Questions

1. (a) Suppose that a quadratic Bézier curve $P(t)$ is given by $P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2, t \in [0;1]$; where $P_0 = (0,0)^T$, $P_1 = (4,0)^T$ and $P_2 = (4,4)^T$. Express the curve segment of this curve $P(t)$ over the interval $t \in [0, 0.5]$ as a Bézier curve.

Solution: By the de Casteljau algorithm, we have

$$\begin{aligned}
P_0^{(0)} &= P_0 = (0,0)^T \\
P_1^{(0)} &= P_1 = (4,0)^T \\
P_2^{(0)} &= P_2 = (4,4)^T \\
P_0^{(1)} &= 0.5P_0^{(0)} + 0.5P_1^{(0)} = (2,0)^T \\
P_1^{(1)} &= 0.5P_1^{(0)} + 0.5P_2^{(0)} = (4,2)^T \\
P_0^{(2)} &= 0.5P_0^{(1)} + 0.5P_1^{(1)} = (3,1)^T
\end{aligned}$$

The points $P_0^{(0)}, P_0^{(1)}, P_0^{(2)}$ form the Bézier control polygon of the curve segment of $P(t)$ over the interval $t \in [0, 0.5]$, and the curve segement can therefore be expressed as the following Bézier curve

$$\tilde{P}(t) = (1-t)^2 P_0^{(0)} + 2t(1-t)P_0^{(1)} + t^2 P_0^{(2)},\ t \in [0,1].$$

(b) Compute the control points of the cubic Bézier curve representing a segment of the cubic curve $y = 2x^3$, $x \in [-3, 3]$.

Solution: The parametric equation of the cubic curve is given by

$$Q(x) = (x, 2x^3), \quad x \in [-3, 3].$$

By reparametrization with $t = (x+3)/6$, we obtain the same curve

$$P(t) = \left(6t-3, 2(6t-3)^3\right), \quad t \in [0,1].$$

Let P_0, P_1, P_2, P_3 be the 4 control points of $P(t)$.
Due to the end-point interpolating property, we have

$$P_0 = P(0) = (-3, -54)^T \quad \text{and} \quad P_3 = P(1) = (3, 54)^T$$

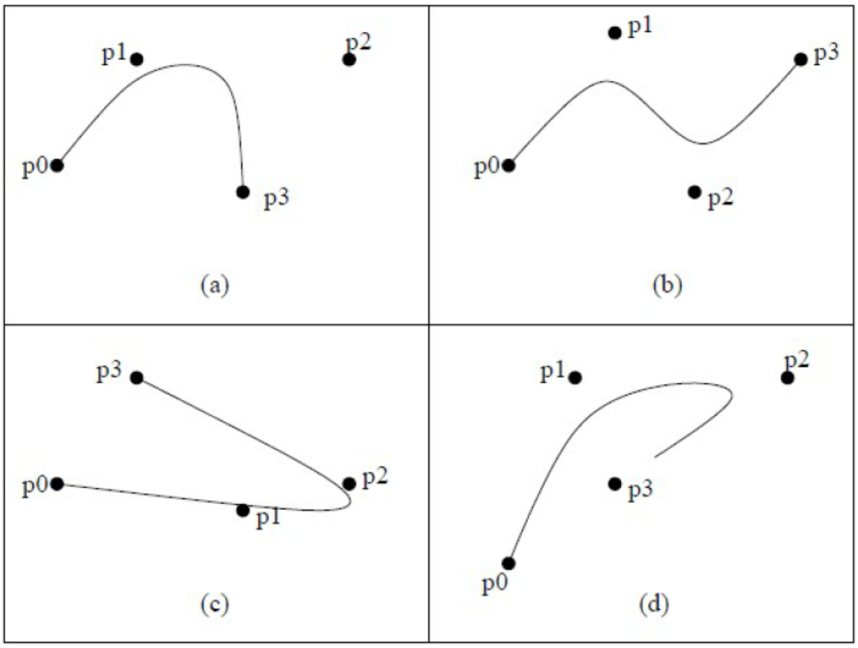
By the end-tangent interpolating property, we have

$$\begin{aligned}
P'(0) &= \frac{dP(t)}{dt}\Big|_{t=0} = 3(P_1 - P_0), \\
P'(1) &= \frac{dP(t)}{dt}\Big|_{t=1} = 3(P_3 - P_2).
\end{aligned}$$

Now, $P'(t) = \big(6, 36(6t-3)^2\big)$ and hence

$$
\begin{aligned}
P'(0) = (6, 36 \times 9)^T = 3(P_1 - P_0) \quad &\Rightarrow P_1 = (-1, 54)^T, \\
P'(1) = (6, 36 \times 9)^T = 3(P_3 - P_2) \quad &\Rightarrow P_2 = (1, -54)^T.
\end{aligned}
$$

(c) Which of the following must not be cubic Bézier curves, and why not?



Solution:

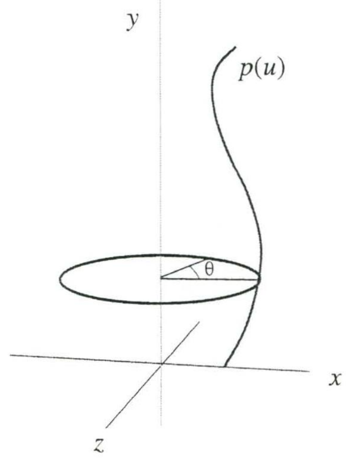
Figures (a), (c) and (d) must not be cubic Bézier curves.

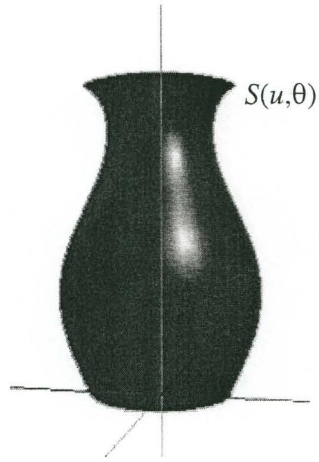
A cubic Bézier curve has the end-tangent interpolation property. In Fig (a), the tangent line at P_3 is not the line P_2P_3, so (a) must not be cubic Bézier curves.

A cubic Bézier curve must be always in the convex hull of the control points. In Fig (c), a segment of curve is outside line P_1P_2.

A cubic Bézier curve has the endpoint interpolation property. In Fig (d), the end point is not P_3, so (d) must not be cubic Bézier curves.

2. Consider the parametric curve $p(u)$ defined on the x-y plane in the figure below. When u varies from 0.0 to 1.0, $p(u)$ moves from the lower end of the curve to the upper end. A sweeping surface $S(u,\theta)$, shown in the right figure, is formed when the curve is revolved about the y-axis. This surface has two parameters: $0 \le u \le 1$ and $0 \le \theta < 2\pi$, where θ denotes the amount of revolution.





(a) Give the formulas for the x, y and z coordinates of a point on $S(u,\theta)$. You may assume that $p(u)$ is given as as $\big(x_p(u), y_p(u)\big)$.
Solution:

$$
\begin{aligned}
x_S(u,\theta) &= x_p(u)\cos\theta \\
y_S(u,\theta) &= y_p(u) \\
z_S(u,\theta) &= -x_p(u)\sin\theta
\end{aligned}
$$

(b) Let $p'(u) = \big(x'_p(u), y'_p(u)\big)$ be the first derivative of $p(u)$. Derive the normal of $S(u,\theta)$. (Do not normalize the normal vector in your answer.)
Solution:

$$
\begin{aligned}
\frac{\partial S(u,\theta)}{\partial u} &= \big(x'_p(u)\cos\theta,\ y'_p(u),\ -x'_p(u)\sin\theta\big) \\
\frac{\partial S(u,\theta)}{\partial \theta} &= \big(-x_p(u)\sin\theta,\ 0,\ -x_p(u)\cos\theta\big)
\end{aligned}
$$

The normal of $S(u,\theta)$ is given by

$$
\begin{aligned}
& \frac{\partial S(u,\theta)}{\partial u} \times \frac{\partial S(u,\theta)}{\partial \theta} \\
= \ & \big(-x_p(u)y'_p(u)\cos\theta,\ x'_p(u)x_p(u)\cos^2\theta + x'_p(u)x_p(u)\sin^2\theta,\ x_p(u)y'_p(u)\sin\theta\big) \\
= \ & \big(-x_p(u)y'_p(u)\cos\theta,\ x'_p(u)x_p(u)\ x_p(u)y'_p(u)\sin\theta\big)
\end{aligned}
$$