Game Theory and Control

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github.com/silvasta/summary-gtc

1 Static games

1.1 Basic Definitions

Game theory studies mathematical models of conflict and cooperation among rational decision-makers. A static game is characterized by:

- **Players**: Rational agents making decisions.
- **Actions** (pure strategies): Finite sets of choices, e.g., $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ for Player 1, $\Sigma = \{\sigma_1, \dots, \sigma_m\}$ for Player 2.
- **Information structure**: Simultaneous moves with no knowledge of opponents' choices.
- **Outcomes/Payoffs**: Cost functions $J_1(\gamma_i, \sigma_j) = a_{ij}$ (Player 1 minimizes) and $J_2(\gamma_i, \sigma_j) = b_{ij}$ (Player 2 minimizes), represented by matrices A and B.

Examples include wireless power control (interference minimization), Deer Hunt (coordination with multiple NE), self-driving cars at intersections (Chicken game variants), and inspection games (no pure NE).

1.1.1 Matrix Representation

Games are often shown as bimatrix form:

	σ_1	σ_2
γ_1	(a_{11}, b_{11})	(a_{12}, b_{12})
γ_2	(a_{21}, b_{21})	(a_{22}, b_{22})

1.1.2 Nash Equilibrium

A pure Nash Equilibrium (NE) is a profile (γ^*, σ^*) where no player benefits from unilateral deviation:

$$J_1(\gamma^*, \sigma^*) \leq J_1(\gamma_i, \sigma^*) \forall i$$

and

$$J_2(\gamma^*, \sigma^*) \leq J_2(\gamma^*, \sigma_j) \forall j$$

Rational players converge to NE as stable outcomes.

1.1.3 Dominated Actions

Action γ_k is strictly dominated by γ_i if

$$J_1(\gamma_i, \sigma_j) < J_1(\gamma_k, \sigma_j) \forall j$$

Rational players avoid dominated actions.

1.1.4 Reduced Game

Iteratively remove dominated actions to simplify the game, revealing NE.

1.1.5 Security Levels and Policies

Pure **security level** for Player 1:

$$\underline{J_1} = \min_i \max_j a_{ij}$$

(worst-case guaranteed cost assuming adversarial opponent)

Security policy

$$\arg \min_i \max_j a_{ij}$$

1.2 Multiple Nash Equilibria

Games may have multiple pure NE, not interchangeable

Partial order:

$(J_1', J_2') \prec (J_1, J_2)$ if $J_1' \leq J_1, J_2' \leq J_2$ (at least one strict)

1.2.1 Admissible Nash Equilibria

A NE is admissible if not Pareto-dominated by another NE (better for one, not worse for other). Represent via poset: outcomes ordered by Pareto dominance.

Hasse diagram visualizes this poset as a graph where nodes are NE outcomes, edges indicate dominance (from worse to better), and minimal elements (with no incoming edges from below) are admissible, representing undominated equilibria that rational players might prefer.

1.3 Mixed Strategies

Extend to probabilities $y \in \Delta^n$ (simplex), $z \in \Delta^m$

Expected costs $J_1(y, z) = y^\top Az$, $J_2(y, z) = y^\top Bz$. Distinguish from pure strategies (deterministic).

1.3.1 Security Levels

Mixed security level $\min_y \max_z y^\top Az = \min_y \max_j (A^\top y)_j$

(linear program, computable efficiently).

Mixed security strategy: $\arg \min_y \max_z y^\top Az$

(robust against worst-case mixed opponent, often lower than pure level)

Computational complexity Polynomial-time solvable via LP, unlike some NE computations.

1.3.2 Mixed Nash Equilibrium

Best responses (y^*, z^*) where:

$$y^\top Az^* \geq (y^*)^\top Az^* \forall y$$

$$(y^*)^\top Bz \geq (y^*)^\top Bz^* \forall z$$

Certify by checking pure deviations.

Completely mixed all probabilities positive, satisfy indifference:

$$Az^* = p^* \mathbf{1} \quad (y^*)^\top B = q^* \mathbf{1}^\top$$

Non-completely mixed exist without indifference

e.g., 3x3 game with $y^* = z^* = (1/3, 2/3, 0)$

1.4 Nash Theorem

Every finite game has at least one mixed NE

Proof via Kakutani's fixed-point theorem: The best-response correspondence $\beta : \Delta^n \times \Delta^m \to 2^{\Delta^n \times \Delta^m}$ maps each strategy profile to the set of best responses (non-empty, convex, upper hemicontinuous due to continuity of expected payoffs and compactness of simplices) Kakutani guarantees a fixed point, which is a NE where each player's strategy is a best response to the other's.

2 Zero-sum games

Zero-sum games model competitive scenarios where one player's gain is the other's loss, applicable in areas like security, economics, and control. Nash equilibria in these games are computationally tractable and possess unique values, making them foundational for game theory.

2.1 Two-Person Zero-sum Games

Games in which the two players have opposite payoffs:

$$J_1(\gamma, \sigma) = -J_2(\gamma, \sigma)$$

Like a static game with Payoff Matrix $B = -A$

Row player (P1, minimizer) pays a_{ij} to column player (P2, maximizer). P1 minimizes expected outcome V, P2 maximizes it.

2.1.1 Example: Rock-Paper-Scissors

2.1.2 Rock, Paper, Scissors

Consider only one round of the game.

$$A = \begin{array}{c} \begin{array}{ccc} \text{Rock} & \text{Paper} & \text{Scissors} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Rock} \\ \text{Paper} \\ \text{Scissors} \end{array}$$

No pure Nash equilibrium exists.

2.2 Security Levels and Policies

	Security level	Security policy
P1	$\bar{V} = \min_i \max_j a_{ij}$	$\underline{i} \in \arg\min_i(\max_j a_{ij})$
P2	$\underline{V} = \max_j \min_i a_{ij}$	$\bar{j} \in \arg\max_j(\min_i a_{ij})$

2.2.1 Min-Max Property

Security levels in **zero-sum games** have a fundamental property that general static games don't have.

For every finite matrix A, the following properties hold:

(i) Security levels are well defined and unique

(ii) Both players have security policies (not necessarily unique)

(iii) The security levels always satisfy: $\underline{V} \leq \bar{V}$

Proof For any matrix A,

$$\max_j \min_i a_{ij} \leq \min_i \max_j a_{ij}$$

as the max of mins cannot exceed the min of maxes.

2.3 Nash equilibrium in zero-sum games

A pure Nash equilibrium (saddle-point) (i^*, j^*) satisfies:

$$a_{i^*j} \leq a_{i^*j^*} \leq a_{ij^*} \quad \forall i, j.$$

Known as **saddle-point equilibrium** with value $V^* := a_{i^*j^*}$

Interpretation: No regret, stable under iteration.

2.4 Saddle-point and security levels

Not all zero-sum games have a saddle point (f.e. Rock-Paper-Scissors). We can exactly characterize the zero-sum games that have a saddle point.

Theorem 1 (Saddle-point and security levels). A zero-sum game defined by A has a saddle-point equilibrium **if and only if**

$$\underline{V} = \bar{V} \quad (= V^* \text{ saddle-point value})$$

Proof: If equal, security strategies form a saddle-point; conversely, saddle-point implies equal security levels. Consequences: Unique value V^* for all equilibria; interchangeability of strategies.

Important consequences follow (only for zero-sum games!)

All saddle-point equilibria (Nash equilibria) of a zero-sum game have the same value V^*, which we denote as the **value of the game**.

2.5 Mixed strategies

2.5.1 Mixed strategies

P1 chooses $y \in \Delta^m$ (simplex),

P2 chooses $z \in \Delta^n$

Expected payoff: $v(y, z) = y^T Az$.

2.5.2 Mixed security levels

$\bar{V}_m = \min_y \max_z y^T Az = \min_y \max_j (A^T y)_j$

$\underline{V}_m = \max_z \min_y y^T Az = \max_z \min_i (Az)_i$.

2.5.3 Computing via Linear Programming

For P1's mixed security level:

$$\min_{y,t} t \quad s.t. \quad A^T y \leq t \cdot \mathbf{1}, \quad \mathbf{1}^T y = 1, \quad y \geq 0.$$

Optimal $t^* = \bar{V}_m$. Dual gives P2's strategy.

2.5.4 Min-Max Theorem (von Neumann)

For finite zero-sum games: $\underline{V}_m = \bar{V}_m$ (unique game value).

Always exists mixed Nash equilibrium.

2.5.5 Nash Equilibrium

2.5.6 Mixed Nash Equilibrium

Theorem 2 (Mixed Nash equilibrium for zero-sum games). Policy $(y^\star, z^\star)$ is called mixed-strategy saddle-point equilibrium (or Nash equilibrium) if

$$y^{\star T} Az^\star \leq y^T Az^\star \quad \forall y \quad \text{(minimizer)}$$

$$y^{\star T} Az^\star \geq y^{\star T} Az \quad \forall z \quad \text{(maximizer)}$$

$y^{\star T} Az^\star$ is called **saddle point value**

Exam-Relevant Extensions and Exercises

From exercises: Skew-symmetric games (e.g., "Pick a Number") have value 0; mixed NE computed via LP, often mixing few strategies (e.g., uniform over subset).
Exam connections (e.g., 2024 exam): Extend to extensive-form zero-sum games with imperfect information, feedback vs. non-feedback, behavioral vs. mixed strategies (Kuhn's theorem: equivalence in perfect recall), backward induction for subgame equilibria, and computing mixed NE in trees.
Key properties for exams: Unique value, LP solvability, no pure NE implies mixed (e.g., Rock-Paper-Scissors: uniform $1/3$); in rectangular games, minimizer mixes at most as many as maximizer's actions.

3 Auctions

Auctions model strategic interactions where N bidders (players) submit bids $x_i \geq 0$, with private true values t_i for the item(s). The outcome is determined by winner selection $w(x)$ and payment $p(x)$, yielding utility for bidder i:

$$J_i(x) = \begin{cases} t_i - p(x) & \text{if } i = w(x) \\ 0 & \text{otherwise.} \end{cases}$$

(Note: Utilities are typically defined as gains, so we maximize $U_i = -J_i$; lectures use cost minimization.)

3.1 First-Price Sealed-Bid Auction

Winner: $w(x) = \arg\max_i x_i$; pays $p(x) = x_{w(x)}$.

- Overbidding ($x_i > t_i$) is strictly dominated by truthful bidding ($x_i = t_i$): winning with overbid risks loss if $p > t_i$.

- Truthful bidding is not dominant: best response depends on others' bids; rational players underbid ($x_i < t_i$) to balance win probability and surplus.

- No dominant strategy equilibrium; Bayes-Nash equilibria require valuation distributions (e.g., symmetric independent private values yield shading: $x_i = \frac{N-1}{N} t_i$ for uniform [0,1]).

Private information (t_i unknown) leads to inefficiency; equilibria are not socially optimal.

3.2 Second-Price Sealed-Bid Auction (Vickrey)

Winner: $w(x) = \arg\max_i x_i$; pays $p(x) = \max_{j \neq w(x)} x_j$.

- **Dominant Strategy**: Truthful bidding ($x_i = t_i$) is weakly dominant.

Proof: Bid x_i affects only win/loss, not payment (fixed at second-highest). Overbidding risks winning unprofitably, underbidding risks losing profitably. Thus, $U_i(t_i, x_{-i}) \geq U_i(x_i, x_{-i})$ for all x_i, x_{-i}

- **Incentive Compatibility (IC)** Truth-telling optimal regardless of others.

- **Social Efficiency**: Allocates to highest t_i if truthful.

- Revenue: Equals second-highest t_i; not maximal but elicits truthfulness.

- IC and efficiency often align: revealing t_i enables optimal allocation.

3.3 Generalized auctions

3.3.1 Bids

Each **bid** is represented by a pair $x_j = (b_j, m_j)$

- b_j is the bidded amount

- m_j describes the object of the bid

(Can be extended to allow multiple bids)

Fungible goods $m_j \in \mathbb{R}_{>0}$ parts of a total quantity M

Non-fungible goods $m_j \in 2^{\mathcal{M}}$ with finite set of items $\mathcal{M}$

3.3.2 Choice function

Choice function w maps bids x into N-dimensional binary vector

$$w_j(x) = \begin{cases} 1 & \text{if bid } j \text{ is accepted} \\ 0 & \text{otherwise} \end{cases}$$

- Choice constraints

3.3.3 Payment function

Payment function p maps bids x into N-dimensional vector where $p_j(x)$ is the payment requested from the player that placed the bid j

3.4 Vickrey-Clarke-Groves (VCG) Mechanism

Generalizes Vickrey for complex allocations.

3.4.1 VCG Choice

$w^*(x) = \arg\max_{w:(w,m)\in C} \sum_j b_j w_j$ (maximizes reported social welfare).

3.4.2 VCG Payment (Clarke pivot)

$$p_j(x) = \max_{w: w_j = 0} \sum_{k \neq j} b_k w_k - \sum_k b_k w_k^*(x) - b_j w_j^*(x)$$

Interpretation p_j is the welfare loss imposed on others by including bid j (externality).

Social Utility: $U(t, w) = \sum_i t_i w_i$

(aggregate true value; payments cancel out).

If truthful ($b_j = t_j$), w^* maximizes U^*.

3.4.3 Properties

Truthful Bidding Dominant Weakly dominant strategy

Proof: payment independent of own bid; misreporting can only worsen allocation relative to truthful externality.

Non-Negative Utility (Individual Rationality)

Truthful bidding yields $U_i \geq 0$

Proof: $U_i = U^*(t) - U^*_{-i}(t_{-i}) \geq 0$, as full optimum $\geq$ optimum without i

Social Efficiency: Achieves max U if truthful.

DSIC: Dominant-strategy incentive compatible.

Exam-Relevant Insights from Exercises

Collusion Vulnerability: Shill bidding (e.g., splitting bids) can manipulate outcomes, increasing payments (e.g., single player posing as multiple to extract higher revenue in reverse auctions)
Low/Non-Monotonic Revenue: Can yield zero revenue despite high values (e.g., two identical bidders cancel externalities); revenue increases if bidders drop out or reduce bids, violating monotonicity.
Computational Challenges: Solving w^* often NP-hard (e.g., combinatorial auctions ≈ knapsack).

3.5 Open Problems in Auction Design

VCG guarantees DSIC, efficiency, and non-negative utilities but suffers:

- Low revenue (prioritizes efficiency over seller profit).

- Collusion (shills, bid coordination).

- Computational intractability for large C.

- Alternatives trade efficiency for revenue (e.g., Myerson optimal auctions maximize expected revenue under Bayesian priors).

These align with exam themes

- equilibria (Nash vs. dominant)
- strategies (truthful vs. strategic)
- repeated/repeated-like manipulations
- efficiency in strategic settings

4 Potential games

Potential games form a subclass of non-zero-sum games with guaranteed pure Nash equilibria (NE) and converging best-response (BR) dynamics, crucial for decentralized control and exam problems involving equilibrium existence, computation, and efficiency (e.g., Price of Anarchy, PoA).

4.1 N-Player Games and Best-Response Dynamics

Each player i selects $\gamma_i \in \Gamma_i$ (finite set) to minimize cost:

$$J_i(\gamma_1, \gamma_2, \ldots \gamma_N) = J_i(\gamma_i, \gamma_{-i})$$

Definition 1 (Pure Nash equilibrium in N-player games). A pure strategy profile $\gamma^* = (\gamma_1^*, \ldots, \gamma_N^*)$ is a pure NE if:

$$\forall i,\ \forall \gamma_i' \in \Gamma_i : J_i(\gamma_i^*, \gamma_{-i}^*) \leq J_i(\gamma_i', \gamma_{-i}^*)$$

Definition 2 (Pure Best Response in N-player games). The best (non-empty, possibly non-singleton) response set is:

$$R_i(\gamma_{-i}) = \arg\min_{\gamma_i \in \Gamma_i} J_i(\gamma_i, \gamma_{-i})$$

γ^* is pure NE iff $\gamma_i^* \in R_i(\gamma_{-i}^*)\ \forall i$

BR dynamics Start at γ^0; iteratively update one player's strategy to a BR, reducing some J_i. May cycle (e.g., rock-paper-scissors) unless structured (e.g., potential games).

Exam relevance

Prove convergence or find cycles in small games.

4.2 Potential Games: Definition and Properties

Definition 3 (Exact Potential Function). A function $P : \Gamma \to \mathbb{R}$ is an **exacct potential** if $\forall i, \forall \gamma_{-i}, \forall \gamma_i', \gamma_i'' \in \Gamma_i$:

$$J_i(\gamma_i', \gamma_{-i}) - J_i(\gamma_i'', \gamma_{-i}) = P(\gamma_i', \gamma_{-i}) - P(\gamma_i'', \gamma_{-i})$$

A game is a **potential game** if it admits a potential function.

Key: Unilateral deviation changes match P changes (same for all players).

Proposition 1 (Existence and Convergence). Finite potential games have ≥ 1 pure NE. BR dynamics converge finitely to a pure NE (since P strictly decreases on non-NE updates; finite Γ prevents infinite descent).

Pure NE are local minima of P (directional: no unilateral descent). Multiple NE possible, initial conditions affect convergence.

Exam tip

Check potential by verifying cost differences or constructing P (e.g., via summation over paths)

4.2.1 Path Improvement Characterization

A path in Γ is a sequence where consecutive profiles differ by one player's unilateral deviation. A closed path returns to start; simple if no repeats.

Proposition 2. A game is potential iff the total improvement $I(\mathcal{P}) = \sum$ (cost changes along path) is zero for every finite simple closed path of length 4 (or all closed paths).

Proof via cycle decomposition

exam-relevant for verifying potential in small games (e.g., check 4-cycles)

4.3 Congestion Games

Canonical example of potential games, modeling resource sharing (e.g., traffic, networks).

Definition 4 (Congestion Game). Resources $\{1, \ldots, M\}$

$\gamma_i \subseteq \{1, \ldots, M\}$ (subset used by i)

load $\ell_j(\gamma) = |\{i : j \in \gamma_i\}|$

cost $J_i(\gamma) = \sum_{j \in \gamma_i} f_j(\ell_j(\gamma))$ with f_j non-decreasing.

Theorem 3 (Rosenthal's Potential). Every congestion game is potential with $P(\gamma) = \sum_{j=1}^{M} \sum_{k=1}^{\ell_j(\gamma)} f_j(k)$.

Proof: Unilateral change by i (add/remove resources) alters P exactly by the marginal $f_j(k)$ terms matching ΔJ_i. Thus, pure NE exist; BR converges.

Exam example

Braess paradox in routing (adding link worsens equilibrium travel time, e.g., from 64 to 70 min)

4.4 Equilibrium Refinements and Examples

Multiple NE: Use Hasse diagram (partial order by Pareto dominance) or admissibility (no weakly dominated strategies). E.g., in renewable energy game (moral hazard with fine f):

For $c/\epsilon < f < b - s$, two pure NE ((Reserve, No Shed), (No Reserve, Shed)); one mixed; Hasse ranks (R,NS) > (NR,S). Security strategies: Maximin (e.g., Reserve for player 1).

Identical-interest games (same $J_i = \phi \forall i$) are potential with $P = \phi$. Zero-sum potential games exist but are trivial or decomposable.

Flight booking: Congestion on tickets; social optimum is NE (minimizes P = total cost) but PoA > 1 (counterexample with PoA ≥ 3.5).

4.5 Social Welfare and Price of Anarchy

Definition 5 (Welfare and PoA). Welfare $W(\gamma) = \sum_i J_i(\gamma)$ (or max, etc.). PoA = $\max_{\gamma \in \text{NE}} W(\gamma) / \min_{\gamma \in \Gamma} W(\gamma)$ (measures inefficiency of worst NE vs optimum).

In potential games, NE minimize P locally, but $P \neq W$ implies PoA > 1 (e.g., Braess: ≈ 1.09; renewable: PoA=1 with proper fine aligning incentives).

Exam

Compute PoA in routing or design mechanisms (fines/lanes) to minimize it

5 Convex games

5.1 Games with Infinite (Continuous) Action Spaces

Many real-world games involve continuous actions (e.g., quantities in auctions, prices, control inputs like force/velocity, waiting times, coverage paths).

Challenges Finite-game results (e.g., Nash's theorem, argmin/argmax existence) fail; algorithms like inspection or linear programming are unsuitable.

5.1.1 Motivating Examples

Cournot Competition

N producers: Choose quantities $x_i \in [0, k]$.
Marginal cost: $c > 0$.
Market price: $p(x) = a - b \sum x_j$ $(a, b > 0)$.
Losses: $J_i = cx_i - p(x)x_i$.
Properties: Convex game; unique NE.
Best response:

$$x_i^* = \max\{0, \min\{k, (a - c - b \sum_{j \neq i} x_j^*)/(2b)\}\}$$

Potential:

$$P(x) = (c - a)^\top x + \frac{1}{2} x^\top (bI + b\mathbf{1}\mathbf{1}^\top) x$$

Minimizing P: Yields the NE.

Bertrand Competition

Producers: Choose prices $x_i \geq 0$.
Marginal cost: $c > 0$.
Demand: Splits to lowest price (tie: half each).
Losses: Discontinuous and non-convex.
Equilibrium: No pure NE without capacities.
With capacity $2/3$: Still no pure NE (mixed NE exists).
Properties: Not convex if action sets unbounded.

5.2 Definitions and Background

Convex Set: Closed under convex combinations.

Convex Function: $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$, $\lambda \in [0, 1]$. Strictly convex if $<$.

For differentiable f: Convex iff $\nabla^2 f \succeq 0$ (Hessian PSD)

First-order: $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$.

Convex Game: N players

Actions $x_i \in K_i \subseteq \mathbb{R}^n$ (compact, convex, nonempty)

Losses $J_i(x_i, x_{-i}) : K \to \mathbb{R}$
continuous in x convex in x_i (fixed x_{-i})

Pure Nash Equilibrium (NE): $x^* \in K$ s.t. $\forall x_i \in K_i$, all i.

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*)$$

Assumptions sufficient but not necessary for existence; relaxing (e.g., open/unbounded/non-convex K_i) may destroy NE (counterexamples: open intervals lead to boundary deviations; unbounded allows infinite descent).

5.3 Existence of Pure NE

Theorem 4 (Existence of Pure NE - Convex Games). If K_i compact convex, J_i continuous, convex in x_i, then $\exists$ pure NE.

Proof: Best-response map $\Omega(x) =$ (arg $\min_{x_1} J_1(x_1, x_{-1}), \ldots$) is upper hemicontinuous, convex-valued. Kakutani fixed-point applies. (Compactness crucial; non-convex K_i may still have NE, e.g., Prisoner's Dilemma.)

5.4 Variational Inequalities (VI)

Game Map $F(x) = \nabla_{x_1} J_1(x), \ldots, \nabla_{x_N} J_N(x)$

VI(K, F) Find $x^* \in K$ s.t. $F(x^*)^\top (y - x^*) \geq 0 \, \forall y \in K$.
Equivalent to first-order optimality in convex optimization.

Theorem (NE $\iff$ VI): In continuously differentiable convex games, x^* is NE iff solves VI(K, F). Advantages: Borrow VI theory (uniqueness) and numerics (solvers) for NE. Applicability requires J_i continuously differentiable (e.g., fails for non-smooth like $\max\{0, x_i\}$).

5.5 Monotonicity and Uniqueness of NE

Monotonicity of F: Monotone if $(F(x) - F(y))^\top (x - y) \geq 0$; strictly if > 0 $(x \neq y)$; strongly if $\geq \mu \|x - y\|^2$, $\mu > 0$. Check via Jacobian: F (strictly/strongly) monotone if $\nabla F + (\nabla F)^\top \succ 0$ (or $\succeq 0$ for monotone).

Theorem (Uniqueness): If F strictly monotone on K, then unique NE (SOL(VI(K, F)) singleton). Strong monotonicity implies uniqueness globally.

Warning: Individual J_i strictly convex $\not\Rightarrow$ unique NE (counterexample: $J_i = (x_1 - x_2)^2$ both; F not monotone). Exam-relevant: Cournot Jacobian positive definite $\implies$ strong monotonicity $\implies$ unique NE.

5.6 Projected Game Map Dynamics

Computing/Learning NE

Discrete-Time Iteration: $x(t+1) = \Pi_K[x(t) - \gamma F(x(t))]$, or per-player: $x_i(t+1) = \Pi_{K_i}[x_i(t) - \gamma \nabla_{x_i} J_i(x(t))]$. Interpretation: Projected gradient step assuming others fixed (best-response-like with conjecture).

Assumptions for Convergence: F μ-strongly monotone, L-Lipschitz ($\gamma \in (0, 2\mu/L^2)$).

Theorem (Convergence): Under assumptions, iteration converges to unique NE (Banach fixed-point: map contractive). Proof: Projection non-expansive (Π_K 1-Lipschitz); strong monotonicity + step-size ensure contractivity.

In potential convex games (e.g., Cournot), best-response dynamics converge to NE (global min of potential). Applications: TCP congestion control, power control (SINR games), ride-hailing pricing

exam-like

oligopoly with substitutability θ)

6 Stackelberg games

Stackelberg games model hierarchical decision-making where the **leader** (Player 1) commits to a strategy first, and the **follower** (Player 2) observes and best-responds.

Let $\mathcal{Y}$ and Σ be the strategy spaces (pure or mixed), and $J_i(y, \sigma)$ the cost for Player i.

6.1 Definition of Stackelberg games

6.2 Definitions and Core Concepts

Rational Reaction Set For leader strategy $y \in \mathcal{Y}$ and $\forall \xi \in \Sigma$ the follower's rational reaction set is:

$$R(y) = \arg\min_{\sigma \in \Sigma} J_2(y, \sigma) = \{\sigma \in \Sigma \mid J_2(y, \sigma) \leq J_2(y, \xi)\}$$

Stackelberg Equilibrium

A pair $(\tilde{y}, \tilde{\sigma}(\cdot))$ with $\tilde{y} \in \mathcal{Y}$ and $\tilde{\sigma} : \mathcal{Y} \to \Sigma$ is a Stackelberg equilibrium if:

- Leader optimizes: $\tilde{y} \in \arg\min_{y \in \mathcal{Y}} J_1(y, \tilde{\sigma}(y))$.
- Follower best-responds: $\tilde{\sigma}(y) \in R(y)$ for all $y \in \mathcal{Y}$.

In zero-sum games, this coincides with the minimax (security) strategy.

Applications include oligopoly models (e.g., Cournot quantity competition where leader gains advantage by committing first, yielding higher profit than simultaneous Nash; Bertrand price competition where leader may face disadvantage due to aggressive follower response).

6.3 Stackelberg Zero-Sum Games

In zero-sum games ($J_2 = -J_1$), Stackelberg equilibria equal mixed Nash equilibria, with value

$$v = \min_y \max_\sigma J_1(y, \sigma) = \max_\sigma \min_y J_1(y, \sigma)$$

No commitment advantage/disadvantage.

6.3.1 Mixed Security Strategy

Leader's mixed security strategy:

$$\tilde{y} \in \arg\min_{y \in \mathcal{Y}} \max_{\sigma \in \Sigma} J_1(y, \sigma)$$

6.3.2 Pure Security Strategy

Restrict to pure strategies:

$$\tilde{y} \in \arg\min_{y \in Y} \max_{\sigma \in S} J_1(y, \sigma)$$

often worse than mixed.

6.3.3 Stackelberg vs. Nash

Mixed Stackelberg and Nash coincide in zero-sum games.
Pure Stackelberg may differ and be suboptimal.

6.4 Stackelberg Non-Zero-Sum Games

Highest Leader Cost (Pessimistic)

$$H = \inf_y \max_{\sigma \in R(y)} J_1(y, \sigma)$$

always achievable in equilibrium.

Theorem 5 (Upper Bound on Stackelberg Cost). No equilibrium achieves $J_1 < H$.

Theorem 6 (Nash vs. Highest Leader Cost). Every Nash equilibrium (y^*, σ^*) satisfies $J_1(y^*, \sigma^*) \geq H$; Stackelberg is never worse for leader than any NE.

Lowest Leader Cost (Optimistic)

$$L = \min_y \min_{\sigma \in R(y)} J_1(y, \sigma)$$

generically $L = H$ for random payoffs.

Theorem 7 (Nash vs. Lowest Leader Cost). Every Nash equilibrium satisfies $J_1(y^*, \sigma^*) \geq L$.

Pure-strategy Stackelberg: H_{pure} can exceed Nash costs if rational reaction sets are non-singleton.

6.4.1 Computation of Stackelberg Equilibrium

For finite games, use divide-and-conquer: For each pure follower strategy σ_j, solve LP

$$\min_y y^\top A_j \quad \text{s.t.} \quad y^\top B_k \geq y^\top B_j \, \forall k \neq j,\ y \in \Delta^m,$$

where A_j, B_k are payoff vectors. Select minimum J_1 among solutions. Easier than Nash computation.

6.5 Security Games

Defender (leader) randomizes coverage over targets; attacker (follower) observes and attacks optimally. Coverage vector $c = X^\top y$, where X is target-schedule matrix.

Definition 6 (Coverage Vector). $c_t = \sum_s x_{s,t} y_s$, probability target t is covered.

Stackelberg solution via LP; provides commitment advantage over Nash.

6.5.1 Stackelberg Solution

Minimize expected damage via LP over mixed y, assuming attacker best-responds.

6.5.2 Nash Equilibria

Equivalent to NE of auxiliary zero-sum game (set defender payoff = -attacker payoff). All NE attacker-equivalent. If feasible coverages downward-closed ($\gamma' \leq \gamma \in \Gamma \implies \gamma' \in \Gamma$), every Stackelberg strategy is Nash.

Exam Relevance

Focus on equilibrium computation (LP), zero/non-zero-sum differences, leader advantage (e.g., Cournot/Bertrand), security games (auxiliary zero-sum, downward-closed sets)

7 Repeated games

7.1 Tragedy of the commons

N players share a single resource. Each can **cooperate** (limit consumption) or **exploit** (overuse).

For 2 players (maximizers), payoff matrix:

	cooperate	exploit
cooperate	(a, a)	$(0, b)$
exploit	$(b, 0)$	(c, c)

For N players:

$$P_i = \begin{cases} a\frac{N_C - 1}{N-1} & \text{(cooperate, } N_C \text{ cooperators)} \\ (b - c)\frac{N_C}{N-1} + c & \text{(exploit)} \end{cases}$$

Exploit is dominant strategy and unique pure Nash equilibrium.

7.1.1 Applications

Overpopulation, pollution, overfishing, antibiotic resistance, Wi-Fi overuse, etc.

7.2 Finitely Repeated Games $G^{(T)}$

Supergame $G^{(T)}$, same game G repeated T times.

Actions $u^t \in \{\text{cooperate, exploit}\}^N$ of N players at stage t

History of the game $\mathcal{H}^t$ actions of N players before stage t

$\mathcal{H}^1 = ()$ empty vector, $\mathcal{H}^T = (u^1, \ldots, u^{T-1})$

Assumption: All agents have perfect recall of the past.

7.3 Strategies in a repeated game

Pure strategy $\gamma_i^{(T)} : \mathcal{H}^t \to \{\text{cooperate, exploit}\}$.

Payoff $P_i^{(T)} = \sum_{t=1}^T P_i(u^t)$, where $u^t = \gamma^{(t)}(\mathcal{H}^t)$.

NE: No player improves by unilaterally changing strategy (sequence of functions, not just actions).

Nash Equilibrium in Finite Repetitions

Theorem 8. If γ^* is a NE of G, then constant strategy $\gamma^t = \gamma^* \, \forall t$ is NE of $G^{(T)}$. If γ^* is unique NE of G, then every NE of $G^{(T)}$ yields $u^t = \gamma^* \, \forall t$ (by backward induction: last stage is G, induct backwards).

In Tragedy of Commons (unique dominant NE = exploit), only equilibrium outcome is repeated exploitation; repetition does not sustain cooperation.

Exercise Insight: Repetition Can Lead to Bad Outcomes

In some games with unique stage NE (e.g., (A,A) payoff 2 each), finite repetition preserves it, but infinite can sustain worse equilibria (e.g., (B,B) payoff 1 each) via triggers, yielding lower payoffs

7.4 Infinitely Repeated Games $G^{(\infty)}$

Discounted payoff For patience factor $0 < \delta < 1$:

$$P_i^{(\infty)}(\gamma^{(\infty)}) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} P_i(u^t)$$

Evil strategy $\gamma_i^t =$ exploit $\forall t, i$ is always NE.

Grim trigger strategy
$\gamma_i^t =$ cooperate if all cooperated $\forall \tau < t$, else exploit forever.
Proposition 3 (Trigger NE in TC). For $a = 2, b = 3, c = 1$, all playing grim trigger is NE iff $\delta \geq 1/2$ (deviation gain 1 today $\leq$ discounted future loss 1 per period: $1 \leq \delta/(1-\delta)$).

Tit-for-tat (2 players): Cooperate first, then copy opponent's previous action; NE for high δ in PD-like games.
Theorem 9 (Cooperation via Repetition (Folk Theorem Variant)). Let $\hat{\gamma}$ yield $\hat{P}_i > \bar{P}_i := \min_{\gamma_{-i}} \max_{\gamma_i} P_i(\gamma_i, \gamma_{-i})$ (minmax value) $\forall i$. Then $\exists \delta_0 \in (0,1)$ s.t. $\forall \delta > \delta_0$, $G^{(\infty)}$ has NE with on-path payoffs $\hat{P}_i$ each period.

Construction: Grim trigger – play $\hat{\gamma}$ if no deviations, else switch forever to minmax profile against deviator.

Infinitely repeated games often have many NE, enabling cooperation if better than minmax.

Exercise Insight: Non-Credible Threats

In games with unique stage NE (e.g., (B,A) payoffs (6,0)), infinite repetition can sustain better outcomes like repeated (A,A) (4 each) via triggers for high δ, but threats may be non-credible (e.g., punisher hurts themselves more).

Relevance to Exams

Focus on: Unique NE preservation in finite games; cooperation sustainability in infinite via grim triggers and high δ; folk theorem for multiple equilibria; potential pitfalls like bad outcomes or non-credible threats (hints at subgame perfection).

8 Multistage games

Multistage games extend repeated games by allowing variable stages, sequential or simultaneous moves, imperfect information, and memory constraints.

The extensive form (game tree) is preferred over the normal (matrix) form due to efficiency, especially for large games like tic-tac-toe (9! pure strategies for Player 1 in simplified form).

8.1 Extensive Form Representation

Game tree Root to leaves, nodes are decision points labeled by players, edges are actions, leaves have payoffs (focus on zero-sum for saddle-points)

Information sets Group indistinguishable nodes (dashed lines) each set must have identical action sets. Imperfect information arises when sets span multiple nodes.

Examples Tic-tac-toe (symmetries create large info sets) surprise quiz paradox (teacher-student game with surprise payoffs).

Conversion to normal form Enumerate pure strategies to form payoff matrix, but size explodes exponentially.

8.2 Strategies in Extensive Form

Pure strategy Maps each info set to one action.

Mixed strategy Probability distribution over pure strategies (dim: $\prod |U_i| - 1$, exponential). Randomizes ex-ante over full plans.

Behavioral strategy Independent distribution over actions per info set (dim: $\sum(|U_i| - 1)$, linear). Local randomization during play.

Kuhn's theorem (feedback games) Every mixed strategy has an equivalent behavioral strategy inducing the same outcome distribution and payoffs.

8.3 Equilibria and Credibility

Many Nash equilibria (NE) exist but may rely on non-credible threats (e.g., nuclear deterrence: "launch if provoked" only credible if automated).

Subgame Perfect Equilibrium (SPE) Strategy profile that is NE in every subgame (refines NE by ensuring credibility).

Zermelo's theorem (perfect info, finite zero-sum, no chance): One player has winning strategy or draw.

8.4 Backward Induction

For perfect info games: Start at leaves, propagate optimal choices upward, yields pure SPE efficiently (vs. matrix enumeration).

Applications: Chess (theoretical draw/win, unsolved), tic-tac-toe (draw with optimal play).

Feedback games Each Player 1 info set roots a separate subgame, enables recursive decomposition.

In feedback games with behavioral strategies:

- At each info set, solve local matrix game for minimax value and behavioral NE (via LP if needed).

- Propagate values backward to root; yields SPE in behavioral strategies. No convergence guarantee for pure NE, but Nash's theorem ensures mixed/behavioral existence in finite games.

8.5 Relevant Examples and Exercises

Surprise test paradox Modeled as non-feedback zero-sum game (K=2 days) teacher maximizes surprise, student minimizes.

NE: Teacher tests day 1 w.p. 1/3, student studies day 1 w.p. 2/3, average study days 4/3 >1 (incentivizes more study). Backward induction fails due to imperfect info, but NE shows surprise possible.

- **Tic-tac-toe**: Extensive form has $9 \times 7^8 \times 5^6 \times 3^4 \times 1^2$ pure strategies for P1; backward induction proves optimal draw.

Exam focus

Distinguish mixed vs. behavioral (equivalence in feedback games); apply backward induction for SPE in zero-sum extensive/feedback games; analyze credibility in non-zero-sum settings; model applications like epidemics or power control as stochastic multistage games with state-dependent equilibria.

9 Dynamic games

9.1 Escape Game Example

The escape game is a zero-sum feedback game where Alice tries to reach the safe zone over K stages without being intercepted by Eve. Alice moves up, middle, or down; Eve blocks one row. The game tree grows exponentially (e.g., for 3 actions, order of $1 + 3 + 9 + \cdots + 3^{K-1}$ nodes), requiring backward induction on the full tree, which is computationally intractable for large K (e.g., 5×10^9 LPs for $K = 20$).

9.1.1 Taming Complexity in Dynamic Games

Model as a state-based *loop model* instead of tree:

states $X = \{U, C, D, L\}$ (up, center, down, lost)

actions $U(x_k)$ and $V(x_k)$ depend on state.

Dynamics: $x_{k+1} = f(x_k, u_k, v_k) = L$ if $u_k = v_k$ or $x_k = L$, else $x_{k+1} = u_k$.

Stage cost $g_k = 0$ for $k < K$, $g_K = +1$ if lost (Eve wins), -1 if safe (Alice wins).

Backward induction on states reduces to $4 \times K$ LPs (scalable). Value function $V_k(x_k)$ computed recursively: $V_K(x_K)$ from last-stage NE, then

$$V_{k-1}(x_{k-1}) = \min_{\gamma} \max_{\sigma}[g_{k-1} + V_k(f(\cdot))]$$

For infinite K, $V_1(C) \to 0$ (Alice's escape probability approaches 0.5 numerically, as Eve can always adapt).

This state-based approach is preferable to tree induction for large K or infinite horizons, enabling efficient computation in control applications.

9.1.2 From Tree to Loop Model

Tree models capture full history, loop models abstract to Markov states evolving via $x_{k+1} = f(x_k, u_k, v_k, \dots)$, with outcomes $g_k(x_k, u_k, v_k, \dots)$. Equivalent for feedback games, allowing control interpretations (strategies as feedback laws).

9.2 Non-Zero-Sum Dynamic Games

In multi-player settings (e.g., three-truck platoon), players minimize individual costs:

$$J_i = \sum_k g_{i,k}(x_k, u_{1,k}, \dots, u_{n,k}) + g_{i,K}(x_K)$$

Subgame-perfect Nash Equilibrium (SPNE): strategies form NE in every subgame.

SPNE corresponds to optimal feedback laws in control theory.

9.2.1 Backward Induction

For finite-horizon games, compute SPNE via backward induction on states:

At stage K - Solve static game for each x_K:
find NE strategies $\gamma_K^*(x_K)$, value $V_K(x_K) = g_K + g_{\text{terminal}}$.

Recursively at stage k - For each x_k:
Solve static game with payoff $g_k(x_k, u_k) + V_{k+1}(f(x_k, u_k))$, yielding $\gamma_k^*(x_k)$, $V_k(x_k)$.

Relevant for exams:

Ensures subgame perfection; extends to stochastic games (with probabilities in dynamics)

9.2.2 One-Player Case: Dynamic Programming

Reduces to Bellman's equation:

$$V_k(x_k) = \min_{u_k}[g_k(x_k, u_k) + V_{k+1}(f(x_k, u_k))]$$

optimal policy $\mu_k^*(x_k)$.

9.2.3 Two-Player Linear-Quadratic (LQ) Games

System: $x_{k+1} = Ax_k + B_1 u_k + B_2 v_k$.

Costs (non-zero-sum):

$J_1 = \sum_{k=0}^{K-1}(x_k^T Q_1 x_k + u_k^T R_1 u_k + v_k^T S_1 v_k) + x_K^T P_1 x_K$,
$J_2 = \sum_{k=0}^{K-1}(x_k^T Q_2 x_k + v_k^T R_2 v_k + u_k^T S_2 u_k) + x_K^T P_2 x_K$
Backward induction yields affine feedback:

Assume $V_{1,k}(x) = x^T P_{1,k} x + q_{1,k}^T x + r_{1,k}$,
similarly for $V_{2,k}$.

At each step, solve for NE:

$\hat{u}_k = -(R_1 + B_1^T P_{1,k+1} B_1)^{-1}(B_1^T P_{1,k+1} A x_k + B_1^T P_{1,k+1} B_2 v_k + \dots)$ (coupled)
Results in Riccati-like equations for $P_{i,k}$, enabling unique NE under positive-definiteness assumptions (convex costs).

Exam relevance

Mirrors uniqueness in convex games; dynamics like projected gradient; backward induction in stochastic/epidemic games

Exercises Insights

Exercises reinforce backward induction in escape game: Compute mixed NE numerically, implement in MATLAB, analyze convergence. Highlights scalability of state-based methods over tree-based for dynamic/repeated games.

10 Stochastic games

Stochastic games extend randomized feedback games to model multi-stage interactions with state-dependent transitions, addressing intractability for large/infinite horizons by reducing complexity via state aggregation.

Key objectives:

- Identify limitations of randomized feedback games for large/infinite stages.

- Introduce stochastic games as a tractable subclass.

- Construct state-transition models.

- Differentiate solution methods for finite vs. infinite-stage games.

- (Bonus) Mitigate curse of dimensionality via stochastic population games.

10.1 Motivating Example: Epidemic Game

Two players: Alice (initially Not Infected, NI) and Eve (initially Infected, I). Actions: H (stay home, boredom cost 1 for NI) or O (go out, cost 0 but infection risk). Eve's illness cost: 3 (base), reduced to 2 if both H (quarantine). Infection prob. α if both O; recovery prob. β if infected stays H. States: (NI,I), (I,I) (absorbing if both infected). Relevant for exams: Model as stochastic game; compute equilibria via backward induction; analyze parameter-dependent pure Nash (e.g., quarantine vs. go out).

10.1.1 1 Stage (One-Shot Game)

- Game tree: Simple extensive form with states.

- Not a full feedback game due to randomization.

- Nash: Depends on α, β; e.g., both O if α low.

- Values: Expected costs for each player.

10.1.2 2 Stages

- Game tree: Branches based on transitions; solve subgames.

- Feedback game with states reducing info sets from 4^{K-i} to $2K$.

- Solve via backward induction: 2 LPs per state (matrix games).

- Equilibria shift with α, β (e.g., cooperation if recovery likely).

10.1.3 Taming Complexity

- subgames dependent of states {(NI,I),(I,I)}

- from 4^{k-i} to $2K$

10.2 State transition model

- **Stages** become **discrete time**

- **Behavioral strategies** (maps from information sets to actions) become **feedback policies** (maps from states to inputs)

- **Kuhn's theorem holds**: NE in behavioral strategies guaranteed to exist

- Can compute (subgame-perfect) behavioral NE using **backward induction**

10.3 Probabilistic state transitions

- from certainty 1 to probability α

10.4 Stochastic game

Basic elements

- Stages $k = 0, \ldots, K$-1
- State space $x_k \in \mathcal{X} = \{x^1, \ldots, x^\ell\}$, initial state x_0
- Action space $u_k \in \mathcal{U} = \{u^1, \ldots, u^n\}$, $v_k \in \mathcal{V} = \{v^1, \ldots, v^m\}$
- Stage outcome functions $g^{(1)}(x_k, u_k, v_k)$, $g^{(2)}(x_k, u_k, v_k)$
- State transition probabilities $\mathbb{P}(x_{k+1}|x_k, u_k, v_k)$

10.4.1 Finite-Stage Stochastic Game

- Strategies: Time-dependent behavioral $\gamma_k^b(x), \sigma_k^b(x)$.

- Payoff: $\mathbb{E}[\sum_{k=0}^{K-1} g|x_0]$.

- Solve: Backward induction over ℓK matrix games per state/-time.

10.4.2 Infinite-Stage Stochastic Game

- Strategies: Stationary $\gamma^b(x), \sigma^b(x)$

- Payoff: $\mathbb{E}[\sum_{k=0}^{\infty} \delta^k g|x_0]$ (discounted)

- Solve: Policy iteration (init strategies; evaluate V via linear system; improve via best-response games); or value iteration.

- Theorem: If converges, yields subgame-perfect behavioral NE.

- Warning: Convergence not guaranteed (unlike MDPs); counterexamples exist.

10.5 Value Functions

- Finite: $V_k^i(x) =$ expected remaining payoff from k, separable as stage $g + V_{k+1}$

- Infinite: $V^i(x) = \mathbb{E}[g^i(x, u, v) + V^i(x')]$ solves Bellman Eq.

10.6 Example: Traffic Game (2 Cars at Crossing)

- Matrix: $\begin{pmatrix} (100, 100) & (0, 1) \\ (1, 0) & (v, v) \end{pmatrix}$

- Actions: Go/Wait; crash if both Go.

- Solve NE via p,q mixtures; value $y^T Gz$.

- Extend to stochastic: Add states (e.g., traffic light), transitions.

10.7 Summary: Stochastic Games

Stochastic games model state-dependent multi-stage play with probabilistic transitions. Finite: Backward induction for subgame-perfect NE. Infinite: Stationary policies, Bellman eq., policy/value iteration (convergence not assured).

Exam focus

Epidemic modeling, parameter-dependent equilibria, backward induction in finite stages, value functions.

11 Bonus Material: Stochastic Population Games

Curse of dimensionality: State space explodes (2^N for N players, e.g., epidemics).

Assumptions for tractability (mean-field limit, N∞): 1. Large symmetric anonymous players. 2. Separable states. 3. Payoffs/transitions depend on own state/action + distributions (d_k, π_k). 4. Continuity in distributions.

Theorem: Exists stationary (d^*, π^*) as NE for δ-discounted infinite game. Reduces dimension to $p + n$ (independent of N); e.g., city epidemic from 2^{400k} to 4 dims.

Uncovered topics: Partial info/Bayesian games, differential games.