Model Predictive Control

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github.com/silvasta/summary-mpc

Contents

1 Introduction

Requirements for MPC

1. A model of the system
2. A state estimator
3. Define the optimal control problem
4. Set up the optimization problem
5. Get the optimal control sequence (solve the optimization problem)
6. Verify that the closed-loop system performs as desired

1.1 Exact ODE solution of a Linear System

$$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}B^c u(\tau)d\tau$$

Problem Most physical systems are nonlinear

Idea use First Order Taylor expansion $f(\bar{x}) + \frac{\partial f}{\partial x^\top}\Big|_{\bar{x}}(x-\bar{x})$

1.2 Linearization

$$\begin{aligned} \dot{x_s} &= g(x_s,u_s) = 0 & \Delta\dot{x} &= \dot{x} - \dot{x_s} = A^c\Delta x + B^c\Delta u \\ y_s &= h(x_s,u_s) & \Delta y &= y - y_s = C\Delta x + D\Delta u \end{aligned}$$

$$A^c = \frac{\partial g}{\partial x^T}\Big|_{\substack{x_s\\u_s}} \quad B^c = \frac{\partial g}{\partial u^T}\Big|_{\substack{x_s\\u_s}} \quad C = \frac{\partial h}{\partial x^T}\Big|_{\substack{x_s\\u_s}} \quad D = \frac{\partial h}{\partial u^T}\Big|_{\substack{x_s\\u_s}}$$

1.3 Discretization

For general nonlinear systems only approximate discretization methods exist, such as Euler, quality depends on sampling time

Approximation **Notation**

$$\dot{x}^c \approx \frac{x^c(t+T_s) - x^c(t)}{T_s} \qquad \begin{aligned} x(k) &:= x^c(t_0+kT_s) \\ u(k) &:= u^c(t_0+kT_s) \end{aligned}$$

Exact Discretization of Linear Time-Invariant Models

$$x(t_{k+1}) = \underbrace{e^{A^cT_s}}_{=A}x(t_k) + \underbrace{\int_0^{T_s} e^{A^c(T_s-\tau)}B^c d\tau}_{B=(A^c)^{-1}(A-\mathbb{I})B^c}\, u(t_k)$$

$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i B u(k+N-1-i)$$

1.4 Analysis of LTI Discrete-Time Systems

Controllabe if $\text{rank}(\mathcal{C}) = n,\ \mathcal{C} = \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}$

$\forall (x(0), x^*) \exists$ finite time N with inputs $\mathcal{U}$, s.t. $x(N) = x^*$

Stabilizable iff all uncontrollable modes stable

Observable if $\text{rank}(\mathcal{O}) = n,\ \begin{bmatrix} C^\top & \cdots & (CA^{n-1})^\top \end{bmatrix}^\top$

$\forall x(0) \exists$ finite time N, s.t. the measurements

$y(0), \ldots, y(N-1)$ uniquely distinguish initial state $x(0)$

Detectablitiy iff all unobservable modes stable

1.5 Lyapunov

Stability is a property of an **equilibrium point** $\bar{\mathbf{x}}$ of a system

Definition 1 (Lyapunov Stability). $\bar{\mathbf{x}}$ is **Lyapunov stable** if:

$\forall\, \epsilon > 0\, \exists\, \delta(\epsilon)$ s.t. $||x(0) - \bar{x}|| < \delta(\epsilon) \rightarrow ||x(k) - \bar{x}|| < \epsilon$

Definition 2 (Globally asymptotic stability). If $\bar{\mathbf{x}}$ is Lyapunov stable and attractive, i.e., $\lim_{k\to\infty} ||x(k) - \bar{x}|| = 0,\ \forall x(0)$ then $\bar{\mathbf{x}}$ is **globally asymptotic stable**.

Definition 3 (Global Lyapunov function). For $\bar{\mathbf{x}} = 0$, function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is called **Lyapunov function** if it is continuous at the origin, finite $\forall\, x \in \mathbb{R}^n$,

$$||x|| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$$

$$V(x) > 0\, \forall\, x \in \mathbb{R}^n \setminus \{0\} \quad V(0) = 0$$

$$V(g(x)) - V(x) \leq -\alpha(x) \quad \forall x \in \mathbb{R}^n$$

where $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ continuous positive definite

Lyapunov Theorem

Theorem 1. If a system admits a Lyapunov function $V(x)$, then $\bar{\mathbf{x}} = 0$ is **globally asymptotically stable**.

Theorem 2 (Lyapunov indirect method). For linearization of system around $\bar{\mathbf{x}} = 0$ and resulting matrix $A = \frac{\partial g}{\partial x^T}\Big|_{x=0}$ with eigenvalues

$$|\lambda_i| := \begin{cases} \forall i := |\lambda_i| < 1 & \text{x=0 is asymptotically stable} \\ \exists i := |\lambda_i| > 1 & \text{origin is unstable} \\ \exists i := |\lambda_i| = 1 & \text{no info about stability} \end{cases}$$

Discrete-Time Lyapunov equation

$$A^T P A - P = -Q, \quad Q \succ 0$$

Theorem 3 (Existence of solution of DT Lyapunov equation). The discrete-time Lyapunov equation (3) has a unique solution P > 0 if and only if A has all eigenvalues inside the unit circle, i.e. if and only if the system x(k + 1) = Ax(k) is stable.

1.6 Optimal Control

Unconstrained Finite Horizon Control Problem

$$\begin{aligned} J^\star(x(0)) := \min_U\ & x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i \\ \text{subject to } & x_{i+1} = Ax_i + Bu_i \quad i = 0, \ldots, N-1 \\ & x_0 = x(0) \end{aligned}$$

$P \succeq 0$, with $P = P^T$ terminal weight

$Q \succeq 0$, with $Q = Q^T$ state weight

$R \succ 0$, with $R = R^T$ input weight

1.7 Batch Approach

expresses cost function in terms of $x(0)$ and input sequence U

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \mathbb{I} \\ A \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & \cdots & 0 \\ B & 0 & 0 \\ AB & B & 0 \\ \vdots & \ddots & 0 \\ A^{N-1}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$\overline{Q} :=$ blockdiag$(Q, \ldots, Q, P)$ $\quad \overline{R} :=$ blockdiag$(R, \ldots, R)$

Optimal Input (from $\nabla_U J(x(0), U) = 2HU + 2F^\top x(0) = 0$)

$$U^\star(x(0)) = -\underbrace{\left((\mathcal{S}^u)^\top \overline{Q}\mathcal{S}^u + \overline{R}\right)}_{H\text{(Hessian)}^{-1}} \underbrace{(\mathcal{S}^u)^\top \overline{Q}\mathcal{S}^x}_{F^\top} x(0)$$

Optimal Cost

$$J^\star(x(0)) = x(0)^\top(\mathcal{S}_x^\top \overline{Q}\mathcal{S}_x - \mathcal{S}_x^\top \overline{Q}\mathcal{S}_u(\mathcal{S}_u^\top \overline{Q}\mathcal{S}_u + \overline{R})^{-1}\mathcal{S}_u^\top \overline{Q}\mathcal{S}_x)x(0)$$

1.8 Recursive Approach

uses dynamic programming to solve problem backwards from N

$$J_j^\star(x(j)) := \min_{U_{j\to N}} x_N^\top P x_N + \sum_{i=j}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$

Ricatti Equations

RDE - Riccati Difference Equation

$$P_i = A^\top P_{i+1} A + Q - A^\top P_{i+1} B (B^\top P_{i+1} B + R)^{-1} B^\top P_{i+1} A$$

RDE - Riccati Difference Equation solved recursively

$$P_i = A^\top P_{i+1} A + Q - A^\top P_{i+1} B (B^\top P_{i+1} B + R)^{-1} B^\top P_{i+1} A$$

ARE - Algebraic Riccati Equation solved analytically

$$P_\infty = A^\top P_\infty A + Q - A^\top P_\infty B (B^\top P_\infty B + R)^{-1} B^\top P_\infty A$$

From Principle Of Optimality **Optimal Cost-To-Go**

$$J_j^\star(x_j) = \min_{u_j} I(x_i, u_i) + J_{j+1}^\star(x_{j+1}) \qquad J_i^\star(x_i) = x_i^\top P_i x_i$$

Optimal Control Policy

$$u_i^\star = F_i x_i = -(B^\top P_{i+1} B + R)^{-1} B^\top P_{i+1} A \cdot x(i)$$

1.9 Comparison of Batch and Recursive Approaches

Batch optimization returns sequence $U^\star(x(0))$ of **numeric values** depending only on x(0), dynamic programming yields **feedback policies** $u_i^\star = F_i x_i$ depending on each x_i.

Choice of P

1. Match infinite solution, use ARE
2. Assume no control needed after N, use Lyapunov Equation (makes only sense when asymptotically stable, otherwise P not positive definite)
3. set constraint $x_{i+N} = 0$

1.10 Infinite Horizon LQR

LQR

$$J_\infty^\star(x(k)) = \min \sum_{i=0}^{\infty} x_i^\top Q x_i + u_i^\top R u_i$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i, \quad x_0 = x(k)$$

Same u as for finite problem but with ARE Constant Feedback Matrix $F\infty$ asymptotically stable for.. Q,R,stabi,detect

1.11 Optimization

A mathematical optimization problem is generally formulated as:

Mathematical Optimization Problem

Decision variable $x \in \mathbb{R}^n$
Objecvtce function $f : \text{dom}(f) \to \mathbb{R}$
Inequality constraints g_i $(i \in \#\text{constraints})$
Equality constraints h_i $(i \in \#\text{constraints})$
Fesable set $\mathcal{X} := \{x | g(x) \leq 0, h(x) = 0\}$

minimize $f(x)$
subject to:
$g_i(x) \leq 0$
$h_i(x) = 0$

Feasible point $x \in \text{dom}(f)$ with $g_i(x) \leq 0,\ h_i(x) = 0$
Strictly feasible point x with strict inequality $g_i(x) < 0$
Optimal value $f^\star$ (or $p^\star$) $= \inf\{f(x) | g_i(x) \leq 0, h_j = 0\}$
$f^\star = +\infty$: OP infeasible, $f^\star = -\infty$: OP unbound below
Optimizer set: $\text{argmin}_{x \in \mathcal{X}}\ f(x) := \{x \in \mathcal{X} | f(x) = f^\star\}$

$x^\star$ is a **Global Minimum** if $f(x^\star) \leq f(x)$
$x^\star$ is a **Local Minimum** if $\exists\ \epsilon > 0$ s.t. $f(x^\star) \leq f(x)$ $\forall x \in \mathcal{X} \cap B_\epsilon(x^\star)$, open ball with center $x^\star$ and radius ϵ

1.12 Convex Sets

Definition 4 (Convex Set). Set $\mathcal{C}$ is convex if and only if

$$\theta x + (1 - \theta) y \in \mathcal{C},\ \forall\, x, y \in \mathcal{C},\ \forall\, \theta \in [0, 1]$$

Definition 5 (Hyperplanes). $\{x \in \mathbb{R}^n \mid a^\mathsf{T} x = b\}$
Definition 6 (Halfspaces). $\{x \in \mathbb{R}^n \mid a^\mathsf{T} x \leq b\}$
can be **open** (strict inequality) or **closed** (non-strict inequality)
Definition 7 (Polyhedra). intersection of **finite** number of closed halfspaces: polyhedra $\{x \in \mathbb{R}^n \mid A^{q \times n} x \preceq b^{q \times 1}, \}$
Definition 8 (Polytope). is a **bounded** polyhedron.
Definition 9 (Convex hull). for $\{v_1, \ldots, v_k\} \in \mathbb{R}^d$ is:
$\text{co}(\{v_1, \ldots, v_k\}) := \{x | x = \sum_i \lambda_i v_i, \lambda \geq 0, \sum_i \lambda_i = 1\}$
Definition 10 (Ellipsoid). set: $\{x | (x - x_c)^\mathsf{T} A^{-1} (x - x_c) \leq 1\}$ where x_c is center of ellipsoid, $A \succ 0$ (i.e. positive definite) (Semi-axis lengths are square roots of eigenvalues of A)
Definition 11 (Norm Ball). $B_r(x) := \{\xi \in \mathbb{R}^n : |\xi - x|_p < r\}$ where p defines the l_p norm, $p = \{1|2|..|\infty\}$
Intersection $\mathcal{C}_1, \mathcal{C}_2$ cv $\Rightarrow \mathcal{C}_1 \cap \mathcal{C}_2$ convex **(cv)**
Image under affine map $\mathcal{C} \subseteq \mathbb{R}^n$ cv $\Rightarrow \{Ax + b \mid x \in \mathcal{C}\}$ cv
Inverse IoaM $\mathcal{C} \subseteq \mathbb{R}^m$ cv $\Rightarrow \{x \in \mathbb{R}^n \mid Ax + b \in \mathcal{C}\}$ cv

1.13 Convex Functions

Definition 12 (Convex Function). $f : \mathcal{C}_{cv} \to \mathbb{R}$ is convex iff

$$f(\theta x + (1 - \theta) y) \leq \theta f(x) + (1 - \theta) f(y),\ \forall\, x, y \in \mathcal{C},\ \forall\, \theta \in [0, 1]$$

f is strictly convex if this inequality is strict.
Definition 13 (Epigraph). $f : \mathbb{R}^n \to \mathbb{R}$ cv $\Leftrightarrow$ epi(f) is cv set

$$\text{epi}(f) := \{(x, t) \in \mathbb{R}^{n+1} | f(x) \leq t\}$$

Check Convexity f is convex if it is composition of simple convex function with convexity preserving operations or if
$f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable, $\partial^2 f / \partial x^2 \succeq 0\ \forall\, x \in \mathbb{R}^n$
$g : \mathbb{R} \to \mathbb{R}$ with $g(t) = f(x + tv)$ convex in $t\ \forall\, x, v \in \mathbb{R}^n$ $\to f$ convex (restriction to a line)
- the point wise maximum of convex functions is convex
- the sum of convex functions is convex
- $f(Ax + b)$ is convex if f is convex

Theorem 4. For a convex optimization problem, **any** locally optimal solution is globally optimal (local optima are global optima).

Linear Programming minimize $c^\mathsf{T} x$ s.t. $Ax - b \geq 0,\ x \geq 0$
Step 1: $\mathcal{L}(x, \lambda_1, \lambda_2) = c^\mathsf{T} x - \lambda_1^\mathsf{T}(Ax - b) - \lambda_2^\mathsf{T} x,\ \lambda_i \geq 0$
Step 2: $\inf_{x \in \mathbb{R}^n} \mathcal{L} = \lambda_1^\mathsf{T} b$, if $c - A^\mathsf{T} \lambda_1 - \lambda_2 = 0$, else $-\infty$
Step 3: Dual, maximize $b^\mathsf{T} \lambda$ s.t. $c - A^\mathsf{T} \lambda \geq 0, \lambda \geq 0$ (again LP)
Quadratic Programming min ...

1.14 Optimality Conditions

Lagrange Duality

Consider $f^\star = \inf_{x \in \mathbb{R}^n} f(x)$ s.t. $g(x) \leq 0,\ h(x) = 0$

Lagrangian $\mathcal{L}(x, \lambda, \nu) = f(x) + \lambda^\mathsf{T} g(x) + \nu^\mathsf{T} h(x)$

Dual Function $d(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \nu)$

Proposition 1 (Weak Duality). $d(\lambda, \nu) \leq f^\star, \forall \lambda \geq 0, \nu \in \mathbb{R}^h$
Definition 14 (Constraint qualification). **Slaters Condition** holds if $\exists$ at least one strictly feasible point $\hat{x}$ $(h(\hat{x}) = 0,\ g(\hat{x}) < 0)$
Proposition 2 (Strong Duality). If Slater's condition holds and OP is convex $\Rightarrow \exists \lambda \geq 0, \nu \in \mathbb{R}^{n_h}$ s.t. $d(\lambda, \nu) = f^\star$

KKT Conditions (Karush-Kuhn-Tucker)

Theorem 5 (KKT Conditions). If Slater's condition holds and (1.14) is convex $\to x^\star \in \mathbb{R}^n$ is a minimizer of the primal (1.14) and $(\lambda^\star \geq 0, \nu^\star) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ is a maximizer of the dual $\Leftrightarrow$ is equivalent to the following statements:

KKT-1 (Stationary Lagrangian) $\nabla_x \mathcal{L}(x^\star, \lambda^\star, \nu^\star) = 0$
KKT-2 (primal feasibility) $g(x^\star) \leq 0, h(x^\star) = 0$
KKT-3 (dual feasibility) $\lambda^\star, \nu^\star \in \mathbb{R}^{n_h} \geq 0$
KKT-4 (compementary slackness) $\lambda^{\star T} g(x^\star) = 0$, $\nu^{\star T} h(x^\star) = 0$

In addition we have: $\sup_{\lambda \geq 0, \nu \in \mathbb{R}^{n_h}} q(\lambda, \nu) = \inf_{x \in \mathcal{C}} f(x)$

Remark Without Slater, KKT1-4 still implies $x^\star$ minimizes (1.14) and λ, ν maximizes dual, but the converse is no longer true. There can be primal-minimizer/dual-maximizer not satisfy KKT.

2 Nominal-MPC

2.1 CFTOC

CFTOC Constrained Finite Time Optimal Control problem

$$J(x(k)) = x_N^\mathsf{T} P x_N + \sum_{i=0}^{N-1} x_i^\mathsf{T} Q x_i + u_i R u_i$$

$$J^\star(x(k)) = \min_U I_f(x_N) + \sum_{i=0}^{N-1} I(x_i, u_i)$$

s.t $x_{i+1} = Ax_i + Bu_i,\ i = 0, \ldots, N - 1$
$x_i \in \mathcal{X},\ u_i \in \mathcal{U},\ \mathcal{X}_N \in \mathcal{X}_f,\ x_0 = x(k)$

N is the time horizon and X , U, Xf are polyhedral regions

2.2 Transform Quadratic Cost CFTOC into QP

Goal $\min_{z \in \mathbb{R}^n} \frac{1}{2} z^\top H z + q^\top z + r$ s.t. $Gz \leq h,\ Az = b$

2.2.1 Substitute without substitution

Idea Keep state equations as equality constraints
Define variable $z = [x_1^\top \quad \ldots x_N^\top \quad u_0^\top \quad \ldots u_{N-1}^\top]^\top$
Equalities from system dynamics $x_{i+1} = Ax_i + Bu_i$

$$G_{eq} = \left[\begin{array}{cccc|ccc} \mathbb{I} & & & & -B & & \\ -A & \mathbb{I} & & & & \ddots & \\ & \ddots & \ddots & & & & \\ & & -A & \mathbb{I} & & & -B \end{array}\right] E_{eq} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Inequalities $G_{in} z \leq w_{in} + E_{in} x(k)$ from $\mathcal{X} = \{x \mid A_x x \leq b_x\}, \mathcal{U} = \{u \mid A_u u \leq b_u\}, \mathcal{X}_f = \{x \mid A_f x \leq b_f\}$

$$G_{in} = \left[\begin{array}{cccc|cccc} 0 & & & & 0 & & & \\ \hline A_x & & & & 0 & & & \\ & \ddots & & & & \ddots & & \\ & & A_x & & & & 0 & \\ & & & A_f & & & & 0 \\ \hline 0 & & & & A_u & & & \\ & \ddots & & & & \ddots & & \\ & & 0 & & & & A_u & \\ & & & 0 & & & & A_u \end{array}\right] w_{in} = \left[\begin{array}{c} b_x \\ \hline b_x \\ \vdots \\ b_x \\ b_f \\ \hline b_u \\ \vdots \\ b_u \\ b_u \end{array}\right]$$

$$E_{in} = \begin{bmatrix} -A_x \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Cost Matrix $\bar{H} = \text{diag}(Q, \ldots, Q, P, R, \ldots, R)$
Finally the resulting quadratic optimization problem

$$J^\star(x(k)) = \min_z \left[z^\top\ x(k)^\top\right] \left[\begin{smallmatrix} \bar{H} & 0 \\ 0 & Q \end{smallmatrix}\right] \left[z^\top\ x(k)^\top\right]^\top$$

s.t $G_{in} z \leq w_{in} + E_{in} x(k) \quad G_{eq} z = E_{eq} x(k)$

2.2.2 Substitute with substitution

Idea Substitute the state equations.
Step 1 Rewrite cost as

$$\begin{aligned} J(x(k)) &= UXXXX \\ &= [U^\top \quad x(k)^\top] \left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] [U^\top \quad x(k)^\top]^\top \end{aligned}$$

Step 2 Rewrite constraints compaclty as $GU \leq w + Ex(k)$
Step 3 Rewrite constrained problem as

$$J^\star(x(k)) = \min_U [U^\top \quad x(k)^\top] \left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] [U^\top \quad x(k)^\top]^\top$$

subj. to $GU \leq w + Ex(k)$

2.3 Invariance

Definition 15 (Positively Invariant Set $\mathcal{O}$). For an autonomous or closed-loop system, the set $\mathcal{O}$ is positively invariant if:

$$x(k) \in \mathcal{O} \Rightarrow x(k+1) \in \mathcal{O}, \quad \forall k \in \{0, 1, \ldots\}$$

Definition 16 (Maximal Positively Invariant Set $\mathcal{O}_\infty$). A set that contains all $\mathcal{O}$ is the maximal positively invariant set $\mathcal{O}_\infty \subset \mathcal{X}$
Definition 17 (Pre-Sets). The set of states that in the dynamic system $x(k+1) = g(x(k))$ in one time step evolves into the target set $\mathcal{S}$ is the **pre-set** of $\mathcal{S} \Rightarrow$ pre$(\mathcal{S}) := \{x \mid g(x) \in \mathcal{S}\}$

Theorem 6 (Geometric condition for invariance). Set $\mathcal{O}$ is positively invariant set iff $\mathcal{O} \subseteq$ pre$(\mathcal{O}) \Leftrightarrow$ pre$(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$

Proof. **Necessary** if $\mathcal{O} \not\subseteq$ pre$(\mathcal{O})$, then $\exists \bar{x} \in \mathcal{O}$ s.t $\bar{x} \notin$ pre$(\mathcal{O}) \rightsquigarrow \bar{x} \in \mathcal{O}, \bar{x} \notin$ pre$(\mathcal{O})$, thus $\mathcal{O}$ not positively invariant
Sufficient if $\mathcal{O}$ not pos invar set, then $\exists \bar{x} \in \mathcal{O}$ s.t $g(\bar{x}) \notin \mathcal{O}$ $\rightsquigarrow \bar{x} \in \mathcal{O}, \bar{x} \notin$ pre$(\mathcal{O})$ thus $\mathcal{O} \notin$ pre$(\mathcal{O})$ $\square$

Computing Invariant Sets

Pre-Set Computation

System with constraints
$x(k+1) = Ax(k) + Bu(k)$
$u(k) \in \mathcal{U} := \{u | Gu \leq g\}$
and set $\mathcal{S} := \{x | Fx \leq f\}$
pre$(\mathcal{S}) := \{x \mid Ax \in \mathcal{S}\}$
$= \{x \mid FAx \leq f\}$

Conceptual Algorithm

first line
$\Omega_0 \leftarrow \mathcal{X}$
loop
$\Omega_{i+1} \leftarrow$ pre$(\Omega_i) \cap \Omega_i$
if $\Omega_{i+1} = \Omega_i$ **then**
return
$\mathcal{O}_\infty = \Omega_i$
end if
end loop

(Same but much harder for control invariat sets)

Conceptual Algorithm

first line
$\Omega_0 \leftarrow \mathcal{X}$
loop
$\Omega_{i+1} \leftarrow$ pre$(\Omega_i) \cap \Omega_i$
if $\Omega_{i+1} = \Omega_i$ **then**
return $\mathcal{O}_\infty = \Omega_i$
end if
end loop

(Same but much harder for control invariat sets)

Conceptual Algorithm

```
@decorator()
# Example Python code
def hello_world():
  # This is a comment
    print("Hello, World!")
```

(Same but much harder for control invariat sets)

2.4 Control Invariance

Definition 18 (Control Invariant Set). $\mathcal{C} \subseteq \mathcal{X}$ control invariant if

$$x(k) \in \mathcal{C} \Rightarrow \ \exists u(k) \in \mathcal{U} \text{ s.t } g(x(k), u(k)) \in \mathcal{C}\ \forall k$$

Definition 19 (Maximal Control Invariant Set $\mathcal{C}_\infty$). A set that contains all $\mathcal{C}$ is the maximal positively invariant set $\mathcal{C}_\infty \subset \mathcal{X}$
Intuition For all states in $\mathcal{C}_\infty$ exists control law s.t constraints are never violated $\rightsquigarrow$ **The best any controller could ever do**
Pre-set pre$(\mathcal{S}) := \{x \mid \exists u \in \mathcal{U}$ s.t $g(x, u) \in \mathcal{S}\}$
Set $\mathcal{C}$ is control invariant iff: $\mathcal{C} \subseteq$ pre$(\mathcal{C}) \Leftrightarrow$ pre$(\mathcal{C}) \cap \mathcal{C} = \mathcal{C}$

Control Law from Control Invariant Set

Let $\mathcal{C}$ control invariant set for $x(k+1) = g(x(k), u(k))$
Control law $\kappa(x(k))$ will **guarantee** that system satisfies constraints **for all time** if: $g(x, \kappa(x)) \in \mathcal{C} \; \forall x \in \mathcal{C}$
We can use this fact to **synthesize** control law κ with f as any function (including $f(x, u) = 0$)

$$\kappa(x) := \text{argmin}\{f(x, u) \mid g(x, u) \in \mathcal{C}\}$$

Does not ensure that system will converge
Difficult because calculating control invariant sets is hard
MPC implicitly describes $\mathcal{C}$ s.t easy to represent/compute

Theorem 7. Minkowski-Weyl
The following statements are equivalent for $\mathcal{P} \subseteq \mathbb{R}^d$

- $\mathcal{P}$ is a polytope and there exists A, b s.t $\mathcal{P} = \{x \mid Ax \le b\}$
- $\mathcal{P}$ finitely generated, $\exists$ finite set $\{v_i\}$ s.t $\mathcal{P} = \text{co}(\{v_1, ..., v_s\})$

MOST COMMON Polytopic

1

Lemma 1. Invariant Sets from Lyapunov Functions
If $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for $x(k+1) = g(x(k))$, then $Y := \{x \mid V(x) \le \alpha\}$ is an invariant set for all $\alpha \ge 0$

Proof. Lyapunov property $V(g(x)) - V(x) < 0$ implies that once $V(x(k)) \le \alpha, V(x(j)) < \alpha, \forall j \ge k \to$ Invariance □

Example System for $x(k+1) = Ax(k)$ with $P \succ 0$ that satisfies $A^\top PA - P \prec 0 \rightsquigarrow$ then $V(x(k)) = x(k)^\top Px(k)$ is Lyap. function

Goal – find largest α s.t set $Y_\alpha \in \mathcal{X}$
$Y_\alpha := \{x \mid x^\top Px \le \alpha\} \subset \mathcal{X} := \{x \mid Fx \le f\}$
Equivalent to $\max_\alpha \alpha$ subj. to $h_{Y_\alpha}(F_i) \le f_i \; \forall i \in \{1 \dots n\}$

2.5 Feasibility and Stability

What can go wrong with "standard" MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution

- No stability guarantee, i.e., trajectories may not converge to the origin

MPC Mathematical Formulation

V1

$$J^\star(x(k)) = \min \sum_{i=0}^{N-1} x_i^\top Qx_i + u_i^\top Ru_i$$

V3

$$J^\star(x(k)) = \min \sum_{i=0}^{N-1} x_i^\top Qx_i + u_i^\top Ru_i$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i \qquad x_0 = x(k) \quad x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}$$

Stability of MPC - Main Result

Theorem 8. The closed-loop system under the MPC control law $u_0^\star(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for the system $x(k+1) = Ax(k) + Bu_0^\star(x(k))$ under the following assumptions:
1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $\kappa_f(x_i)$:

$$x_{i+1} = Ax_i + B\kappa_f(x_i) \in \mathcal{X}_f \forall x_i \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \in X, \kappa_f(x_i) \in U \forall x_i \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$I_f(x_{i+1}) - I_f(x_i) \le -I(x_i, \kappa_f(x_i)) \quad \forall x_i \in \mathcal{X}_f$$

Finite-horizon MPC may not satisfy constraints for all time!
Finite-horizon MPC may not be stable!
- An infinite-horizon provides stability and invariance.
- Infinite-horizon *faked* by forcing final state into an invariant set for which there exists invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- Extends to non-linear systems, but compute sets is difficult!

3 Practical-MPC

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4 Robust-MPC

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5 Implementation

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