Large-Scale Convex Optimization

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1 Introduction

Large Scale Problem of dimension n but iterations $\ll n$ desired
Convex One of the only problem classes that are "solvable"
Optimization with decision variable x, objective function f and feasible set $\mathcal{C} = \{\xi \in \mathbb{R}^n : g(\xi) \le 0,\ h(\xi) = 0\}$
Local minimum $x^\star$ if $\exists \epsilon > 0$ s.t. $f(x^\star) \le f(x)$, $\forall x \in \mathcal{C} \cap B_\epsilon(x^\star)$, $B_\epsilon(x^\star) := \{\xi \in \mathbb{R}^n : |\xi - x^\star| < \epsilon\}$
Proposition 1. f (lower-semi-)continuous, $f(x) \to \infty$ for $|x| \to \infty$, $\mathcal{C}$ closed $\Rightarrow \exists$ of OP with: $\min_{x\in\mathcal{C}} f(x)$ and $x^\star \in \operatorname{argmin}_{x\in\mathcal{C}} f(x)$
Definition 1 (Lipschitz continuity). $q : \mathbb{R}^n \to \mathbb{R}^m$ is **Lipschitz** with constant L if: $|q(x) - q(y)| \le L|x - y| \forall x, y \in \mathbb{R}^m$

$$f \text{ is Lipschitz with constant } L \Leftrightarrow |\nabla f(x)|_2 \le L$$

OP class $\mathcal{P}$ with $\mathcal{C} = [0,1]^n$, f is l^∞-Lipschitz with constant L
Proposition 2. For any algorithm $\exists$ problem in $\mathcal{P}$, s.t. achieving $|f(x_N)$-$f(x)| < \epsilon$ requires $N \ge (\lfloor \frac{L}{2\epsilon} \rfloor)^n - 1$
Definition 2. OP convex if, f and g_i convex functions, h affine.
Definition 3. $q : \mathbb{R}^n \to \mathbb{R}$ convex (affine) if $\forall x, y \in \mathbb{R}^n$

$$q(\theta x + (1\text{-}\theta)y) \le \theta q(x) + (1\text{-}\theta)q(y) \quad \forall \theta \in [0,1]$$

Proposition 3. If OP convex, local minimum == global minimum

2 Convex Optimization Problem

Definition 4 (Convex Set). A set $\mathcal{C}$ is convex if and only if $\theta x + (1-\theta)y \in \mathcal{C}, \forall x, y \in \mathcal{C}, \forall \theta \in [0,1]$
(hyperplane || half-space) $\{x \in \mathbb{R}^n \mid a^\mathsf{T} x (= || \le) b\}$
polyhedra $\{x \in \mathbb{R}^n \mid A^{q\times n}x \preceq b^{q\times 1}, C^{r\times n}x = d^{r\times 1}\}$

Operations that preserve convexity (sets)

Intersection $\mathcal{C}_1, \mathcal{C}_2$ cv $\Rightarrow \mathcal{C}_1 \cap \mathcal{C}_2$ convex **(cv)**
Image under affine map $\mathcal{C} \subseteq \mathbb{R}^n$ cv $\Rightarrow \{Ax + b \mid x \in \mathcal{C}\}$ cv
Inverse IoaM $\mathcal{C} \subseteq \mathbb{R}^m$ cv $\Rightarrow \{x \in \mathbb{R}^n \mid Ax + b \in \mathcal{C}\}$ cv

Separating Hyperplane Theorem

Theorem 1. $\mathcal{C} \subseteq \mathbb{R}^n$ non-empty closed **(cl)** convex set, $y \notin \mathcal{C}$ $\to \exists\, a \ne 0, b \in \mathbb{R}$ s.t. $a^\mathsf{T} x + b < a^\mathsf{T} y + b, \forall x \in \mathcal{C}$
Corollary 1. $\mathcal{C}_{\text{cl,cv}}$: intersection of cl half-spaces that contain $\mathcal{C}$

Support function

Idea represent any cl,cv set by its supporting hyperplanes

$$\sigma_{\mathcal{C}}(a) = \sup_{x\in\mathcal{C}} a^\mathsf{T} x \quad \text{if known, one can construct}$$
$$\mathcal{C} = \bigcap_{a\in\mathbb{R}^n} \{x \in \mathbb{R}^n \mid a^\mathsf{T} x - \sigma_c(a) \le 0\}$$
$$= \{x \in \mathbb{R}^n \mid \sup_{a\in\mathbb{R}^n} a^\mathsf{T} x - \sigma_c(a) \le 0\}$$

Definition 5. $f : \mathbb{R}^n \to \mathbb{R}$ cv $\Leftrightarrow$ epigraph of f is cv set

$$\operatorname{epi}(f) := \{(x,t) \in \mathbb{R}^{n+1} | f(x) \le t\}$$

$\to$ this provides a link between convex sets and functions

Operations that preserve convexity (functions)

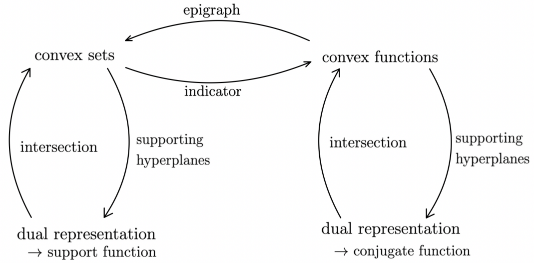
- the point wise maximum of convex functions is convex
- the sum of convex functions is convex
- $f(Ax + b)$ is convex if f is convex

Check Convexity f is convex if it is composition of simple convex function with convexity preserving operations or if

$f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable, $\partial^2 f/\partial x^2 \succeq 0 \,\forall\, x \in \mathbb{R}^n$
$g : \mathbb{R} \to \mathbb{R}$ with $g(t) = f(x + tv)$ convex in $t \,\forall\, x, v \in \mathbb{R}^n$ $\to f$ convex (restriction to a line)
Extended real numbers $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

Indicator function $\psi_{\mathcal{C}}(x) := \begin{cases} +\infty & \text{if } x \notin \mathcal{C} \ge 0 \\ 0 & \text{if } x \in \mathcal{C} \end{cases}$

$\to$ this provides another link between convex sets and functions
We can write $\min_{x\in\mathcal{C}} f(x)$ as $\min_{x\in\mathbb{R}^n} f(x) + \psi_{\mathcal{C}}(x)$
Definition 6 (3). $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is called proper if f is bounded below and if $\exists\, x \in \mathbb{R}^n$ s.t. $f(x) < \infty$
Definition 7 (Legendre Transformation). The **conjugate function** of $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is defined as $f^\star(y) = \sup_{x\in\mathbb{R}^n} y^\mathsf{T} x - f(x)$
Concave $\nabla_x^2 f^\star \prec 0 \Rightarrow$ maximizer of sup satisfies $\nabla_x f^\star = 0$



Theorem 2 (Conjugate of Conjugate). $f : \mathbb{R}^n \to \bar{\mathbb{R}}$
(i) f proper, cv, epi(f) closed $\Rightarrow f^{**} = f$
(ii) $f(x) \ge f^{**}(x), \forall x \in \mathbb{R}^n$

3 KKT and Lagrange Duality

Basic 2d Example for derivation: $\min_{x\in\mathbb{R}^2} f(x)$ s.t. $h(x) = 0$ $\to \nabla f(x^\star), \nabla h(x^\star)$ co-linear $\Leftrightarrow \exists\, \nu^\star \in \mathbb{R} : \nabla f(x^\star) + \nu^\star \nabla h(x^\star) = 0 \Leftrightarrow f(x) + \nu^\star h(x)$ is stationary at $x^\star$

Generalization for $n \to \infty$ and with constraints

We consider $f^\star = \inf_{x\in\mathbb{R}^n} f(x)$ s.t. $g(x) \le 0, h(x) = 0$ (1)

Lagrange $\mathcal{L}(x, \lambda, \nu) = f(x) + \lambda^\mathsf{T} g(x) + \nu^\mathsf{T} h(x)$
Dual Function $d(\lambda, \nu) = \inf_{x\in\mathbb{R}^n} \mathcal{L}(x, \lambda, \nu)$

Proposition 4 (Weak Duality). $d(\lambda, \nu) \le f^\star, \forall \lambda \ge 0, \nu \in \mathbb{R}^h$
Definition 8 (Constraint qualification). $\mathcal{C}$ convex, **Slaters Condition** holds if $\exists\, \hat{x} \in \mathbb{R}^n$ s.t. $h(\hat{x}) = 0$ and $g(\hat{x}) < 0$
Proposition 5 (Strong Duality). If Slater's condition holds and (1) is convex $\Rightarrow \exists \lambda \ge 0, \nu \in \mathbb{R}^{n_h}$ s.t. $d(\lambda, \nu) = f^\star$

KKT

Theorem 3 (KKT Conditions). Slater's condition holds and (1) is convex $\to x^\star \in \mathbb{R}^n$ is a minimizer of the primal (1) and $(\lambda^\star \ge 0, \nu^\star) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ is a maximizer of the dual $\Leftrightarrow$

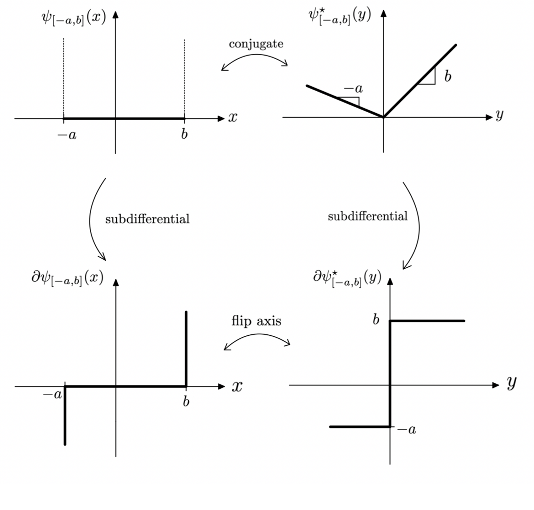
$\nabla_x \mathcal{L}(x^\star, \lambda^\star, \nu^\star) = 0$ KKT-1 (Stationary Lagrangian)
$g(x^\star) \le 0, h(x^\star) = 0$ KKT-2 (primal feasibility)
$\lambda^\star \ge 0, \nu^\star \in \mathbb{R}^{n_h}$ KKT-3 (dual feasibility)
${\lambda^\star}^\mathsf{T} g(x^\star) = 0 = {\nu^\star}^\mathsf{T} h(x^\star)$ KKT-4 (compementary slackness)

In addition we have: $\sup_{\lambda\ge 0, \nu\in\mathbb{R}^{n_h}} q(\lambda, \nu) = \inf_{x\in\mathcal{C}} f(x)$
Remark Without Slater, KKT1-4 still implies $x^\star$ minimizes (1) and λ, ν maximizes dual, but the converse is no longer true. There can be primal-minimizer/dual-maximizer not satisfy KKT.

Subdifferential

For cv f we have $f(x) \ge f(\bar{x}) + \nabla f(\bar{x})^\mathsf{T}(x - \bar{x}),\ \forall x, \bar{x} \in \mathbb{R}^n$
Definition 9. $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ cv, the subdifferential of f at $\bar{x}$ is: $\partial f(\bar{x}) := \{\lambda \in \mathbb{R}^n \mid f(x) \ge f(\bar{x}) + \lambda^\mathsf{T}(x - \bar{x}), \forall x \in \mathbb{R}^n\}$
EXAMPLE?

Proposition 6. f (like D9), $x^\star \in \operatorname{argmin}_x f(x) \Leftrightarrow 0 \in \partial f(x^\star)$
Proposition 7 (Relation to conjugate functions). For convex f with epi(f) closed: $y \in \partial f(x) \leftrightarrow x \in \partial f^\star(y)$



4 Convex Optimization Problems

Optimal value $f^\star = \inf\{f(x) \mid g_i(x) \le 0, h_j = 0\}$
$f^\star = +\infty$ OP is infeasible, $f^\star = -\infty$ OP is unbound below

Feasibility Problem

Special case $f(x) = 0, \forall x \Leftrightarrow \min_s$ s.t. $g_i(x) \le s, h_j(x) = 0$
Linear Programming minimize $c^\mathsf{T} x$ s.t. $Ax - b \ge 0,\ x \ge 0$
Step 1: $\mathcal{L}(x, \lambda_1, \lambda_2) = c^\mathsf{T} x - \lambda_1^\mathsf{T}(Ax - b) - \lambda_2^\mathsf{T} x,\ \lambda_i \ge 0$
Step 2: $\inf_{x\in\mathbb{R}^n} \mathcal{L} = \lambda_1^\mathsf{T} b$, if $c - A^\mathsf{T}\lambda_1 - \lambda_2 = 0$, else $-\infty$
Step 3: Dual, maximize $b^\mathsf{T}\lambda$ s.t. $c - A^\mathsf{T}\lambda \ge 0, \lambda \ge 0$ (again LP)
Proposition 8. The optimal solution of a linear program (if it exists) lies always on the boundary of the feasible set and there exists an optimal solution that is a vertex of the feasible set.
Quadratic Programming convex if $P = P^\mathsf{T}$ positiv semi-definite minimize $\frac{1}{2} x^\mathsf{T} P x + q^\mathsf{T} x$ s.t. $Gx \le h,\ Ax = b$

Second-Order Cone Program

minimize $f^\mathsf{T} x$ s.t. $|A_i x + b| \le c_i^\mathsf{T} x + d_i, Fx = g$
Second-order cone $C_{n+1} = \{(x,t) \mid x \in \mathbb{R}^n, t \in \mathbb{R}, |x| \le t\}$
$|A_i x + b| \le c_i^\mathsf{T} x + d_i \Leftrightarrow (\mathrm{A}_i x + b, c_i^\mathsf{T} x + d_i) \in C_{n+1}$
Semi-Definite Programming with symmetric F_i, X, A_i
minimize $c^\mathsf{T} x$ s.t. $\sum_{i=1}^n x_i F_i + G \preceq 0, Ax = b$
Standard form minimize tr(CX) s.t. $X \ge 0$, tr$(A_i X) = b_i$
tr$(CX) = \sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$, $C \in \mathbb{R}^{n\times n}, i = 1, .., m$
LP$\subset$ QP $\subset$ QCQP (Quadratically Constrained QP) $\subset$ SOCP $\subset$ SDP

5 Gradient methods - Part I

Definition 10 (smoothness). $f : \mathbb{R}^n \to \mathbb{R}$ is L-smooth **(L-sm)** if $\nabla f(x)$ satisfies $|\nabla f(x) - \nabla f(y)| \le L|x - y| \,\forall x, y \in \mathbb{R}^n$
Taylor $\to f(y) \le f(x) + \nabla f(x)^\mathsf{T}(y - x) + \frac{L}{2}|x - y|^2$
Definition 11 (strong convexity). $f : \mathbb{R}^n \to \mathbb{R}$ is μ-strongly convex **(μ-scv)** if $f(y) \ge f(x) + \nabla f(x)^\mathsf{T}(y - x) + \frac{\mu}{2}|x - y|^2$
How to find μ/L, Spectra of Hessian $\nabla^2 f$, min/max eigenvalue

Gradient Descent

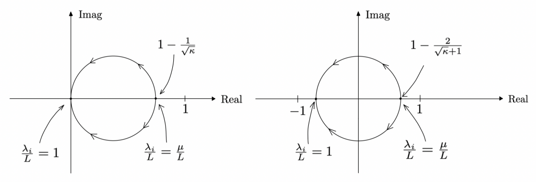
$x_{k+1} = x_k - T\nabla f(x_k)$ for $k = (k_0, \dots, k_N)$ given x_0, T
Assume $f(x) = c_0 + b^\mathsf{T} x + \frac{1}{2} x^\mathsf{T} H x$, $H \succ 0 \Rightarrow Hx^\star = -b$
$x_{k+1} - x^\star = x_k - x^\star - T(b + Hx_k) = (I - TH)(x_k - x^\star)$

Convergence given by eigenvalues of $I - TH$, use $H = U\Lambda U^\mathsf{T}$
$x_N - x^\star = U(I - T\Lambda)^N U^\mathsf{T}(x_0 - x^\star) \to$ conv-rate $1 - T\lambda_i$
f: L-sm, μ-scv $\to \mu \le \min \lambda_i$, $\max \lambda_i \le L$, $\to$ conv-rate $\rho(T)$ $=: \max_{\mu\le h\le L} |1 - Th| \to |x_N - x^\star| \le \rho(T)^N |x_0 - x^\star|$
$T^\star = \frac{2}{L+\mu}$, with **condition number** $\kappa := \frac{L}{\mu}$ and $1 - \xi \le e^{-\xi}$
$\rho(T^\star) = \frac{L-\mu}{L+\mu} = \frac{\kappa-1}{\kappa+1} = (1 - \frac{2}{\kappa+1}) \le e^{-\frac{2}{\kappa+1}} \to$ algebraic complexity $N \ge \frac{\kappa+1}{2} \ln(\frac{|x_0 - x^\star|}{\epsilon})$ to achieve $|x_N - x^\star| \le \epsilon$

Momentum-based methods

$$q_{k+1} = q_k + Tp_{k+1}$$
$$p_{k+1} = (1 - 2dT)p_k - T\nabla f(q_k + \beta p_k)/L$$

Nesterov accelerated gradient $T = 1, d = \frac{1}{\sqrt{k}+1}, \beta = \frac{\sqrt{k}-1}{\sqrt{k}+1}$
Heavy Ball tuned on quadratic $T = \frac{2\sqrt{k}}{\sqrt{k}+1}, d = \frac{1}{\sqrt{k}+1}, \beta = 0$



Nesterov Heavy-ball (tuned on quadratics)

$C_{\text{Nesterov}}(1 - \frac{1}{\sqrt{\kappa}})^N \approx \frac{|q_N - q^\star|}{|q_0 - q^\star|} \approx C_{\text{HeavyBall}}(1 - \frac{2}{\sqrt{\kappa+1}})^N$
Theorem 4. f: L-sm, μ-scv $\to$ Nesterov's method satisfies:

$$|q_N - q^\star| \le \sqrt{\kappa + 1}(1 - 1/\sqrt{\kappa})^{N/2}|q_0 - q^\star|$$
$$f(q_N) - f^\star \le \frac{L + \mu}{2}(1 - 1/\sqrt{\kappa})^N |q_0 - q^\star|^2$$

Requires $N \ge 2\sqrt{\kappa} \ln(\frac{|q_0 - q^\star|}{\epsilon})$ to achieve $|x_N - x^\star| \le \epsilon$
Theorem 5. For any first-order method $\exists f : \mathbb{R}^\infty \to \mathbb{R}$, μ-scv, L-sm, s.t. $|x_k - x^\star| \ge (1 - \frac{2}{\sqrt{\kappa}+1})^k |x_0 - x^\star| \forall k \ge 0$
Line search optimal step $\nu_t^* = \operatorname{argmin}_{\nu\in\mathbb{R}} f(x_t - \nu\nabla f(x_t))$
Adaptive Methods gradient, momentum to adjust stepsize

6 Gradient Methods - Part II

Definition 12. $\operatorname{prox}_{\mathcal{C}}(x) = \operatorname{argmin}_{y\in\mathcal{C}} \frac{1}{2}|x - y|^2$ with $\mathcal{C} \subset \mathbb{R}^n$
Lemma 1. cl, cv $\mathcal{C} \subset \mathbb{R}^n \to |\operatorname{prox}_{\mathcal{C}}(x) - \operatorname{prox}_{\mathcal{C}}(y)| \le |x - y|$ $\leftarrow |\operatorname{prox}_{\mathcal{C}}(x) - \operatorname{prox}_{\mathcal{C}}(y)|^2 \le (\operatorname{prox}_{\mathcal{C}}(x) - \operatorname{prox}_{\mathcal{C}}(y))^\mathsf{T}(x - y)$

Projected Gradient Descent

$x_{k+1} = \operatorname{prox}_{\mathcal{C}}(x_k - T\nabla f(x_k))$, for $x_0, k_{0..N}, T \in (0, 2/L)$
Proposition 9. f: L-sm, μ-scv $\to$ projected GD with $T = \frac{2}{L+\mu}$ satisfies $|x_N - x^\star| \le |x_0 - x^\star|(1 - \frac{2}{\kappa+1})^N$ (κ still $\frac{L}{\mu}$)
Lemma 2. $f : \mathbb{R}^n \to \mathbb{R}$, L-sm, **cv** $\to \tilde{f}$ strongly-cv
$\hat{f}(x) = f(x) + \frac{\mu}{2}|x - x_0|^2$ and $|\tilde{x}^\star - x_0| \le |x^\star - x_0|$ and $f(x) - f(x^\star) \le \tilde{f}(x) - \tilde{f}(\tilde{x}^\star) + \frac{\mu}{2}|x^\star - x_0|^2, \mu > 0$
$\to$ from here one can apply GD or Nesterov, which results in: $f(x_N) - f(x_0) \le \epsilon$ after $N \sim L|x^\star - x_0|^2/\epsilon$ iterations
Proposition 10 (Subgradient Method). cl, cv $\mathcal{C}$ contained in ball of radius R, $x_{0..N-1}$ satisfy $f(\frac{1}{N}\sum_{k=0}^{N-1}) - f(x^\star) \le \frac{RL_f}{\sqrt{N}}$ under $x_{k+1} = \operatorname{prox}_{\mathcal{C}}(x_k - Tg_k)$ $g_k \in \partial f(x_k)$, $T = \frac{R}{L_f\sqrt{N}}$

Assumptions on f	Method	$N : f(x_N) - f(x^\star) \leqslant \varepsilon$	Optimal
μ-strongly convex	gradient descent	$N \sim \kappa \ln(1/\varepsilon)$	No
L-smooth	Nesterov	$N \sim \sqrt{\kappa} \ln(1/\varepsilon)$	Yes
L-smooth	gradient descent	$N \sim 1/\varepsilon$	No
	Nesterov (varying stepsize)	$N \sim 1/\sqrt{\varepsilon}$	Yes
L_f-Lipschitz, compact set	subgradient method	$N \sim 1/\varepsilon^2$	Yes

7 Stochastic Gradient Descent (SGD)

Stochastic Gradient $(\tilde{y}_i - \phi(\tilde{x}_i; \theta))^\mathsf{T} \frac{\partial\phi}{\partial\theta}\Big|_{\tilde{x}_i;\theta}, i \sim \{1, ..., m\}$

Problem formulation: $\min_{x\in\mathbb{R}^n} F(x) = \min_{x\in\mathbb{R}^n} \mathbb{E}[f(x,\xi)]$

$$\mathbb{E}_\xi[f(x,\xi)] = \begin{cases} \int_{\mathbb{R}^q} f(x,\bar\xi)p_\xi(\bar\xi)d\bar\xi & \text{continuous Random V} \\ \sum_{\bar\xi} f(x,\bar\xi)p_\xi(\bar\xi) & \text{discrete R Variable}\end{cases}$$

Step 1: $\xi_k \leftarrow$ generate realization of ξ
Step 2: $x_{k+1} = x_k - T_k g(x_k,\xi_k)$, step size T_k, SG $g(.)$
$\nabla_x f(x,\bar\xi),\ \bar\xi \sim p_\epsilon$ or $\frac{1}{n_{mb}}\sum_{i=1}^{n_{mb}} \nabla_x f(x,\bar\xi_i), \xi_i \sim p_\epsilon$
$\Rightarrow$ The iterate x_k is now a random variable! Assumptions:
A1 $F(x)$ is bounded below, ensures $\exists \min_x F(x)$ for $F:$ L-sm
A2 $\mathbb{E}_\xi[g(x,\xi)] = \nabla F(x), \forall x\in\mathbb{R}^n$, ensures SG unbiased.
A3 $\exists M, M_v \ge 0$ s.t. $\mathrm{Var}_\xi[g(x,\xi)] \le M + M_v|\nabla F(x)|^2$ $\forall x \in\mathbb{R}^n$, ensures that variance is bounded.
Proposition 11. F μ-scv L-sm, SGD const. $T < \frac{1}{L(M_v+1)}$

$$\mathbb{E}[F(x_k)] - F(x^\star) \le \frac{TLM}{2\mu} + (1-T\mu)^k(F(x_0)-F(x^\star))$$

$T = \frac{ln(N)}{\mu N} \to N \sim \left(\frac{LM}{2\mu^2} + F(x_0) - F(x^\star)\right)/\epsilon$
to ensure $\mathbb{E}[F(x_N)] - F(x^\star) \le \epsilon$
$(1-T\mu)^N \le e^{-T\mu N}$ this in EQ
The role of mini batches $M \to M/n_{mb}$, $M_v \to M_v/n_{mb}$
Same analysis holds, But run SGD with T/nmb to get same result... Advantage in computation if paralellization possible!
Can we do non-(strongly-)convex functions?
Proposition 12. F, L-sm, SGD with $T \le \frac{1}{L(1+M_v)}$ achieves
$\mathbb{E}[\sum_{k=0}^{N-1}|\nabla F(x_k)^2|] \le NTLM + \frac{2}{T}(F(x_0)-F_{\text{inf}})$
$F_{\text{inf}} = \inf_{x\in\mathbb{R}^n} F(x)$

F	Criterion $\le \epsilon$	$N \sim$	$T_k \sim$
μ-sc L-sm	$\mathbb{E}[F(\bar x_N)] - F(x_0)$	$1/\epsilon$	$1/k$
L-sm	$\mathbb{E}[\frac{1}{N}\sum_{k=0}^{N-1}\|\nabla F(x_k)\|^2]$	$1/\epsilon^2$	$1/\sqrt{k}$
Lip, cv	$\mathbb{E}[F(\bar x_N)] - F(x_0)$	$1/\epsilon^2$	$1/\sqrt{k}$

8 ADMM

Parallelization $\min_{x\in\mathbb{R}^n}\sum_{i=1}^m f_i(x_i)$ s.t. $x_1 = \cdots = x_m$

Dual ascent

Consider: $\min_{x\in\mathbb{R}^n} f(x)$ s.t. $Ax=b, A\in\mathbb{R}^{m\times n}$
Derive dual: $\mathcal{L}(x,\lambda) = f(x) + \lambda^\top Ax - \lambda^\top b$

$\inf_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda) = \underbrace{-\sup_{x\in\mathbb{R}^n}\{(-\lambda^\top A)x - f(x)\}}_{-f^\star(-A^\top\lambda)} - \lambda^\top b$

Dual can be stated as: $\sup_{\lambda\in\mathbb{R}^m} \underbrace{-f^\star(-A^\top\lambda) - \lambda^\top b}_{:=d(\lambda)}$

Subgradient of d given by: $\partial d(\lambda) = A\partial f^\star(-A^\top\lambda) - b$
Recall $v\in\partial f^\star(u) \Leftrightarrow u \in \partial f(v)$ which means that the optimizer in $-\sup_{x\in\mathbb{R}^n}\{(-\lambda^\top A)x - f(x)\}$ satisfies:

$$-A^\top\lambda \in \partial f(x^\star) \Leftrightarrow x^\star\in\partial f^\star(-A^\top\lambda)$$

As a Result, the subgradient $\partial d(\lambda)$ can be expressed via

$$\partial d(\lambda) = Ax - b, \text{where } x \in \mathrm{argmin}_{\hat x\in\mathbb{R}^n}\{f(\hat x) + \hat x^\top A^\top\lambda\}$$

Dual Subgradient Method

$$\begin{aligned} x_k &\in \mathrm{argmin}_{\hat x\in\mathbb{R}^n}\{f(\hat x) + \hat x^\top A^\top\lambda_k\} \\ \lambda_{k+1} &= \lambda_k + T_k(Ax_k - b), \quad T_k > 0\end{aligned}$$

Example 1

XXX
$\lambda_{k+1,i} = \lambda_{k,i} + T_k(x_{k_i} - x_{k_{i+1}})$

Example 2

$f(x=)\sum_{i=1}^m f_i(x_i)$ with $Ax=b$
$x=(x_1,\dots,x_n)$ and $A = [A_1,\dots,A_m]$
Dual subgradient becomes
$x_{k_i} \in \underset{\hat{x_i}}{\mathrm{argmin}}\{f_i(\hat{x_i}) + \lambda_k^\top A_i \hat{x_i}\}$ (local minimization)
$\lambda_{k+1} = \lambda_k + T_k(\sum_{i=1}^m A_i x_{k_i} - b)$ (broadcasting)
IMAGE
Proposition 13. f convex with closed epigraph, f is μ-strongly convex if and only if $f^\star$ is $1/\mu$-smooth.

Derive ADMM

$$\text{Idea: } \min_{x\in\mathbb{R}^n} f(x) + \frac{\rho}{2}|Ax-b|^2 \quad \text{s.t. } Ax=b \text{ with } \rho>0$$

Leads to this Augmented Lagrangian

$$\begin{aligned} x_k &= \mathrm{argmin}_{x\in\mathbb{R}^n} f(x) + \lambda_k^\top Ax + \frac{\rho}{2}|Ax-b|^2 \\ \lambda_{k+1} &= \lambda_k + T_k(Ax_k - b) \quad (\text{typically } T_k=\rho)\end{aligned}$$

Advantage Improved convergence properties even if f non-scv
Disadvantage Loose of decomposability/parallelization due to augmentation with quadratic term.
This motivates ADMM which tries to combine the best of both worlds. (Well conditioned minimization and parallelization)
Consider: $\min_{x\in\mathbb{R}^n, z\in\mathbb{R}^m} f(x)+g(z)$ s.t. $Ax+Bz=c$
augmented objective: min $f(x)+g(z)+\frac{\rho}{2}|Ax+Bz-c|^2$
augmented lagrangian: objective + $\lambda^\top(Ax+Bz-c)$

Alternating direction method of multipliers

$$\begin{aligned} x_k &= \underset{x\in\mathbb{R}^n}{\mathrm{argmin}}\mathcal{L}_p(x, z_{k-1}, \lambda_k) \\ z_k &= \underset{z\in\mathbb{R}^m}{\mathrm{argmin}}\mathcal{L}_p(x_k, z, \lambda_k) \\ \lambda_{k+1} &= \lambda_k + \rho(Ax_k + Bz_k - c)\end{aligned}$$

EXAMPLE Images Low/High rank

9 Distributed optimization with ADMM

Goal Solve s.t. each term can be handled by its own processor.

$$\min_{x_1..,x_N,z\in\mathbb{R}^n} \sum_{i=1}^N f_i(x_i) \quad \text{s.t.} \quad x_i = z \quad (f_i \text{ convex}) \quad (2)$$

Global Consensus Problem

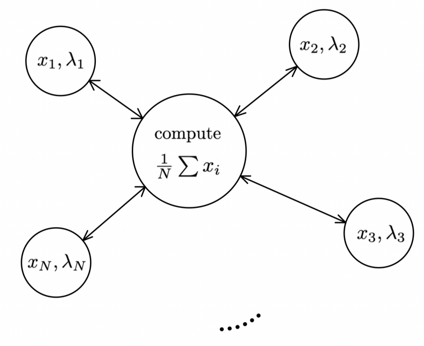
Step 1: Augmented Lagrangian $i\in[1,N]$ to solve (2) with ADMM.

$$\begin{aligned}\mathcal{L}_p(x_i,..,\lambda_i) &= \sum_{i=1}^N f_i(x_i) + \lambda_i^\top(x_i - z) + \frac{\rho}{2}|x_i - z|^2 \\ &= \sum_{i=1}^N f_i(x_i) + \frac{\rho}{2}|x_i - z + \frac{1}{\rho}\lambda_i|^2 - \frac{1}{2\rho}|\lambda_i|^2\end{aligned}$$

Step 2: Formulate ADMM

$$\begin{aligned} x_i^{k+1} &= \underset{x_i\in\mathbb{R}^n}{\mathrm{argmin}} f_i(x_i) + \frac{\rho}{2}|x_i - z^k + \frac{1}{\rho}\lambda_i^k|^2 \\ z^{k+1} &= \underset{z\in\mathbb{R}^n}{\mathrm{argmin}} \frac{\rho}{2}\sum_{i=1}^N |x_i^{k+1} - z + \frac{1}{\rho}\lambda_i^k|^2 \\ &= \frac{1}{N}\sum_{i=1}^N(x_i^{k+1} + \frac{1}{\rho}\lambda_i^k) \\ \lambda_i^{k+1} &= \lambda_i^k + \rho(x_i^{k+1} - z^{k+1})\end{aligned}$$

FURTHER REFORMULATIONS...



Sharing Problem

$$\min_{x_1,..,x_N\in\mathbb{R}^n}\sum_{i=1}^N f_i(x_i) + g\left(\sum_{i=1}^N x_i\right) \quad (3)$$

$\to$ copy all the variables $x_i = z_i$
$\to$ formulate augmented Lagrangian
$\to$ state ADMM dynamics

Optimization over Graphs

$g=(V,E)$ undirected graph with vertices V and edges E

$$\begin{aligned}\min_{x\in\mathbb{R}^n}\sum_{i\in V} f_i(x) \Rightarrow &\min_{x_i\in|V|, z_i\in|E|}\sum_{i\in V}^N f_i(x_i) \\ &\text{s.t. } x_i = z_{ij}, x_j = z_{ij} \quad \forall(i,j)\in E\end{aligned}$$

Step 1: Augmented Lagrangian
Step 2: Form the Algorithm
ALGORITHM

10 Signal denoising and regression

Linear equation $y = Ax, y\in\mathbb{R}^n, x\in\mathbb{R}^m, A\in\mathbb{R}^{m\times n}$
classic setting $m \gg n$ | **modern** setting $m \ll n$ or $m\sim n$

Classic setting with outliers

l_2-norm: $\min_{x\in\mathbb{R}^n}|Ax-y|_2^2$ results in heavy impact for outliers
l_1-norm: $\min_{x\in\mathbb{R}^n}|Ax-y|$ can be reformulated and solved as
LP: $\min_{z\in\mathbb{R}^m} z^\top\mathbf{1}$ s.t. $-z\le Ax-y\le z, 0\le z$

Combined: $\phi_{\text{Hub}}(u) = \begin{cases} u^2 & \text{if } |u|\le M \\ 2Mu - M^2 & \text{if } |u| > M\end{cases}$

Modern setting

$Ax=y$ infinite many solutions $\to$ add regulizer to find best
Tikhonov regression: $\min_{x\in\mathbb{R}^n}|Ax-y|_2^2 + \lambda|x|_2^2$
Least Absolute Shrinkage and Selection Operator
LASSO: $\min_{x\in\mathbb{R}^n}|Ax-y|_2^2 + \lambda|x|_1$
$\Leftrightarrow \min_{x\in\mathbb{R}^n}|Ax-y|_2^2$ s.t. $|x|_1\le c$ results in sparse solution

11 Classification

Setup Dataset with pairs of $(\tilde x_i, \tilde y_i),\ i = 1,...,N$ with data $\tilde x_i\in\mathbb{R}^n$ and class $\tilde y_i\in\{1,2,...,K\}$
Naive approach, linear regression
Slightly improved with probabilistic approach
Linear Discriminant Analysis
SVM aims to maximize margin of decision boundary

12 Adaptive decision-making

:D