Game Theory and Control

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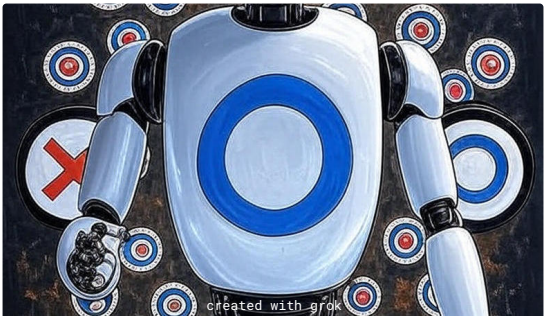
github.com/silvasta/summary-gtc



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1 Static games

1.1 Basic Definitions

Game theory is the study of mathematical models of conflict and cooperation between rational decision-makers.

Several things are needed to characterize a game:

- The **players** are the agents that make decisions

- The **actions** available to each player at each decision point

- The **information structure** specifies what each player knows before making each decision, in particular with respect to other players' decisions

- The **outcome** for each player, which depends on all players' decisions

1.1.1 Matrix representation

1.1.2 Nash Equilibrium

1.1.3 Dominated actions

1.1.4 Reduced game

1.1.5 Security levels and policies

1.2 Multiple Nash Equilibria

1.2.1 Admissible Nash Equilibria

Poset

Hasse diagram

Minimal element

1.3 Mixed strategies

Careful!Refer to deterministic strategies as pure strategies!

1.3.1 Security levels

Mixed security level

Mixed security strategy

Computational complexity

1.3.2 Mixed Nash Equilibrium

1.4 Nash Theorem

2 Zero-sum games

- Zero-sum games model a large number of practical applications

- Nash equilibria in zero-sum games have many useful properties

- Nash equilibria in zero-sum games are much easier to compute

2.1 Two-Person Zero-sum Games

Games in which the two players have opposite payoffs:

$$J_1(\gamma, \sigma) = -J_2(\gamma, \sigma)$$

Static game with $B = -A$ (we only indicate one matrix, A)

2.1.1 Payoff Matrix

- Row player loses aij

- Column player gains aij

- Row player minimizes outcome V

- Column player maximizes outcome V

2.1.2 Rock, Paper, Scissors

Consider only one round of the game.

$$A = \begin{matrix} \\ \text{Rock} \\ \text{Paper} \\ \text{Scissors} \end{matrix} \begin{matrix} \begin{matrix} \text{Rock} & \text{Paper} & \text{Scissors} \end{matrix} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{matrix}$$

2.2 Security levels and policies

- Security level Player 1,2

- Security policy Player 1,2

2.2.1 Min-Max Property

Security levels in **zero-sum games** have a fundamental property that general static games don't have.

For every finite matrix A, the following properties hold:

(i) Security levels are well defined and unique

(ii) Both players have security policies (not necessarily unique)

(iii) The security levels always satisfy:

2.3 Nash equilibrium in zero-sum games

Intepretation: As in static games: no regret, stable strategy when iterated, etc.

Also known as **saddle-point equilibrium**

$$a_{i^\star j} \leq a_{i^\star j^\star} \leq a_{ij^\star} \quad \forall i \in 1, ..., n, \forall j \in 1, ..., m$$

Saddle-point value $V^\star := a_{i^\star j^\star}$

2.4 Saddle-point and security levels

Not all zero-sum games have a saddle point (f.e. Rock-Paper-Scissors). We can exactly characterize the zero-sum games that have a saddle point.

Theorem 1 (Saddle-point and security levels)**.** A zero-sum game defined by A has a saddle-point equilibrium **if and only if**

$$\underline{V} = \bar{V} \quad (= V^\star \text{ saddle-point value})$$

Important consequences follow (only for zero-sum games!)

All saddle-point equilibria (Nash equilibria) of a zero-sum game have the same value $V^\star$, which we denote as the value of the game.

2.5 Mixed strategies

2.5.1 Computing the mixed security level via LP

pseudo, matlab code:

m = 5; n = 10; A = rand (m, n) ; y = sdpvar (m, 1) ; t = sdpvar (1 , 1) ; obj = t ; c o n s t r a i n t s = [A ' ⊠ y <= ones (n , 1) ⊠ t , sum (y) == 1 , y >= 0] ; optimize (constraints , obj) ; S e c u r i t y L e v e l = double (t) ; S e c u r i t y P o l i c y = double (y) ;

2.5.2 Min-Max Property

2.5.3 Nash Equilibrum

Theorem 2 (Mixed Nash equilibrium for zero-sum games)**.** Policy $(y^\star, z^\star)$ is called mixed-strategy saddle-point equilibrium (or Nash equilibrium) if

$$y^{\star T} A z^\star \leq y^T A z^\star \quad \forall y \quad \text{(minimizer)}$$

$$y^{\star T} A z^{\star} \geq y^{\star T} A z \quad \forall z \quad \text{(maximizer)}$$

$y^{\star T} A z^{\star}$ is called **saddle point value**

3 Auctions

- N Players: bidders

- Action: bid $x_i \geq 0$

Outcome

- $w(x)$ the winner of the auction

- $p(x)$ the price that the winner has to pay

- t_i the true value of the item for agent i

The outcome (cost) for agent i is:

$$J_i = \begin{cases} p(x) - t_i & \text{if } i = w(x) \\ 0 & \text{otherwise} \end{cases}$$

3.1 First-price auctions

In first-price auctions, the best-response bid depends on the bids of the other agents and your own true value.

- Underbidding $x_i < t_i$

- Truthful bidding $x_i = t_i$

- Overbidding $x_i > t_i$

Proposition 1 (first-price auctions)**.** .

- any **overbidding** strategy is dominated by **truthful bidding**
- **truthful bidding** is not a dominant strategy

3.2 The problem of private information

Until now, we always assumed that the cost functions of the agents is known to the other agents. Auctions are an application of game theory in which this is not true, and we encode all the **private information** in the true value t_i.

Lack of information leads to inefficiency!

3.3 Second-price auctions

Winner selection $w(x) = \text{argmax}_i \, x_i$

(the bidder with the highest bid wins)

Payment rule $p(x) = \max_{i \neq w(x)} x_i$

(the winner pays the second-largest bid)

3.3.1 Dominant Strategy

In second-price auctions, a Nash equilibrium exists and **can be computed by each agent based on their own private information.**

Truthful bidding is a weakly dominant strategy in a second-price auction

Intuition: your bid determines whether you win, not how much you pay.

- Incentive compatibility

3.3.2 Properties of second-price auction

Social efficiency

Does the auctioneer achieve the highest return?

No (cost of eliciting truthful bidding...)

Incentive compatibility and **social efficiency** often go together (the true value needs to be disclosed in order to be used for efficient allocation).

4 Generalized auctions

4.1 Bids

Each **bid** is represented by a pair $x_j = (b_j, m_j)$

- b_j is the bidded amount

- m_j describes the object of the bid

(Can be extended to allow multiple bids)

Fungible goods $m_j \in \mathbb{R}_{>0}$ parts of a total quantity M

Non-fungible goods $m_j \in 2^{\mathcal{M}}$ with finite set of items $\mathcal{M}$

4.2 Choice function

Choice function w maps bids x into N-dimensional binary vector

$$w_j(x) = \begin{cases} 1 & \text{if bid } j \text{ is accepted} \\ 0 & \text{otherwise} \end{cases}$$

- Choice constraints

4.3 Payment function

Payment function p maps bids x into N-dimensional vector where $p_j(x)$ is the payment requested from the player that placed the bid j

5 VCG auctions

- VCG choice function

- VCG payment function

In order to compute each payment p_j, we need to evaluate the choice function twice: with and without the bid j

5.1 Social utility

The social utility is the aggregate utility of all players and the auctioneer

$$U(t, w) = \sum_i t_i w_i$$

The social utility - depends on the true value of the goods according to the player that receives it

- does not depend on the entity of the payments

If players bid truthfully $b_j = t_j$ then VCG choice function achieves maximal social utility $U^\star$

Interpretation of the VCG payment if agents bid truthfully:

5.1.1 Non-negative utility

When an agent bids truthfully, his utility is non-negative.

5.2 Dominant bidding strategy

Truthful bidding is a weakly dominant strategy in a VCG auction.

6 Open problem of auction design

We saw how to design an auction which guarantees

- incentive compatibility / truthful bidding

- optimal social efficiency

- non-negative payments

Unfortunately, it comes with drawbacks. For example

- it yields low returns

- it can be manipulated by colluding agents

- it is computationally challenging to solve.

7 Potential games

8 Convex games

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