Assignment 1: Using solve_ivp to study the plasma sheath

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Semester 1

1 Overview

In this assignment you are going to use Python and the SciPy library to study plasma Debye sheaths. These are formed when plasma meets material surfaces. Since electrons move much faster than ions, they are absorbed first. This creates a charge imbalance and so an electric field forms which accelerates ions into the surface so that the flow of ions matches the flow of electrons. Good places to start if you want to read more on the background/context are the Debye sheath Wikipedia page and lecture notes by R.Fitzpatrick.

In a 1-D sheath, call x the distance from the plasma into the sheath. Inside the plasma sheath the basic equations are for conservation of energy of the ions, ion continuity, and Boltzmann relation for the electrons.

Conservation of energy for the ions is:

$$\frac{1}{2}m_i v_i\left(x\right)^2 = \frac{1}{2}m_i v_s^2 - e\phi\left(x\right)$$

where m_i is the mass of the ions, v_i is the velocity of the ions, and v_s is the ion velocity at the start of the sheath (which must be sonic according to the Bohm sheath criterion), and ϕ is the electrostatic potential.

The **ion continuity** equation in 1D is

$$n_i\left(x\right) v_i\left(x\right) = n_s v_s$$

where $n_i\left(x\right)$ is the ion density and n_s is the density at the beginning of the sheath.

The **electron density** is assumed to follow a Boltzmann distribution as electrons move quickly back and forwards through the sheath

$$n_e\left(x\right) = n_s \exp\left(\phi\left(x\right)/T_e\right)$$

where T_e is the electron temperature in units of eV (in SI units the factor in the exponential would be written $e\phi/k_B T_e$), and $n_e\left(x\right)$ is the electron density. Finally, the electric potential ϕ is given by **Poisson's equation**

$$\frac{d^2\phi}{dx^2} = e\left(n_e\left(x\right) - n_i\left(x\right)\right)/\epsilon_0$$

Tasks: The assignment is split into three tasks, with a total of 10 marks.

2 Normalisation

The first thing to do is to normalise these equations, to make all quantities dimensionless with magnitudes ~ 1 and to identify the important control parameters (if any). A reasonable starting place is to pick a typical length scale and velocity (or time scale). In this case I've chosen the Debye length λ_D and ion sound speed c_s

$$\lambda_D = \sqrt{\frac{\epsilon_0 T_e}{e n_s}} \qquad c_s = \sqrt{\frac{e T_e}{m_i}}$$

(note in SI units we would have $\lambda_D = \sqrt{\epsilon_0 k_B T_{e,k}/e^2 n_s}$ and $c_s = \sqrt{k_B T_{e,k}/m_i}$ where $T_{e,k}$ is the temperature in Kelvin). We then normalise all quantities to these values. Since T_e is in eV and we can expect some sort of equipartition, it's reasonable to normalise the electrostatic potential ϕ to T_e. **Note**: Values with hats on are normalised

$$\hat{x} = x/\lambda_D \qquad \hat{n}_i = n_i/n_s \qquad \hat{n}_e = n_e/n_s$$
$$\hat{\phi} = \phi/T_e \qquad \hat{E} = E\lambda_D/T_e \qquad \hat{v}_i = v_i/c_s$$

where the electric field is

$$E = -\frac{d\phi}{dx}$$

For example substituting normalised quantities into Poisson's equation gives

$$\frac{T_e}{\lambda_D^2}\frac{d^2\hat{\phi}}{d\hat{x}^2} = e n_s \left(\hat{n_e} - \hat{n_i}\right)/\epsilon_0$$

which simplifies to

$$\frac{d^2\hat{\phi}}{d\hat{x}^2} = \hat{n_e} - \hat{n_i}$$

Task [2 marks]: Perform the same procedure with the energy conservation, ion continuity, and electron density equations to get a set of equations which don't contain any physical constants like e, m_i and ϵ_0

$$\hat{v}_i\left(\hat{x}\right) = \ldots \qquad \hat{n}_i\left(\hat{x}\right) = \ldots \qquad \hat{n}_e\left(\hat{x}\right) = \ldots$$

3 Solving with SciPy

In the Computational Techniques lectures we will study how to solve initial value problems. We can use the same methods to solve this sheath problem, except rather than time we're integrating with respect to the distance x from the plasma to the wall.

The only x derivatives in this problem are in Poisson's equation. Solve these using SciPy's solve_ivp function by first writing the equations as first order ODE's

$$\frac{d\hat{\phi}}{d\hat{x}} = -\hat{E}$$
$$\frac{d\hat{E}}{d\hat{x}} = \ldots$$

where $\frac{d\hat{E}}{d\hat{x}}$ will depend on the electron and ion density which will change with ϕ.

Task [5 marks] At $x = 0$, use initial conditions $\hat{\phi} = 0$ and $\hat{E} = 0.001$ and integrate the equations for $\hat{v}_s = 1$ up to $x = 40$ using 100 points. You may find the solveivp2.py example code from Lecture 4 useful.

Task [3 marks] Calculate the normalised current, starting from the following expression (see R.Fitzpatrick notes)

$$j = en_s c_s \left[\sqrt{\frac{m_i}{2\pi m_e}} \exp\left(\phi/T_e\right) - 1\right]$$

You can assume a pure hydrogen plasma, so $m_i/m_e \simeq 1840$. **Hint**: You'll first need to normalise the equation, and when ϕ is very negative $\hat{j} \to -1$. You must clearly use this information to determine the normalisation.

Plot the normalised current against the distance x.

Points to consider: Please pay attention to how you present your figures, part of the marks available are for the quality of the presentation. As this is the first programming assignment I will be lenient with regards to the style of the programming. In future assignments the quality (as determined by readability, robustness etc.) of the code will also contribute towards the marks so please do pay attention to the feedback on your submission.

4 Example plots

To help you check your implementation the following figure shows an example result. It should be noted that whilst the figure shows "correct" data the presentation would need to be improved for full marks (so don't just copy this!).

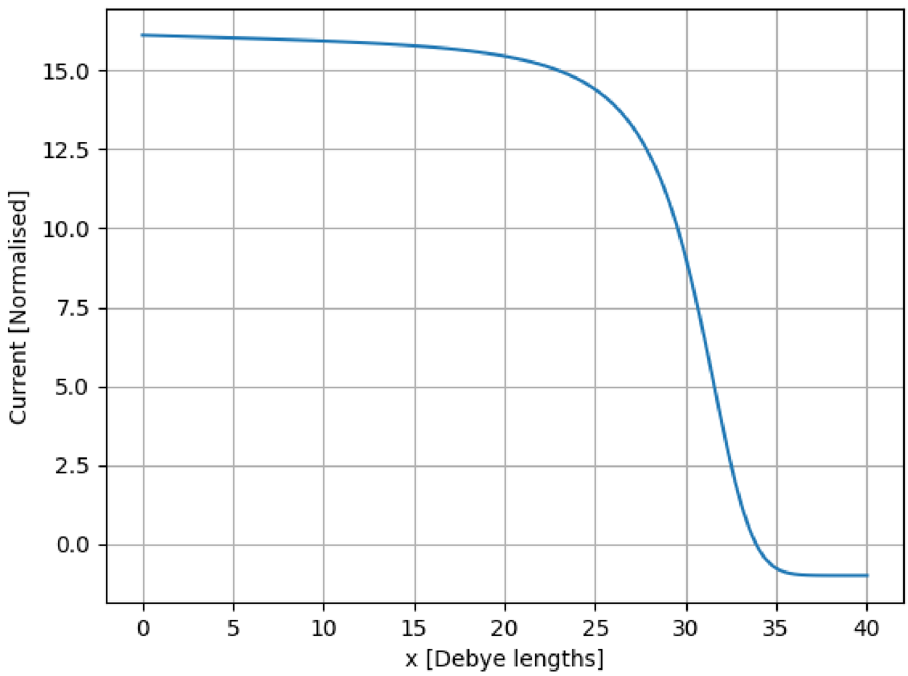

Figure 1: Figure showing plot of normalised current as a function of the normalised distance $\hat{x}$. Shows current drops from around 17 down to -1.