COMP3271 Computer Graphics

# Geometry

2019-20

# Objectives

Introduce the elements of geometry

- Scalars
- Vectors
- Points

Develop mathematical operations among them in a coordinate-free manner

Define basic primitives

- Line segments
- Polygons

# Basic Elements

Geometry is the study of the relationships among objects in an n-dimensional space

- In computer graphics, we are interested in objects that exist in three dimensions

Want a minimum set of primitives from which we can build more sophisticated objects

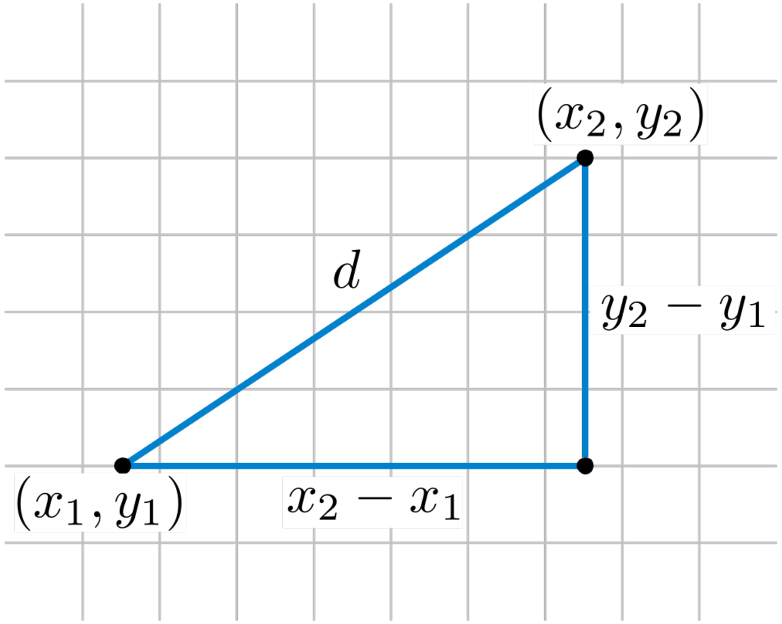
We will need three basic elements

- Scalars
- Vectors
- Points

# Cartesian Geometry

Points were at locations in space $\boldsymbol{p}=(x, y, z)$

We derived results by algebraic manipulations involving these coordinates




Example:
A line is given by the equation $x + 2y = 4$

Example:
To find intersection of two lines by solving a system of two linear equations

$$x + 2y = 4$$
$$3x - y = 1$$

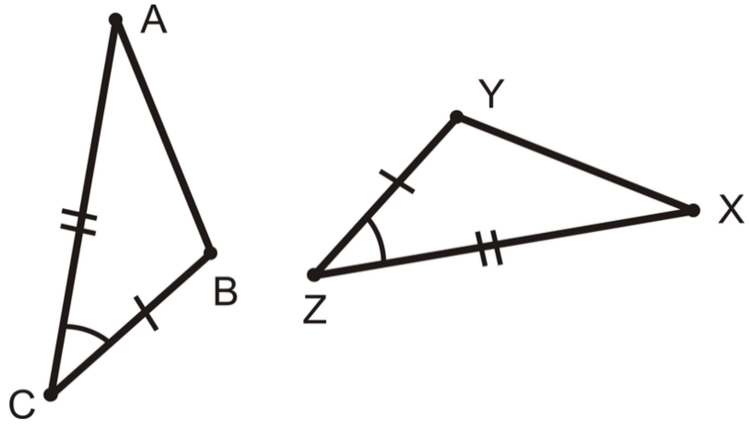
# Cartesian Geometry

The Cartesian based approach was nonphysical

- Physically, points exist regardless of the location of an arbitrary coordinate system

However, most geometric results are independent of the coordinate system

- Example Euclidean geometry: two triangles are identical if two corresponding sides and the angle between them are identical



# Cartesian Geometry

Example: To add two points $P$ and $Q$

Let's try using coordinates to add two points.

(1) $P = (0,0)$ and $Q = (1,1)$, then $P+Q$ is $(1,1)$ which is the same as $Q$

(2) $P = (1,1)$ and $Q = (2,2)$, then $P+Q$ is $(3,3)$ which is a totally different point.

So it depends on the coordinate system and we cannot derive clear geometric interpretation from it.

Question: what is the geometric meaning in adding two points?

# Coordinate Free Geometry

The three basic elements in CFG

- **Scalars**, **Vectors**, **Points**

**Scalars**

Scalars are simply real numbers which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutativity, inverses)

Scalars alone have no geometric properties

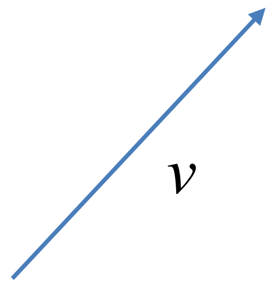
# Vectors

Physical definition: a vector is a quantity with two attributes

- Direction
- Magnitude

Examples include

- Force
- Velocity
- Directed line segments
  - Most important example for graphics



# Vector Operations

There is a zero vector

- Zero magnitude, undefined orientation

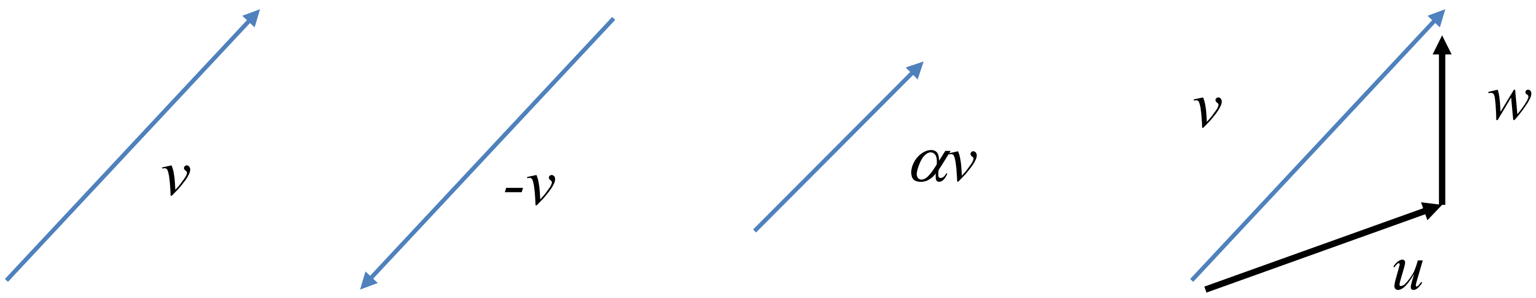
Every vector has an **inverse**

- Same magnitude but points in opposite direction

Every vector can be **multiplied by a scalar**

The sum of any two vectors is a vector

- Use head-to-tail axiom




inverse scaling addition

# Vector Operation

Magnitude of vector: $||v||$

Dot product: $||v||\,||w||\cos(\theta)$, where θ is the angle between the two vectors $v$ and $w$

Cross product: $v \times w$, where $v$ and $w$ are 3D vectors. The cross product is a new vector perpendicular to $v$ and $w$ with magnitude $||v||\,||w||\sin(\theta)$

# Linear Vector Spaces

Mathematical system for manipulating vectors

Operations

- Scalar-vector multiplication $u = \alpha v$
- Vector-vector addition: $w = u + v$

Expressions such as

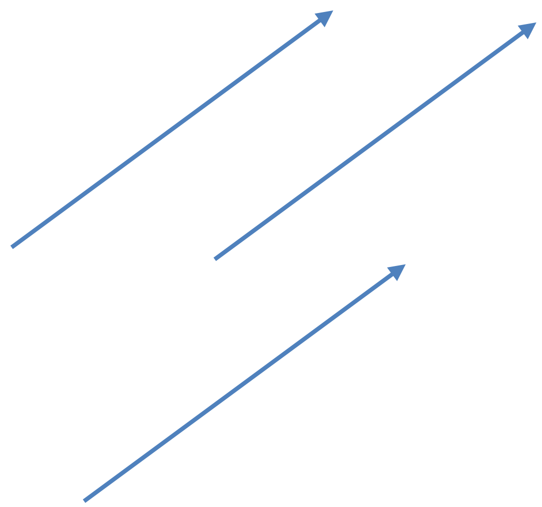
$$v = u + 2w - 3r$$

make sense in a vector space

# Vectors Lack Position

These vectors are identical

- Same length and magnitude



Vectors spaces insufficient for geometry

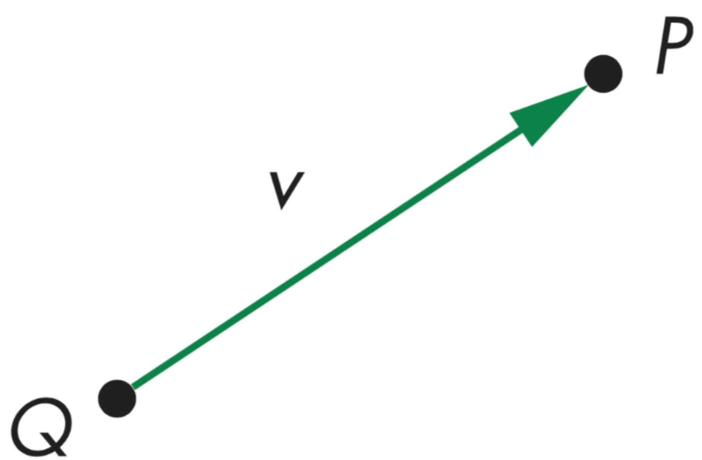
- Need points

# Points

Location in space

Operations allowed between points and vectors

- Point-point subtraction yields a vector
- Equivalent to point-vector addition




$$\boldsymbol{v} = \mathrm{P} - \mathrm{Q}$$

$$\mathrm{P} = \boldsymbol{v} + \mathrm{Q}$$

# Affine Spaces

Point + a vector space

Operations

- Vector-vector addition
- Scalar-vector multiplication
- Point-vector addition
- Scalar-scalar operations

For any point, define

- $1 \bullet P = P$
- $0 \bullet P = \mathbf{0}$ (zero vector)

# Coordinate-Free Geometry

Why CFG?

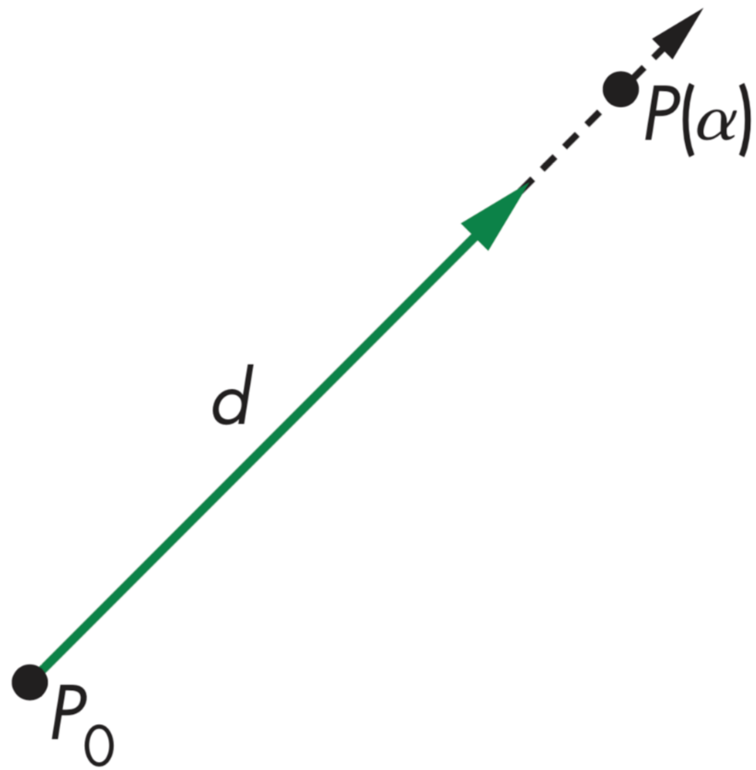
- We only care about the intrinsic geometric properties of the objects when reasoning about the geometric objects. e.g., congruent triangles
- CFG derivations usually provide much more geometric intuition and are simpler than using coordinates. e.g., dot product
- CFG provides kind-of "type checking" for geometric reasoning. e.g., point + vector

# Lines

Consider all points of the form

- $P(\alpha) = P_0 + \alpha\, \mathbf{d}$
- Set of all points that pass through $P_0$ in the direction of the vector $\mathbf{d}$

Parametric equation



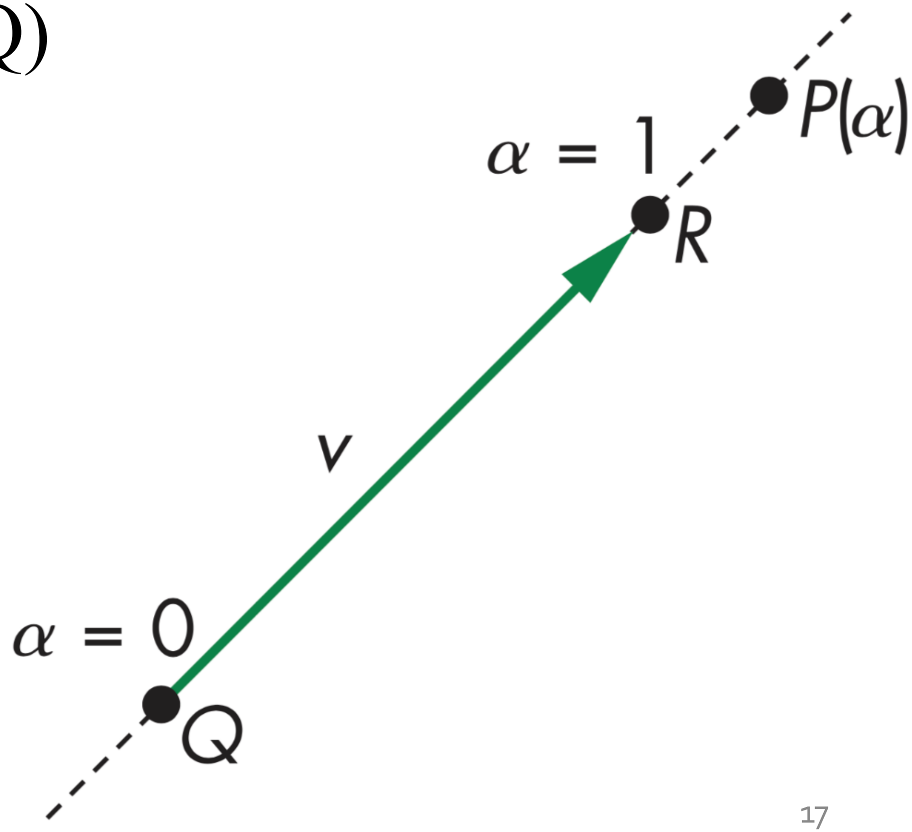
# Rays and Line Segments

If $\alpha$ >= 0, then $P(\alpha)$ is the ray (or half-line) leaving $P_0$ in the direction $\mathbf{d}$

If we use two points to define $\mathbf{v}$, then

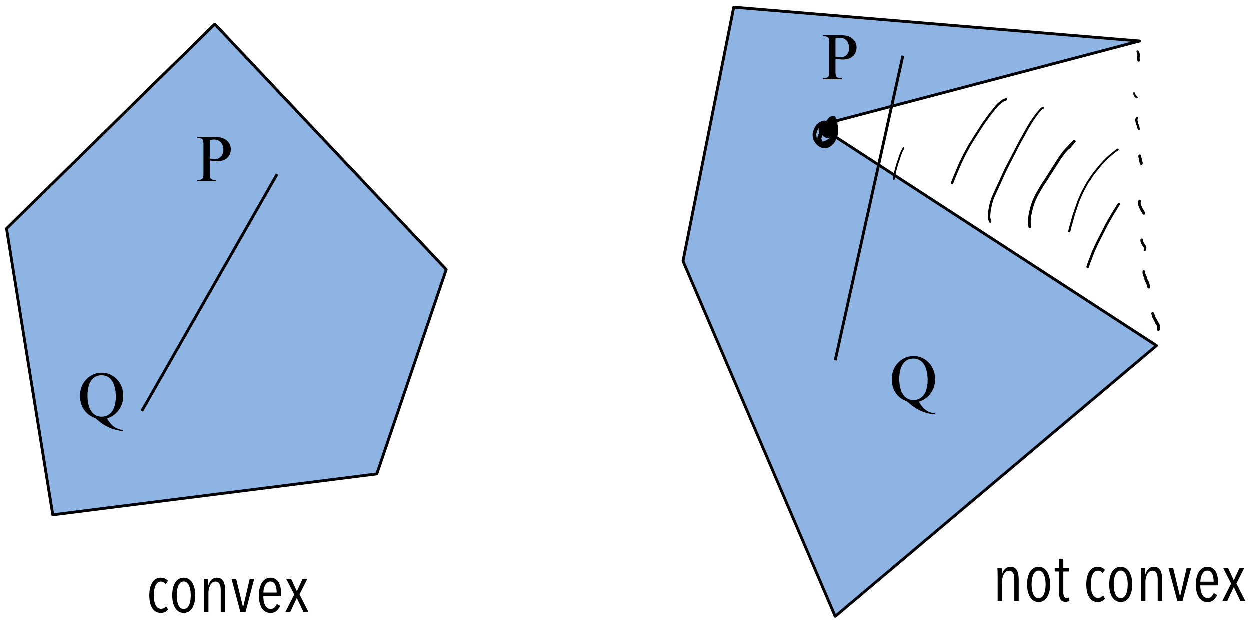
$$P(\alpha) = Q + \alpha\mathbf{v} = Q + \alpha\,(R - Q)$$
$$= \alpha R + (1 - \alpha)Q$$

For $0 <= \alpha <= 1$, we get all the points on the line segment joining $R$ and $Q$



# Convexity

An object is convex iff for any two points in the object all points on the line segment between these points are also in the object




This definition applies no matter what the object is or what the number of dimensions is

# Affine Sums

The affine sum of $n$ points $P_1, P_2, \ldots..P_n$ is defined as

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n$$

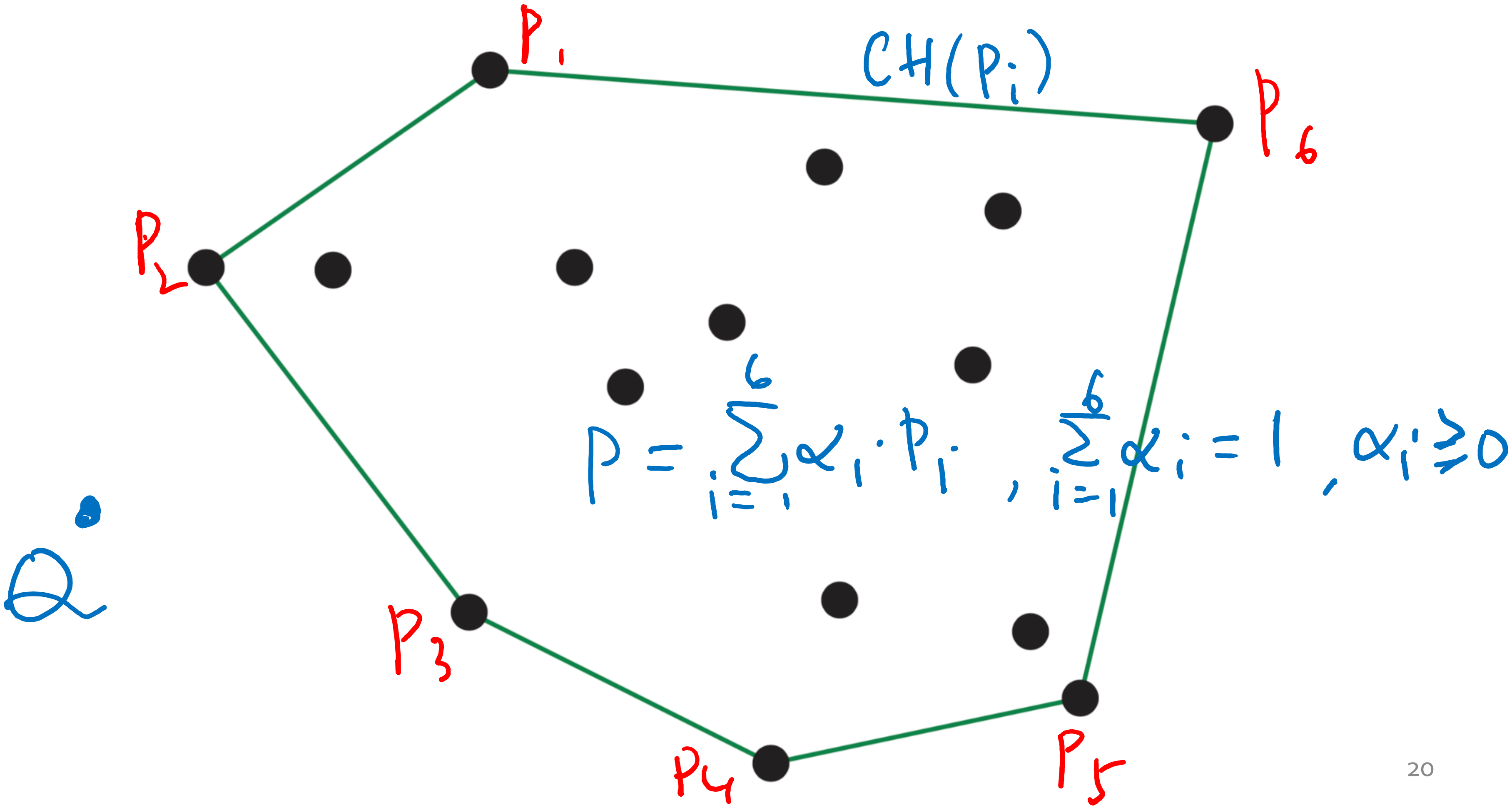
where $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$

If, in addition, $\alpha_i >= 0$ for all $i$, we have the **convex hull** of $P_1, P_2, \ldots, P_n$

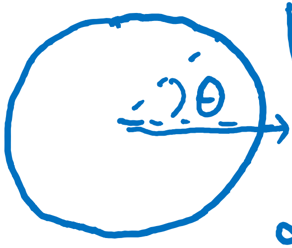
# Convex Hull

Smallest convex object containing $P_1, P_2, \ldots, P_n$

Formed by "shrink wrapping" points



# Curves and Surfaces




$P(\alpha) = (\cos\alpha, \sin\alpha)$

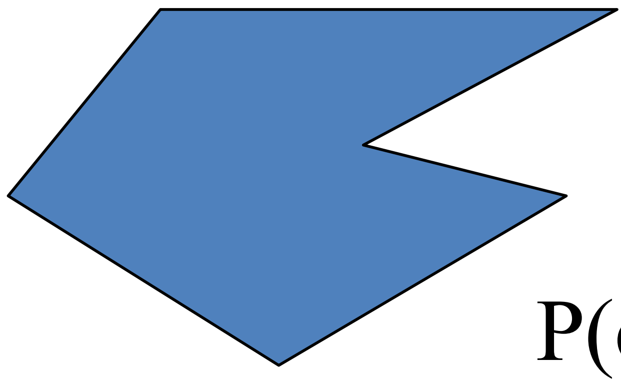
$\alpha = [0, 2\pi]$

Curves are one parameter entities of the form $\mathrm{P}(\alpha)$ where the function is nonlinear

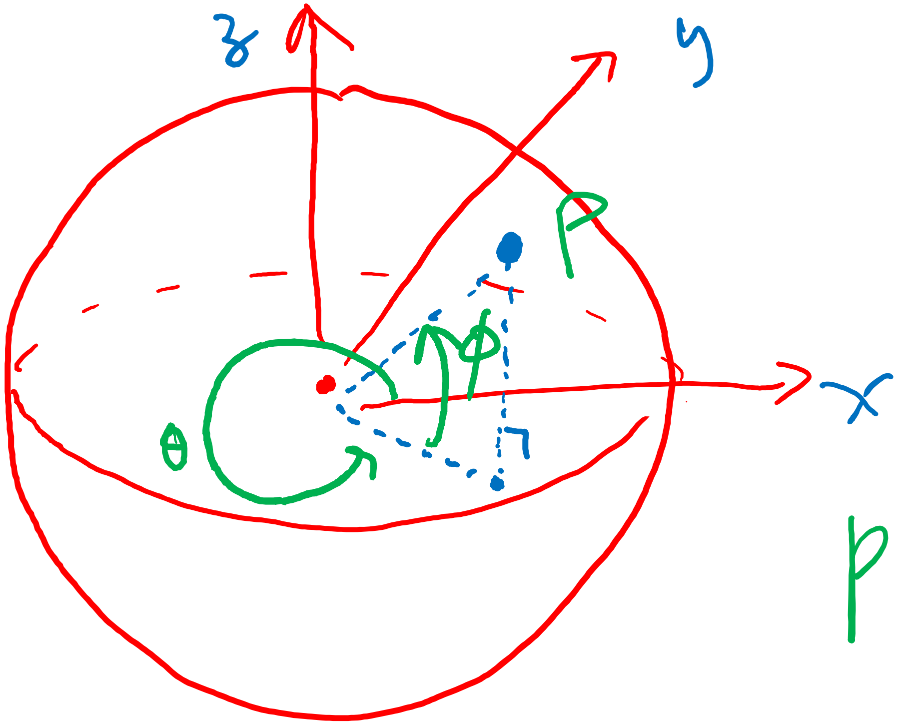
Surfaces are formed from two-parameter functions $\mathrm{P}(\alpha, \beta)$

- Linear functions give planes and polygons

$\mathrm{P}(\alpha)$



$\mathrm{P}(\alpha, \beta)$

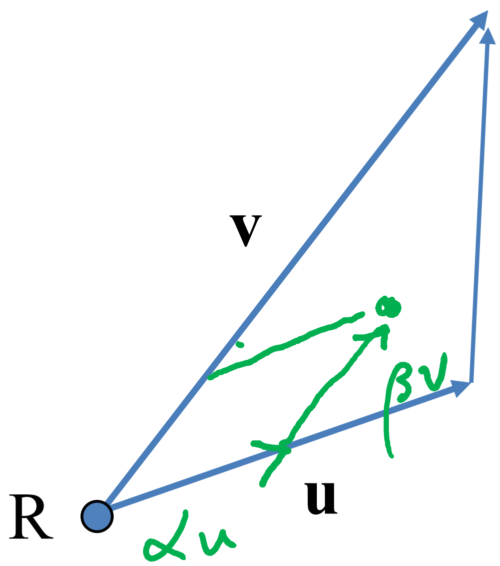


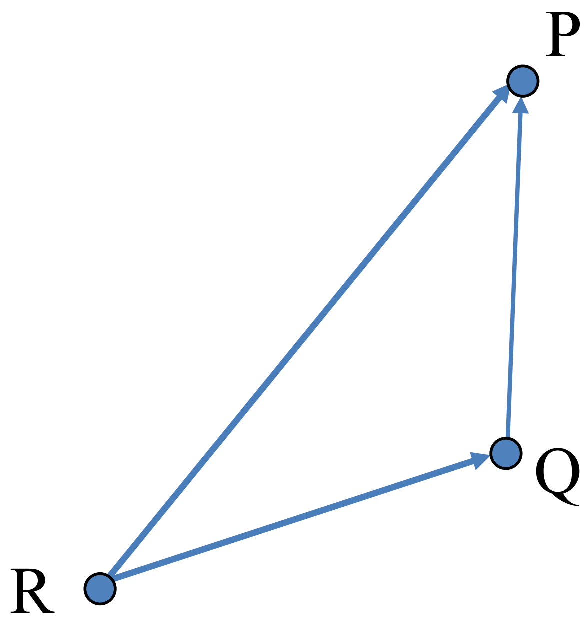

$\cancel{P =}$

$$P(\theta, \phi) = \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix}$$

# Planes

A plane can be defined by a point and two vectors, or by three points




$$N(\alpha, \beta) = R + \alpha\mathbf{u} + \beta\boldsymbol{v}$$

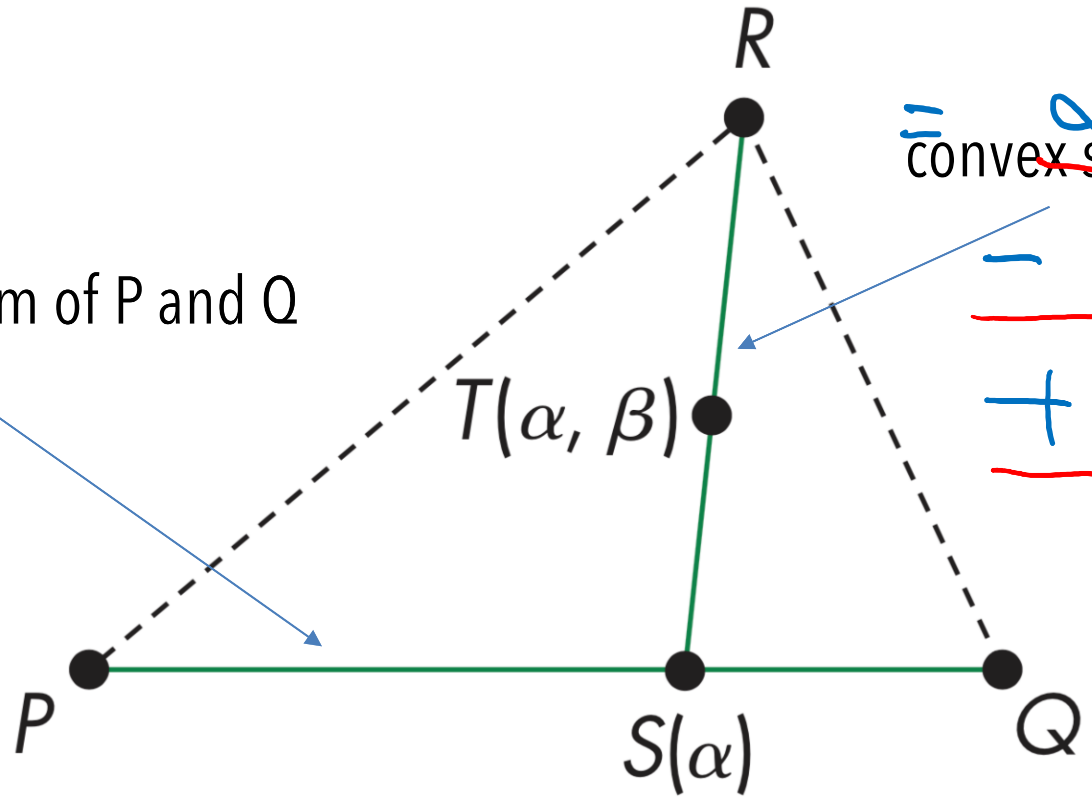
$$N(\alpha,\beta) = R + \alpha(Q-R) + \beta(P-R)$$

# Triangles

$$T(\alpha,\beta) = \beta\, S(\alpha) + (1-\beta) R$$
$$= \beta[\alpha P + (1-\alpha) Q] + (1-\beta) R, \quad \beta\in[0,1]$$
$$= \alpha\beta P + (1-\alpha)\beta Q + (1-\beta) R$$

convex sum of S($\alpha$) and R

convex sum of P and Q




for $0 <= \alpha, \beta <= 1$, we get all points in triangle

$$S(\alpha) = \alpha P + (1-\alpha) Q, \quad \alpha\in[0,1]$$

# Barycentric Coordinates

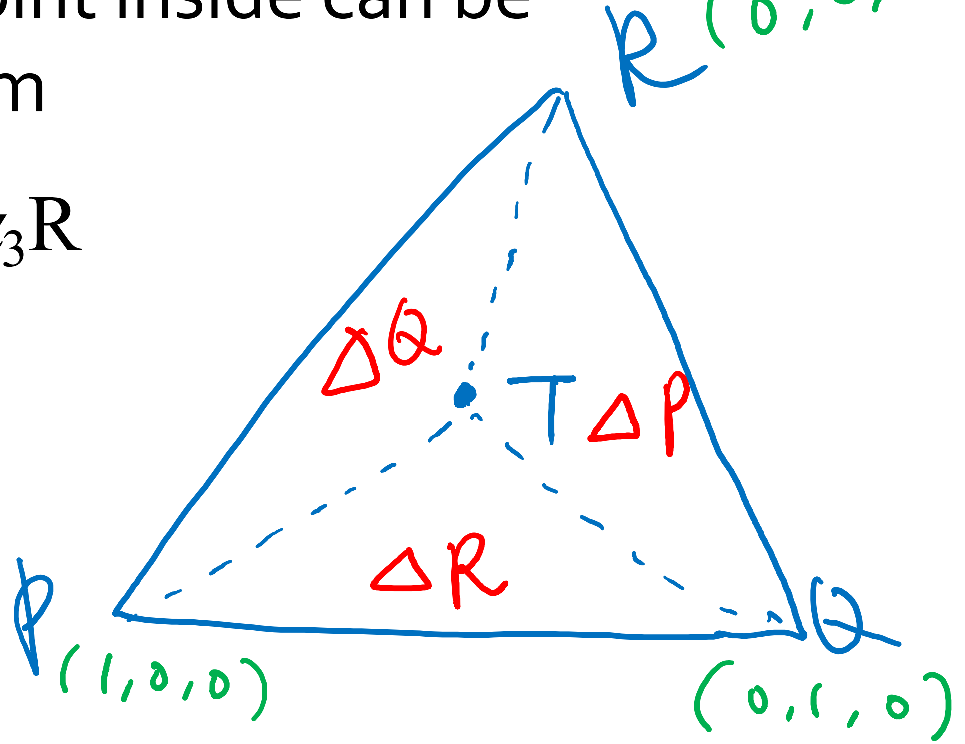
Triangle is convex so any point inside can be represented as an affine sum

$P(\alpha_{1,}\ \alpha_{2,}\ \alpha_3)=\alpha_1 P + \alpha_2 Q + \alpha_3 R$

where

$\alpha_1 + \alpha_2 + \alpha_3 = 1,$

$\alpha_i >= 0$




The representation is called the **barycentric coordinate** representation of P

$\alpha_1 = \frac{\Delta P}{\Delta PQR}, \quad \alpha_2 = \frac{\Delta Q}{\Delta PQR}, \quad \alpha_3 = \frac{\Delta R}{\Delta PQR}$

# Normal to a Plane

Every plane has a vector $\mathbf{n}$ perpendicular (or orthogonal) to it, and we called it the normal to the plane

parametric equation

From $P(\alpha,\beta) = R+\alpha\mathbf{u}+\beta\mathbf{v}$, we know we can use the cross product to find $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ and the equivalent form

$$(P(\alpha,\beta) - R) \cdot \mathbf{n} = 0$$

implicit equation