Conic Sections

A common class of curves with a very long history

- defined by intersections of a plane with a cone
- defined implicitly by the 2^{nd} degree polynomial

$$F(x,y) = ax^2 + 2bxy + 2cx + dy^2 + 2ey + f$$

$(Ax+By+C)(Ax+By+C)=0$ → A^2

In matrix form

$$F(x,y) = \mathbf{v}^\top \mathbf{Q} \mathbf{v} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

This describes several generally useful kinds of curves

- circles, ellipses, parabolas, hyperbolas, and lines

Quadric Surfaces

Quadrics are the 3D analogue of conics

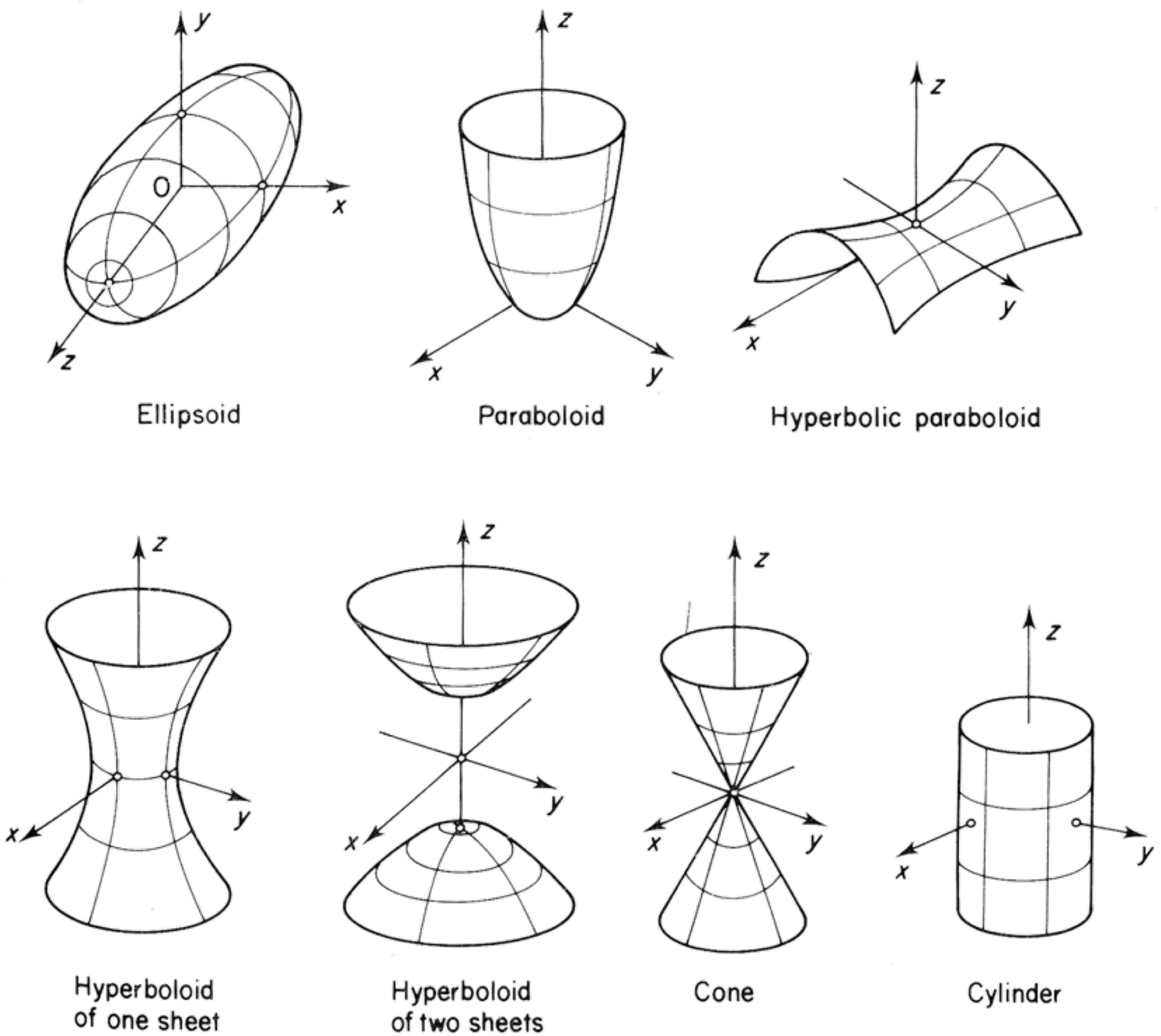
- defined by the 2nd degree polynomial

$$F(x,y,z) = ax^2 + 2bxy + 2cxz + 2dx + ey^2 + 2fyz + 2gy + hz^2 + 2iz + j$$

- or in matrix form

$$F(x,y,z) = \mathbf{v}^{\top}\mathbf{Q}\mathbf{v} = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

Quadric Surfaces

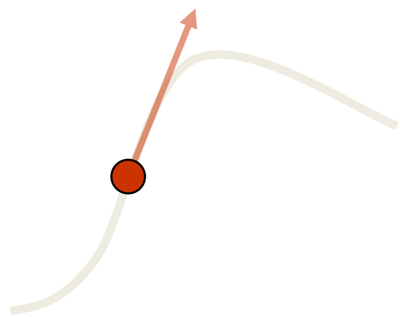


Quadric classes (from Paul Heckbert).

Sweeping out Surfaces

We view space curves as being swept out by a moving point

$$\mathbf{p}(u) = \begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix}$$



- as we vary u the point moves through space
- the curve is the path the point takes

Essentially looked at surfaces the same way

$$\mathbf{p}(u,v) = \begin{bmatrix} x(u,v) & y(u,v) & z(u,v) \end{bmatrix}$$

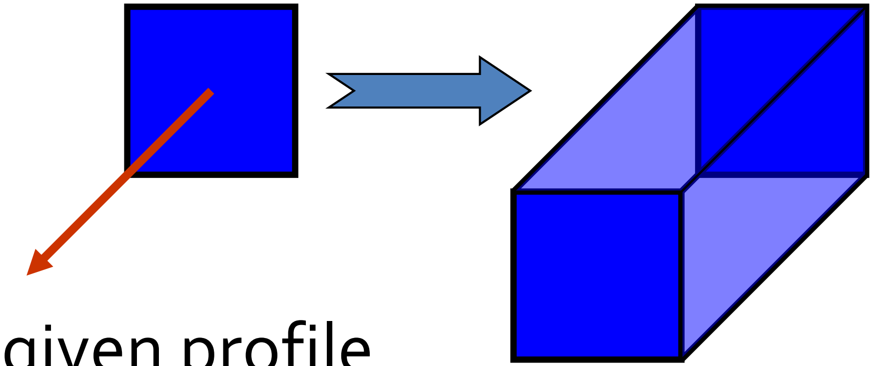
Now let's think about sweeping curves through space instead

- this will define a surface
- the set of all points visited by the curve during its motion

Extrusion Surfaces

Here's a particularly simple method

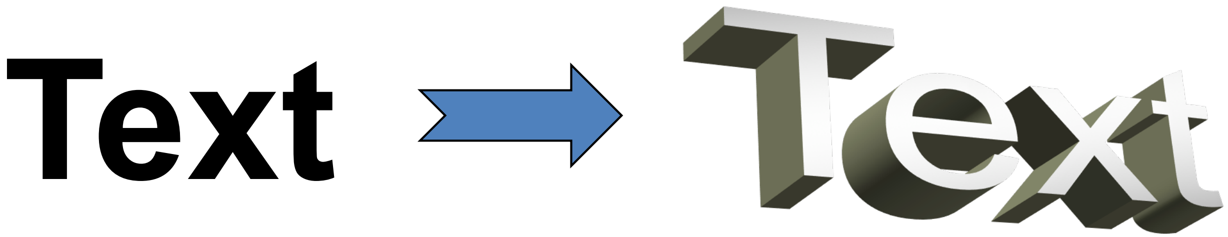
- specify initial (closed) curve
- pick an axis to move along
- and a distance to move



Sweeps out something with the given profile

- open curve defines a surface with an open boundary
- closed curve defines something like a cylinder

This is a common technique used to create 3D text



Extrusion Surfaces

Can be generated by translating a 2D cross-section curve along a fixed direction.

Let the cross-section curve be $C(u)$, and the given direction vector be D. Then the surface is defined by

$$E(u,v) = vD + C(u), \qquad v \in [v_0, v_1].$$

$C(\theta) = (\cos\theta, \sin\theta, 0), \ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Surfaces of Revolution

$S(\theta, \phi) = R_y(\phi)\, C(\theta)$

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

Extrusion moves curves via translation

- we can just as easily use rotation

Start with some curve

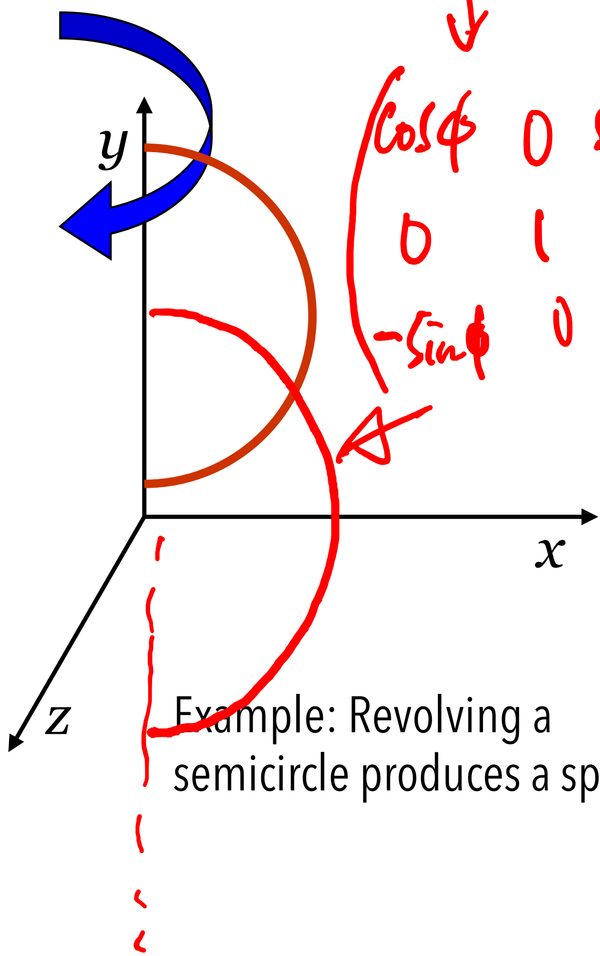
- pick an axis of rotation
- rotate about axis by 360°

Characteristics of revolved surfaces

- closed if endpoints on axis
- open otherwise
- but we can always fill in top/bottom
- by construction, they're symmetric

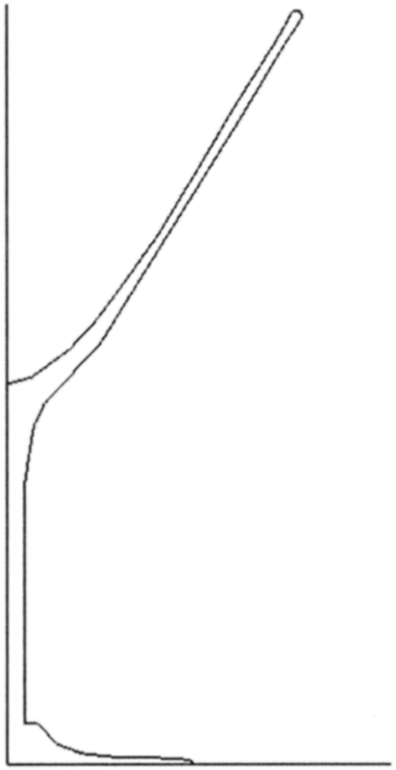
Lots of other easy examples:

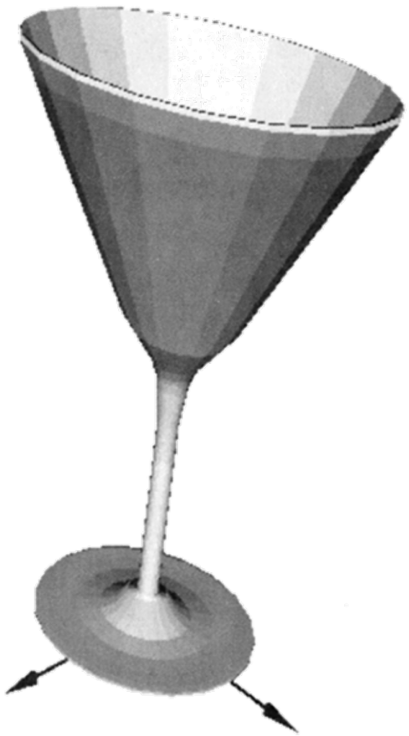
- cylinder, cone, paraboloid, …

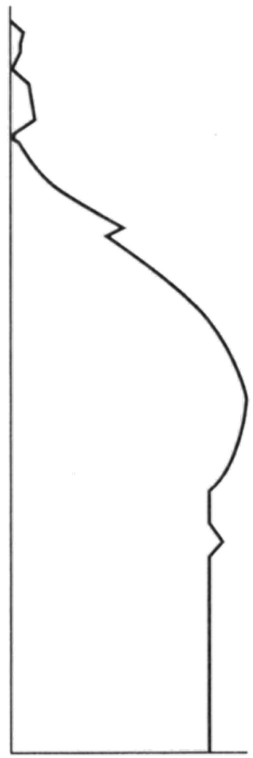


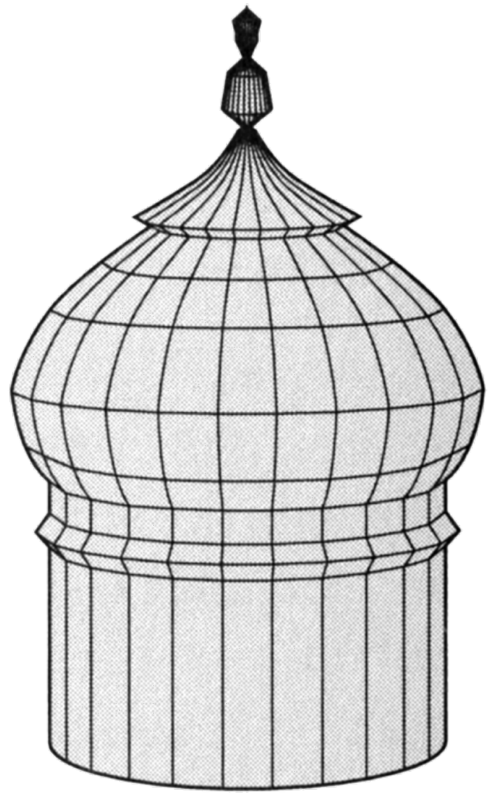
Example: Revolving a semicircle produces a sphere.

More Complex Examples of Revolution





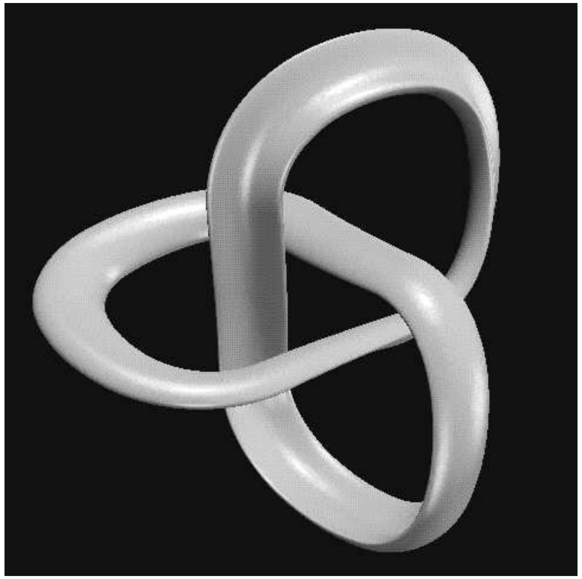




Sweep Surfaces

The sweep surface is generated by sweeping a 2D cross-section curve along an axis curve.

The plane of the cross-section curve is usually kept perpendicular to the tangent of the axis curve.



A sweep surface along a cubic spline curve.

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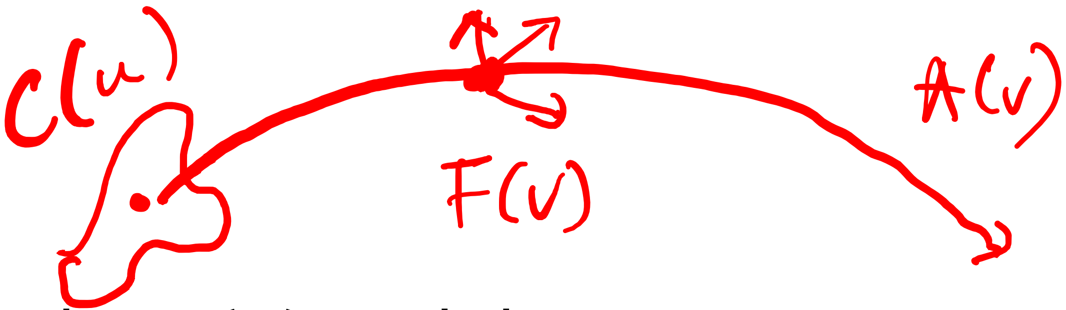
Sweep Surfaces



Let the cross-section curve be $C(u)$ and the axis curve be $A(v)$.
Let $F(v)$ be the matrix representing a rotation frame attached at $A(v)$. Then the sweep surface is defined by

$$S(u,v) = A(v) + F(v)C(u), \qquad v \in [v_0, v_1].$$

To use $s(v)$ to change the size of the cross-section curve while sweeping it, we have

$$S(u,v) = A(v) + s(v)F(v)C(u), \qquad v \in [v_0, v_1].$$