COMP3271 Computer Graphics

Curves & Surfaces (I)

2019-20

Objectives

Different representations for curves and surfaces

Design criteria

Parametric curves & surfaces

Interpolation

Escaping Flatland

Until now we have worked with flat entities such as lines and flat polygons

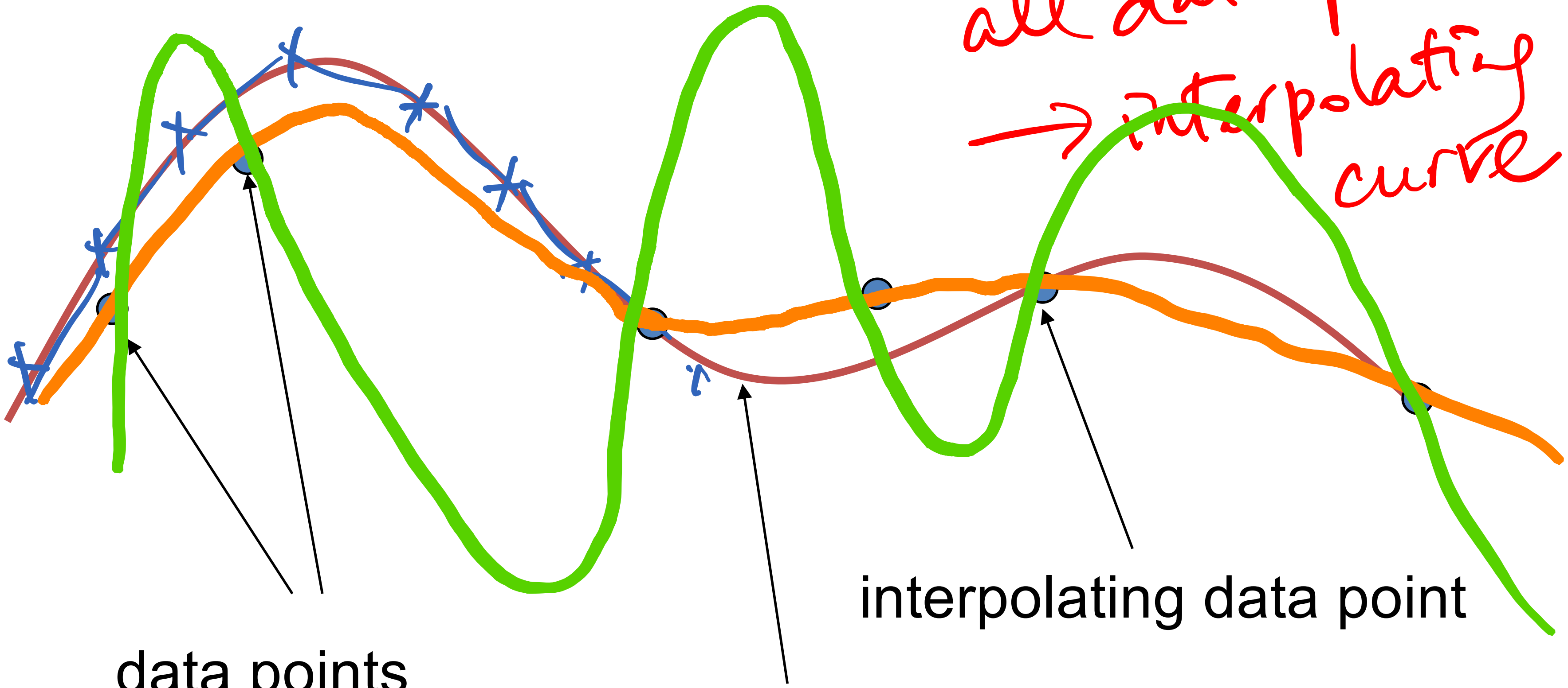
- Fit well with graphics hardware
- Mathematically simple

But the world is not composed of flat entities

- Need curves and curved surfaces
- Implementation can render them approximately with flat primitives

Modeling with Curves

if curve passes thru all data points → interpolating curve



What Makes a Good Representation?

There are many ways to represent curves and surfaces

Some design criteria

- Local control of shape
- Stability
- Smoothness and continuity (in terms of derivatives)
- Ability to evaluate derivatives
- Ease of evaluation
- Ease of rendering
- Must we interpolate or can we just come close to data?

Explicit Representation

$y = 3$

Most familiar form of curve in 2D

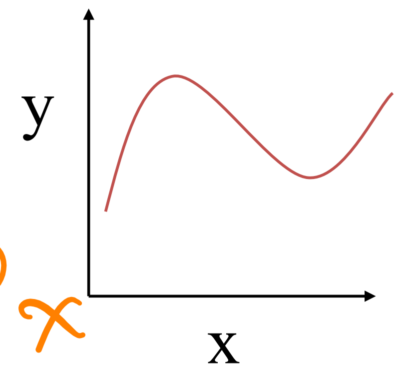
$$y = \mathrm{f}(x)$$

Express a variable in terms of other variables

Cannot represent all curves

- Vertical lines
- Circles

$y = ?\, x$

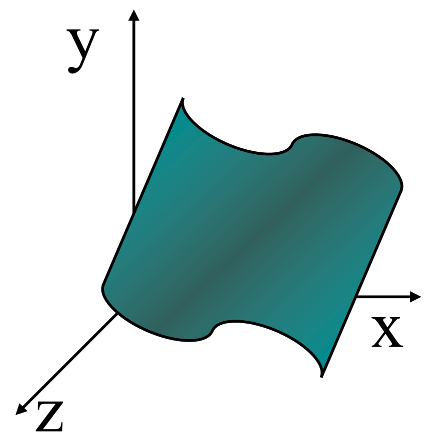


Extension to 3D

- $y = \mathrm{f}(x)$, $z = \mathrm{g}(x)$
- The form $z = \mathrm{f}(x,y)$ defines a surface

3D curve

Cannot represent a sphere in the form of $z = \mathrm{f}(x, y)$. Why?



Implicit Representation

Two dimensional curve(s)

$$g(x, y)=0$$

Represents the membership of points on curve

Much more robust

- All lines $ax + by + c = 0$
- Circles $x^2 + y^2 - r^2 = 0$

Not unique

- $(x^2 + y^2 - r^2)^2 = 0$ and $\sqrt{x^2 + y^2} - 1 = 0$ represent the same circle as $x^2 + y^2 - r^2 = 0$.

In general, no analytic way to solve for points that satisfy the equation

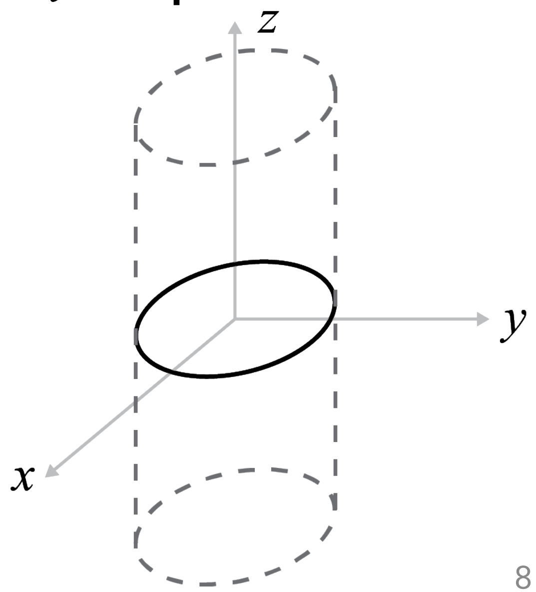
Implicit Representation

$g(x, y) = 0$

Three dimensions $g(x, y, z)=0$ defines a surface

- E.g., $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ represents the unit sphere

Question: how to represent the unit circle centered at the origin in the xy-plane implicitly in the xyz-space?



To represent a 3D curve

- Intersect two surfaces to get a curve

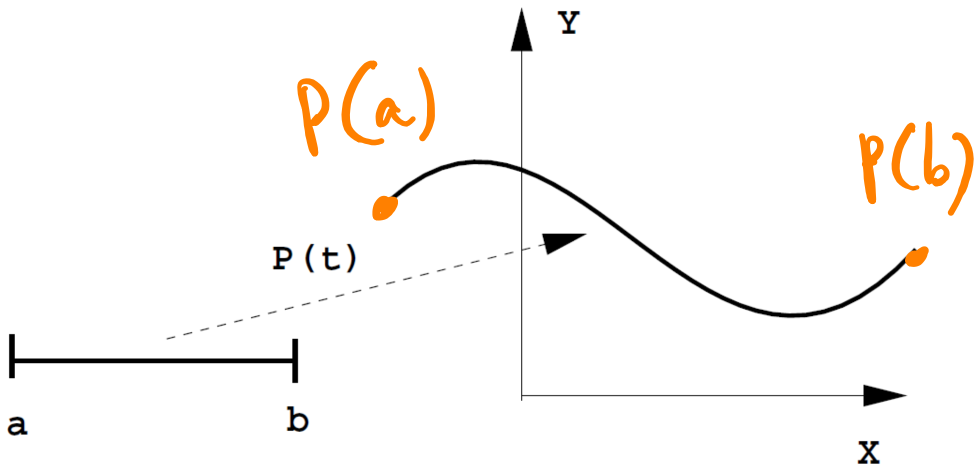
$$\begin{cases} \text{cylinder} & x^2 + y^2 - 1 = 0 \\ \text{plane} & z = 0 \end{cases}$$

Parametric Representation

Two dimensional curves:

$$x = x(t), y = y(t), \quad t\epsilon[a, b]$$

Express the *x,y* values of each point on the curve explicitly in terms of an independent variable, *t*, i.e., the **parameter**, with a domain [a,b]



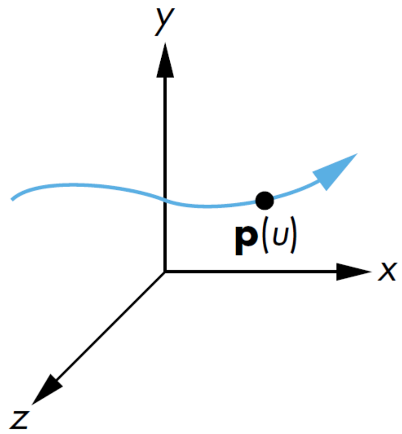
Example: Unit circle

$$P(\theta) = \big(x(\theta), y(\theta)\big) = (\cos\theta, \sin\theta), \theta\epsilon[0,2\pi)$$

Easily extended to three dimensional curves:

$$x = x(u), y = y(u), z = z(u), \ u\epsilon[a, b]$$

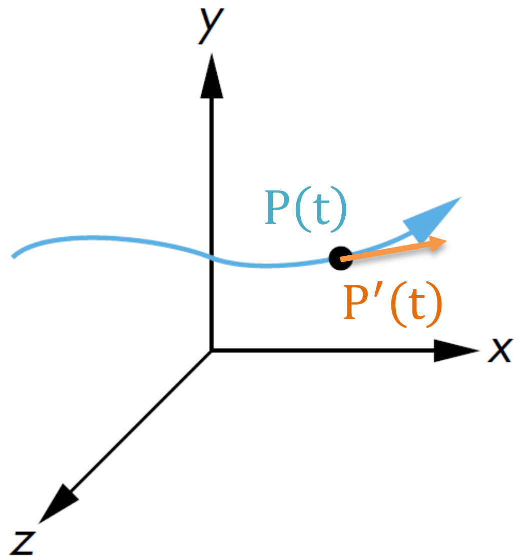
Still in one parameter, hence a curve



Parametric Representation

We trace the curve $P(t) = (x(t), y(t), z(t))$ as t varies.
Hence, we can talk of the velocity of $P(t)$:

$$\mathrm{P}'(\mathrm{t}) = \frac{dP(t)}{dt} = \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{bmatrix}$$



This gives the tangent direction of the curve.

The speed of $P(t)$ is then $|\mathrm{P}'(\mathrm{t})|$.

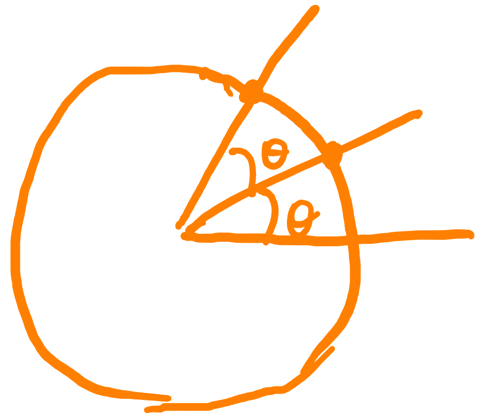
Parametric Representation

When the speed of $P(t)$ is constant or nearly constant, the computed points $P(t_i)$ on $P(t)$ are evenly or nearly evenly spaced if the parameters t_i , $i = 0, 1, 2, \ldots$, are evenly sampled.

Example. The following parametric equation of the unit circle has a constant speed.

$$P(\theta) = \big(x(\theta), y(\theta)\big) = (\cos\theta , \sin\theta) , \theta \epsilon [0,2\pi)$$

$$P'(\theta) = (-\sin\theta , \cos\theta)$$

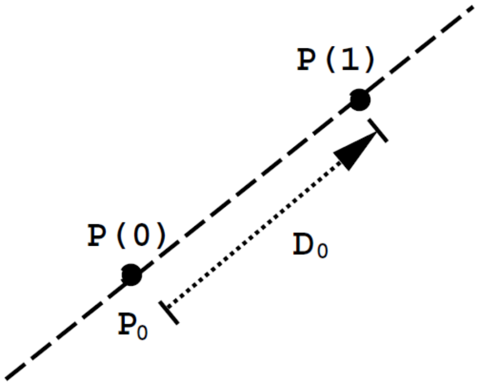


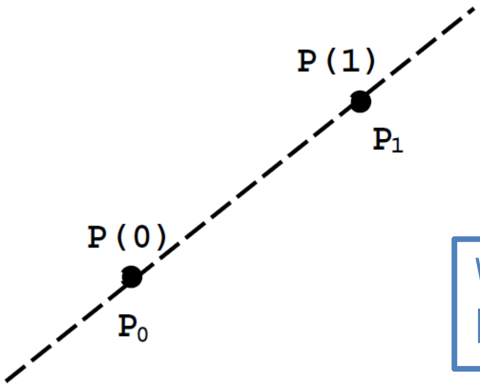
$$|P'(\theta)| = 1$$

Parametric Lines

Example. A straight line passing through the point P_0 with the direction vector D_0 can be represented by

$$P(t) = \big(x(t), y(t)\big) = P_0 + tD_0,\ \ t \in \ (-\infty, \infty).$$





With $t \in [0,1]$, we get the straight line segment P_0 and P_1

Example. A straight line passing through two distinct points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ is commonly represented by

$$P(t) \ = \ (1-t)P_0 \ + \ tP_1, t \in \ (-\infty, \infty).$$

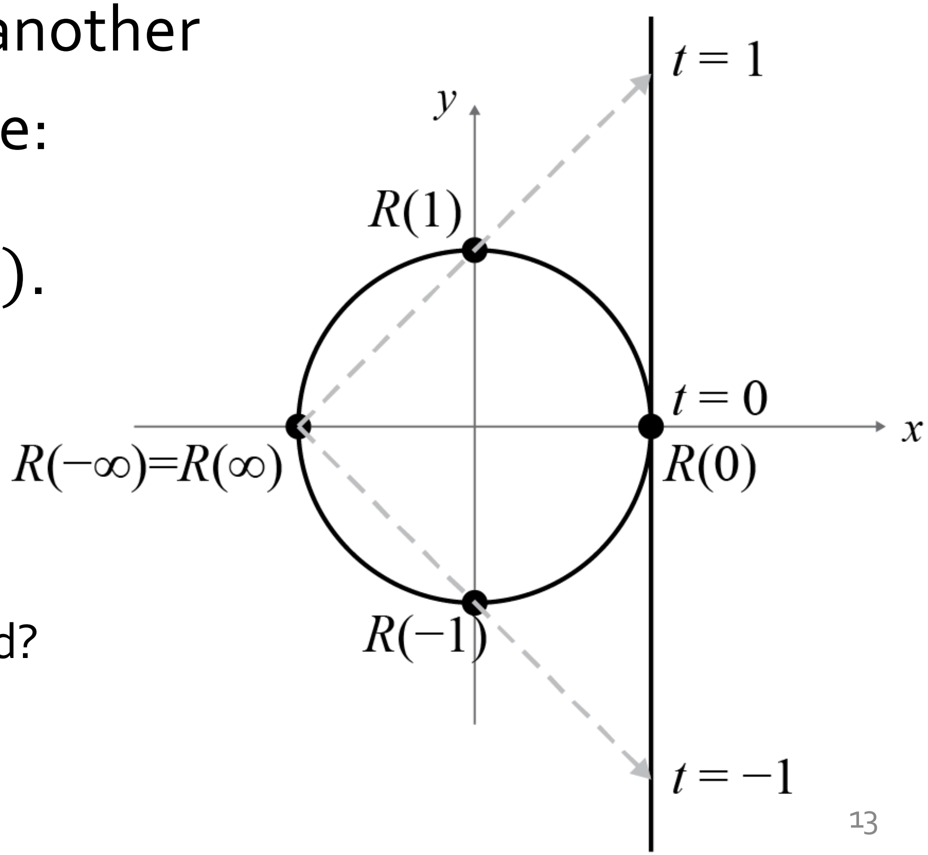
Unit Circle in Parametric Form

$$P(\theta) = \big(x(\theta), y(\theta)\big) = (\cos\theta, \sin\theta), \theta \epsilon [0,2\pi)$$

Since $\cos\theta = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$ and $\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}$

Substituting $t = \tan\frac{\theta}{2}$, we have another parametric from for the unit circle:

$$R(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), t \in (-\infty, \infty).$$



Is this parameterization with constant speed?

Parametric Surfaces

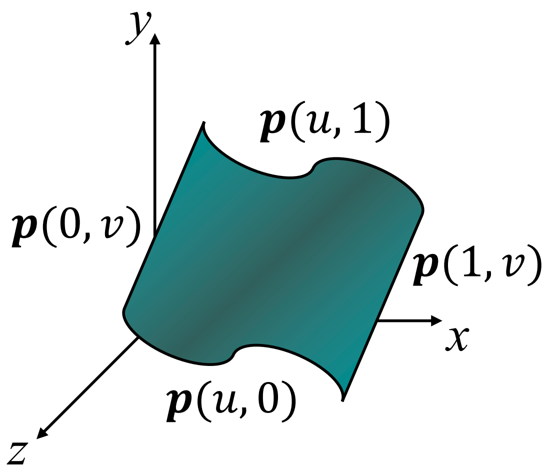
Surfaces require 2 parameters

$$x = x(u, v)$$

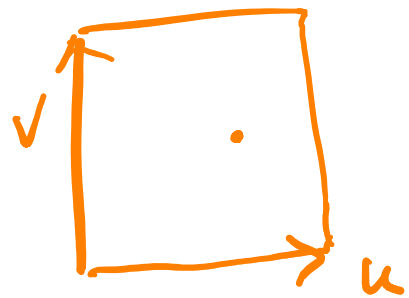
$$y = y(u, v)$$

$$z = z(u, v)$$

$$\boldsymbol{p}(u, v) = [x(u, v), y(u, v), z(u, v)]^T$$



the four boundary curves of a patch



Want same properties as curves:

- Smoothness
- Differentiability
- Ease of evaluation

Surface Normals

We can differentiate with respect to u and v to obtain the normal at any point $\boldsymbol{p}$

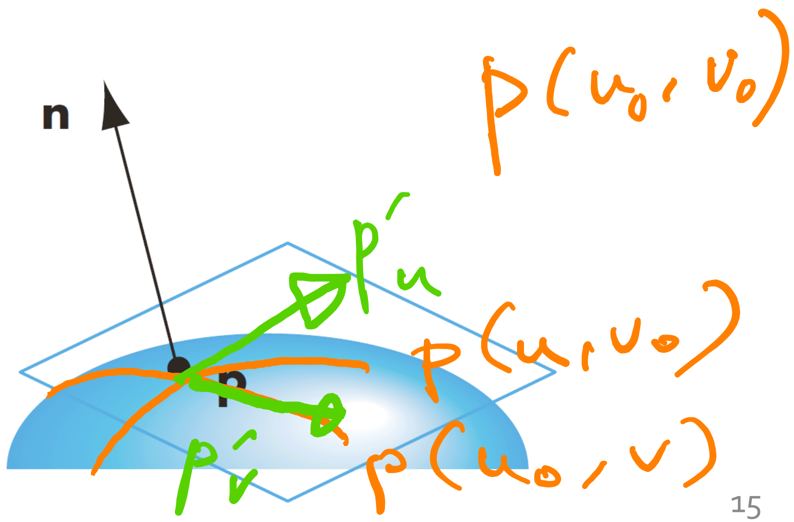
p'_u

$$\frac{\partial \mathbf{p}(u,v)}{\partial u} = \begin{bmatrix} \partial x(u,v)/\partial u \\ \partial y(u,v)/\partial u \\ \partial z(u,v)/\partial u \end{bmatrix}$$

p'_v

$$\frac{\partial \mathbf{p}(u,v)}{\partial v} = \begin{bmatrix} \partial x(u,v)/\partial v \\ \partial y(u,v)/\partial v \\ \partial z(u,v)/\partial v \end{bmatrix}$$

$$\mathbf{n} = \frac{\partial \mathbf{p}(u,v)}{\partial u} \times \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

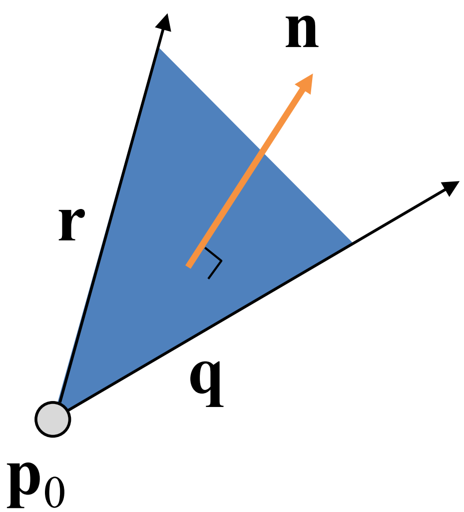


Parametric Planes

Point-vector form

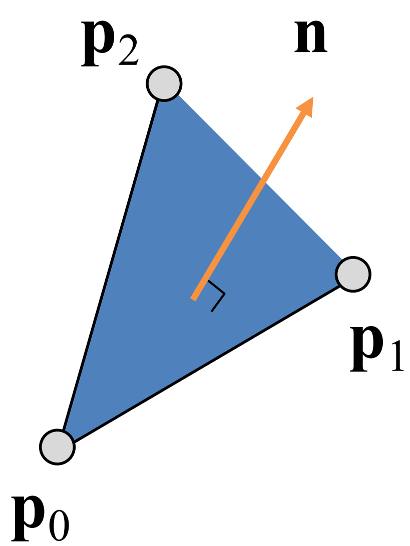
$$\mathbf{p}(u,v) = \mathbf{p}_0 + u\mathbf{q} + v\mathbf{r}$$

$$\mathbf{n} = \mathbf{q} \times \mathbf{r}$$



Three-point form

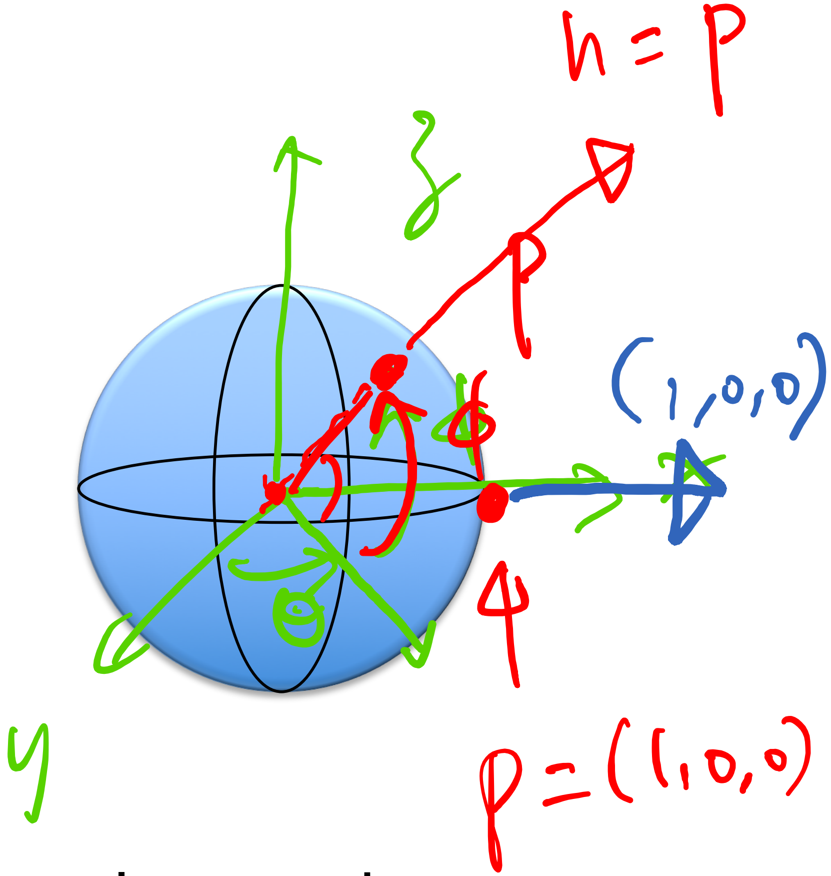
$$\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_0$$
$$\mathbf{r} = \mathbf{p}_2 - \mathbf{p}_0$$



Parametric Spheres

$$x(\theta, \varphi) = r\cos\theta\sin\varphi$$
$$y(\theta, \varphi) = r\sin\theta\sin\varphi$$
$$z(\theta, \varphi) = r\cos\varphi$$

$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi$$



θ: constant; circles of constant longitude
φ: constant; circles of constant latitude

Exercise: differentiate to show $\mathbf{n} = \mathbf{p}$