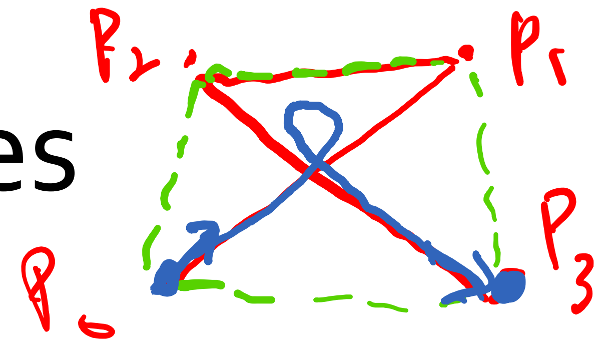
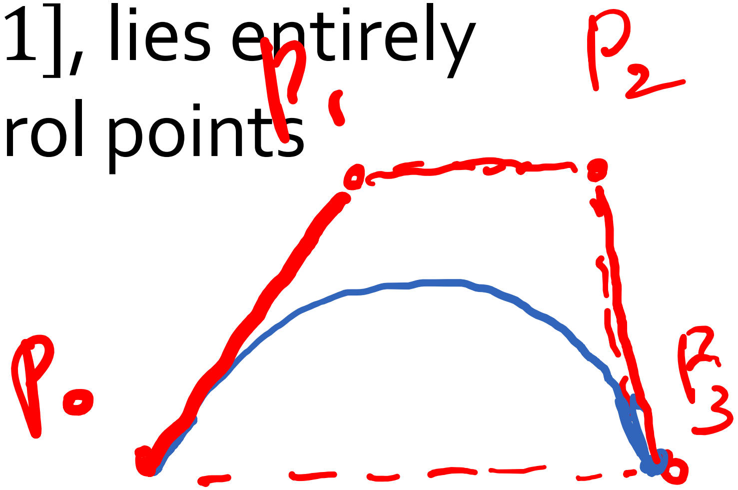
Properties of Bézier Curves



4. Convex hull property
   The curve segment $P(t), t \in [0,1]$, lies entirely inside the convex hull of all control points



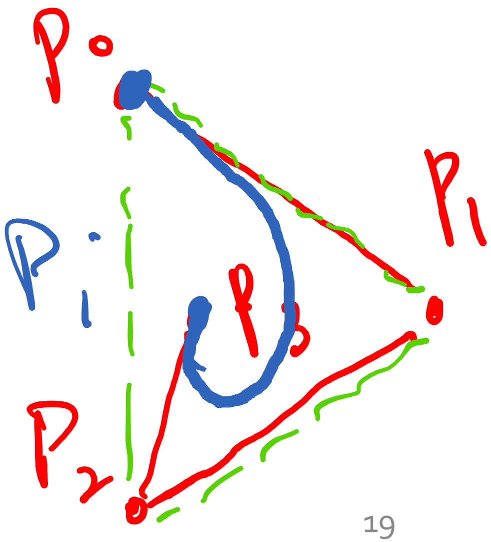
Proof:

This property follows from

$$\binom{n}{i} t^i (1-t)^{n-i} \qquad \sum_{i=0}^{n} B_{i,n}(t) \equiv 1,$$

$$B_{i,n}(t) \geq 0, \quad t \in [0,1]$$

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t) P_i$$



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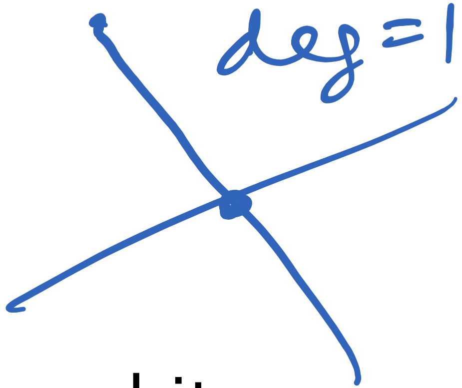
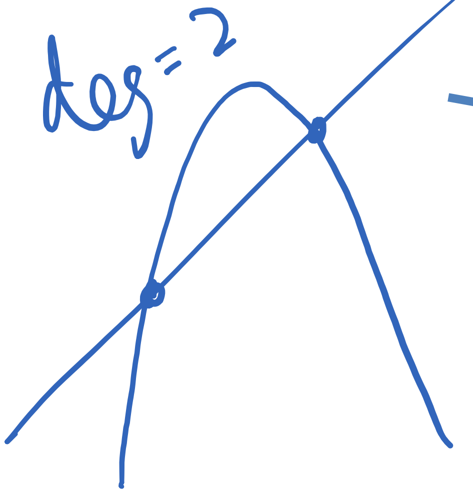
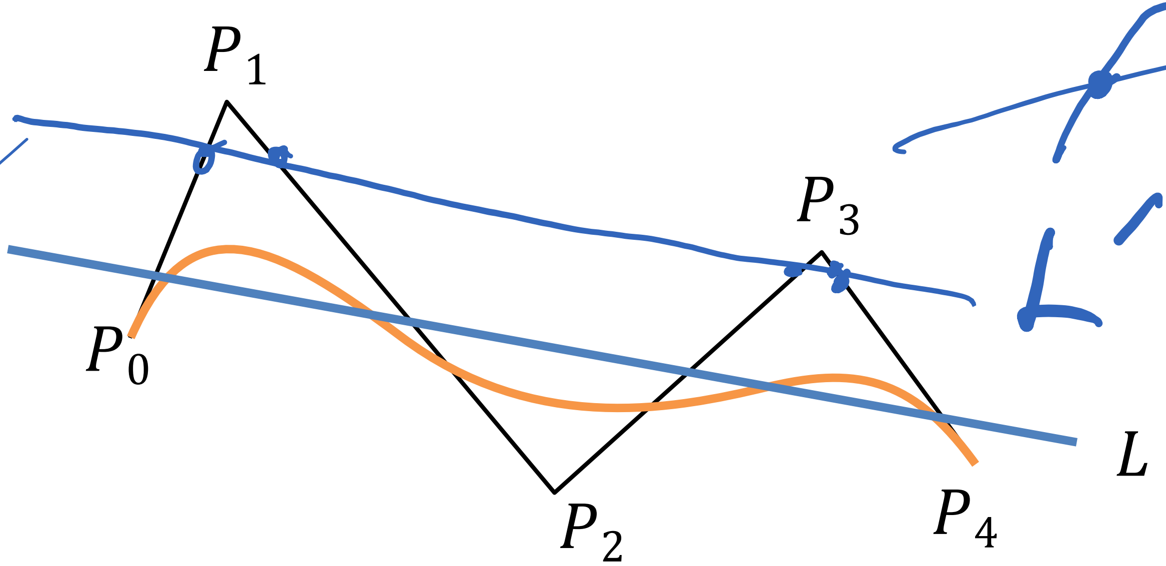
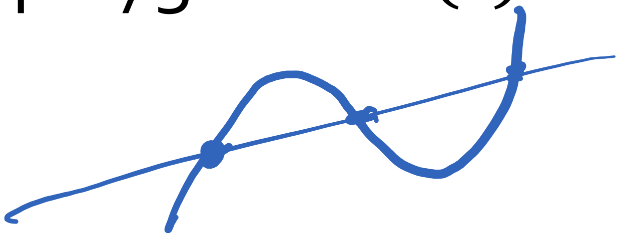


5. Variation diminishing property
   The number of intersections between an arbitrary straight line L and the Bézier curve $P(t), t \in [0,1]$, is NO greater than the number of intersections between the line L and the control polygon of $P(t)$.







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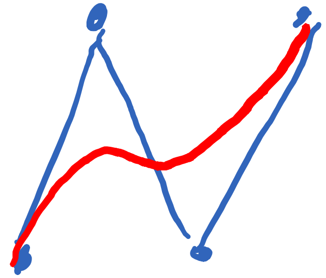


6. Invariant form under affine transformations
   For any affine transformation $\mathbf{M}$: $X' = AX + b$, there is

$$\mathbf{M}\big(P(t)\big) = \sum_{i=0}^{n} B_{i,n}(t)\mathbf{M}(P_i).$$

Proof:

$$\begin{aligned}
\mathbf{M}\big(P(t)\big) &= AP(t) + b \\
&= \sum_{i=0}^{n} B_{i,n}(t)AP_i + \sum_{i=0}^{n} B_{i,n}(t)b \\
&= \sum_{i=0}^{n} B_{i,n}(t)(AP_i + b) \\
&= \sum_{i=0}^{n} B_{i,n}(t)M(P_i)
\end{aligned}$$

Properties of Bézier Curves

6. Invariant form under affine transformations

Thanks to this property, when a Bézier curve is transformed affinely, we can

a) transform many sampled points on the curve directly, or

b) transform the control points only and use the transformed control points to generate a new Bézier curve

These two ways yield the same transformed curve, but the latter can be more efficient when there are many points to be evaluated on the curve.

Composite Bézier Curves

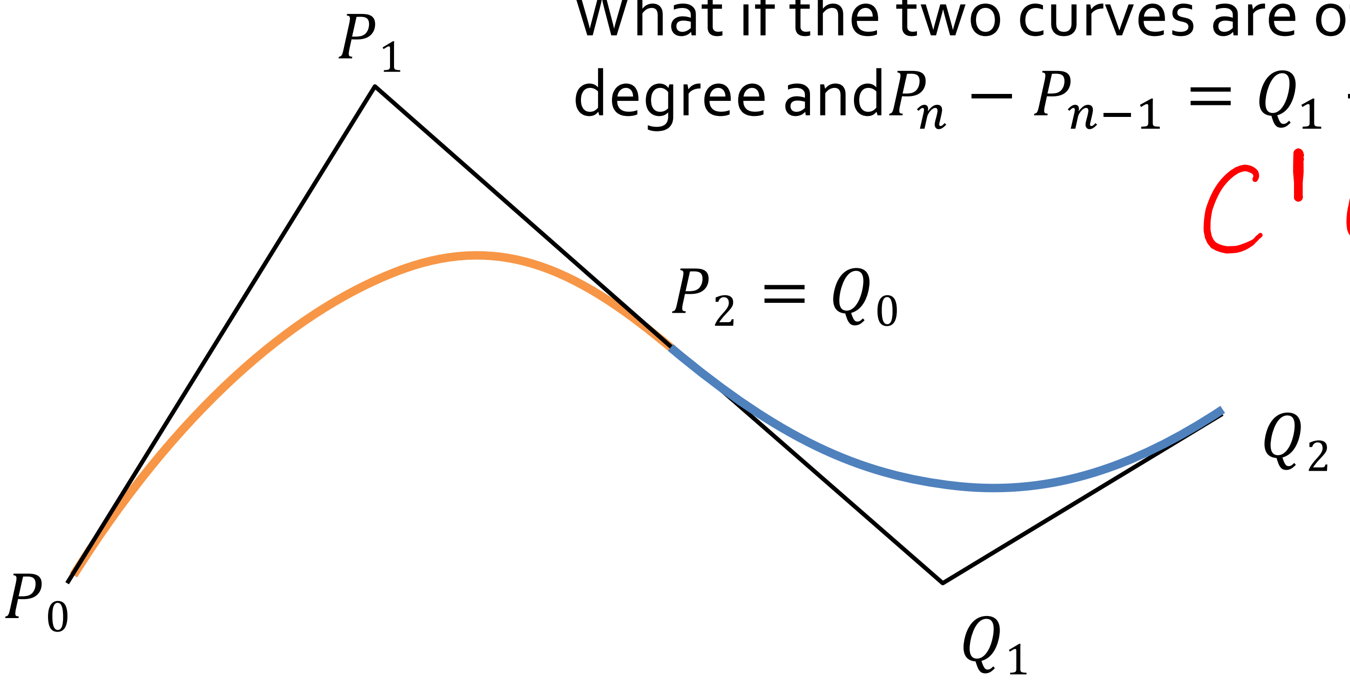
By placing the end control points of two Bézier curve collinear, we can obtain a smooth curve comprising two Bézier curves.

The curves then have G^1 continuity at P_2. Why?

$|P_1P_2| \neq |Q_1P_2|$

What if the two curves are of the same degree and $P_n - P_{n-1} = Q_1 - Q_0$?

C¹ continuity



Bézier Matrix

$$\mathbf{p}(u) = \mathbf{b}(u)^T\mathbf{p} = \mathbf{u}^T\mathbf{M}_B\mathbf{p}$$

blending functions

For cubic Bézier curves

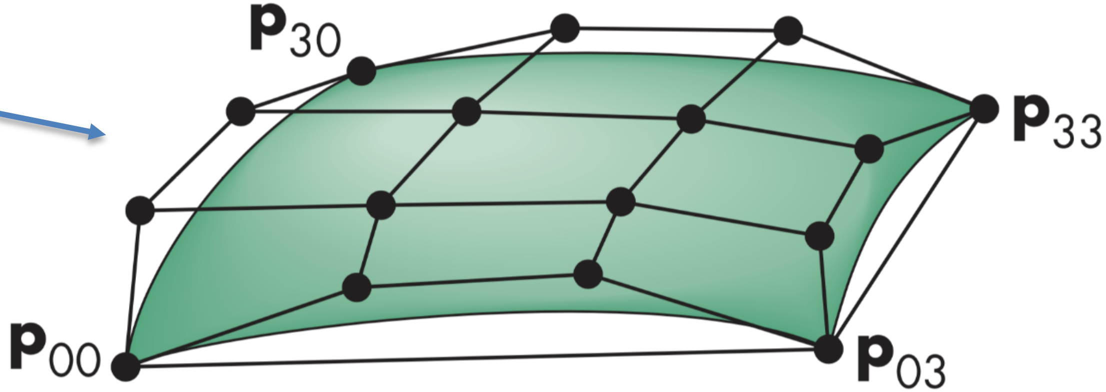
$$\mathbf{M}_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

Bézier Patches

Using same data array $\mathbf{P} = [\mathbf{p}_{ij}]$ as with interpolating form

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_i(u)\, b_j(v)\, \mathbf{p}_{ij} = \mathbf{u}^T \mathbf{M}_B \mathbf{P} \mathbf{M}_B{}^T \mathbf{v}$$

Patch lies in
convex hull



Bézier Bicubic Surface Patches

A 2 x 1 bicubic surface patches