The Determinant and Arc Indices of θ-Curves and Handcuff-Graphs

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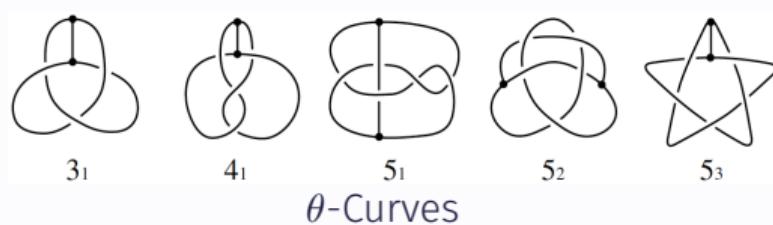
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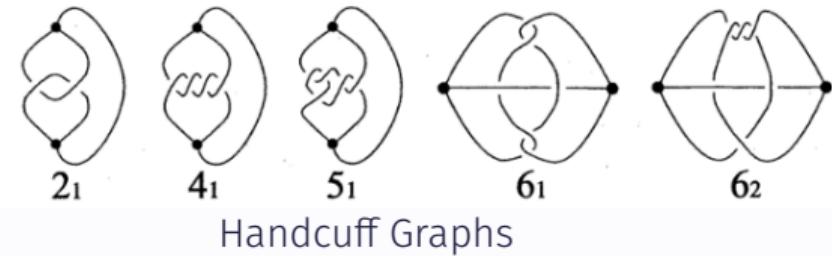
Introduction

θ-Curves and Handcuff Graphs

- A **θ-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A constituent knot T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.

- A handcuff graph H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.
- A constituent link H_{12}, is a subgraph of H that consists of two vertices v_1, v_2 and two edges e_1, e_2.

3_1 4_1 5_1 5_2 5_3

θ-Curves

2_1 4_1 5_1 6_1 6_2

Handcuff Graphs

2

Arc Presentation

- Arc presentation of a θ-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called pages).
- The embedding is with the common boundary line (called axis).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- Arc index, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is minimal arc presentation.

Trefoil
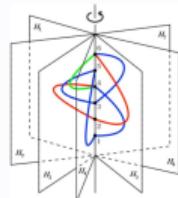
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Grid Diagram

$\theta_{5,2}$

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Grid Diagram

$\Phi_{2,1}$
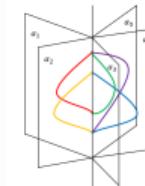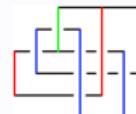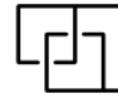
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Grid Diagram

- The **grid diagram** of θ-curve or handcuff graph is a diagram with only vertical and horizontal strands.
- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Arc Presentation of the θ-Curve and Handcuff Graph

- A grid diagram gives rise to an arc presentation and vice versa.

Theorem

Every θ-curve and handcuff graph admit a grid diagram.

Corollary
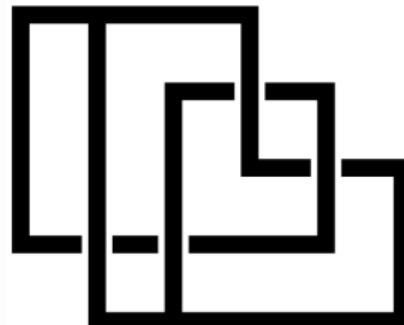
Every θ-curve and handcuff graph admit an arc presentation.

Determinant of θ-curve and Handcuff graph

THC-cromwell matrix

- The **Cromwell Matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ-curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \longmapsto \qquad \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Determinants of the cromwell matrices of Knot

Theorem

Let K be a knot. Then the determinant of a cromwell matrix of K is 0 or ± 2.

PROOF

$$\xmapsto{\text{Grid diagram}}$$

$$\xmapsto{\text{Cromwell}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$
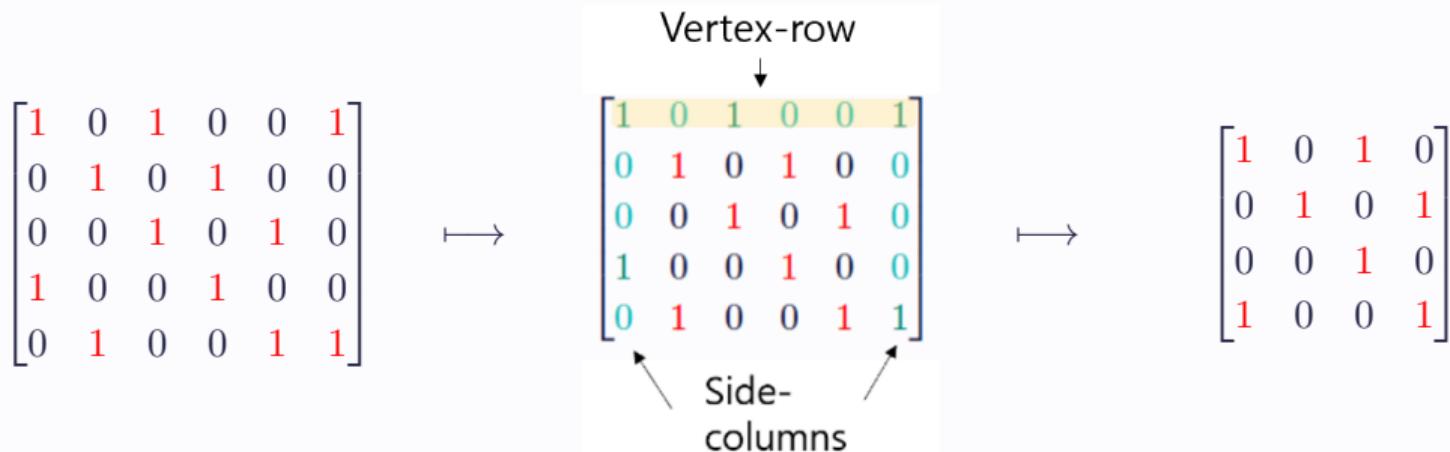
$$\xmapsto{\text{row/column operations}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \xmapsto{\text{row/column operations}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

H-deletion of THC-cromwell matricies

- The **vertex-row** of THC-cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic}, where $a < b < c$, as its elements.
- The **side-column** of THC-cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-cromwell matrix G is $(n-1) \times (n-1)$ matrix which deleted vertex-row and its two side-columns from the matrix G.

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\longmapsto
\begin{matrix}
\text{Vertex-row} \\
\downarrow \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix} \\
\text{Side-} \\
\text{columns}
\end{matrix}
\longmapsto
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
$$

Determinants of the THC-cromwell matrices

Theorem
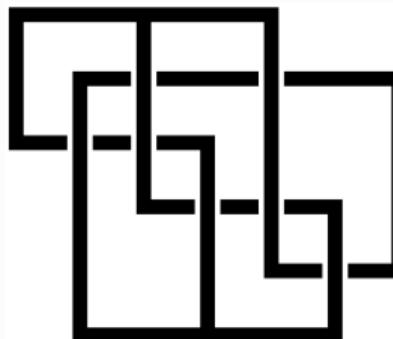
*Let M be a THC-cromwell matrice of θ-curve or handcuff graph.*

- *det*(M) = ±1 ⟺ M represents θ-curve*
- *det*(M) = 0 or ±2 ⟺ M represents handcuff graph*
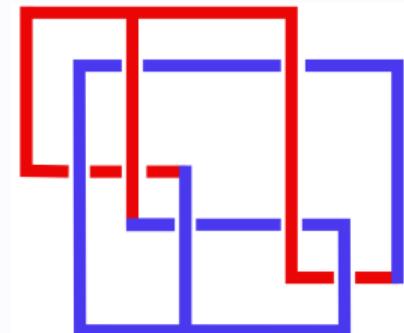
where det*(M) = determinant of H-deletion matrix of M

PROOF

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

⟷

$\xmapsto{H-deletion}$
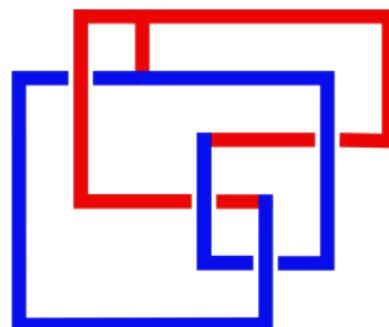
CASE 1. When M represents θ-curve

ⓘ Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$

$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xmapsto{\text{subtracting}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So det*(M) = ± 1
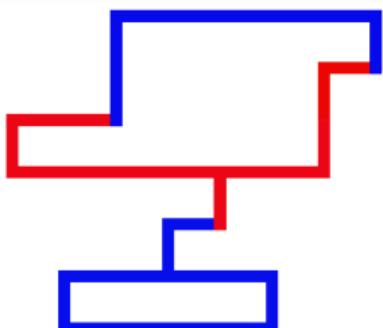
CASE 2. When M represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$

$$\xrightarrow[only\,T-loop]{H-deletion} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[operations]{row/column} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \xrightarrow{seperating} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So det(M) = 0 or ± 2

12

CASE 2. When M represents handcuff graph

ⓘ T-loop & Line-shape

cromwell matrix $\xrightarrow{\;H-deletion\;}$ H-deletion matrix $\xrightarrow{\;seperating\;operations\;}$

$$\begin{bmatrix} T-loop & * \\ O & Line-shape \end{bmatrix} \xrightarrow{\;seperating\;operations\;}$$

$$\begin{bmatrix} Knot & * & * \\ 0 & Line-shape & * \\ 0 & 0 & Line-shape \end{bmatrix}$$

ⓘⓘ Knot & Line-shape

cromwell matrix $\xrightarrow{\;H-deletion\;}$ H-deletion matrix $\xrightarrow{\;seperating\;operations\;}$ $\begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$

So det(M) = 0 or ± 2

Lower and Upper Bounds of Arc Index

Theorem
Let T be any θ-curve. Then,

$$\alpha(T) \leq c(T) + 3.$$

Theorem
Let H be any handcuff graph. Then,

$$\alpha(H) \leq c(H) + 5.$$

Especially, if the constituent link of H is non-split,

$$\alpha(H) \leq c(H) + 3.$$

Theorem

Let T be any non-trivial prime θ-curve or handcuff graph. Then,

$$\alpha(T) \leq c(T) + 3.$$

Result

| 3_1 | 4_1 | 5_1 | 5_2 | 5_3 | 5_4 | 5_5 | 5_6 | 5_7 | 6_1 | 6_2 | 6_3 | 6_4 | 6_5 | 6_6 | 6_7 | 6_8 |

| 6_9 | 6_{10} | 6_{11} | 6_{12} | 6_{13} | 6_{14} | 6_{15} | 6_{16} | 7_1 | 7_2 | 7_3 | 7_4 | 7_5 | 7_6 | 7_7 | 7_8 | 7_9 |

| 7_{10} | 7_{11} | 7_{12} | 7_{13} | 7_{14} | 7_{15} | 7_{16} | 7_{17} | 7_{18} | 7_{19} | 7_{20} | 7_{21} | 7_{22} | 7_{23} | 7_{24} | 7_{25} | 7_{26} |

| 7_{27} | 7_{28} | 7_{29} | 7_{30} | 7_{31} | 7_{32} | 7_{33} | 7_{34} | 7_{35} | 7_{36} | 7_{37} | 7_{38} | 7_{39} | 7_{40} | 7_{41} | 7_{42} | 7_{43} |

| 7_{44} | 7_{45} | 7_{46} | 7_{47} | 7_{48} | 7_{49} | 7_{50} | 7_{51} | 7_{52} | 7_{53} | 7_{54} | 7_{55} | 7_{56} | 7_{57} | 7_{58} | 7_{59} | 7_{60} |

| 7_{61} | 7_{62} | 7_{63} | 7_{64} | 7_{65} |
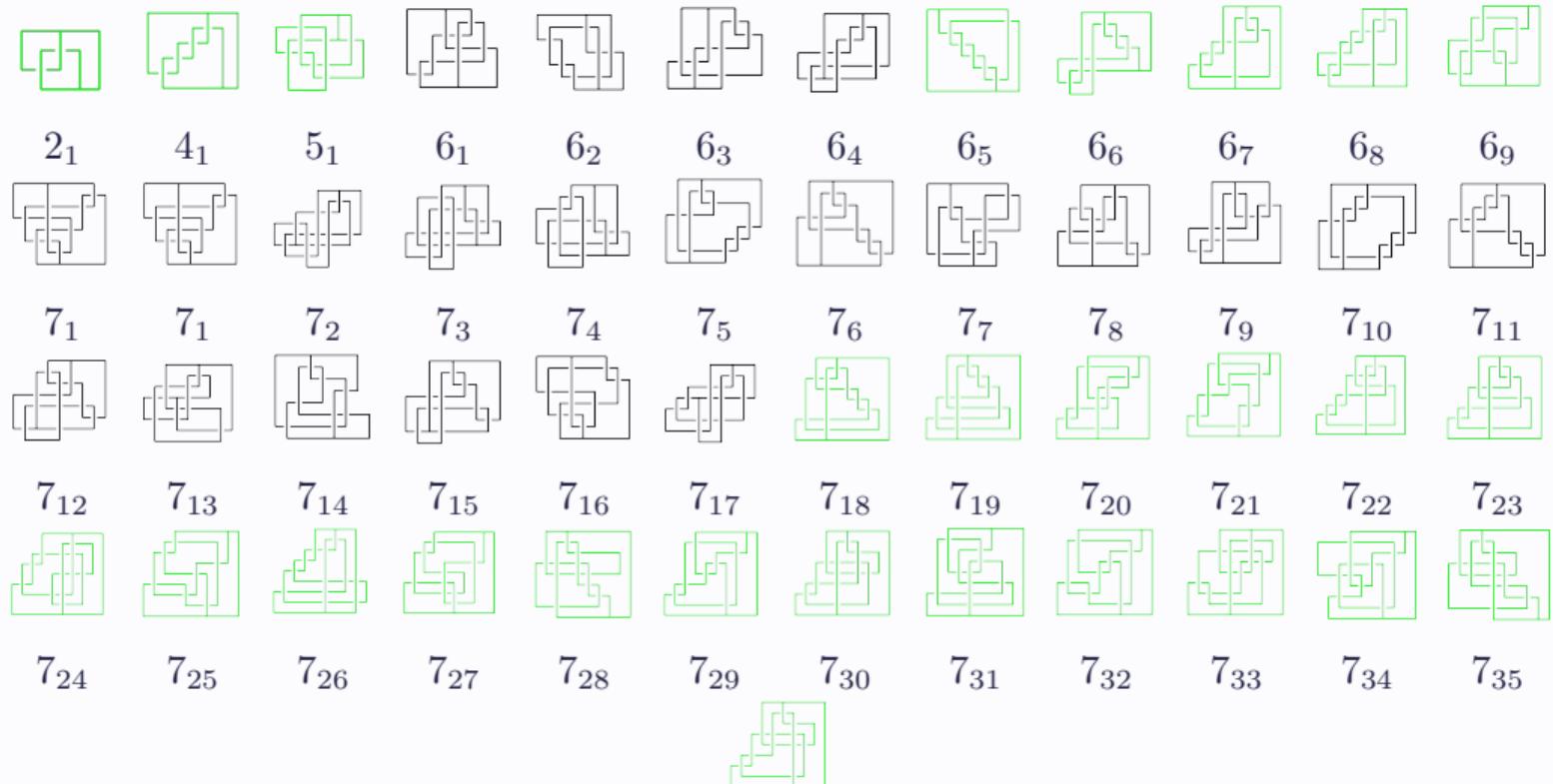
Grid Diagrams of Handcuff-Graphs

| 2_1 | 4_1 | 5_1 | 6_1 | 6_2 | 6_3 | 6_4 | 6_5 | 6_6 | 6_7 | 6_8 | 6_9 |

| 7_1 | 7_1 | 7_2 | 7_3 | 7_4 | 7_5 | 7_6 | 7_7 | 7_8 | 7_9 | 7_{10} | 7_{11} |

| 7_{12} | 7_{13} | 7_{14} | 7_{15} | 7_{16} | 7_{17} | 7_{18} | 7_{19} | 7_{20} | 7_{21} | 7_{22} | 7_{23} |

| 7_{24} | 7_{25} | 7_{26} | 7_{27} | 7_{28} | 7_{29} | 7_{30} | 7_{31} | 7_{32} | 7_{33} | 7_{34} | 7_{35} |

7_{36}

Further Research

Further Research

- We tried to use the Python for determining arc indices. For the Python program, we used the Topoly package, but the package had an error. We would like to use another tool to completely determine the arc indices of θ-curves and handcuff-graphs, such as Yamada package.
- Applying the result of our research to θ-curves or handcuff-graphs of higher crossings or to other spatial graphs can be researched further.