Model Predictive Control

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github.com/silvasta/summary-mpc

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Introduction

Requirements for MPC

1. A model of the system
2. A state estimator
3. Define the optimal control problem
4. Set up the optimization problem
5. Get the optimal control sequence (solve the optimization problem)
6. Verify that the closed-loop system performs as desired

1 System Theory Basics

1.1 Learning Objectives

- Describe dynamics with continuous-time state-space models
- Derive linearized model and understand limitations of linear description
- Discretize nonlinear and linear systems and contrast model properties
- Analyse stability, controllability, observability of linear systems
- Understand Lyapunov stability and prove stability of nonlinear systems
- Construct a Lyapunov function for stable linear systems

1.2 Model of Dynamic Systemes

Goal Introduce mathematical models to be used in Model Predictive Control (MPC) for describing the behavior of dynamic systems

- If not stated differently, we use deterministic models
- Models of physical systems derived from first principles are mainly:
nonlinear, time-invariant, continuous-time, state space models (*)
- Target models for standard MPC are mainly:
linear, time-invariant, discrete-time, state space models (†)
- Focus of this section is on how to 'transform' (*) to (†)

1.2.1 Physical Model

Nonlinear Time-Invariant Continuous-Time State Space Models
$\dot{x} = g(x, u)$
$y = h(x, u)$
$g(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ system dynamics
$h(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ output funtion
$x \in \mathbb{R}^n$ state vector
$u \in \mathbb{R}^m$ input vector
$y \in \mathbb{R}^p$ output vector
- Very general class of models
- Higher order ODEs can be easily brought to this form
- Analysis and control synthesis generally hard $\to$ linearization to bring it to linear, time-invariant (LTI), continuous-time, state space form

1.2.2 Continuous LTI-Model

$\dot{x} = A^c x + B^c u$
$y = Cx + Du$
$A^c \in \mathbb{R}^{n\times n}$ system matrix
$B^c \in \mathbb{R}^{n\times m}$ input matrix
$C \in \mathbb{R}^{p\times n}$ output matrix
$D \in \mathbb{R}^{p\times m}$ troughput matrix
$x \in \mathbb{R}^n$ state vector
$u \in \mathbb{R}^m$ input vector
$y \in \mathbb{R}^p$ output vector

Solution to linear ODEs

$$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}B^c u(\tau)d\tau$$

Problem Most physical systems are nonlinear but linear systems are much better understood
Nonlinear systems can be well approximated by a linear system in a *small neighborhood* around a point in state space
Idea Control keeps the system around some operating point $\to$ replace nonlinear by a linearized system around operating point
TAYLOR

Linearization

Stationary operating point: x_s, u_s
$\dot{x_s} = g(x_s, u_s) = 0$
$y_s = h(x_s, u_s)$

$$\dot{x} = \underbrace{g(x_s, u_s)}_{=0} + \underbrace{\frac{\partial g}{\partial x^T}\bigg|_{\substack{x_s\\u_s}}}_{=A^c} \underbrace{(x - x_s)}_{=\Delta x} + \underbrace{\frac{\partial g}{\partial u^T}\bigg|_{\substack{x_s\\u_s}}}_{=B^c} (u$$

$$\Rightarrow \dot{x} - \underbrace{\dot{x_s}}_{=0} = \Delta\dot{x} = A^c\Delta x + B^c\Delta u$$

$$y = \underbrace{h(x_s, u_s)}_{y_s} + \underbrace{\frac{\partial h}{\partial x^T}\bigg|_{\substack{x_s\\u_s}}}_{=C} (x - x_s) + \underbrace{\frac{\partial h}{\partial u^T}\bigg|_{\substack{x_s\\u_s}}}_{=D} (u$$

$$\Rightarrow y - y_s = \Delta y = C\Delta x + D\Delta u$$

Subsequently, instead of Δx, Δu and Δy , x, u and y are used for brevity.

1.2.3 Discrete Models

Discrete-Time systems are describey by difference equations
$x(k + 1) = g(x(k), u(k))$
$y(k) = h(x(k), u(k))$
$g(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ system dynamics
$h(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ output funtion
$x \in \mathbb{R}^n$ state vector

$u \in \mathbb{R}^m$ input vector
$y \in \mathbb{R}^p$ output vector
- Inputs and outputs of a discrete-time system are defined only at discrete time points, i.e. its inputs and outputs are sequences defined for $k \in \mathbb{Z}$
- Discrete time systems describe either
1. Inherently discrete systems, eg. bank savings account balance at the k-th month $x(k+1) = (1+\alpha)x(k) + u(k)$
2. *Transformed* continuous-time system
- Vast majority of controlled systems not inherently discrete-time systems
- Controllers almost always implemented using microprocessors
- Finite computation time must be considered in the control system design $\rightarrow$ discretize the continuous-time system
- Discretization is the procedure of obtaining an 'equivalent' discrete-time system from a continuous-time system
- The discrete-time model describes the state of the continuous-time system only at particular instances $t_k, k \in \mathbb{Z}^+$ in time, where $t_{k+1} = t_k + T_s$ and T_s is called the sampling time
- Usually $u(t) = u(t_k) \forall t \in [t_k, t_k+1)$ is assumed (and implemented)

1.2.4 Discretization

1.2.5 Recap

1.3 Analysis of Discrete-Time LTI-Systems

1.3.1 Coordinate Transform

1.3.2 Stability

1.3.3 Controllability

1.3.4 Observability

1.4 Analysis of Discrete-Time Nonlinear-Systems

1.4.1 Stability

1.4.2 Lyapunov

1.5 Recap

2 Unconstrained LQ Optimal Control

2.1 Learning Objectives

- Learn to compute finite horizon unconstrained linear quadratic optimal controller in two ways
- Understand principle of optimality
- Learn to compute infinite horizon unconstrained linear quadratic optimal controller
- Understand impact of horizon length
- Prove stability of infinite horizon unconstrained linear quadratic optimal control
- Learn how to 'simulate' quasi-infinite horizon

2.2 Introduction to Optimal Control

2.2.1 Optimal Control

Discrete-time **optimal control** is concerned with choosing an optimal input sequence $U := [u_0^T, u_1^T, ...]^T$ (as measured by some objective function), over a finite or infinite time horizon, in order to apply it to a system with a given initial state x(0).
The objective, or cost, function is often defined as a **sum of stage costs** $l(x_i, u_i)$and, when the horizon has finite length N, terminal cost $I_f(x_N)$:

$$J(x_0, U) := I_f(x_N) + \sum_{i=0}^{N-1} I((x_i, u_i)$$

The states $\{x_i\}_{i=0}^N$ must satisfy the system dynamics

$$x_{i+1} = g(x_i, u_i) \quad i = 0, \dots, N-1$$
$$x_0 = x(0)$$

and there may be state and/or input constraints

$$h(x_i, u_i) \leq 0 \quad i = 0, \dots, N-1$$

In the finite horizon case, there may also be a constraint that the final state x_N lies in a set $\mathcal{X}_f$
A general finite horizon optimal control formulation for discrete-time systems is therefore

$$\begin{aligned} J^\star(x(0)) &:= \min_U J(x(0), U) \\ \text{subject to } & x_{i+1} = g(x_i, u_i) \quad i = 0, \dots, N-1 \\ & h(x_i, u_i) \leq 0 \quad i = 0, \dots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(0) \end{aligned}$$

2.2.2 Linear Quadratic Optimal Control

In this section, only **linear** discrete-time time-invariant systems

$$x(k+1) = Ax(k) + Bu(k)$$

and quadratic cost functions

$$J(x(0)) := x_N^\top P x_N + \sum_{i=0}^{N-1} (x_i^\top Q x_i + u_i^\top R u_i)$$

are considered, and we consider only the problem of regulating the state to the origin, **without state or input constraints**.
The two most common solution approaches will be described here
1. **Batch Approach**, which yields a series of **numerical values** for the input
2. **Recursive Approach**, which uses Dynamic Programming to compute control **policies** or **laws**, i.e. functions that describe how the control decisions depend on the system states.

2.2.3 Unconstrained Finite Horizon Control Problem

Goal Find a sequence of inputs $U := [u_0^T, ..., u_{N-1}^T]^T$ that minimizes the objective function

$$\begin{aligned} J^\star(x(0)) &:= \min_U x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i \\ \text{subject to } & x_{i+1} = Ax_i + Bu_i \quad i = 0, \dots, N-1 \\ & x_0 = x(0) \end{aligned}$$

$P \succeq 0$, with $P = P^T$ terminal weight
$Q \succeq 0$, with $Q = Q^T$ state weight
$R \succ 0$, with $R = R^T$ input weight
N horizon length
Note that $x(0)$ is the current state, whereas $x_0, \dots, x_N$ and $x_0, \dots, x_{N-1}$ are optimization variables that are constrained to obey the system dynamics and the initial condition.

2.2.4 Batch Approach

The batch solution explicitly represents all future states x_i in terms of initial condition x_0 and inputs $u_1, \dots, u_{N-1}$
Starting with $x_0 = x(0)$, we have $x_1 = Ax(0) + Bu_0$ and $x_2 = Ax_1 + Bu_1 = A^2x(0) + ABu_0 + Bu_1$, by substitution for $x1$, and so on. Continuing up to x_N we obtain:

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \mathbb{I} \\ A \\ \vdots \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}$$

The equation above can be represented as
$X := S^x x(0) + S^u U$
Cost
$\overline{Q} :=$ blockdiag$(Q, \dots, Q, P)$
$\overline{R} :=$ blockdiag$(R, \dots, R)$
Summary
The Batch Approach expresses the cost function in terms of the initial state $x(0)$ and input sequence U by eliminating the states x_i.
Because the cost $J(x(0), U)$ is a positive definite quadratic function of U, its minimizer $U^\star$ is unique and can be found by setting $\nabla_U J(x(0), U) = 0$
This gives the optimal input sequence $U^\star$ as a linear function of the initial state $x(0)$.
Optimal Input

$$U^\star(x(0)) = -\underbrace{\left((\mathcal{S}^u)^\top \overline{Q} \mathcal{S}^u + \overline{R}\right)}_{H\text{(Hessian)}^{-1}} \underbrace{(\mathcal{S}^u)^\top \overline{Q} \mathcal{S}^x}_{F^\top} x(0)$$

Optimal Cost
$J^\star(x(0)) = x(0)^\top (\mathcal{S}_x^\top \overline{Q} \mathcal{S}_x - \mathcal{S}_x^\top \overline{Q} \mathcal{S}_u (\mathcal{S}_u^\top \overline{Q} \mathcal{S}_u + \overline{R})^{-1} \mathcal{S}_u^\top \overline{Q} \mathcal{S}_x) x(0)$
Note If there are state or input constraints, solving this problem by matrix inversion is not guaranteed to result in a feasible input sequence

2.2.5 Recursive Approach

Alternatively, we can use dynamic programming to solve the same problem in a recursive manner.
Define the j-step optimal cost-to-go as the optimal cost attainable for the step j problem:

$$\begin{aligned} J_j^\star(x(j)) &:= \min_{U_{j \to N}} x_N^\top P x_N + \sum_{i=j}^{N-1} x_i^\top Q x_i + u_i^\top R u_i \\ \text{subject to } & x_{i+1} = Ax_i + Bu_i \quad i = 0, \dots, N-1 \\ & x_j = x(j) \end{aligned}$$

This is the minimum cost attainable for the remainder of the horizon after step j
Start at time $N-1$
Iterate backwards
Optimal Control Policy

$$u_i^\star = -(B^\top P_{i+1} B + R)^{-1} B^\top P_{i+1} A \cdot x(i) := F_i x_i$$

Optimal Cost-To-Go

$$J_i^\star(x_i) = x_i^\top P_i x_i$$

Riccati Difference Equation (RDE)

$$P_i = A^\top P_{i+1} A + Q - A^\top P_{i+1} B (B^\top P_{i+1} B + R)^{-1} B^\top P_{i+1} A$$

Find any P_i by recursive evaluation from $P_N = P$ (the given terminal weight)
Evaluating down to P_0, we obtain the N-step cost-to-go

$$J^\star(x(0)) = J_0^\star(x(0)) = x(0)^\top P_0 x(0)$$

The recursive solution method used from here relies on Bellman's **Principle of Optimality**
For any solution for steps 0 to N to be optimal, any solution for steps j to N with $j \geq 0$, taken from the 0 to N solution, must itself be optimal for the j-to-N problem
Therefore we have, for any $j = 0, \dots, N$

$$\begin{aligned} J_j^\star(x_j) &= \min_{u_j} I(x_i, u_i) + J_{j+1}^\star(x_{j+1}) \\ \text{subject to } & x_{j+1} = Ax_j + Bu_j \end{aligned}$$

Interpretation
Suppose that the fastest route from Los Angeles to Boston passes through Chicago. Then the principle of optimality formalizes the obvious fact that the Chicago to Boston portion of the route is also the fastest route for a trip that starts from Chicago and ends in Boston.

2.2.6 Comparison of Batch and Recursive Approaches

- Fundamental difference: Batch optimization returns a sequence $U^\star(x(0))$ of **numeric values** depending only on the initial state x(0), while dynamic programming yields **feedback policies** $u_i^\star = F_i x_i, i = 0, \dots, N-1$ depending on each x_i.
- If the state evolves exactly as modelled, then the sequences of control actions obtained from the two approaches are identical.
- The recursive solution should be more robust to disturbances and model errors, because if the future states later deviate from their predicted values, the exact optimal input can still be computed.
- The Recursive Approach is computationally more attractive because it breaks the problem down into single-step problems. For large horizon length, the Hessian H in the Batch Approach, which must be inverted, becomes very large.
- Without any modification, both solution methods will break down when inequality constraints on x_i or u_i are added.
- The Batch Approach is far easier to adapt than the Recursive Approach when constraints are present: just perform a constrained minimization for the current state.

2.3 Receding Horizon

Receding horizon strategy introduces feedback.
For unconstrained systems, this is a **constant linear controller**
However, can extend this concept to much more complex systems (MPC)

2.4 Infinite Horizon LQR

2.4.1 Infinite Horizon Control Problem

2.4.2 Stability of Infinite Horizon LQR

2.4.3 Choices of Terminal Weight P in Finite Horizon Control

2.4.4 Choices of Terminal Weight P in Finite Horizon Control

Goal Control law to minimize relative *energy* of input and state/output
Why?
- Easy to describe objective / tune controller
- Simple to compute and implement
- Proven and effective
Why infinite-horizon?
- Stable
- Optimal solution (doesn't usually matter)
In MPC we normally cannot have an infinite horizon because it results in an infinite number of optimization variables.
Use *tricks* to *simulate* quasi-infinite horizon.

3 Optimization

3.1 Learning Objectives

- Learn to 'read' and define optimization problems
- Understand property of convexity of sets and functions
- Understand benefit of convex optimization problems
- Learn and contrast properties of LPs and QPs
- Pose the dual problem to a given primal optimization problem
- Test optimality of a primal and dual solution by means of KKT conditions
- Understand meaning of dual solution for the cost function

3.2 Main Concepts

A mathematical optimization problem is generally formulated as:

$$\begin{aligned} &\min_{x \in \mathrm{dom}(f)} f(x) \\ &\text{s.t. } g_i(x) \leq 0, \quad i = 1, \dots, m \\ &\qquad h_i(x) = 0, \quad i = 1, \dots, p \end{aligned} \tag{1}$$

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ decision variable
- $f : \mathrm{dom}(f) \to \mathbb{R}$ objectivce function
- $\mathcal{X} = \{x \in \mathbb{R}^n : g(\xi) \leq 0,\ h(\xi) = 0\}$ fesabile set

3.3 Convex Sets

Definition 1 (Convex Set). A set $\mathcal{C}$ is convex if and only if

$$\theta x + (1-\theta) y \in \mathcal{C} \forall\, x, y \in \mathcal{C}, \quad \forall\, \theta \in [0,1]$$

(hyperplane || half-space) $\{x \in \mathbb{R}^n \mid a^\top x (=||\leq) b\}$
polyhedra $\{x \in \mathbb{R}^n \mid A^{q\times n} x \preceq b^{q\times 1}, C^{r\times n} x = d^{r\times 1}\}$
polytope
Intersection $\mathcal{C}_1, \mathcal{C}_2$ cv $\Rightarrow \mathcal{C}_1 \cap \mathcal{C}_2$ convex **(cv)**
Image under affine map $\mathcal{C} \subseteq \mathbb{R}^n$ cv $\Rightarrow \{Ax + b \mid x \in \mathcal{C}\}$ cv
Inverse IoaM $\mathcal{C} \subseteq \mathbb{R}^m$ cv $\Rightarrow \{x \in \mathbb{R}^n \mid Ax + b \in \mathcal{C}\}$ cv

3.4 Convex Functions

Check Convexity f is convex if it is composition of simple convex function with convexity preserving operations or if
$f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable, $\partial^2 f / \partial x^2 \succeq 0\ \forall\, x \in \mathbb{R}^n$
$g : \mathbb{R} \to \mathbb{R}$ with $g(t) = f(x + tv)$ convex in $t\ \forall\, x, v \in \mathbb{R}^n$
$\to f$ convex (restriction to a line)

3.4.1 Operations that preserve convexity (functions)

- the point wise maximum of convex functions is convex
- the sum of convex functions is convex
- $f(Ax+b)$ is convex if f is convex

3.5 Convex Optimization Problems

Optimal value $f^\star = \inf\{f(x) \mid g_i(x) \leq 0, h_j = 0\}$
$f^\star = +\infty$ OP is infeasible, $f^\star = -\infty$ OP is unbound below

3.5.1 Linear Programming

minimize $c^\top x$ s.t. $Ax - b \geq 0,\ x \geq 0$
Step 1: $\mathcal{L}(x, \lambda_1, \lambda_2) = c^\top x - \lambda_1^\top (Ax - b) - \lambda_2^\top x,\ \lambda_i \geq 0$
Step 2: $\inf_{x \in \mathbb{R}^n} \mathcal{L} = \lambda_1^\top b$, if $c - A^\top \lambda_1 - \lambda_2 = 0$, else $-\infty$
Step 3: Dual, maximize $b^\top \lambda$ s.t. $c - A^\top \lambda \geq 0, \lambda \geq 0$ (again LP)

3.5.2 Quadratic Programming

3.6 Optimality Conditions

We consider $f^\star = \inf_{x \in \mathbb{R}^n} f(x)$ s.t. $g(x) \leq 0, h(x) = 0$ (2)

Lagrange $\mathcal{L}(x, \lambda, \nu) = f(x) + \lambda^\top g(x) + \nu^\top h(x)$
Dual Function $d(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \nu)$

3.6.1 Weak and Strong Duality

Proposition 1 (Weak Duality). $d(\lambda, \nu) \leq f^\star, \forall \lambda \geq 0, \nu \in \mathbb{R}^h$
Definition 2 (Constraint qualification). $\mathcal{C}$ convex, **Slaters Condition** holds if $\exists\, \hat{x} \in \mathbb{R}^n$ s.t. $h(\hat{x}) = 0$ and $g(\hat{x}) < 0$
Proposition 2 (Strong Duality). If Slater's condition holds and (4) is convex $\Rightarrow \exists \lambda \geq 0, \nu \in \mathbb{R}^{n_h}$ s.t. $d(\lambda, \nu) = f^\star$

3.6.2 KKT (Karush-Kuhn-Tucker) Conditions

Theorem 1 (KKT Conditions). Slater's condition holds and (4) is convex $\to x^\star \in \mathbb{R}^n$ is a minimizer of the primal (4) and $(\lambda^\star \geq 0, \nu^\star) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ is a maximizer of the dual $\Leftrightarrow$

$\nabla_x \mathcal{L}(x^\star, \lambda^\star, \nu^\star) = 0$	KKT-1 (Stationary Lagrangian)
$g(x^\star) \leq 0, h(x^\star) = 0$	KKT-2 (primal feasibility)
$\lambda^\star \geq 0, \nu^\star \in \mathbb{R}^{n_h}$	KKT-3 (dual feasibility)
$\lambda^{\star T} g(x^\star) = 0 = \nu^{\star T} h(x^\star)$	KKT-4 (compementary slackness)

In addition we have: $\sup_{\lambda \geq 0, \nu \in \mathbb{R}^{n_h}} q(\lambda, \nu) = \inf_{x \in \mathcal{C}} f(x)$
Remark Without Slater, KKT1-4 still implies $x^\star$ minimizes (4) and λ, ν maximizes dual, but the converse is no longer true. There can be primal-minimizer/dual-maximizer not satisfy KKT.

3.7 Summary

- Convex optimization problem:
- Convex cost function
- Convex inequality constraints
- Affine equality constraints
- Benefit of convex problems: Local = Global optimality
- Only need to find one minimum, it is the global minimum!
- For convex optimization problem: If slater condition holds, x □ optimal iff □(λ□ , v □) satisfying KKT conditions - Convex optimization problems can be solved efficiently
- Many problems can be written as convex opt. problems (with some effort)

Note: Duality and optimality conditions similarly extend to Convex Cone Programs

3.7.1 Why did we consider the dual problem?

- The dual problem is convex, even if the primal is not –> can be 'easier' to solve than primal
- The dual problem provides a lower bound for the primal problem: d □ ≤ p□ (and d(λ, v) ≤ p(x) for all feasible x, λ, v) (provides suboptimality bound)
- The dual provides a certificate of optimality via the KKT conditions for convex problems
- KKT conditions lead to efficient optimization algorithms
- Lagrange multipliers provide information about active constraints at the optimal solution: if λ□i > 0, then gi (x □) = 0
- Lagrange multipliers provide information about sensitivity of optima

4 Introduction

4.1 Exact Solution

$x(t) = e^{A^c(t-t_0)} x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)} B^c u(\tau) d\tau$
TAYLOR

4.2 Linearization

$\dot{x_s} = g(x_s, u_s) = 0\ y_s = h(x_s, u_s)$
$\Delta \dot{x} = \dot{x} - \dot{x_s} = A^c \Delta x + B^c \Delta u$
$\Delta y = y - y_s = C \Delta x + D \Delta u$
$A^c = \frac{\partial g}{\partial x^T}\Big|_{\substack{x_s \\ u_s}}\ B^c = \frac{\partial g}{\partial u^T}\Big|_{\substack{x_s \\ u_s}}\ C = \frac{\partial h}{\partial x^T}\Big|_{\substack{x_s \\ u_s}}\ D = \frac{\partial h}{\partial u^T}\Big|_{\substack{x_s \\ u_s}}$

4.3 Discretization

4.4 Lyapunov

4.5 Optimal Control

Unconstrained Finite Horizon Control Problem

$$\begin{aligned} J^\star(x(0)) := \min_U x_N^\top P x_N + \textstyle\sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i \\ \text{subject to } x_{i+1} = A x_i + B u_i \quad i = 0, \dots, N-1 \\ x_0 = x(0) \end{aligned}$$

4.6 Batch Approach

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \mathbb{I} \\ A \\ \vdots \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & \cdots & 0 \\ B & 0 & 0 \\ AB & B & 0 \\ \vdots & \ddots & 0 \\ A^{N-1}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}$$

Cost
$\overline{Q} := \mathrm{blockdiag}(Q, \dots, Q, P)$
$\overline{R} := \mathrm{blockdiag}(R, \dots, R)$
Optimal Input

$$U^\star(x(0)) = -\underbrace{\left((\mathcal{S}^u)^\top \overline{Q} \mathcal{S}^u + \overline{R}\right)}_{H(\text{Hessian})^{-1}} \underbrace{(\mathcal{S}^u)^\top \overline{Q} \mathcal{S}^x}_{F^\top} x(0)$$

Optimal Cost
$J^\star(x(0)) = x(0)^\top (\mathcal{S}_x^\top \overline{Q} \mathcal{S}_x - \mathcal{S}_x^\top \overline{Q} \mathcal{S}_u (\mathcal{S}_u^\top \overline{Q} \mathcal{S}_u + \overline{R})^{-1} \mathcal{S}_u^\top \overline{Q} \mathcal{S}_x) x(0)$

4.7 Recursive Approach

4.8 Infinite Horizon LQR

$$\begin{aligned} J_\infty^\star(x(k)) = \min \sum_{i=0}^{\infty} x_i^\top Q x_i + u_i^\top R u_i \\ \text{subj. to } x_{i+1} = A x_i + B u_i \\ x_0 = x(k) \end{aligned}$$

5 Optimization

A mathematical optimization problem is generally formulated as:

$$\begin{aligned} &\min_{x \in \mathrm{dom}(f)} f(x) \\ &\text{s.t. } g_i(x) \leq 0, \quad i = 1, \dots, m \\ &\qquad h_i(x) = 0, \quad i = 1, \dots, p \end{aligned} \tag{3}$$

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ decision variable
- $f : \mathrm{dom}(f) \to \mathbb{R}$ objectivce function
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5.1 Convex Sets

Definition 3 (Convex Set). A set $\mathcal{C}$ is convex if and only if

$$\theta x + (1-\theta) y \in \mathcal{C} \forall\, x, y \in \mathcal{C}, \quad \forall\, \theta \in [0,1]$$

(hyperplane || half-space) $\{x \in \mathbb{R}^n \mid a^\top x (=||\leq) b\}$
polyhedra $\{x \in \mathbb{R}^n \mid A^{q\times n} x \preceq b^{q\times 1}, C^{r\times n} x = d^{r\times 1}\}$
polytope
Intersection $\mathcal{C}_1, \mathcal{C}_2$ cv $\Rightarrow \mathcal{C}_1 \cap \mathcal{C}_2$ convex **(cv)**
Image under affine map $\mathcal{C} \subseteq \mathbb{R}^n$ cv $\Rightarrow \{Ax + b \mid x \in \mathcal{C}\}$ cv
Inverse IoaM $\mathcal{C} \subseteq \mathbb{R}^m$ cv $\Rightarrow \{x \in \mathbb{R}^n \mid Ax + b \in \mathcal{C}\}$ cv

5.2 Convex Functions

Check Convexity f is convex if it is composition of simple convex function with convexity preserving operations or if
$f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable, $\partial^2 f / \partial x^2 \succeq 0\ \forall\, x \in \mathbb{R}^n$
$g : \mathbb{R} \to \mathbb{R}$ with $g(t) = f(x + tv)$ convex in $t\ \forall\, x, v \in \mathbb{R}^n$
$\to f$ convex (restriction to a line)

5.2.1 Operations that preserve convexity (functions)

- the point wise maximum of convex functions is convex
- the sum of convex functions is convex
- $f(Ax+b)$ is convex if f is convex

5.3 Convex Optimization Problems

Optimal value $f^\star = \inf\{f(x) \mid g_i(x) \leq 0, h_j = 0\}$
$f^\star = +\infty$ OP is infeasible, $f^\star = -\infty$ OP is unbound below

5.3.1 Linear Programming

minimize $c^\top x$ s.t. $Ax - b \geq 0,\ x \geq 0$
Step 1: $\mathcal{L}(x, \lambda_1, \lambda_2) = c^\top x - \lambda_1^\top (Ax - b) - \lambda_2^\top x,\ \lambda_i \geq 0$
Step 2: $\inf_{x \in \mathbb{R}^n} \mathcal{L} = \lambda_1^\top b$, if $c - A^\top \lambda_1 - \lambda_2 = 0$, else $-\infty$
Step 3: Dual, maximize $b^\top \lambda$ s.t. $c - A^\top \lambda \geq 0, \lambda \geq 0$ (again LP)

5.3.2 Quadratic Programming

5.4 Optimality Conditions

Lagrange Duality

We consider ...

$$f^\star = \inf_{x \in \mathbb{R}^n} f(x) \ \text{ s.t. } \ g(x) \leq 0, h(x) = 0 \tag{4}$$

Lagrange $\mathcal{L}(x, \lambda, \nu) = f(x) + \lambda^\top g(x) + \nu^\top h(x)$
Dual Function $d(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \nu)$

Lagrange $\mathcal{L}(x, \lambda, \nu) = f(x) + \lambda^\top g(x) + \nu^\top h(x)$
Dual Function $d(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \nu)$

5.4.1 Weak and Strong Duality

Proposition 3 (Weak Duality). $d(\lambda, \nu) \leq f^\star, \forall \lambda \geq 0, \nu \in \mathbb{R}^h$
Definition 4 (Constraint qualification). $\mathcal{C}$ convex, **Slaters Condition** holds if $\exists\, \hat{x} \in \mathbb{R}^n$ s.t. $h(\hat{x}) = 0$ and $g(\hat{x}) < 0$
Proposition 4 (Strong Duality). If Slater's condition holds and (4) is convex $\Rightarrow \exists \lambda \geq 0, \nu \in \mathbb{R}^{n_h}$ s.t. $d(\lambda, \nu) = f^\star$

KKT Conditions (Karush-Kuhn-Tucker)

Theorem 2 (KKT Conditions). If Slater's condition holds and (4) is convex $\to x^\star \in \mathbb{R}^n$ is a minimizer of the primal (4) and $(\lambda^\star \geq 0, \nu^\star) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ is a maximizer of the dual $\Leftrightarrow$ is equivalent to the following statements:

KKT-1 (Stationary Lagrangian)	$\nabla_x \mathcal{L}(x^\star, \lambda^\star, \nu^\star) = 0$
KKT-2 (primal feasibility)	$g(x^\star) \leq 0, h(x^\star) = 0$
KKT-3 (dual feasibility)	$\lambda^\star, \nu^\star \in \mathbb{R}^{n_h} \geq 0$
KKT-4 (compementary slackness)	$\lambda^{\star T} g(x^\star) = 0$ $\nu^{\star T} h(x^\star) = 0$

In addition we have: $\sup_{\lambda \geq 0, \nu \in \mathbb{R}^{n_h}} q(\lambda, \nu) = \inf_{x \in \mathcal{C}} f(x)$

Remark Without Slater, KKT1-4 still implies $x^\star$ minimizes (4) and λ, ν maximizes dual, but the converse is no longer true. There can be primal-minimizer/dual-maximizer not satisfy KKT.

6 Nominal-MPC

6.1 Constrained Infinite Time Optimal Control

what we would like to solve

$$J^\star_\infty(x(0)) = \min_U \sum_{i=0}^{\infty} I(x_i, u_i)$$
$$\text{s.t} \quad x_{i+1} = Ax_i + Bu_i,\ i = 0, \dots, \infty$$
$$x_i \in \mathcal{X},\ u_i \in \mathcal{U},\ x_0 = x(0)$$

- Stage cost $I(x, u)$ *cost* of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties... ... but we can't compute it: there are an **infinite number of variables**

what we can sometimes solve

$$J^\star(x(k)) = \min_U I_f(x_N) + \sum_{i=0}^{N-1} I(x_i, u_i)$$
$$\text{s.t} \quad x_{i+1} = Ax_i + Bu_i,\ i = 0, \dots, N-1$$
$$x_i \in \mathcal{X},\ u_i \in \mathcal{U},\ \mathcal{X}_N \in \mathcal{X}_f,\ x_0 = x(k)$$

Truncate after a finite horizon:
- $I_f(x_N)$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

6.2 Learning Objectives

- Understand feasible set of constrained finite horizon optimal control (CFTOC) problem
- Write quadratic cost CFTOC as QP
- Write 1-$/\infty$-norm cost CFTOC as LP
- Contrast properties of LP and QP solution

6.3 Constrained Linear Optimal Control

Cost Funcion

Squared Euclidian Norm

$$J(x(k)) = x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i R u_i$$

N is the time horizon and X , U, Xf are polyhedral regions

Feasible Set

Set of states for which the optimal control problem is feasible: XN = x0 □ Rn | □(u0 , . . . , uN−1) such that xi □ X , ui □ U, i = 0, . . . , N − 1, xN □ Xf , where xi+1 = Axi + Bui

Note: XN is independent of the cost.

6.4 Constrained Optimal Control: Quadratic cost

6.4.1 Transform Quadratic Cost CFTOC into QP

Transform CFTOC as defined into QP of the following form:

$$\min_{z \in \mathbb{R}^n} \tfrac{1}{2} z^\top H z + q^\top z + r \quad \text{s.t. } Gz \le h,\ Az = b$$

6.4.2 Without Substitution

Idea Keep state equations as equality constraints (often more efficient)

$$J^\star(x(k)) = \min_z \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix}^\top$$
$$\text{s.t} \quad G_{in} z \le w_{in} + E_{in} x(k)$$
$$G_{eq} z = E_{eq} x(k)$$

Define variable $z = \begin{bmatrix} x_1^\top & \dots x_N^\top & u_0^\top & \dots u_{N-1}^\top \end{bmatrix}^\top$

Equalities from system dynamics $x_{i+1} = Ax_i + Bu_i$

$$G_{eq} = \left[\begin{array}{ccc|ccc} \mathbb{I} & & & -B & & \\ -A & \mathbb{I} & & & -B & \\ & -A & \mathbb{I} & & & -B \\ & \cdots & \cdots & & & \cdots \end{array}\right], E_{eq} = \begin{bmatrix} A \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

Inequalities $G_{in} z \le w_{in} + E_{in} x(k)$ for $\mathcal{X}, \mathcal{U}, \mathcal{X}_f$

$$\mathcal{X} = \{x \mid A_x x \le b_x\}, \mathcal{U} = \{u \mid A_u u \le b_u\}, \mathcal{X}_f = \{x \mid A_f x \le b_f\}$$

$$G_{in} = \left[\begin{array}{cccc|cccc} 0 & & & & 0 & & & \\ \hline A_x & & & & 0 & & & \\ & A_x & & & & 0 & & \\ & & \dots & & & & \dots & \\ & & & A_f & & & & 0 \\ \hline 0 & & & & A_u & & & \\ & 0 & & & & A_u & & \\ & & \dots & & & & \dots & \\ & & & 0 & & & & A_u \end{array}\right] w_{in} = \begin{bmatrix} b_x \\ \hline b_x \\ b_x \\ \cdots \\ b_f \\ \hline b_u \\ b_u \\ \cdots \\ b_u \end{bmatrix} E_{in} = \begin{bmatrix} -A_x^T \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

6.4.3 Without Substitution

Step 1

Step 2

Step 3

6.4.4 Quadratic Cost State Feedback Solution

6.5 Constrained Optimal Control: 1-Norm and ∞-Norm Cost

6.6 Receding Horizon Control Notation

7 Invariance

7.1 Objectives of Constrained Control

7.2 Learning Objectives

- Learn definition and meaning of invariance

(Region in which an autonomous system satisfies the constraints for all time)

- Learn definition and meaning of control invariance

(Region for which there exists a controller so that the system satisfies the constraints for all time)

- Learn how to (conceptually) compute these sets
- Learn how to compute polytopic and ellipsoidal invariant sets

7.3 Invariance

Invariance: Which states are "good"?

7.4 Control Invariance

Control Invariance: Does a good input exist?

7.4.1 Relation to MPC

- A control invariant set is a powerful object
- If one can compute one, it provides a direct method for synthesizing a control law that will satisfy constraints - The maximal control invariant set is the best any controller can do!!!

So why don't we always compute them? We can't...

- Constrained linear systems : Often too complex - (Constrained) nonlinear system : (Almost) always too complex

$\Rightarrow$ MPC implicitly describes a control invariant set such that it's easy to represent and compute.

7.5 Summary Invariant Set

- Core component of MPC problem
- Special case: Linear System / Polyhedral Constraints
– Polyhedral invariant set
– Can represent the maximum invariant set
– Can be complex (many inequalities) for more than □ 5 - 10 states
– Resulting MPC optimization will be a quadratic program
– Ellipsoidal invariant set
– Smaller than polyhedral (not the maximal invariant set)
– Easy to compute for large dimensions
– Fixed complexity
– Resulting MPC optimization will be a quadratically constrained quadratic program

Special case: Linear system, polyhedral constraints.

- Very difficult to compute
- Very complex
- Very useful

7.6 Practical Computation of Invariant Sets

7.6.1 Polytopes

7.6.2 Ellipsoids

8 Feasibility and Stability

8.1 Learning Objectives

- Contrast stability properties of LQR and MPC for constrained problems
- Understand why MPC by itself does not provide guarantees on stability and constraint satisfaction
- Design MPC with closed-loop stability and constraint satisfaction
- State sufficient conditions
- Engineer terminal ingredients
- Understand main proof idea

8.2 MPC: Key Points Illustrated

MPC Mathematical Formulation

$$J^\star(x(k)) = \min \textstyle\sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$
$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$
$$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}$$
$$x_0 = x(k)$$

Example Saturation

LQR with saturated inputs unstable

MPC with input constraint not convergent to steady state but to limit cycle

MPC + with rate constraints converges

8.3 Loss of Feasibility and Stability in MPC

What can go wrong with "standard" MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution
- No stability guarantee, i.e., trajectories may not converge to the origin

Example: Feasibility and stability are function of tuning

8.3.1 Summary: Feasibility and Stability

Infinite-Horizon

If we solve the RHC problem for $N = \infty$ (as done for LQR), then the open loop trajectories are the same as the closed loop trajectories. Hence

- If problem is feasible, the closed loop trajectories will be always feasible
- If the cost is finite, then states and inputs will converge asymptotically to the origin

Finite-Horizon

RHC is "short-sighted" strategy approximating infinite horizon controller. But

- Feasibility. After some steps the finite horizon optimal control problem may become infeasible. (Infeasibility occurs without disturbances and model mismatch!)
- Stability. The generated control inputs may not lead to trajectories that converge to the origin.

8.3.2 Feasibility and stability in MPC - Solution

Main idea Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

8.4 Feasibility and Stability Guarantees in MPC

8.4.1 Lyapunov Stability

8.4.2 Feasibility and Stability of MPC: Proof

Main steps:
- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Two cases:

1. Terminal constraint at zero: $x_N = 0$
2. Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

General notation: (terminal,stage cost)

Zero terminal constraint

Recursive feasibility

Lyapunov Stability

Terminal set constraint

Problem: The terminal constraint $x_N = 0$ reduces the size of the feasible set

Goal: Use convex set $\mathcal{X}_f$ to increase the region of attraction

8.4.3 Stability of MPC - Main Result

Theorem 3. The closed-loop system under the MPC control law $u_0^\star(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for the system $x(k+1) = Ax(k) + Bu_0^\star(x(k))$ under the following assumptions:

1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $\kappa_f(x_i)$:

$$x_{i+1} = Ax_i + B\kappa_f(x_i) \in \mathcal{X}_f \text{ for all } x_i \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \in X, \kappa_f(x_i) \in U \text{ for all } x_i \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$I_f(x_{i+1}) - I_f(x_i) \le -I(x_i, \kappa_f(x_i)) \quad \forall x_i \in \mathcal{X}_f$$

8.4.4 Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for feasibility and stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attraction for small N
- Solutions available for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N, region of attraction approaches maximum control invariant set

8.5 Extension to Nonlinear MPC

Consider nonlinear system dynamics: $x(k+1) = g(x(k), u(k))$
- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems

$\Rightarrow$ Results can be directly extended to nonlinear systems. However, computing the sets $\mathcal{X}_f$ and function I_f can be very difficult!

8.6 Summary

Finite-horizon MPC may not be stable!
Finite-horizon MPC may not satisfy constraints for all time!
- An infinite-horizon provides stability and invariance.
- We 'fake' infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.

9 Nominal-MPC

9.1 CFTOC

CFTOC Constrained Finite Time Optimal Control problem

$$J(x(k)) = x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i R u_i$$

$$J^\star(x(k)) = \min_U I_f(x_N) + \sum_{i=0}^{N-1} I(x_i, u_i)$$

$$\text{s.t} \quad x_{i+1} = Ax_i + Bu_i,\ i = 0, \dots, N-1$$
$$x_i \in \mathcal{X},\ u_i \in \mathcal{U},\ \mathcal{X}_N \in \mathcal{X}_f,\ x_0 = x(k)$$

N is the time horizon and X , U, Xf are polyhedral regions

9.2 Invariance

9.3 Feasibility and Stability

The first reason is that re-optimization provides robustness to any noise or modeling errors, while the second is that the solution at time k = 0 is sub-optimal because it is over a finite horizon. Re-optimizing can provide a control law with better performance.

MPC Mathematical Formulation

$$J^\star(x(k)) = \min \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$
$$x_0 = x(k) \quad x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}$$

9.3.1 Stability of MPC - Main Result

Theorem 4. The closed-loop system under the MPC control law $u_0^\star(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for the system $x(k+1) = Ax(k) + Bu_0^\star(x(k))$ under the following assumptions:
1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $\kappa_f(x_i)$:

$$x_{i+1} = Ax_i + B\kappa_f(x_i) \in \mathcal{X}_f \forall x_i \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \in X, \kappa_f(x_i) \in U \forall x_i \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$I_f(x_{i+1}) - I_f(x_i) \leq -I(x_i, \kappa_f(x_i)) \quad \forall x_i \in \mathcal{X}_f$$

Finite-horizon MPC may not satisfy constraints for all time!
Finite-horizon MPC may not be stable!
- An infinite-horizon provides stability and invariance.
- Infinite-horizon *faked* by forcing final state into an invariant set for which there exists invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- Extends to non-linear systems, but compute sets is difficult!

10 Practical-MPC

11 Practical-MPC

12 Robust-MPC

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13 Robust-MPC

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14 Implementation

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15 Implementation

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