The Determinant and Arc Indices of θ-Curves and Handcuff-Graphs

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NOV 22, 2025
R&E 2025

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Upper Bounds of Arc Index

Upper Bounds of Arc Index

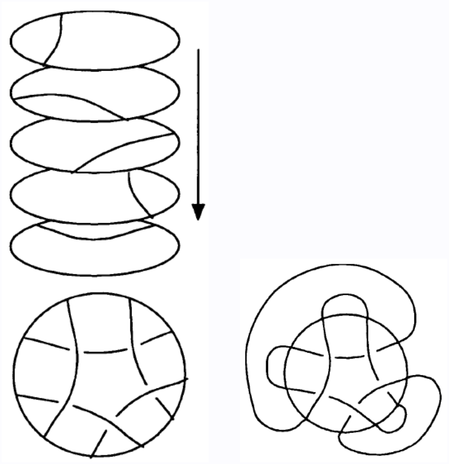
Theorem

Let T be any non-trivial prime θ-curve or handcuff graph. Then,

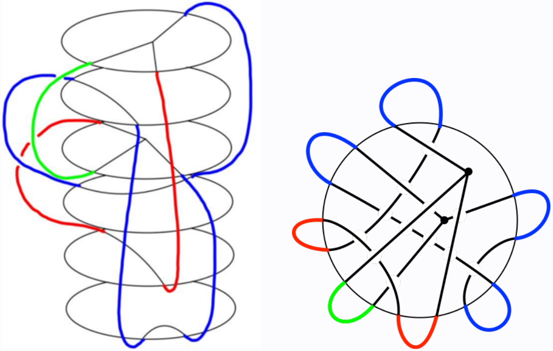
$$\alpha(T) \leq c(T) + 3.$$

Lower Bounds of Arc Index

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link



Stacked Tangle of a θ-Curve

Figure from [?]

Yamada Polynomials

Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$** is calculated by the following properties:

- **Y6:** $R(\;) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R(\;) = 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** $R(\;) - R(\;) = (x - x^{-1})\left[R(\;) - R(\;)\right]$
- **Y10:** $R(\;) = x^2 R(\;)$, $R(\;) = x^{-2} R(\;)$
- **Y11:** $R(\;) = R(\;)$ **Y12:** $R(\;) = R(\;)$
- **Y13:** $R(\;) = R(\;)$, $R(\;) = R(\;)$
- **Y14:** $R(\;) = -xR(\;)$, $R(\;) = -x^{-1}R(\;)$

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Lower Bounds from Yamada Polynomial

Proposition

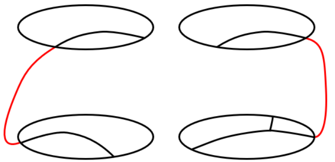
Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T ***without nested caps****. Then*

$$\max \deg_x R(S_T) \leq c + n, \quad \min \deg_x R(S_T) \geq -(c+n),$$

where c, n *is the number of caps and crossings in S_T, respectively.*

Proof

- Let c_s, c_{ss} be the number of **simlpe caps** or **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T. Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

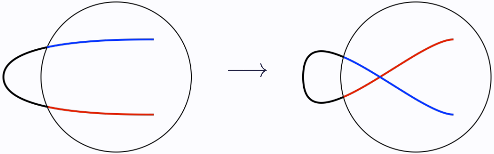
where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

$$\min \deg_x R(S_T) - 2p = \min \deg_x R\left(S_T^{neg}\right) \geq -c + -(n+p)$$
$$\max \deg_x R(S_T) + 2(c-p) = \max \deg_x R\left(S_T^{pos}\right) \leq c + [n + (c-p)]$$
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq c + 2n \qquad \square$$

Proof of Theorem

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proof

Let S_T be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

Consider the maximum number of crossings in S_T, and use the previous theorem.