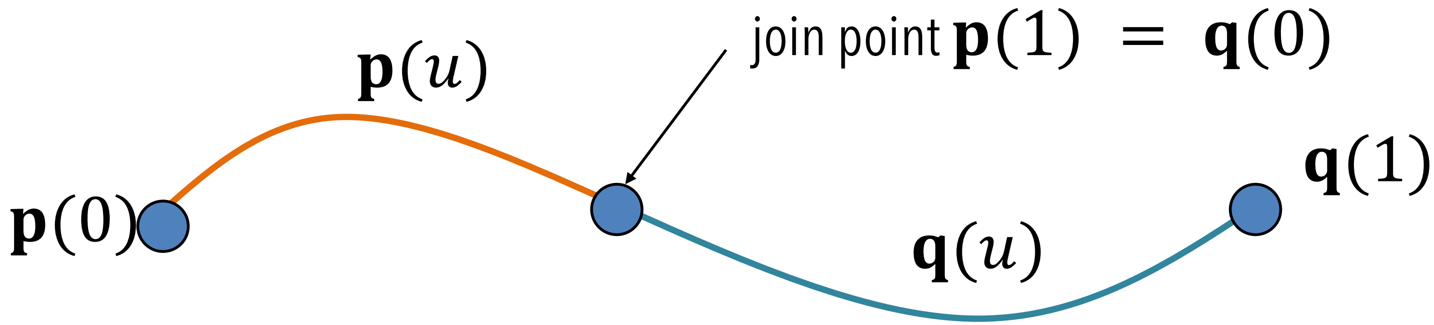
Curve Segments

After normalizing u, each curve can be written as

$$\mathbf{p}(u) = [x(u), y(u), z(u)]^T, \quad 0 \leq u \leq 1$$

In classical numerical methods, we design a single global curve

In computer graphics and CAD, it is better to design small connected curve segments

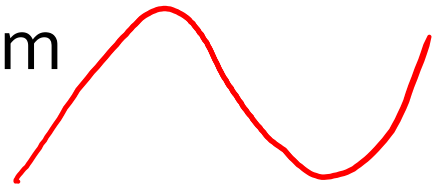


$a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0 = \sum_{i=0}^{n} a_i t^i$

Parametric Polynomial Curves

$\mathbf{p}(t) = (x(t), y(t), z(t))^T$ is called a polynomial curve if $x(t)$, $y(t)$ and $z(t)$ are polynomial functions of t.

A polynomial curve of degree n is of the form



$$\mathbf{p}(t) = \sum_{k=0}^{n} \mathbf{c}_k t^k \quad \text{where} \quad \mathbf{c}_k = \begin{pmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{pmatrix}.$$

also a polynomial curve of **order $n+1$**

Hence,

$$x(t) = \sum_{k=0}^{n} c_{xk} t^k,\ y(t) = \sum_{k=0}^{n} c_{yk} t^k,\ z(t) = \sum_{k=0}^{n} c_{zk} t^k$$

$t \in [0,1]$.

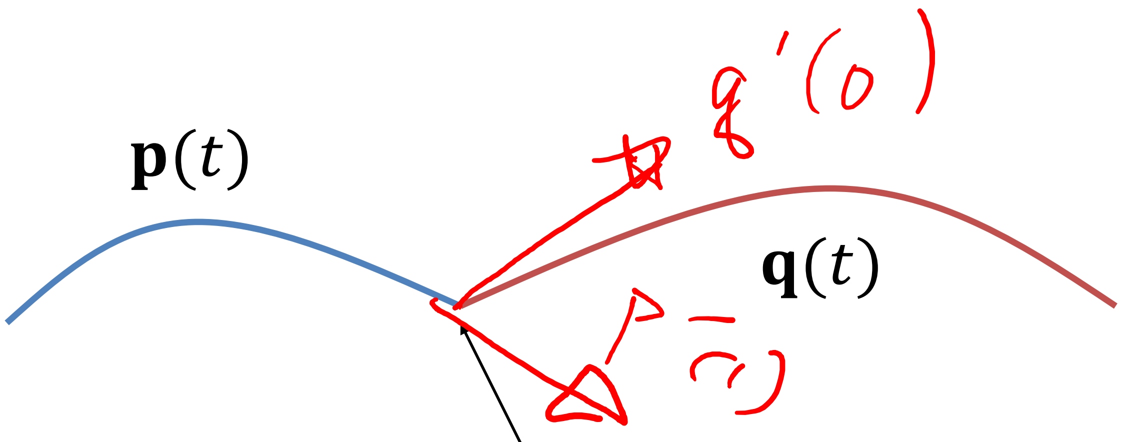
We need to determine $3(n+1)$ coefficients
Equivalently we need $3(n+1)$ independent conditions

Why Polynomials

Easy to evaluate – need only $+, -, \times, \div$ (e.g., no sin/cos)

Continuous and differentiable everywhere

- Still need to worry about continuity at join points including continuity of derivatives



join point $\mathbf{p}(1) = \mathbf{q}(0)$
but $\mathbf{p}'(1) \neq \mathbf{q}'(0)$

Cubic Parametric Polynomials

We mostly use cubic curves which gives balance between ease of evaluation and flexibility in design

$$\mathbf{p}(t) = \sum_{k=0}^{3} \mathbf{c}_k t^k$$

$$\sum_{k=0}^{3} c_{kx} t^k$$

Four coefficients to determine for each of x, y and z

Seek four independent conditions for various values of t resulting in 4 equations in 4 unknowns for each of x, y and z

- Conditions are a mixture of continuity requirements at the join points and conditions for fitting the data

Representation in Matrix-Vector Form

$$\mathbf{p}(u) = \sum_{k=0}^{3} \mathbf{c}_k u^k = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 u^2 + \mathbf{c}_3 u^3$$

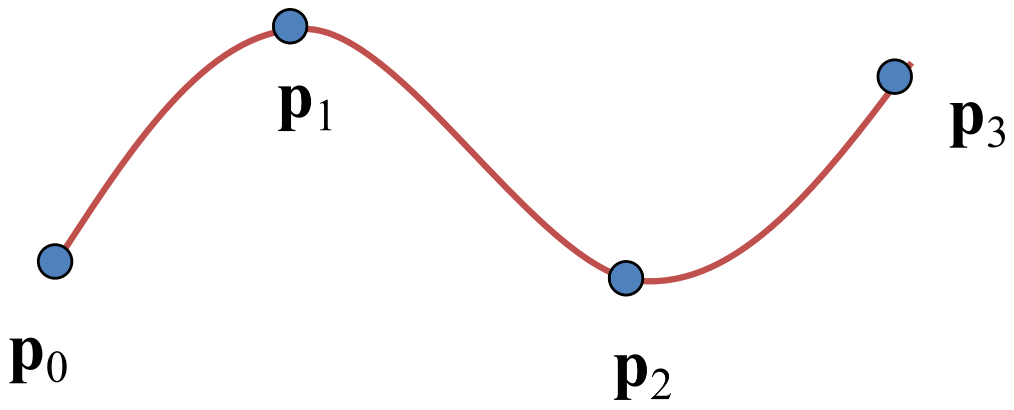
a cubic curve

$$= \mathbf{u}^T \mathbf{c} = \mathbf{c}^T \mathbf{u}$$

where

$$\mathbf{c} = \begin{pmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}, \quad \mathbf{c}_k = \begin{pmatrix} c_{kx} \\ c_{ky} \\ c_{kz} \end{pmatrix}$$

Interpolating Curve

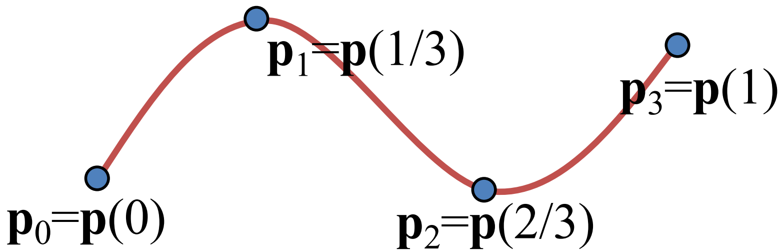


Given four data (control) points $\mathbf{p}_0$, $\mathbf{p}_1$,$\mathbf{p}_2$, $\mathbf{p}_3$
determine cubic $\mathbf{p}(u)$ which passes through them

Need to find $\mathbf{c}_0$,$\mathbf{c}_1$,$\mathbf{c}_2$, $\mathbf{c}_3$

Interpolation Equations

$$\mathbf{p}(u) = \sum_{k=0}^{3} \mathbf{c}_k u^k = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 u^2 + \mathbf{c}_3 u^3$$



apply the interpolating conditions at $u = 0,\ 1/3,\ 2/3,\ 1$

$$\begin{aligned}
\mathbf{p}_0 &= \mathbf{p}(0) = \mathbf{c}_0 \\
\mathbf{p}_1 &= \mathbf{p}(1/3) = \mathbf{c}_0 + (1/3)\mathbf{c}_1 + (1/3)^2\mathbf{c}_2 + (1/3)^3\mathbf{c}_2 \\
\mathbf{p}_2 &= \mathbf{p}(2/3) = \mathbf{c}_0 + (2/3)\mathbf{c}_1 + (2/3)^2\mathbf{c}_2 + (2/3)^3\mathbf{c}_2 \\
\mathbf{p}_3 &= \mathbf{p}(1) = \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_2
\end{aligned}$$

or in matrix form

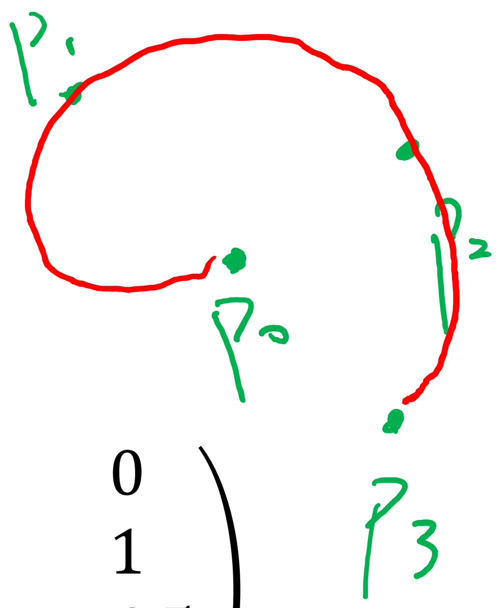
$$\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)^T = \mathbf{A}\mathbf{c}, \quad \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Interpolation Matrix $p(u) = \sum_{i=0}^{3} c_i u^i$

$u \in [0,1]$

$$\mathbf{p} = \mathbf{A}\mathbf{c}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$A^{-1} p = c$

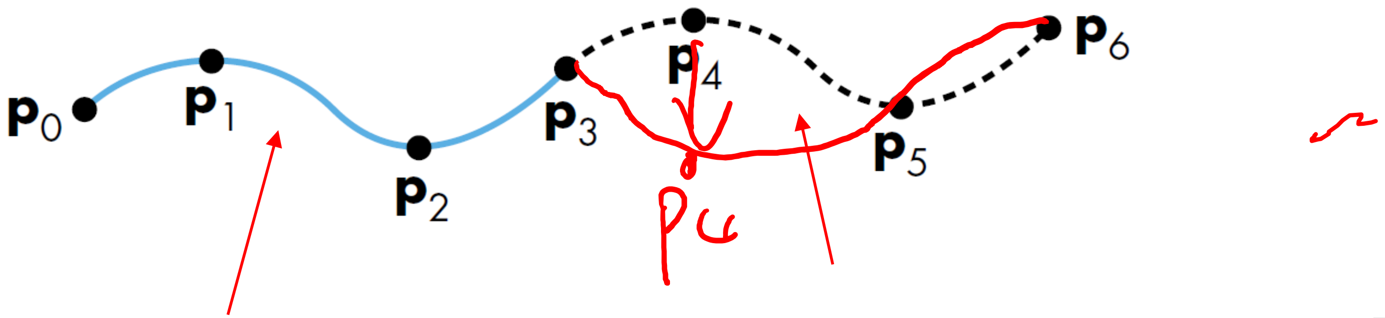


Let $\mathbf{M}_I = \mathbf{A}^{-1}$ and solving for $\mathbf{c}$,

$$\mathbf{c} = \mathbf{M}_I\mathbf{p}, \qquad \mathbf{M}_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & 22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{pmatrix}$$

Note that $\mathbf{M}_I$ does not depend on input data and can be used for each segment in x, y, and z

Interpolating Multiple Segments



use $\mathbf{p} = [\mathbf{p}_0\ \mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^T$

use $\mathbf{p} = [\mathbf{p}_3\ \mathbf{p}_4\ \mathbf{p}_5\ \mathbf{p}_6]^T$

Get continuity at join points but not continuity of derivatives

$p_0 \quad (1-t)p_0 + tp_1 \quad p_1$

Blending Functions

$\mathbf{u} = \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}$

$u^T M = \begin{pmatrix} 1 & u & u^2 & u^3 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

Rewriting the equation for p(u)

$$\sum_{i=0}^{3} c_i u^i = \mathbf{p}(u) = \mathbf{u}^T\mathbf{c} = \mathbf{u}^T\mathbf{M}_I\mathbf{p} \;=\; \mathbf{b}(u)^T\mathbf{p}$$

where $\mathbf{b}(u) \;=\; [b_0(u)\;\; b_1(u)\;\; b_2(u)\;\; b_3(u)]^T$ is an array of **blending polynomials** such that

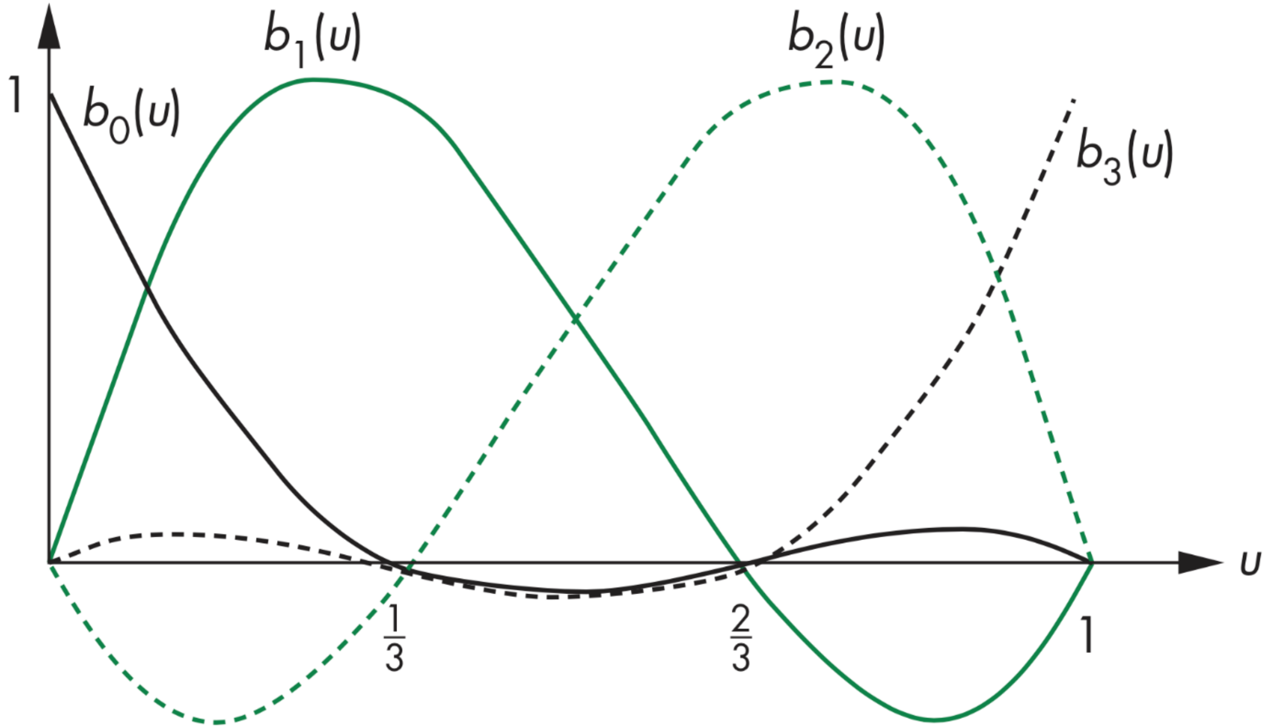
$$\mathbf{p}(u) \;=\; b_0(u)\mathbf{p}_0 + \; b_1(u)\mathbf{p}_1 + \; b_2(u)\mathbf{p}_2 + \; b_3(u)\mathbf{p}_3$$

$$\begin{aligned} b_0(u) \;&=\; -4.5(u-\tfrac{1}{3})(u-\tfrac{2}{3})(u-1) \\ b_1(u) \;&=\; 13.5u\,(u-\tfrac{2}{3})(u-1) \\ b_2(u) \;&=\; -13.5u\,(u-\tfrac{1}{3})(u-1) \\ b_3(u) \;&=\; 4.5u\,(u-\tfrac{1}{3})(u-\tfrac{2}{3}) \end{aligned}$$

Blending Functions

These functions are not smooth

- Hence the interpolation polynomial is not smooth



Cubic Polynomial Surfaces

A degree $m \times n$ parametric polynomial surface is defined as

order : $(m+1) \times (n+1)$

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \sum_{i=0}^{\overset{m}{\cancel{n}}} \sum_{j=0}^{n} \mathbf{c}_{ij}\, u^i\, v^j , \qquad (u,v) \in [0,1]^2$$

More specifically, a cubic polynomial surface is given by

$$\mathbf{p}(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} \mathbf{c}_{ij} u^i v^j , \qquad (u,v) \in [0,1]^2.$$

Need 48 coefficients (4 x 4 x 3) to determine a surface patch

Interpolating Patch

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{c}_{ij}\, u^i\, v^j$$

a bicubic surface

Define $\mathbf{u} = [1\ u\ u^2\ u^3]^T$, $\mathbf{v} = [1\ v\ v^2\ v^3]^T$, $\mathbf{C} = [\mathbf{c}_{ij}]$

4x4 matrix, each element is a 3-vector

We may write

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{c}_{ij}\, u^i\, v^j = \mathbf{u}^T\mathbf{C}\mathbf{v}$$

Interpolating Patch in Matrix Form

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{c}_{ij}\, u^i v^j \quad = \mathbf{u}^T\mathbf{C}\mathbf{v}$$

$u = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$

Need 16 conditions to determine the 16 coefficients $\mathbf{c}_{ij}$

a 3-vector

Consider $u = v = 0$, and we have

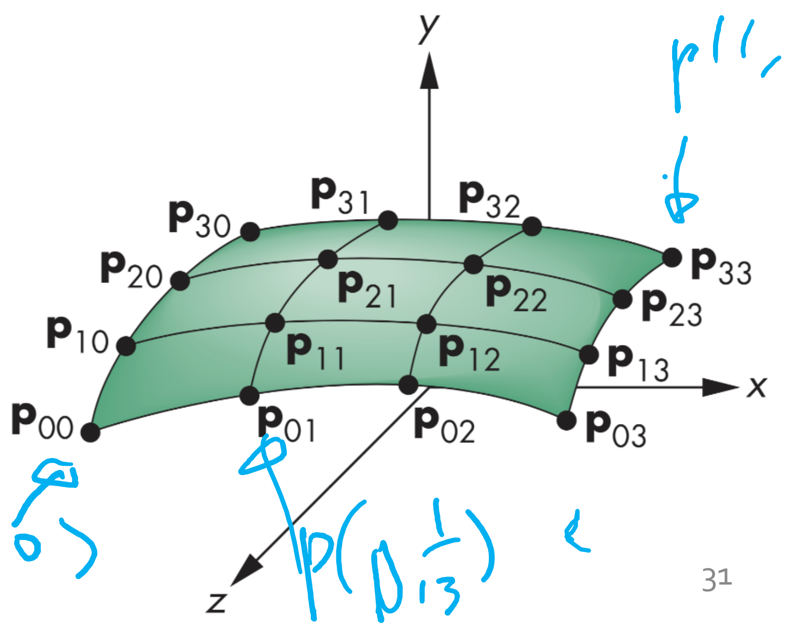
$u = 0$

$v = \frac{1}{3}$

$$\mathbf{p}_{00} = [1 \quad 0 \quad 0 \quad 0]\,\mathbf{C}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \mathbf{c}_{00}$$

$p_{01} = [1 \ 0 \ 0 \ 0]\, C \begin{bmatrix} 1 \\ \frac{1}{3} \\ (\frac{1}{3})^2 \\ (\frac{1}{3})^3 \end{bmatrix}$

Choose $u, v = 0, \frac{1}{3}, \frac{2}{3}, 1$
and we have the 16 conditions



$p(1,1)$

$p(0,0)$

$p(0, \frac{1}{3})$

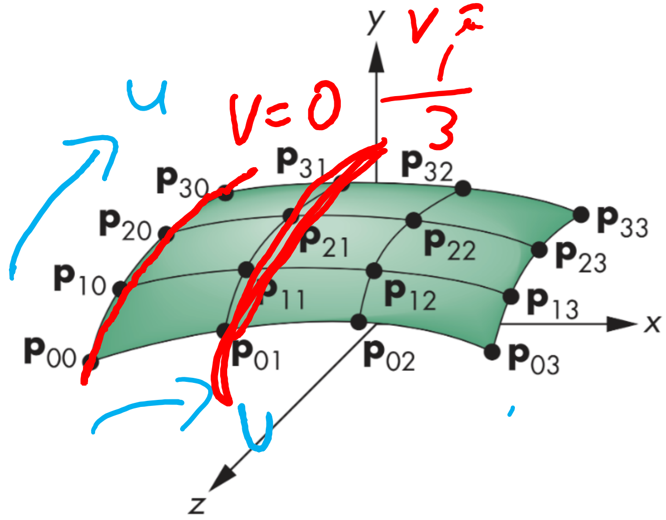
Interpolating Patch in Matrix Form

Instead of writing the 16 equations one by one and solve for $\mathbf{C}$, let's consider $v = 0$,

$$\mathbf{p}(u,0) = \sum_{i=0}^{3} \mathbf{c}_{i0} u^i = \mathbf{u}^T \mathbf{C} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{u}^T \mathbf{M}_I \begin{bmatrix} \mathbf{p}_{00} \\ \mathbf{p}_{10} \\ \mathbf{p}_{20} \\ \mathbf{p}_{30} \end{bmatrix}$$

A cubic curve interpolating $\mathbf{p}_{00}$, $\mathbf{p}_{10}$, $\mathbf{p}_{20}$, $\mathbf{p}_{30}$

$$u^T C \begin{bmatrix} \frac{1}{3} \\ (\frac{1}{3})^2 \\ (\frac{1}{3})^3 \end{bmatrix} = u^T M_I \begin{bmatrix} p_{01} \\ p_{11} \\ p_{21} \\ p_{31} \end{bmatrix}$$



Doing the same for $v = \frac{1}{3}, \frac{2}{3}, 1$, and we have 3 more equations similar to the above. Hence we have

$$\mathbf{u}^T \mathbf{C} \mathbf{A}^T = \mathbf{u}^T \mathbf{M}_I \mathbf{P} \quad \text{where } \mathbf{P} = [\mathbf{p}_{ij}] \qquad \text{and therefore } \mathbf{C} = \mathbf{M}_I \mathbf{P} {\mathbf{M}_I}^T$$

$$\boxed{\mathbf{M}_I = \mathbf{A}^{-1}}$$

The bicubic surface is given by $\mathbf{p}(u,v) = \mathbf{u}^T \mathbf{M}_I \mathbf{P} {\mathbf{M}_I}^T \mathbf{v}$

Blending Patches

We can build and analyze surfaces from our knowledge of curves

$$\mathbf{p}(u, v) \ = \ \mathbf{u}^T \mathbf{M}_I \mathbf{P} \mathbf{M}_I{}^T \mathbf{v}$$

Note that $\mathbf{M}_I \mathbf{u}^T$ are the blending functions

$$\mathbf{b}(u) \ = \ [b_0(u) \ \ b_1(u) \ \ b_2(u) \ \ b_3(u)]^T$$

Hence we may rewrite the surface equation as

$$\sum_{i=0}^{3} \sum_{j=0}^{3} b_i(u) b_j(\cancel{u}\,v) \mathbf{p}_{ij}$$

Each $b_i(u)\ b_j(v)$ describes a blending patch