Model Predictive Control

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github.com/silvasta/summary-mpc

Requirements and Steps to MPC

1. **Model of the System** dynamics to state space
2. **State Estimator** track trajectory and disturbance
3. **Optimal Control Problem** define strategy
4. **Optimization problem** mathematical formulation
5. **Get Optimal Control Sequence** solve optimization
6. **Verify Closed-Loop Performance** iterative tests

1 Introduction to Systems and Controls

Idea Create a model by solving the systems physical equations

$$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}B^c u(\tau)d\tau$$

(Exact Solution to ODE of a Linear System)

Problem Most physical systems are nonlinear

Trick use First Order Taylor expansion $f(\bar{x}) + \frac{\partial f}{\partial x^T}\Big|_{\bar{x}}(x-\bar{x})$

1.1 Linearization

Idea Nonlinear system stable enough around an equilibrium

System equations $\dot{x}_s = g(x_s, u_s) = 0, y_s = h(x_s, u_s)$

Find stationary operating point x_s, u_s and plug in derivative:

$$\begin{aligned} \Delta\dot{x} &= \dot{x} - \dot{x}_s & A^c &= \frac{\partial g}{\partial x^T}\Big|_{\substack{x_s\\u_s}} & B^c &= \frac{\partial g}{\partial u^T}\Big|_{\substack{x_s\\u_s}} \\ &= A^c\Delta x + B^c\Delta u \\ \Delta y &= y - y_s & C &= \frac{\partial h}{\partial x^T}\Big|_{\substack{x_s\\u_s}} & D &= \frac{\partial h}{\partial u^T}\Big|_{\substack{x_s\\u_s}} \\ &= C\Delta x + D\Delta u \end{aligned}$$

1.2 Discretization

For general nonlinear systems only approximate discretization methods exist, such as Euler, quality depends on sampling time

Approximation

$$\dot{x}^c \approx \frac{x^c(t+T_s) - x^c(t)}{T_s}$$

Notation

$$x(k) := x^c(t_0 + kT_s) \qquad u(k) := u^c(t_0 + kT_s)$$

Exact Discretization of Linear Time-Invariant Models

$$x(t_{k+1}) = \underbrace{e^{A^cT_s}}_{=A} x(t_k) + \underbrace{\int_0^{T_s} e^{A^c(T_s-\tau)}B^c d\tau}_{B=(A^c)^{-1}(A-I)B^c}\, u(t_k)$$

$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i Bu(k+N-1-i)$$

1.3 Analysis of Discrete-Time LTI Systems

Controllable if $\text{rank}(\mathcal{C}) = n$, $\mathcal{C} = [B \quad \cdots \quad A^{n-1}B]$

$\forall(x(0), x^*)\exists$ finite time N with inputs $\mathcal{U}$, s.t. $x(N) = x^*$

Stabilizable iff all uncontrollable modes stable

Observable if $\text{rank}(\mathcal{O}) = n$, $[C^\top \quad \cdots \quad (CA^{n-1})^\top]^\top$

$\forall x(0)\exists$ finite time N, s.t. the measurements

$y(0), \ldots, y(N-1)$ uniquely distinguish initial state $x(0)$

Detectable iff all unobservable modes stable

2 Optimization

Mathematical Optimization Problem

Decision variable $x \in \mathbb{R}^n$
Objective function $f : \text{dom}(f) \to \mathbb{R}$
Inequality constraints g_i ($i\in$#constraints)
Equality constraints h_i ($i\in$#constraints)
Feasible set $\mathcal{X} := \{x | g(x) \le 0, h(x) = 0\}$

minimize $f(x)$
subject to:
$g_i(x) \le 0$
$h_i(x) = 0$

Feasible point $x \in \text{dom}(f)$ with $g_i(x) \le 0,\ h_i(x) = 0$
Strictly feasible point x with strict inequality $g_i(x) < 0$
Optimal value $f^\star$ (or $p^\star$) $= \inf\{f(x) | g_i(x) \le 0, h_j = 0\}$
$f^\star = +\infty$: OP infeasible, $f^\star = -\infty$: OP unbound below
Optimizer set: $\text{argmin}_{x\in\mathcal{X}} f(x) := \{x \in \mathcal{X} | f(x) = f^\star\}$

$x^\star$ is a **Global Minimum** if $f(x^\star) \le f(x)$
$x^\star$ is a **Local Minimum** if $\exists\, \epsilon > 0$ s.t. $f(x^\star) \le f(x)$
$\forall x \in \mathcal{X} \cap B_\epsilon(x^\star)$, open ball with center $x^\star$ and radius ϵ

2.1 Convex Sets

Definition 1 (Convex Set). Set $\mathcal{C}$ is convex if and only if

$$\theta x + (1-\theta)y \in \mathcal{C},\ \forall x, y \in \mathcal{C},\ \forall \theta \in [0,1]$$

Intersection $\mathcal{C}_1, \mathcal{C}_2$ cv $\Rightarrow \mathcal{C}_1 \cap \mathcal{C}_2$ convex **(cv)**

Image under affine map $\mathcal{C} \subseteq \mathbb{R}^n$ cv $\Rightarrow \{Ax + b \mid x \in \mathcal{C}\}$ cv

Inverse IoaM $\mathcal{C} \subseteq \mathbb{R}^m$ cv $\Rightarrow \{x \in \mathbb{R}^n \mid Ax + b \in \mathcal{C}\}$ cv

Definition 2 (Hyperplanes). $\{x \in \mathbb{R}^n \mid a^\mathsf{T}x = b\}$

Definition 3 (Halfspaces). $\{x \in \mathbb{R}^n \mid a^\mathsf{T}x \le b\}$

can be **open** (strict inequality) or **closed** (non-strict inequality)

Definition 4 (Polyhedra). intersection of **finite** number of closed halfspaces: polyhedra $\{x \in \mathbb{R}^n \mid A^{q\times n}x \preceq b^{q\times 1}, \}$

Definition 5 (Polytope). is a **bounded** polyhedron.

Definition 6 (Convex hull). for $\{v_1, \ldots, v_k\} \in \mathbb{R}^d$ is:

$\text{co}(\{v_1, \ldots, v_k\}) := \{x | x = \sum_i \lambda_i v_i, \lambda \ge 0, \sum_i \lambda_i = 1\}$

Definition 7 (Ellipsoid). set: $\{x | (x-x_c)^\top A^{-1}(x-x_c) \le 1\}$ where x_c is center of ellipsoid, $A \succ 0$ (i.e. positive definite) (Semi-axis lengths are square roots of eigenvalues of A)

Definition 8 (Norm Ball). $B_r(x) := \{\xi \in \mathbb{R}^n : |\xi - x|_p < r\}$ where p defines the l_p norm, $p = \{1|2|..|\infty\}$

Theorem 1. Minkowski-Weyl

The following statements are equivalent for $\mathcal{P} \subseteq \mathbb{R}^d$

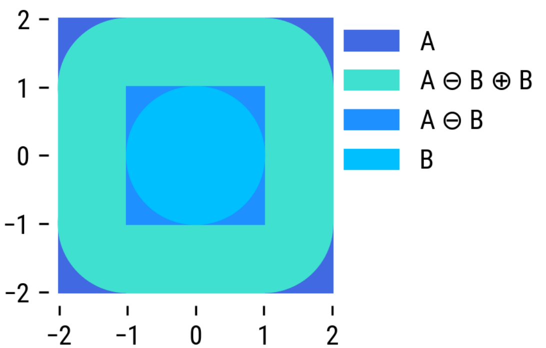
$\mathcal{P}$ is a polytope and there exists A, b s.t $\mathcal{P} = \{x \mid Ax \le b\}$

$\mathcal{P}$ finitely generated, $\exists$ finite set $\{v_i\}$ s.t $\mathcal{P} = \text{co}(\{v_1, \ldots, v_s\})$

Definition 9. **Minkowski Sum** for $A, B \subset \mathbb{R}^n$

$$A \oplus B := \{x + y | x \in A, y \in B\}$$

Example $[a, b] \oplus [c, d] = [a + c, b + d]$



Definition 10. **Pontryagin Difference** for $A, B \subset \mathbb{R}^n$

$$A \ominus B := \{x | x + e \in A, \forall e \in B\}$$

Example $[a, b] \ominus [c, d] = [a - c, b - d]$

2.2 Convex Functions

Definition 11 (Convex Function). $f : \mathcal{C}_{\text{convex}} \to \mathbb{R}$ is convex iff

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y),\ \forall x, y\ \forall \theta \in [0,1]$$

f is strictly convex if this inequality is strict.

Definition 12 (Epigraph). $f : \mathbb{R}^n \to \mathbb{R}$ cv $\Leftrightarrow$ epi(f) is cv set

$$\text{epi}(f) := \{(x,t) \in \mathbb{R}^{n+1} | f(x) \le t\}$$

Check Convexity f is convex if it is composition of simple convex function with convexity preserving operations or if

$f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable, $\partial^2 f/\partial x^2 \succeq 0\ \forall x \in \mathbb{R}^n$

$g : \mathbb{R} \to \mathbb{R}$ with $g(t) = f(x + tv)$ convex in $t\ \forall x, v \in \mathbb{R}^n$ $\to f$ convex (restriction to a line)

- the point wise maximum of convex functions is convex
- the sum of convex functions is convex
- $f(Ax + b)$ is convex if f is convex

2.3 Optimality Conditions

Lagrange Duality

Consider $f^\star = \inf_{x\in\mathbb{R}^n} f(x)$ s.t. $g(x) \le 0, h(x) = 0$ (1)

Lagrangian $\mathcal{L}(x, \lambda, \nu) = f(x) + \lambda^\mathsf{T}g(x) + \nu^\mathsf{T}h(x)$

Dual Function $d(\lambda, \nu) = \inf_{x\in\mathbb{R}^n} \mathcal{L}(x, \lambda, \nu)$

Proposition 1 (Weak Duality). $d(\lambda, \nu) \le f^\star, \forall \lambda \ge 0, \nu \in \mathbb{R}^h$

Definition 13 (Constraint qualification). **Slater's Condition** holds if $\exists$ at least one **strictly feasible point** $\hat{x}$ ($h(\hat{x}) = 0,\ g(\hat{x}) < 0$)

Proposition 2 (Strong Duality). If Slater's condition holds and OP is convex $\Rightarrow \exists \lambda \ge 0, \nu \in \mathbb{R}^{n_h}$ s.t. $d(\lambda, \nu) = f^\star$

KKT Conditions (Karush-Kuhn-Tucker)

Theorem 2 (KKT Conditions). If Slater's condition holds and (1) is convex $\to x^\star \in \mathbb{R}^n$ is a minimizer of the primal (1) and $(\lambda^\star \ge 0, \nu^\star) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ is a maximizer of the dual $\Leftrightarrow$ is equivalent to the following statements:

KKT-1 (Stationary Lagrangian)	$\nabla_x \mathcal{L}(x^\star, \lambda^\star, \nu^\star) = 0$
KKT-2 (primal feasibility)	$g(x^\star) \le 0, h(x^\star) = 0$
KKT-3 (dual feasibility)	$\lambda^\star, \nu^\star \in \mathbb{R}^{n_h} \ge 0$
KKT-4 (complementary slackness)	$\lambda^{\star T} g(x^\star) = 0$ $\nu^{\star T} h(x^\star) = 0$

In addition we have: $\sup_{\lambda\ge 0, \nu\in\mathbb{R}^{n_h}} q(\lambda, \nu) = \inf_{x\in\mathcal{C}} f(x)$

Remark Without Slater, KKT1-4 still implies $x^\star$ minimizes (1) and λ, ν maximizes dual, but the converse is no longer true. There can be primal-minimizer/dual-maximizer not satisfy KKT.

2.4 Convex Optimization Problems

Theorem 3. For a convex optimization problem, **any** locally optimal solution is globally optimal (local optima are global optima).

Linear Programming minimize $c^\mathsf{T}x$ s.t. $Ax - b \ge 0,\ x \ge 0$

Step 1: $\mathcal{L}(x, \lambda_1, \lambda_2) = c^\mathsf{T}x - \lambda_1^\mathsf{T}(Ax - b) - \lambda_2^\mathsf{T}x,\ \lambda_i \ge 0$

Step 2: $\inf_{x\in\mathbb{R}^n} \mathcal{L} = \lambda_1^\mathsf{T}b$, if $c - A^\mathsf{T}\lambda_1 - \lambda_2 = 0$, else $-\infty$

Step 3: Dual, maximize $b^\mathsf{T}\lambda$ s.t. $c - A^\mathsf{T}\lambda \ge 0, \lambda \ge 0$ (again LP)

Proposition 3. The optimal solution of a linear program (if it exists) lies always on the boundary of the feasible set and there exists an optimal solution that is a vertex of the feasible set.

Quadratic Programming convex if $P = P^\mathsf{T}$ positive semi-definite minimize $\frac{1}{2}x^\mathsf{T}Px + q^\mathsf{T}x$ s.t. $Gx \le h,\ Ax = b$

Second-Order Cone Program

minimize $f^\mathsf{T}x$ s.t. $|A_ix + b| \le c_i^\mathsf{T}x + d_i, Fx = g$

Second-order cone $C_{n+1} = \{(x,t) \mid x \in \mathbb{R}^n, t \in \mathbb{R}, |x| \leq t\}$
$|A_i x + b| \leq c_i^\top x + d_i \Leftrightarrow (A_i x + b, c_i^\top x + d_i) \in C_{n+1}$
Semi-Definite Programming with symmetric F_i, X, A_i
minimize $c^\top x$ s.t. $\sum_{i=1}^n x_i F_i + G \preceq 0, Ax = b$

3 Optimal Control

Discrete-Time Optimal Control Problem

Cost Function

$$J(x_0, U) = \sum_{i=0}^{N-1} l(x_i, u_i) + l_f(x_N) \qquad (2)$$

Stage Cost $l_f(x_i, u_i)$

Terminal Cost $l_f(x_N)$

Constraints

$$x_{i+1} = g(x_i, u_i)$$
$$x_0 = x(k)$$
$$h(x_i, u_i) \leq 0 \quad \text{(optional)}$$

Ricatti Equations

Riccati Difference Equation - RDE solved recursively
$P_i = A^\top P_{i+1} A + Q - A^\top P_{i+1} B (B^\top P_{i+1} B + R)^{-1} B^\top P_{i+1} A$

Algebraic Riccati Equation - ARE solved analytically
$P_\infty = A^\top P_\infty A + Q - A^\top P_\infty B (B^\top P_\infty B + R)^{-1} B^\top P_\infty A$

3.1 Unconstrained Finite Horizon Control Problem

Linear Quadratic Optimal Control

Cost Function

$$J^\star(x(0)) := \min_U \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + x_N^\top P x_N \qquad (3)$$

Terminal weight $P \succeq 0$ symmetric
State weight $Q \succeq 0$ symmetric
Input weight $R \succ 0$ symmetric

Constraints
No input or state constraints!
$$x(k+1) = Ax_k + Bu_k$$
Only dynamics matter.

Batch Approach
express cost function in terms of $x(0)$ and input sequence U

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \mathbb{I} \\ A \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & \cdots & 0 \\ B & 0 & 0 \\ AB & B & 0 \\ \vdots & \ddots & 0 \\ A^{N-1}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$X := \mathcal{S}^x x(0) + \mathcal{S}^u U \quad J(x(0), U) = X^\top \overline{Q} X + U^\top \overline{R} U$
$\overline{Q} :=$ blockdiag$(Q, \ldots, Q, P) \quad \overline{R} :=$ blockdiag$(R, \ldots, R)$
Optimal Input set $\nabla_U J(x(0), U) = 2HU + 2F^\top x(0) = 0$

$$U^\star(x(0)) = -\Big(\underbrace{(\mathcal{S}^u)^\top \overline{Q} \mathcal{S}^u + \overline{R}}_{H\text{(Hessian)}^{-1}}\Big) \underbrace{(\mathcal{S}^u)^\top \overline{Q} \mathcal{S}^x}_{F^\top} x(0)$$

Optimal Cost $(x_0 = x(0))$

$$J^\star(x_0) = x_0^\top (\mathcal{S}_x^\top \overline{Q} \mathcal{S}_x - \mathcal{S}_x^\top \overline{Q} \mathcal{S}_u (\mathcal{S}_u^\top \overline{Q} \mathcal{S}_u + \overline{R})^{-1} \mathcal{S}_u^\top \overline{Q} \mathcal{S}_x) x_0$$

Recursive Approach
use dynamic programming to solve problem backwards from N

$$J_j^\star(x(j)) := \min_{U_{j \to N}} x_N^\top P x_N + \sum_{i=j}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$

From Principle Of Optimality

$$J_j^\star(x_j) = \min_{u_j} I(x_i, u_i) + J_{j+1}^\star(x_{j+1})$$

Optimal Cost-To-Go

$$J_i^\star(x_i) = x_i^\top P_i x_i$$

Optimal Control Policy use Riccati

$$u_i^\star = F_i x_i = -(B^\top P_{i+1} B + R)^{-1} B^\top P_{i+1} A \cdot x(i)$$

Comparison of Batch and Recursive Approach
Dynamic programming yields **feedback policies** $u_i^\star = F_i x_i$ depending on each x_i. Batch optimization returns sequence $U^\star(x(0))$ of **numeric values** depending only on x(0),

3.2 Infinite Horizon Control Problem

Linear Quadratic Regulator

Cost Function

$$J_\infty^\star(x(k)) = \min_{u(\cdot)} \sum_{i=0}^{\infty} x_i^\top Q x_i + u_i^\top R u_i \qquad (4)$$

Constraints $\quad x_{i+1} = Ax_i + Bu_i$
$\quad x_0 = x(k)$

Optimal Input $\quad u^\star(k) = F_\infty x(k) \qquad$ (use Riccati)

$$F_\infty x(k) = -(B^\top P_\infty B + R)^{-1} B^\top P_\infty A x(k)$$

Lemma 1 (Lyapunov function for LQR). If (A, B) is stabilizable and $Q, R \succ 0$ then $J^\star(x) = x^\top P_\infty x$ is a **Lyapunov function** for the system $x^+ = (A + BF_\infty)x$ where F_∞ is the constant feedback matrix and $P_\infty \succ 0$ solves the Riccati equation..

Lyapunov

Stability is a property of an **equilibrium point** $\bar{\mathbf{x}}$ of a system

Definition 14 (Lyapunov Stability). $\bar{\mathbf{x}}$ is **Lyapunov stable** if: $\forall \epsilon > 0 \, \exists \, \delta(\epsilon)$ s.t. $|x(0) - \bar{x}|_2 < \delta(\epsilon) \to |x(k) - \bar{x}|_2 < \epsilon$

Definition 15 (Globally asymptotic stability). If $\bar{\mathbf{x}}$ is attractive, i.e., $\lim_{k \to \infty} ||x(k) - \bar{x}|| = 0$, $\forall x(0)$ and Lyapunov stable then $\bar{\mathbf{x}}$ is **globally asymptotic stable.**

Definition 16 (Global Lyapunov function). For the equilibrium $\bar{\mathbf{x}} = 0$ of a system $x(k+1) = g(x(k))$, a function V, continuous at the origin, finite and such that $\forall \, x \in \mathbb{R}^n$:

$$|x| \to \infty \Rightarrow V(x) \to \infty$$
$$V(x) = 0 \text{ if } x = 0 \quad \text{else} \quad V(x) > 0$$
$$V(g(x)) - V(x) \leq -\alpha(x)$$

for continuous positive definite $\alpha : \mathbb{R}^n \to \mathbb{R}$
then $V : \mathbb{R}^n \to \mathbb{R}$ is called **Lyapunov function.**

Theorem 4. If a system admits a Lyapunov function $V(x)$, then $\bar{\mathbf{x}} = 0$ is **globally asymptotically stable.**

Theorem 5 (Lyapunov indirect method). System linearized around $\bar{\mathbf{x}} = 0$ with resulting matrix A and eigenvalues λ_i.
If $\forall |\lambda_i| < 1$ then the origin is asymptotically stable.
if $\exists |\lambda_i| > 1$ then origin is unstable.
If $\exists |\lambda_i| = 1$ we can't conclude anything about stability.

Discrete-Time Lyapunov equation $\quad A^T P A - P = -Q, \quad Q > 0 \quad (5)$

Theorem 6 (Existence of solution, DT Lyapunov equation). The discrete-time Lyapunov equation has a unique solution $P > 0$ iff the system $x(k+1) = Ax(k)$ is stable.

3.3 Constrained Finite Time Optimal Control Problem

CFTOC

Cost Function equal to Linear Quadratic Optimal Control (3)
Constraints as in (3) **+** $x_N \in \mathcal{X}_f, x_{i<N} \in \mathcal{X}, u_i \in \mathcal{U}$

Quadratic Cost CFTOC

$$J(x(k)) = x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i R u_i$$

Goal: Transform into QP

$$\min_{z \in \mathbb{R}^n} \tfrac{1}{2} z^\top H z + q^\top z + r \quad \text{s.t. } Gz \leq h, \ Az = b$$

Construction of QP without Substitution
Idea Keep state equations as equality constraints

Define variable $z = \begin{bmatrix} x_1^\top & \ldots x_N^\top & u_0^\top & \ldots u_{N-1}^\top \end{bmatrix}^\top$
Equalities from system dynamics $x_{i+1} = Ax_i + Bu_i$

$$G_{eq} = \left[\begin{array}{cccc|ccc} \mathbb{I} & & & & -B & & \\ -A & \mathbb{I} & & & & \ddots & \\ & \ddots & \ddots & & & & \\ & & -A & \mathbb{I} & & & -B \end{array}\right] \quad E_{eq} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Inequalities $G_{in} z \leq w_{in} + E_{in} x(k)$ for $\mathcal{X}, \mathcal{U}$

$$\begin{aligned} \mathcal{X} &= \{x \mid A_x x \leq \mathcal{X}_f q b_x\} \\ \mathcal{U} &= \{u \mid A_u u \leq b_u\} \\ \mathcal{X}_f &= \{x \mid A_f x \leq b_f\} \end{aligned} \qquad E_{in} = \begin{bmatrix} -A_x \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$G_{in} = \left[\begin{array}{cccc|cccc} 0 & & & & 0 & & & \\ \hline A_x & & & & 0 & & & \\ & \ddots & & & & \ddots & & \\ & & A_x & & & & 0 & \\ & & & A_f & & & & 0 \\ \hline 0 & & & & A_u & & & \\ & \ddots & & & & \ddots & & \\ & & 0 & & & & A_u & \\ & & & 0 & & & & A_u \end{array}\right] \quad w_{in} = \left[\begin{array}{c} b_x \\ \hline b_x \\ \vdots \\ b_x \\ b_f \\ \hline b_u \\ \vdots \\ b_u \\ b_u \end{array}\right]$$

Cost Matrix $\bar{H} =$ diag$(Q, \ldots, Q, P, R, \ldots, R)$

Finally the resulting quadratic optimization problem

$$J^\star(x(k)) = \min_z \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix}^\top$$
$$\text{s.t} \quad G_{in} z \leq w_{in} + E_{in} x(k) \quad G_{eq} z = E_{eq} x(k)$$

Construction of QP with substitution
Idea Substitute the state equations.
Step 1 Rewrite cost as

$$\begin{aligned} J(x(k)) &= U^\top H U + 2x(k)^\top F U + x(k)^\top Y x(k) \\ &= \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} H & F^\top \\ F & Y \end{bmatrix} \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix}^\top \end{aligned}$$

Step 2 Rewrite constraints compaclty as $GU \leq w + Ex(k)$

$$G = \begin{bmatrix} A_u & 0 & \ldots & 0 \\ 0 & A_u & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_u \\ 0 & 0 & \ldots & 0 \\ A_x B & 0 & \ldots & 0 \\ A_x A B & A_x B & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_f A^{N-1} B & A_f A^{N-2} B & \ldots & A_f B \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix}, w = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Step 3 Rewrite constrained problem as

$$J^\star(x(k)) = \min_U \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} H & F^\top \\ F & Y \end{bmatrix} \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix}^\top$$
subj. to $GU \leq w + Ex(k)$

Choice of P

1 Match the infinite solution $P_N = P_\infty$
2 Use solution of the Lyapunov Equation (5) if the system is asymptotically stable (otherwise P not positive definite), assumes no control needed after end of horizon.
3 Set $P = 0$ und use instead constraint $x_{i+N} = 0$

4 Invariance

Definition 17 (Positively Invariant Set $\mathcal{O}$). For an autonomous or closed-loop system, the set $\mathcal{O}$ is positively invariant if:

$$x(k) \in \mathcal{O} \Rightarrow x(k+1) \in \mathcal{O}, \quad \forall k \in \{0, 1, \ldots\}$$

Definition 18 (Maximal Positively Invariant Set $\mathcal{O}_\infty$). A set that contains all $\mathcal{O}$ is the maximal positively invariant set $\mathcal{O}_\infty \subset \mathcal{X}$
Definition 19 (Pre-Sets). The set of states that in the dynamic system $x(k+1) = g(x(k))$ in one time step evolves into the target set $\mathcal{S}$ is the **pre-set** of $\mathcal{S} \Rightarrow \text{pre}(\mathcal{S}) := \{x \mid g(x) \in \mathcal{S}\}$
Lemma 2. **Invariant Sets from Lyapunov Functions**
If $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for $x(k+1) = g(x(k))$, then $Y := \{x \mid V(x) \leq \alpha\}$ is an invariant set for all $\alpha \geq 0$

Proof. Lyapunov property $V(g(x)) - V(x) < 0$ implies that once $V(x(k)) \leq \alpha, V(x(j)) < \alpha, \forall j \geq k \to$ Invariance □

Example System $x(k+1) = Ax(k), A^\top P A - P \prec 0 \prec P$ and resulting Lyapunov function $V(x(k)) = x(k)^\top P x(k)$
Goal Find the largest α s.t the invarinat set $Y_\alpha \in \mathcal{X}$

$$Y_\alpha := \{x \mid x^\top P x \leq \alpha\} \subset \mathcal{X} := \{x \mid Fx \leq f\}$$

Equivalent to $\max_\alpha \alpha \quad$ s.t. $h_{Y_\alpha}(F_i) \leq f_i \, \forall i \in \{1 \ldots n\}$

Theorem 7 (Geometric condition for invariance). Set $\mathcal{O}$ is positively invariant set iff $\mathcal{O} \subseteq \text{pre}(\mathcal{O}) \Leftrightarrow \text{pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$

Proof. **Necessary** if $\mathcal{O} \nsubseteq \text{pre}(\mathcal{O})$, then $\exists \bar{x} \in \mathcal{O}$ s.t $\bar{x} \notin \text{pre}(\mathcal{O}) \rightsquigarrow \bar{x} \in \mathcal{O}, \bar{x} \notin \text{pre}(\mathcal{O})$, thus $\mathcal{O}$ not positively invariant

Sufficient if $\mathcal{O}$ not positive invariant set, then $\exists \bar{x} \in \mathcal{O}$ s.t $g(\bar{x}) \notin \mathcal{O} \rightsquigarrow \bar{x} \in \mathcal{O}, \bar{x} \notin \text{pre}(\mathcal{O})$ thus $\mathcal{O} \nsubseteq \text{pre}(\mathcal{O})$ □

4.1 Control Invariance

Definition 20 (Control Invariant Set). $\mathcal{C} \subseteq \mathcal{X}$ control invariant if

$$x(k) \in \mathcal{C} \Rightarrow \exists u(k) \in \mathcal{U} \text{ s.t } g(x(k), u(k)) \in \mathcal{C}\ \forall k$$

Definition 21 (Maximal Control Invariant Set $\mathcal{C}_\infty$). A set that contains all $\mathcal{C}$ is the maximal positively invariant set $\mathcal{C}_\infty \subset \mathcal{X}$

Intuition For all states in $\mathcal{C}_\infty$ exists control law s.t constraints are never violated $\rightsquigarrow$ **The best any controller could ever do**

Pre-set $\text{pre}(\mathcal{S}) := \{x \mid \exists u \in \mathcal{U} \text{ s.t } g(x, u) \in \mathcal{S}\}$

Set $\mathcal{C}$ is control invariant iff: $\mathcal{C} \subseteq \text{pre}(\mathcal{C}) \Leftrightarrow \text{pre}(\mathcal{C}) \cap \mathcal{C} = \mathcal{C}$

Control Law from Control Invariant Set

Control law $\kappa(x(k))$ will **guarantee** that the system with control invariant set $\mathcal{C}$ satisfies constraints **for all time** if

$$x(k+1) = g(x(k), u(k)) \rightarrow g(x, \kappa(x)) \in \mathcal{C}\ \forall x \in \mathcal{C}$$

We can use this fact to **synthesize** control law κ

$$\kappa(x) := \text{argmin}\{f(x, u) \mid g(x, u) \in \mathcal{C}\}$$

with f as any function (including $f(x, u) = 0$)

Does not ensure that system will converge
Difficult because calculating control invariant sets is hard
MPC implicitly describes $\mathcal{C}$ s.t easy to represent/compute

4.2 Robust Invariance

Definition 22 (Robust Positive Invariant Set $\mathcal{O}^{\mathcal{W}}$). For the autonomous system $x(k+1) = g(x(k), w(k))$, the set $\mathcal{O}^{\mathcal{W}}$ is robust positive invariant if:

$$x \in \mathcal{O}^{\mathcal{W}} \Rightarrow g(x, w) \in \mathcal{O}^{\mathcal{W}}, \quad \forall w \in \mathcal{W}$$

Given set Ω and dynamic system $x(k+1) = g(x(k), w(k))$,

$$\text{pre}^{\mathcal{W}}(\Omega) := \{x \mid g(x, w)\} \in \Omega\ \forall w \in \mathcal{W}$$

Definition 23 (Robust Pre-Sets). The set of states that in the dynamic system $x(k+1) = g(x(k), w(k))$ for all disturbance $w \in \mathcal{W}$ in one time step evolves into the target set Ω is the **pre-set** of $\Omega \Rightarrow \text{pre}^{\mathcal{W}}(\Omega) := \{x | g(x, w) \in \Omega\ \forall w \in \mathcal{W}\}$

Theorem 8 (Geometric condition for robust invariance). Set $\mathcal{O}^{\mathcal{W}}$ is robust positive invariant iff $\mathcal{O}^{\mathcal{W}} \subseteq \text{pre}^{\mathcal{W}}(\mathcal{O}^{\mathcal{W}})$

Minimum Robust Invariant Set

$$\mathcal{F}_\infty = \bigoplus_{j=0}^{\infty} A_K^j \mathcal{W}, \mathcal{F}_0 := \{0\} \Rightarrow \mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}_\infty$$

4.3 Computing Invariant Sets and Pre-sets

Conceptual Algorithm

$\Omega_0 \leftarrow \mathcal{X}$
loop
$\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$
if $\Omega_{i+1} = \Omega_i$ **then**
return $\mathcal{O}_\infty = \Omega_i$
end if
end loop

Pseudo algorithm
Inputs, outputs, especially pre-set calculation will be different, the overall concept not, used for:
Invariant Sets
Control Invariant Sets
Robust Invariant Sets

Replace Intersection with Minkowski Sum and Pre-Set with $A^i\mathcal{W}$ to get an Idea for **Minimum Robust Invariant set**

System for Pre-Set Computation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ u(k) \in \mathcal{U} &:= \{u | Gu \leq g\} \\ \mathcal{S} &:= \{x | Fx \leq f\} \end{aligned}$$

Invariant Pre-Set

$$\begin{aligned} \text{pre}(S) &:= \{x \mid Ax \in S\} \\ &= \{x \mid FAx \leq f\} \end{aligned}$$

Control Invariant Pre-Set

$$\begin{aligned} \text{pre}(S) &:= \{x \mid \exists u \in \mathcal{U}, Ax + Bu \in S\} \\ &= \{x \mid \exists u \in \mathcal{U}, FAx + FBu \leq f\} \\ &= \left\{x \mid \exists u \in \mathcal{U}, \begin{bmatrix} FA & FB \\ 0 & G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} f \\ g \end{bmatrix} \right\} \end{aligned}$$

This is a **projection** operation

System for Robust Pre-Set Computation

$$\begin{aligned} x(k+1) &= Ax(k) + w(k) \\ \Omega &:= \{x \mid Fx \leq f\} \end{aligned}$$

Robust Invariant Pre-Set

$$\begin{aligned} \text{pre}^{\mathcal{W}}(\Omega) &= \{x \mid FAx + Fw \leq f\} \\ &= \{x \mid FAx \leq f - \max_{w \in \mathcal{W}} Fw\} \\ &= \{x \mid FAx \leq f - h_{\mathcal{W}^i}(F)\} \end{aligned}$$

where $h_{\mathcal{W}^i}(F)$ is the **support function** $\sigma_{\mathcal{C}}(a) = \sup_{x \in \mathcal{C}} a^\top x$

5 Nominal MPC

MPC Mathematical Formulation

$$\underset{U}{\text{argmin}} \sum_{i=0}^{N-1} l(x_i, u_i) + l_f(x_N) \tag{6}$$

Constraints

$$\begin{aligned} x_0 &= x(k) \\ x_{i+1} &= Ax_i + Bu_i \\ x_i &\in \mathcal{X} \\ u_i &\in \mathcal{U} \\ x_N &\in \mathcal{X}_f \end{aligned}$$

$l_f(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon.

What can go wrong with *standard* MPC?

- No feasibility guarantee, the problem may not have a solution
- No stability guarantee, trajectories may not converge to origin

Stability of MPC - Main Result

Assumptions

1 Stage cost is strictly positive and only zero at the origin

2 Terminal set is **invariant** under local control law $\kappa_f(x_i)$:

$$x_{i+1} = Ax_i + B\kappa_f(x_i) \in \mathcal{X}_f\ \forall x_i \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \in X, \kappa_f(x_i) \in U\ \forall x_i \in \mathcal{X}_f$$

3 Terminal cost is a continuous **Lyapunov function** s.t.

$$l_f(x_{i+1}) - l_f(x_i) \leq -l(x_i, \kappa_f(x_i)) \quad \forall x_i \in \mathcal{X}_f$$

Theorem 9. Under the previous assumptions, the closed-loop system under the MPC control law $u_0^\star(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for

$$x(k+1) = Ax(k) + Bu_0^\star(x(k))$$

Finite-horizon MPC may not satisfy constraints for all time!

Finite-horizon MPC may not be stable!

- An infinite-horizon provides stability and invariance.
- Infinite-horizon *faked* by forcing final state into an invariant set for which there exists invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- Extends to non-linear systems, but compute sets is difficult!

6 Practical MPC

6.1 Steady-state Target Problem

- Reference is achieved by the target state x_s if $z_s = Hx_s = r$
- Target state should be a steady-state, i.e. $x_s = Ax_s + Bu_s$

$$\begin{aligned} x_s &= Ax_s + Bu_s \\ z_s &= Hx_s = r \end{aligned} \iff \begin{bmatrix} \mathbb{I} - A & -B \\ H & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$\nexists$ solution $\rightarrow \min (Hx_s - r)^\top Q_s (Hx_s - r)$ (closest x to r)
If $\exists$ multiple feasible $u_s \rightarrow$ compute $\min u_s^\top R_s u_s$ (cheapest)

$$\min_U |z_N - Hx_s|_{P_z}^2 + \sum_{i=1}^{N-1} |z_i - Hx_s|_{Q_z}^2 + |u_i - u_s|_R^2$$

6.2 Offset-free Reference Tracking

Reference Tracking

$$\begin{aligned} \Delta x_{k+1} &= x_{k+1} - x_s \\ &= A\Delta x_k + Bu_k - (Ax_s + Bu_s) \\ &= A\Delta x_k + B\Delta u_k \end{aligned}$$

$$\begin{aligned} G_x x &\leq h_x \\ G_u u &\leq h_u \end{aligned} \Rightarrow \begin{aligned} G_x \Delta x &\leq h_x - G_x x_s \\ G_u \Delta u &\leq h_u - G_u u_s \end{aligned}$$

Convergence

Assume feasible target with $x_s \in \mathcal{X}, u_s \in \mathcal{U}$, choose terminal weight $V_f(x)$ and constraint $\mathcal{X}_f$ as in regulation case satisfying

$$V_f(x(k+1)) - V_f(x(k)) \leq -l(x(k), Kx(k))$$

and $(A + BK)x \in \mathcal{X} \quad \forall x \in \mathcal{X}_f$ for both

If in addition the target reference x_s, u_s is such that

$$x_s \oplus \mathcal{X}_f \subseteq \mathcal{X}, K\Delta x + u_s \in \mathcal{U}, \quad \forall \Delta x \in \mathcal{X}_f$$

then the closed loop system converges to the target reference.

Proof. Invariance under local control law is inherited from regulation case. Constraint satisfaction is provided by extra conditions and convergence comes from the asymptotic stability of the regulation problem: $\Delta x(k) \rightarrow 0$ for $k \rightarrow \infty$ □

Terminal set use $\mathcal{X}_f^{\text{scaled}} = \alpha \mathcal{X}_f$ (s.t. constraints satisfied)

Disturbance Cancellation

Approach Model the disturbance, use the measurements and model to estimate the state and disturbance and find control inputs that use the disturbance estimate to remove offset.

Augmented Model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + B_d d_k \\ y_k &= Cx_k + C_d d_k \end{aligned}$$

Constant disturbance $d_{k+1} = d_k$

Observable iff $\begin{bmatrix} A-I & B_d \\ C & C_d \end{bmatrix}$ has full rank (assuming $n_x = n_d$)

Observer For Augmented Model

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} L_x \\ L_d \end{bmatrix} (C\hat{x}_k + C_d \hat{d}_k - y_k)$$

Error Dynamics $\Rightarrow$ choose L s.t error dynamics converge to 0

$$\begin{bmatrix} x_{k+1} - \hat{x}_{k+1} \\ d_{k+1} - \hat{d}_{k+1} \end{bmatrix} = \left(\begin{bmatrix} A & B_d \\ 0 & \mathbb{I} \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C & C_d \end{bmatrix} \right) \begin{bmatrix} x_k - \hat{x}_k \\ d_k - \hat{d}_k \end{bmatrix}$$

Lemma 3. Steady-state of an asym. stable observer satisfies:

$$\begin{bmatrix} A - \mathbb{I} & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_\infty \\ y_\infty - C_d \hat{d}_\infty \end{bmatrix} \quad (\text{for } n_y = n_d)$$

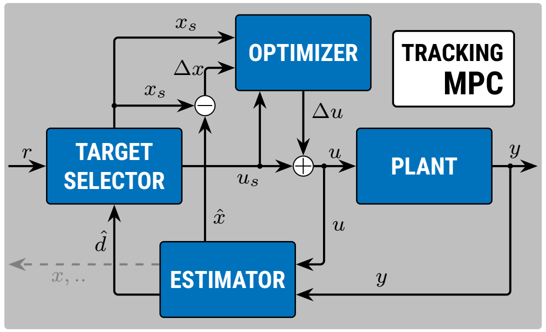
$\Rightarrow$ Observer output $C\hat{x}_\infty + C_d \hat{d}_\infty$ tracks y_∞ without offset

Reference Tracking with Disturbance Cancellation

Goal Track constant reference: $Hy(k) = z(k) \rightarrow r, k \rightarrow \infty$

$$\begin{aligned} x_s &= Ax_s + Bu_s + B_d \hat{d}_\infty \\ z_s &= H(Cx_s + C_d \hat{d}_\infty) = r \end{aligned}$$

$$\begin{bmatrix} A - I & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r - HC_d \hat{d} \end{bmatrix}$$



Offset-free Tracking - Main Result

Theorem 10. Assuming RHC recursively feasible, $n_d = n_y$, unconstrained for $k \geq j$, and the closed loop system

$$x(k+1) = Ax(k) + B\kappa(\cdot) + B_d d \quad \text{with } (\cdot) = (\hat{x}, \hat{d}, r)$$
$$\hat{x}(k+1) = (A + L_x C)\hat{x}(k) + (B_d + L_x C_d)\hat{d}(k) + B\kappa(\cdot) - L_x y(k)$$
$$\hat{d}(k+1) = L_d C\hat{x}(k) + (\mathbb{I} + L_d C_d)\hat{d}(k) - L_d y(k)$$

converges, then $z(k) = Hy(k) \to r$ as $k \to \infty$

6.3 Soft Constraints

Input constraints are dictated by physical constraints on the actuators and are **usually hard**

State/output constraints arise from practical restrictions on the allowed operating range and are **rarely hard**

Hard state/output constraints always lead to **complications in the controller implementation**

Soft Constrained MPC

$$\min_u \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + l_\epsilon(\epsilon_i) + x_N^\top P x_i + l_\epsilon(\epsilon_N) \qquad (7)$$

Quadratic penalty
$l_\epsilon(\epsilon_i) = \epsilon_i^\top S \epsilon_i$
(e.g $S = Q$)
+ linear norm penalty
$l_\epsilon(\epsilon_i) += v|\epsilon_i|_{1/\infty}$

Constraints
$$x_{i+1} = Ax_i + Bu_i$$
$$H_x x_i \leq k_x + \epsilon_i$$
$$H_u u_i \leq k_u$$
$$\epsilon_i \geq 0 \quad \text{slack variable}$$

Original $\min_z f(z)$ s.t. $g(z) \leq 0$

Softened $\min_{z,\epsilon} f(z) + l_\epsilon(\epsilon)$ s.t. $g(z) \leq \epsilon$, $\epsilon \geq 0$

Requirement on $l_\epsilon(\epsilon)$ If the original problem has a feasible solution $z^\star$, then the softened problem should have the same solution $z^\star$, and $\epsilon = 0$.

Theorem 11 (Exact Penalty Funtcion). $l_\epsilon(\epsilon) = v \cdot \epsilon$ satisfies requirement for any $v > \lambda^\star \geq 0$, where $\lambda^\star$ is optimal Lagrange multiplier for original problem

7 Robust MPC

Uncertain System $x(k+1) = g(x(k), u(k), w(k); \theta)$

Robust Constraint Satisfaction

Idea Compute a set of tighter constraints such that if the nominal system meets these constraints, then the uncertain system will too. We then do MPC on the nominal system.

Goal Ensure constraints satisfied for the MPC sequence.

Disturbance reachable set $\quad \mathcal{F}_i = \bigoplus_{j=0}^{i-1} A^j \mathcal{W}$

Robust Open-Loop MPC

$$\min_U \sum_{i=0}^{N-1} l(x_i, u_i) + l_f(x_N) \qquad \begin{aligned} x_0 &= x(k) \\ x_i &\in \mathcal{X} \ominus \mathcal{F}_i \\ u_i &\in \mathcal{U} \\ x_N &\in \mathcal{X}_f \ominus \mathcal{F}_N \end{aligned}$$
s.t. $x_{i+1} = Ax_i + Bu_i$

Closed Loop Robust MPC

Idea Separate the available control authority into two parts:

$$z(k+1) = Az(k) + Bv(k)$$

steers noise-free *nominal* system to origin

$$u_i = K(x_i - z_i) + v_i$$

compensates for deviations, i.e. a *tracking* controller, to keep the real trajectory close to the nominal system.

$\Rightarrow$ We fix the linear feedback controller K offline, and optimize over the nominal inputs $\{v_0, ..., v_{N-1}\}$ and nominal trajectory $\{z_0, ..., z_N\}$, which results in a convex problem.

$e_{i+1} = x_{i+1} - z_{i+1} = (A + BK)e_i + w_i$

7.1 Robust Constraint-Tightening MPC

Robust Constraint-Tightening MPC

$$\min_{Z,V} \sum_{i=0}^{N-1} l(z_i, v_i) + l_f(z_N) \qquad \begin{aligned} z_0 &= x(k) \\ z_i &\in \mathcal{X} \ominus \mathcal{F}_i \\ u_i &\in \mathcal{U} \ominus K(\mathcal{F}_i) \\ z_N &\in \mathcal{X}_f^{ct} \ominus \mathcal{F}_N \end{aligned}$$
subj. to $z_{i+1} = Az_i + Bv_i$

$\mathcal{F}_0 := 0 \qquad F_i := \mathcal{W} \oplus A_K \mathcal{W} \oplus \ldots A_K^{i-1}\mathcal{W},$

Control Law $u(k) = v_0^\star + K(x(k) - z_0) = v_0^\star$

7.2 Robust Tube MPC

Idea Ignore noise and plan the nominal trajectory, bound maximum error at any time with RPI set $\mathcal{E} : \epsilon_i \in \mathcal{E}\epsilon_{i+1} \in \mathcal{E}$

Ideally $\mathcal{E}$ is selected as the minimum RPI set F_∞

We know that the real trajectory stays 'nearby' the nominal one $x_i \in z_i \oplus \mathcal{E}$ because we plan to apply the controller $u_i = K(x_i$-$z_i) + v_i)$ in the future.

(we won't actually do this, but it's a valid sub-optimal plan)

We must ensure that all possible state trajectories satisfy the constraints This is now equivalent to ensuring that $x_i \in z_i \oplus \mathcal{E}$

(address input constraints later)

What do we need to make this work?

Compute the set $\mathcal{E}$ that the error will remain inside

Previously we wanted the **maximum robust invariant set**, or the largest set in which our terminal control law works.

We now want the **minimum robust invariant set**, or the smallest set that the state will remain inside despite the noise.

Modify constraints on nominal trajectory $\{z_i\}$

$x_i \in z_i \oplus \mathcal{E} = \{z_i + e | e \in \mathcal{E}\}$

Tube MPC

Cost function $\quad J(Z, V) := \sum_{i=0}^{N-1} l(z_i, v_i) + l_f(z_N)$

Feasible set $\quad \mathcal{Z}(x_0) := \begin{cases} z_{i+1} &= Az_i + Bv_i \\ z_i &\in \mathcal{X} \ominus \mathcal{E} \\ v_i &\in \mathcal{U} \ominus K\mathcal{E} \\ z_N &\in \mathcal{X}_f \\ x_0 &\in z_0 \oplus \mathcal{E} \end{cases}$

Optimization $\quad (V^\star(x_0), Z^\star(x_0)) = \text{argmin}_{V,Z}\{J(Z,V) | (Z,V) \in \mathcal{Z}(x_0)\}$

Control law $\quad \mu_{\text{tube}}(x) := K(x - z_0^\star(x)) + v_0^\star(x)$

Theorem 12 (Robust Invariance of Tube MPC). The set $\mathcal{Z} := \{x | \mathcal{Z}(x) \neq \emptyset\}$ is a robust invariant set of the system $x(k+1) = Ax(k) + B\mu_{\text{tube}}(x(k)) + w(k)$ subject to the constraints $x, u \in \mathcal{X} \times \mathcal{U}$.

Theorem 13 (Robust Stability of Tube MPC). The state $x(k)$ of the system $x(k+1) = Ax(k) + B\mu_{\text{tube}}(x(k)) + w(k)$ converges to the limit of the set $\mathcal{E}$.

Tube MPC - Quick Summary

To implement tube MPC:

– Offline –

1. Stabilizing controller K so that $A + BK$ is (Schur) stable
2. Compute the minimal robust invariant set $E = F_\infty$ for the system $x(k+1) = (A + BK)x(k) + w(k), w \in \mathcal{W}^1$
3. Compute tightened constraints $\bar{\mathcal{X}} := \mathcal{X} \ominus \mathcal{E}, \bar{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{E}$
4. Choose terminal weight function l_f and constraint $\mathcal{X}_f$ satisfying assumptions*

– Online –

1. Measure / estimate state x
2. Solve optimization problem for $(V^\star(x_0), Z^\star(x_0))$
3. Set the input to $u = K(x - z_0^\star(x)) + v_0^\star(x)$

8 Implementation

CFTOC problem is **multiparametric quadratic program** **mp-QP**

Two options to solve it:

- Iterative optimization methods
- Explicit solution

Let $I := 1, ..., m$ be the set of constraint indices.

Definition 24 (Active Set). $A(x)$ and it's complement $NA(x)$

$$A(x) := \{j \in I : G_j z^\star(x) - S_j x = w_j\}$$
$$NA(x) := \{j \in I : G_j z^\star(x) - S_j x < w_j\}$$

Definition 25 (Critical Region). CR_A is set of parameters x for which set $A \subseteq I$ of constraints i active at the optimum. For given $\bar{x} \in \mathcal{K}^\star$ let $(A, NA) := (A(\bar{x}), NA(\bar{X}))$. Then

$$CR_A := \{x \in \mathcal{K}^\star : A(x) = A\} \quad \text{(states share active set)}$$

Online evaluation: Point location

Sequential search

Logarithmic search

Large Scale Convex Optimization

Definition 26 (Lipschitz continuity). $q : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz with constant L if: $|q(x) - q(y)| \leq L|x - y| \forall x, y \in \mathbb{R}^m$

f is **Lipschitz** (Lip) with constant $L \Leftrightarrow |\nabla f(x)|_2 \leq L$

Theorem 14. $\mathcal{C} \subseteq \mathbb{R}^n$ non-empty closed (**cl**) convex set, $y \notin \mathcal{C} \to \exists\, a \neq 0, b \in \mathbb{R}$ s.t. $a^\top x + b < a^\top y + b, \forall x \in \mathcal{C}$

Definition 27 (smoothness). $f : \mathbb{R}^n \to \mathbb{R}$ is L-smooth (**L-sm**) if $\nabla f(x)$ satisfies $|\nabla f(x) - \nabla f(y)| \leq L|x - y| \; \forall x, y \in \mathbb{R}^n$

Taylor $\to f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{L}{2}|x - y|^2$

Definition 28 (strong convexity). $f : \mathbb{R}^n \to \mathbb{R}$ is μ-strongly convex (**μ-scv**) if $f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{\mu}{2}|x - y|^2$

How to find μ/L, Spectra of Hessian $\nabla^2 f$, min/max eigenvalue

Gradient Descent

$x_{k+1} = x_k - T\nabla f(x_k)$ for $k = (k_0, \ldots, k_N)$ given x_0, T

Line search optimal step $\nu_t^* = \text{argmin}_{\nu \in \mathbb{R}} f(x_t - \nu \nabla f(x_t))$

Definition 29. $\text{prox}_{\mathcal{C}}(x) = \text{argmin}_{y \in \mathcal{C}} \frac{1}{2}|x - y|^2$ $\mathcal{C} \subset \mathbb{R}^n$

Lemma 4. cl, cv $\mathcal{C} \subset \mathbb{R}^n \to |\text{prox}_{\mathcal{C}}(x) - \text{prox}_{\mathcal{C}}(y)| \leq |x - y| \leftarrow |\text{prox}_{\mathcal{C}}(x) - \text{prox}_{\mathcal{C}}(y)|^2 \leq (\text{prox}_{\mathcal{C}}(x) - \text{prox}_{\mathcal{C}}(y))^\top (x - y)$

Projected Gradient Descent

$x_{k+1} = \text{prox}_{\mathcal{C}}(x_k - T\nabla f(x_k))$, for $x_0, k_{0..N}, T \in (0, 2/L)$

```
def get_next_u(y: Measurement, r: Reference):
    """
    System handler for offset-free tracking
    """
    # approximate state, disturbance
    x, d = estimator(y)
    # find steady state und generate delta
    x_s, u_s = target_selector(x, r, d)
    x_delta = x - x_s
    # call solver with new parameter
    u_delta = mpc_regulator(x_delta, x_s, u_s)
    u = u_delta + u_s

    return u
```

TRANSFORM P-Nj

ℓ_∞-Minimization

$\min_{x \in \mathbb{R}^m} ||x||_\infty$ subj. to $Fx \leq g$ $\quad \Longleftrightarrow \quad$ $\min_{x,t} t$ subj. to $-\mathbf{1}_m t \leq x \leq \mathbf{1}_m t,\; F$

ℓ_1-Minimization

$\min_{x \in \mathbb{R}^m} ||x||_1$ subj. to $Fx \leq g$ $\quad \Longleftrightarrow \quad$ $\min_{x \in \mathbb{R}^m, t \in \mathbb{R}^m} \mathbf{1}_m^\top t$ subj. to $-t \leq x \leq t,\; Fx \leq$