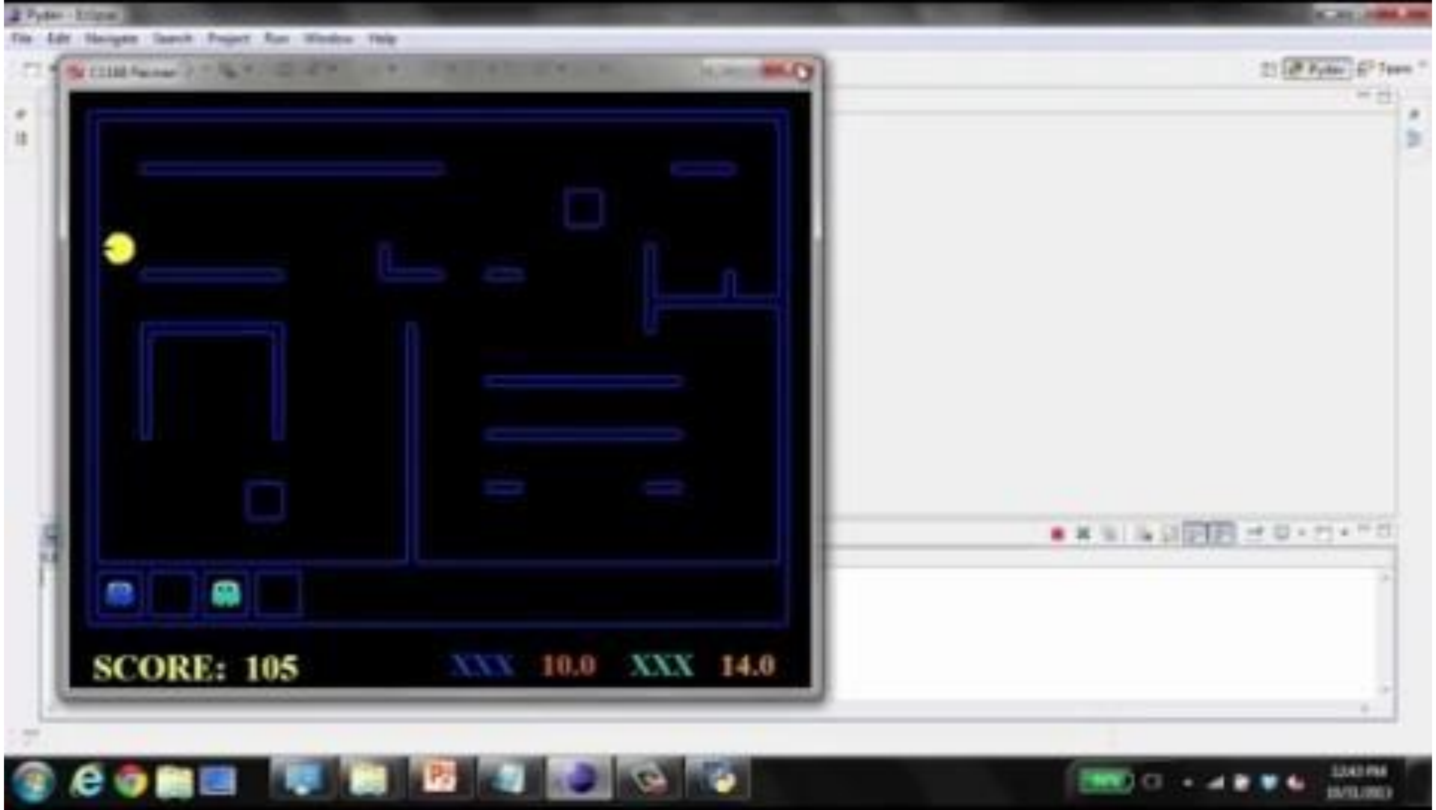
Most materials in this chapter are based on materials from http://aima.cs.berkeley.edu/ and http://ai.berkeley.edu

# Chapter 5
# Hidden Markov Models

## COMP 3270
## Artificial Intelligence

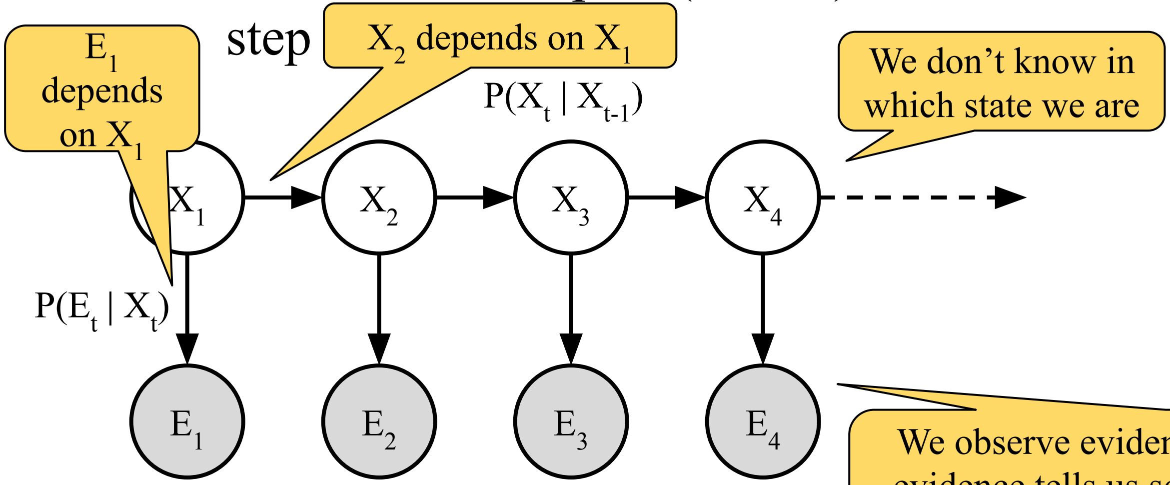
Dirk Schnieders

# Hidden Markov Models


SCORE: 105
XXX 10.0
XXX 14.0

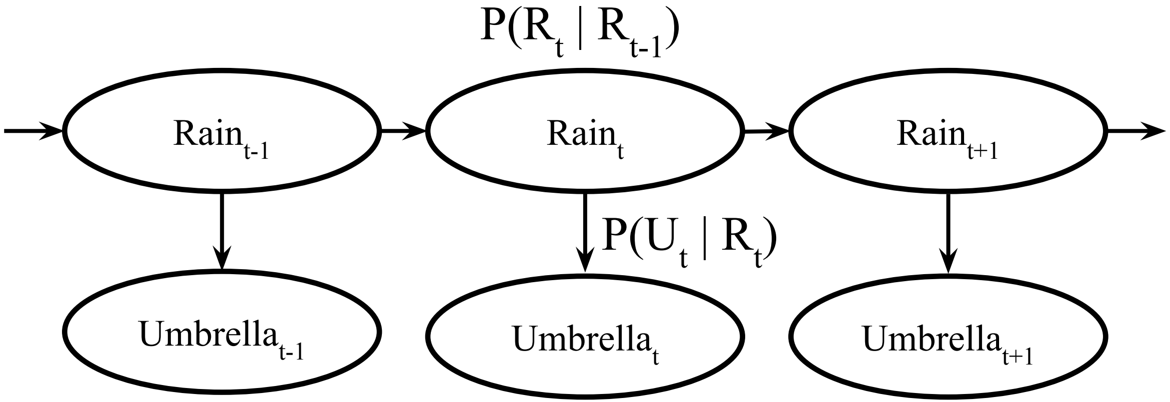
# Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step




We observe evidence at every time step. The evidence tells us something about the state at that time

# Example: Weather HMM




| $R_t$ | $R_{t+1}$ | $P(R_{t+1}\|R_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $R_t$ | $U_t$ | $P(U_t\|R_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

- An HMM is defined by
  - Initial distribution $P(R_1)$
  - Transitions $P(R_t \mid R_{t-1})$
  - Emissions $P(U_t \mid R_t)$

# Example: Ghostbusters HMM

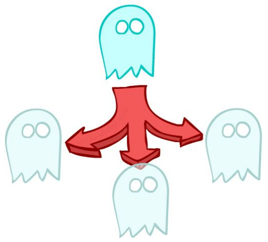
- $P(X_1)$ = uniform

Where is the ghost?

| | | | | |
|---|---|---|---|---|
| | | ⋮ | | |
| | 1/9 | 1/9 | 1/9 | |
| … | 1/9 | 1/9 | 1/9 | … |
| | 1/9 | 1/9 | 1/9 | |
| | | ⋮ | | |

$P(X_1)$

- $P(X|X')$ = usually move clockwise, but sometimes move in a random direction or stay in place

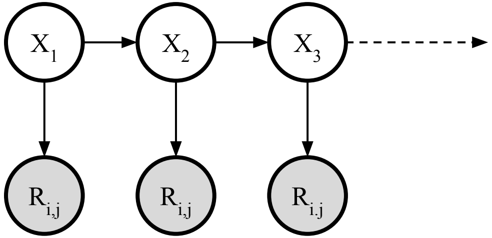


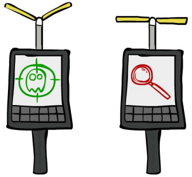
| | | |
|---|---|---|
| 1/6 | 1/6 | 1/2 |
| 0 | 1/6 | 0 |
| 0 | 0 | 0 |

$P(X|X'=\langle 1,2\rangle)$

X: next state
X': current state

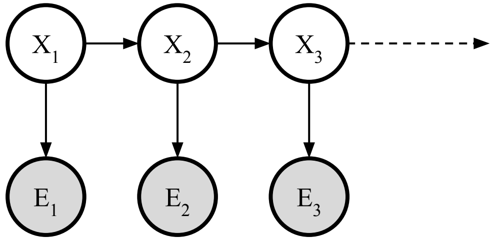
- $P(R_{ij}|X)$ = same sensor model as before
  - red means close, green means far away



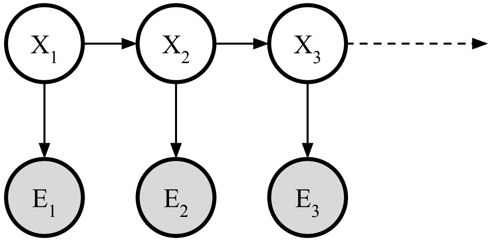

0.03 <0.01 0.02 <0.01 <0.01 <0.01 0.03 0.03 <0.01 <0.01
0.03 0.29 0.03 <0.01 <0.01 <0.01 0.03 <0.01 <0.01 <0.01
0.03 0.29 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01
<0.01 <0.01 0.03 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01
<0.01 <0.01 <0.01 0.03 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01
<0.01 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01 <0.01
BUST
TIME+1

# Joint Distribution of an HMM




- Joint distribution:
  - $P(X_1, E_1, X_2, E_2, X_3, E_3) = ?$

# Joint Distribution of an HMM



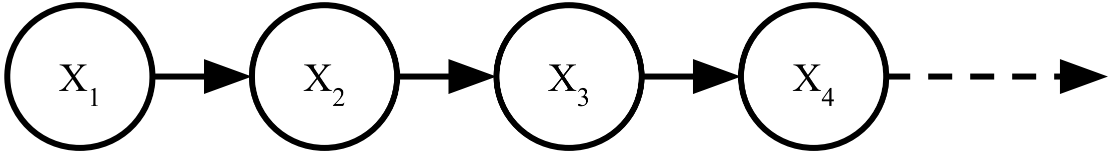

- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

- More generally:

$$P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t)$$

# Joint Distribution of a Markov Model




$P(X_1)$ $\quad$ $P(X_t|X_{t-1})$

- Joint distribution:

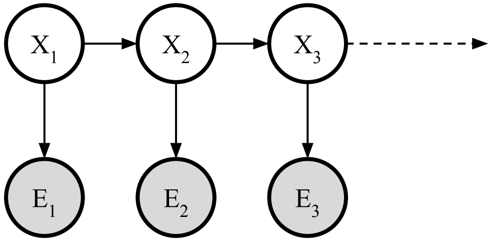
$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

- More generally:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$
$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Questions to be resolved:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Quiz 1: Joint Distribution of an HMM



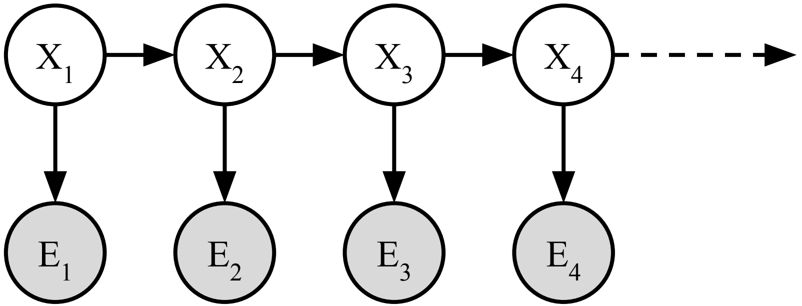

- Joint distribution:

Why?

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

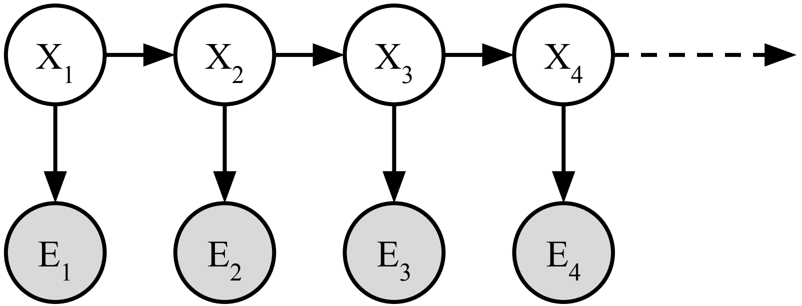
# Conditional Independence

- HMMs have two important independence properties
  - Hidden Markov process: future depends on past via present
  - Current observation independent of all else given current state



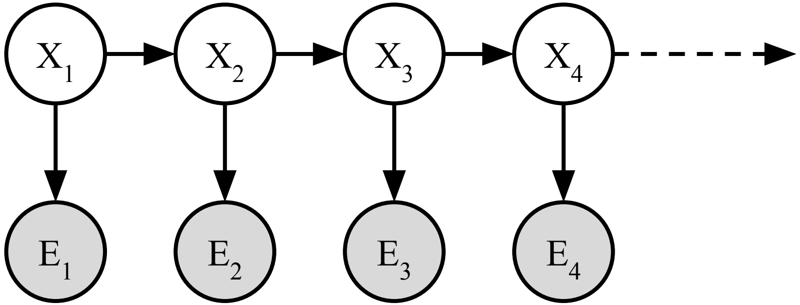
# Quiz 2: Conditional Independence

- HMMs have two important independence properties
  - Markov hidden process: future depends on past via present
  - Current observation independent of all else given current state




- Quiz
  - Does this mean that evidence variables are guaranteed to be independent?

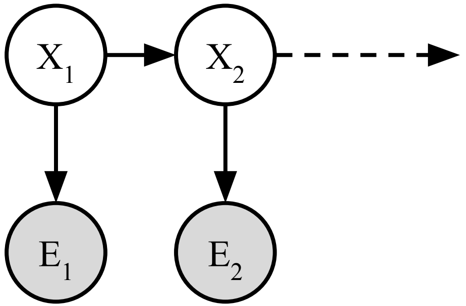
# Implied Conditional Independencies




- Many implied conditional independencies, e.g.,

$$E_1 \perp\!\!\!\perp X_2, E_2, X_3, E_3 \mid X_1$$

# Quiz 3: Joint Distribution of an HMM



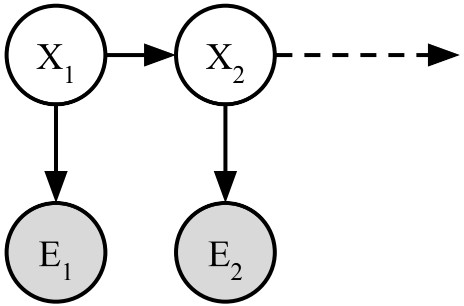

| $X_1$ | $P(X_1)$ |
|---|---|
| $+x_1$ | 0.7 |
| $-x_1$ | 0.3 |

| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| $+x_t$ | $+x_{t+1}$ | 0.2 |
| $+x_t$ | $-x_{t+1}$ | 0.8 |
| $-x_t$ | $+x_{t+1}$ | 0.6 |
| $-x_t$ | $-x_{t+1}$ | 0.4 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| $+x_t$ | $+e_t$ | 0.5 |
| $+x_t$ | $-e_t$ | 0.5 |
| $-x_t$ | $+e_t$ | 0.9 |
| $-x_t$ | $-e_t$ | 0.1 |

- $P(X_1, X_2, E_1{=}{+}e_1, E_2{=}{-}e_2) = ?$

# Quiz 4: Joint Distribution of an HMM




| $X_1$ | $P(X_1)$ |
|---|---|
| $+x_1$ | 0.7 |
| $-x_1$ | 0.3 |

| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| $+x_t$ | $+x_{t+1}$ | 0.2 |
| $+x_t$ | $-x_{t+1}$ | 0.8 |
| $-x_t$ | $+x_{t+1}$ | 0.6 |
| $-x_t$ | $-x_{t+1}$ | 0.4 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| $+x_t$ | $+e_t$ | 0.5 |
| $+x_t$ | $-e_t$ | 0.5 |
| $-x_t$ | $+e_t$ | 0.9 |
| $-x_t$ | $-e_t$ | 0.1 |

- $P(X_2, E_1=+e_1, E_2=-e_2) = ?$

# HMM Examples

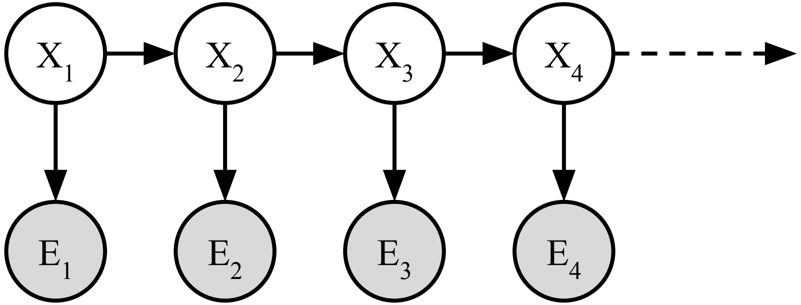
- Speech recognition
    - Observations are acoustic signals (continuous valued)
    - States are specific positions in specific words (so, tens of thousands)
- Machine translation
    - Observations are words (tens of thousands)
    - States are translation options
- Robot tracking
    - Observations are range readings (continuous)
    - States are positions on a map (continuous)

# Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time

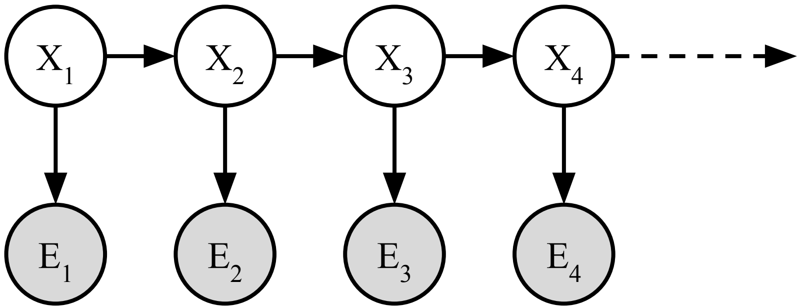
New notation!

Keep track of what you believe about a variable X as evidence comes in and time passes

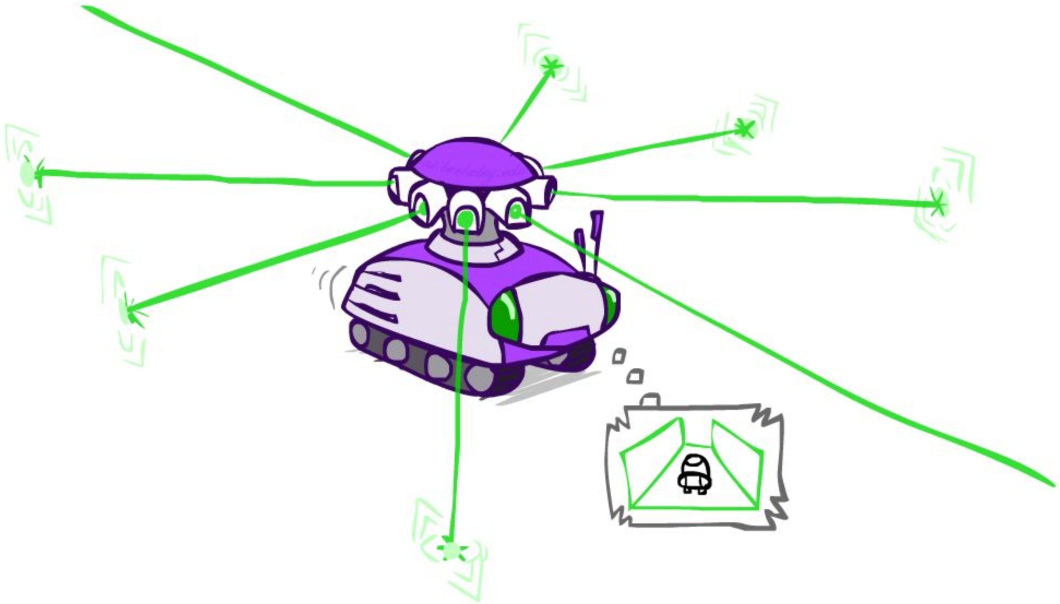


# Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time
- We start with $B_1(X)$ in an initial setting, usually uniform
- As time passes, or we get observations, we update $B(X)$
- The Kalman filter was invented in the 60’s and first implemented as a method of trajectory estimation for the Apollo program



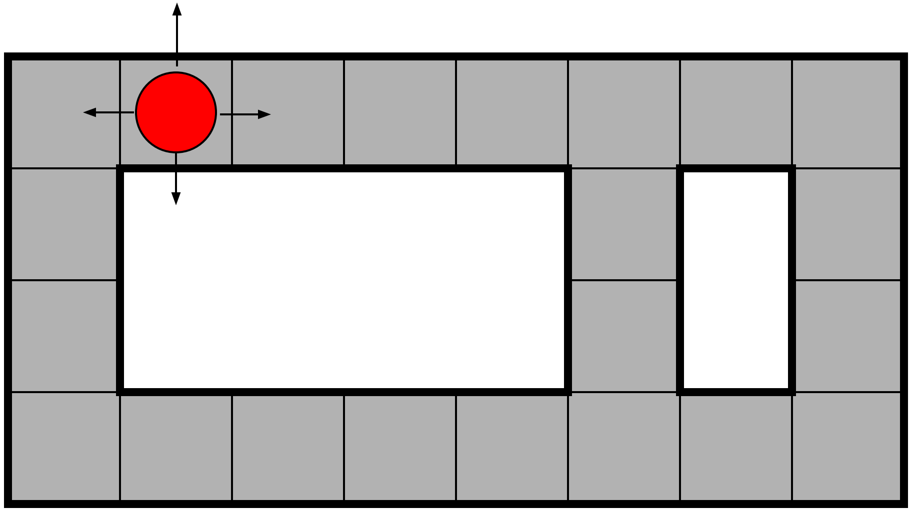
# Example: Robot Localization



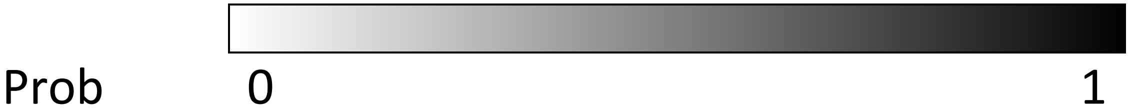
Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

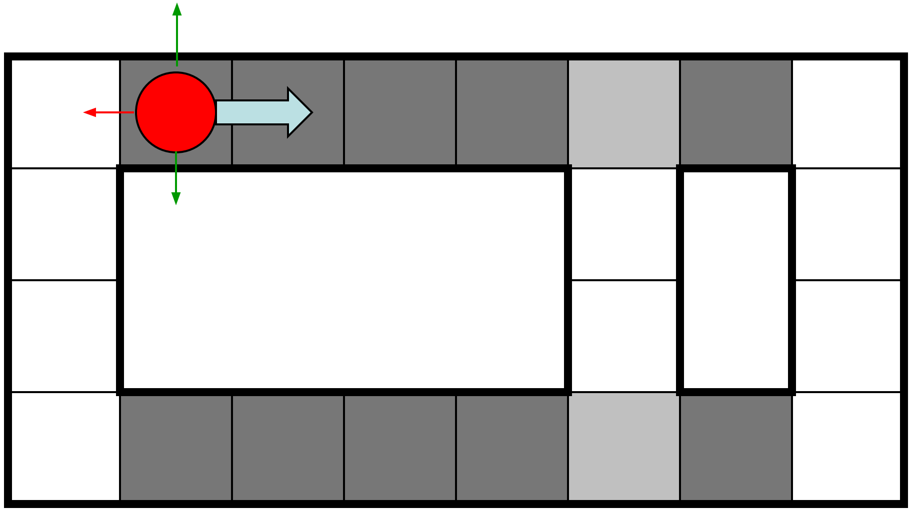
# Example: Robot Localization



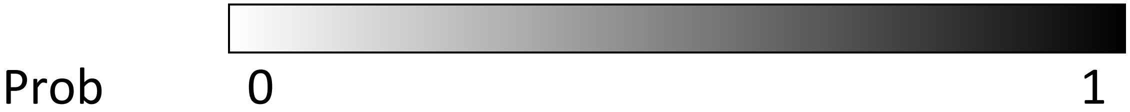
t=0



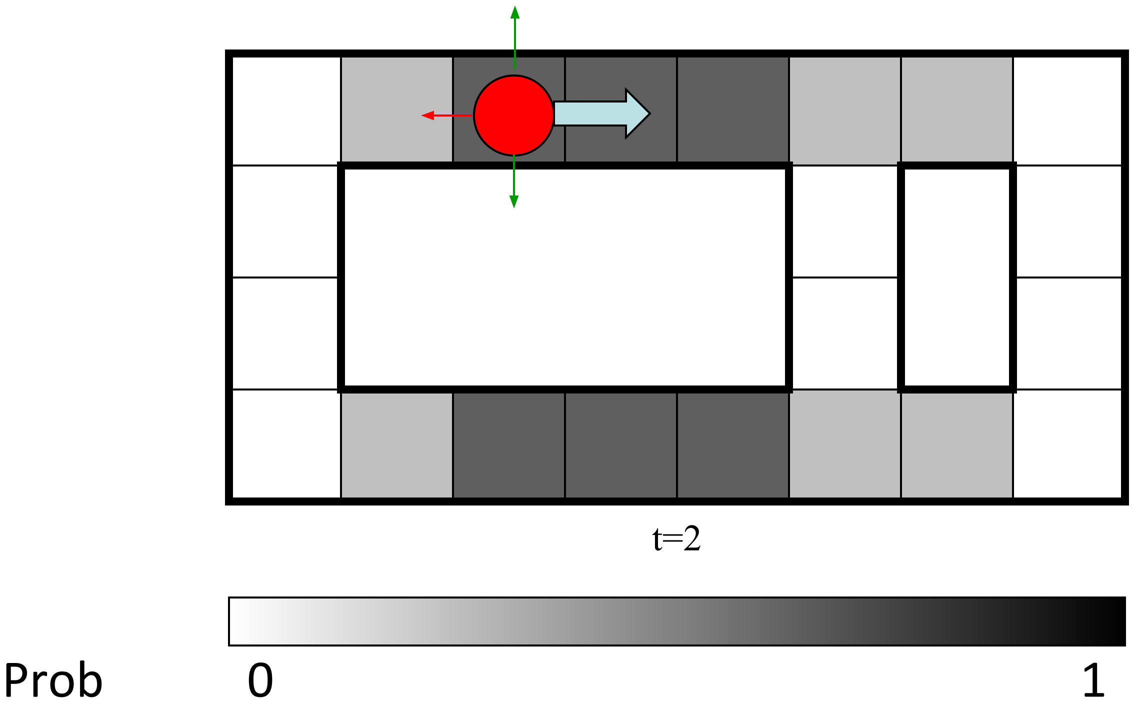
# Example: Robot Localization



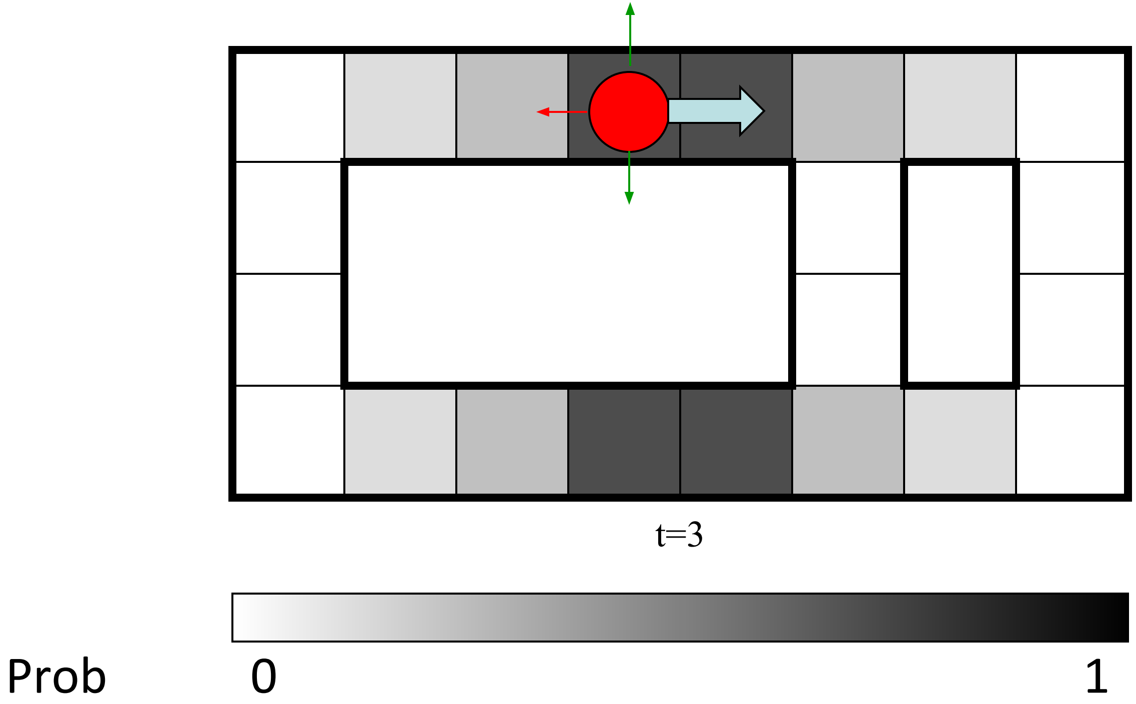
t=1



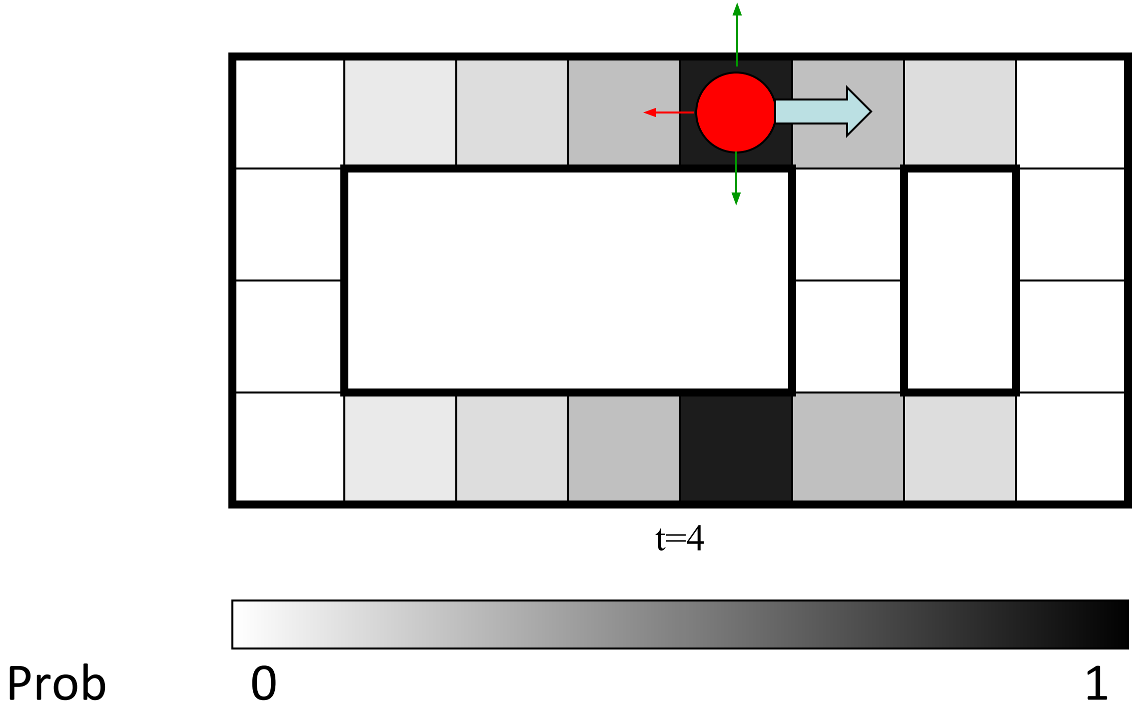
# Example: Robot Localization



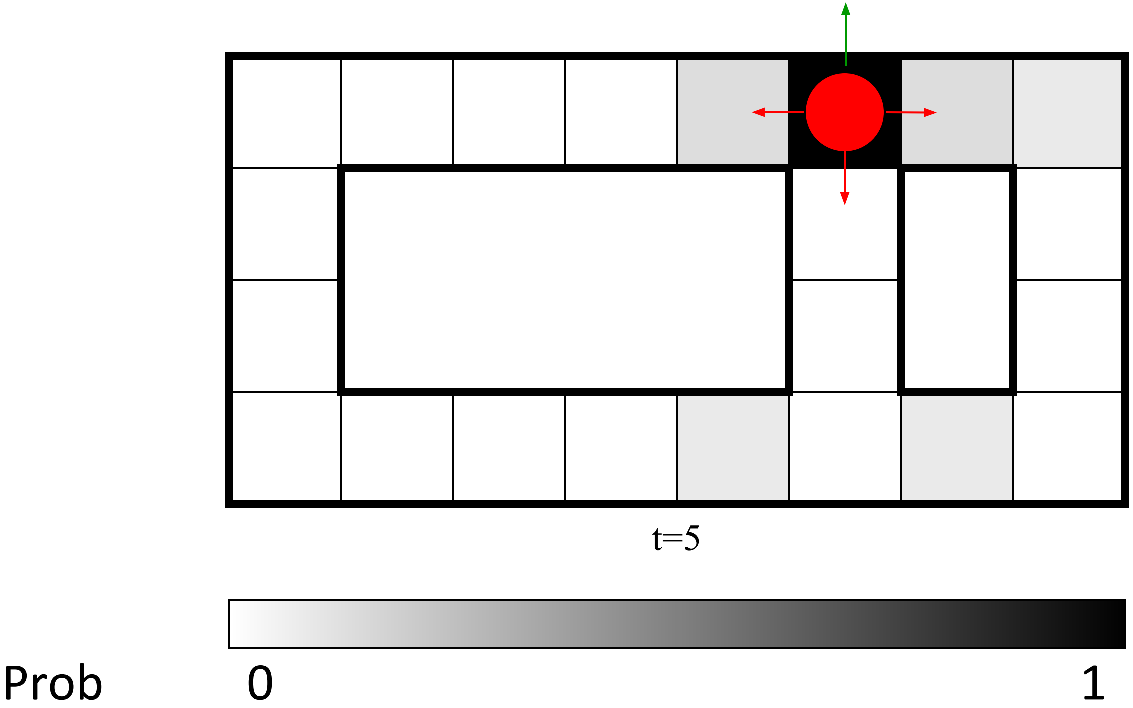
# Example: Robot Localization



# Example: Robot Localization

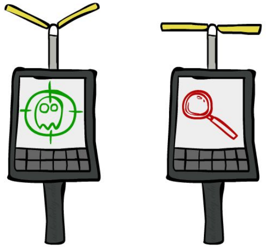


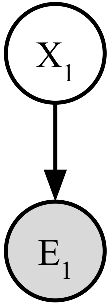
# Example: Robot Localization



# Inference: Base Cases

Seeing evidence:





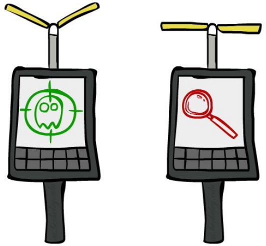

$P(X_1)$

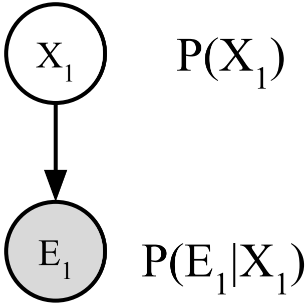
$P(E_1|X_1)$

$P(X_1|e_1) = ?$

# Inference: Base Cases

Seeing evidence:





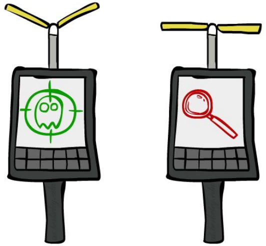

$$P(X_1|e_1)$$

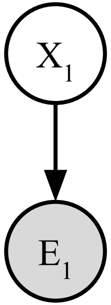
$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$

Definition of conditional probability

# Inference: Base Cases

Seeing evidence:






$P(X_1)$

$P(E_1|X_1)$

$$P(X_1|e_1)$$

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$

We don't know this one

We know this one because $P(x_1,e_1) = P(x_1)P(e_1|x_1)$

# Ghostbusters, Revisited

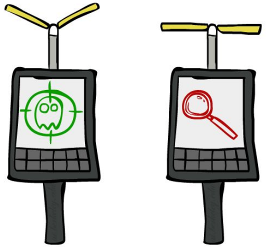
- Let's say we have two distributions:
  - Prior distribution over ghost location: P(G)
    - Let this be uniform
  - Sensor reading model: P(R|G)
    - Given: we know what our sensors do
    - R = reading color measured at (1,1)
    - E.g. P(R = yellow | G=(1,1)) = 0.1
- We can calculate the posterior distribution P(G|r) over ghost locations given a reading using Bayes' rule:

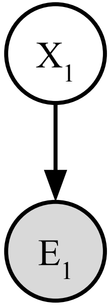
$$P(g|r) \propto P(r|g)P(g)$$

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

# Inference: Base Cases

Seeing evidence:





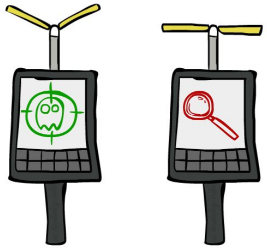

$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$

We take our current propabilities and multiply with the evidence probability. Then renormalize

# Inference: Base Cases

Seeing evidence:

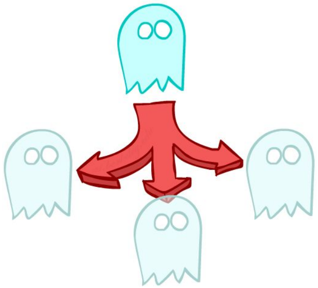


$$X_1 \rightarrow E_1$$

$$P(X_1|e_1)$$

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$
$$\propto_{X_1} P(x_1, e_1)$$
$$= P(x_1)P(e_1|x_1)$$

Time passes:



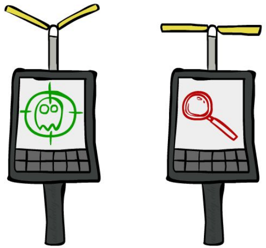
$P(X_1)$ $P(X_2|X_1)$

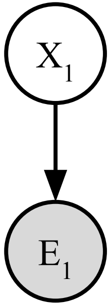
$$X_1 \rightarrow X_2$$

$$P(X_2)$$

# Inference: Base Cases

Seeing evidence:

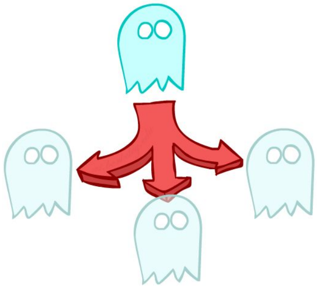




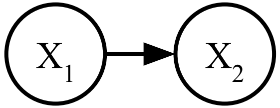

$$P(X_1|e_1)$$

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$
$$\propto_{X_1} P(x_1, e_1)$$
$$= P(x_1)P(e_1|x_1)$$

Time passes:



$P(X_1)$ $P(X_2|X_1)$




$$P(X_2)$$

$$P(x_2) = \sum_{x_1} P(x_1, x_2)$$
$$= \sum_{x_1} P(x_1)P(x_2|x_1)$$

Same as Markov Model

# Passage of Time

- Assume we have current believe P(X | evidence to date)

$$B(X_t) = P(X_t | e_{1:t})$$

$e_{1:t} = e_1, e_2, \ldots, e_t$

# Passage of Time

- Assume we have current believe P(X | evidence to date)

  $B(X_t) = P(X_t|e_{1:t})$

- Then, after one time step passes:

  $P(X_{t+1}|e_{1:t})$

$$P(x_2) = \sum_{x_1} P(x_1, x_2)$$
$$= \sum_{x_1} P(x_1)P(x_2|x_1)$$

We can still apply the formula, even with all the | $e_{1:t}$ in the back

We are going to apply this formula

# Passage of Time

- Assume we have current believe P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$

- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$
$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$

$$P(x_2) = \sum_{x_1} P(x_1, x_2)$$
$$= \sum_{x_1} P(x_1)P(x_2|x_1)$$

Use conditional independence assumption to get rid of this

# Passage of Time

- Assume we have current believe P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$

- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$
$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$
$$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

We have not seen $e_{t+1}$ yet

- Or, compactly

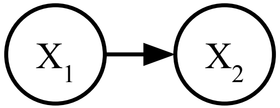
$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

New notation! Note that $B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$ is different from $B(X_{t+1}) = P(X_{t+1} | e_{1:t+1})$

# Passage of Time

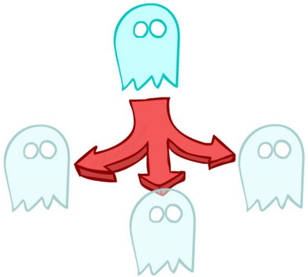
- Assume we have current believe P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$




- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$
$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$
$$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$



- Or, compactly

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

Basic idea
Beliefs get "pushed" through the transitions

# Example: Passage of Time

- As time passes, uncertainty "accumulates"

(Transition model: ghosts usually go clockwise)

| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | 1.00 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |

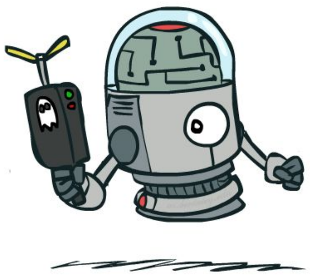
T = 1

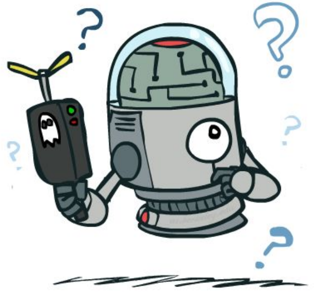
| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | 0.06 | <0.01 | <0.01 | <0.01 |
| <0.01 | 0.76 | 0.06 | 0.06 | <0.01 | <0.01 |
| <0.01 | <0.01 | 0.06 | <0.01 | <0.01 | <0.01 |

T = 2

| | | | | | |
|---|---|---|---|---|---|
| 0.05 | 0.01 | 0.05 | <0.01 | <0.01 | <0.01 |
| 0.02 | 0.14 | 0.11 | 0.35 | <0.01 | <0.01 |
| 0.07 | 0.03 | 0.05 | <0.01 | 0.03 | <0.01 |
| 0.03 | 0.03 | <0.01 | <0.01 | <0.01 | <0.01 |

T = 5





# Observation

- Assume we have current belief P(X | previous evidence):

  $B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$

- Then, after evidence comes in

  $P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t})$

We are going to apply this formula

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$
$$\propto_{X_1} P(x_1, e_1)$$
$$= P(x_1)P(e_1|x_1)$$

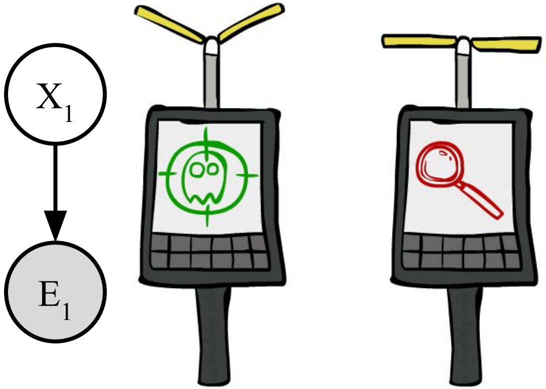
# Observation

- Assume we have current belief P(X | previous evidence):

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

- Then, after evidence comes in

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t})$$
$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t})$$
$$= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t})$$
$$= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$




- Or, compactly

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

- Basic idea: beliefs "reweighted" by likelihood of evidence
  - Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

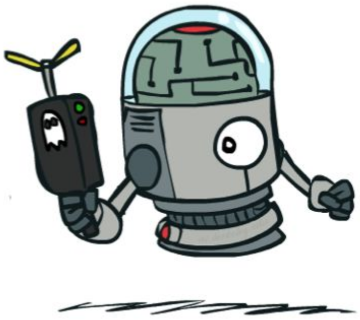
| | | | | | |
|---|---|---|---|---|---|
| 0.05 | 0.01 | 0.05 | <0.01 | <0.01 | <0.01 |
| 0.02 | 0.14 | 0.11 | 0.35 | <0.01 | <0.01 |
| 0.07 | 0.03 | 0.05 | <0.01 | 0.03 | <0.01 |
| 0.03 | 0.03 | <0.01 | <0.01 | <0.01 | <0.01 |

Before observation

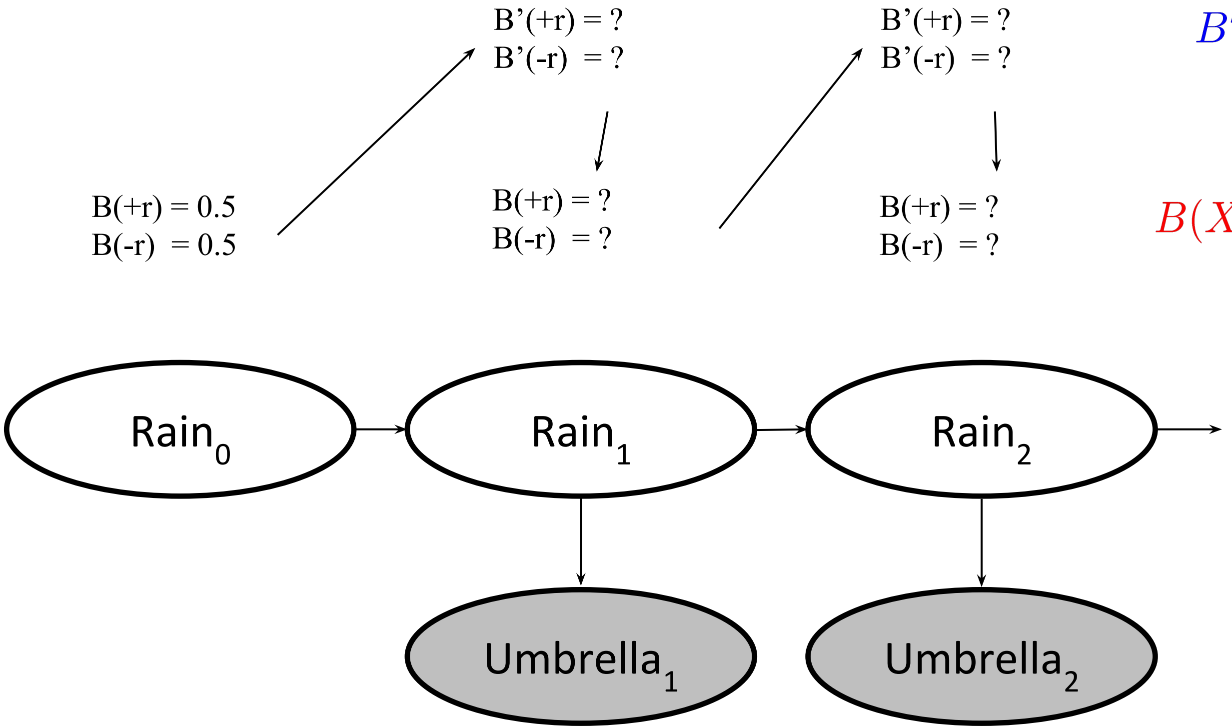
| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | 0.02 | <0.01 |
| <0.01 | <0.01 | <0.01 | 0.83 | 0.02 | <0.01 |
| <0.01 | <0.01 | 0.11 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |

After observation

$$B(X) \propto P(e|X)B'(X)$$



# Example: Weather




$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Example: Weather

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = ?
B'(-r) = ?

B(+r) = ?
B(-r) = ?

B'(+r) = ?
B'(-r) = ?

B(+r) = ?
B(-r) = ?

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Example: Weather

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = 0.5
B'(-r) = 0.5

B(+r) = ?
B(-r) = ?

B'(+r) = ?
B'(-r) = ?

B(+r) = ?
B(-r) = ?

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

$B'(X_{t+1} = +r) = P(X_{t+1} = +r \mid x_t = +r)\, B(x_t = +r) + P(X_{t+1} = +r \mid x_t = -r)\, B(x_t = -r)$
$= 0.7 * 0.5 + 0.3 * 0.5 = 0.5$

$B'(X_{t+1} = -r) = P(X_{t+1} = -r \mid x_t = +r)\, B(x_t = +r) + P(X_{t+1} = -r \mid x_t = -r)\, B(x_t = -r)$
$= 0.3 * 0.5 + 0.7 * 0.5 = 0.5$

| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Example: Weather

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = 0.5
B'(-r) = 0.5

B(+r) = ?
B(-r) = ?

B'(+r) = ?
B'(-r) = ?

B(+r) = ?
B(-r) = ?

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Example: Weather

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = 0.5
B'(-r) = 0.5

B(+r) = 0.8182
B(-r) = 0.1818

B'(+r) = ?
B'(-r) = ?

B(+r) = ?
B(-r) = ?

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

$B(X_{t+1} = +r) \propto P(E_{t+1}=+u|X_{t+1}=+r)B'(X_{t+1}=+r) = 0.9 * 0.5 = 0.45$

$B(X_{t+1} = -r) \propto P(E_{t+1}=+u|X_{t+1}=-r)B'(X_{t+1}=-r) = 0.2 * 0.5 = 0.1$

$z = 0.55$

$B(X_{t+1} = +r) = 0.45 / z = 0.8182$

$B(X_{t+1} = -r) = 0.1 / z = 0.1818$

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Quiz 5: Weather

Calculate these values!

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = 0.5
B'(-r) = 0.5

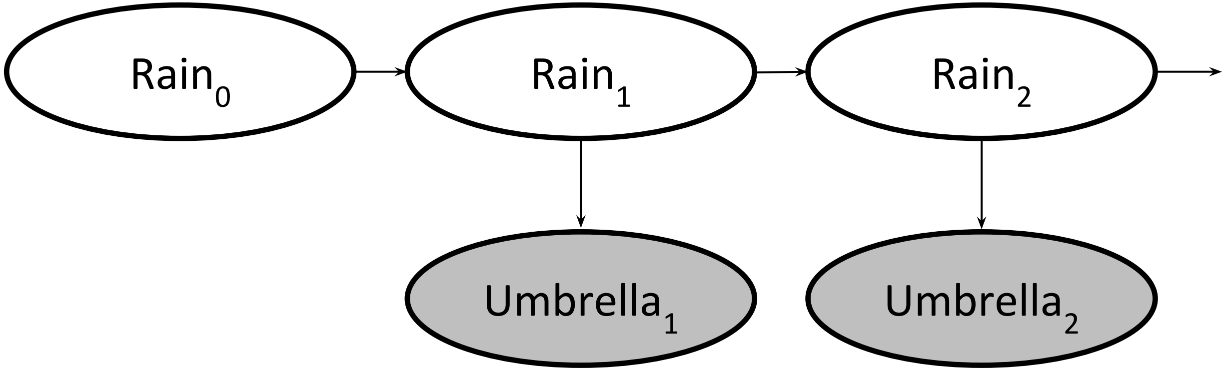
B(+r) = 0.8182
B(-r) = 0.1818

B'(+r) = ?
B'(-r) = ?

B(+r) = ?
B(-r) = ?

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$




| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Quiz 6: Weather

Calculate these values!

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = 0.5
B'(-r) = 0.5

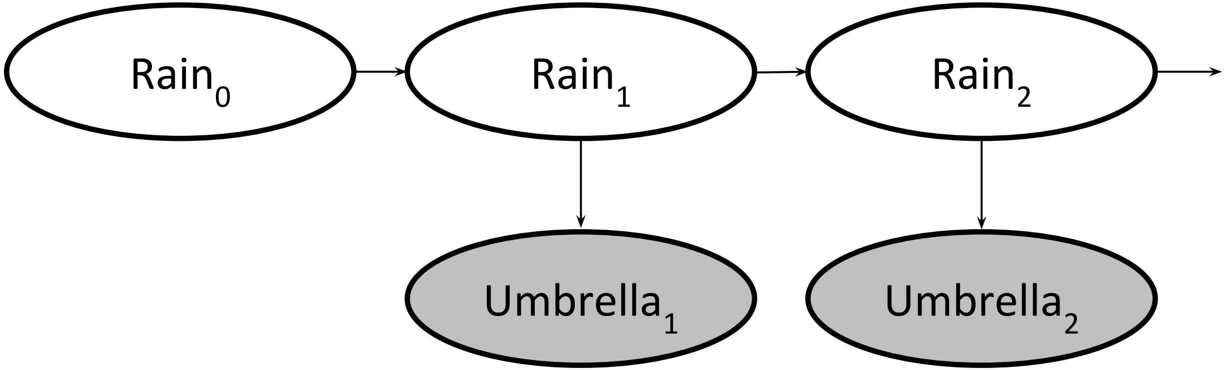
B(+r) = 0.8182
B(-r) = 0.1818

B'(+r) = ?
B'(-r) = ?

B(+r) = ?
B(-r) = ?

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$




| $X_t$ | $X_{t+1}$ | $P(X_{t+1}\|X_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $X_t$ | $E_t$ | $P(E_t\|X_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# The Forward Algorithm

- Alternatively, we can also just do a single update
- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

- We can derive the following updates

$$P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t})$$

We can normalize as we go if we want to have P(x|e) at each time step, or just once at the end…

$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t)$$

$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1})$$

observation update

Time update

# Online Belief Updates

- Every time step, we start with current P(X | evidence)
- We update for time:

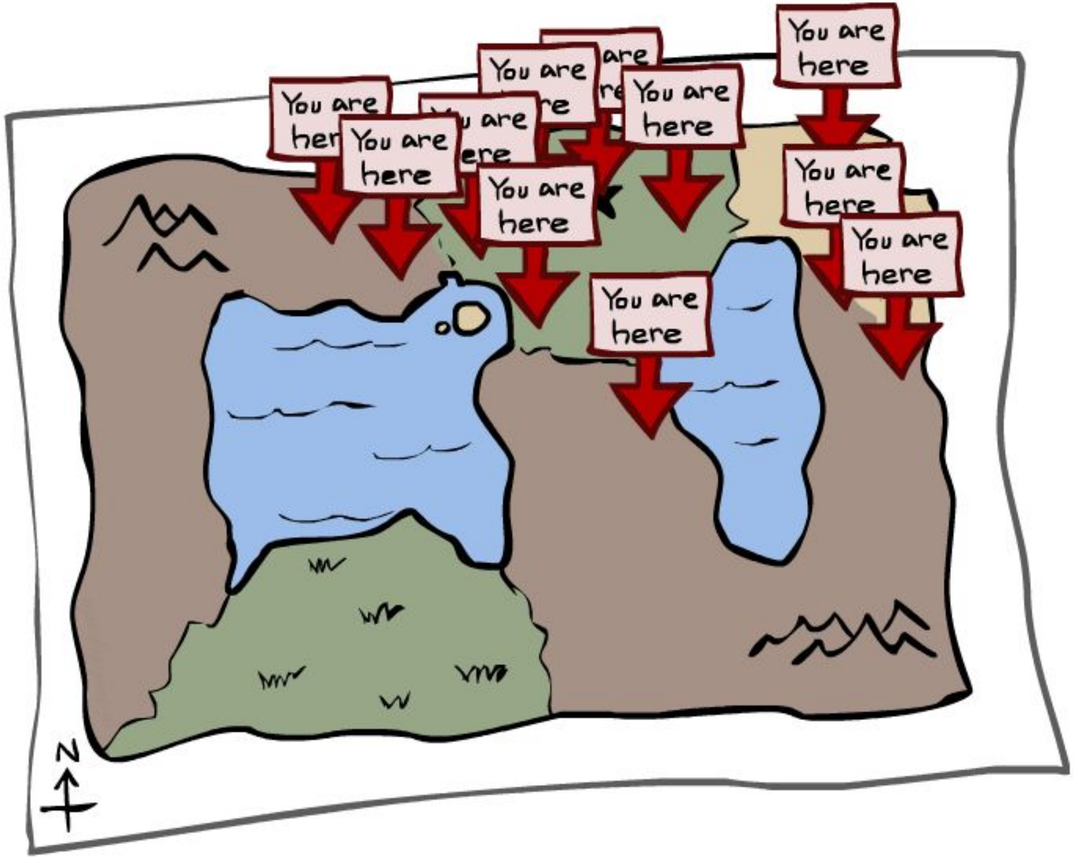
$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

- We update for evidence:

$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

- The forward algorithm does both at once (and doesn't normalize)

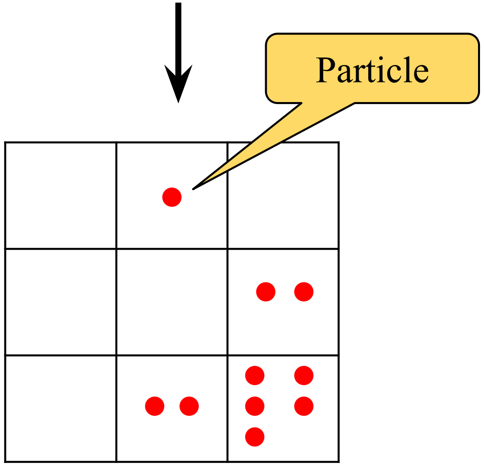
# Particle Filtering



# Particle Filtering

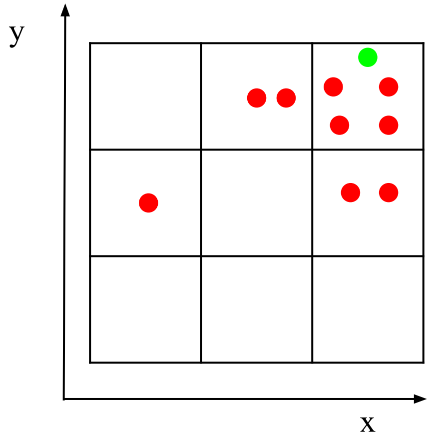
- Filtering: approximate solution
- Sometimes |X| is too big to use exact inference
    - |X| may be too big to even store B(X)
    - E.g. X is continuous
- Solution: approximate inference
    - Track samples of X, not all values
    - Samples are called particles
    - Time per step is linear in the number of samples
    - But: number needed may be large
    - In memory: list of particles, not states
- This is how robot localization works in practice
    - Particle is just new name for sample

| | | |
|---|---|---|
| 0.0 | 0.1 | 0.0 |
| 0.0 | 0.0 | 0.2 |
| 0.0 | 0.2 | 0.5 |



# Representation: Particles

- Our representation of P(X) is now a list of N particles (samples)
  - Generally, N << |X|
  - Storing map from X to counts would defeat the point
- P(x) approximated by number of particles with value x
  - So, many x may have P(x) = 0!
  - More particles, more accuracy
- For now, all particles have a weight of 1




Particles (x,y):
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)

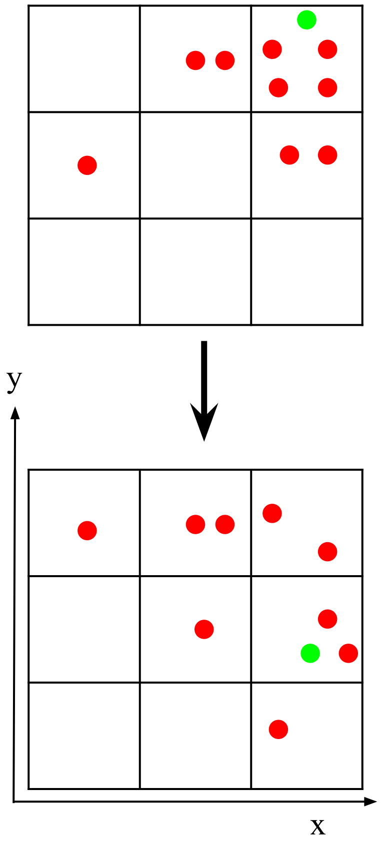
# Particle Filtering: Elapse of Time

- Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

  - This is like prior sampling – samples' frequencies reflect the transition probabilities
  - Here, most samples move clockwise, but some move in another direction or stay in place
- This captures the passage of time
  - If enough samples, close to exact values before and after (consistent)

Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)




Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,3)
(1,3)
(2,3)
(3,2)
(2,2)

# Particle Filtering: Observe

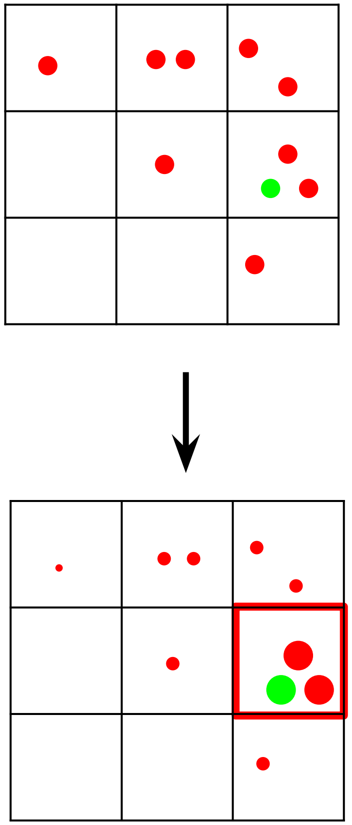
- Slightly trickier:
  - Don't sample observation, fix it
  - Similar to likelihood weighting, downweight samples based on the evidence

$$w(x) = P(e|x)$$
$$B(X) \propto P(e|X)B'(X)$$

- As before, the probabilities don't sum to one, since all have been downweighted (in fact they now sum to (N times) an approximation of P(e))

Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,3)
(1,2)
(3,3)
(3,3)
(2,3)

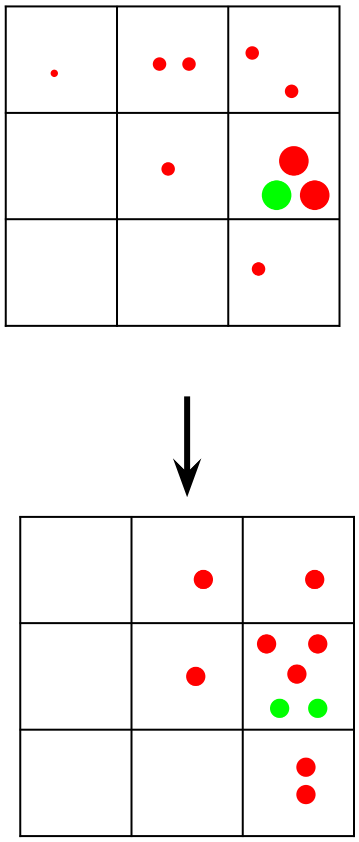
Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,3) w=.4
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4



# Particle Filtering: Resample

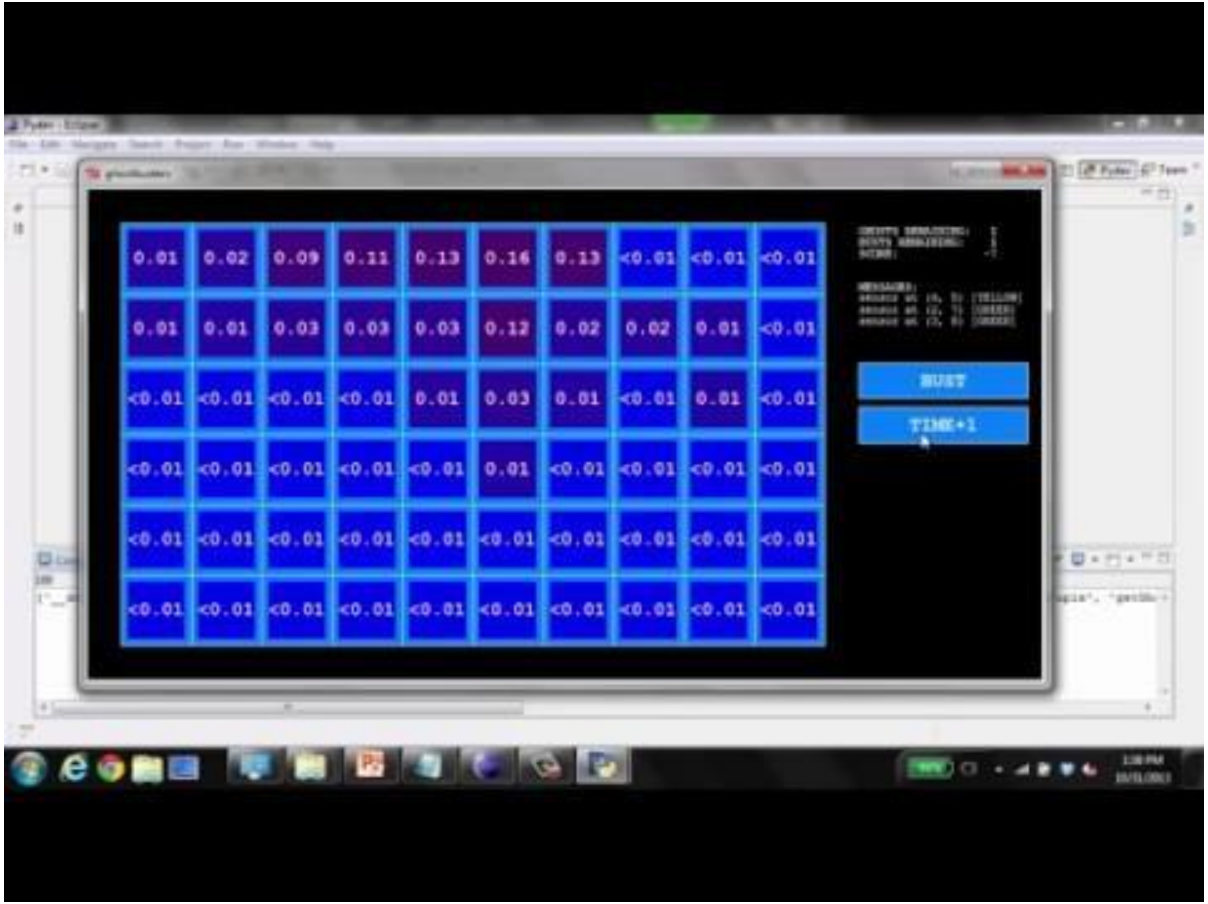
- Rather than tracking weighted samples, we resample
  - N times, we choose from our weighted sample distribution (i.e. draw with replacement)
  - This is equivalent to renormalizing the distribution
  - Now the update is complete for this time step, continue with the next one

Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,3) w=.4
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

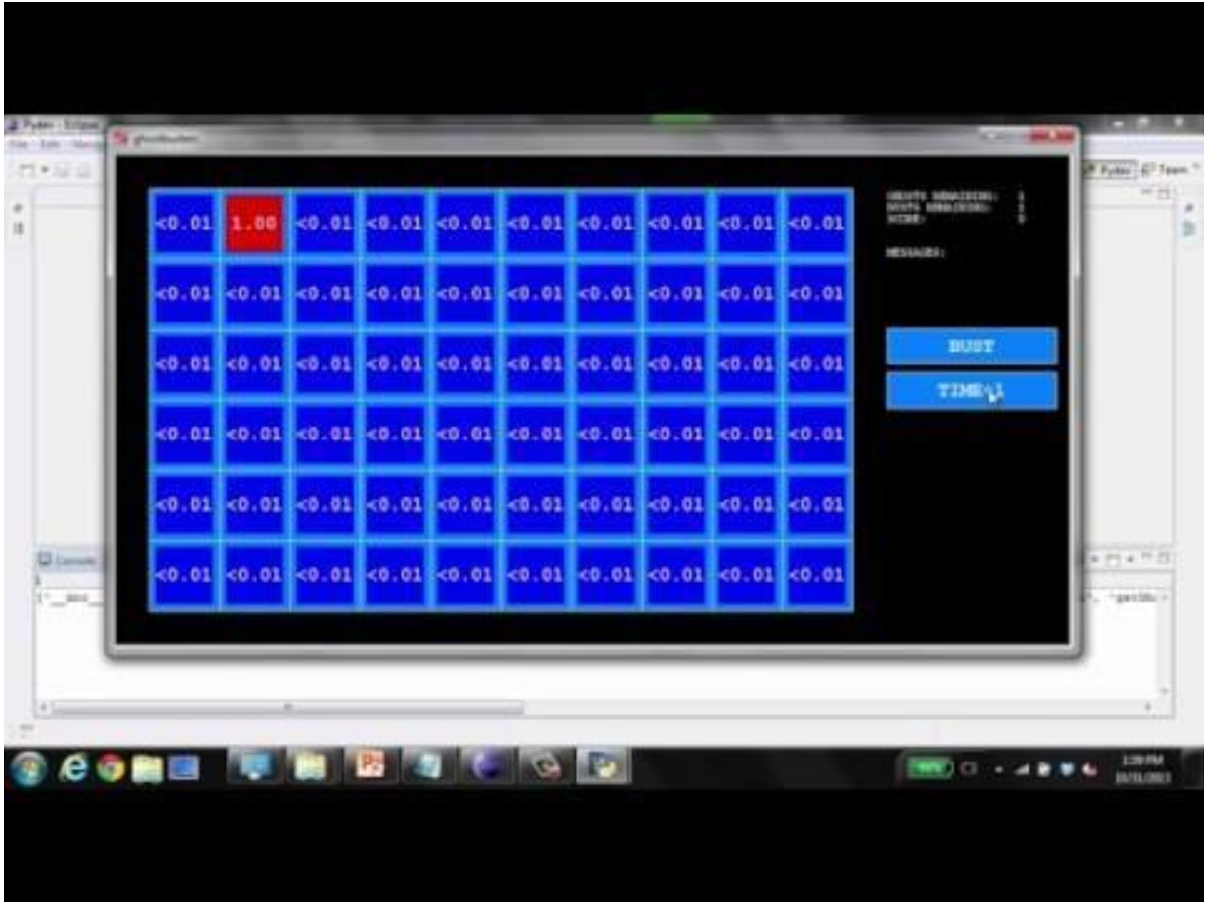


(New) Particles:
(3,2)
(2,2)
(3,2)
(3,1)
(3,3)
(3,2)
(3,1)
(2,3)
(3,2)
(3,2)

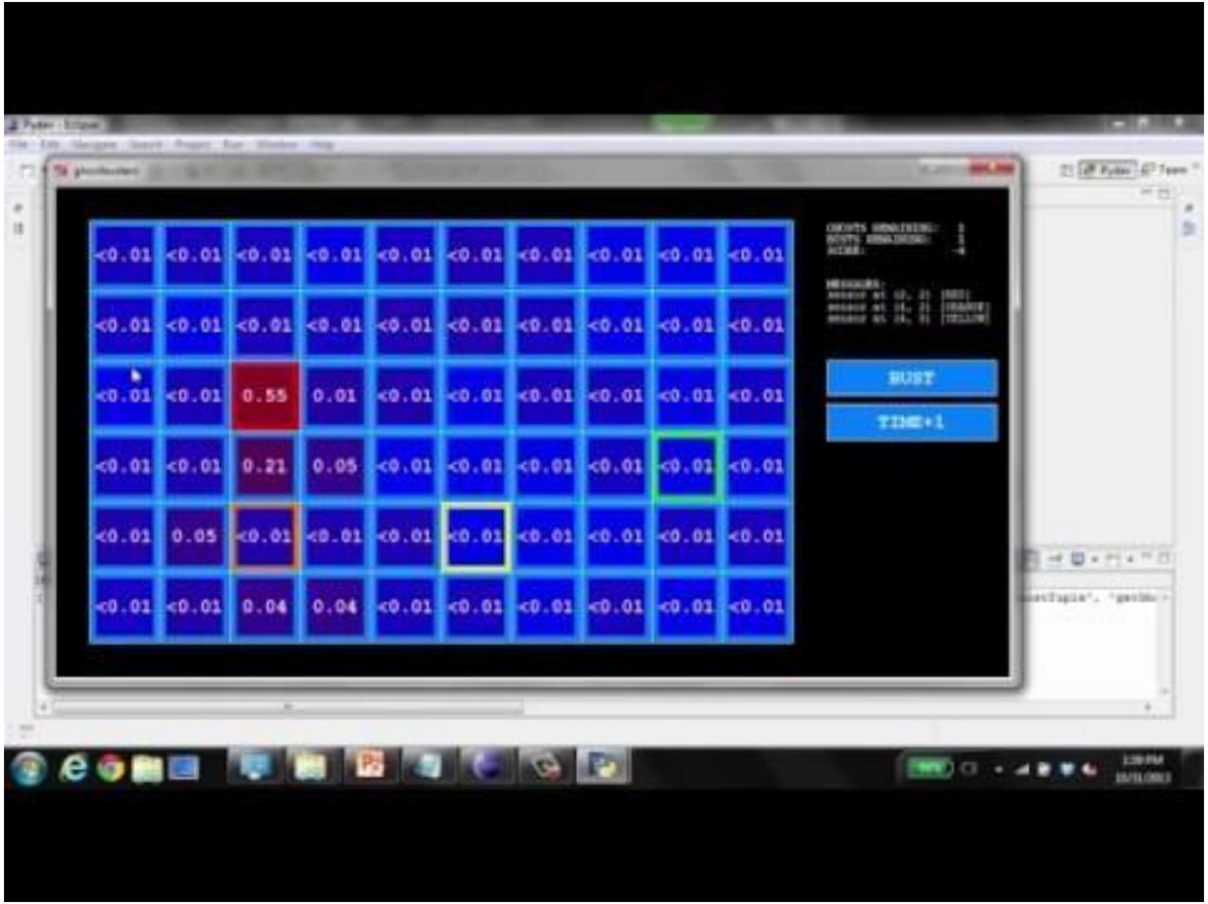
# Particle Filtering in Ghostbusters - Few Particles



# Particle Filtering in Ghostbusters - One Particle

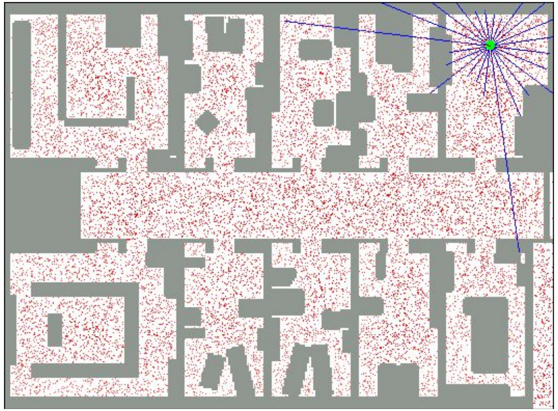


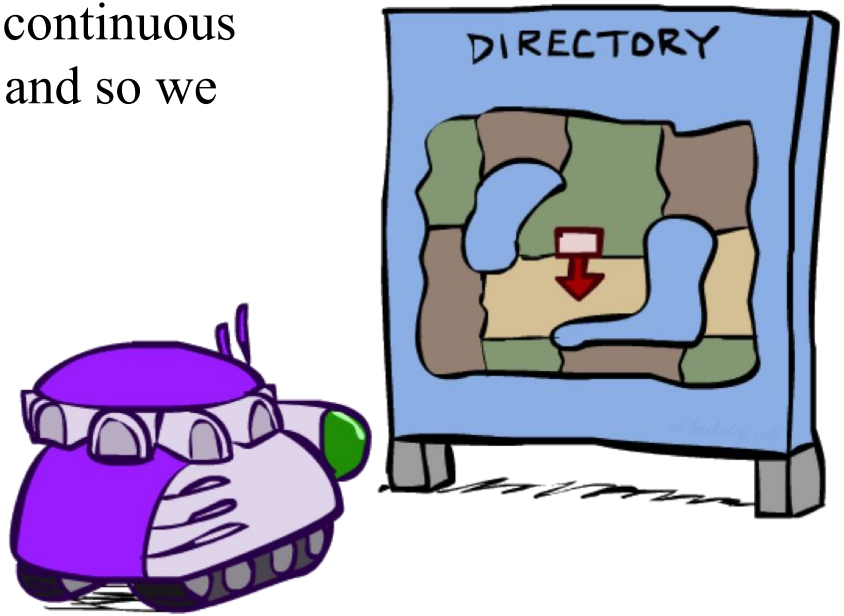
# Particle Filtering in Ghostbusters - Many Particles



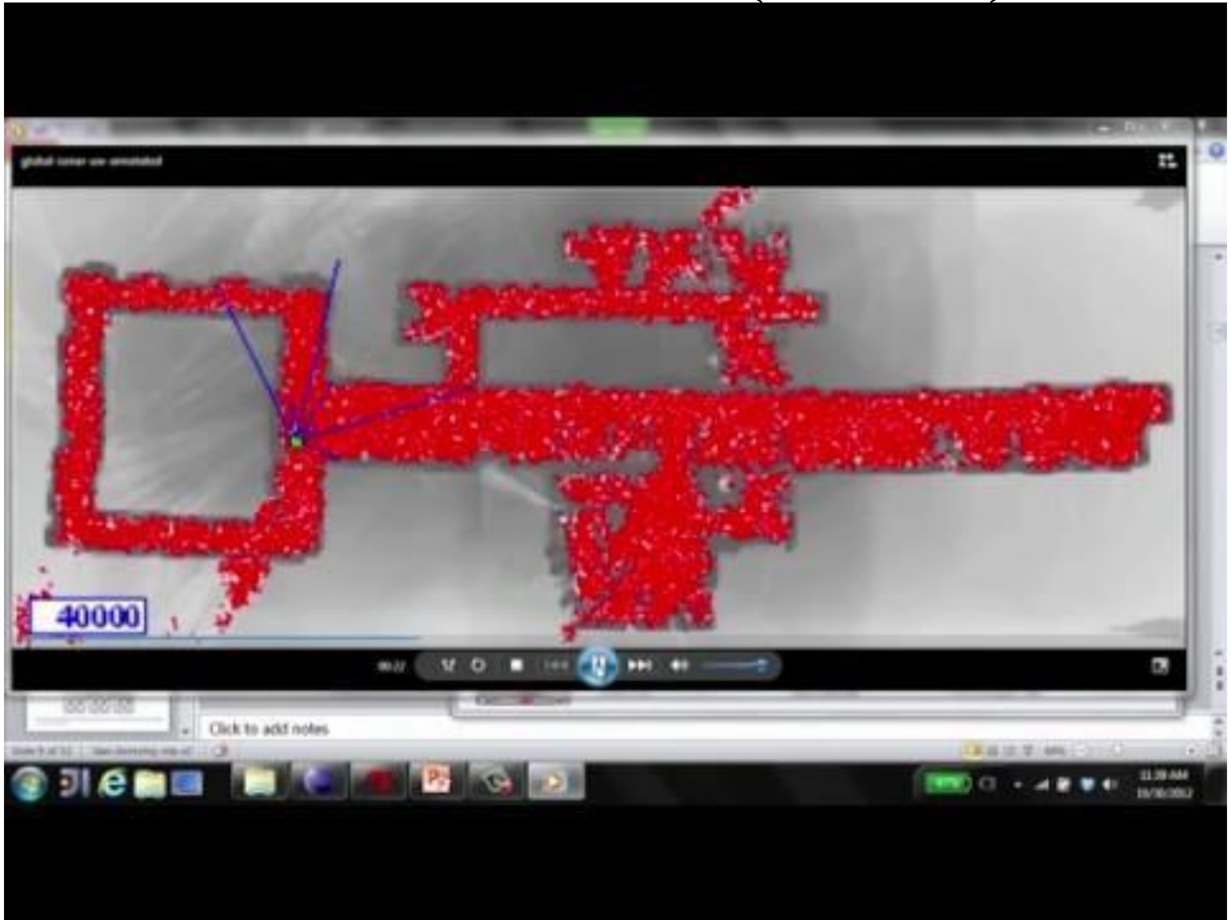
# Robot Localization

- In robot localization:
  - We know the map, but not the robot's position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store B(X)
  - Particle filtering is a main technique

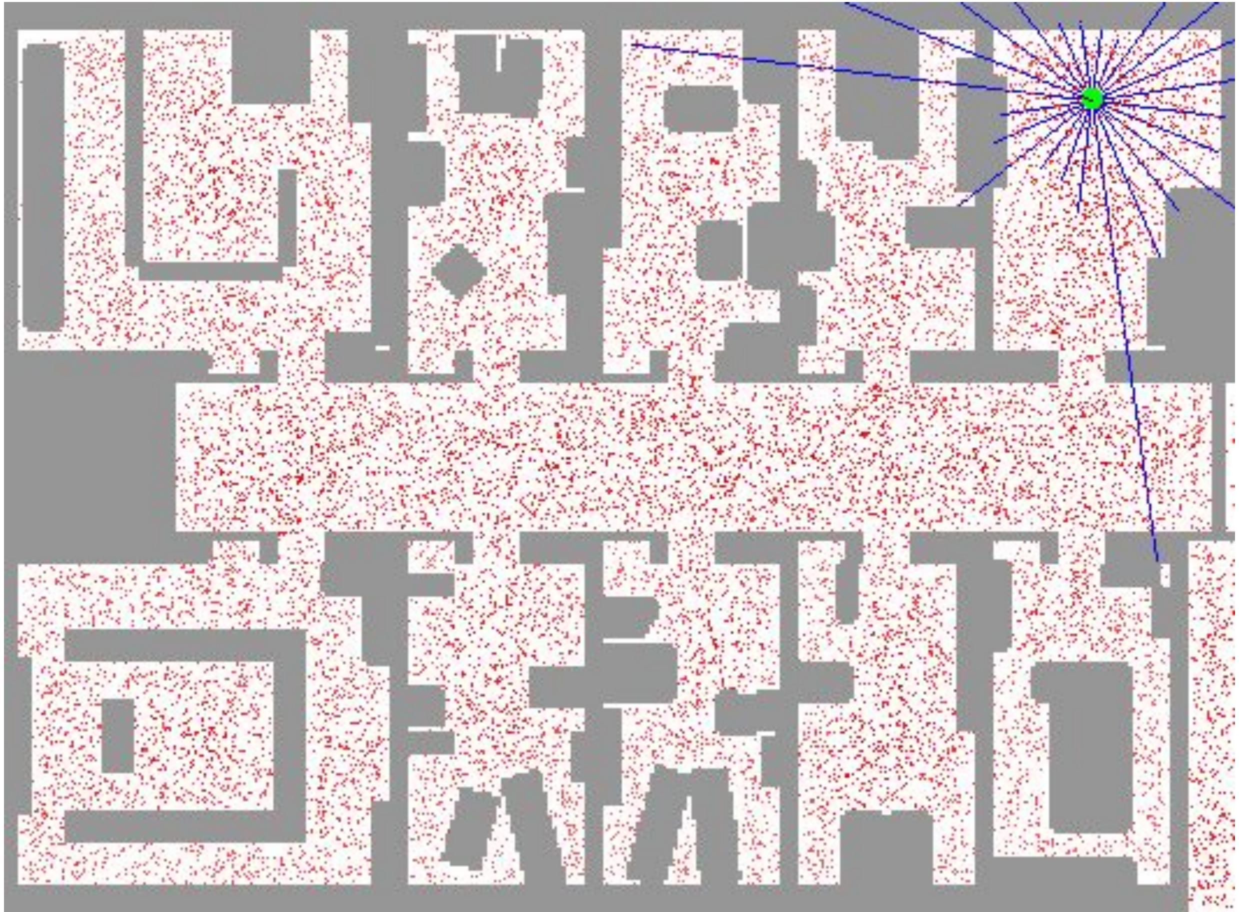




# Particle Filter Localization (Sonar)

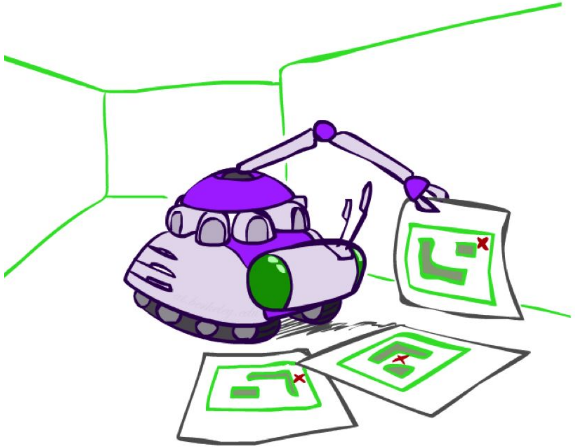


# Particle Filter Localization (Laser)

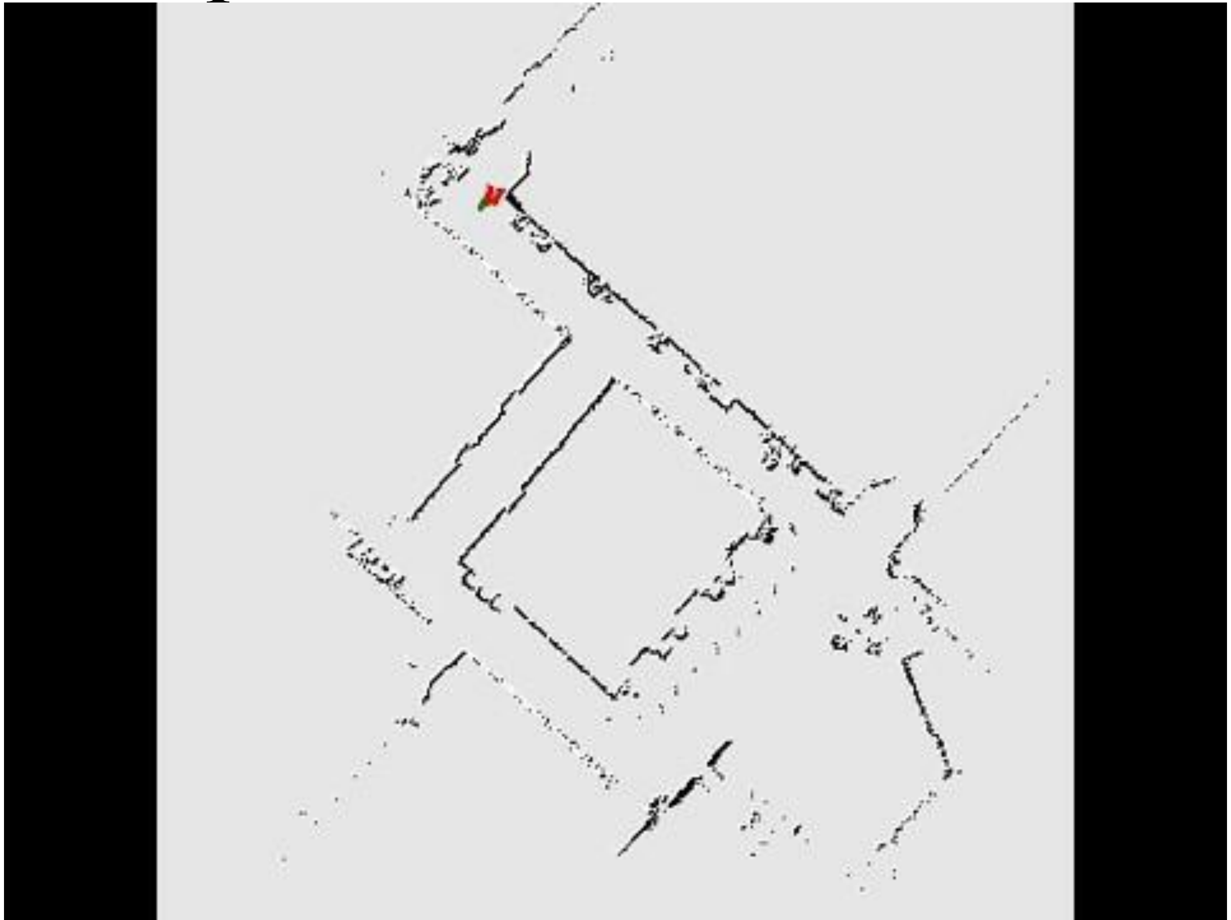


# Robot Mapping

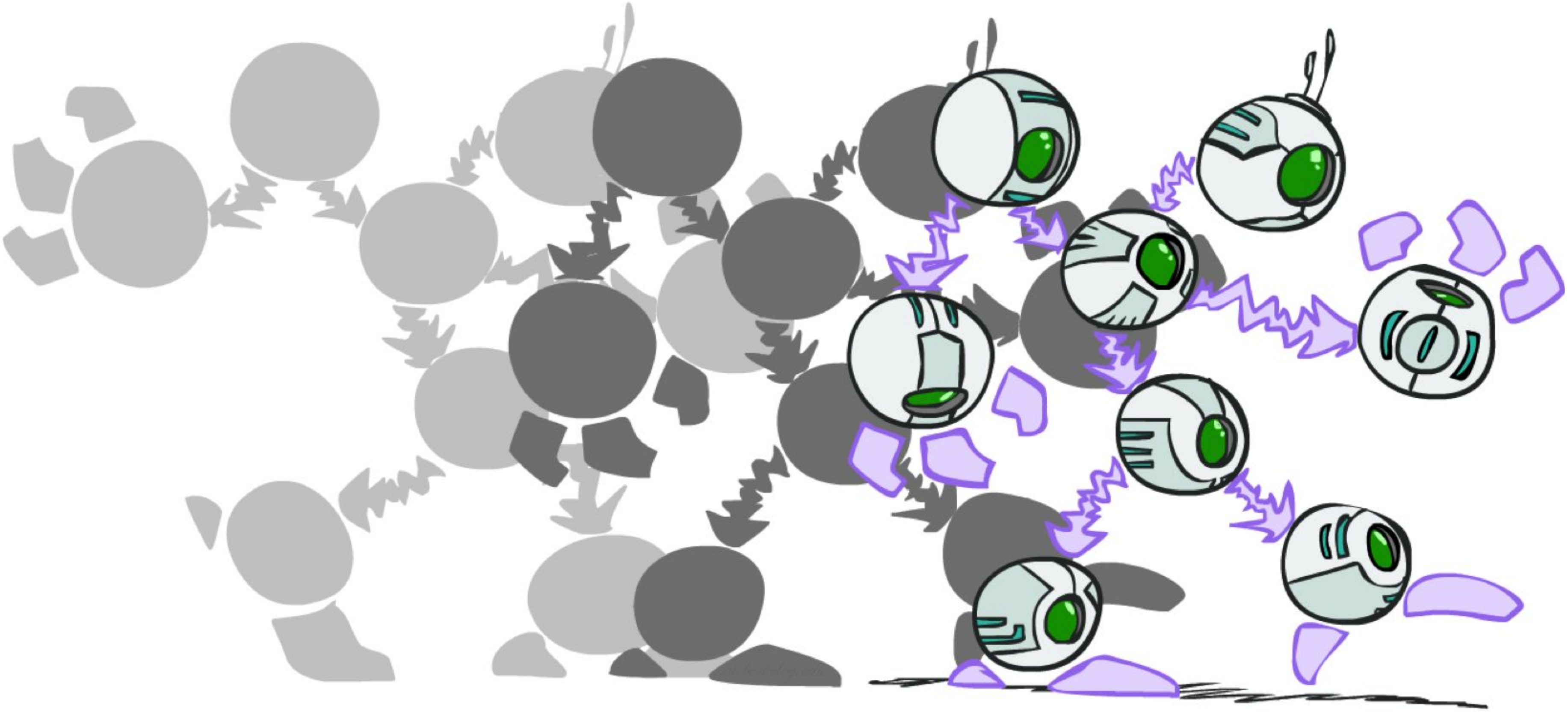
- SLAM: Simultaneous Localization And Mapping
  - We do not know the map or our location
  - State consists of position AND map!
  - Main techniques: Kalman filtering (Gaussian HMMs) and particle methods



# SLAM Example

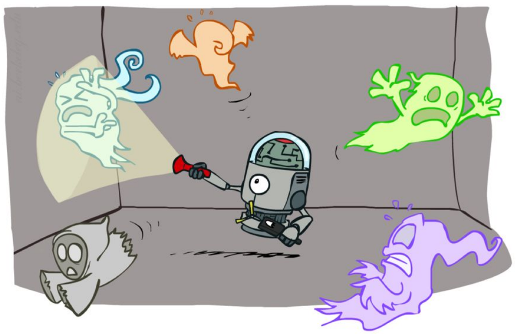


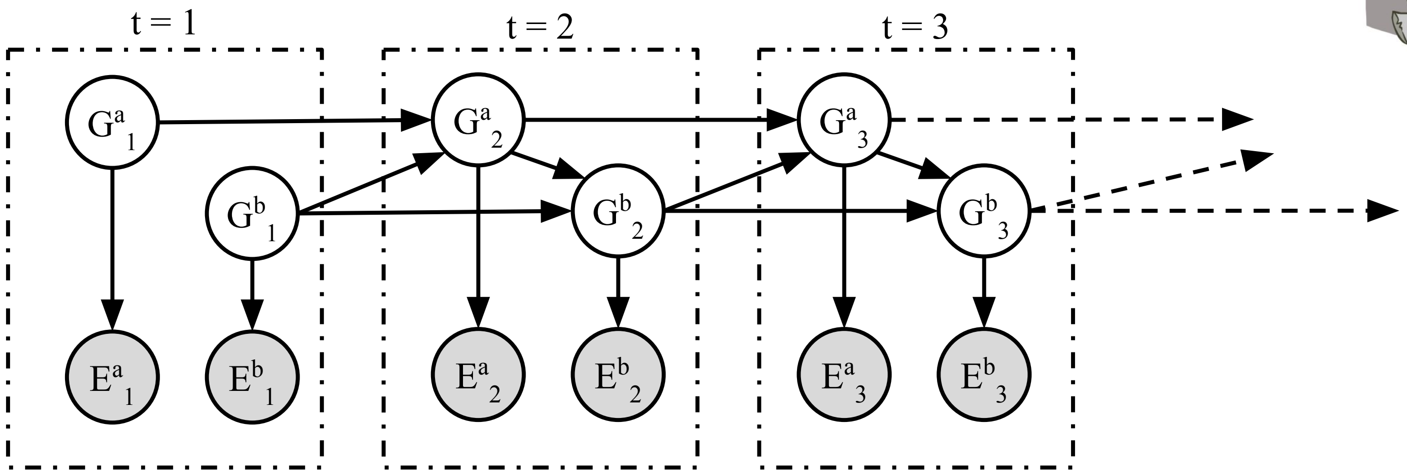
# Dynamic Bayes Nets

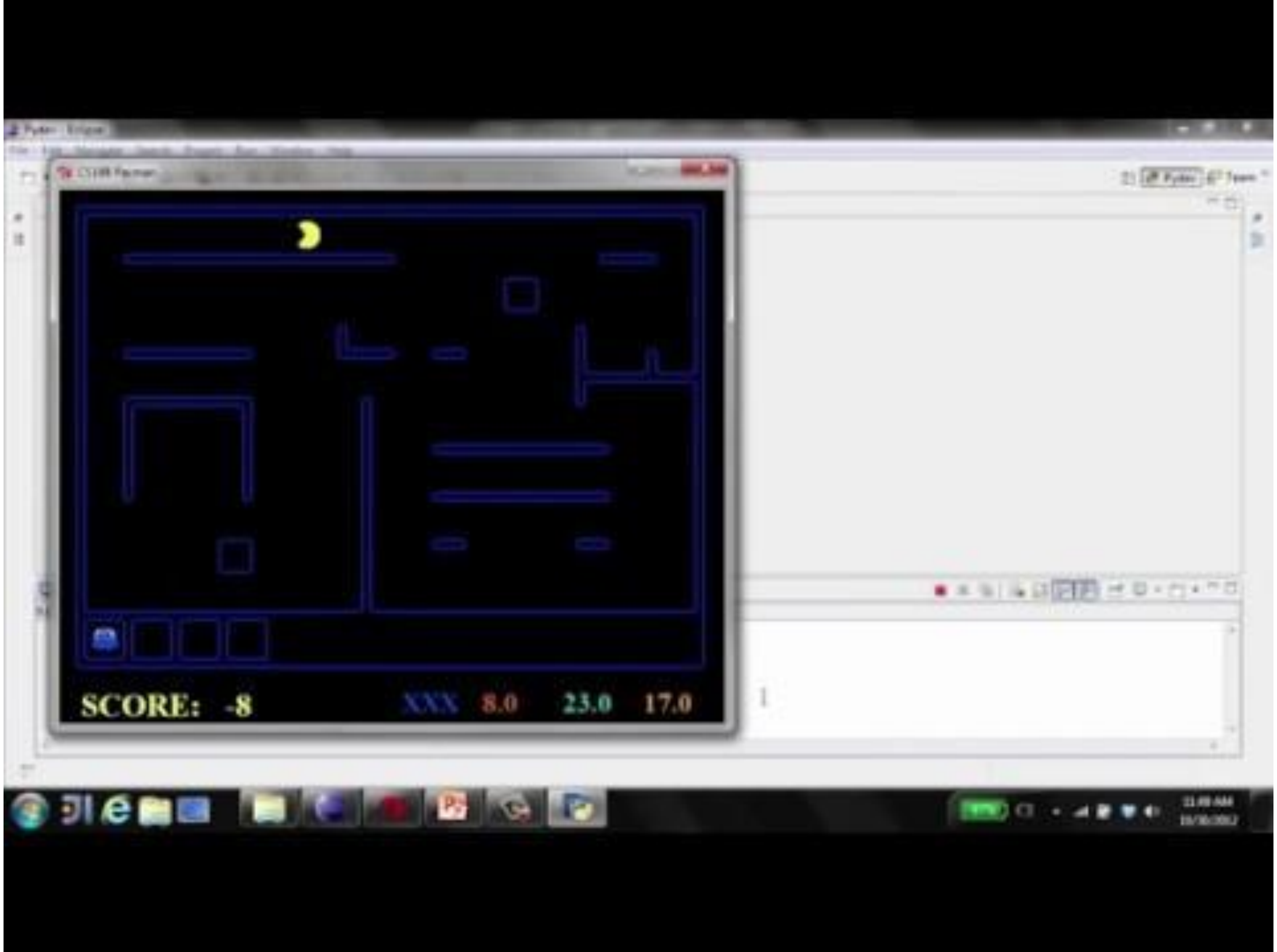


# Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net (BN) structure at each time
  - More on BNs later in the course
- Variables from time t can condition on those from t-1
- Dynamic Bayes nets are a generalization of HMMs

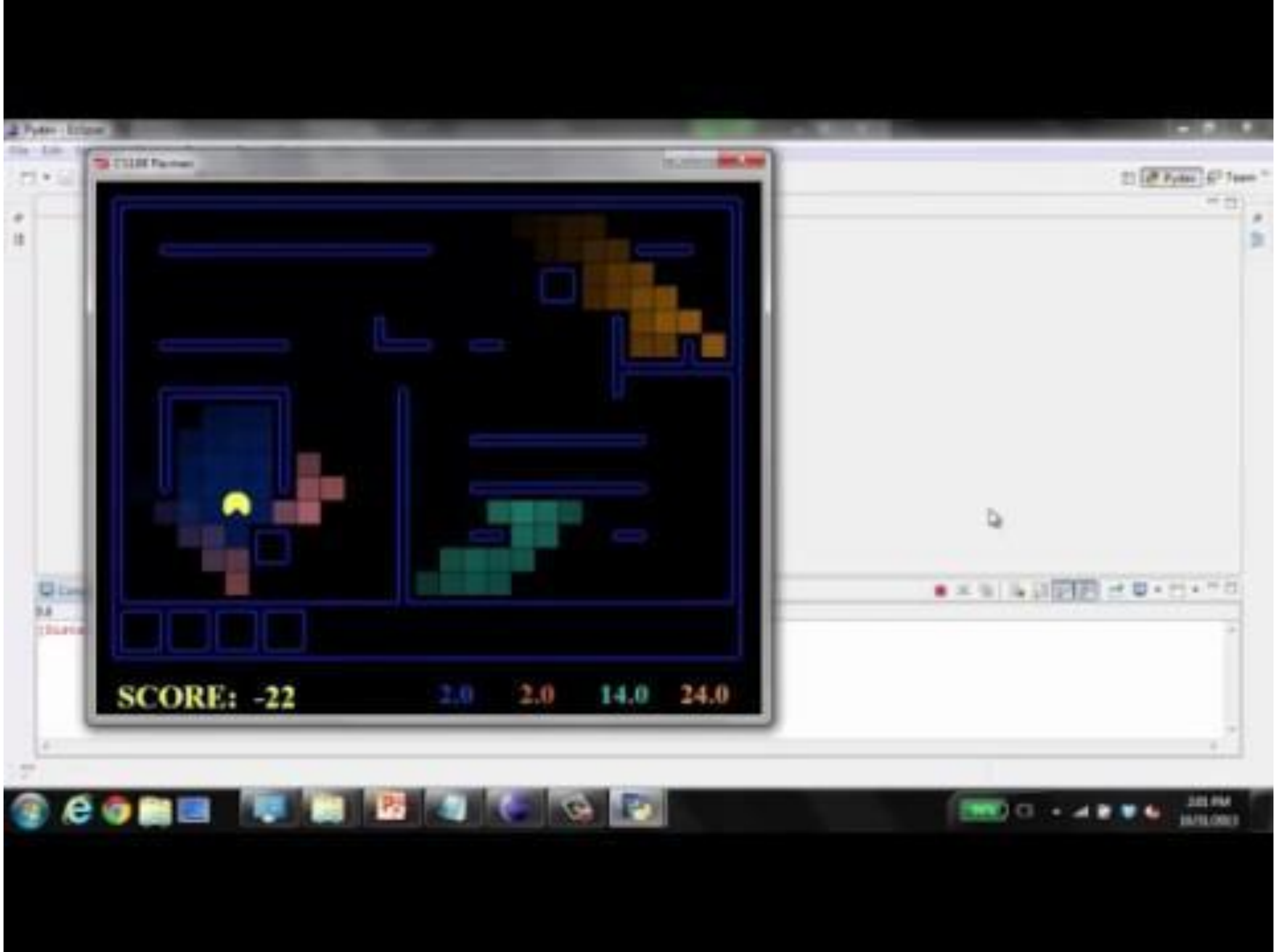





SCORE: -8
XXX 8.0 23.0 17.0

# DBN Particle Filter

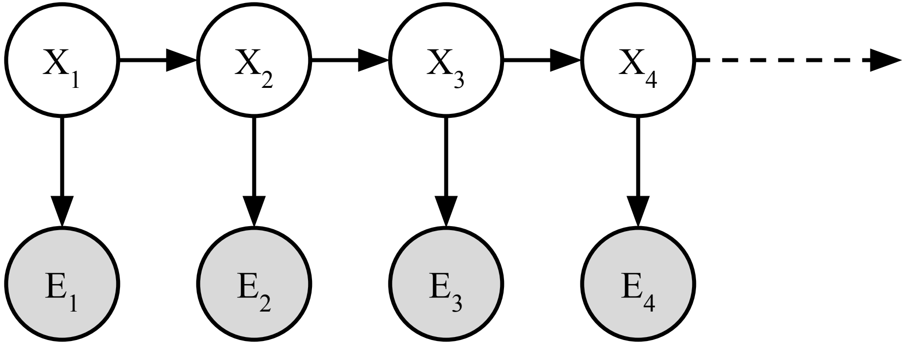
- A particle is a complete sample for a time step
- Initialize: Generate prior samples for the t=1 Bayes net
  - Example particle: $G^a_1 = (3,3)$ $G^b_1 = (5,3)$
- Elapse time: Sample a successor for each particle
  - Example successor: $G^a_2 = (2,3)$ $G^b_2 = (6,3)$
- Observe: Weight each entire sample by the likelihood of the evidence conditioned on the sample
  - Likelihood: $P(E^a_1|G^a_1) * P(E^b_1 |G^b_1 )$
- Resample: Select prior samples (tuples of values) in proportion to their likelihood


SCORE: -22
2.0
2.0
14.0
24.0

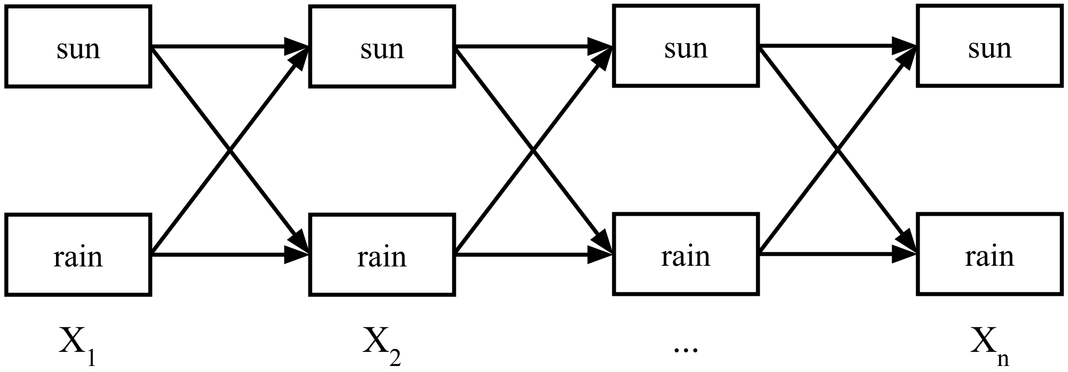
# Most Likely Explanation

# HMMs: MLE Queries

- HMMs defined by
  - States X
  - Observations E
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X|X_{-1})$
  - Emissions: $P(E|X)$
- New query: most likely explanation: $\arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$
- New method: the Viterbi algorithm

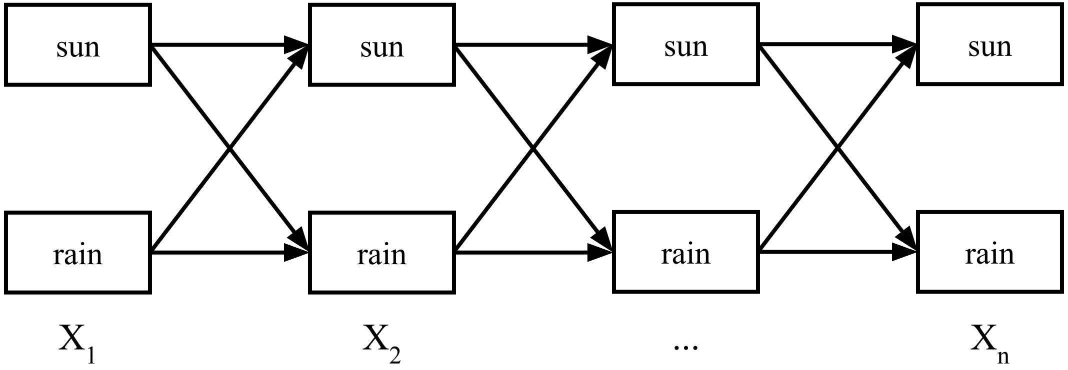


# State Trellis




- State trellis: graph of states and transitions over time
- Each arc represents some transition $x_{t-1} \rightarrow x_t$
- Each arc has weight $P(x_t|x_{t-1})P(e_t|x_t)$
- Each path is a sequence of states
- The product of weights on a path is that sequence's probability along with the evidence
- Forward algorithm computes sums of paths, Viterbi computes best paths

# Forward / Viterbi Algorithms




Forward Algorithm (Sum)

$$f_t[x_t] = P(x_t, e_{1:t})$$

$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) f_{t-1}[x_{t-1}]$$

Viterbi Algorithm (Max)

$$m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t})$$

$$= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}]$$

# Quiz 7

- Solve the problem on the handout sheet 1

# Quiz 8

- Solve the problem on the handout sheet 2