# The Determinant and an Arc Index of Theta Curve and Handcuff-Graph

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# Lower Bounds of Arc Index

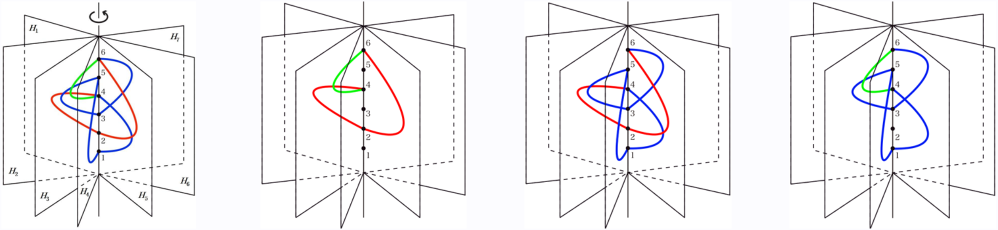
# Lower Bounds from Constituent Knots

**Theorem**

*Let $T$ be any $\theta$-curve and $K_1$, $K_2$, $K_3$ be three constituent knots of $T$. Then*

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

## Proof




$\square$

### Theorem

*Let $T$ be any $\theta$-curve and $K_1$, $K_2$, $K_3$ be three constituent knots of $T$. Then*

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

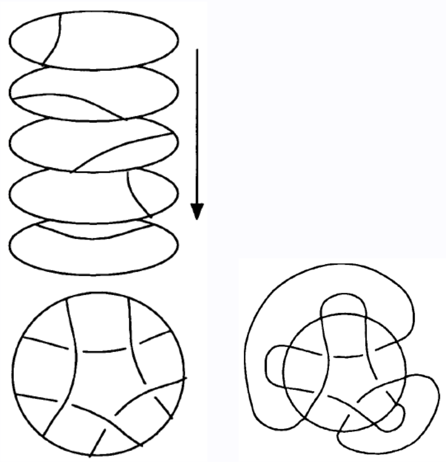
## Proof

- A minimal arc presentation of $T$ is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- $S_i$ be the set of half plane corresponding the edge $e_i$.
- $S_i \cup S_{i+1}$ form an arc presentation of the knot $K_i$.
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

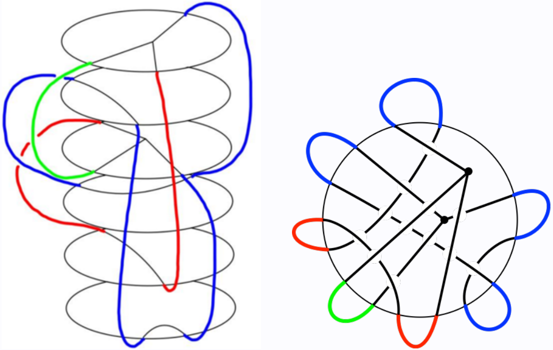
$$\sum_{i=1}^{3} \alpha(K_i) \leq 2 \sum_{i=1}^{3} |S_i| = 2\alpha(T)$$

$\square$

# Stacked Tangle of an $\theta$-Curve



Stacked Tangle of a Link

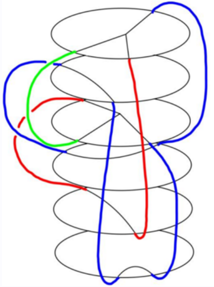


Stacked Tangle of a $\theta$-Curve

Figure from [?]

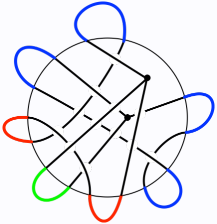
Stacked tangle of an $\theta$-curve is stacked disks each with the frame as boundary with following properties:

- Only two disk called non-simple disks contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called simple disks contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point(by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Simple closure of stacked tangle is a stacked tangle with caps satisfying following properties:

- A cap is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.



Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

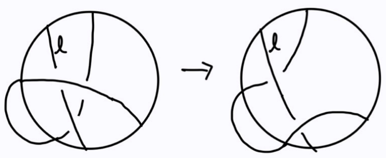
A reduced simple closure of a stacked tangle is

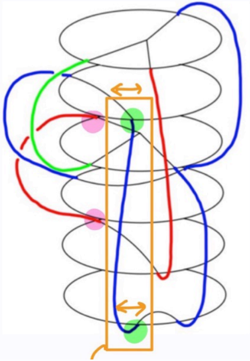
- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

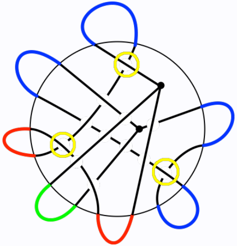
## Proposition

*A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.*

### Proof



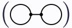
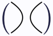
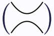
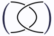
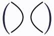
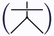
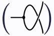
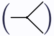
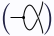
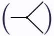





□

## Yamada Polynomials

Let $D_T$ be a diagram of an $\theta$-curve $T$. Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** $R\left(\right.$$\left.\right) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R\left(\right.$$\left.\right) = 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary $\theta$-curve diagram $T'$
- **Y9:** $R\left(\right.$$\left.\right) - R\left(\right.$$\left.\right) = (x - x^{-1})\left[R\left(\right.\right.$$\left.\right) - R\left(\right.$$\left.\left.\right)\right]$
- **Y10:** $R\left(\right.$$\left.\right) = x^2 R\left(\right.$$\left.\right)$, $R\left(\right.$$\left.\right) = x^{-2} R\left(\right.$$\left.\right)$
- **Y11:** $R\left(\right.$$\left.\right) = R\left(\right.$$\left.\right)$ **Y12:** $R\left(\right.$$\left.\right) = R\left(\right.$$\left.\right)$
- **Y13:** $R\left(\right.$$\left.\right) = R\left(\right.$$\left.\right)$, $R\left(\right.$$\left.\right) = R\left(\right.$$\left.\right)$
- **Y14:** $R\left(\right.$$\left.\right) = -xR\left(\right.$$\left.\right)$, $R\left(\right.$$\left.\right) = -x^{-1}R\left(\right.$$\left.\right)$

**Proposition ([?])**

*$R(D_T)$ is an ambient isotopy invariant of $T$ up to multiplying $(-x)^n$ for some integer $n$.*

# Lower Bounds from Yamada Polynomial

**Theorem**

*Let $T$ be any $\theta$-curve or handcuff graph. Then*

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

*where $R(T)$ is a Yamada Polynomial of $T$.*

**Proposition**

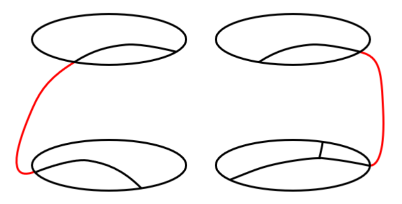
*Let $S_T$ be a simple closure of stacked tangle of a $\theta$-curve or handcuff graph $T$* ***without nested caps****. Then*

$$\max\deg_x R(S_T) \leq c+n, \quad \min\deg_x R(S_T) \geq -(c+n),$$

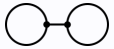
*where* $\boldsymbol{c, n}$ *is the number of caps and crossings in $S_T$, respectively.*

## Proof

- Use double mathematical induction of $(c_s + c_{ss}, n)$.



## Basis Step:

- If $c_s + c_{ss} = 0$, then $S_T$ has no simple disks and is equivalent to the result of applying Y14 to .
  $\therefore R(S_T) = -x^{\pm 3}\left[-x^2 - x - 2 - x^{-1} - x^{-2}\right] \implies 5 \le c + n.$
- If $n = 0$, then $S_T$ is equivalent to  $\cup \bigcirc \cup \cdots \cup \bigcirc$.
  $\therefore R(S_T) = 0 \implies 0 < 2 \le c + n.$

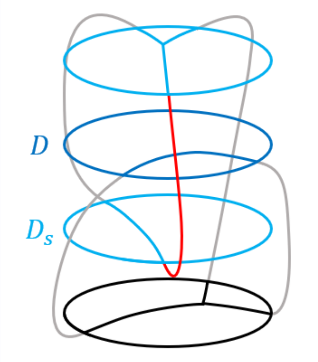
All base cases satisfy the inequality.

## Inductive Step:

Assume that it holds for any $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$, and $c_s + c_{ss} > 0$.

Let $S_T$ be a **simple closure of stacked tangle** of a $\theta$-curve or handcuff graph $T$ such that the number of simple caps, semi-simple caps, and crossings are $c_s$, $c_{ss}$, $n$, respectively.

Take the topmost **simple disk** $D_s$ connected to the top disk, and a **disk** $D$ directly above $D_s$.



## CASE 1. Suppose that there is no cap between $D_s$ and $D$.

### ① Suppose that there is no intersection between $D_s$ and $D$ in $S_T$.

- $D_s$ and $D$ do not affect each other.
- We can swap the position of $D_s$ and $D$ without affecting the rest of the diagram.

### ② Suppose that there is an intersection between $D_s$ and $D$ in $S_T$.

- Let $S_T^-$, $S_T^0$ and $S_T^\infty$ be the simple closure of stacked tangle which is obtained by replacing $\times$ with $\times$, $)($ and $\asymp$, respectively.
- The simple caps, semi-simple caps, and crossings of the both are $c_s, c_{ss}, n-1$.
- Applying Y9

$$R(\times) - R(\times) = (x - x^{-1})\left[R()() - R(\asymp)\right],$$

then

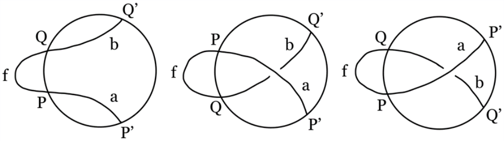
$$R(S_T) - R(S_T^-) = (x - x^{-1})(R(S_T^0) - R(S_T^\infty)).$$

- Then, it is sufficient to show that the interchanged one holds.

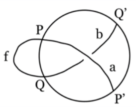
## CASE 2. Suppose that there is a cap between $D_s$ and $D$.

### ① Suppose that $D$ is a simple disk.

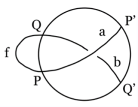
- When view from above, there are three cases:




- After applying Y10, the second and third cases can be regarded as the first case, and the cap can be reduced.

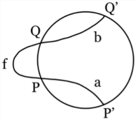



or




$S_T$

$\longrightarrow$




$S'_T$

- $S'_T$ has $c-1$ caps, $c_s-1$ simple caps, $c_{ss}$ semi-simple caps and $n-1$ crossings.

- By induction hypothesis,

$$\begin{aligned}
\max\deg_x R(S_T) &= \max\deg_x R(S'_T) \pm 2 \\
&\leq [(c-1)+(n-1)] \pm 2 \\
&\leq c+n, \\
\min\deg_x R(S_T) &= \min\deg_x R(S'_T) \pm 2 \\
&\geq -[(c-1)+(n-1)] \pm 2 \\
&\geq -(c+n).
\end{aligned}$$

## ② *D* is not a simple disk.

- When viewed from above, all the cases can be reduced as follows.


$S_T$ $S'_T$ $S_T$ $S''_T$

- $R(S_T) = -x^{\pm 1}R(S'_T)$ and $R(S_T) = x^{\pm 2}R(S''_T)$ by Y14 and Y10, respectively.
- Both of $S'_T$ and $S''_T$ have $c-1$ caps, $c_s$ simple caps, $c_{ss}-1$ semi-simple caps, and $n-1$ crossing.

- By induction hypothesis, in the first case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S'_T) \pm 1\\ &\le [(c-1)+(n-1)] \pm 1\\ &\le c+n.\end{aligned}$$

- Similarly, in the second case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S''_T) \pm 2\\ &\le [(c-1)+(n-1)] \pm 2\\ &\le c+n.\end{aligned}$$

- It holds for $\min\deg_x R(S_T)$ in the same way.

$\square$

### Proposition

*Let $S_T$ be a reduced simple closure of stacked tangle of a $\theta$-curve or handcuff graph $T$ corresponding to minimal arc presentation of $T$. Then*

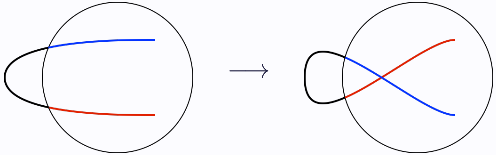
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

*where $n$ is the number of crossings in $S_T$.*

## Proof

- $S_T$ is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps $c$ in $S_T$ is exactly arc index of $T$, $\alpha(T)$.

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by $1$.
- $p$ of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the $p$ negative($c - p$ positive) curls.

| | $S_T^{neg}$ | $S_T^{pos}$ |
|---|---|---|
| Number of Caps | $c$ | $c$ |
| Number of Crossings | $n+p$ | $n+(c-p)$ |

- $R\left(S_T^{neg}\right) = x^{-2p}R(S_T)$ and $R\left(S_T^{pos}\right) = x^{2(c-p)}R(S_T)$

$$
\begin{aligned}
\min\deg_x R(S_T) - 2p &= \min\deg_x R\left(S_T^{neg}\right) \\
&\geq -c + -(n+p) \\
\max\deg_x R(S_T) + 2(c-p) &= \max\deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n + (c-p)] \\
\min\deg_x R(S_T) &\geq -c - n + p \\
\max\deg_x R(S_T) &\leq n + p \\
\max\deg_x R(S_T) - \min\deg_x R(S_T) &\leq c + 2n
\end{aligned}
$$

□

**Theorem**

*Let $T$ be any $\theta$-curve or handcuff graph. Then*

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

*where $R(T)$ is a Yamada Polynomial of $T$.*

**Proof**

Let $S_T$ be a reduce simple closure of stacked tangle of a $\theta$-curve or handcuff graph $T$ corresponding to minimal arc presentation of $T$.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : $2$
- The number of simple disks : $\alpha(T) - 3$

## ① Let $T$ be any $\theta$-curve.

Consider the maximum number of crossings in $S_T$.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)$
- number of crossings by a simple disk and non-simple disk : $2(\alpha(T)-3)$
- number of crossings by two non-simple disks : $2$
- number of crossings counted by disks joined by cap : $\alpha(T)-2$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)+2(\alpha(T)-3)+2-(\alpha(T)-2) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

$$\max\deg_x R(S_T) - \min\deg_x R(S_T) \leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8$$

$$2 + \sqrt{\max\deg_x R(S_T) - \min\deg_x R(S_T) - 4} \leq \alpha(T)$$

## ② Let $T$ be any handcuff graph.

Consider the maximum number of crossings in $S_T$.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings by two non-simple disks : $1$
- number of crossings counted by disks joined by cap : $\alpha(T)-1-2=\alpha(T)-3$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)+2\left(\alpha(T)-3\right)+1-\left(\alpha(T)-3\right) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

$$\max\deg_x R(S_T)-\min\deg_x R(S_T)\leq 2n+\alpha(T)\leq \alpha(T)^2-4\alpha(T)+8$$

$$2+\sqrt{\max\deg_x R(S_T)-\min\deg_x R(S_T)-4}\leq \alpha(T)$$

$\square$