COMP3271 Computer Graphics

Curves & Surfaces (II)

2019-20

Objectives

Introduce the Bézier curves and surfaces

Derive the required matrices

Other Types of Curves and Surfaces

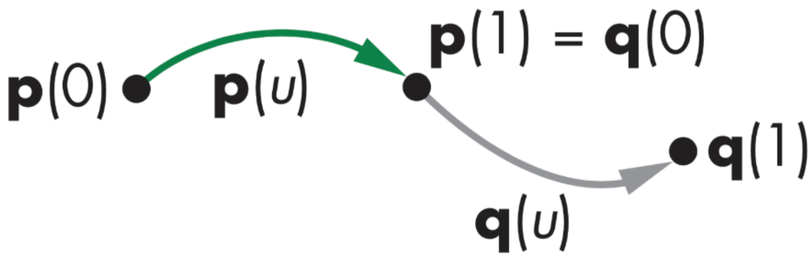
How can we get around the limitations of the interpolating form

- Lack of smoothness
- Discontinuous derivatives at join points

We have four conditions (for cubics) that we can apply to each segment

- Not necessarily using interpolating points as conditions
- Need only come close to the data

Parametric and Geometric Continuity



If $\mathbf{p}(1) = \mathbf{q}(0)$, then we have C^0 parametric continuity.

If $\mathbf{p}'(1) = \mathbf{q}'(0)$, i.e., each of the derivatives of x, y, and z components are continuous at join points, then we have C^1 parametric continuity.

Parametric and Geometric Continuity

Or we can only require that the tangents of the resulting curve be continuous, i.e., $\mathbf{p}'(1) = \alpha\mathbf{q}'(0)$ then we have G^1 geometric continuity)

This gives more flexibility then C^1 continuity as we need satisfy only two conditions rather than three at each join point

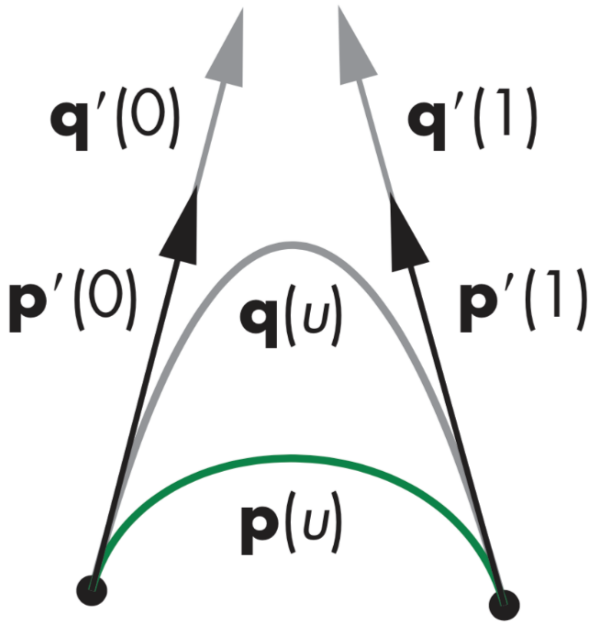
Note that G^0 is the same as C^0 continuity.

Example

Here the curves $\mathbf{p}$ and $\mathbf{q}$ have the same tangents at the ends of the segment but different derivatives

Can generate different curves by changing the "magnitude" (α) in G^1 continuity

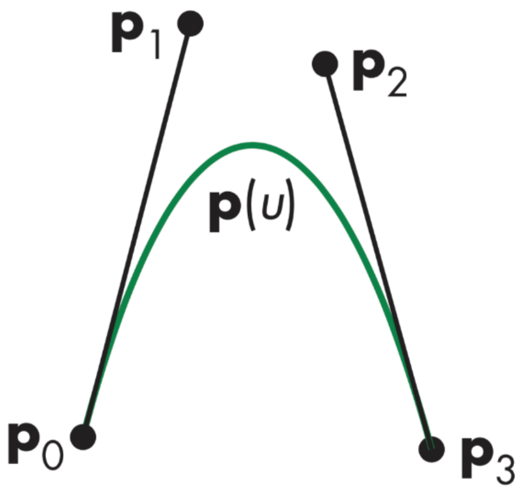
This techniques is used in drawing applications



Bézier's Idea

In graphics and CAD, we do not usually have derivative data

Bézier suggested using the same 4 data points (for cubic curve) as with the cubic interpolating curve to approximate the derivatives at the two end points



Bézier Curves

The B' ezier curve is just another representation of polynomial curves.

It is given by

$$P(t) = B_{0,n}(t)P_0 + B_{1,n}(t)P_1 + \cdots + B_{n,n}(t)P_n, t \in [0,1]$$

where

$$B_{i,n}(t) = \frac{n!}{(n-i)!\, i!}(1-t)^{n-i}t^i, \qquad i = 0,1,\ldots,n$$

Bernstein polynomials of degree n

Bézier Curve of Degree 1 $\sum_{i=0}^{1} B_{i,1}(t)\, p_i$

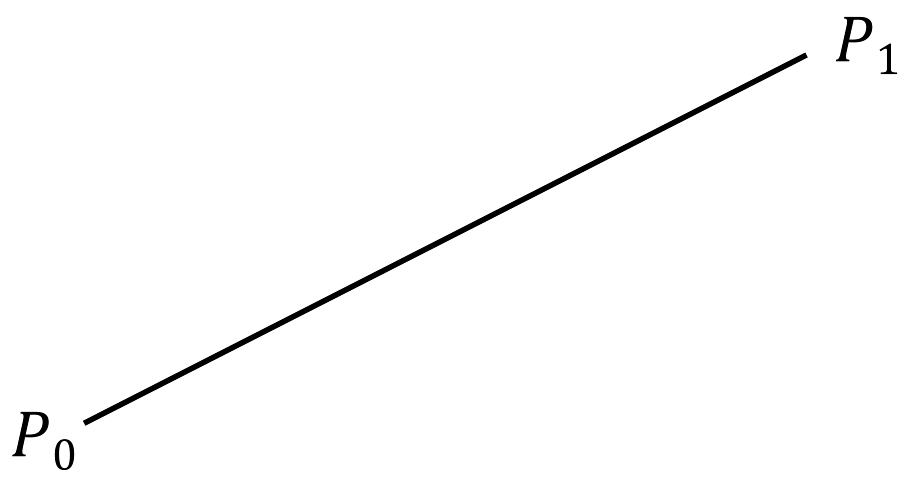
$$n = 1$$

$$P(t) = B_{0,1}(t)P_0 + B_{1,1}(t)P_1$$

$$B_{0,1}(t) = 1 - t,\ B_{1,1}(t) = t$$

$$p(t) = (1-t)P_0 + tP_1$$

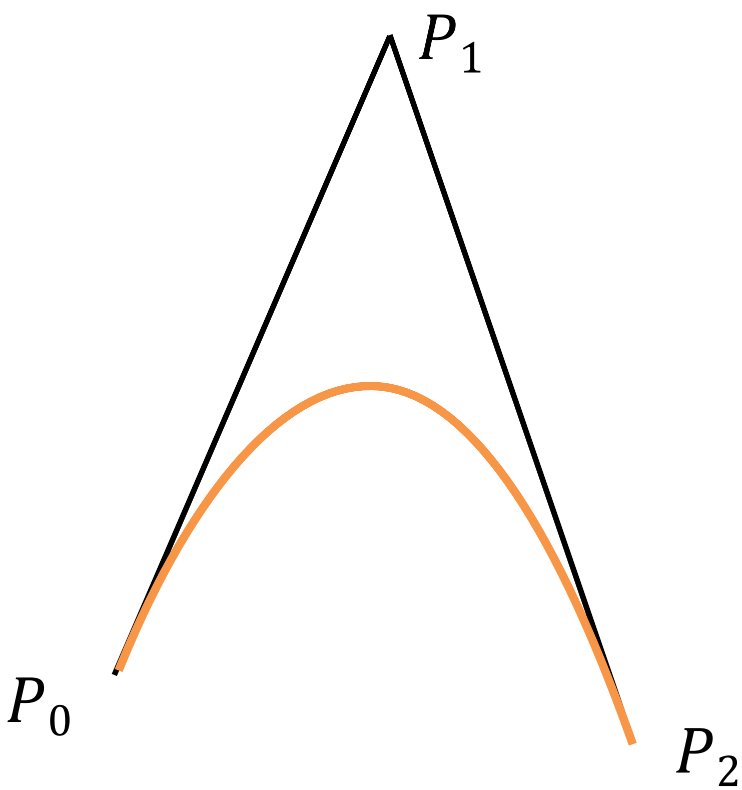
What is the curve?



Bézier Curve of Degree 2

The quadratic Bézier curve:

$$P(t) = (1-t)^2\, P_0 + 2(1-t)tP_1 + t^2P_2, \qquad t \in [0,1].$$

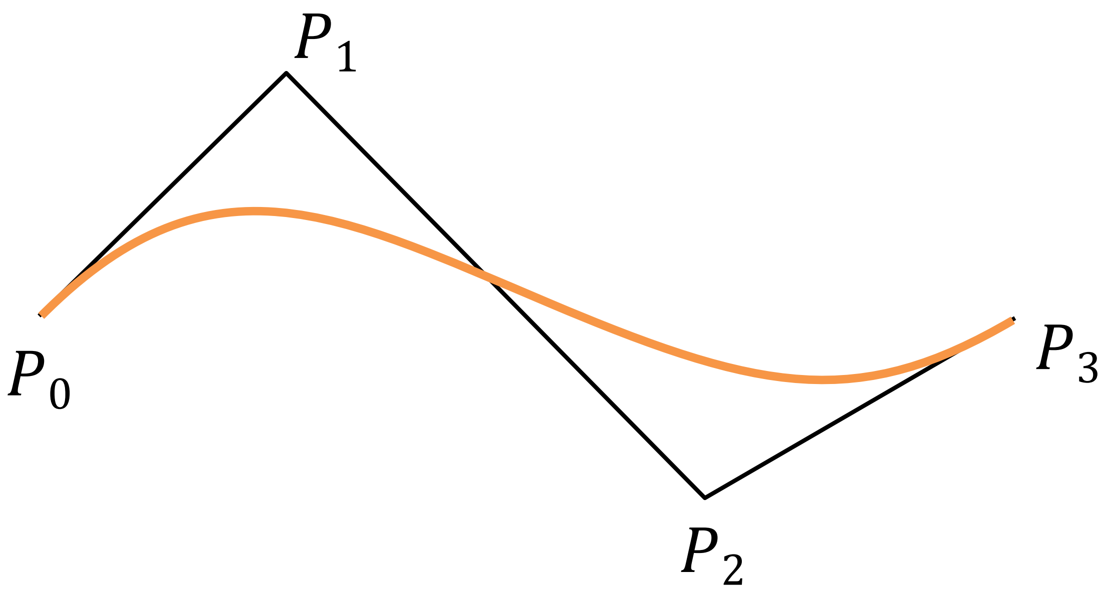


Bézier Curve of Degree 3

The cubic Bézier curve

$$P(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t) t^2 P_2 + t^3 P_3,$$

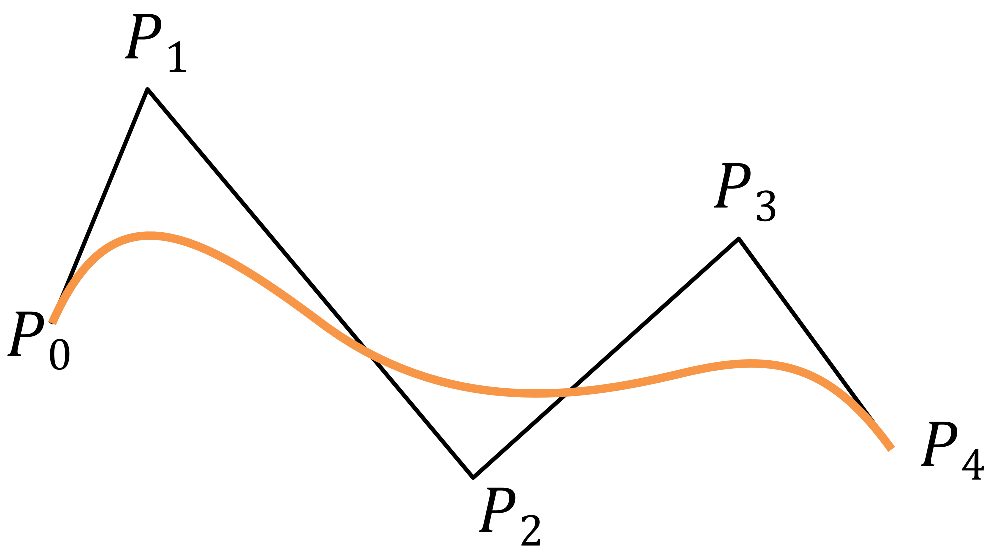
$t \in [0, 1]$.



Bézier Curve of Degree 4

The quartic Bézier curve

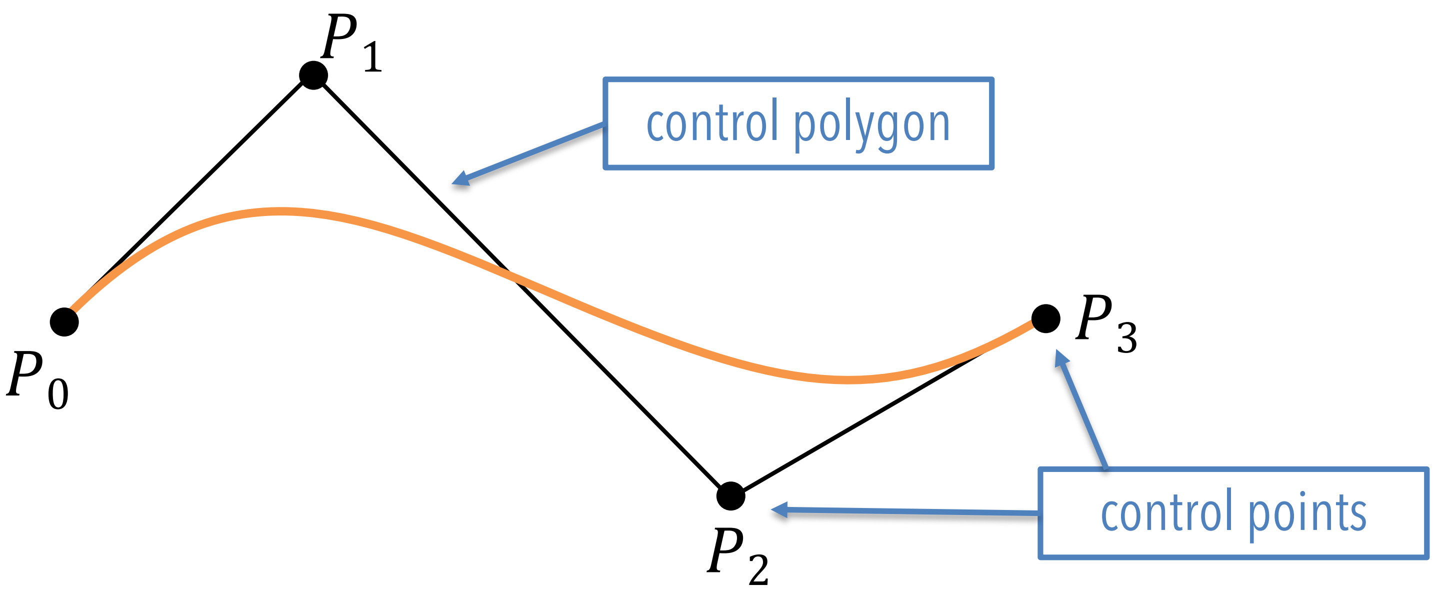
$$P(t) = (1-t)^4 P_0 + 4(1-t)^3 t P_1 + 6(1-t)^2 t^2 P_2$$
$$+4(1-t)t^3 P_3 + t^4 P_4, \ \ t \in \ [0,1].$$



Terminologies

The points P_i are called the **control points** or control vertices of the Bézier curve $P(t)$.

The polygon connecting $P_0, P_1, \dots, P_n$, in this order, is called the **control polygon** of $P(t)$.



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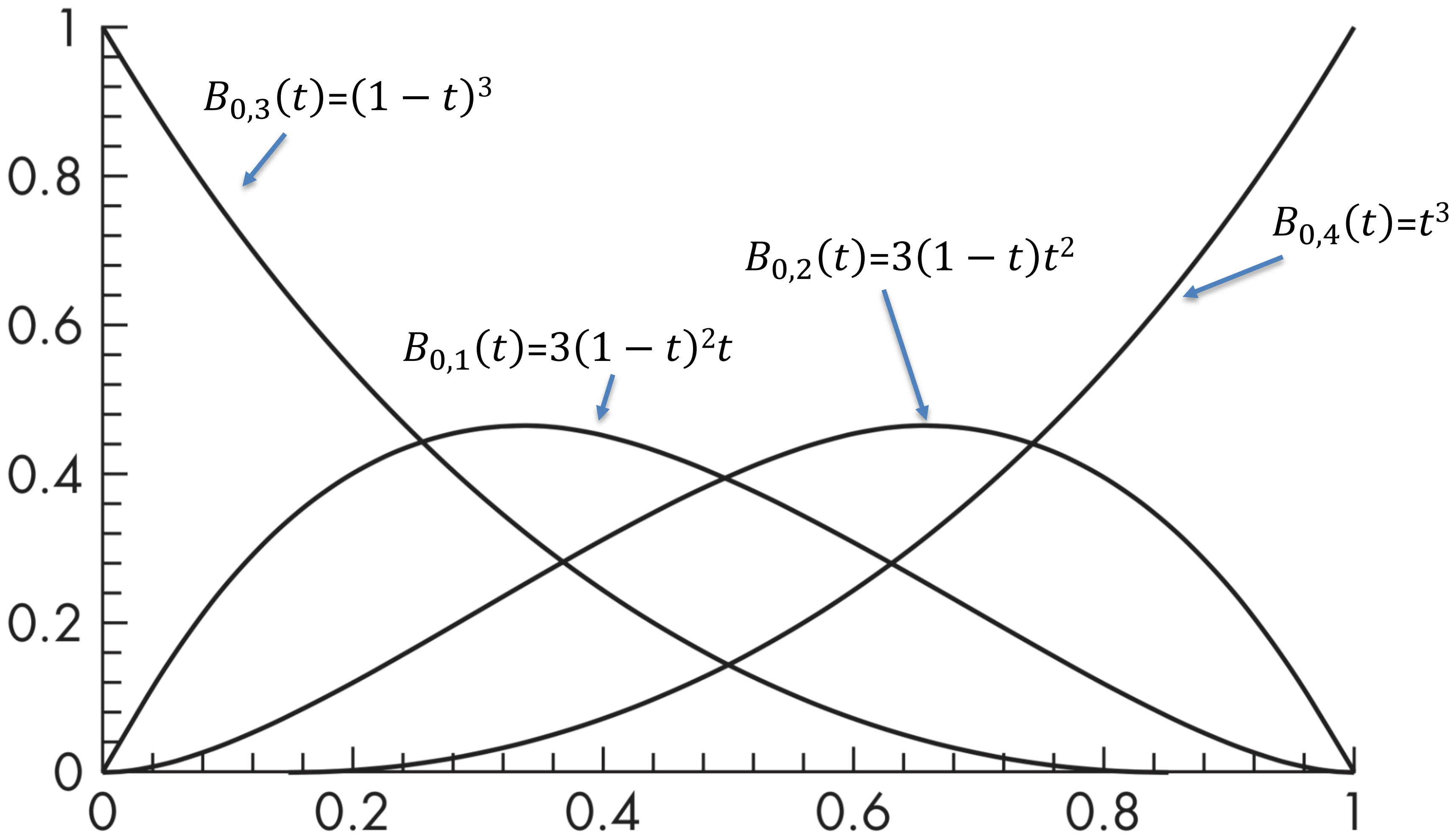
The polygon connecting $P_0, P_1, \dots, P_n$, in this order, is called the **control polygon** of $P(t)$.

The polynomials $B_{i,n}(t)$ are called the **blending functions** or **basis functions**.

Bézier Cubic Basis Functions



Properties of Bézier Curves

1. Any polynomial curve can be put in the Bézier form.

Proof:

The polynomials

$$\{B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)\}$$

$$\{1, t, t^2, \dots, t^n\}$$

span the same space.

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Example $\begin{pmatrix} B_{0,3}(t) \\ B_{1.3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix} = \begin{pmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$

Properties of Bézier Curves

2. A Bézier curve $P(t)$ of degree n interpolates the two endpoints P_0 and P_n.

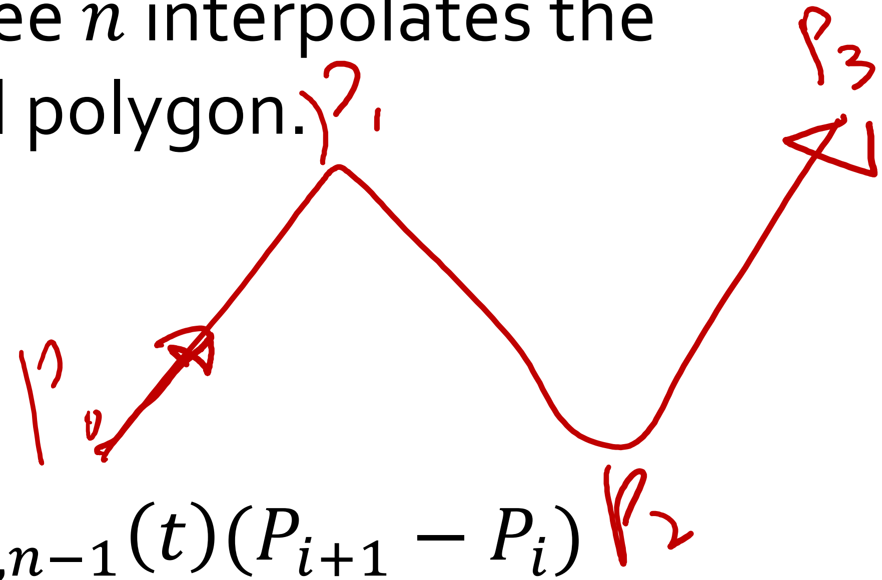
$\sum_{i=0}^{n} B_{i,n}(t) P_i$

Proof:

$$P(0) = P_0, \qquad P(1) = P_n$$

Properties of Bézier Curves

3. A Bézier curve $P(t)$ of degree n interpolates the two end sides of the control polygon.



Proof:

$$P'(t) = \frac{dP(t)}{dt} = n \sum_{i=0}^{n-1} B_{i,n-1}(t)(P_{i+1} - P_i)$$

$$P'(t)|_{t=0} = n(P_1 - P_0)$$

$$P'(t)|_{t=1} = n(P_n - P_{n-1})$$