COMP3271 Computer Graphics

Curves & Surfaces (II)

2019-20

Objectives

Introduce the Bézier curves and surfaces

Derive the required matrices

Other Types of Curves and Surfaces

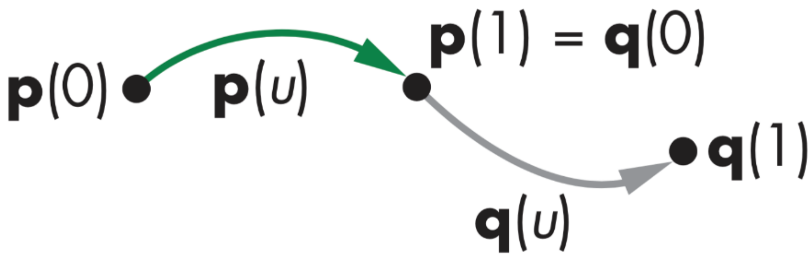
How can we get around the limitations of the interpolating form

- Lack of smoothness
- Discontinuous derivatives at join points

We have four conditions (for cubics) that we can apply to each segment

- Not necessarily using interpolating points as conditions
- Need only come close to the data

Parametric and Geometric Continuity



If $\mathbf{p}(1) = \mathbf{q}(0)$, then we have C^0 parametric continuity.

If $\mathbf{p}'(1) = \mathbf{q}'(0)$, i.e., each of the derivatives of x, y,and z components are continuous at join points, then we have C^1 parametric continuity.

Parametric and Geometric Continuity

Or we can only require that the tangents of the resulting curve be continuous, i.e., $\mathbf{p}'(1) = \alpha\mathbf{q}'(0)$ then we have G^1 geometric continuity)

This gives more flexibility then C^1 continuity as we need satisfy only two conditions rather than three at each join point

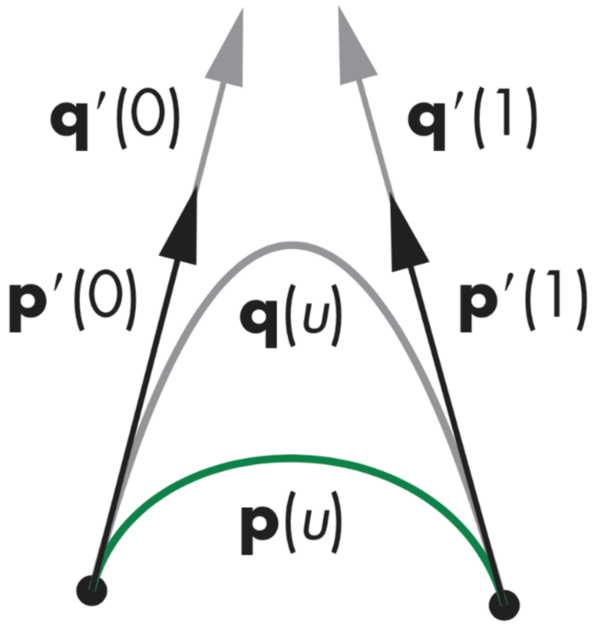
Note that G^0 is the same as C^0 continuity.

Example

Here the curves $\mathbf{p}$ and $\mathbf{q}$ have the same tangents at the ends of the segment but different derivatives

Can generate different curves by changing the "magnitude" (α) in G^1 continuity

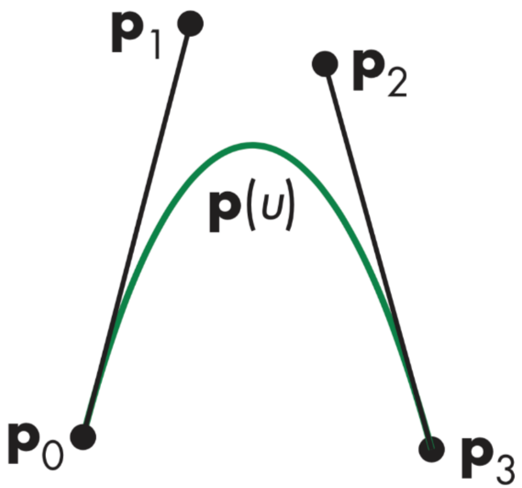
This techniques is used in drawing applications



Bézier's Idea

In graphics and CAD, we do not usually have derivative data

Bézier suggested using the same 4 data points (for cubic curve) as with the cubic interpolating curve to approximate the derivatives at the two end points



Bézier Curves

The B′ezier curve is just another representation of polynomial curves.

It is given by

$$P(t) = B_{0,n}(t)P_0 + B_{1,n}(t)P_1 + \cdots + B_{n,n}(t)P_n, t \in [0,1]$$

where

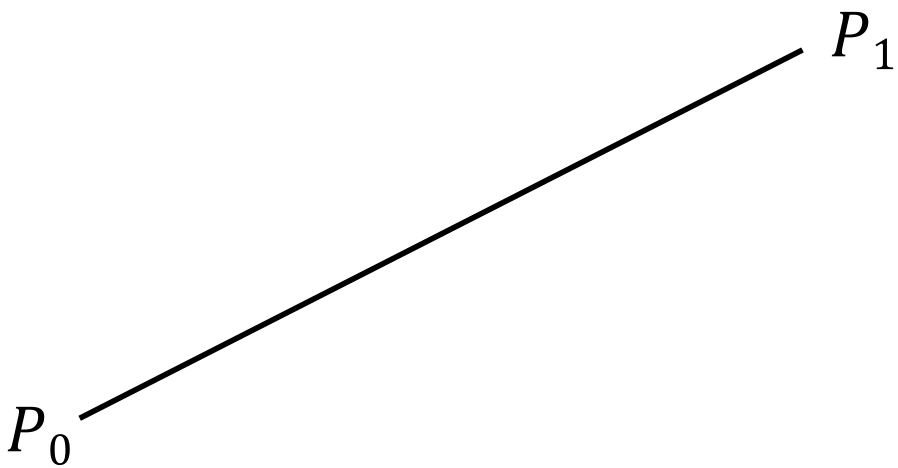
$$B_{i,n}(t) = \frac{n!}{(n-i)!\,i!}(1-t)^{n-i}t^i, \qquad i = 0,1,\ldots,n$$

Bernstein polynomials of degree n

Bézier Curve of Degree 1

$$n = 1$$
$$P(t) = B_{0,1}(t)P_0 + B_{1,1}(t)P_1$$
$$B_{0,1}(t) = 1 - t,\ B_{1,1}(t) = t$$

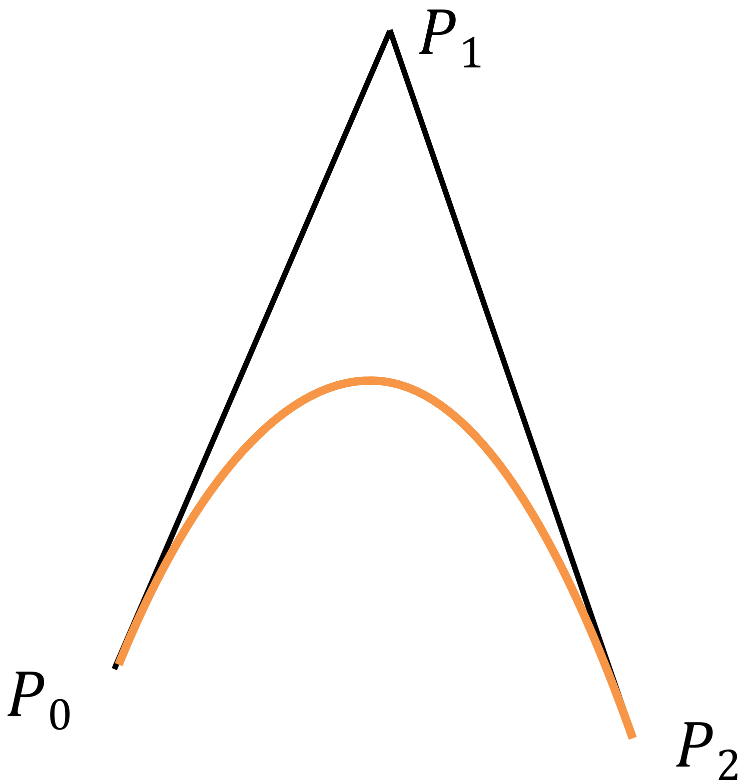
What is the curve?



Bézier Curve of Degree 2

The quadratic Bézier curve:

$$P(t) = (1-t)^2\, P_0 + 2(1-t)tP_1 + t^2P_2, \qquad t \in [0,1].$$

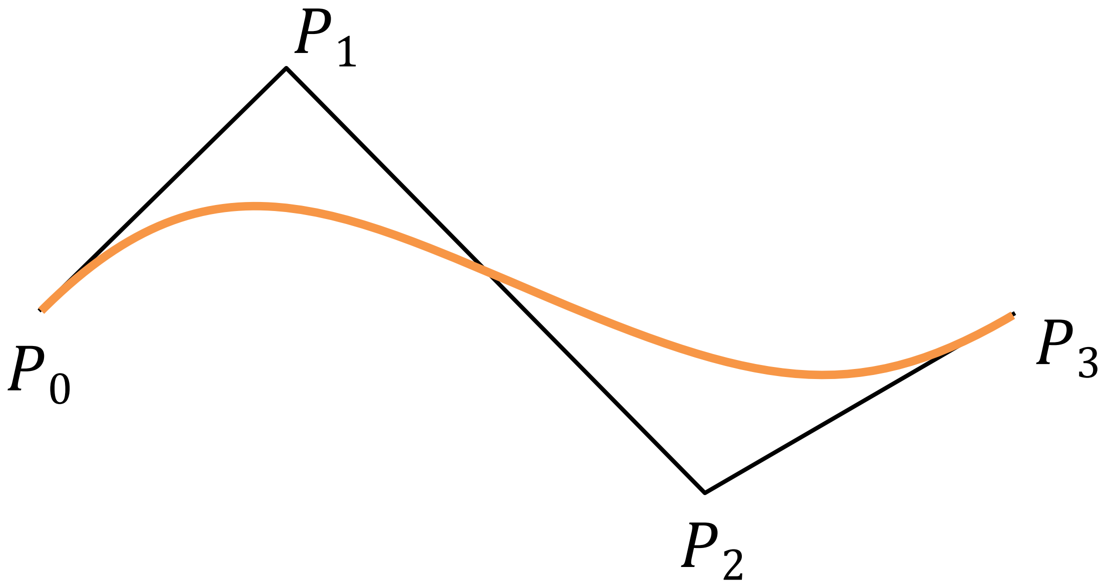


Bézier Curve of Degree 3

The cubic Bézier curve

$$P(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t) t^2 P_2 + t^3 P_3,$$

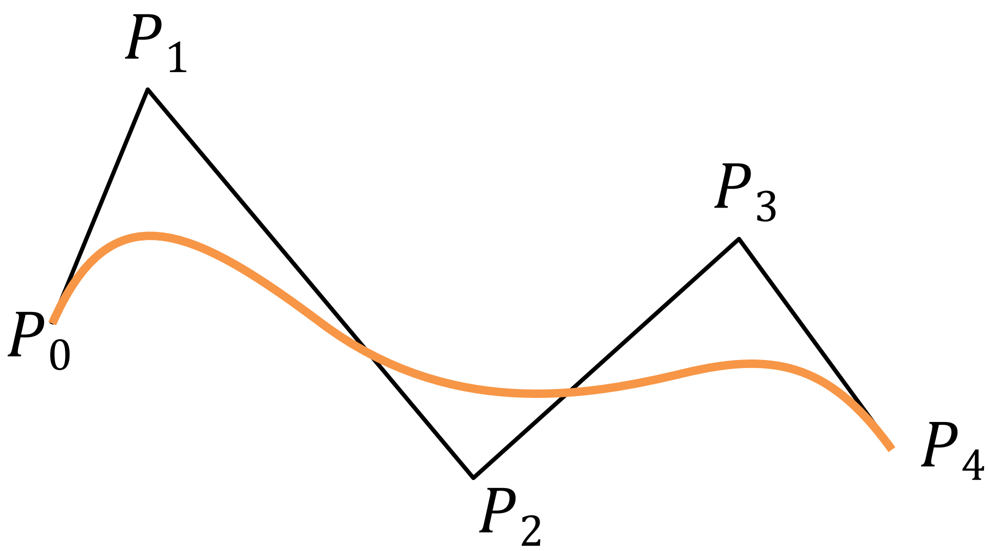
$t \in [0,1]$.



Bézier Curve of Degree 4

The quartic Bézier curve

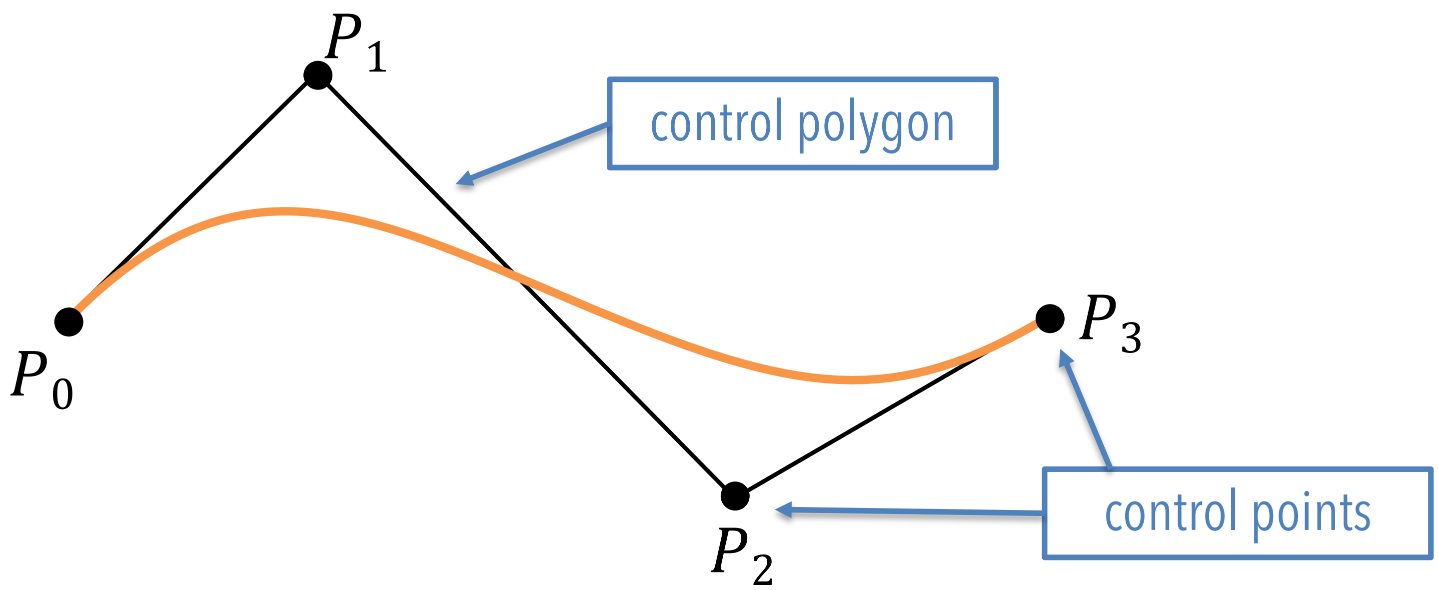
$$P(t) = (1-t)^4 P_0 + 4(1-t)^3 t P_1 + 6(1-t)^2 t^2 P_2$$
$$+4(1-t)t^3 P_3 + t^4 P_4,\ \ t \in\ [0,1].$$



Terminologies

The points P_i are called the **control points** or control vertices of the Bézier curve $P(t)$.

The polygon connecting $P_0, P_1, \ldots, P_n$, in this order, is called the **control polygon** of $P(t)$.



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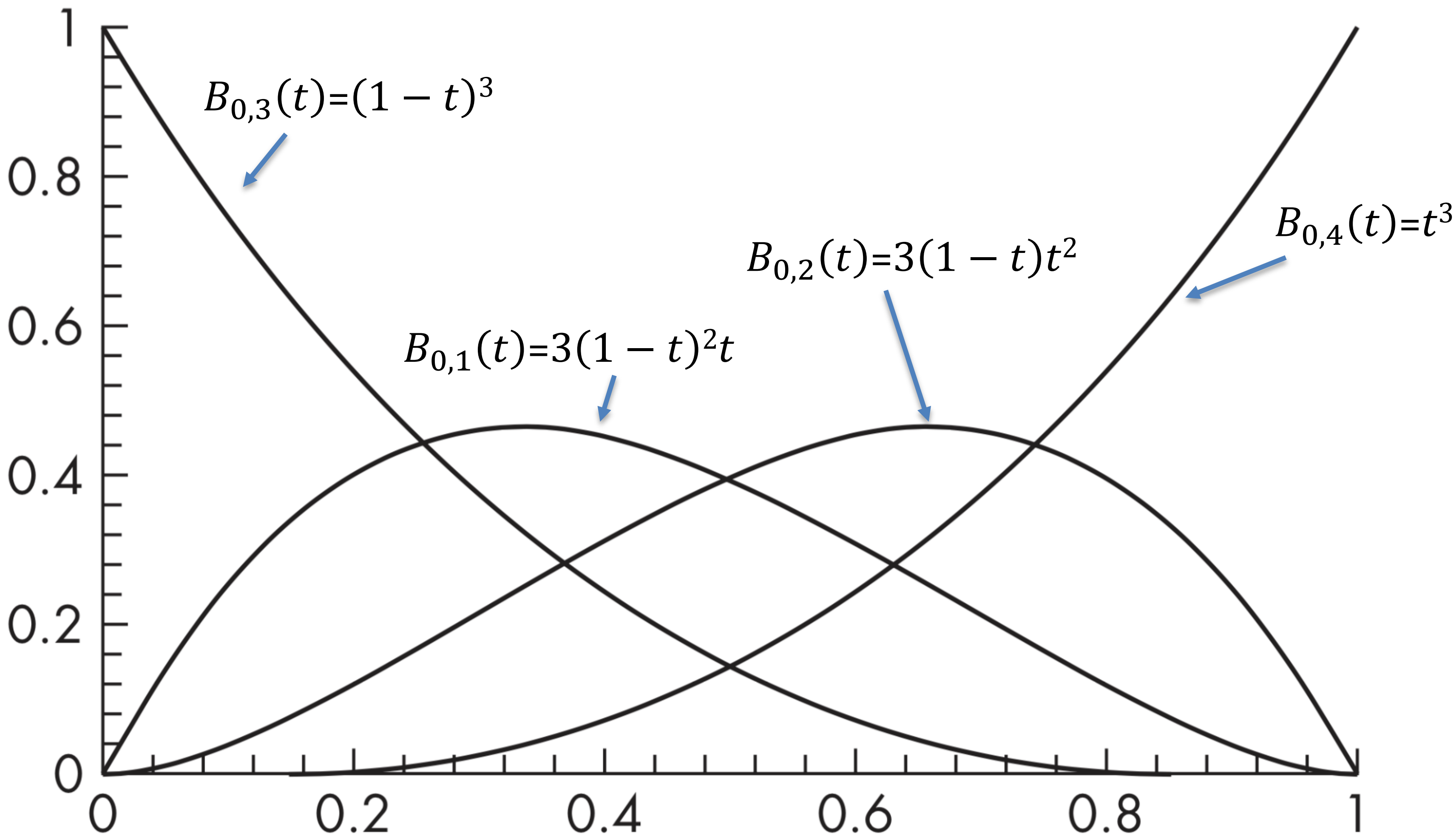
The polynomials $B_{i,n}(t)$ are called the **blending functions** or **basis functions**.

Bézier Cubic Basis Functions



Properties of Bézier Curves

1. Any polynomial curve can be put in the Bézier form.

Proof:

The polynomials

$$\{B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)\}$$
$$\{1, t, t^2, \dots, t^n\}$$

span the same space.

Example $$\begin{pmatrix} B_{0,3}(t) \\ B_{1.3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix} = \begin{pmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Properties of Bézier Curves

2. A Bézier curve $P(t)$ of degree n interpolates the two endpoints P_0 and P_n.

Proof:

$$P(0) = P_0, \qquad P(1) = P_n$$

Properties of Bézier Curves

3. A Bézier curve $P(t)$ of degree n interpolates the two end sides of the control polygon.

Proof:

$$P'(t) = \frac{dP(t)}{dt} = n \sum_{i=0}^{n-1} B_{i,n-1}(t)(P_{i+1} - P_i)$$

$$P'(t)|_{t=0} = n(P_1 - P_0)$$

$$P'(t)|_{t=1} = n(P_n - P_{n-1})$$

Properties of Bézier Curves

4. Convex hull property
   The curve segment $P(t), t \in [0,1]$, lies entirely inside the convex hull of all control points

Proof:

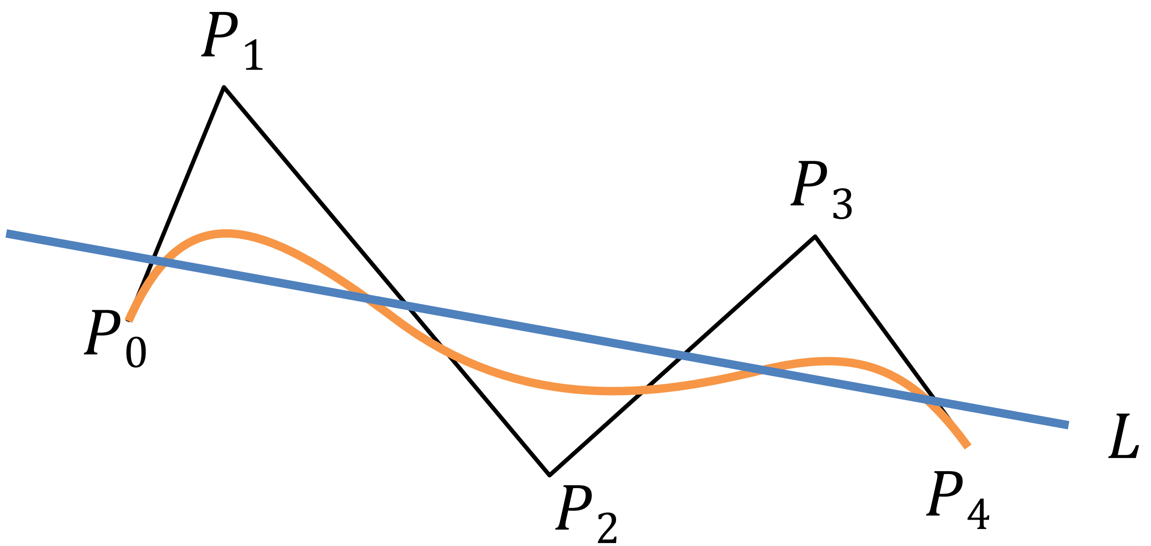
This property follows from

$$\sum_{i=0}^{n} B_{i,n}(t) \equiv 1,$$

$$B_{i,n}(t) \geq 0, \qquad t \in [0,1]$$

Properties of Bézier Curves

5. Variation diminishing property
The number of intersections between an arbitrary straight line L and the Bézier curve $P(t), t \in [0,1]$, is NO greater than the number of intersections between the line L and the control polygon of $P(t)$.



Properties of Bézier Curves

6. Invariant form under affine transformations
   For any affine transformation $\mathbf{M}$: $X' = AX + b$,
   there is
$$\mathbf{M}\big(P(t)\big) = \sum_{i=0}^{n} B_{i,n}(t)\mathbf{M}(P_i).$$

Proof:

$$\begin{aligned}\mathbf{M}\big(P(t)\big) &= AP(t) + b \\ &= \sum_{i=0}^{n} B_{i,n}(t)AP_i + \sum_{i=0}^{n} B_{i,n}(t)b \\ &= \sum_{i=0}^{n} B_{i,n}(t)(AP_i + b) \\ &= \sum_{i=0}^{n} B_{i,n}(t)M(P_i)\end{aligned}$$

Properties of Bézier Curves

6. Invariant form under affine transformations

Thanks to this property, when a Bézier curve is transformed affinely, we can

a) transform many sampled points on the curve directly, or

b) transform the control points only and use the transformed control points to generate a new Bézier curve

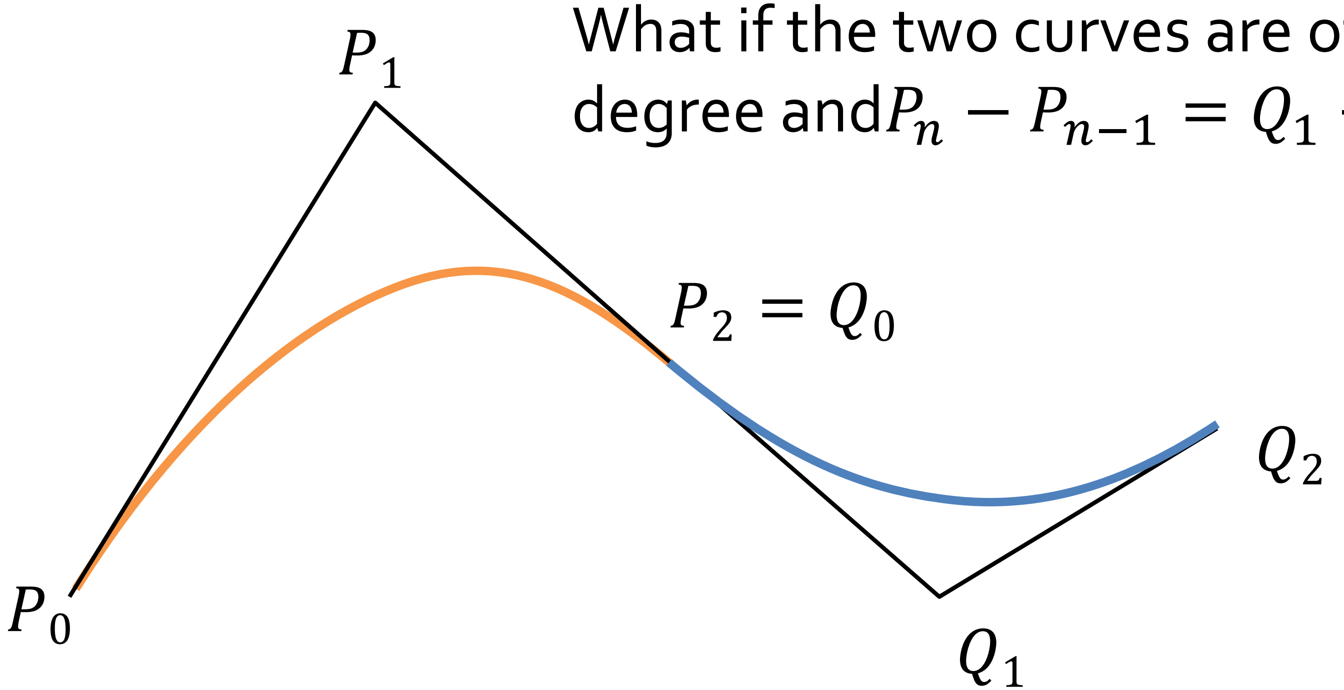
These two ways yield the same transformed curve, but the latter can be more efficient when there are many points to be evaluated on the curve.

Composite Bézier Curves

By placing the end control points of two Bézier curve collinear, we can obtain a smooth curve comprising two Bézier curves.

The curves then have G^1 continuity at P_2. Why?

What if the two curves are of the same degree and $P_n - P_{n-1} = Q_1 - Q_0$?



Bézier Matrix

$$\mathbf{p}(u) = \mathbf{b}(u)^T\mathbf{p} = \mathbf{u}^T\mathbf{M}_B\mathbf{p}$$

blending functions

For cubic Bézier curves

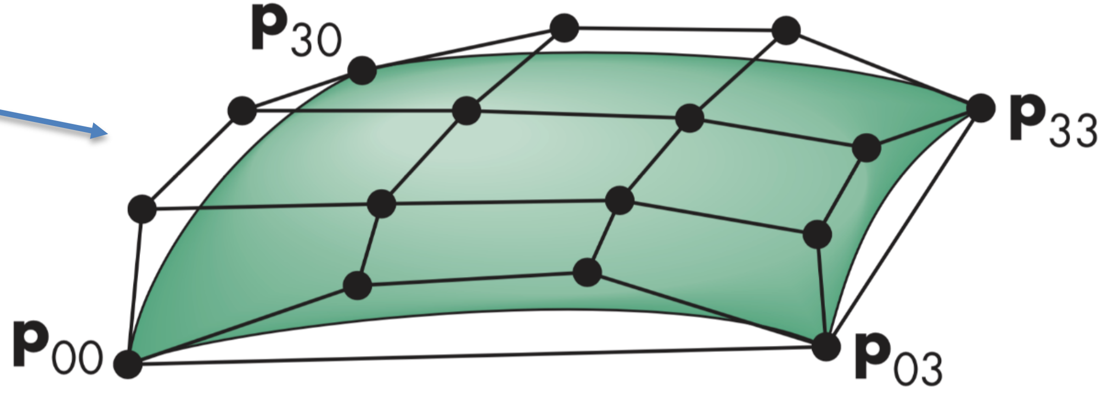
$$\mathbf{M}_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

Bézier Patches

Using same data array $\mathbf{P} = [\mathbf{p}_{ij}]$ as with interpolating form

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_i(u)\, b_j(v)\, \mathbf{p}_{ij} = \mathbf{u}^T\mathbf{M}_B\mathbf{P}{\mathbf{M}_B}^T\mathbf{v}$$

Patch lies in
convex hull



Bézier Bicubic Surface Patches

A 2 x 1 bicubic surface patches