The Determinant and Arc Indices of θ-Curves and Handcuff-Graphs

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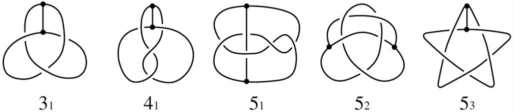
Introduction

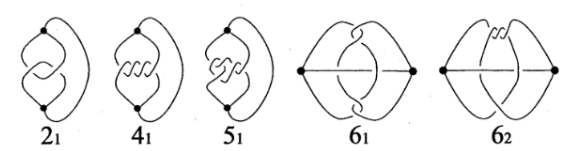
θ-Curves and Handcuff Graphs

- A **θ-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot T_{ij}**, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.

- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.
- A **constituent link H_{12}**, is a subgraph of H that consists of two vertices v_1, v_2 and two edges e_1, e_2.



θ-Curves



Handcuff Graphs

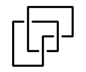
Arc Presentation

- Arc presentation of a θ-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called pages).
- The embedding is with the common boundary line (called axis).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- Arc index, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is minimal arc presentation.

Arc Presentation

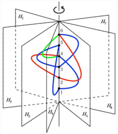
Trefoil

Open Book

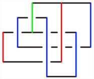


Grid Diagram

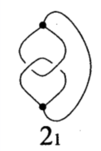
$\theta_{5,2}$



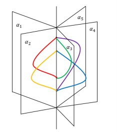
Open Book



Grid Diagram



$\Phi_{2,1}$



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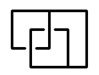


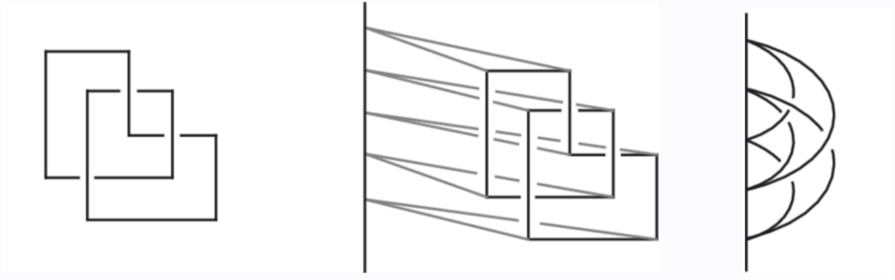
Grid Diagram

Grid Diagram

- The **grid diagram** of θ-curve or handcuff graph is a diagram with only vertical and horizontal strands.
- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Arc Presentation of the θ-Curve and Handcuff Graph

- A grid diagram gives rise to an arc presentation and vice versa.



Theorem

Every θ-curve and handcuff graph admit a grid diagram.

Corollary

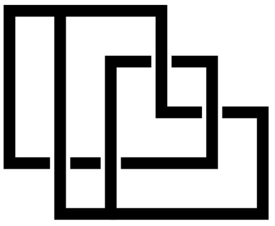
Every θ-curve and handcuff graph admit an arc presentation.

Determinant of θ-curve and Handcuff graph

THC-cromwell matrix

- The **Cromwell Matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ-curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\longmapsto \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

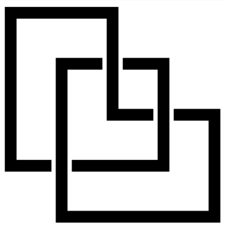
Determinants of the cromwell matrices of Knot

Theorem

Let K be a knot. Then the determinant of a cromwell matrix of K is 0 or ± 2.

Proof

$$\xmapsto[\text{diagram}]{\text{Grid}}$$



$$\xmapsto{\text{Cromwell}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

H-deletion of THC-cromwell matricies

- The **vertex-row** of THC-cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic}, where $a < b < c$, as its elements.
- The **side-column** of THC-cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-cromwell matrix G is $(n-1)\times(n-1)$ matrix which deleted vertex-row and its two side-columns from the matrix G.

Vertex-row

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Side-columns

Determinants of the THC-cromwell matrices

Theorem

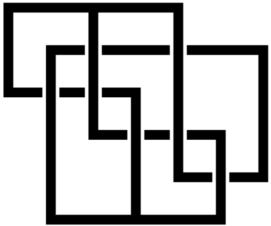
Let M be a THC-cromwell matrice of θ-curve or handcuff graph.

- $det^*(M) = \pm 1 \iff M$ *represents* θ*-curve*
- $det^*(M) = 0$ *or* $\pm 2 \iff M$ *represents handcuff graph*

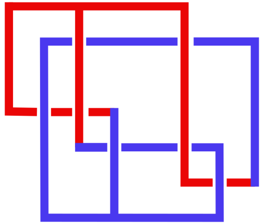
where $det^*(M)$ = *determinant of H-deletion matrix of* M

Proof

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



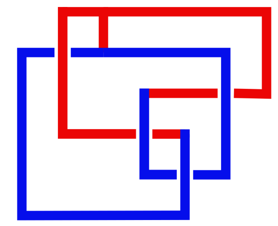
$\xmapsto{H-deletion}$



CASE 1. When M represents θ-curve

Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

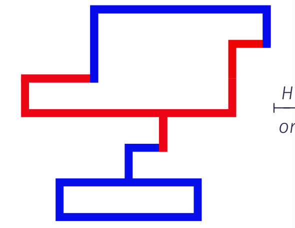
$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xmapsto{\text{subtracting}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det^*(M) = \pm 1$

CASE 2. When M represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto[\text{only } T-loop]{H-deletion} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xmapsto{\text{seperating}} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So $\det(M) = 0$ or ± 2

CASE 2. When M represents handcuff graph

ⓘ T-loop & Line-shape

$$\text{cromwell matrix} \xmapsto{H-deletion} \text{H-deletion matrix} \xmapsto[operations]{seperating}$$

$$\begin{bmatrix} T-loop & * \\ O & Line-shape \end{bmatrix} \xmapsto[operations]{seperating} \begin{bmatrix} Knot & * & * \\ 0 & Line-shape & * \\ 0 & 0 & Line-shape \end{bmatrix}$$

ⓘⓘ Knot & Line-shape

$$\text{cromwell matrix} \xmapsto{H-deletion} \text{H-deletion matrix} \xmapsto[operations]{seperating} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So det(M) = 0 or ± 2

Upper Bounds of Arc Index

Theorem

Let T be any non-trivial prime θ-curve or handcuff graph. Then,

$$\alpha(T) \leq c(T) + 3.$$

Lower Bounds of Arc Index

Lower Bounds from Constituent Knots/Links

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

Lower Bounds from Constituent Knots/Links

Theorem

Let H be any handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then

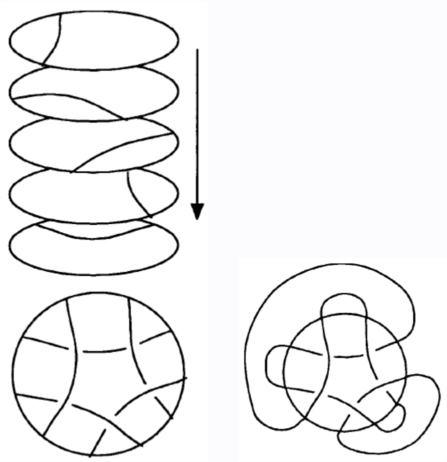
$$\alpha(H) \geq c(L) + 3.$$

Corollary

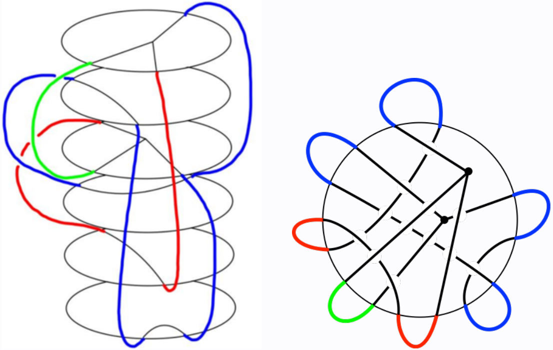
Let H be any handcuff graph, and L be a constituent link of T. If L is alternating and non-split,

$$\alpha(H) = c(L) + 3.$$

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link

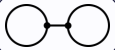
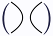
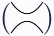
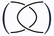
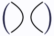
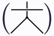
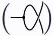
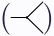
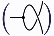
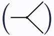


Stacked Tangle of a θ-Curve

Figure from [?]

Yamada Polynomials

Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$** is calculated by the following properties:

- **Y6:** R () $= -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** R () $= 0$
- **Y8:** $R(T' \cup$ $) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** R () $- R$ () $= (x - x^{-1})\left[R\right.$ () $- R$ ()$\left.\right]$
- **Y10:** R () $= x^2 R$ (), R () $= x^{-2} R$ ()
- **Y11:** R () $= R$ () **Y12:** R () $= R$ ()
- **Y13:** R () $= R$ (), R () $= R$ ()
- **Y14:** R () $= -xR$ (), R () $= -x^{-1}R$ ()

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proposition

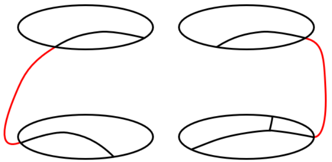
Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T ***without nested caps****. Then*

$$\max\deg_x R(S_T) \leq c + n, \quad \min\deg_x R(S_T) \geq -(c+n),$$

where $\boldsymbol{c}, \boldsymbol{n}$ *is the number of caps and crossings in S_T, respectively.*

Proof

- Let c_s, c_{ss} be the number of **simlpe caps** or **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T. Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

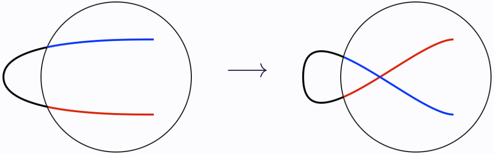
where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

$$\min \deg_x R(S_T) - 2p = \min \deg_x R\left(S_T^{neg}\right) \geq -c + -(n+p)$$
$$\max \deg_x R(S_T) + 2(c-p) = \max \deg_x R\left(S_T^{pos}\right) \leq c + [n+(c-p)]$$
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq c + 2n \qquad \square$$

Proof of Theorem

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proof

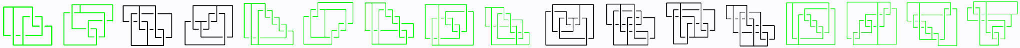
Let S_T be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

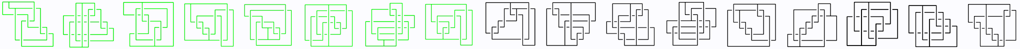
Consider the maximum number of crossings in S_T, and use the previous theorem.

Result

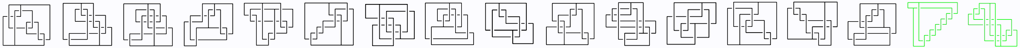
Grid Diagrams of θ-Curves



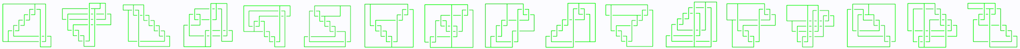
3_1 4_1 5_1 5_2 5_3 5_4 5_5 5_6 5_7 6_1 6_2 6_3 6_4 6_5 6_6 6_7 6_8



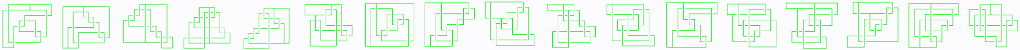
6_9 6_{10} 6_{11} 6_{12} 6_{13} 6_{14} 6_{15} 6_{16} 7_1 7_2 7_3 7_4 7_5 7_6 7_7 7_8 7_9



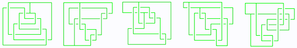
7_{10} 7_{11} 7_{12} 7_{13} 7_{14} 7_{15} 7_{16} 7_{17} 7_{18} 7_{19} 7_{20} 7_{21} 7_{22} 7_{23} 7_{24} 7_{25} 7_{26}



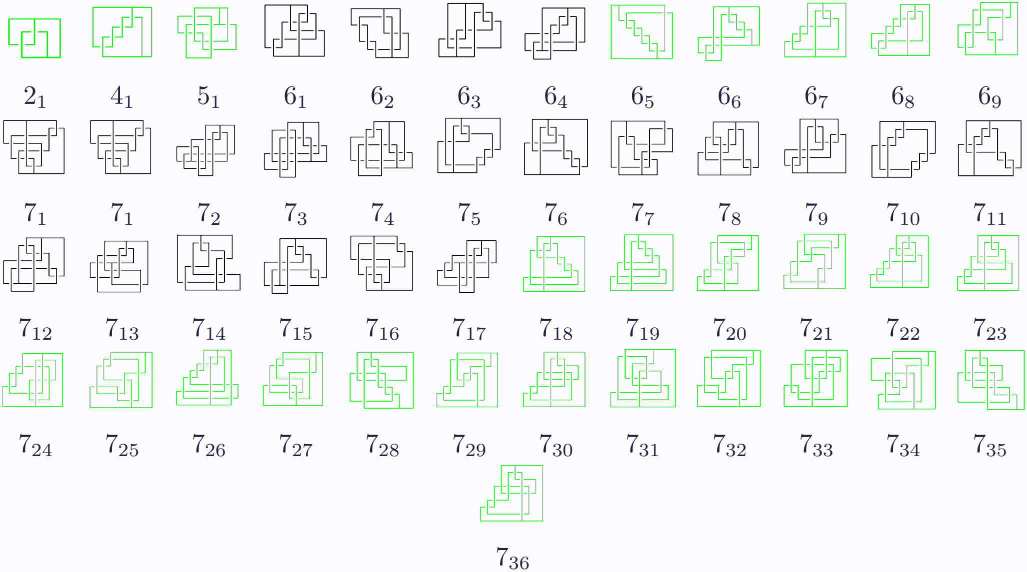
7_{27} 7_{28} 7_{29} 7_{30} 7_{31} 7_{32} 7_{33} 7_{34} 7_{35} 7_{36} 7_{37} 7_{38} 7_{39} 7_{40} 7_{41} 7_{42} 7_{43}



7_{44} 7_{45} 7_{46} 7_{47} 7_{48} 7_{49} 7_{50} 7_{51} 7_{52} 7_{53} 7_{54} 7_{55} 7_{56} 7_{57} 7_{58} 7_{59} 7_{60}



7_{61} 7_{62} 7_{63} 7_{64} 7_{65}


2_1 4_1 5_1 6_1 6_2 6_3 6_4 6_5 6_6 6_7 6_8 6_9
7_1 7_1 7_2 7_3 7_4 7_5 7_6 7_7 7_8 7_9 7_{10} 7_{11}
7_{12} 7_{13} 7_{14} 7_{15} 7_{16} 7_{17} 7_{18} 7_{19} 7_{20} 7_{21} 7_{22} 7_{23}
7_{24} 7_{25} 7_{26} 7_{27} 7_{28} 7_{29} 7_{30} 7_{31} 7_{32} 7_{33} 7_{34} 7_{35}
7_{36}

Further Research

Further Research

- We tried to use the Python for determining arc indices. For the Python program, we used the Topoly package, but the package had an error. We would like to use another tool to completely determine the arc indices of θ-curves and handcuff-graphs, such as Yamada package.
- Applying the result of our research to θ-curves or handcuff-graphs of higher crossings or to other spatial graphs can be researched further.