**1. (20 marks)**

(a) Show that the Mandelbrot set is symmetric about the x-axis.

(b) Show that any Julia set associated with $f(z) = z^2 + c$ is symmetric about the origin.

**Solution**

(a) Assume $c_0$ is a point in Mandelbrot set, then we have $z_n = z_{n-1}^2 + c_0$,$z_0 = 0$, $z_n \nrightarrow \infty$. As $\overline{z^2} = \overline{z}^2$, we have $\overline{z_n} = \overline{z_{n-1}^2 + c_0} = \overline{z_{n-1}^2} + \overline{c_0} = \overline{z}_{n-1}^2 + \overline{c_0}$. It means $\overline{c_0}$ is also a point in Mandelbrot set. So Mandelbrot set is symmetric about x-axis. ( $c = a + ib$ and $\overline{c} = a - ib$)

(b) Assume $z_0$ is a point in Julia set, from the definition$\{z | z_n \nrightarrow \infty, z_n = z_{n-1}^2 + c_0\}$, we have $z_1 = z_0^2 + c_0 = (-z_0)^2 + c_0$, which means that $-z_0$ is also a point in Julia set. Hence, Julia set is symmetric about the origin. ($z = x + iy$ and $-z = -x - iy$)

**2. (10 marks)**

Derive the rotation transformation of $\theta$ degrees about the point $(a, b)^T$ in the 2D plane.

(a) Write the transformation in the form of $X' = MX + B$.

(b) Write the transformation in the form of $X' = NX$ in homogeneous coordinates.

**Solution**

The idea is:

(1) translate $(a, b)^T$ to the origin;

(2) rotate $\theta$ degrees about the origin;

(3) translate the point back.

(a)$X' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a - a*\cos\theta + b*\sin\theta \\ b - a*\sin\theta - b*\cos\theta \end{bmatrix}$

(b) $X' = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} \cos\theta & -\sin\theta & a - a*\cos\theta + b*\sin\theta \\ \sin\theta & \cos\theta & b - a*\sin\theta - b*\cos\theta \\ 0 & 0 & 1 \end{bmatrix} X$

**3. (10 marks)**

Derive the $2 \times 2$ transformation matrix for the reflection about the line $ax + by = 0$ in the $2D$ plane.

**Solution**

The idea is :

(1) rotate the line to x-axis about the origin;

(2) reflect across x-axis;

(3) rotate the line back.

In case that b=0, line $ax + by = 0$ is the y-axis, and the transformation matrix is: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

If $b \neq 0$, suppose $tan(\theta) = -\frac{a}{b}$, the transformation matrix M is,

$$\begin{aligned} M &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -\cos^2\theta + \sin^2\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \end{aligned}$$

Because $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{b^2-a^2}{b^2+a^2}$, $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{-2ab}{a^2+b^2}$, we get the result as

$$M = \begin{bmatrix} \frac{b^2-a^2}{b^2+a^2} & \frac{-2ab}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & -\frac{b^2-a^2}{b^2+a^2} \end{bmatrix}$$

**4. (10 marks)**

Given two triangles $\Delta ABC$ and $\Delta A'B'C'$ in 2D plane with $A = (0,0)^T$,$B = (1,0)^T$,$C = (0,1)^T$, $A' = (0,2)^T$, $B' = (-1,-1)^T$, and $C' = (2,0)^T$. Derive the affine transformation matrix for the transformation $T$ that maps $\Delta ABC$ to $\Delta A'B'C'$ such that $T(A) = A'$, $T(B) = B'$, $T(C) = C'$.

**Solution**

We represent $\triangle ABC$ in the matrix form.

$$M = \begin{bmatrix} A & B & C \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The matrix form of $\triangle A'B'C'$ is

$$M' = \begin{bmatrix} A' & B' & C' \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The transformation can be represented as

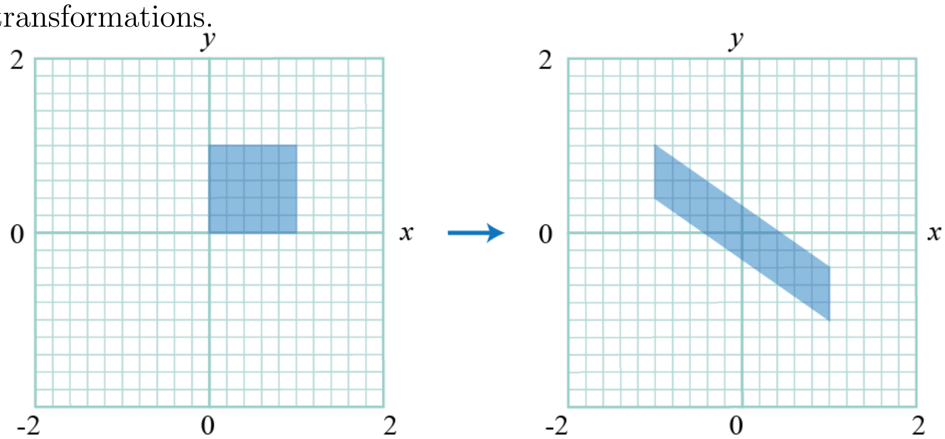
$$M' = TM$$

So we can compute the transformation matrix as follows.

$$
\begin{aligned}
T &= M'M^{-1} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 2 & 0 \\ -3 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

**5. (10 marks)**

Give a sequence of 4 x 4 matrices that transforms the unit square in the left figure to the parallelogram in the right. Find a sequence of OpenGL function calls that implements these transformations.




**Solution**

The transformation matrix is,

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.7 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -0.3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
Shearing \qquad \times \qquad Translation \qquad \times Scaling
$$

The sequence of OpenGL function calls is:

GLfloat $M[16] = \{1, -0.7, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1\}$;

glMultMatrix(M);

glTranslatef(-1.0,-0.3,0.0);

glScalef(2.0,0.6,1.0);

**6. (10 marks)** Show that any sequence of rotations and translations can be replaced by a single rotation about the origin followed by a translation.

**Solution**

We can show by simply multiplying $4\times 4$ matrices that the concatenation of two rotations yields a rotation and than the concatenation of two translations yields a translation. By looking at the product of a rotation and a translation, we find left three columns of $RT$ are the left three columns of $R$ and the right column of $RT$ is the right column of the translation matrix. If we now consider $RTR'$ where $R'$ is a $4\times 4$ rotation matrix, the left three columns are exactly the same as the left three columns of $RR'$ and the right column still has 1 as its bottom element. Thus, the form is the same as $RT$ with an altered rotation (which is the concatenation of the two rotations) and an altered translation. Inductively, we can see that any further concatenations with rotations anf translations do no alter this form.

**7. (10 marks)** Consider the line in $\mathbb{R}^3$ given by

$$L(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Let $Q_1$ and $Q_2$ be two points on this line that are at distance $\sqrt{2}$ apart from each other. We now transform the line by applying transformation $T$ given by

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What is the distance between $T(Q_1)$ and $T(Q_2)$?

**Solution 1**

Suppose

$$Q_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Because the distance of $Q_1Q_2$ is $\sqrt{2}$, we have $\|a-b\|$=1.

$$T(Q_1) = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a+1 \\ 1 \\ a+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a+4 \\ 4 \\ a+3 \\ 1 \end{pmatrix}$$

$$T(Q_2) = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b+1 \\ 1 \\ b+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2b+4 \\ 4 \\ b+3 \\ 1 \end{pmatrix}$$

distance of $T(Q_1)T(Q_2)$ is $\sqrt{(2a-2b)^2+(a-b)^2} = \sqrt{5}$

**Solution 2**

The unit vector of the direction of $Q_1Q_2$ is $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)^T$ and the length of $Q_1Q_2$ is $\sqrt{2}$. Apply the transformation matrix $T$ to the unit vector:

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

So we have the length of the transformed unit vector as $\sqrt{(\sqrt{2})^2 + (\frac{1}{\sqrt{2}})^2} = \sqrt{\frac{5}{2}}$.

Since the original length of $Q_1Q_2$ is $\sqrt{2}$, we have the length of the transformed $Q_1Q_2$ as $\sqrt{\frac{5}{2}} \times \sqrt{2} = \sqrt{5}$.

8. **(20 marks)**

Given an affine transformation $X' = MX + B$, where M is a 2 by 2 matrix and B is a 2D vector, find the equation of the image E of the circle $x^2 + y^2 - 1 = 0$ under this transformation. Show that E is an ellipse.

**Solution**

We denote the unit circle as $X^TX = 1$, where $X = (x, y)^T$, and define an arbitrary 2D affine transformation $X' = MX + B$, where $M$ is a nonsingular $2 \times 2$ matrix, and $B$ is a translation vector of $2 \times 1$. So we have $X = M^{-1}(X' - B)$. Substitute it into $X^TX = 1$, and we have

$$[M^{-1}(X'-B)]^T[M^{-1}(X'-B)] = 1$$
$$\Leftrightarrow (X'-B)^T(M^{-1})^TM^{-1}(X'-B) = 1.$$

Now we show that $(M^{-1})^TM^{-1}$ is a symmetric positive definite matrix. This is because $((M^{-1})^TM^{-1})^T = (M^{-1})^TM^{-1}$ and for any nonzero vector $Y$,

$$Y^T(M^{-1})^TM^{-1}Y = (M^{-1}Y)^TM^{-1}Y = \|M^{-1}Y\|^2 > 0,$$

since $M$ is nonsingular and $M^{-1}Y \neq \mathbf{0}$.
And one definition of ellipse is

$$\{X|(X - X_c)^T A(X - X_c) = 1\},$$

with $X, X_c \in R^n$, $X_c$ the center of ellipse, and $A_{n\times n}$ a symmetric positive definite matrix. Therefore $X'$ is on an ellipse and $B$ is the center.