COMP3271 Computer Graphics

# Curves & Surfaces (I)

2019-20

# Objectives

Different representations for curves and surfaces

Design criteria

Parametric curves & surfaces

Interpolation

# Escaping Flatland

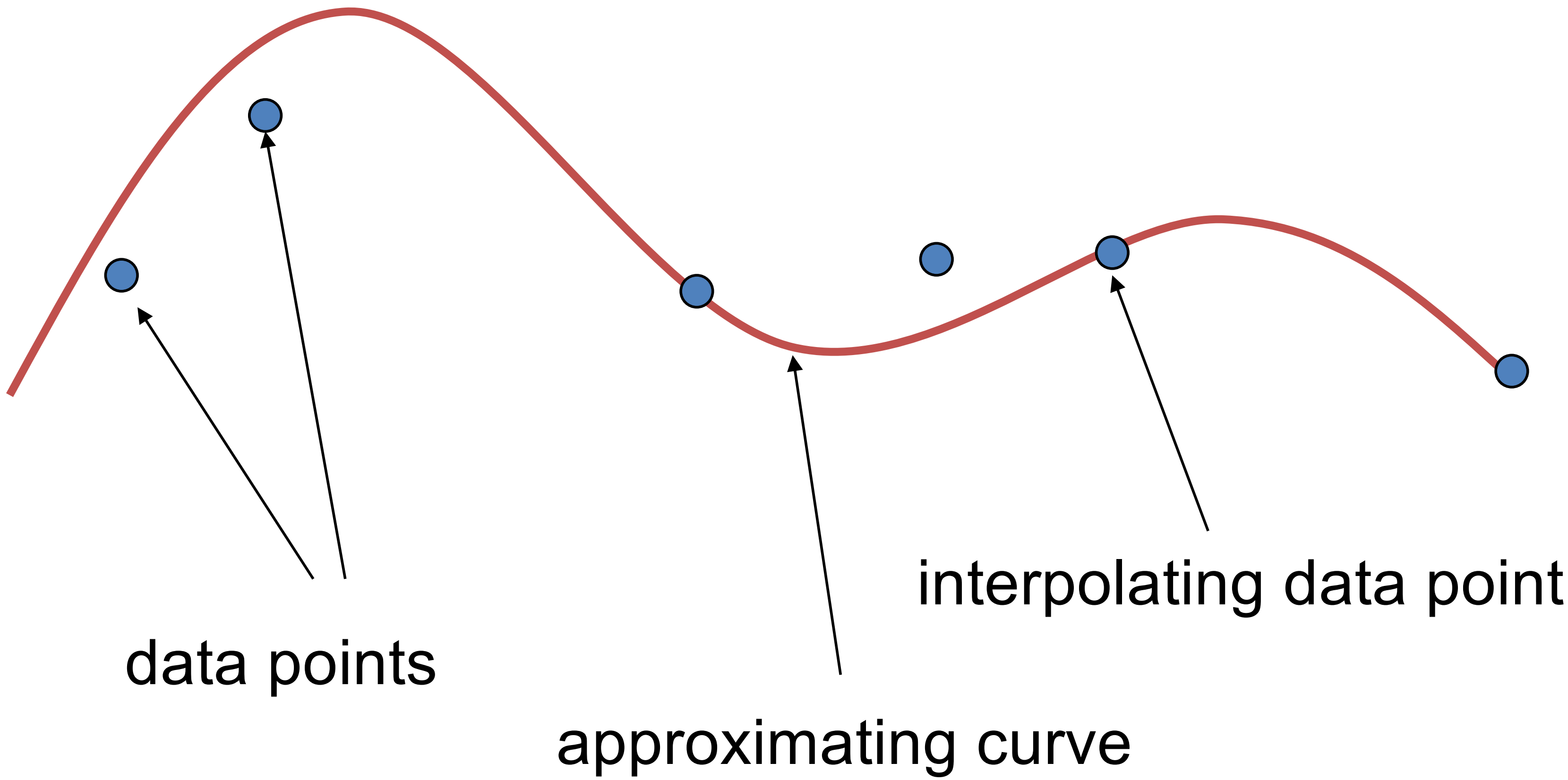
Until now we have worked with flat entities such as lines and flat polygons

- Fit well with graphics hardware
- Mathematically simple

But the world is not composed of flat entities

- Need curves and curved surfaces
- Implementation can render them approximately with flat primitives

# Modeling with Curves



# What Makes a Good Representation?

There are many ways to represent curves and surfaces

Some design criteria

- Local control of shape
- Stability
- Smoothness and continuity (in terms of derivatives)
- Ability to evaluate derivatives
- Ease of evaluation
- Ease of rendering
- Must we interpolate or can we just come close to data?

# Explicit Representation

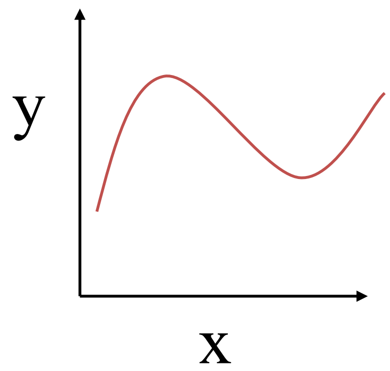
Most familiar form of curve in 2D

$$y = \mathrm{f}(x)$$

Express a variable in terms of other variables

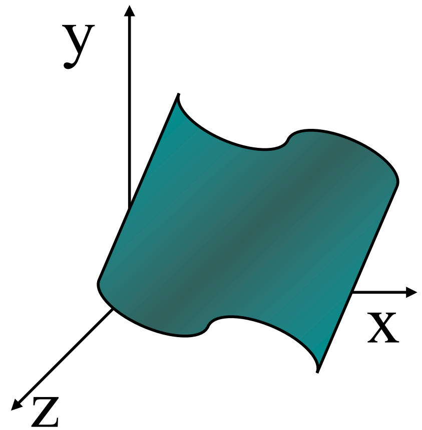
Cannot represent all curves

- Vertical lines
- Circles




Extension to 3D

- $y = \mathrm{f}(x),\ z = \mathrm{g}(x)$
- The form $z = \mathrm{f}(x,y)$ defines a surface




Cannot represent a sphere in the form of $z = \mathrm{f}(x, y)$. Why?

# Implicit Representation

Two dimensional curve(s)

$$g(x, y)=0$$

Represents the membership of points on curve

Much more robust

- All lines $ax + by + c = 0$
- Circles $x^2 + y^2 - r^2 = 0$

Not unique

- $(x^2 + y^2 - r^2)^2 = 0$ and $\sqrt{x^2 + y^2} - 1 = 0$ represent the same circle as $x^2 + y^2 - r^2 = 0$.

In general, no analytic way to solve for points that satisfy the equation

# Implicit Representation

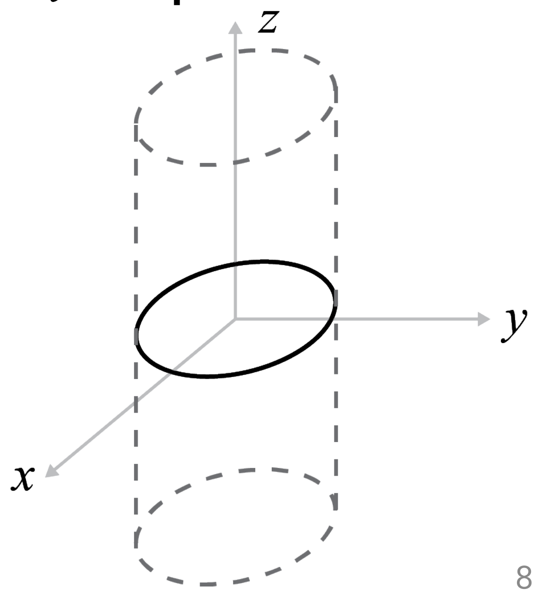
Three dimensions $g(x, y, z)=0$ defines a surface

- E.g., $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ represents the unit sphere

**Question**: how to represent the unit circle centered at the origin in the $xy$-plane implicitly in the $xyz$-space?

To represent a 3D curve

- Intersect two surfaces to get a curve

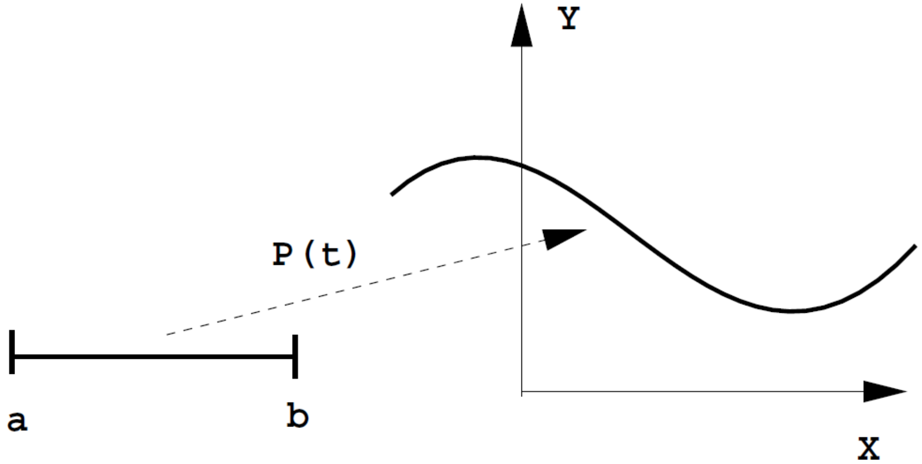


# Parametric Representation

Two dimensional curves:

$$x = x(t), y = y(t), \quad t \epsilon [a, b]$$

Express the *x*,*y* values of each point on the curve explicitly in terms of an independent variable, *t*, i.e., the **parameter**, with a domain [a,b]



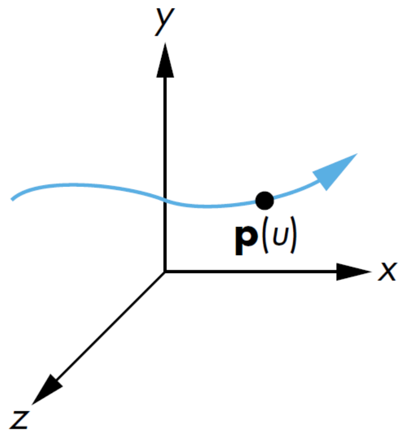

Example: Unit circle

$$P(\theta) = \big(x(\theta), y(\theta)\big) = (\cos\theta, \sin\theta), \theta\epsilon[0,2\pi)$$

Easily extended to three dimensional curves:

$$x = x(u), y = y(u), z = z(u), \ u\epsilon[a, b]$$

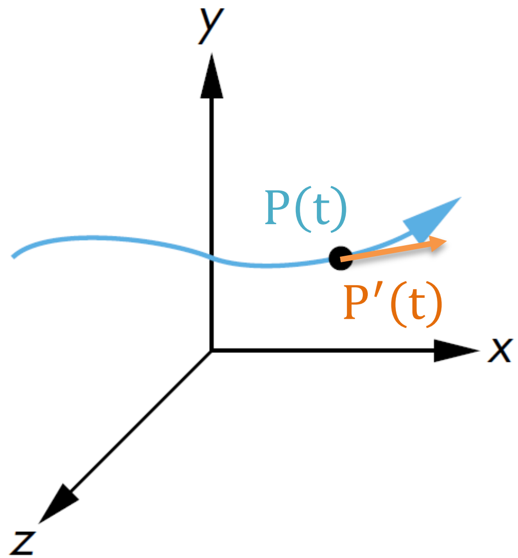
Still in one parameter, hence a curve



# Parametric Representation

We trace the curve $P(t) = (x(t), y(t), z(t))$ as $t$ varies.
Hence, we can talk of the velocity of $P(t)$:

$$\mathrm{P}'(\mathrm{t}) = \frac{dP(t)}{dt} = \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{bmatrix}$$




This gives the tangent direction of the curve.

The speed of $P(t)$ is then $|\mathrm{P}'(\mathrm{t})|$.

# Parametric Representation

When the speed of $P(t)$ is constant or nearly constant, the computed points $P(t_i)$ on $P(t)$ are evenly or nearly evenly spaced if the parameters $t_i$ , $i\ =\ 0, 1, 2, \ldots$, are evenly sampled.

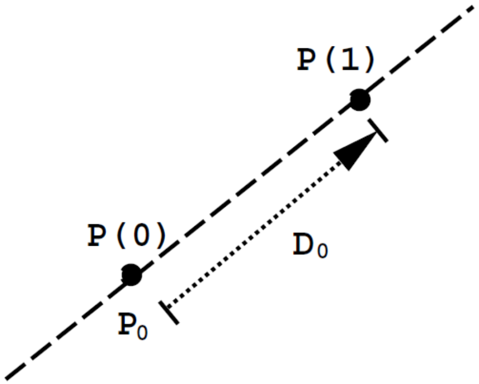
Example. The following parametric equation of the unit circle has a constant speed.

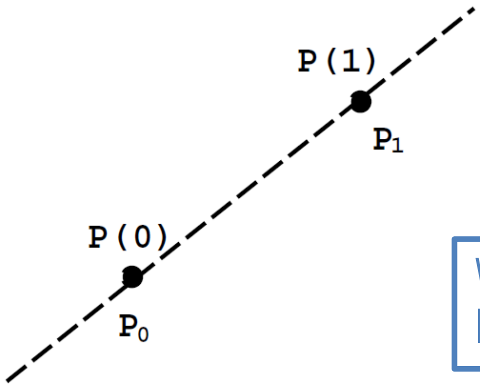
$$P(\theta) = \big(x(\theta), \mathrm{y}(\theta)\big) = (\cos\theta\ , \sin\theta)\ , \theta \epsilon [0{,}2\pi)$$

# Parametric Lines

**Example**. A straight line passing through the point $P_0$ with the direction vector $D_0$ can be represented by

$$P(t) = \big(x(t), y(t)\big) = P_0 + tD_0,\ \ t \in\ (-\infty, \infty).$$




With $t\ \in\ [0,1]$, we get the straight line segment $P_0$ and $P_1$

**Example**. A straight line passing through two distinct points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ is commonly represented by

$$P(t)\ =\ (1-t)P_0\ +\ tP_1, t \in\ (-\infty, \infty).$$

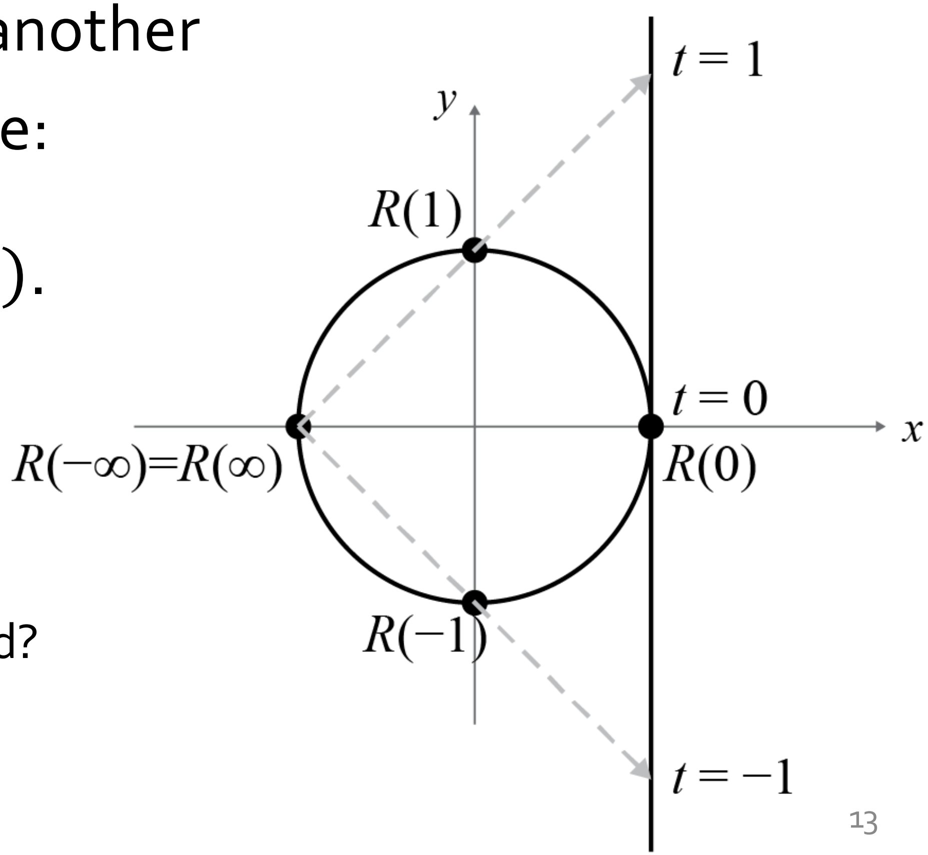
# Unit Circle in Parametric Form

$$P(\theta) = \big(x(\theta), \mathrm{y}(\theta)\big) = (\cos\theta\,, \sin\theta)\,, \theta \epsilon [0{,}2\pi)$$

Since $\cos\theta = \dfrac{1-\tan^{2}(\theta/2)}{1+\tan^{2}(\theta/2)}$ and $\sin\theta = \dfrac{2\tan(\theta/2)}{1+\tan^{2}(\theta/2)}$

Substituting $t = \tan\dfrac{\theta}{2}$, we have another parametric from for the unit circle:

$$R(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), t \in \ (-\infty, \infty).$$




Is this parameterization with constant speed?

# Parametric Surfaces

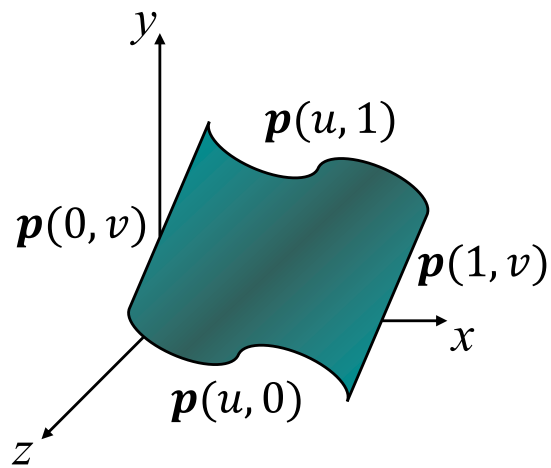
Surfaces require 2 parameters

$$x = x(u, v)$$

$$y = y(u, v)$$

$$z = z(u, v)$$

$$\boldsymbol{p}(u, v) = [x(u, v), y(u, v), z(u, v)]^T$$




the four boundary curves of a patch

Want same properties as curves:

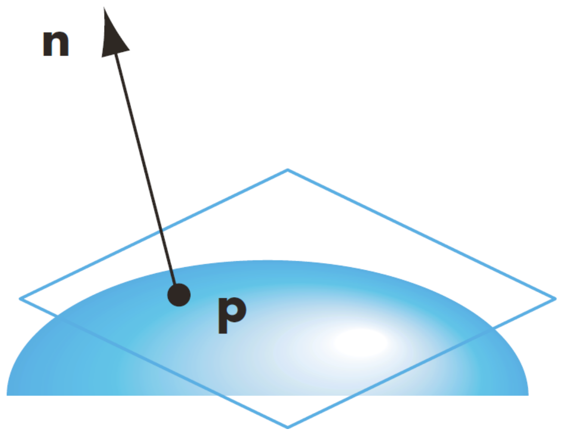
- Smoothness
- Differentiability
- Ease of evaluation

# Surface Normals

We can differentiate with respect to $u$ and $v$ to obtain the normal at any point $\boldsymbol{p}$

$$\frac{\partial \mathbf{p}(u,v)}{\partial u} = \begin{bmatrix} \partial \mathrm{x}(u,v)/\partial u \\ \partial \mathrm{y}(u,v)/\partial u \\ \partial \mathrm{z}(u,v)/\partial u \end{bmatrix} \qquad \frac{\partial \mathbf{p}(u,v)}{\partial v} = \begin{bmatrix} \partial \mathrm{x}(u,v)/\partial v \\ \partial \mathrm{y}(u,v)/\partial v \\ \partial \mathrm{z}(u,v)/\partial v \end{bmatrix}$$

$$\mathbf{n} = \frac{\partial \mathbf{p}(u,v)}{\partial u} \times \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

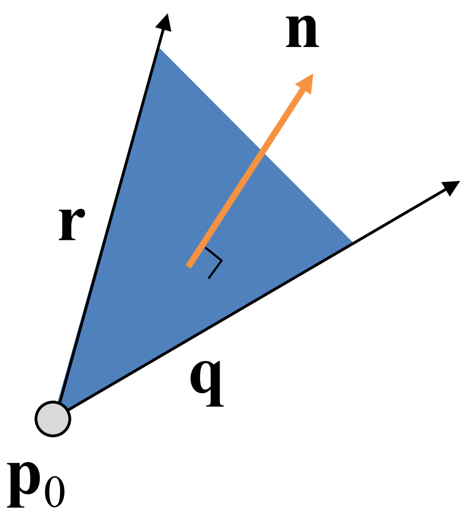


# Parametric Planes

Point-vector form

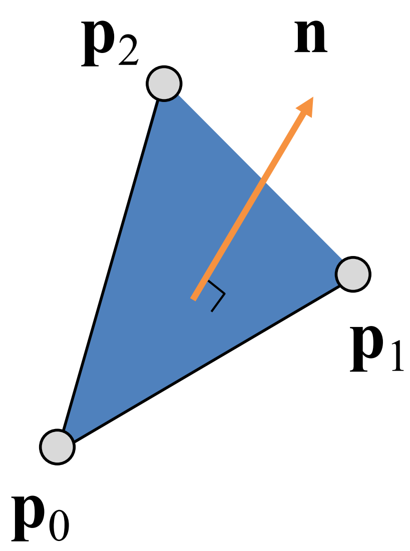
$$\mathbf{p}(u,v) = \mathbf{p}_0 + u\mathbf{q} + v\mathbf{r}$$

$$\mathbf{n} = \mathbf{q} \times \mathbf{r}$$



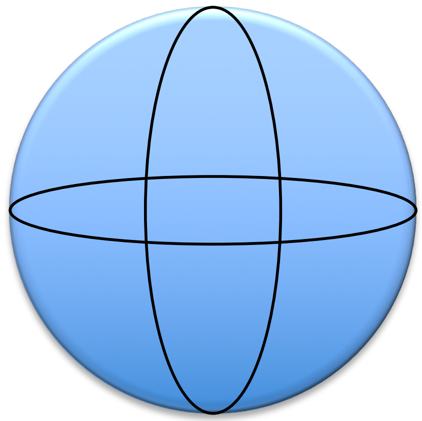

Three-point form

$$\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_0$$
$$\mathbf{r} = \mathbf{p}_2 - \mathbf{p}_0$$



# Parametric Spheres

$$x(\theta, \varphi) = \ r\cos\theta\sin\varphi$$
$$y(\theta, \varphi) = \ r\sin\theta\sin\varphi$$
$$z(\theta, \varphi) = \ r\cos\varphi$$



$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi$$

$\theta$: constant; circles of constant longitude

$\varphi$: constant; circles of constant latitude

Exercise: differentiate to show $\mathbf{n} = \mathbf{p}$

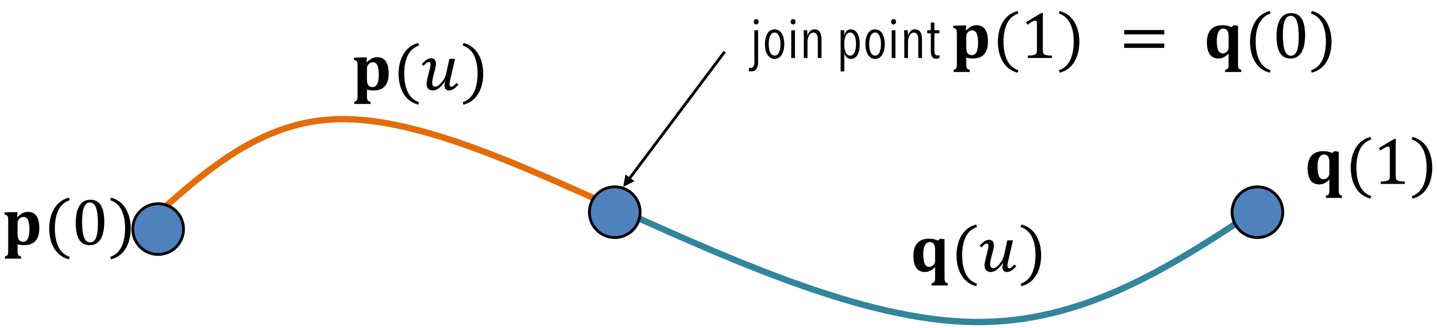
# Curve Segments

After normalizing $u$, each curve can be written as

$$\mathbf{p}(u) = [x(u), y(u), z(u)]^T, \quad 0 \leq u \leq 1$$

In classical numerical methods, we design a single global curve

In computer graphics and CAD, it is better to design small connected curve segments



# Parametric Polynomial Curves

$\mathbf{p}(t) = (x(t), y(t), z(t))^T$ is called a polynomial curve if $x(t)$, $y(t)$ and $z(t)$ are polynomial functions of $t$.

A polynomial curve of degree $n$ is of the form

$$\mathbf{p}(t) = \sum_{k=0}^{n} \mathbf{c}_k t^k \quad \text{where} \quad \mathbf{c}_k = \begin{pmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{pmatrix}.$$

also a polynomial curve of **order $n$+1**

Hence,

$$x(t) = \sum_{k=0}^{n} \mathrm{c}_{xk} t^k,\ y(t) = \sum_{k=0}^{n} \mathrm{c}_{yk} t^k,\ z(t) = \sum_{k=0}^{n} \mathrm{c}_{zk} t^k$$

$t \in [0,1]$.

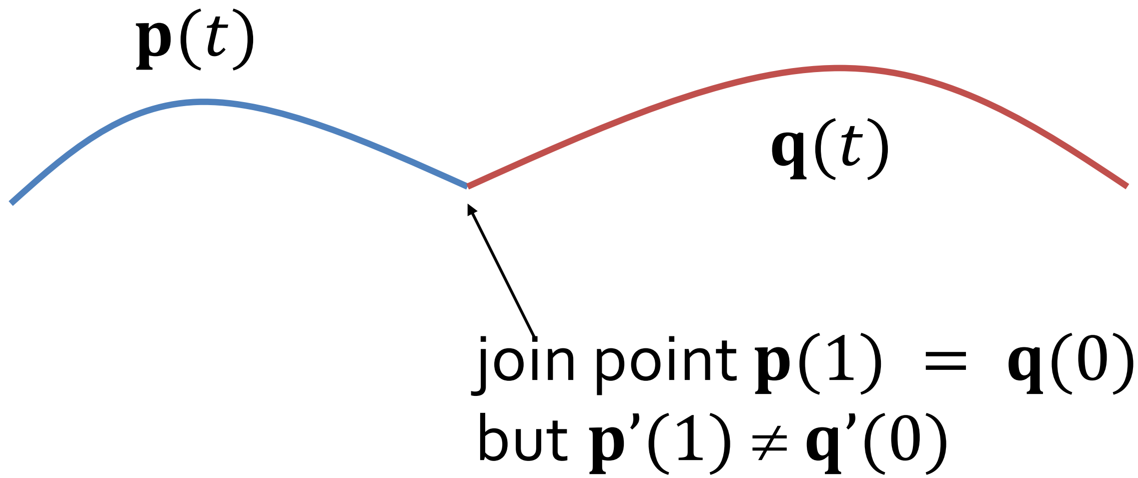
We need to determine 3($n$+1) coefficients
Equivalently we need 3($n$ +1) independent conditions

# Why Polynomials

Easy to evaluate – need only $+, -, \times, \div$ (e.g., no sin/cos)

Continuous and differentiable everywhere

- Still need to worry about continuity at join points including continuity of derivatives



# Cubic Parametric Polynomials

We mostly use cubic curves which gives balance between ease of evaluation and flexibility in design

$$\mathbf{p}(t) = \sum_{k=0}^{3} \mathbf{c}_k t^k$$

Four coefficients to determine for each of $x$, $y$ and $z$

Seek four independent conditions for various values of $t$ resulting in 4 equations in 4 unknowns for each of $x$, $y$ and $z$

- Conditions are a mixture of continuity requirements at the join points and conditions for fitting the data

# Representation in Matrix-Vector Form

$$\mathbf{p}(u) = \sum_{k=0}^{3} \mathbf{c}_k u^k = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 u^2 + \mathbf{c}_3 u^3$$

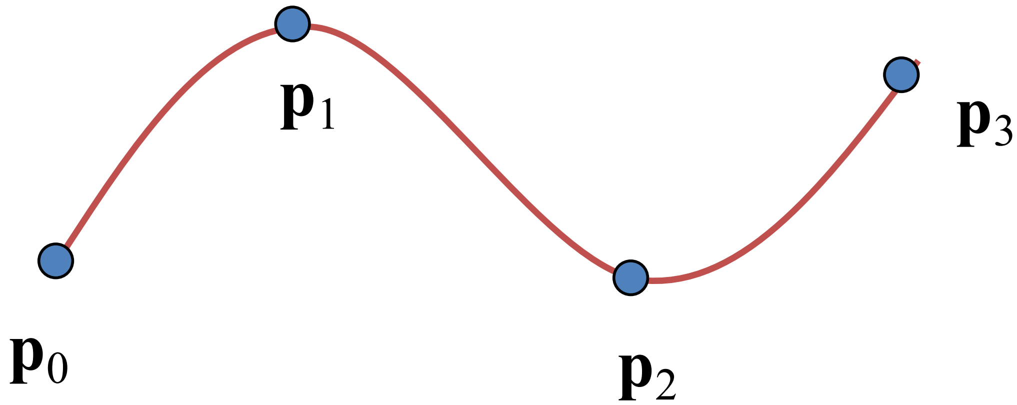
a cubic curve

$$= \mathbf{u}^T \mathbf{c} = \mathbf{c}^T \mathbf{u}$$

where

$$\mathbf{c} = \begin{pmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}, \quad \mathbf{c}_k = \begin{pmatrix} c_{kx} \\ c_{ky} \\ c_{kz} \end{pmatrix}$$

# Interpolating Curve

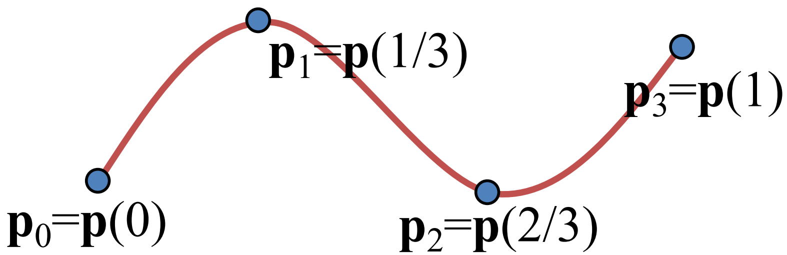



Given four data (control) points $\mathbf{p}_0$ , $\mathbf{p}_1$ ,$\mathbf{p}_2$ , $\mathbf{p}_3$
determine cubic $\mathbf{p}(u)$ which passes through them

Need to find $\mathbf{c}_0$ ,$\mathbf{c}_1$ ,$\mathbf{c}_2$ , $\mathbf{c}_3$

# Interpolation Equations

$$\mathbf{p}(u) = \sum_{k=0}^{3} \mathbf{c}_k u^k = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 u^2 + \mathbf{c}_3 u^3$$




apply the interpolating conditions at $u = 0,\ 1/3,\ 2/3,\ 1$

$$\begin{aligned}
\mathbf{p}_0 &= \mathbf{p}(0) = \mathbf{c}_0 \\
\mathbf{p}_1 &= \mathbf{p}(1/3) = \mathbf{c}_0 + (1/3)\mathbf{c}_1 + (1/3)^2\mathbf{c}_2 + (1/3)^3\mathbf{c}_2 \\
\mathbf{p}_2 &= \mathbf{p}(2/3) = \mathbf{c}_0 + (2/3)\mathbf{c}_1 + (2/3)^2\mathbf{c}_2 + (2/3)^3\mathbf{c}_2 \\
\mathbf{p}_3 &= \mathbf{p}(1) = \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_2
\end{aligned}$$

or in matrix form

$$\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)^T = \mathbf{Ac}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

# Interpolation Matrix

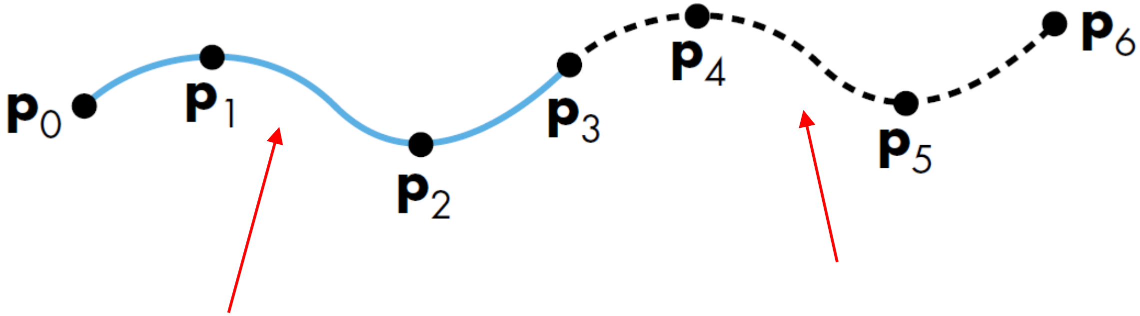
$$\mathbf{p} = \mathbf{A}\mathbf{c}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & (\frac{1}{3}) & (\frac{1}{3})^2 & (\frac{1}{3})^3 \\ 1 & (\frac{2}{3}) & (\frac{2}{3})^2 & (\frac{2}{3})^3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Let $\mathbf{M}_I = \mathbf{A}^{-1}$ and solving for $\mathbf{c}$,

$$\mathbf{c} = \mathbf{M}_I\mathbf{p}, \qquad \mathbf{M}_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & 22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{pmatrix}$$

Note that $\mathbf{M}_I$ does not depend on input data and can be used for each segment in $x$, $y$, and $z$

# Interpolating Multiple Segments




use $\mathbf{p} = [\mathbf{p}_0\ \mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^T$

use $\mathbf{p} = [\mathbf{p}_3\ \mathbf{p}_4\ \mathbf{p}_5\ \mathbf{p}_6]^T$

Get continuity at join points but not continuity of derivatives

# Blending Functions

Rewriting the equation for $\mathrm{p(u)}$

$$\mathbf{p}(u) = \mathbf{u}^T\mathbf{c} = \mathbf{u}^T\mathbf{M}_I\mathbf{p} \;=\; \mathbf{b}(u)^T\mathbf{p}$$

where $\mathbf{b}(u) \;=\; [b_0(u) \;\; b_1(u) \;\; b_2(u) \;\; b_3(u)]^T$ is an array of **blending polynomials** such that

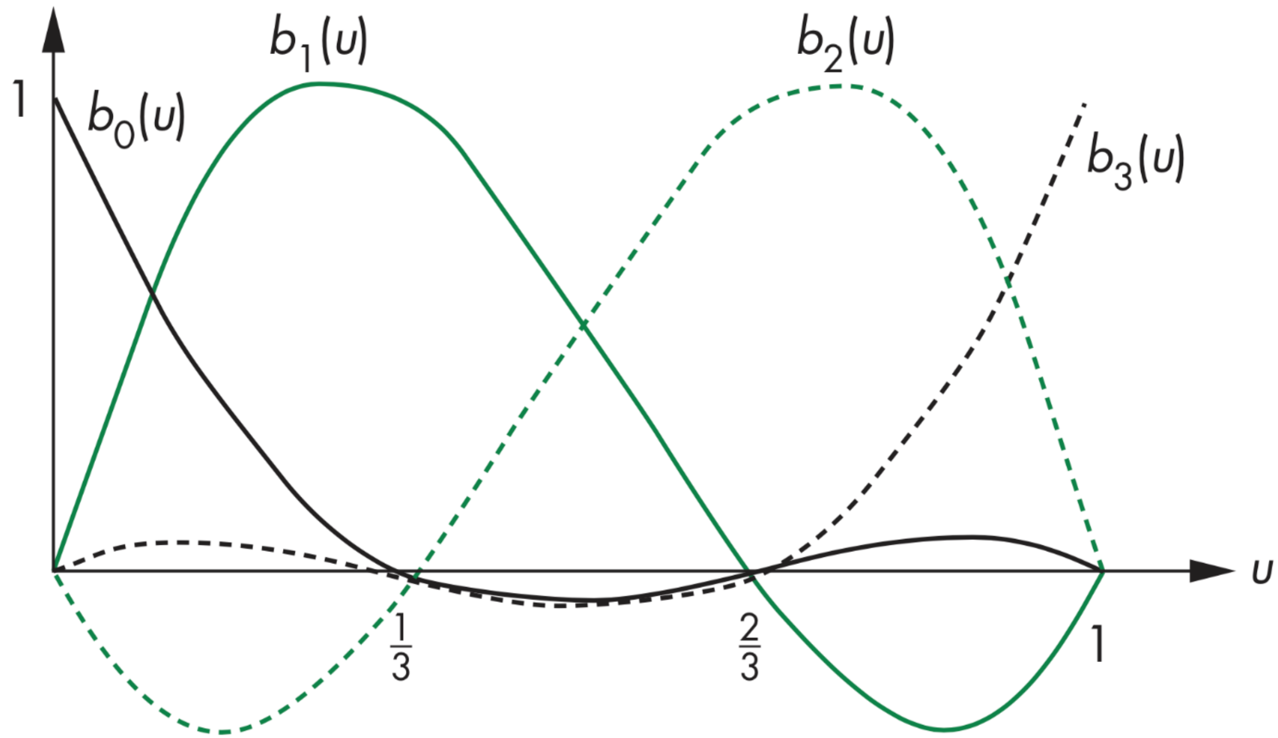
$$\mathbf{p}(u) \;=\; b_0(u)\mathbf{p}_0 + \; b_1(u)\mathbf{p}_1 + \; b_2(u)\mathbf{p}_2 + \; b_3(u)\mathbf{p}_3$$

$$\begin{aligned}
b_0(u) &\;=\; -4.5(u - \tfrac{1}{3})(u - \tfrac{2}{3})(u - 1) \\
b_1(u) &\;=\; 13.5u\,(u - \tfrac{2}{3})(u - 1) \\
b_2(u) &\;=\; -13.5u\,(u - \tfrac{1}{3})(u - 1) \\
b_3(u) &\;=\; 4.5u\,(u - \tfrac{1}{3})(u - \tfrac{2}{3})
\end{aligned}$$

# Blending Functions

These functions are not smooth

- Hence the interpolation polynomial is not smooth



# Cubic Polynomial Surfaces

A degree $m \times n$ parametric polynomial surface is defined as

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \sum_{i=0}^{n} \sum_{j=0}^{n} \mathbf{c}_{ij}\, u^i\, v^j\,, \qquad (u,v) \in [0,1]^2$$

More specifically, a cubic polynomial surface is given by

$$\mathbf{p}(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} \mathbf{c}_{ij} u^i v^j\,, \qquad (u,v) \in [0,1]^2.$$

Need 48 coefficients ( 4 x 4 x 3) to determine a surface patch

# Interpolating Patch

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{c}_{ij}\, u^i\, v^j$$

a bicubic surface

Define $\mathbf{u} = [1 \;\; u \;\; u^2 \;\; u^3]^T,\; \mathbf{v} = [1 \;\; v \;\; v^2 \;\; v^3]^T,\; \mathbf{C} = [\mathbf{c}_{ij}]$

4x4 matrix, each element is a 3-vector

We may write

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{c}_{ij}\, u^i\, v^j \;=\; \mathbf{u}^T\mathbf{C}\mathbf{v}$$

# Interpolating Patch in Matrix Form

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{c}_{ij}\, u^i\, v^j \qquad = \mathbf{u}^T\mathbf{C}\mathbf{v}$$

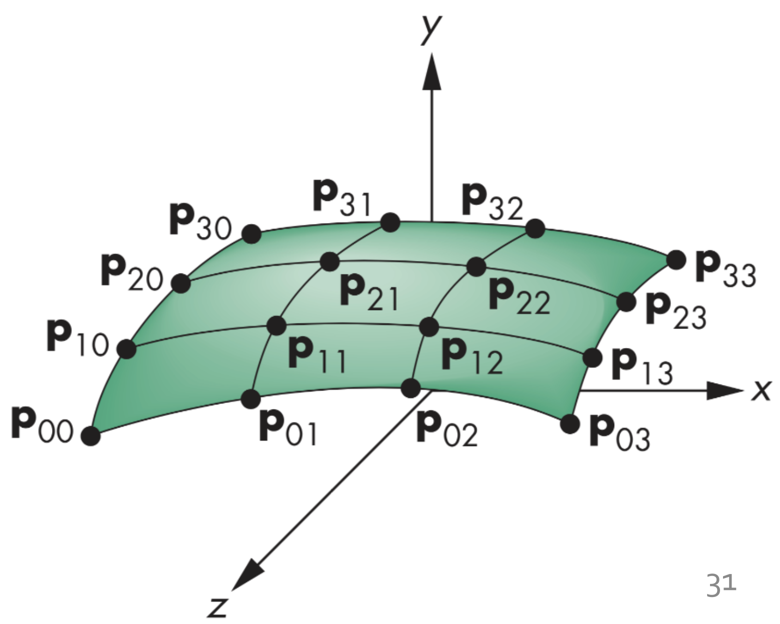
Need 16 conditions to determine the 16 coefficients $\mathbf{c}_{ij}$

a 3-vector

Consider $u = v = 0$, and we have

$$\mathbf{p}_{00} = [1 \quad 0 \quad 0 \quad 0]\,\mathbf{C}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \mathbf{c}_{00}$$

Choose $u, v = 0, \frac{1}{3}, \frac{2}{3}, 1$
and we have the 16 conditions

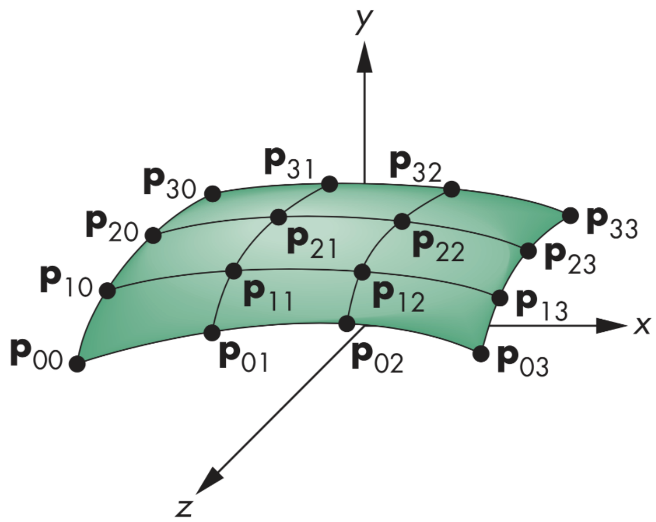


# Interpolating Patch in Matrix Form

Instead of writing the 16 equations one by one and solve for $\mathbf{C}$, let's consider $v = 0$,

$$\mathbf{p}(u,0) = \sum_{i=0}^{3} \mathbf{c}_{i0} u^i = \mathbf{u}^T \mathbf{C} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{u}^T \mathbf{M}_I \begin{bmatrix} \mathbf{p}_{00} \\ \mathbf{p}_{10} \\ \mathbf{p}_{20} \\ \mathbf{p}_{30} \end{bmatrix}$$

A cubic curve interpolating $\mathbf{p}_{00}$, $\mathbf{p}_{10}$, $\mathbf{p}_{20}$, $\mathbf{p}_{30}$




Doing the same for $v = \frac{1}{3}, \frac{2}{3}, 1$, and we have 3 more equations similar to the above. Hence we have

$\mathbf{u}^T \mathbf{C} \mathbf{A}^T = \mathbf{u}^T \mathbf{M}_I \mathbf{P}$ where $\mathbf{P} = [\mathbf{p}_{ij}]$ and therefore $\mathbf{C} = \mathbf{M}_I \mathbf{P} {\mathbf{M}_I}^T$

$\boxed{\mathbf{M}_I = \mathbf{A}^{-1}}$

The bicubic surface is given by $\mathbf{p}(u,v) = \mathbf{u}^T \mathbf{M}_I \mathbf{P} {\mathbf{M}_I}^T \mathbf{v}$

# Blending Patches

We can build and analyze surfaces from our knowledge of curves

$$\mathbf{p}(u,v) \;=\; \mathbf{u}^T\mathbf{M}_I\mathbf{P}\mathbf{M}_I{}^T\mathbf{v}$$

Note that $\mathbf{M}_I\mathbf{u}^T$ are the blending functions

$$\mathbf{b}(u) \;=\; [b_0(u) \;\; b_1(u) \;\; b_2(u) \;\; b_3(u)]^T$$

Hence we may rewrite the surface equation as

$$\sum_{i=0}^{3}\sum_{j=0}^{3} b_i(u)b_j(u)\mathbf{p}_{ij}$$

Each $b_i(u)\ b_j(v)$ describes a blending patch