# Path Tracing Renderer Using Monte Carlo Methods

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October 21, 2024

**Abstract**

This report presents an expository on the foundational mathematical knowledge and implementation of a path tracing renderer using Monte Carlo methods to simulate realistic lighting in a 3D scene. Various sampling techniques and variance reduction methods are explored to enhance image quality and convergence speed. Experimental results demonstrate the effectiveness of these techniques in reducing noise and improving rendering efficiency.

# Contents

# 1 Introduction

## 1.1 Purpose and Scope

### 1.1.1 Goals of the Document

This expository aims to achieve the following objectives:

1. To provide a comprehensive analysis of the *mathematical* foundations underlying path tracing, creating a scene, and Monte Carlo integration techniques.

2. To demonstrate the *visual* impact of Monte Carlo methods in path tracing for realistic light simulation, exploring their impact on image fidelity, convergence rates, and rendering efficiency.

3. To elucidate the *design* and implementation of a path tracing system, with particular emphasis on acceleration structures like Bounding Volume Hierarchies (BVH), real-time visualisation with OpenGL and adhering to software modelling principles.

Through these objectives, this document seeks to offer a thorough exploration of path tracing, from its mathematical underpinnings to its practical realization in computer graphics, serving as a valuable resource for researchers and practitioners in the field of physically-based rendering.

### 1.1.2 Overview of the Methodological Approach

This expository adopts a progressive approach to the principles and implementation of path tracing. We begin by examining the foundational elements of a generic path tracer, including ray representation, intersection testing, and basic shading models. For each of these components, we present the underlying mathematical principles, followed by their corresponding code implementations. This foundational phase establishes the core concepts upon which more advanced techniques are built.

Building upon these foundational elements, we then abstract the entire lighting simulation process into a comprehensive mathematical framework. This transition allows us to view the path tracing problem as an integration task. We introduce Monte Carlo methods as a powerful tool for numerical integration, exploring how these statistical techniques can be applied to solve the rendering equation efficiently.

As we introduce each component, we demonstrate its effect on the rendered image, allowing for a clear understanding of how theoretical concepts translate to visual outcomes. This iterative approach provides immediate feedback on the impact of each implemented feature, reinforcing the connection between mathematical principles and their applications in computer graphics.

## 1.2 Background

### 1.2.1 Overview of Ray Tracing

Ray tracing is a rendering technique that simulates the physical behavior of light to create realistic images. At its core, ray tracing involves tracing the path of light rays as they interact with objects in a virtual scene [2]. The fundamental concept is to cast rays from a virtual camera through each pixel of an image plane into the scene. These rays interact with objects, potentially reflecting, refracting, or being absorbed, mimicking the behavior of light in the real world [9]. By accurately modeling these interactions, ray tracing can produce highly realistic effects such as shadows, reflections, and refractions.

Path tracing, an advanced form of ray tracing, extends this concept by using Monte Carlo methods to solve the rendering equation [4]. It traces numerous light paths through the scene, accounting for multiple bounces and complex light interactions. This approach allows for the accurate simulation of global illumination effects, including soft shadows, color bleeding, and caustics [8].

The power of ray tracing and path tracing lies in their ability to naturally handle a wide range of lighting phenomena, producing physically accurate images. However, this accuracy comes at the cost of increased computational complexity, often requiring sophisticated optimization techniques to achieve reasonable rendering times [6].

### 1.2.2 Importance in Computer Graphics

Path tracing has become a cornerstone technique in modern computer graphics, particularly in applications demanding high levels of photorealism. Its ability to accurately simulate complex light interactions makes it invaluable

in industries such as film and television visual effects, architectural visualization, product design, and video game development [5].

In the film industry, path tracing is used to create photorealistic CGI that seamlessly blends with live-action footage [1]. Moreover, with the advent of real-time ray tracing in consumer hardware, path tracing techniques are increasingly being adopted in interactive applications like video games, pushing the boundaries of real-time graphics fidelity [7].

# 2 Ray Tracing Fundamentals

## 2.1 Rays

### 2.1.1 A Ray of Light as a Vector

In physics, light is an electromagnetic wave that propagates through space. However, in many scenarios, particularly in computer graphics, we can approximate light behavior using the concept of rays. This simplification, known as geometric optics, is valid when the wavelength of light is much smaller than the objects it interacts with.

A ray of light can be thought of as an idealized narrow beam of light traveling in a straight line. This approximation allows us to model light propagation without dealing with the complexities of wave optics, making it computationally feasible for rendering purposes.

In mathematics, a vector is a quantity that has both magnitude and direction. It can be represented as an arrow in space, defined by its starting point and its direction. This concept aligns perfectly with our need to represent a ray of light, which has a point of origin and a direction of propagation.

Formally, we can define a ray $\mathbf{R}(t)$ in 3D space using a parametric equation:

$$\mathbf{R}(t) = \mathbf{O} + t\mathbf{D}$$

where:

- $\mathbf{O}$ is the origin point of the ray (a 3D vector)
- $\mathbf{D}$ is the direction vector of the ray (a 3D unit vector)
- $t$ is a scalar parameter ($t \geq 0$)

This equation describes all points along the ray, starting from the origin and extending infinitely in the direction of $\mathbf{D}$. In practice, we often constrain $t$ to an interval $[t\text{min}, t\text{max}]$ to define a specific segment of the ray.

### 2.1.2 Definition in Code

In code, we can represent this mathematical concept using a Ray class and an Interval class. Here's how these classes are implemented:

```
class Ray
{
public:
    Ray(vec3 &origin, vec3 &direction) {}

    vec3 at(float t) const
    {
        return orig + (dir * t);
    }

private:
    glm::vec3 orig;
    glm::vec3 dir;
};

class Interval
{
public:
    double min;
    double max;
};
```

The Ray class encapsulates the origin and direction of the ray, while the Interval class can be used to represent the valid range of the parameter $t$. The at method in the Ray class implements the parametric equation, allowing us to compute points along the ray.

This representation allows us to efficiently model light paths in our path tracing system, forming the foundation for intersection tests and light transport calculations.

## 2.2 Intersection Testing

Having established the mathematical and programmatic representation of a ray of light, we now turn our attention to determining how these rays interact with objects in our scene. Intersection testing is a crucial component of ray tracing, allowing us to simulate the behavior of light as it encounters various surfaces. To perform these tests efficiently, we leverage the geometric properties of vectors.

### 2.2.1 Mathematical Tools

Two fundamental vector operations are essential for intersection testing: the dot product and the cross product.

**Dot Product** The dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a scalar quantity defined as:

$$\mathbf{A} \cdot \mathbf{B} = AxBx + AyBy + AzBz$$

This operation has several useful properties for intersection testing:

- It can be used to calculate the angle between two vectors:

$$\cos(\theta) = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}$$

- It allows us to determine orthogonality (when the dot product is zero)
- It enables the computation of vector projections

In the context of ray tracing, the dot product is particularly useful for determining the angle between a ray and a surface normal, which is crucial for calculating reflection and refraction.

**Cross Product** The cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$ results in a third vector $\mathbf{C}$ that is perpendicular to both:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AyBz - AzBy, AzBx - AxBz, AxBy - AyBx)$$

The magnitude of the resulting vector is given by:

$$\|\mathbf{C}\| = \|\mathbf{A}\|\|\mathbf{B}\| \sin(\theta)$$

where $\theta$ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

In intersection testing, the cross product serves several important purposes:

- It can be used to compute surface normals for triangles or polygons
- It helps in determining the orientation of surfaces relative to the ray
- It's useful in calculating barycentric coordinates for triangle intersection tests
- It can be employed to find perpendicular vectors, which is helpful in constructing coordinate systems for shading calculations

These mathematical tools form the foundation for implementing various intersection tests. For instance, when testing ray-triangle intersections, we can use the cross product to compute the triangle's normal and the dot product to determine if the ray is facing the correct side of the triangle. Similarly, for ray-plane intersections, the dot product helps us calculate the distance along the ray at which the intersection occurs.

In the following sections, we will explore how these tools are applied to specific geometric primitives, starting with spheres and planes, and then extending to more complex shapes and acceleration structures. By leveraging these vector operations, we can efficiently determine not only if a ray intersects an object, but also the exact point of intersection and the surface properties at that point, which are crucial for accurate light transport simulation in our path tracer.

### 2.2.2 Spheres

A sphere is a common geometric object in ray tracing, defined by its center $\mathbf{C}$ and radius $R$. To determine the intersection of a ray with a sphere, we substitute the ray equation $\mathbf{P}(t) = \mathbf{O}+t\mathbf{D}$ into the sphere's implicit equation:

$$\|\mathbf{P}(t) - \mathbf{C}\|^2 = R^2$$

Expanding and simplifying this equation yields:

$$\|\mathbf{O} + t\mathbf{D} - \mathbf{C}\|^2 = R^2$$

$$(\mathbf{O} - \mathbf{C} + t\mathbf{D}) \cdot (\mathbf{O} - \mathbf{C} + t\mathbf{D}) = R^2$$

$$(\mathbf{O} - \mathbf{C}) \cdot (\mathbf{O} - \mathbf{C}) + 2t\,(\mathbf{O} - \mathbf{C}) \cdot \mathbf{D} + t^2 \mathbf{D} \cdot \mathbf{D} = R^2$$

This is a quadratic equation in $t$:

$$t^2 \mathbf{D} \cdot \mathbf{D} + 2t\,(\mathbf{O} - \mathbf{C}) \cdot \mathbf{D} + \big((\mathbf{O} - \mathbf{C}) \cdot (\mathbf{O} - \mathbf{C}) - R^2\big) = 0$$

Letting $\mathbf{L} = \mathbf{O} - \mathbf{C}$, $a = \mathbf{D} \cdot \mathbf{D}$, $b = 2\mathbf{L} \cdot \mathbf{D}$, and $c = \mathbf{L} \cdot \mathbf{L} - R^2$, we solve the quadratic equation:

$$at^2 + bt + c = 0$$

The solutions for $t$ are given by:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $b^2 - 4ac$ determines the nature of the intersection:

- If $b^2 - 4ac < 0$, the ray does not intersect the sphere.
- If $b^2 - 4ac = 0$, the ray tangentially intersects the sphere at one point.
- If $b^2 - 4ac > 0$, the ray intersects the sphere at two points.

The following image illustrates a ray-sphere intersection:

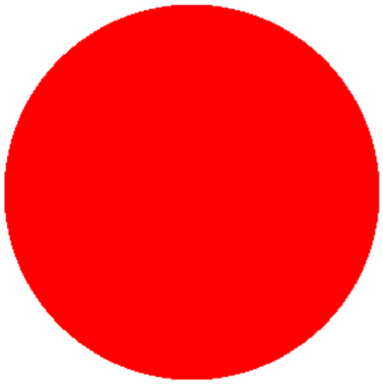


Figure 1: Ray-sphere intersection demonstration

In Figure 1, we can see a red circle representing a sphere on a white background. Notice the jagged pixelated border. This is because at one ray per pixel, a pixel can only be fully red (intersecting the sphere) or fully white (missing the sphere). We will fix this issue later.

### 2.2.3 Planes

A plane is defined by a point $\mathbf{P}_0$ on the plane and a normal vector $\mathbf{N}$. To find the intersection of a ray with a plane, we use the plane equation:

$$\mathbf{N} \cdot (\mathbf{P}(t) - \mathbf{P}_0) = 0$$

Substituting the ray equation $\mathbf{P}(t) = \mathbf{O} + t\mathbf{D}$:

$$\mathbf{N} \cdot (\mathbf{O} + t\mathbf{D} - \mathbf{P}_0) = 0$$

$$\mathbf{N} \cdot \mathbf{O} + t\,(\mathbf{N} \cdot \mathbf{D}) - \mathbf{N} \cdot \mathbf{P}_0 = 0$$

Solving for $t$:

$$t = \frac{\mathbf{N} \cdot (\mathbf{P}_0 - \mathbf{O})}{\mathbf{N} \cdot \mathbf{D}}$$

provided $\mathbf{N} \cdot \mathbf{D} \neq 0$. If $\mathbf{N} \cdot \mathbf{D} = 0$, the ray is parallel to the plane and does not intersect it.

The following image illustrates a ray-plane intersection:

Figure 2: Ray-plane intersection demonstration

In Figure 2, we can see a ray intersecting a plane (represented as a quad for visualization purposes). The image demonstrates how a ray intersects the plane at a single point. This visual representation helps to understand the mathematical concept described above, where we solve for a single value of $t$ to find the intersection point.

### 2.2.4 Extension to Object Meshes

Intersection tests can be extended to complex object meshes using triangle intersection algorithms. This is a crucial step in ray tracing, as most 3D models are represented as meshes composed of triangles.

**Triangle Intersection** The most common method for triangle intersection is the Möller–Trumbore algorithm. This algorithm is efficient and doesn't require precomputation of the triangle plane equation.

Given a ray $\mathbf{R}(t) = \mathbf{O} + t\mathbf{D}$ and a triangle defined by vertices $\mathbf{V}0$, $\mathbf{V}1$, and $\mathbf{V}_2$, we can express any point on the triangle as:

$$\mathbf{T}(u, v) = (1 - u - v)\mathbf{V}0 + u\mathbf{V}1 + v\mathbf{V}_2$$

where $u$ and $v$ are barycentric coordinates satisfying $u \geq 0$, $v \geq 0$, and $u + v \leq 1$.

The intersection point must satisfy:

$$\mathbf{O} + t\mathbf{D} = (1 - u - v)\mathbf{V}0 + u\mathbf{V}1 + v\mathbf{V}_2$$

This can be rewritten as a system of linear equations:

$$\begin{bmatrix} -\mathbf{D} & \mathbf{V}1 - \mathbf{V}0 & \mathbf{V}2 - \mathbf{V}0 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} = \mathbf{O} - \mathbf{V}_0$$

Solving this system using Cramer's rule gives us $t$, $u$, and $v$. If $0 \leq u \leq 1$, $0 \leq v \leq 1$, $u + v \leq 1$, and $t > 0$, then the ray intersects the triangle.

**Mesh Intersection** For a mesh composed of many triangles, we perform the following steps:

For each triangle in the mesh: a. Perform the triangle intersection test. b. If an intersection is found, store the intersection point and distance.

After testing all triangles, return the closest intersection point (smallest positive $t$).

## 2.3 Placing Objects into the World

### 2.3.1 Translation, Rotation, and Scaling

In computer graphics, assembling a scene involves not only defining the objects within it but also positioning them appropriately. Consider the classic demonstration scene known as the *Cornell Box* [3], which is a standard test for rendering algorithms to showcase accurate light transport and shading. From the previous sections, we've discussed how to test intersections with various geometric primitives. However, to build a complex scene like the Cornell Box, we need a systematic way to place and manipulate these objects in three-dimensional space.

Matrices are fundamental tools in linear algebra that we can use to efficiently represent and compute transformations in space. We can represent operations such as:

- **Translation** moves an object by a certain distance along the axes.
- **Rotation** turns an object around an axis passing through the origin.
- **Scaling** resizes an object by stretching or compressing it along the axes.

and put them all into mathematical objects that can be easily combined and applied to points and vectors in 3D space (and computed quickly on a GPU).

In our path tracer we can view a point or vector as a column vector (a matrix of N $\times$ 1):

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transforming this vector involves multiplying it by a transformation matrix:

$$\mathbf{v}' = \mathbf{M}\mathbf{v}$$

where $\mathbf{v}'$ is the transformed vector and $\mathbf{M}$ is the transformation matrix. If this transformation matrix is then applied to all points in our object, then we have successfully transformed our object.

### Translation

We will start with translation as though it may appear the most simple: $\mathbf{v}' = \mathbf{v} + \mathbf{d}$ where $\mathbf{d} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$ is the translation vector. Unfortunately, this breaks the laws of linearity (see subsection A.1).

This is a problem because translation cannot be represented with a 3×3 matrix in 3D space.

To unify translation with rotation and scaling in a single matrix operation, we introduce homogeneous coordinates. By adding an extra dimension to our vectors and matrices, we can represent all affine transformations, including translation, with 4×4 matrices.

In homogeneous coordinates, a point becomes:

$$\mathbf{v}_h = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

A translation matrix in homogeneous coordinates is:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transformation is then applied via:

$$\begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + T_x \\ y + T_y \\ z + T_z \\ 1 \end{pmatrix}$$

While we primarily use 3×3 matrices for rotation and scaling in our renderer, working with homogeneous coordinates and 4×4 matrices becomes essential for incorporating translation and perspective transformations, especially in more advanced rendering techniques.

### 2.3.2 Scaling Matrices

Scaling *is* a linear transformation which means that it can be represented using a 3×3 matrix, however as we would like all our transformations to multiply with each other we add a 4th scaling value of 1.

$$\begin{bmatrix} S_1 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} S_x \cdot x \\ S_y \cdot y \\ S_z \cdot z \\ 1 \end{pmatrix}$$

where $S_x$, $S_y$, and $S_z$ are the scaling factors along the $x$, $y$, and $z$ axes, respectively.

#### Rotation Matrices

Rotation can also be represented using a 3×3 matrix, but again, in practice they are represented as 4×4 matrices. Rotations in 3D are specified with an angle (in radians) and a rotation axis, with which you rotate the object around.

**Rotation Around the X-axis** Rotating a point around the $x$-axis by an angle $\theta$:

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Rotation Around the Y-axis** Rotating a point around the $y$-axis by an angle $\theta$:

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Rotation Around the Z-axis** Rotating a point around the $z$-axis by an angle $\theta$:

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

POTENTIALLY A SECTION HERE TALKING ABOUT PERFORMANCE AND TRANSFORMING VECTORS INTO MODEL SPACE

## 2.4 Color and Shading Models

### 2.4.1 Light and Material Interaction

The appearance of objects in a scene is influenced by how they interact with light. When light strikes a surface, it can be absorbed, reflected, or transmitted. These interactions are governed by the material properties of the surface.

### 2.4.2 Diffuse Reflection

A perfectly diffuse (Lambertian) surface scatters incident light uniformly in all directions. The intensity $I$ of the reflected light is proportional to the cosine of the angle $\theta$ between the light direction $\mathbf{L}$ and the surface normal $\mathbf{N}$:

$$I = I_0 \cdot \max(\mathbf{L} \cdot \mathbf{N}, 0)$$

where $I_0$ is the intensity of the incoming light. This cosine dependency ensures that light hitting the surface at a shallow angle contributes less to the reflected intensity.

### 2.4.3 Specular Reflection

Specular reflection occurs on shiny surfaces where light is reflected in a specific direction. The reflection vector $\mathbf{R}$ is computed as:

$$\mathbf{R} = \mathbf{L} - 2(\mathbf{L} \cdot \mathbf{N})\mathbf{N}$$

The intensity of the reflected light depends on the angle between the reflection vector $\mathbf{R}$ and the view direction $\mathbf{V}$:

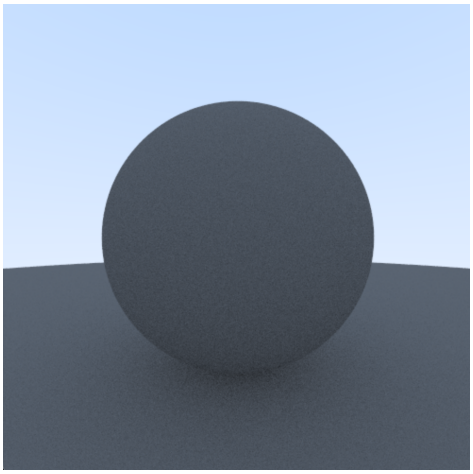
$$I = I_0 \cdot \max(\mathbf{V} \cdot \mathbf{R}, 0)^n$$

where $n$ is the shininess coefficient, determining the sharpness of the reflection.

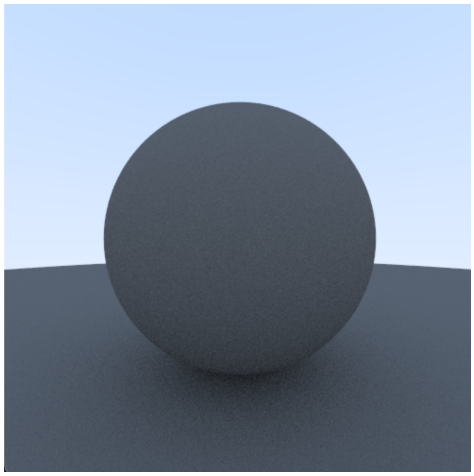
THIS PART IS THE PUTTING YOUR EYE CLOSE TO THE SURFACE MEANS IT WASHES OUT WITH LIGHT

### 2.4.4 Technical Implementation

GAMMA STUFF HERE Color models in code often represent colors as vectors with three components: red, green, and blue (RGB). Each component ranges from 0 to 1. In a Lambertian model, the scattering of rays is uniformly distributed over the hemisphere centered on the surface normal. This can be implemented using Monte Carlo techniques to randomly sample directions within the hemisphere.



(a) Rendered image using Uniform Diffuse Renderer



(b) Rendered image using Lambertian Diffuse Renderer

Figure 3: Comparison of rendering techniques

For specular materials, the direction of the reflected ray is computed using the reflection equation. This is crucial for simulating shiny surfaces such as metals and mirrors.

METAL SPHERE HERE

# 3 Camera Dynamics

## 3.1 Camera and Viewing

### 3.1.1 Camera Model and Rays

The camera in ray tracing simulates the viewpoint from which the scene is rendered. It projects rays from its origin through each pixel on the image plane, capturing the scene from a specific perspective.

#### Basis Vectors and Camera Orientation

**Basis Vectors** The camera's orientation is defined by three orthonormal basis vectors: $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

- $\mathbf{w}$: Vector pointing from the camera position to the view direction. It is computed as:
$$\mathbf{w} = \frac{\mathbf{C} - \mathbf{L}}{\|\mathbf{C} - \mathbf{L}\|}$$
where $\mathbf{C}$ is the camera position and $\mathbf{L}$ is the focal point.
- $\mathbf{u}$: Vector pointing to the right of the camera, computed using the cross product of the up vector $\mathbf{v}_{\text{up}}$ and $\mathbf{w}$:
$$\mathbf{u} = \frac{\mathbf{v}_{\text{up}} \times \mathbf{w}}{\|\mathbf{v}_{\text{up}} \times \mathbf{w}\|}$$
- $\mathbf{v}$: Vector pointing upwards relative to the camera, computed as:
$$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

These basis vectors form a right-handed coordinate system and define the camera's orientation.

**Viewport and Field of View** The camera's field of view (FOV) determines the extent of the observable world. The vertical FOV (vertFov) is the angle of the observable world in the vertical direction:
$$\text{viewportHeight} = 2 \cdot \tan\left(\frac{\text{vertFov}}{2}\right)$$
The viewport width is adjusted according to the aspect ratio (aspectRatio):
$$\text{viewportWidth} = \text{aspectRatio} \cdot \text{viewportHeight}$$

**Perspective Projection** To generate rays from the camera through each pixel, we calculate the position of each pixel in the camera's coordinate system. The pixel positions are offset from a reference point (the upper-left corner of the viewport) by the basis vectors **u** and **v**.

### 3.1.2 Technical Implementation

**Initialization and Basis Vector Calculation** The 'initialize' method sets up the camera's coordinate system using the basis vectors:

```
void Camera::initialize() {
    float theta = glm::radians(vert_fov);
    float h = tan(theta / 2);
    float viewport_height = 2.0f * h;
    float viewport_width = viewport_height * float(image_width) / float(image_heig

    w = glm::normalize(center - look_at);
    u = glm::normalize(glm::cross(vup, w));
    v = glm::cross(w, u);

    pixel00_loc = center - (u * viewport_width / 2.0f) - (v * viewport_height / 2.
    pixel_delta_u = u * viewport_width / float(image_width);
    pixel_delta_v = v * viewport_height / float(image_height);
}
```

Here,

- **w** is calculated as the normalized vector from the camera position to the look-at point.
- **u** is the normalized cross product of the up vector and **w**.
- **v** is the cross product of **w** and **u**.

**Ray Generation** The method 'getRandomRay' generates rays from the camera through each pixel, considering anti-aliasing by randomly offsetting the ray within the pixel:

```
Ray Camera::getRandomRay(int x, int y) const {
    float u_offset = (x + random_float()) * recip_sqrt_sppf;
    float v_offset = (y + random_float()) * recip_sqrt_sppf;
    glm::vec3 ray_origin = center;
    glm::vec3 ray_direction = glm::normalize(pixel00_loc + u_offset * pixel_delta_
    return Ray(ray_origin, ray_direction);
}
```

This method uses the basis vectors to compute the direction of the rays passing through each pixel, incorporating random offsets to achieve anti-aliasing. This process ensures that each pixel's color is the result of multiple samples, reducing aliasing and producing a smoother image.

# 4 Acceleration Structures

## 4.1 Bounding Volume Hierarchies (BVH)

### 4.1.1 Overview of BVH

An advanced structure for efficient intersection tests.

### 4.1.2 Technical Implementation

- **Efficient Ray-Object Intersection Tests:** Using BVH for faster intersection calculations.
- **Building and Traversing BVH:** Methods for constructing and navigating BVH structures.

### 4.1.3 Algorithmic Analysis

Performance analysis of BVH algorithms for various scene complexities.

# 5 Monte Carlo Methods for Ray Tracing

## 5.1 Probability and Monte Carlo Methods

### 5.1.1 Improved Random Sampling

Stochastic methods improve sampling efficiency in rendering.

### 5.1.2 Monte Carlo Integration

Using random sampling to approximate integrals in light transport.

### 5.1.3 Importance Sampling

Advanced sampling techniques enhance efficiency and image quality. reference 3.3

## 5.2 Integration for Lighting Models

### 5.2.1 Radiance and Light Transport Equations

Mathematical models for light behavior in rendering.

### 5.2.2 Numerical Integration Techniques

Applying numerical methods to solve lighting equations.

### 5.2.3 Technical Implementation

Implementing global illumination and integrating direct and indirect lighting.

# 6 Technical Description of the System

Put prerequistes here, opengl cpp etc. NOT TRUE come up with a MORE ACCURATE DESCRIPTION In this section, we will focus on how the discussed ele

## 6.1 System Architecture

### 6.1.1 Design Principles Behind the Modular Architecture

I have found that when architechting a complex system it is essential to keep several non-functional attributes in mind and to always assess your architectural decisions against these. A path tracer requires several key non-functional attributes to be effective and maintainable (ordered by descending importance):

1. Performance: The system must be able to handle complex scenes and produce high-quality renders in a reasonable time frame.

2. Extensibility: As rendering techniques evolve, the system should be able to incorporate new algorithms and features easily.

3. Maintainability: The codebase should be easy to understand, debug, and modify.

With these requirements in mind it naturally leads to a modular monolith approach. I find that Modular design is suited to our non-functional attributes as:

- Encapsulation of Functionality: Each module encapsulates a specific set of related functions, allowing for optimized performance within modules and clear interfaces between them.

- Loose Coupling: Modules interact through well-defined interfaces, making it easier to modify or replace individual components without affecting the entire system (imagine you want to switch to the Vulkan API, this should be quite an easy task!).

- Parallel Development: Different teams or developers can work on separate modules simultaneously, speeding up development and integration of new features.

- Testability: Individual modules can be tested in isolation, facilitating more thorough and efficient testing processes.

The modular monolith approach strikes a balance between the simplicity of a monolithic architecture and the flexibility of a fully distributed system. This approach is particularly well-suited for a path tracer, where tight integration between components is necessary for performance, but clear separation of concerns is crucial for ongoing development and maintenance.

### 6.1.2 Overview of Core Components and Their Interactions

The following C4 diagram provides an overview of the path tracer's modules, along with their responsibilities and interactions:

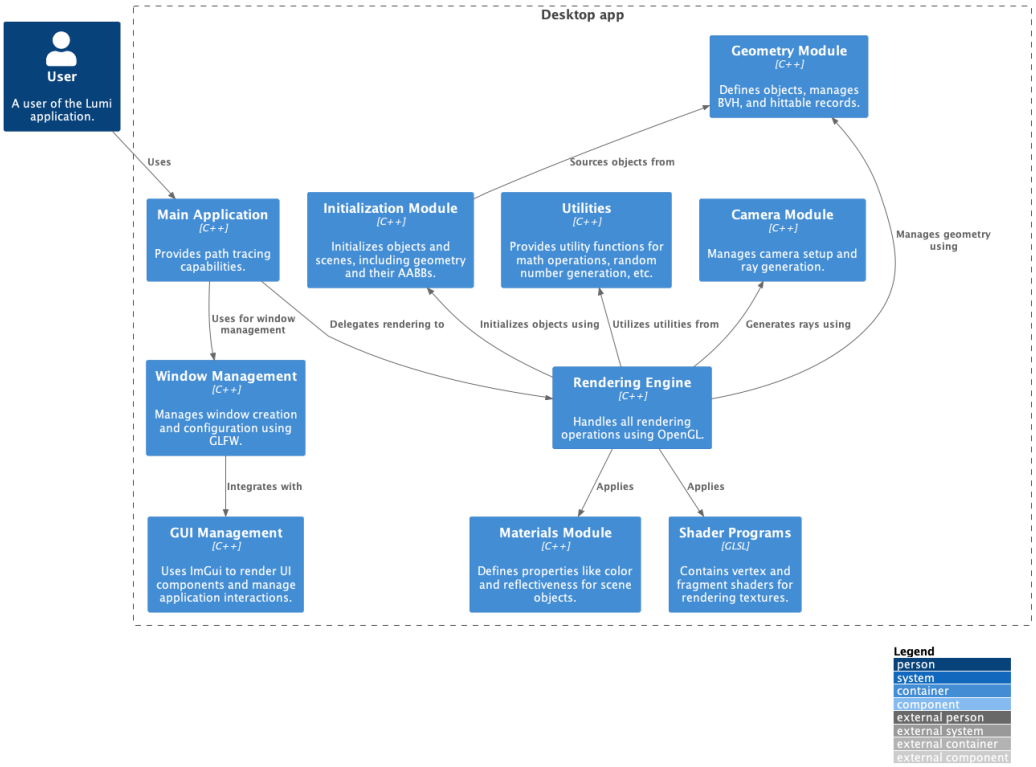



Figure 4: C4 diagram of the path tracer system architecture

The Rendering Engine acts as the central hub, coordinating the activities of other modules. It delegates specific tasks to specialized modules (e.g., ray generation to the Camera Module, object intersection tests to the Geometry Module) and applies materials and shaders as needed.

### 6.1.3 Main Control Loop's Role in Managing Rendering Operations

Here talk about call stack, main loop is the top level which sets up stuff, and summons the renderer to start. The main control loop orchestrates rendering, coordinating ray generation, shading, and image synthesis.

## 6.2 Core Components

### 6.2.1 Geometry: An Abstraction for Hittable Objects

Provides a unified interface for diverse geometrical shapes.

### 6.2.2 Bounding Structures: Use of BVH for Efficient Intersection Tests

Employs BVH for accelerated intersection tests, enhancing performance.

### 6.2.3 Scene Manager: Object Storage and Scene Graph Management

The Scene Manager organizes objects in a scene graph, optimizing rendering operations.

### 6.2.4 Renderer: Ray Generation and Intersection Handling

Handles ray generation and calculates light interactions with surfaces.

### 6.2.5 Camera Module: Virtual Camera Settings and Primary Ray Generation

Simulates camera properties and generates primary rays for rendering.

## 6.3 Utilising OpenGL for Visualising Convergence

In traditional path tracing implementations, the final image is typically output once the required number of samples per pixel has been reached. However, this approach doesn't provide insight into the convergence process. To address this limitation, we've implemented a progressive rendering system using OpenGL, allowing for real-time visualization of the path tracing process.

### 6.3.1 Progressive Rendering

The core of this visualization technique is the accumulation buffer. Instead of directly outputting the final pixel colors, we continuously update an accumulation buffer:

```
vector<vec3> accumulationBuffer(IMAGE_SIZE, 0);
vector<int> sampleCount(IMAGE_SIZE, 0);
```

The `accumulationBuffer` stores the sum of all samples for each pixel, while `sampleCount` keeps track of the number of samples per pixel.

The rendering process is modified to update the accumulation buffer after each sample:

```
accumulationBuffer[index] += rayColor(r, world, maxDepth);
sampleCount[index] += 1;
```

This incremental approach enables us to visualize the gradual convergence of the image over time. To display the progressively rendered image, we utilize OpenGL's texture mapping capabilities. The current state of the render is mapped to a texture that covers the entire window:

```
void updateTexture(unsigned int texture, const vector<vec3> &image,
const int &WIDTH, const int &HEIGHT)
    {
        glBindTexture(GL_TEXTURE_2D, texture);
        glTexSubImage2D(WIDTH, HEIGHT, image.data());
    }
```

MENTION MAPPING TO A UNIFORM 2D IN FRAGMENT SHADER AND OPENGL PIPELINE HERE This texture is then rendered onto a quad that fills the entire screen, allowing for real-time updates of the rendering progress.

### 6.3.2 Benefits of Real-time Visualization

This approach offers several advantages:

1. Parameter Tuning: Real-time visualization allows for quick iterations when adjusting rendering parameters, as the effects of changes are immediately visible.

2. Debugging Aid: It becomes easier to identify and debug issues in the path tracing algorithm as they manifest visually during rendering.

3. Progress Monitoring: Users can make informed decisions about when to terminate the rendering based on the current image quality.

#### 6.3.3 Architecture Considerations

The integration of OpenGL for visualization required careful consideration of the system architecture:

1. Separation of Concerns: The core path tracing logic remains separate from the visualization code, maintaining modularity.

2. Efficient Data Transfer: The accumulation buffer serves as an intermediate representation, minimizing data transfer between the CPU and GPU. MENTION HOW SLOW IT IS/ CPU PATH TRACING VS

3. Flexible Rendering Pipeline: The system is designed to allow easy switching between progressive rendering and traditional batch rendering if needed. CODE HERE?

By leveraging OpenGL for real-time visualization, we've enhanced the path tracer's utility as both a rendering tool and a development aid, providing deeper insights into the rendering process and facilitating more efficient algorithm refinement.

# 7 Conclusion

## 7.1 Summary of Key Points

Monte Carlo methods effectively enhance path tracing and image synthesis, improving quality and efficiency.

## 7.2 Future Work

Future exploration of advanced sampling strategies and optimizations for real-time rendering.

## 7.3 Final Thoughts

Statistical methods are crucial for high-quality rendering solutions.

# 8 References

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# A Detailed Mathematical Derivations

## A.1 Linearity of Transformations

To demonstrate why translation is not a linear transformation, we consider the following properties:

Law of Additivity: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
Law of Homogeneity: $T(c\mathbf{u}) = cT(\mathbf{u})$
If we test the linearity conditions:
Additivity:

$$T(\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) + \mathbf{b} \neq T(\mathbf{u}) + T(\mathbf{v}) = (\mathbf{u} + \mathbf{b}) + (\mathbf{v} + \mathbf{b})$$

The right-hand side becomes $\mathbf{u}+\mathbf{v}+2\mathbf{b}$, which is not equal to the left-hand side.

Homogeneity:

$$T(c\mathbf{u}) = c\mathbf{u} + \mathbf{b} \neq cT(\mathbf{u}) = c(\mathbf{u} + \mathbf{b}) = c\mathbf{u} + c\mathbf{b}$$

The translation vector $\mathbf{b}$ remains unchanged and doesn't scale with $c$, breaking this property as well.

Thus, the translation is an affine transformation, not a linear transformation, because it doesn't preserve these linearity conditions.

# B Code Snippets and Pseudocode

# C Additional Figures and Diagrams