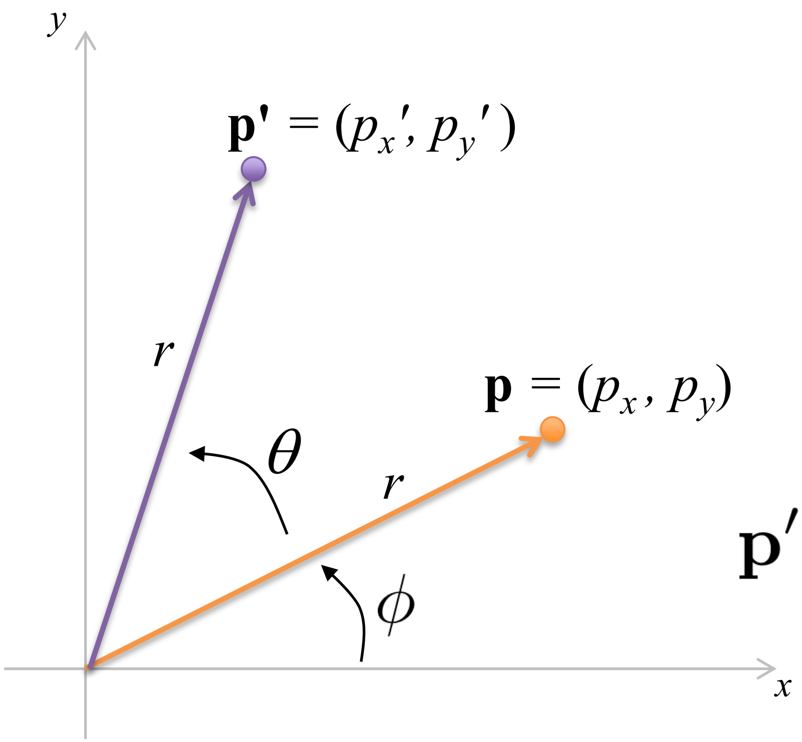
2D Rotation

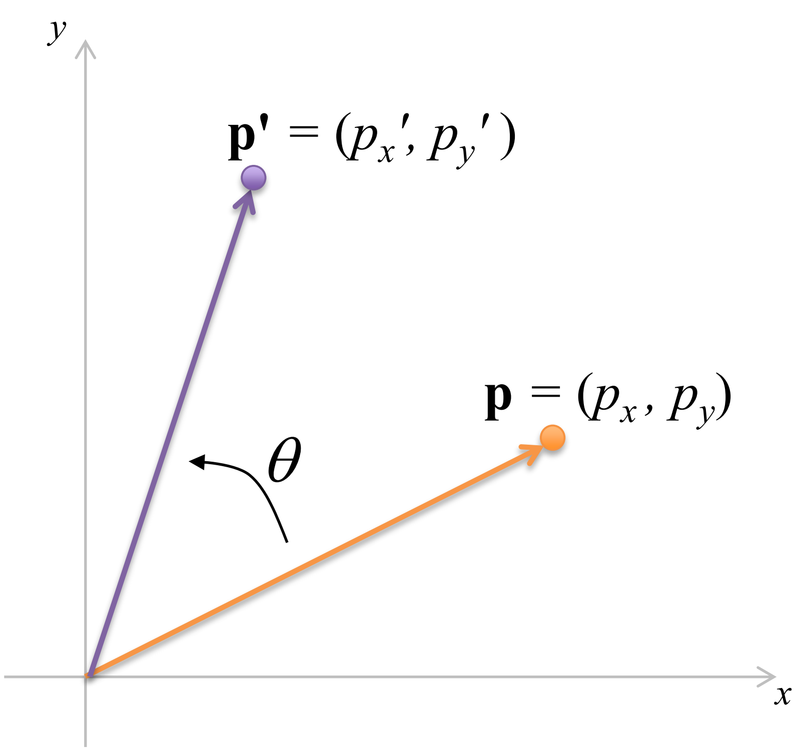
$$p_x = r\cos\phi,\ \ p_y = r\sin\phi$$
$$p'_x = r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$p'_y = r\sin(\phi+\theta) = r\sin\phi\cos\theta + r\cos\phi\sin\theta$$



$$p'_x = p_x\cos\theta - p_y\sin\theta$$
$$p'_y = p_x\sin\theta + p_y\cos\theta$$

$$\mathbf{p}' = \begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} p_x\cos\theta - p_y\sin\theta \\ p_x\sin\theta + p_y\cos\theta \end{pmatrix}$$

2D Rotation



$$\mathbf{p}' = \begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} p_x \cos\theta - p_y \sin\theta \\ p_x \sin\theta + p_y \cos\theta \end{pmatrix}$$

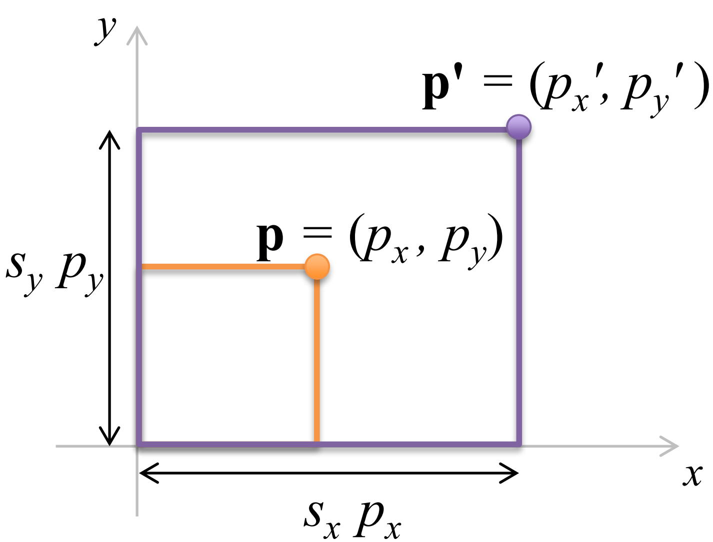
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

$$\mathbf{p}' = \underbrace{\mathbf{R}(\theta)}_{\text{rotation}}\mathbf{p}$$

Note that the rotation is about the origin

2D Scaling

$$\begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} s_x p_x \\ s_y p_y \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

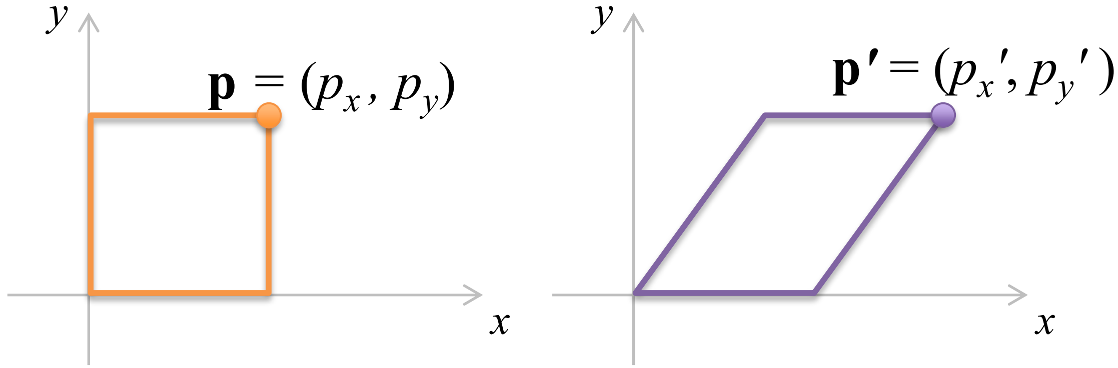


$$\mathbf{p}' = \underbrace{\mathbf{S}(s_x, s_y)}_{\text{scaling}}\mathbf{p}$$

We have a uniform scaling if $s_x = s_y$

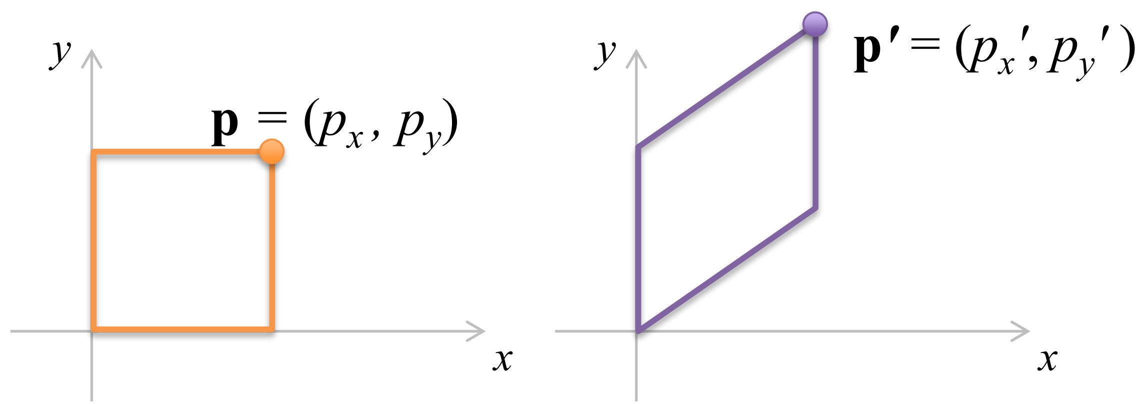
2D Shearing

X-Shear



$$\begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} p_x + m p_y \\ p_y \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} \qquad \mathbf{p}' = \underbrace{\mathbf{Sh}_x(m)}_{\text{shearing}}\mathbf{p}$$

Y-Shear

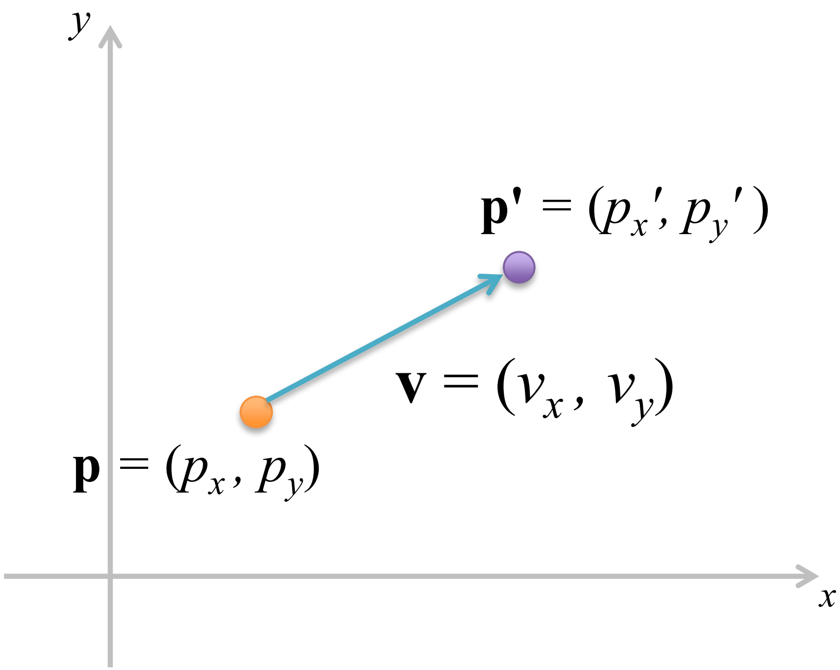


$$\begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} p_x \\ m p_x + p_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} \qquad \mathbf{p}' = \underbrace{\mathbf{Sh}_y(m)}_{\text{shearing}}\mathbf{p}$$

2D Translation

$$\begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} p_x + v_x \\ p_y + v_y \end{pmatrix}$$

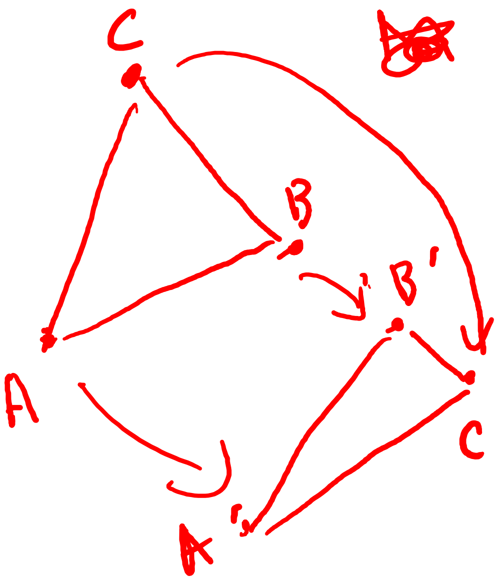
$$\mathbf{p}' = \underbrace{\mathbf{p} + \mathbf{v}}_{\text{translation}}$$



The Basic Transformations

$x' = Mx + b$

M: 2×2 $\quad$ b: 2×1



Rotation	$\mathbf{p}' = \mathbf{R}(\theta)\mathbf{p}$
Scaling	$\mathbf{p}' = \mathbf{S}(s_x, s_y)\mathbf{p}$
Shearing	$\mathbf{p}' = \mathbf{Sh}_x(m)\mathbf{p}$ $\quad$ $\mathbf{p}' = \mathbf{Sh}_y(m)\mathbf{p}$
Translation	$\mathbf{p}' = \mathbf{p} + \mathbf{v}$

$A' = MA + b$

$B' = MB + b$

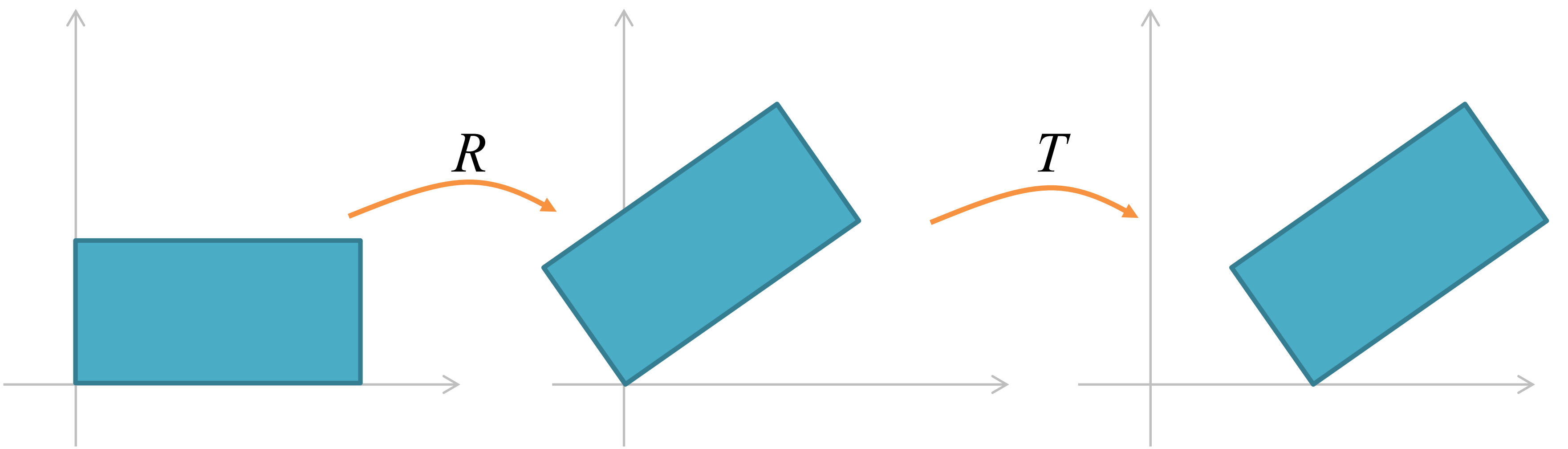
$C' = Mc + b$

- can represent any affine transformation as a sequence of these 4
- A general rotation / scaling / shearing transformation has exactly one fixed point.
- A translation has no fixed point
- $\det(L) \neq 0$ is the scale factor of the area of a transformed region by the affine transformation
- A 2D affine transformation is uniquely determined by correspondences between three pairs of non-collinear points

Composition of Transformations

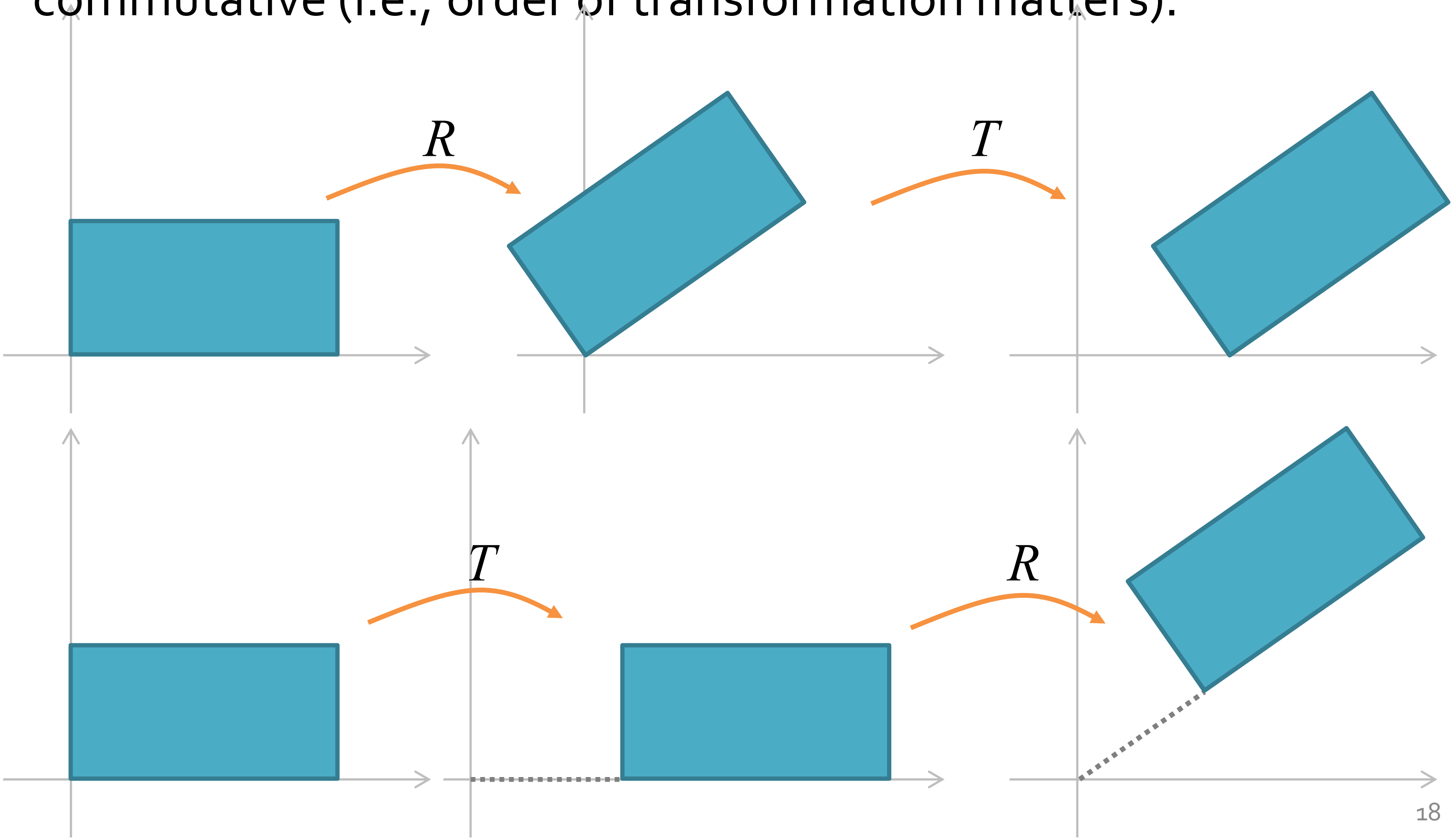
The composition of two affine transformation is also an affine transformation

Example: a rotation R followed by a translation T



Composition of Transformations

In general, the composition of affine transformations is non-commutative (i.e., order of transformation matters).



Matrix Compositions

$p' = R_1 p$

Suppose we want to first rotate a point by $\mathbf{R}_1$, then scale by $\mathbf{S}$, then rotate again by $\mathbf{R}_2$, we have:

$$\mathbf{p}' = \underline{\mathbf{R}_2 \cdot \mathbf{S} \cdot \mathbf{R}_1}\, \mathbf{p} = \mathbf{M}\,\mathbf{p}$$

Transformation matrices are applied from right to left

Now, consider first rotate a point by $\mathbf{R}_1$, then translate by $\mathbf{v}$, then rotate again by $\mathbf{R}_2$, we have:

$$\mathbf{p}' = \mathbf{R}_2 \cdot (\mathbf{R}_1 \mathbf{p} + \mathbf{v}) = \mathbf{R}_2 \cdot \mathbf{R}_1 \mathbf{p} + \mathbf{R}_2 \mathbf{v}$$

Oops! We don't have a nice matrix composition by multiplication for the transformation because translation is not a linear transformation

Homogeneous Coordinates

A 2D point (x, y) is represented as (wx, wy, w), for any real number $w \neq 0$.

Therefore, any given point has infinitely many different homogeneous coordinate representations.

- Two points (wx, wy, w) and (ux, uy, u) in homogeneous coordinates are the same $\rightarrow (2, 4)$ $\rightarrow (2, 4)$
- E.g.: the 2D points (4, 8, 2) and (20, 40, 10) are the same.

Conversely, given any homogeneous coordinates (x, y, w), $w \neq 0$, of a 2D point, the Cartesian coordinates of the point are given by $(x/w, y/w)$.

$$(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$$

Homogeneous Coordinates

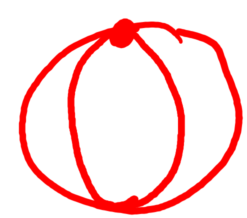
(∞, ∞)

Homogeneous coordinates allow us to define points at infinity.

$\left(\frac{x}{0}, \frac{y}{0}\right)$

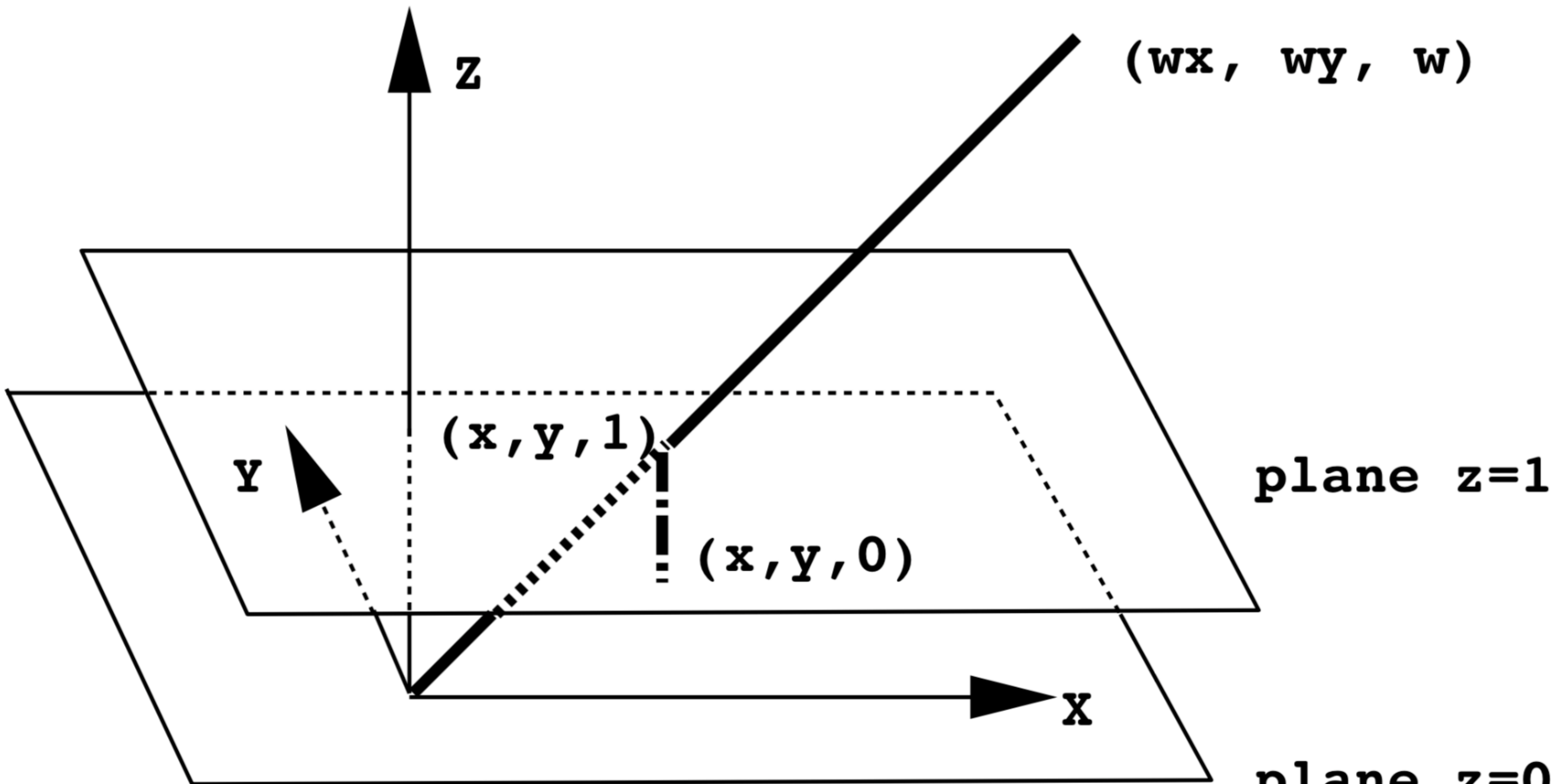
- $(0, 0, 0)$ does not represent a well-defined point
- When $x \neq 0$ or $y \neq 0$, $(x, y, 0)$ are the homogeneous coordinates of a point at infinity; in particular, $(x, y, 0)$ and $(-x, -y, 0)$ stand for the same point at infinity.
- A point at infinity indicates a direction and thus a 2D vector can be represented in homogeneous coordinates as $(x, y, 0)$.

$(-x, -y, 0) \longleftrightarrow (x, y, 0)$



Homogeneous Coordinates

A geometric interpretation



Reading: A.4 of online appendices: “Linear Algebra & Trigonometry” from the book “ Real-Time Rendering”

Homogeneous Representation of 2D Transformations

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ w \end{pmatrix}$$

We represent 2D transformations using 3x3 matrices

Rotation

$$\mathbf{p}' = \mathbf{S}\mathbf{p}$$

$$\begin{pmatrix} p'_x \\ p'_y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} p_x \cos\theta - p_y \sin\theta \\ p_x \sin\theta + p_y \cos\theta \\ 1 \end{pmatrix}$$

This is the same point as in slide 12

Homogeneous Representation of 2D Transformations

Scaling $\mathbf{p}' = \mathbf{S}\mathbf{p}$

$$\begin{pmatrix} p'_x \\ p'_y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x p_x \\ s_y p_y \\ 1 \end{pmatrix}$$

Translation $\mathbf{p}' = \mathbf{T}\mathbf{p}$

$$\begin{pmatrix} p'_x \\ p'_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + v_x \\ p_y + v_y \\ 1 \end{pmatrix}$$

Now a translation can be represented as a matrix as well

Vector Transformations

Note that translation has no effect on vectors.

Translating a vector:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix}$$

Translation has no effect

Composition of Transformations

With homogeneous transformations, composition of transformations can be represented by multiplication of matrices

Rotate, scale then rotate again

$$\mathbf{p}' = \underline{\mathbf{R}_2} \cdot \underline{\mathbf{S}} \cdot \underline{\mathbf{R}_1}\,\mathbf{p} = \underline{\mathbf{M}}\,\mathbf{p}$$

3x3 matrices for 2D transformations

Rotate, translate then rotate again

$$\mathbf{p}' = \mathbf{R}_2 \cdot \mathbf{T} \cdot \mathbf{R}_1\,\mathbf{p} = \mathbf{N}\,\mathbf{p}$$

3D Transformations

We represent 3D transformations using 4x4 matrices:

Rotation about x-axis

$$\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation about y-axis

$$\mathbf{R}_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation about z-axis

$$\mathbf{R}_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Fixed point of a 3D rotation is a straight line

Scaling

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & v_x \\ 0 & 1 & 0 & v_y \\ 0 & 0 & 1 & v_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse Transformations

Suppose we apply a transformation **M** to a point **p**:

$$\mathbf{p}' = \mathbf{M}\mathbf{p}$$

What is the transformation needed to bring **p'** back to **p**?

$$\mathbf{p} =? \ \mathbf{p}'$$

We need the inverse of **M** to bring **p'** back to **p**:

$$\mathbf{M}^{-1}\mathbf{p}' = (\mathbf{M}^{-1}\mathbf{M})\mathbf{p} = \mathbf{I}\,\mathbf{p} = \mathbf{p}$$

Inverse of M

Identity transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A 3x3 identity matrix

Inverse Transformations

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & v_x \\ 0 & 1 & 0 & v_y \\ 0 & 0 & 1 & v_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

inverse of **T**?

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -v_x \\ 0 & 1 & 0 & -v_y \\ 0 & 0 & 1 & -v_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

inverse of **S**?

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

inverse of $\mathbf{R}_z$?

$$\mathbf{R}_z^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \mathbf{R}_z^T$$

Inverse of a rotation equals its transpose

Inverse of Transformation

Suppose we have a transformation **M** given by a composition of transformations:

$$\mathbf{M} = \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}$$

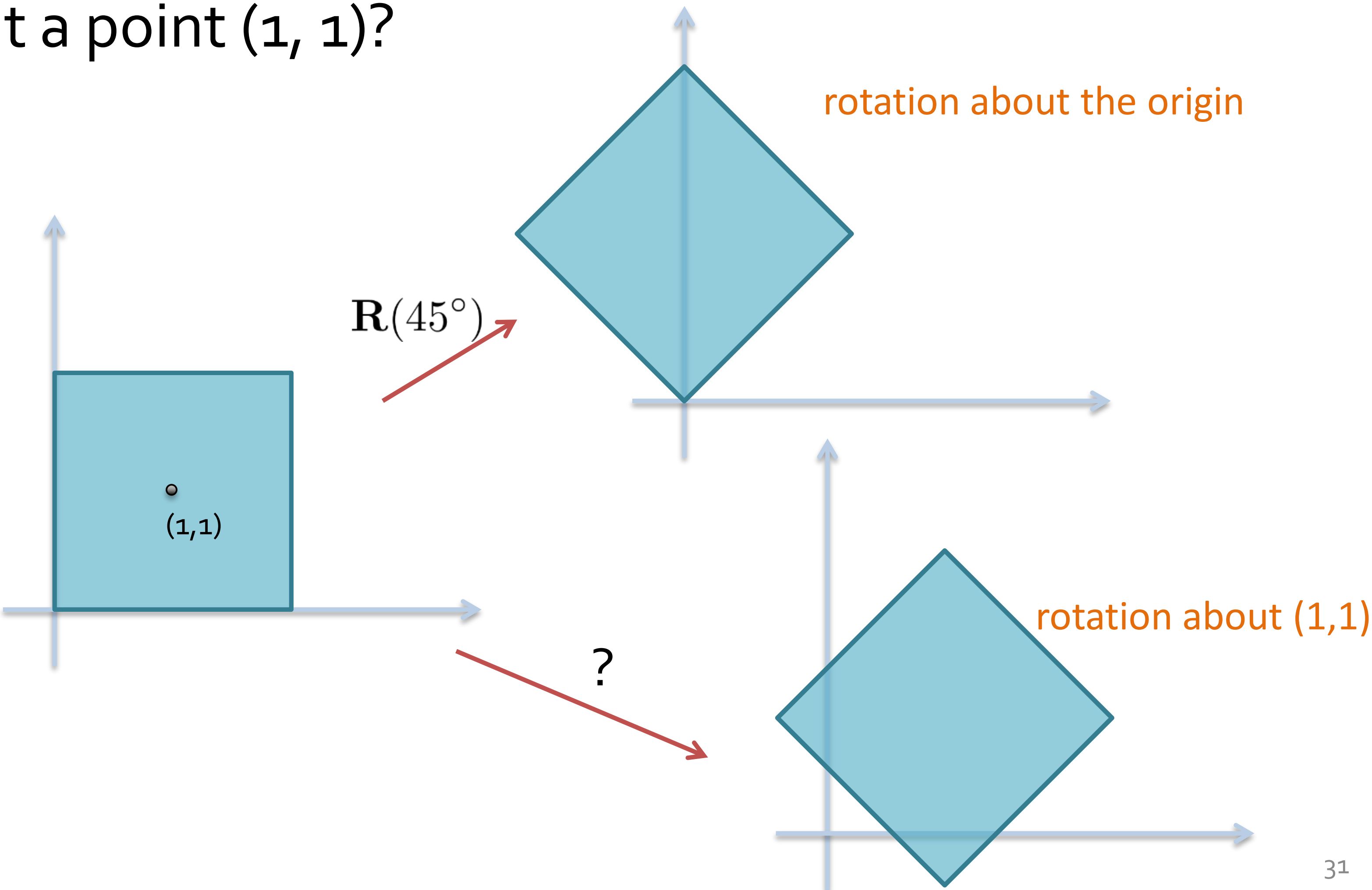
- How to find the inverse of M?

$$\begin{aligned} \mathbf{M}^{-1} &= (\mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T})^{-1} \\ &= \mathbf{T}^{-1} \cdot \mathbf{S}^{-1} \cdot \mathbf{R}^{-1} \end{aligned}$$

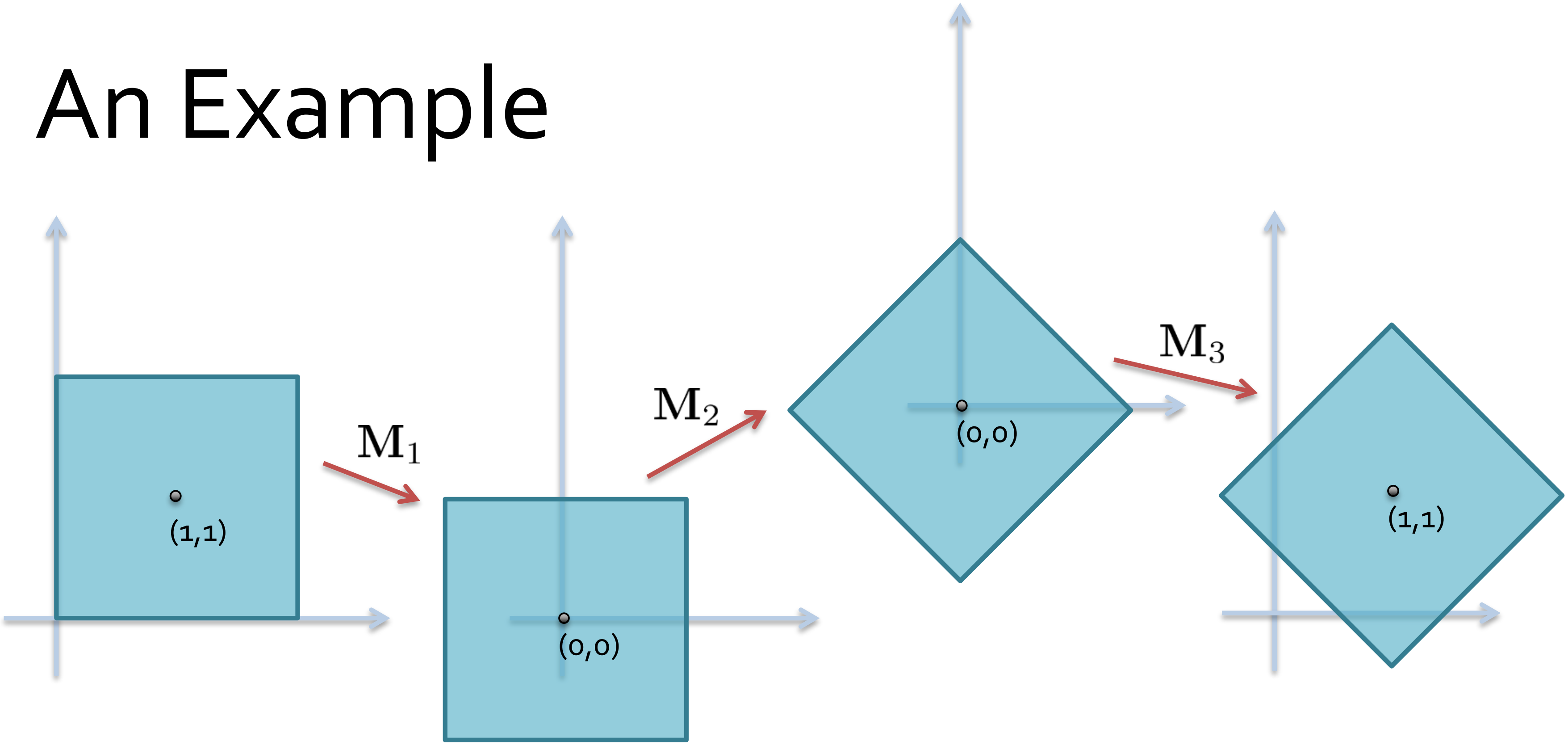
$$\begin{aligned} \mathbf{M}^{-1} \cdot \mathbf{M} &= (\mathbf{T}^{-1} \cdot \mathbf{S}^{-1} \cdot \mathbf{R}^{-1}) \cdot (\mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}) \\ &= \mathbf{T}^{-1} \cdot \mathbf{S}^{-1} \cdot (\mathbf{R}^{-1} \cdot \mathbf{R}) \cdot \mathbf{S} \cdot \mathbf{T} \\ &= \mathbf{T}^{-1} \cdot \mathbf{S}^{-1} \cdot \mathbf{I} \cdot \mathbf{S} \cdot \mathbf{T} \\ &= \cdots = \mathbf{I} \end{aligned}$$

An Example

What is the matrix representing a 2D rotation of 45° about a point (1, 1)?



An Example



translate by (1,1)

Rotate 45°

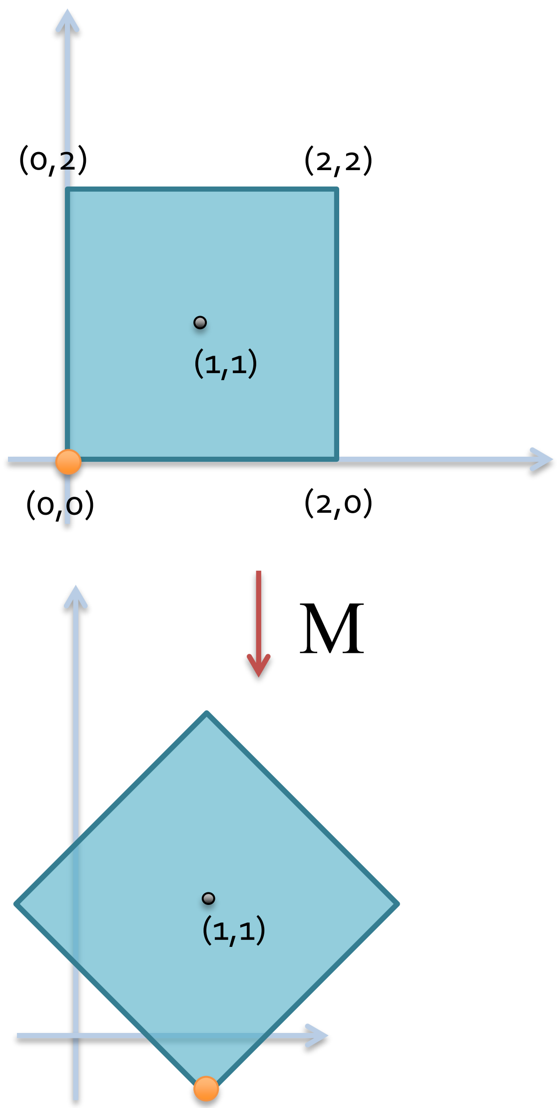
translate by (-1,-1)

$$\mathbf{M} = \mathbf{M}_3 \cdot \mathbf{M}_2 \cdot \mathbf{M}_1$$
$$= \mathbf{T}(1,1) \cdot \mathbf{R}(45^\circ) \cdot \mathbf{T}(-1,-1)$$

An Example

$$
\begin{aligned}
\mathbf{M} &= \mathbf{T}(1,1) \cdot \mathbf{R}(45^\circ) \cdot \mathbf{T}(-1,-1) \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & -\cos 45^\circ + \sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ & -\sin 45^\circ - \cos 45^\circ \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & -\cos 45^\circ + \sin 45^\circ + 1 \\ \sin 45^\circ & \cos 45^\circ & -\sin 45^\circ - \cos 45^\circ + 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 - \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Where will the four vertices of the square be transformed to?



Let's try the point $p = (0,0)^T$

$$p' = \mathbf{M}p$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1-\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 1 \end{pmatrix}$$