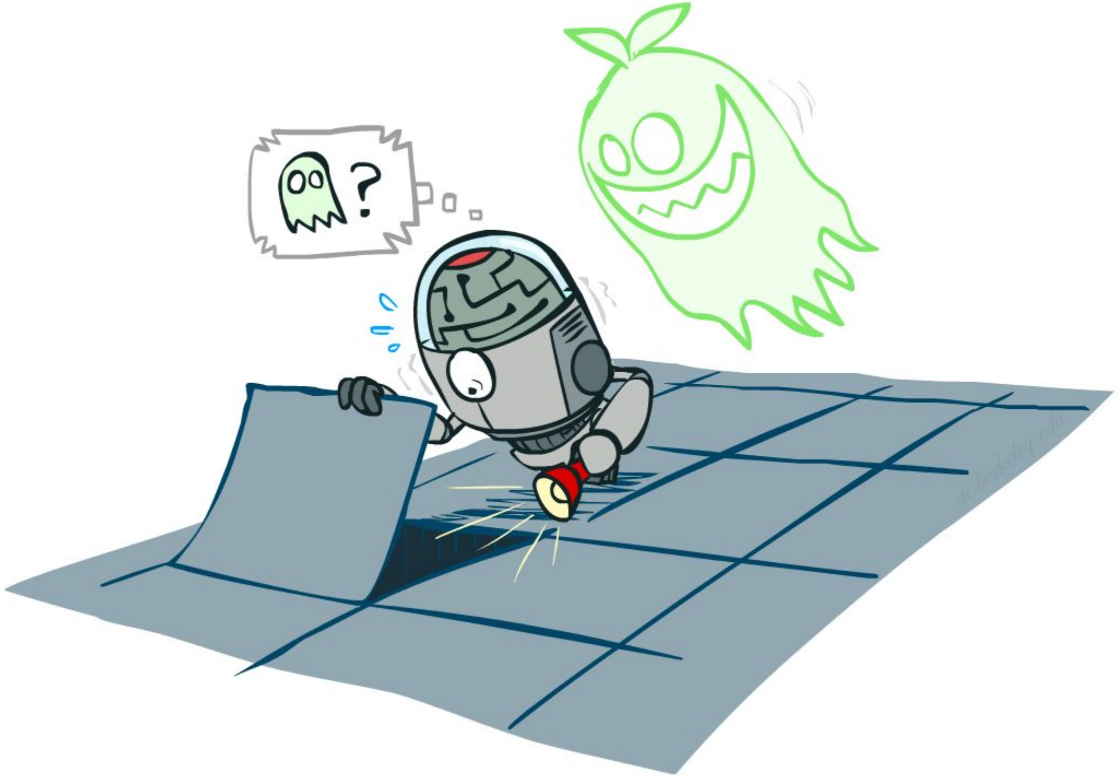
Most materials in this chapter are based on materials from http://aima.cs.berkeley.edu/ and http://ai.berkeley.edu

Chapter 4
Markov Models

COMP 3270
Artificial Intelligence

Dirk Schnieders

Uncertainty

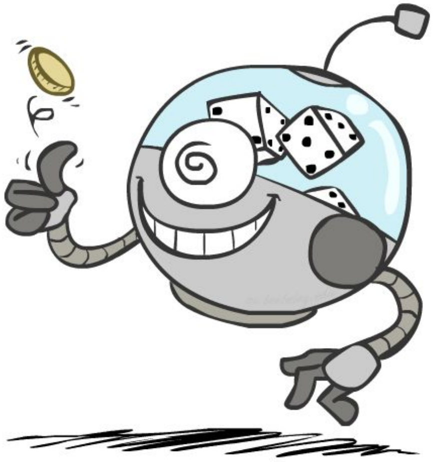


Probabilistic Reasoning

- This lecture is a bit dry
- Motivation:
  - AI History
  - Applications
    - Diagnosis
      - Physician measures symptoms and infer disease
      - IT experts diagnose printer problem
    - Speech recognition: Noisy signal, people have different pronunciations of words
    - Tracking objects: Combine noisy measurements form multiple sensors to estimate position
    - Genetics: Reproduction
    - Error correcting codes: Electrical signals are noisy

Refresher: Probability

- Random Variables
- Joint and Marginal Distributions
- Conditional Distribution
- Product Rule, Chain Rule, Bayes' Rule
- Inference
- Independence



The content in this refresher is important. In this and some of the following chapters we assume that you have the above background.

Running Example: Ghostbusters

Running Example: Ghostbusters

- A single ghost is hiding in the grid somewhere
- Noisy sensor readings tell how close a certain square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green
- We know P (color | distance)
  - Example:

Each of the distances has a distribution similar to this one

P(red | 3)	P(orange | 3)	P(yellow | 3)	P(green | 3)
0.05	0.15	0.5	0.3

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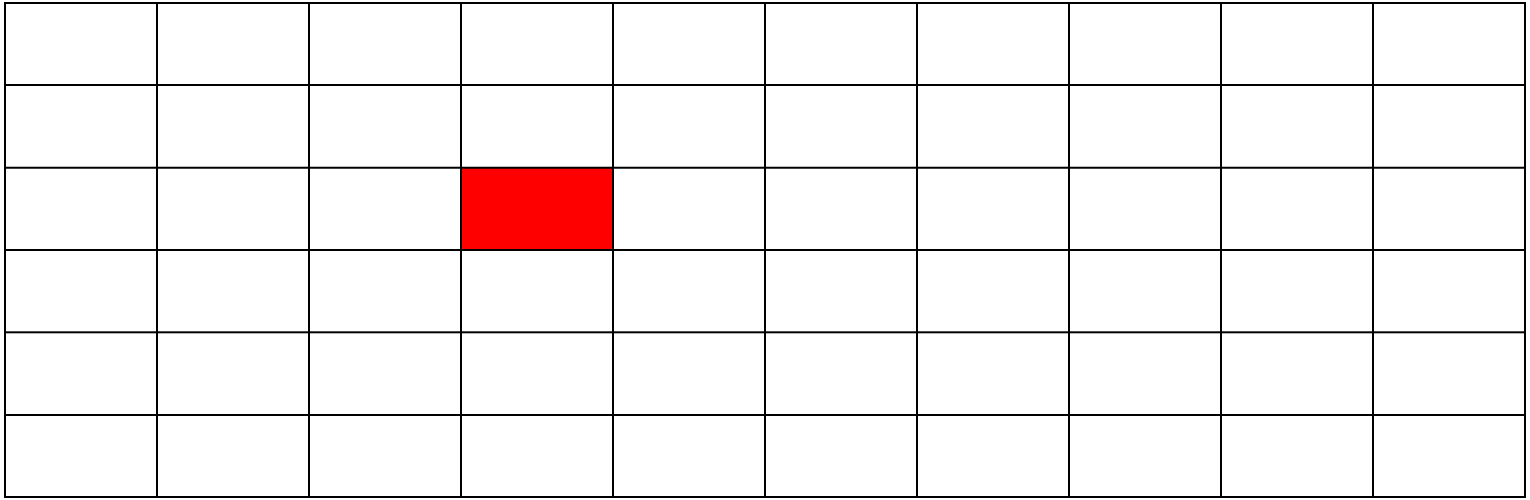

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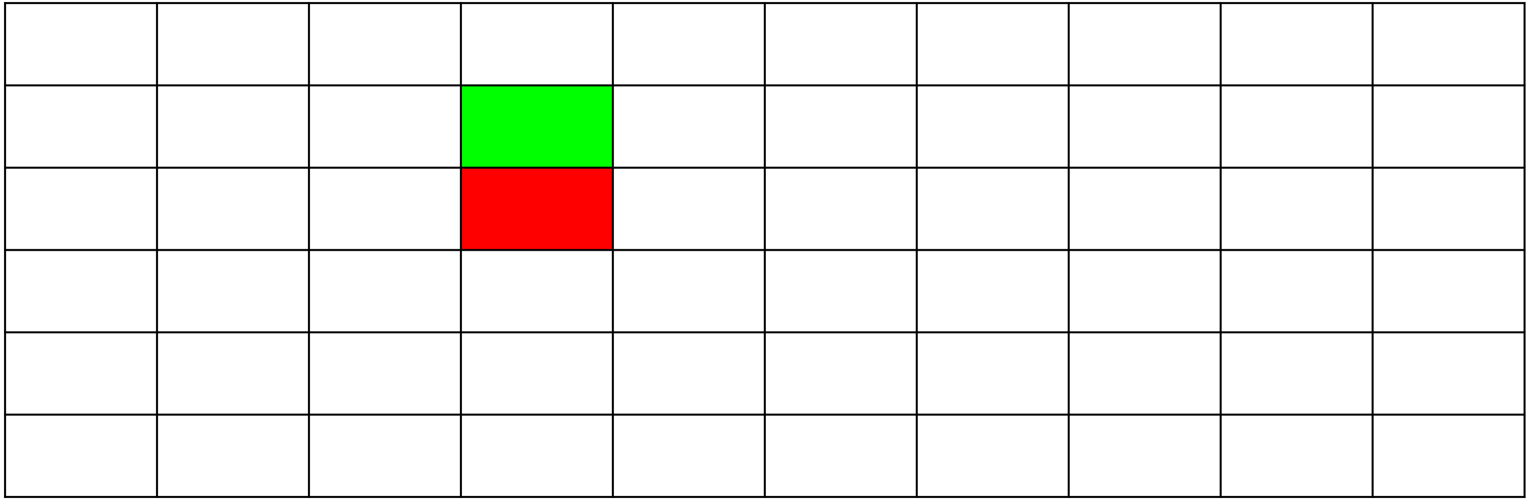

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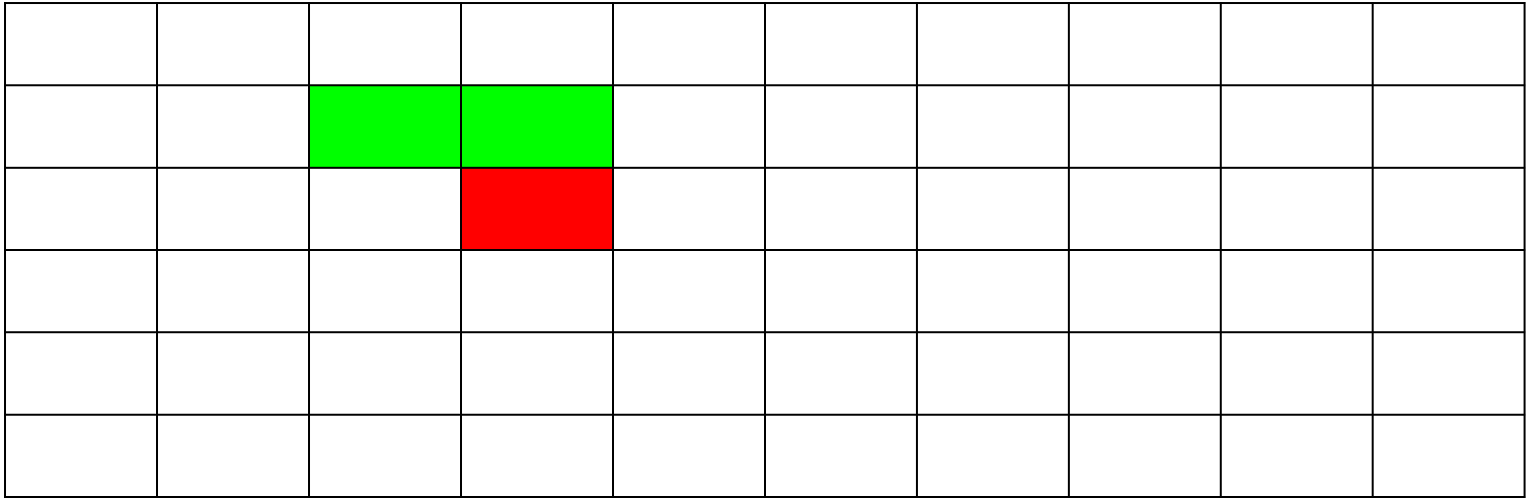

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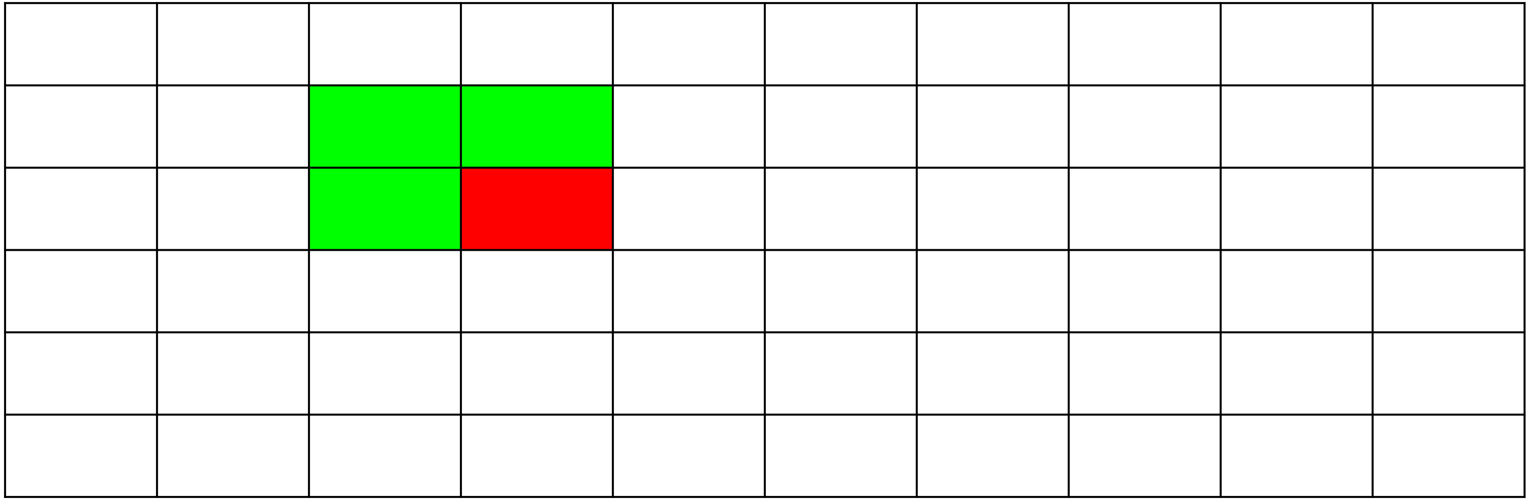

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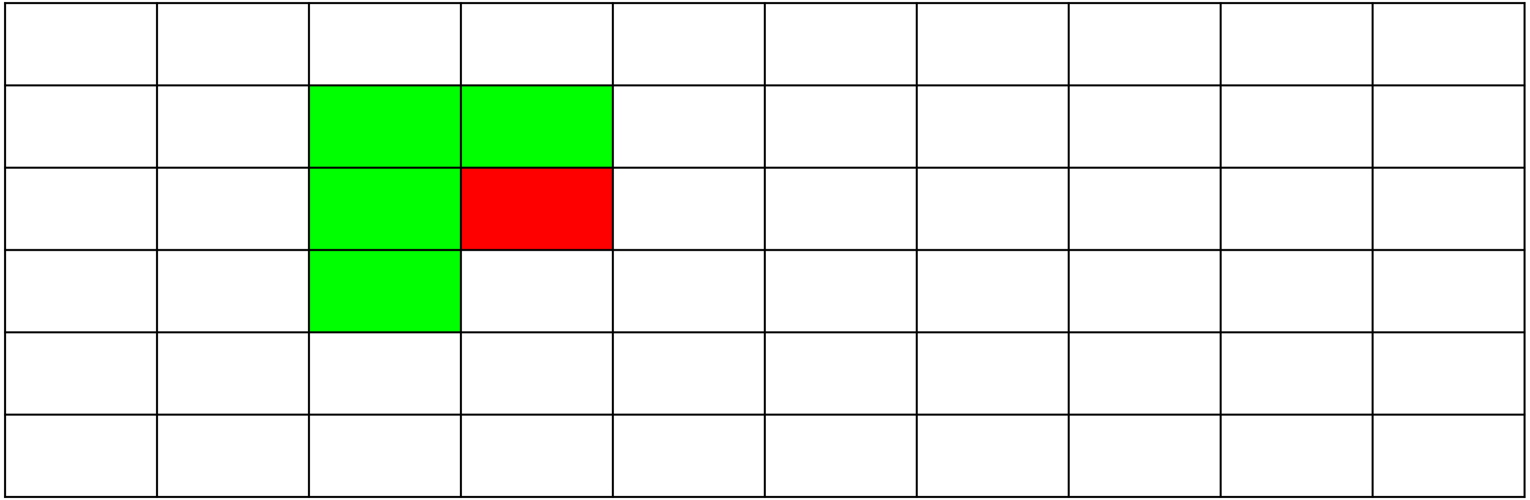

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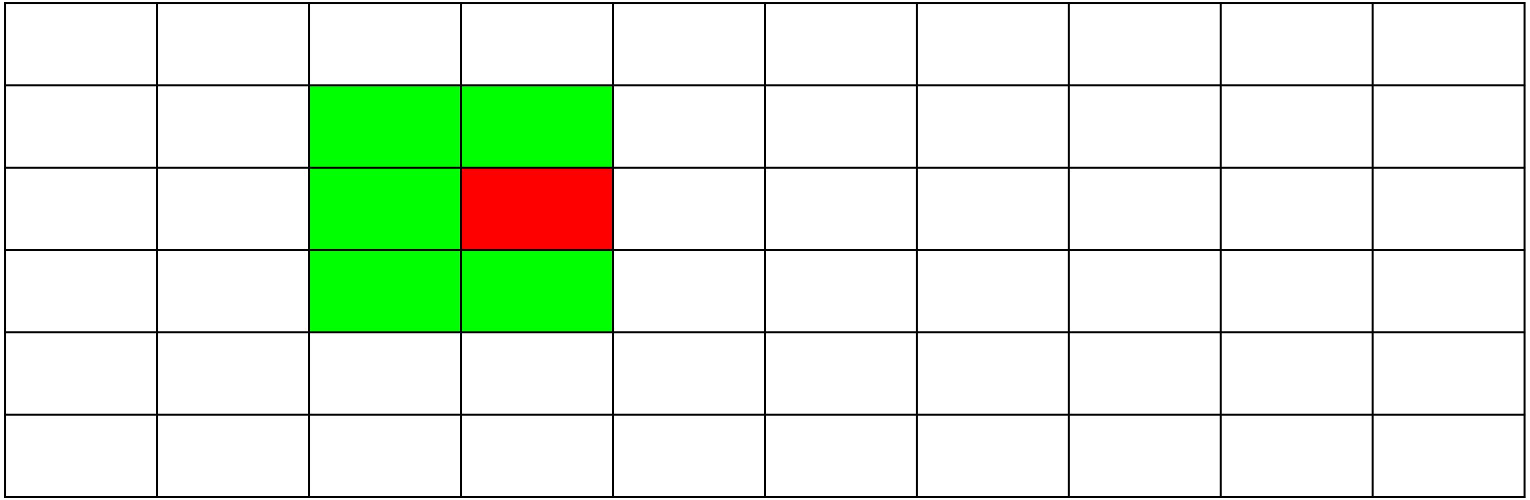

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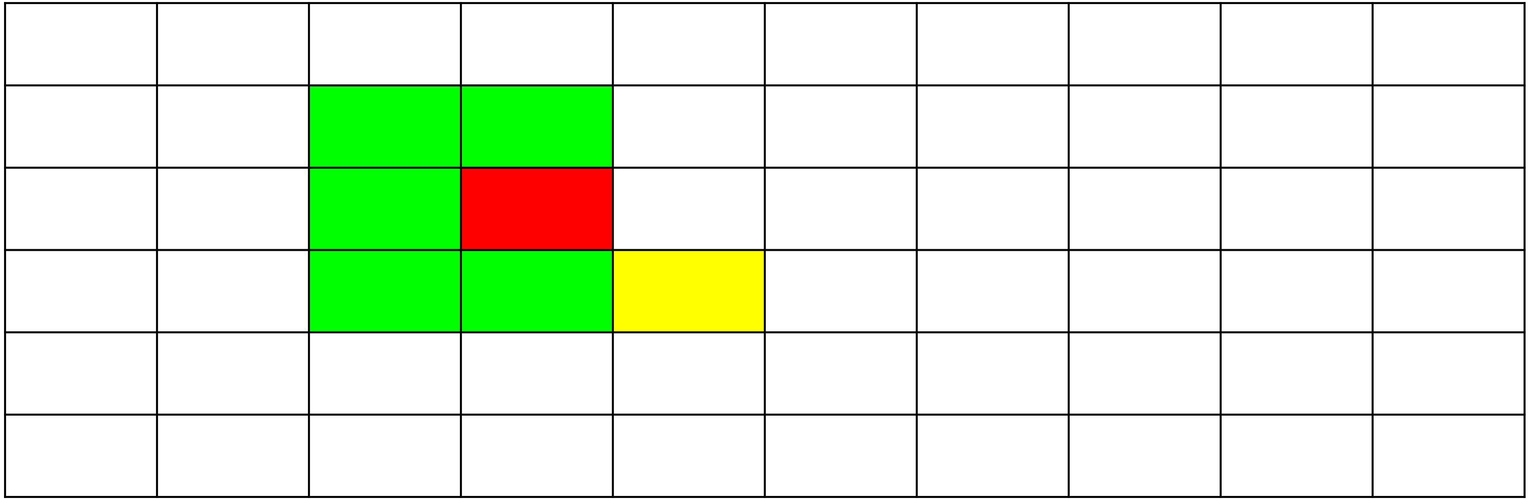

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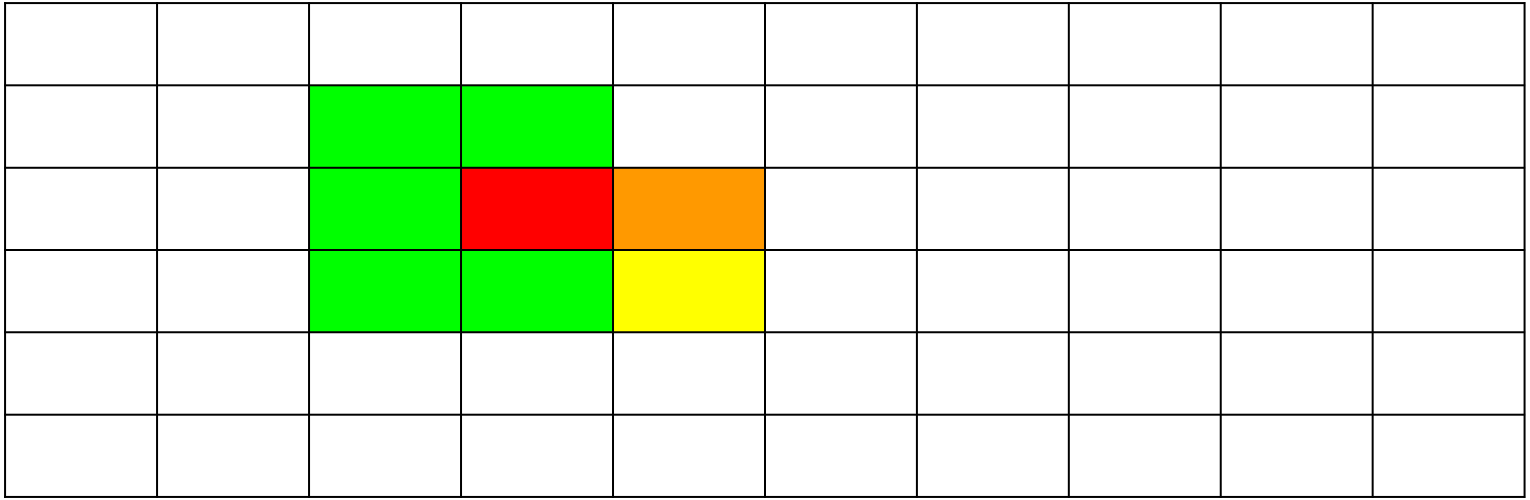

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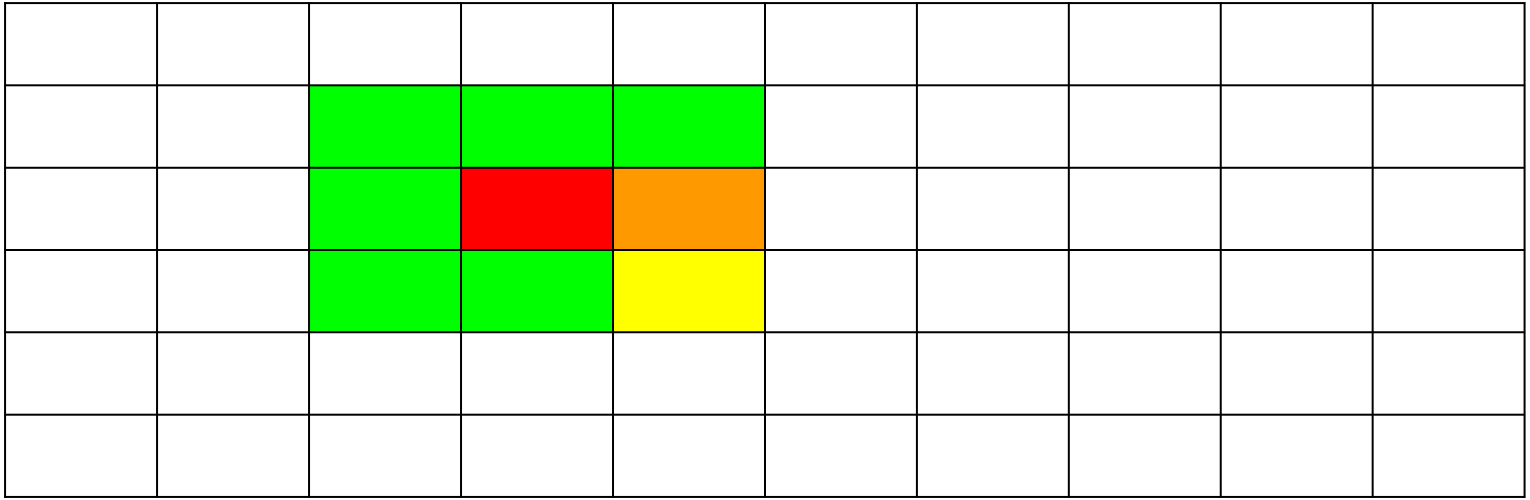

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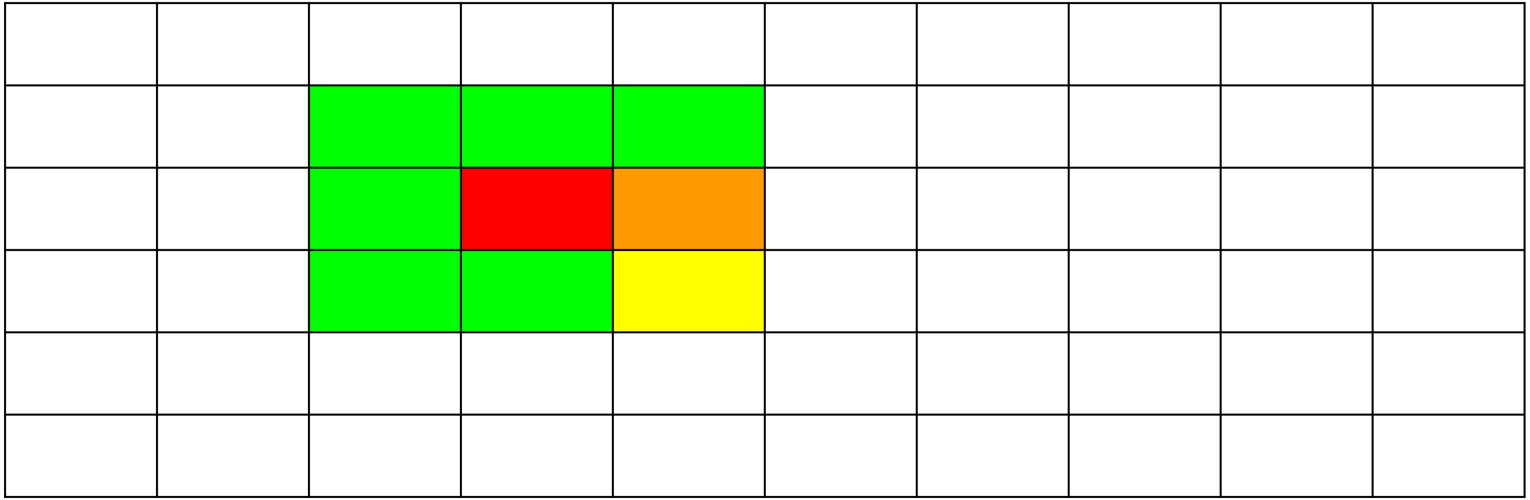

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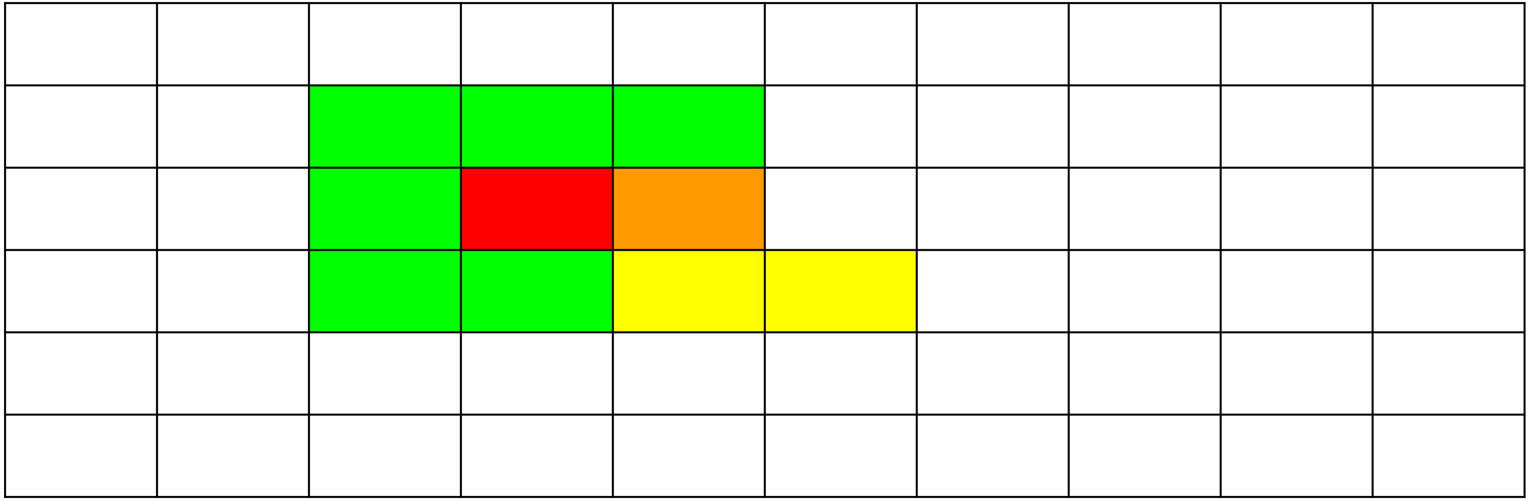

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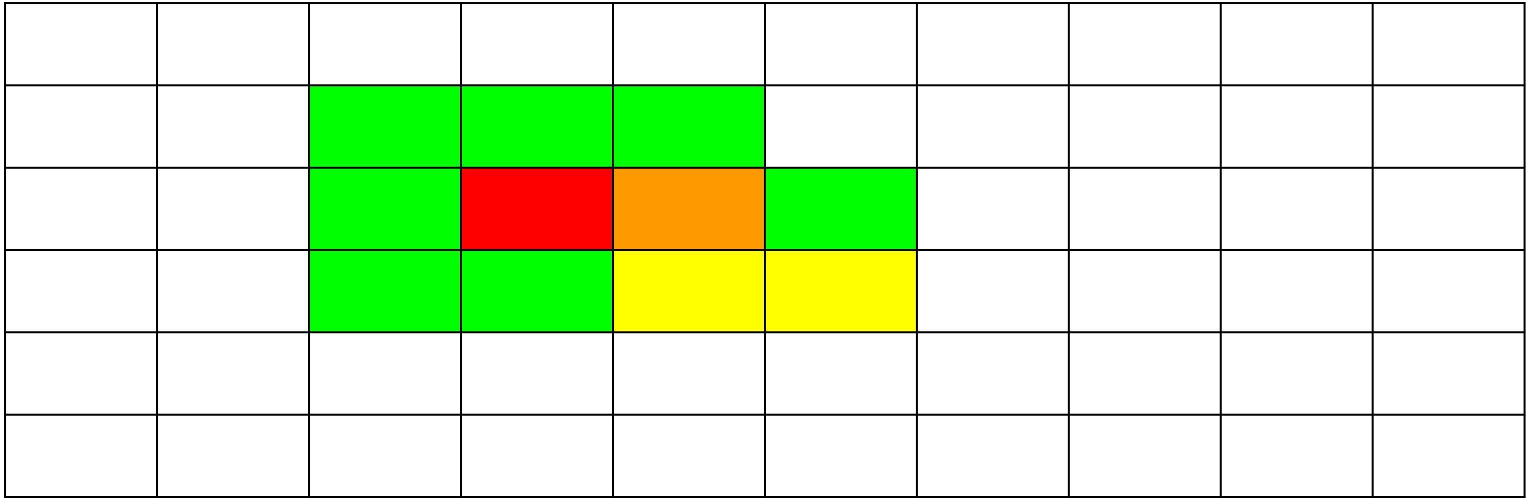

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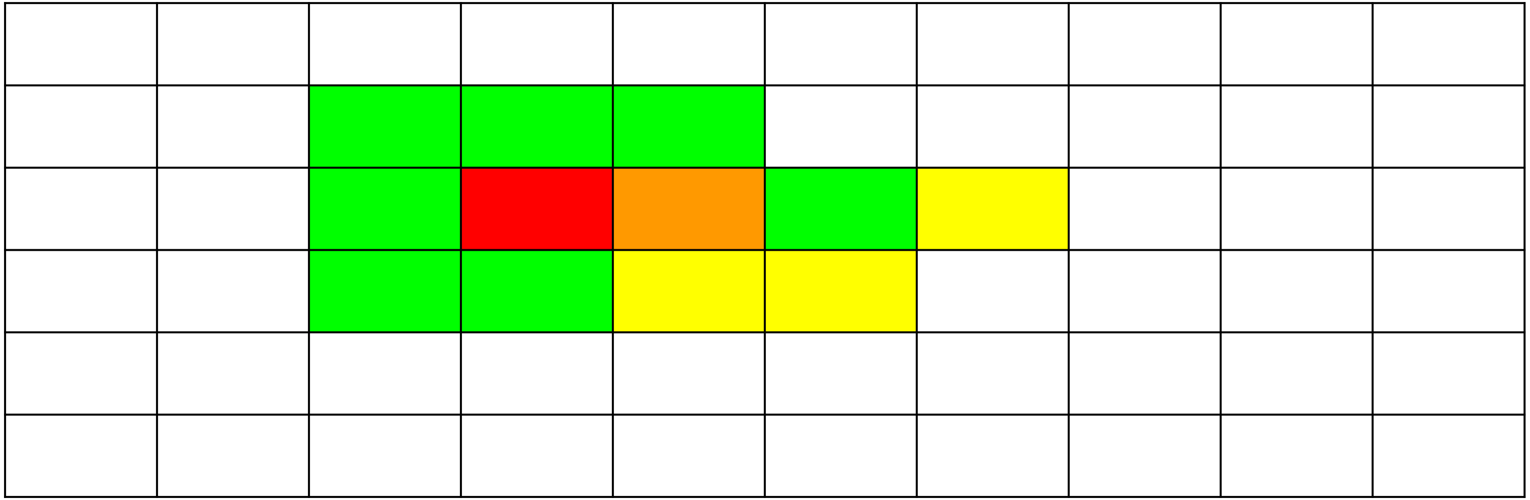

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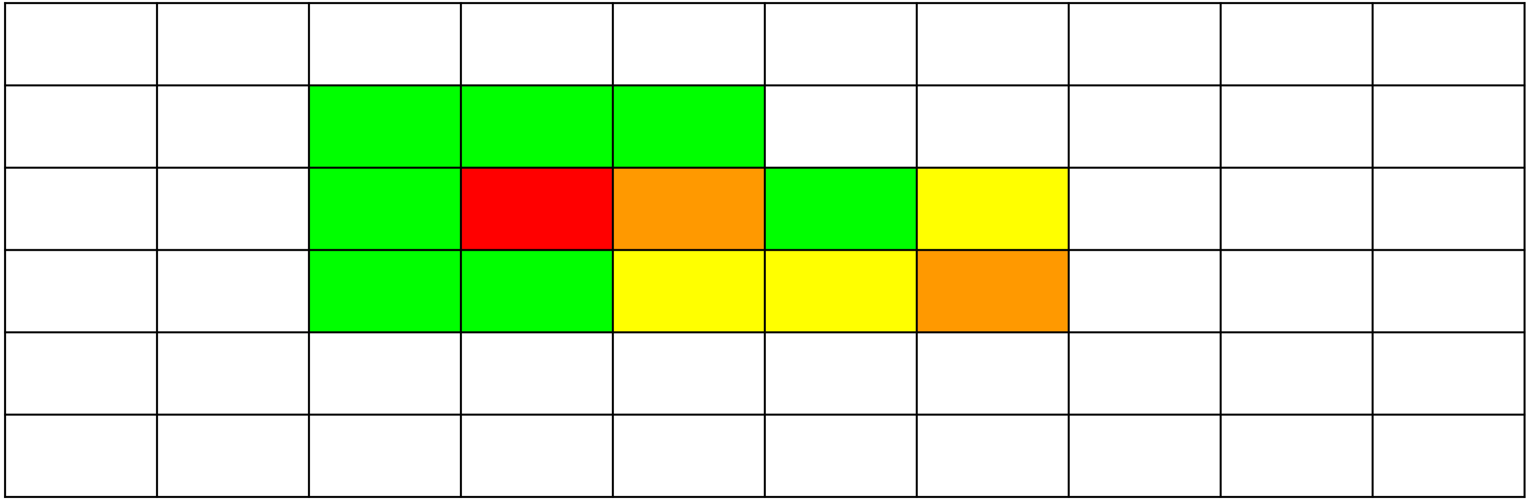

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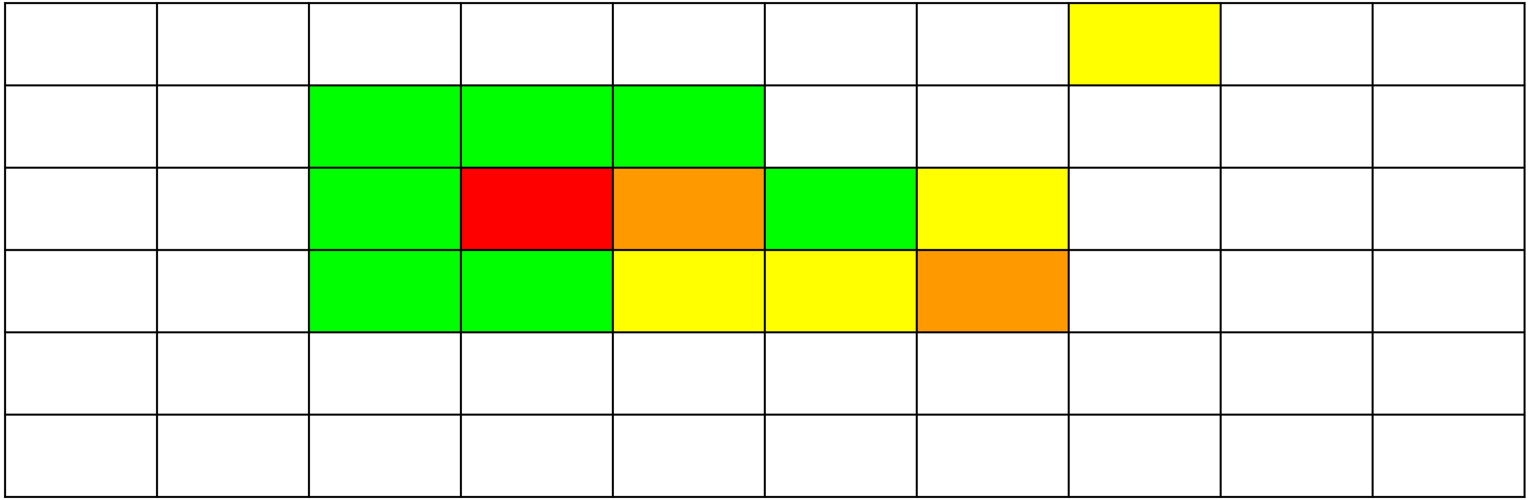

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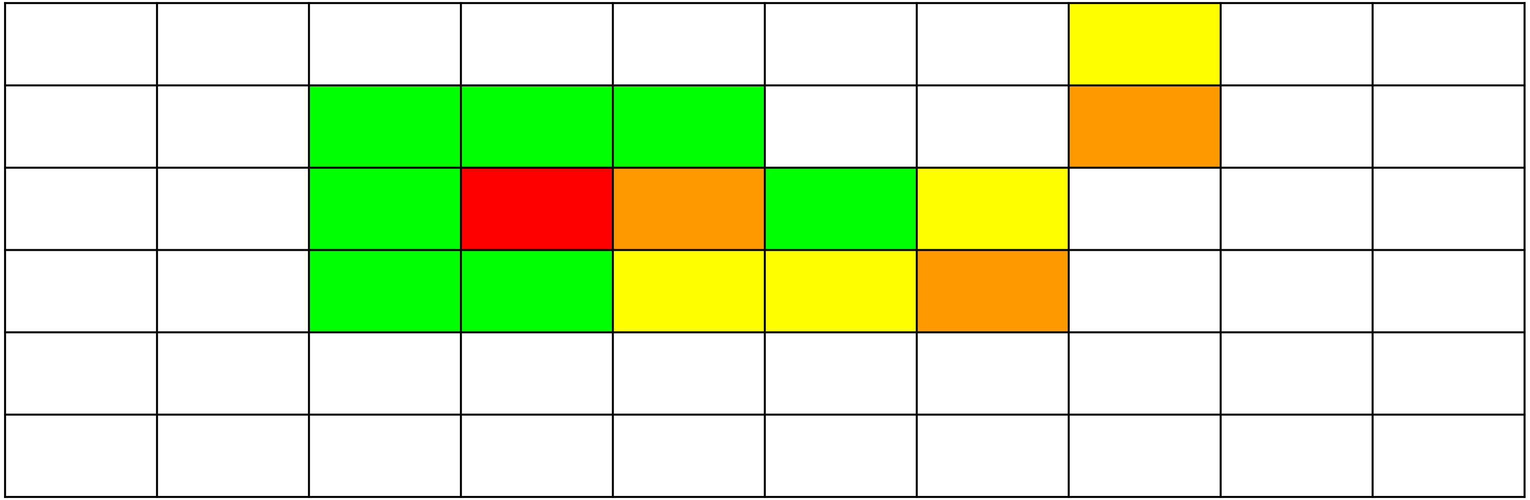

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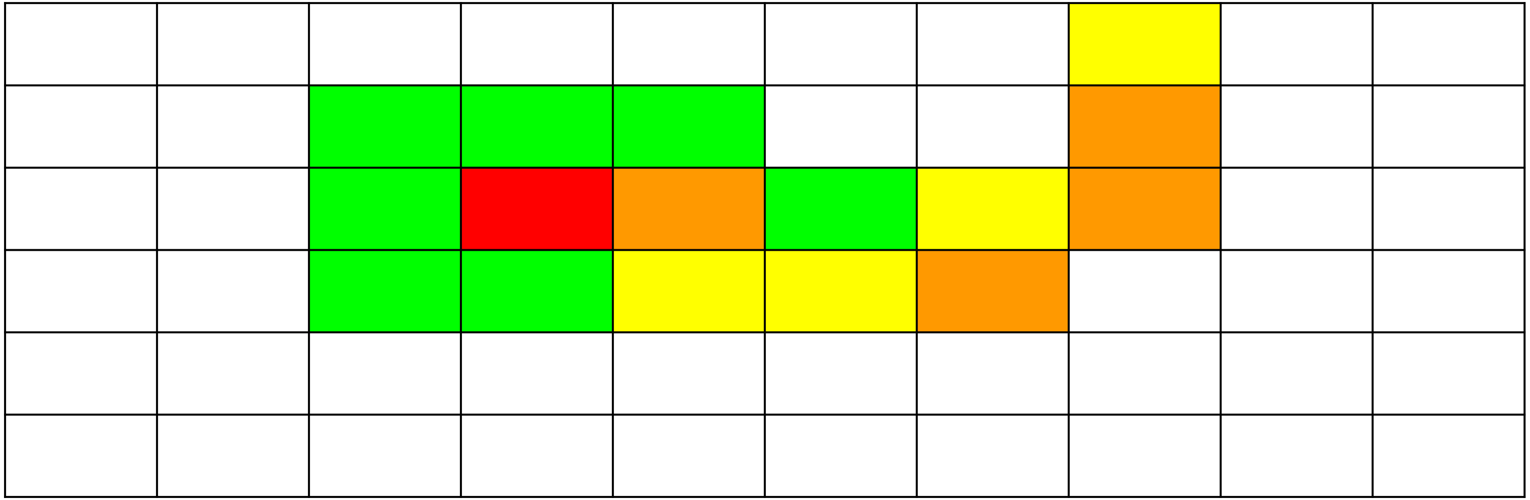

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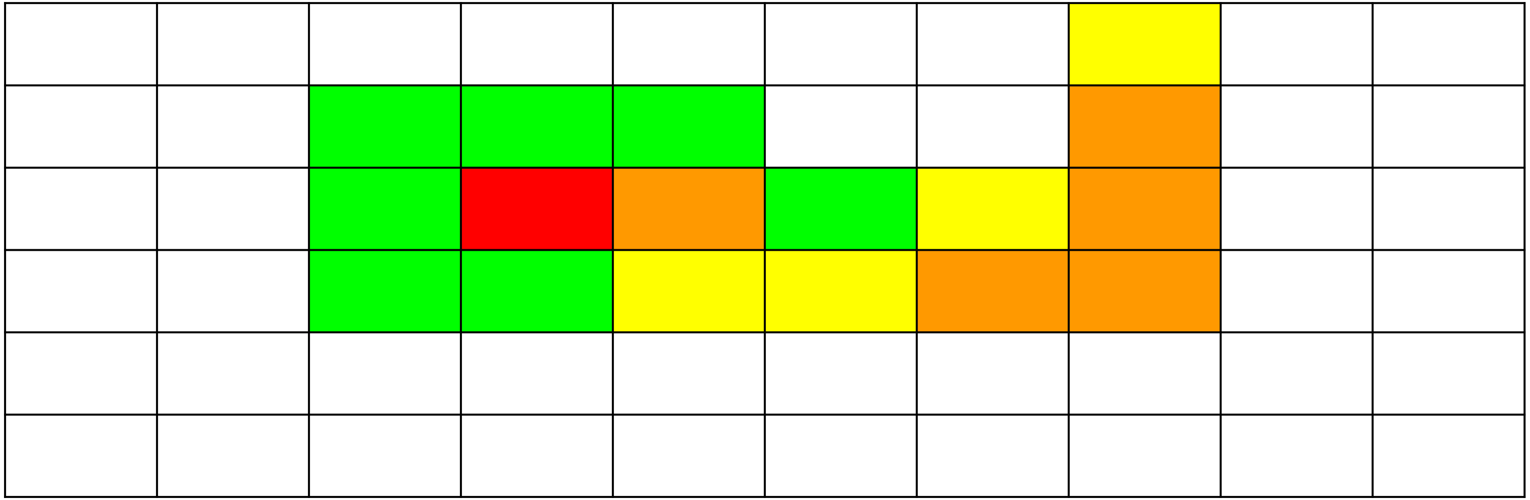

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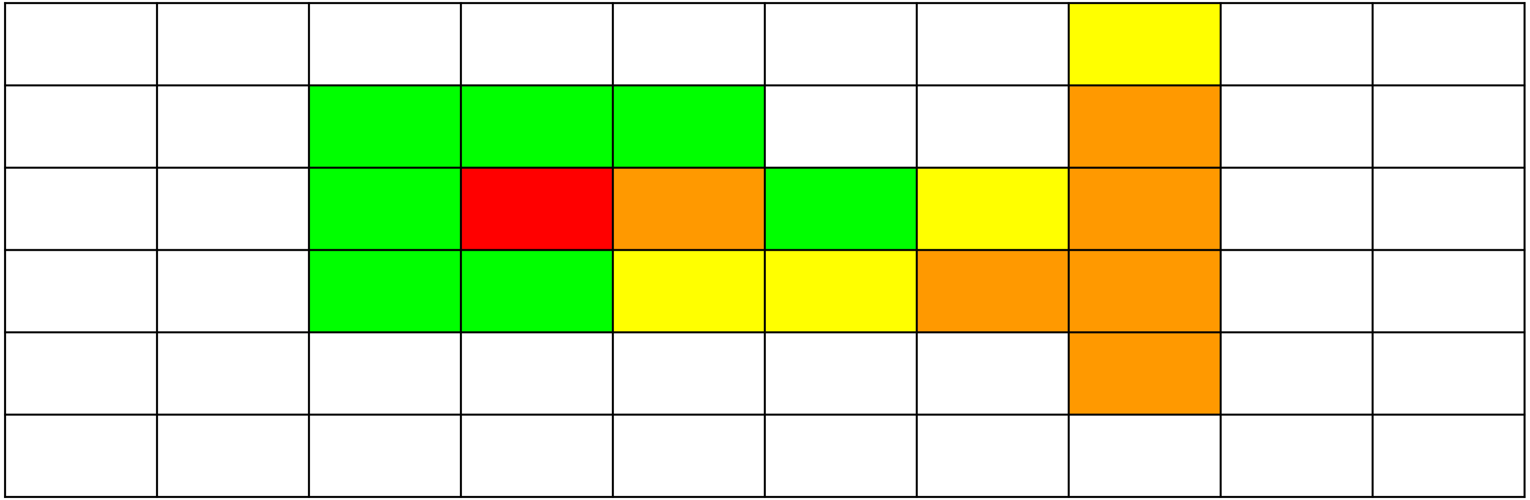

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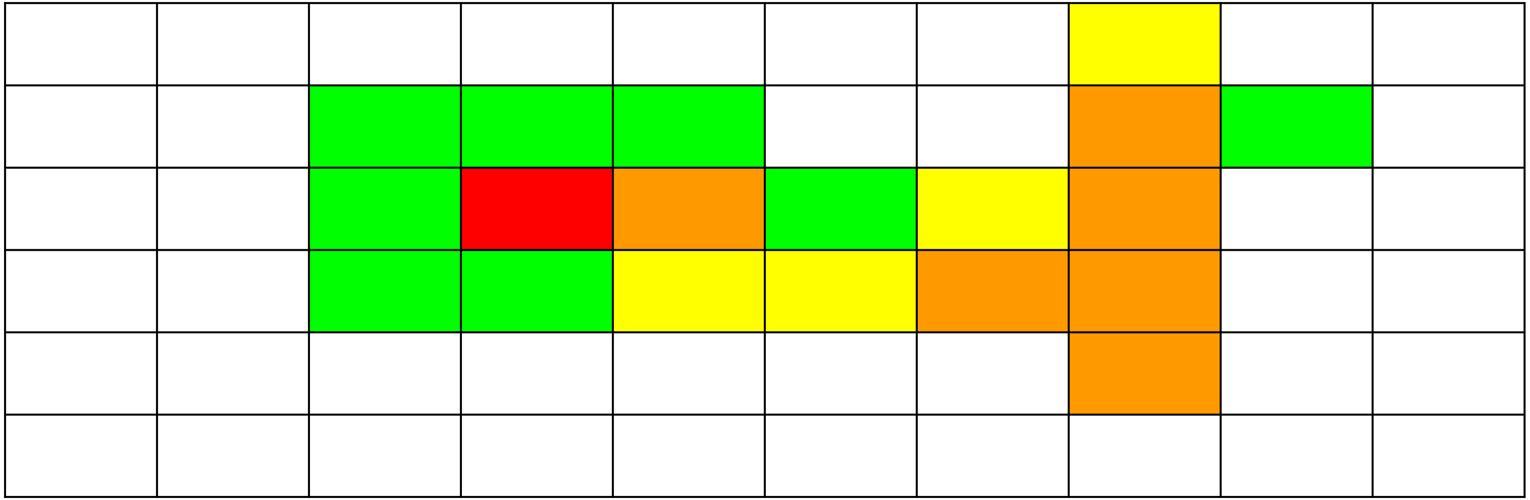

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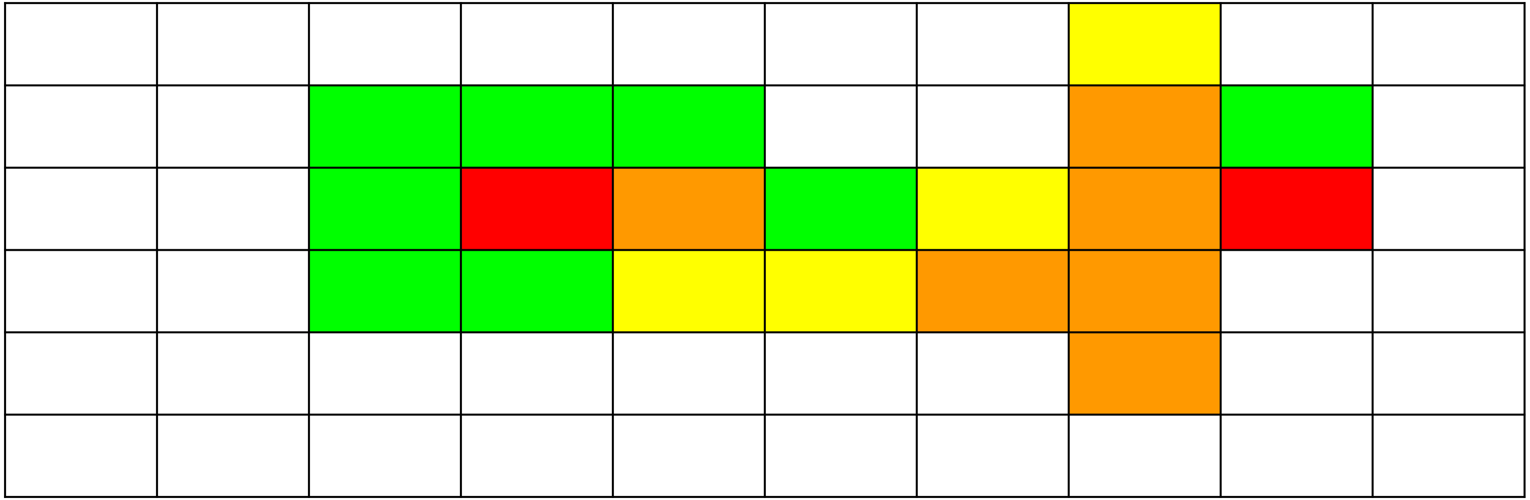

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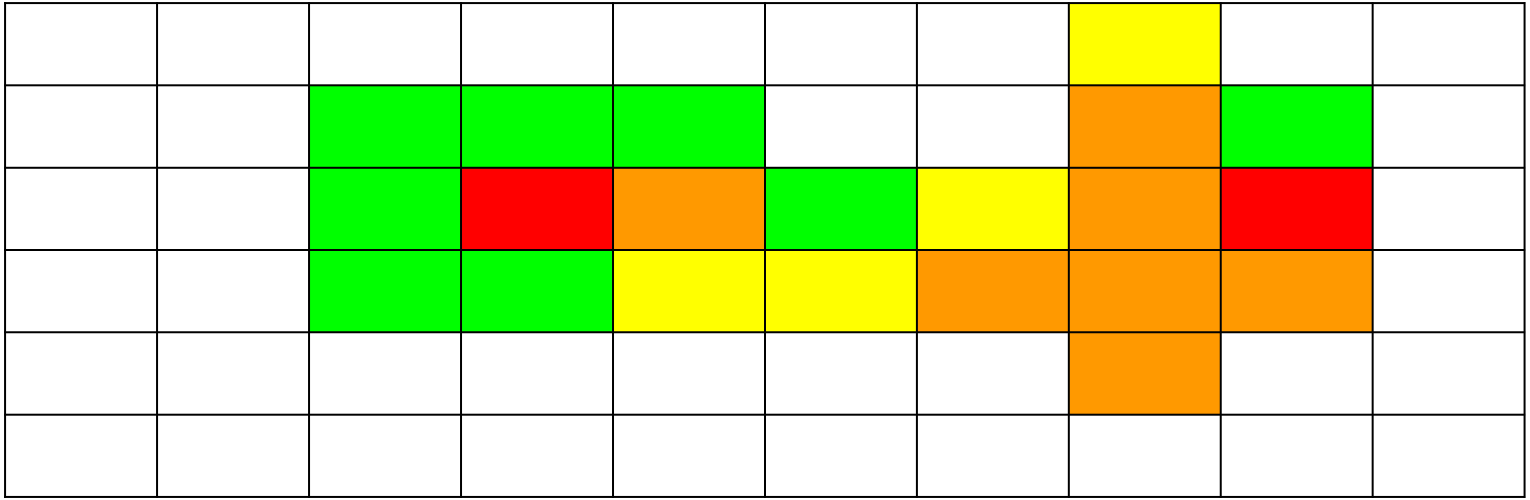

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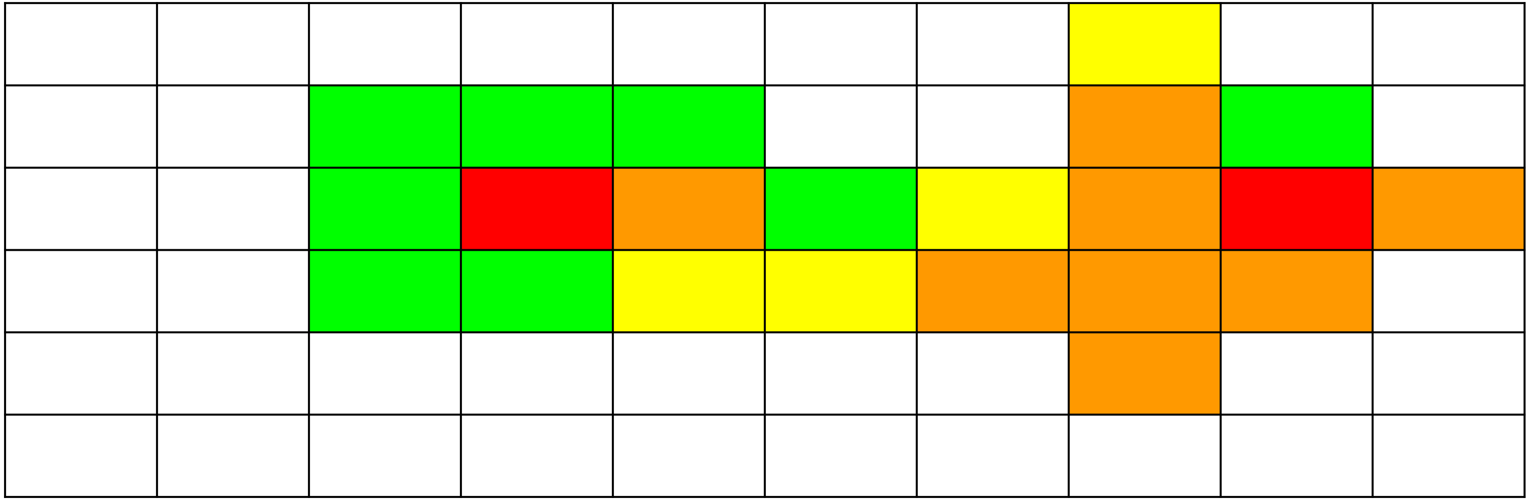

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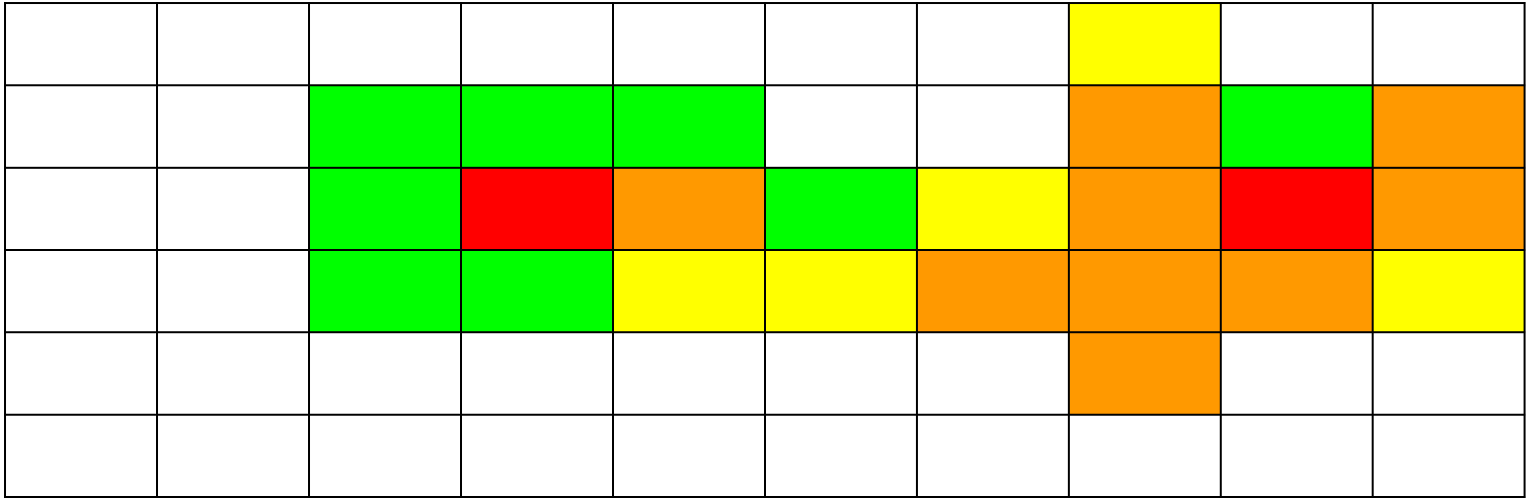

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- We know P (color | distance)
  - Example:

P(red | 3)	P(orange | 3)	P(yellow | 3)	P(green | 3)
0.05	0.15	0.5	0.3

Uncertainty

- General situation:
  - Observed variables (evidence)
    Agent knows certain things about the state of the world
    (e.g., sensor readings or symptoms)
  - Unobserved variables
    Agent needs to reason about other aspects
    (e.g. where an object is or what disease is present)
  - Model
    Agent knows something about how the known variables relate to the unknown variables
- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge
- Assume model is given (in this chapter)

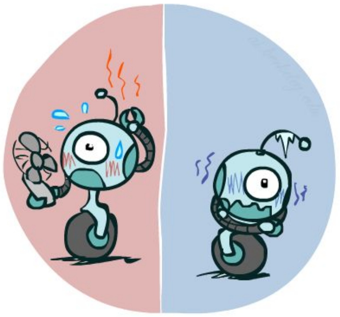
Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters
- Like variables in a CSP, random variables have domains
  - R in {true, false} (often write as {+r, -r})
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), …}

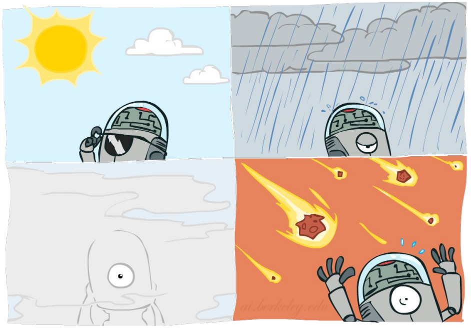
Probability Distributions

- Associate a probability with each value

Temperature	Weather



T	P(T)
hot	0.5
cold	0.5



W	P(W)
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions
  - A distribution is a table of probabilities of values
- A probability (lower case value) is a single number
  - P(W = rain) = 0.1
- The following must be true

$$\forall x \;\; P(X = x) \geq 0 \qquad\qquad \sum_x P(X = x) = 1$$

- Shorthand notation (ok if all domain entries are unique)
  - P(hot) = P(T = hot)
  - P(cold) = P(T = cold)
  - P(rain) = P(W = rain)

W	P(W)
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

T	P(T)
hot	0.5
cold	0.5

Joint Distributions

- A joint distribution over a set of random variables: $X_1, X_2, \ldots, X_n$ specifies a real number for each assignment (or outcome)
  - $P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$
  - $P(x_1, x_2, \ldots, x_n)$
- Must obey:

$$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

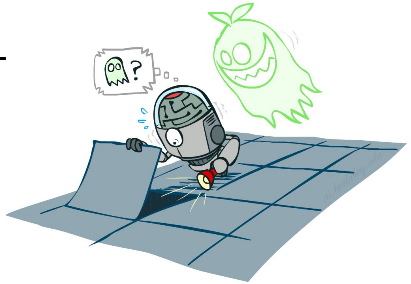
- Size of distribution if n variables with domain sizes d?
  - For all but the smallest distributions, impractical to write out!

Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called outcomes
  - Joint distributions: say whether assignments (outcomes) are likely
  - Normalized: sum to 1.0
  - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: state whether assignments are possible
  - Ideally: only certain variables directly interact

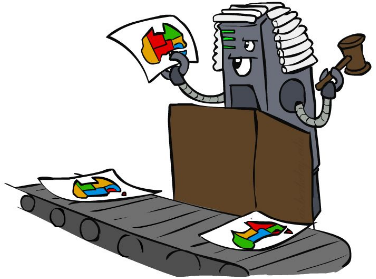
Distribution over T,W

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3



Constraint over T,W

T	W	
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T



Events

- An event is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot and sunny
  - Probability that it's hot
  - Probability that it's hot or sunny
- Typically, the events we care about are partial assignments, like P(T = hot)

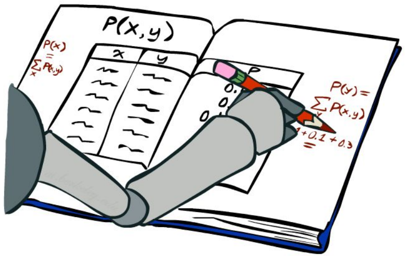
T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Events

- P(+x, +y) =
- P(+x) =
- P(-y OR +x) =

X	Y	P(X, Y)
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions



- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

T	W	P(T, W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

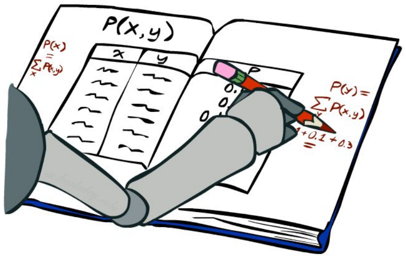
$$\longrightarrow P(t) = \sum_s P(t, s) \longrightarrow$$

T	P(T)
hot	0.5
cold	0.5

$$\longrightarrow P(s) = \sum_t P(t, s) \longrightarrow$$

W	P(W)
sun	0.6
rain	0.4

Quiz: Marginal Distributions



X	Y	P(X, Y)
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$\longrightarrow P(x) = \sum_y P(x,y) \longrightarrow$$

X	P(X)
+x	
-x	

$$\longrightarrow P(y) = \sum_x P(x,y) \longrightarrow$$

Y	P(Y)
+y	
-y	

Conditional Probabilities

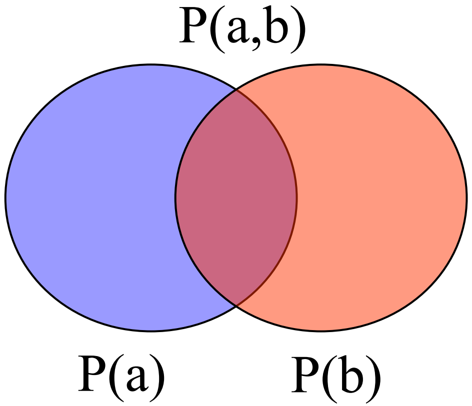
- A simple relation between joint and conditional probabilities
  - Important!
    - If we measure something, what does that tell us about the things we don't measure?

The probability of having A=a given B=b

The probability of having A=a and B=b

$$P(a|b) = \frac{P(a,b)}{P(b)}$$

The probability of having B=b



Conditional Probabilities - Example

$$P(a|b) = \frac{P(a,b)}{P(b)}$$

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = ???$$

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$

$$= 0.2 + 0.3 = 0.5$$

Quiz: Conditional Probabilities

$$P(a|b) = \frac{P(a,b)}{P(b)}$$

X	Y	P(X,Y)
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

P(+x | +y) =

P(-x | +y) =

P(-y | +x) =

Conditional Distributions

$$P(a|b) = \frac{P(a,b)}{P(b)}$$

- Conditional distributions are probability distributions over some variables given fixed values of others

Set of tables!

$P(W|T)$

Conditional Distributions

W	P(W|T = hot)
sun	0.8
rain	0.2

W	P(W|T = cold)
sun	0.4
rain	0.6

$\longleftarrow$

Joint Distribution

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

- What we have done so far:

Joint Distribution

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

Conditional Distribution

W	P(W|T = c)
sun	0.4
rain	0.6

Normalization Trick

Joint Distribution

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$$P(W = s|T = c) =$$
$$P(W = r|T = c) =$$

T	W	P(c, W)
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

Conditional Distribution

W	P(W|T = c)
sun	0.4
rain	0.6

Quiz: Normalization Trick

X	Y	P(X, Y)
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence →

P(X| -y) =

NORMALIZE the selection (make it sum to one) →

To Normalize

Sum to one

- (Dictionary) To bring or restore to a normal condition
- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

W	P
sun	0.2
rain	0.3

Normalize

Z = 0.5

W	P
sun	0.4
rain	0.6

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize

Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)
- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's beliefs given the evidence
- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes beliefs to be updated

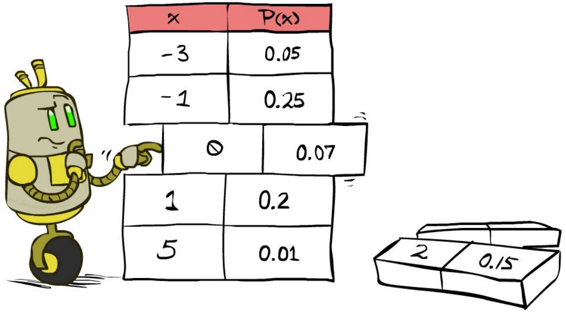
Inference by Enumeration

- General case
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query variable: Q
  - Hidden variables: $H_1 \dots H_r$

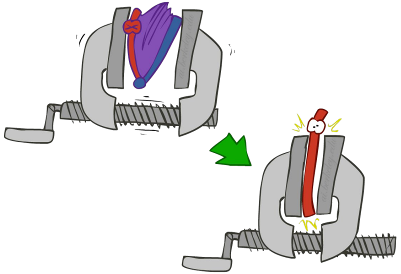
$\left.\begin{matrix} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{matrix}\right\} X_1, X_2, \dots X_n$ *All variables*

We want: $P(Q|e_1 \dots e_k)$

Step 1: Select the entries consistent with the evidence



Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} P(\underbrace{Q, h_1 \dots h_r, e_1 \dots e_k}_{X_1, X_2, \dots X_n})$$

Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration - Example

- P(W) = ?
  - Q = W, E = ∅, H = {S, T}

1

S	T	W	P(S, T, W)
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

2

W	P(W)
sun	0.65
rain	0.35

3

W	P(W)
sun	0.65
rain	0.35

Inference by Enumeration - Example

- P(W | winter) = ?
  - Q = W, E = S, H = T

2

W	P(W)
sun	0.25
rain	0.25

3

W	P(W)
sun	0.5
rain	0.5

	S	T	W	P(S, T, W)
	summer	hot	sun	0.30
	summer	hot	rain	0.05
	summer	cold	sun	0.10
	summer	cold	rain	0.05
1	winter	hot	sun	0.10
	winter	hot	rain	0.05
	winter	cold	sun	0.15
	winter	cold	rain	0.20

Inference by Enumeration - Example

- P(W | winter, hot) = ?
  - Q = W, E = S, T, H = ∅

2

W	P(W)
sun	0.10
rain	0.05

3

W	P(W)
sun	0.67
rain	0.33

1

S	T	W	P(S, T, W)
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

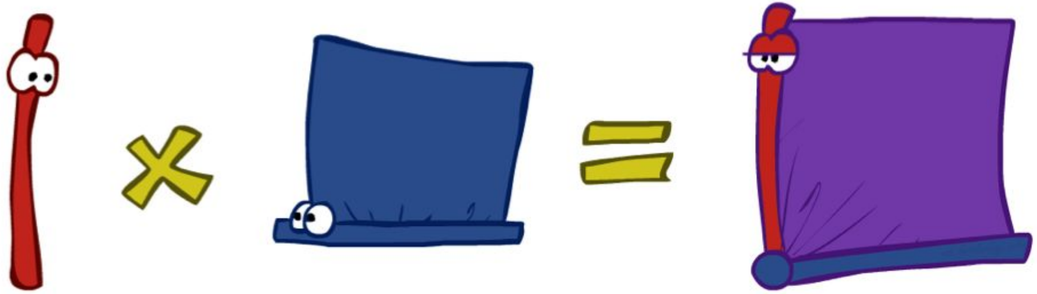
- Worst-case time complexity $O(d^n)$
- Space complexity $O(d^n)$ to store the joint distribution

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(x|y) = \frac{P(x,y)}{P(y)} \quad \Longleftrightarrow \quad P(y)P(x|y) = P(x,y)$$

Definition of Conditional Probability



The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example

T	P(W)
hot	0.5
cold	0.5

W	P(W|hot)
sun	0.8
rain	0.2

W	P(W|cold)
sun	0.4
rain	0.6

W	T	P(W,T)
hot	sun	.5 * .8 = .4
hot	rain	.5 * .2 = .1
cold	sun	.5 * .4 = .2
cold	rain	.5 * .6 = .3

Quiz: Product Rule

$$P(y)P(x|y) = P(x, y)$$

W	P(W)
sun	0.8
rain	0.2

D	P(D|sun)
wet	0.1
dry	0.9

D	P(D|rain)
wet	0.7
dry	0.3

D	W	P(D,W)
wet	sun	
dry	sun	
wet	rain	
dry	rain	

$$P(x|y) = \frac{P(x,y)}{P(y)}$$

The Chain Rule

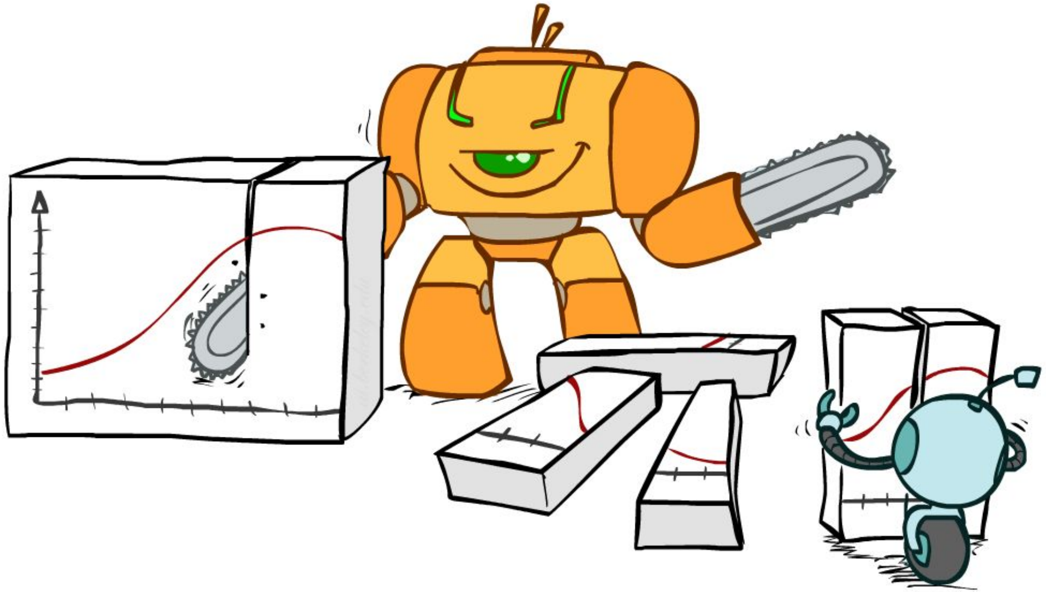
- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this always true?

Bayes Rule



Bayes' Rule

$$P(y)P(x|y) = P(x,y)$$

- Two ways to factor a joint distribution over two variables:

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later

Example: Inference with Bayes' Rule

$$P(x|y) = \frac{P(y|x)}{P(y)} P(x)$$

- Diagnostic probability from causal probability

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- M: meningitis, S: stiff neck
- Given:
  - P(+m) = 0.0001
  - P(+s | +m) = 0.8
  - P(+s) = 0.01
- P(+m | +s) = ?

Quiz: Bayes' Rule

$$P(x|y) = \frac{P(y|x)}{P(y)} P(x)$$

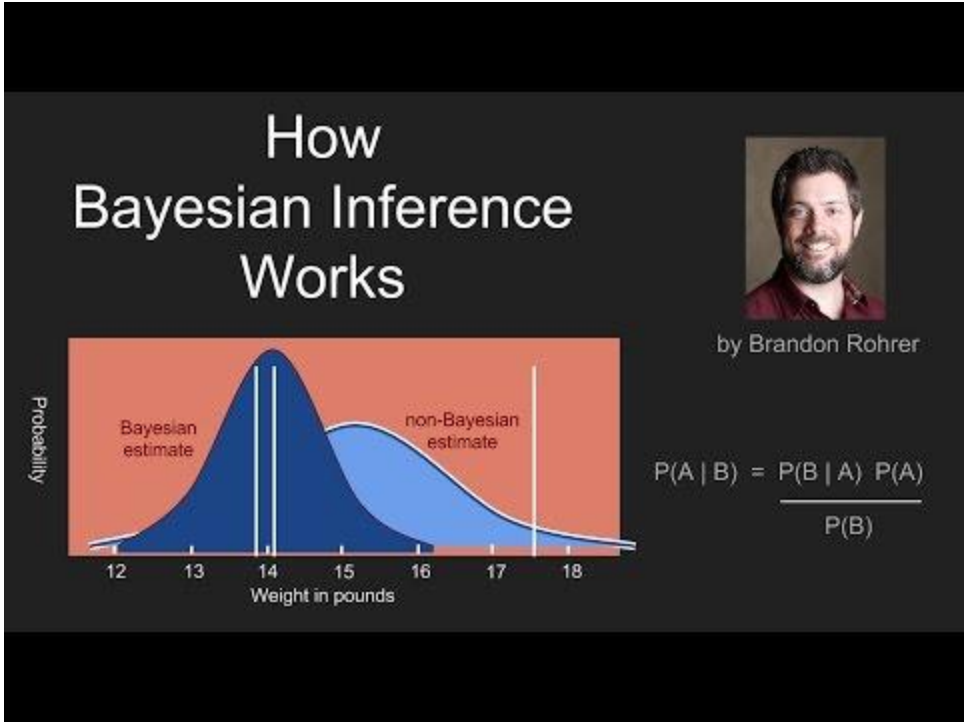
W	P(W)
sun	0.8
rain	0.2

D	W	P(D|W)
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

Distribution!

P(W | dry) = ?

More on Bayes' Rule



Ghostbusters, Revisited

- Let's say we have two distributions:
  - Prior distribution over ghost location: P(G)
    - Let this be uniform
  - Sensor reading model: P(R|G)
    - Given: we know what our sensors do
    - R = reading color measured at (1,1)
    - E.g. P(R = yellow | G=(1,1)) = 0.1
- We can calculate the posterior distribution P(G|r) over ghost locations given a reading using Bayes' rule:

$$P(g|r) \propto P(r|g)P(g)$$

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

Ghostbusters with Probability

- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green

0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02

Ghostbusters with Probability

- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green

0.04	0.04	0.04	<0.01	<0.01	<0.01	0.04	0.04	0.04	0.04
0.04	0.04	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04	0.04
0.04	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04
0.04	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04
0.04	0.04	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04	0.04

Ghostbusters with Probability

- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green

0.02	0.02	0.02	<0.01	<0.01	0.01	0.13	0.01	<0.01	<0.01
0.02	0.02	<0.01	<0.01	<0.01	<0.01	0.01	0.13	0.01	<0.01
0.02	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.01	0.13	0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.01	0.13
0.02	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.02	0.13
0.02	0.02	<0.01	<0.01	<0.01	<0.01	<0.01	0.02	0.02	0.02

Ghostbusters with Probability

- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.02	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.33	0.03	0.02
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04	0.03
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.33
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.02
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

Ghostbusters with Probability

- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.97	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

Independence

- Two variables are independent in a joint distribution if

$$P(X,Y) = P(X)P(Y)$$

$$\forall x, y \, P(x,y) = P(x)P(y)$$

$$X \perp\!\!\!\perp Y$$

  - Says the joint distribution factors into a product of two simple ones
  - Note: Usually, in practice, variables are not independent

- Can use independence as a modeling assumption
  - Simplifying assumption
  - Empirical joint distributions: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity}?
- Independence is like something from CSPs: what?

Example: Independence check

- Given: Distribution
- Task: Check for independence

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

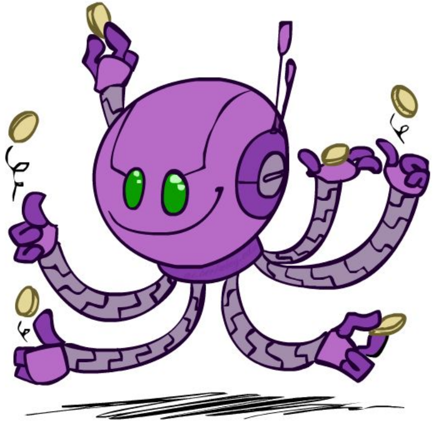
T	P(T)
hot	0.5
cold	0.5

W	P(W)
sun	0.6
rain	0.4

T	W	P = P(T) P(W)
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

Example: Independence

- N fair, independent coin flips



$P(X_1)$

H	0.5
T	0.5

$P(X_2)$

H	0.5
T	0.5

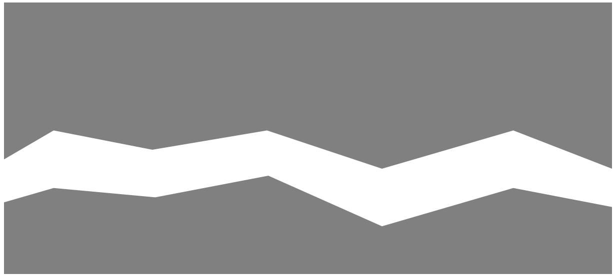
$\cdots$

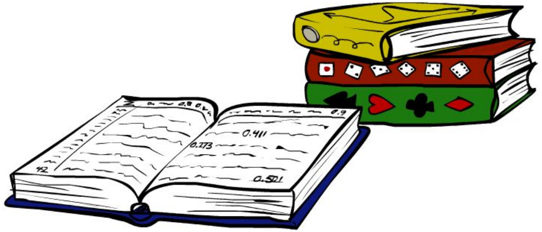
$P(X_n)$

H	0.5
T	0.5

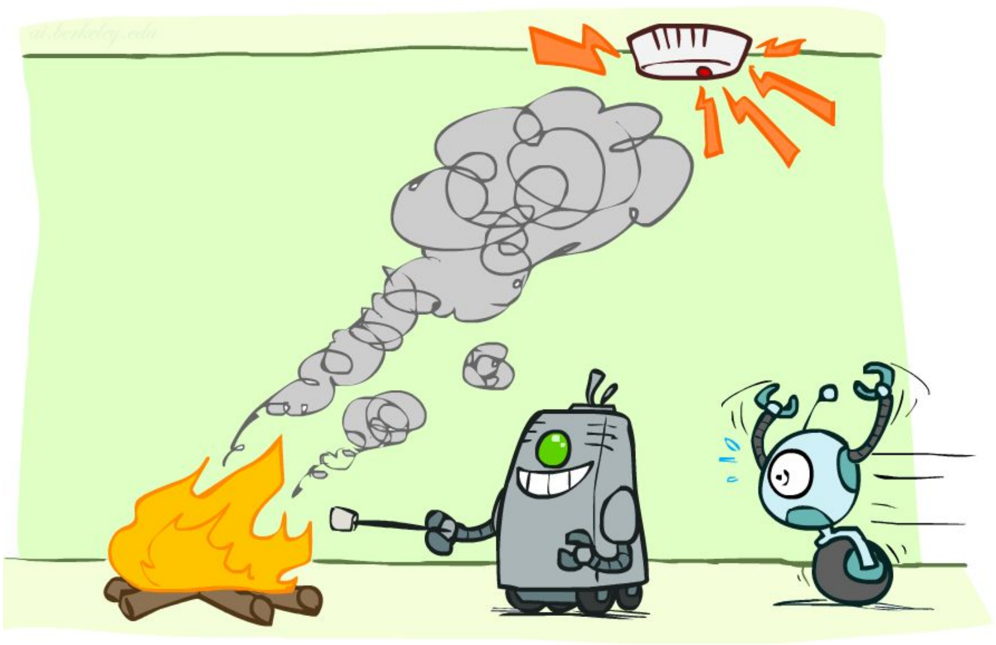
$P(X_1, X_2, \ldots X_n)$

2^n

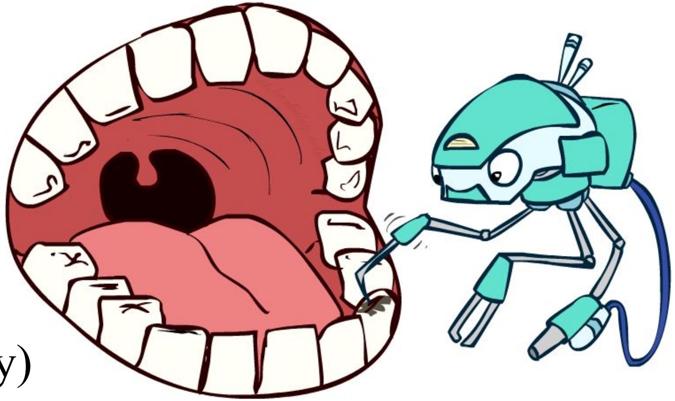




Conditional Independence



Conditional Independence



- P(Toothache, Cavity, Catch)
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)
- The same independence holds if I don't have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)
- Catch is conditionally independent of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)
- Equivalent statements:
  - P(Toothache | Catch , Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily

Conditional Independence

- Unconditional (absolute) independence very rare (why?)
- Conditional independence is our most basic and robust form of knowledge about uncertain environments

$$X \perp\!\!\!\perp Y | Z$$

- X is conditionally independent of Y given Z iff

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

- or, equivalently, iff

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

Quiz - Conditional Independence Proof

- Show that the statement

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

is equivalent to

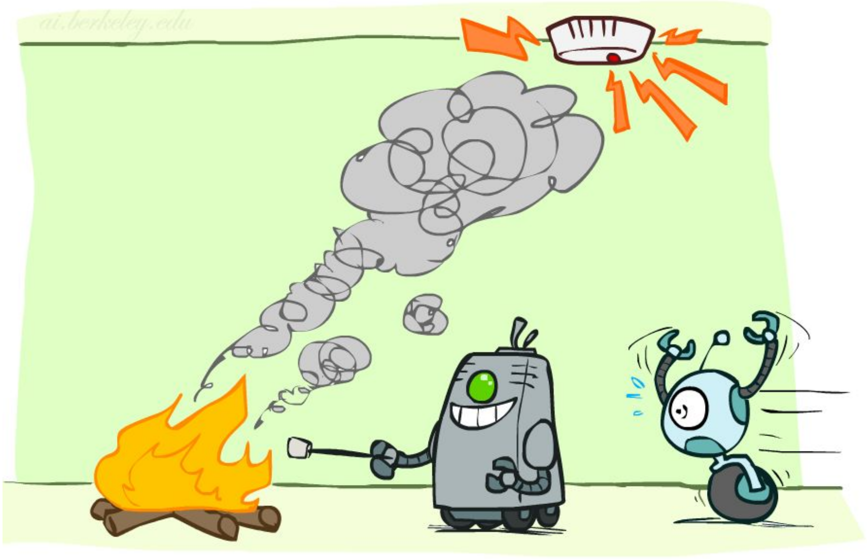
$$\forall x, y, z : P(x|z, y) = P(x|z)$$

Quiz: Conditional Independence

- What about this domain
  - Traffic (T)
  - Umbrella (U)
  - Raining (R)

Quiz: Conditional Independence

- What about this domain
  - Fire (F)
  - Smoke (S)
  - Alarm (A)



Probability Recap

- Conditional probability $P(x|y) = \dfrac{P(x,y)}{P(y)}$

- Product rule $P(x,y) = P(x|y)P(y)$

- Chain rule
$$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \end{aligned}$$

- X, Y independent if and only if: $\forall x, y : P(x,y) = P(x)P(y)$

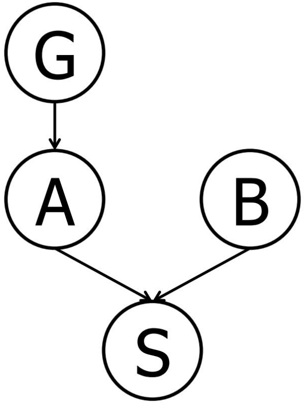
- X and Y are conditionally independent given Z if and only if:
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$
$$\forall x, y, z : P(x|z, y) = P(x|z)$$
$$X \perp\!\!\!\perp Y | Z$$

Quiz - G, S, A, B

- Suppose that a patient can have a symptom (S) that can be caused by two different diseases (A and B)
- It is known that the variation of gene G plays a big role in the manifestation of disease A
- The conditional probability tables for this situation are shown below
- Calculate the following
    - 1. P(+g, +a, +b, +s)
    - 2. P(+a)
    - 3. P(+a| + b)
    - 4. P(+a| + s, +
    - 5. P(+g| + a)
    - 6. P(+g| + b)

$\mathbb{P}(G)$	
$+g$	0.1
$-g$	0.9

$\mathbb{P}(A\|G)$		
$+g$	$+a$	1.0
$+g$	$-a$	0.0
$-g$	$+a$	0.1
$-g$	$-a$	0.9



$\mathbb{P}(B)$	
$+b$	0.4
$-b$	0.6

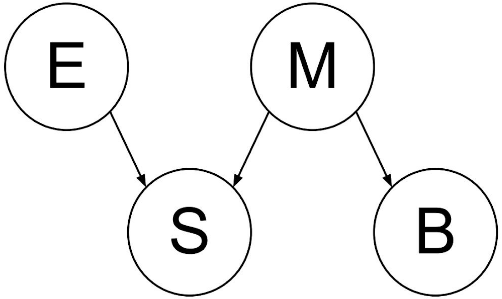
$\mathbb{P}(S\|A,B)$			
$+a$	$+b$	$+s$	1.0
$+a$	$+b$	$-s$	0.0
$+a$	$-b$	$+s$	0.9
$+a$	$-b$	$-s$	0.1
$-a$	$+b$	$+s$	0.8
$-a$	$+b$	$-s$	0.2
$-a$	$-b$	$+s$	0.1
$-a$	$-b$	$-s$	0.9

Quiz - E, S, M, B

- A smell of sulphur (S) can be caused either by rotten eggs (E) or as a sign of the doom brought by the Mayan Apocalypse (M)
- The Mayan Apocalypse also causes the oceans to boil (B)
- The conditional probability tables for this situation are shown below
- Calculate the following
  - P(−e, −s, −m, −b)
  - P(+b)
  - P(+m| + b)
  - P(+m| + s, +b, +e)
  - P(+e| + m)

$P(E)$	
$+e$	0.4
$-e$	0.6

$P(S \mid E, M)$			
$+e$	$+m$	$+s$	1.0
$+e$	$+m$	$-s$	0.0
$+e$	$-m$	$+s$	0.8
$+e$	$-m$	$-s$	0.2
$-e$	$+m$	$+s$	0.3
$-e$	$+m$	$-s$	0.7
$-e$	$-m$	$+s$	0.1
$-e$	$-m$	$-s$	0.9

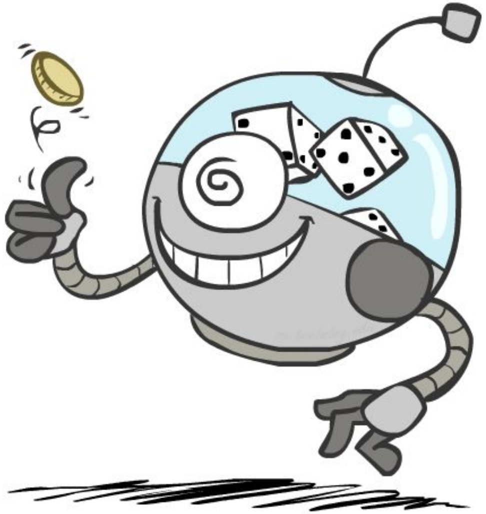


$P(M)$	
$+m$	0.1
$-m$	0.9

$P(B \mid M)$		
$+m$	$+b$	1.0
$+m$	$-b$	0.0
$-m$	$+b$	0.1
$-m$	$-b$	0.9

Refresher: Probability

- Random Variables
- Joint and Marginal Distributions
- Conditional Distribution
- Product Rule, Chain Rule, Bayes’ Rule
- Inference
- Independence



Markov Model

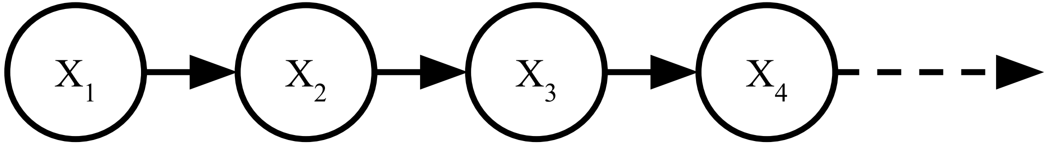
Introducing time (or space) into our models

Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Robot localization
    - From noisy readings of sensors
  - User attention
    - When is a good time to interrupt (popup-ad) a user?
  - Speech recognition
    - Inference on the acoustic sequence of words
  - Medical monitoring
    - Is there a medical emergency occurring ?
- Need to introduce time (or space) into our models

Markov Model

- Chain structure

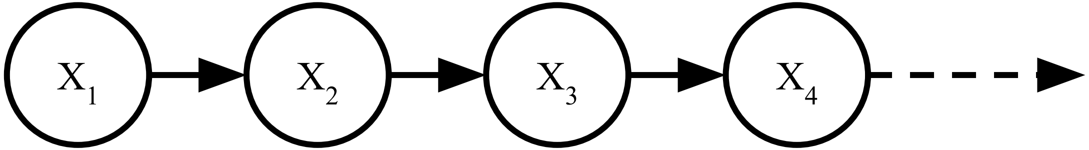


- Value of X at a given time is called the state
- We assume that future states depend only on the current state
  - not on previous states

$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
  - Stationarity assumption
    - transition probabilities **the same at all times**
  - Same as MDP transition model, but no choice of action

Joint Distribution of a Markov Model



$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Joint distribution:

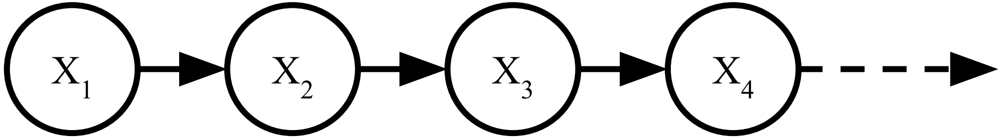
$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

- More generally:

$$\begin{aligned} P(X_1, X_2, \ldots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1}) \\ &= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1}) \end{aligned}$$

- Questions to be resolved:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

Chain Rule and Markov Models



- From the chain rule, every joint distribution over X_1, X_2, X_3, X_4 can be written as:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)P(X_4|X_1, X_2, X_3)$$

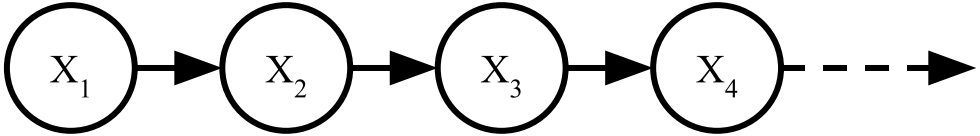
- Assuming that

$$X_3 \perp\!\!\!\perp X_1 \mid X_2 \quad \text{and} \quad X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$$

results in the expression

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

Chain Rule and Markov Models



- From the chain rule, every joint distribution over $X_1, X_2, \ldots, X_T$ can be written as:

$$P(X_1, X_2, \ldots, X_T) = P(X_1) \prod_{t=2}^{T} P(X_t | X_1, X_2, \ldots, X_{t-1})$$

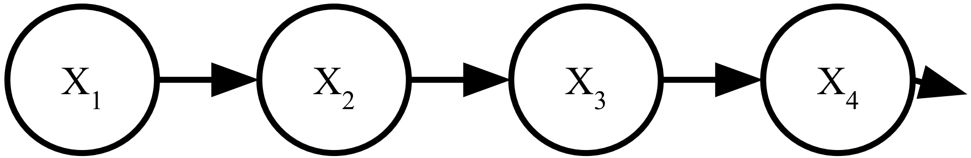
- Assuming that

$$X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$

results in the expression

$$P(X_1, X_2, \ldots, X_T) = P(X_1) \prod_{t=2}^{T} P(X_t | X_{t-1})$$

Quiz - Chain Rule and Markov Models



- We assumed $X_3 \perp\!\!\!\perp X_1 \mid X_2$ and $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$
- Do we also have $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$?

Markov Models Recap

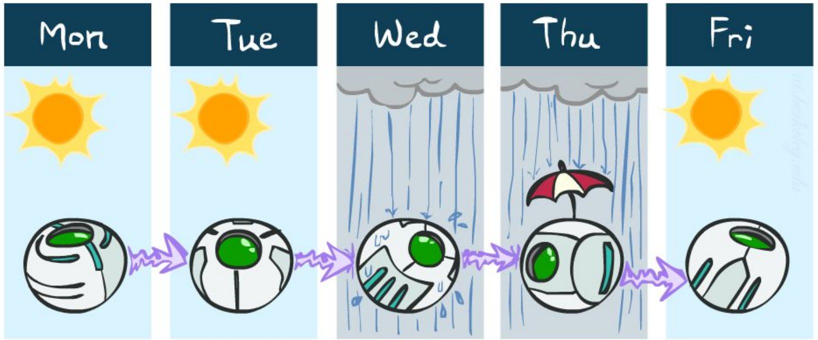
- Explicit assumption for all t: $X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$
$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Implied conditional independencies:
  - Past variables independent of future variables given the present
  - i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption: $P(X_t \mid X_{t-1})$ is the same for all t

Example Markov Chain: Weather

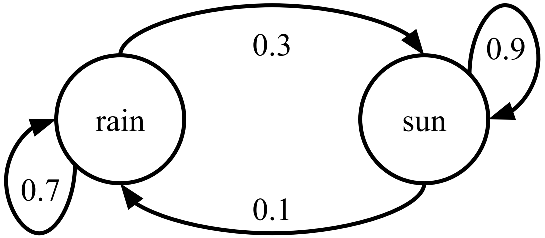
- States: X = {rain, sun}
- Initial distribution: 1.0 sun

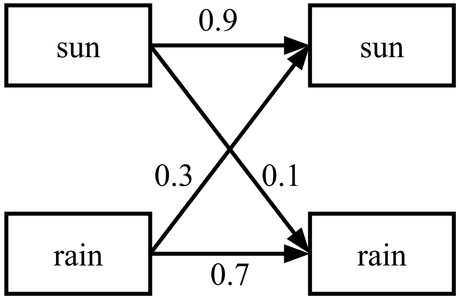


Conditional Probability Table (CPT):

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Two new ways of representing the same CPT



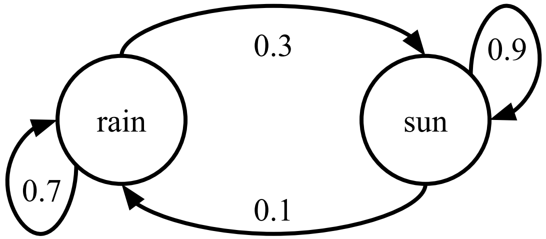


Example Markov Chain: Weather

- Initial distribution: 1.0 sun
- What is the probability distribution after one step?

Initial distribution: 1.0 sun

$P(X_2 = sun) = ?$
$P(X_2 = rain) = ?$



$P(X_2 = sun) = P(X_2 = sun \mid X_1 = sun)P(X_1 = sun) + P(X_2 = sun \mid X_1 = rain)P(X_1 = rain)$

$P(X_2 = sun) = .9 * 1.0 + 0.3 * 0.0 = 0.9$

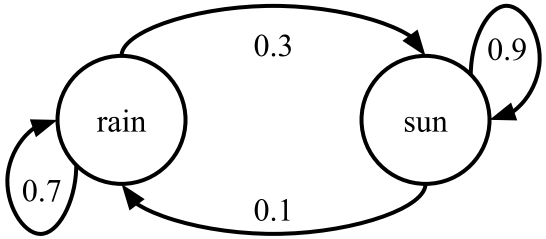
$P(X_2 = rain) = P(X_2 = rain \mid X_1 = sun)P(X_1 = sun) + P(X_2 = rain \mid X_1 = rain)P(X_1 = rain)$

$P(X_2 = rain) = .1 * 1.0 + 0.7 * 0.0 = 0.1$

Quiz - Markov Chain: Weather

- Initial distribution: 1.0 sun
- What is the probability distribution after two steps?

Distribution after one step:
0.9 sun
0.1 rain



$P(X_3 = \text{sun}) = ?$

$P(X_3 = \text{rain}) = ?$

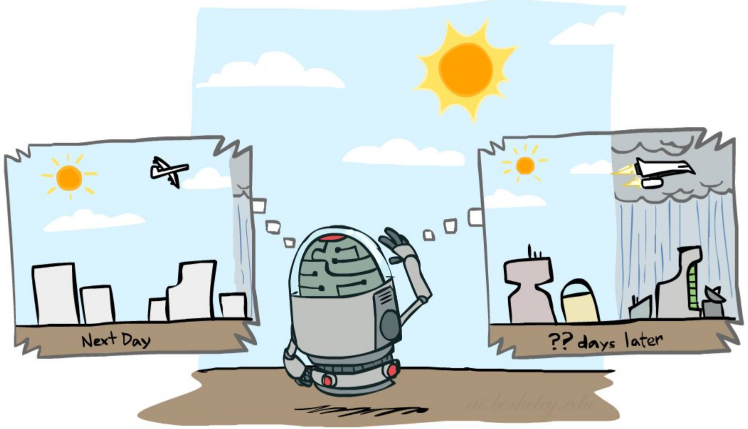
Mini-Forward Algorithm

- Question: What's P(X) on some day t?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$

$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$
$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$



Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\underset{P(X_1)}{\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle} \quad \underset{P(X_4)}{\left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle} \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

- From initial observation of rain

$$\underset{P(X_1)}{\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle} \quad \underset{P(X_4)}{\left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle} \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

- From initial observation of rain

$$\underset{P(X_1)}{\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle} \quad \ldots \quad \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time
  - The distribution we end up in is independent of the initial distribution

- Stationary distribution:
  - The distribution we end up with is called the stationary distribution of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

Example: Stationary Distributions

- Question: What's P(X) at time t = infinity?

$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

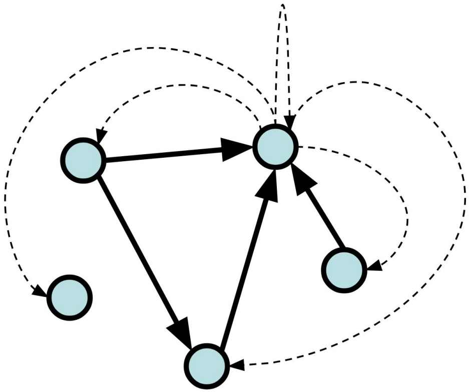
$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

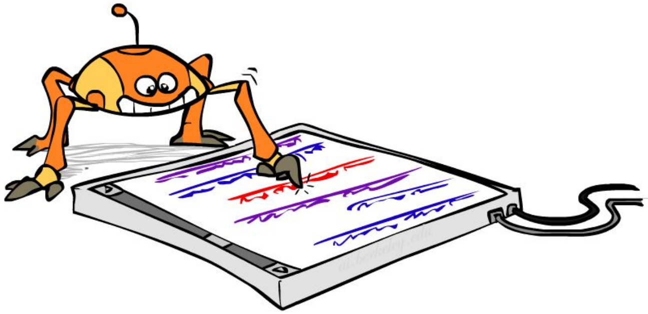
X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

$$P_\infty(sun) + P_\infty(rain) = 1 \Rightarrow \begin{aligned} P_\infty(sun) &= 3/4 \\ P_\infty(rain) &= 1/4 \end{aligned}$$

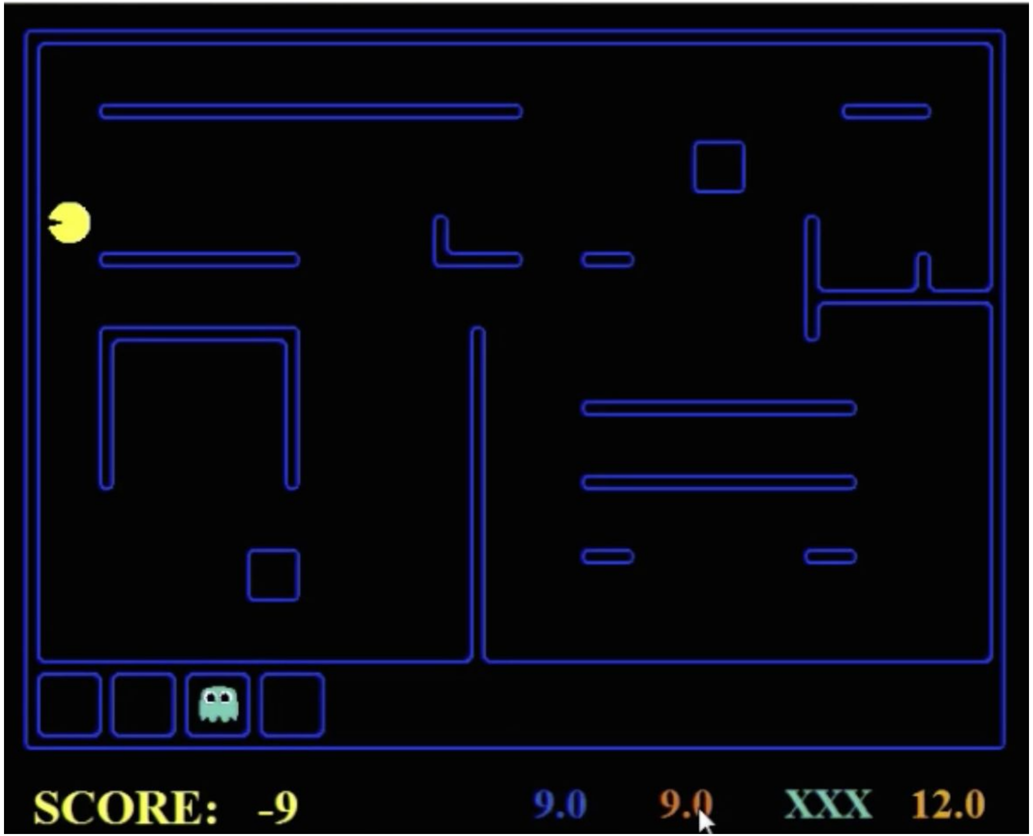
Application of Stationary Distribution: Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
- Transitions:
  - With prob. c, uniform jump to a random page (dotted lines, not all shown)
  - With prob. 1-c, follow a random outlink (solid lines)
- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the hku.hk page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



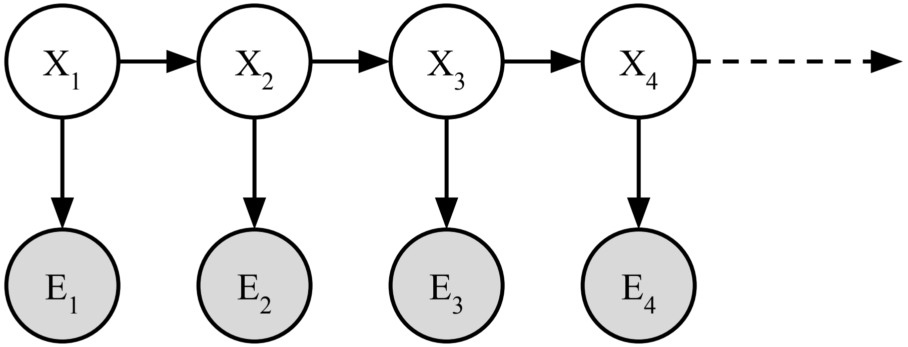


Next time: Hidden Markov Models

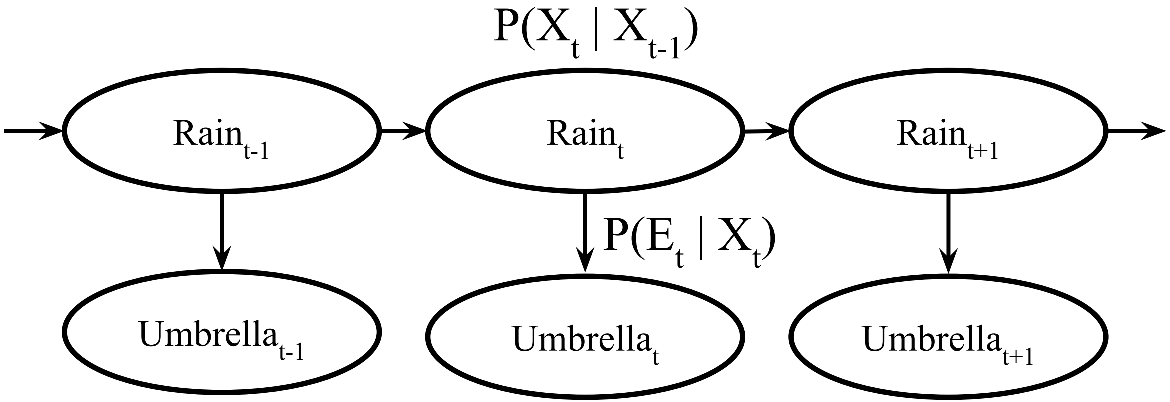

SCORE: -9
9.0
9.0
XXX
12.0

Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step



Example: Weather HMM



- An HMM is defined by
  - Initial distribution $P(X_1)$
  - Transitions $P(X_t \mid X_{t-1})$
  - Emissions $P(E_t \mid X_t)$

R_t	R_{t+1}	$P(R_{t+1} \mid R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8