Journal of Knot Theory and Its Ramifications
Vol. 18, No. 2 (2009) 167–197

AN ENUMERATION OF THETA-CURVES WITH UP TO SEVEN CROSSINGS

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Accepted 30 November 2007

ABSTRACT

We enumerate all the prime θ-curves with up to seven crossings to confirm a table made by Litherland. These θ-curves are mutually distinguished by the Yamada polynomial.

Keywords: Prime θ-curve; prime basic θ-polyhedron; Yamada polynomial.

Mathematics Subject Classification 2000: 57M15, 05C10, 57M25

1. Introduction

A θ-curve is a graph in the 3-sphere S^3 which consists of two vertices and three edges, each edge joining the two vertices. We consider unoriented θ-curves, so we call two θ-curves Θ_1 and Θ_2 are *equivalent* if there is a homeomorphism $h : (S^3, \Theta_1) \to (S^3, \Theta_2)$. In a letter [12], Litherland announced a table of prime θ-curves with up to seven crossings (see Fig. 1), where the completeness of his table has not been proved. We enumerate prime θ-curves with up to seven crossings by extending Conway's method in order to confirm his table. In [2], Conway made an enumeration of prime knots and links by introducing the concept of a *tangle* and a *basic polyhedron*. Here, a polyhedron is a connected 4-regular graph embedded in 2-sphere. In this paper, we construct a *prime basic θ-polyhedron* to enumerate prime θ-curves. Here, a θ-polyhedron is a connected graph embedded in 2-sphere, whose two vertices are 3-valent, and the others are 4-valent. Thus, our θ-polyhedron is different from Conway's polyhedron. We can obtain a θ-curve diagram from a θ-polyhedron by substituting algebraic tangles for their 4-valent vertices.

This paper is organized as follows: In Sec. 2, we give some definitions related to a θ-curve. In Sec. 3, we review Conway's method. In Sec. 4, we construct prime basic θ-polyhedra. In Sec. 5, we give a conclusion. In Sec. 6, we list the Yamada polynomial of each θ-curve. Finally, in Appendix A, we give a table of θ-curve diagrams using our notation.

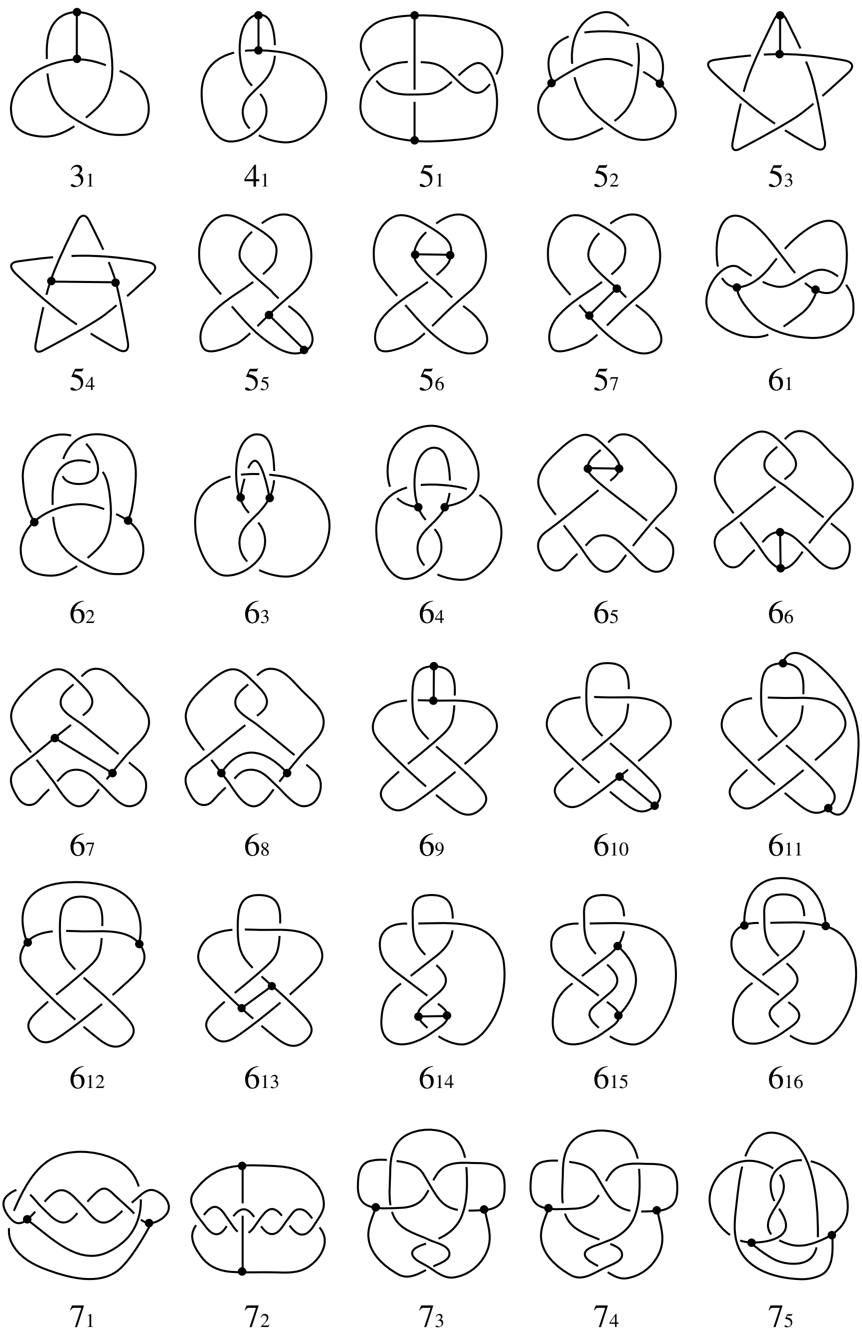


Fig. 1. Prime θ-curves with up to seven crossings.

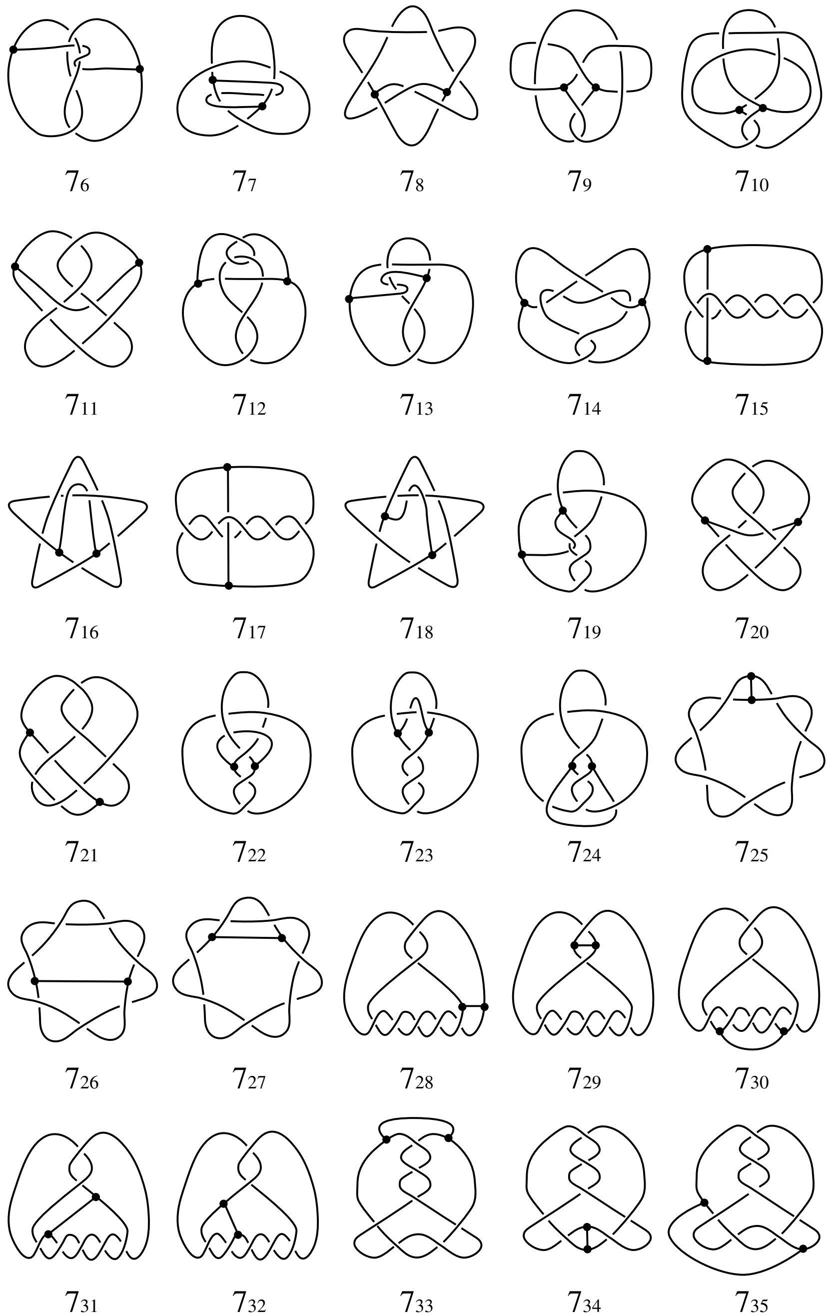


Fig. 1. (*Continued*)

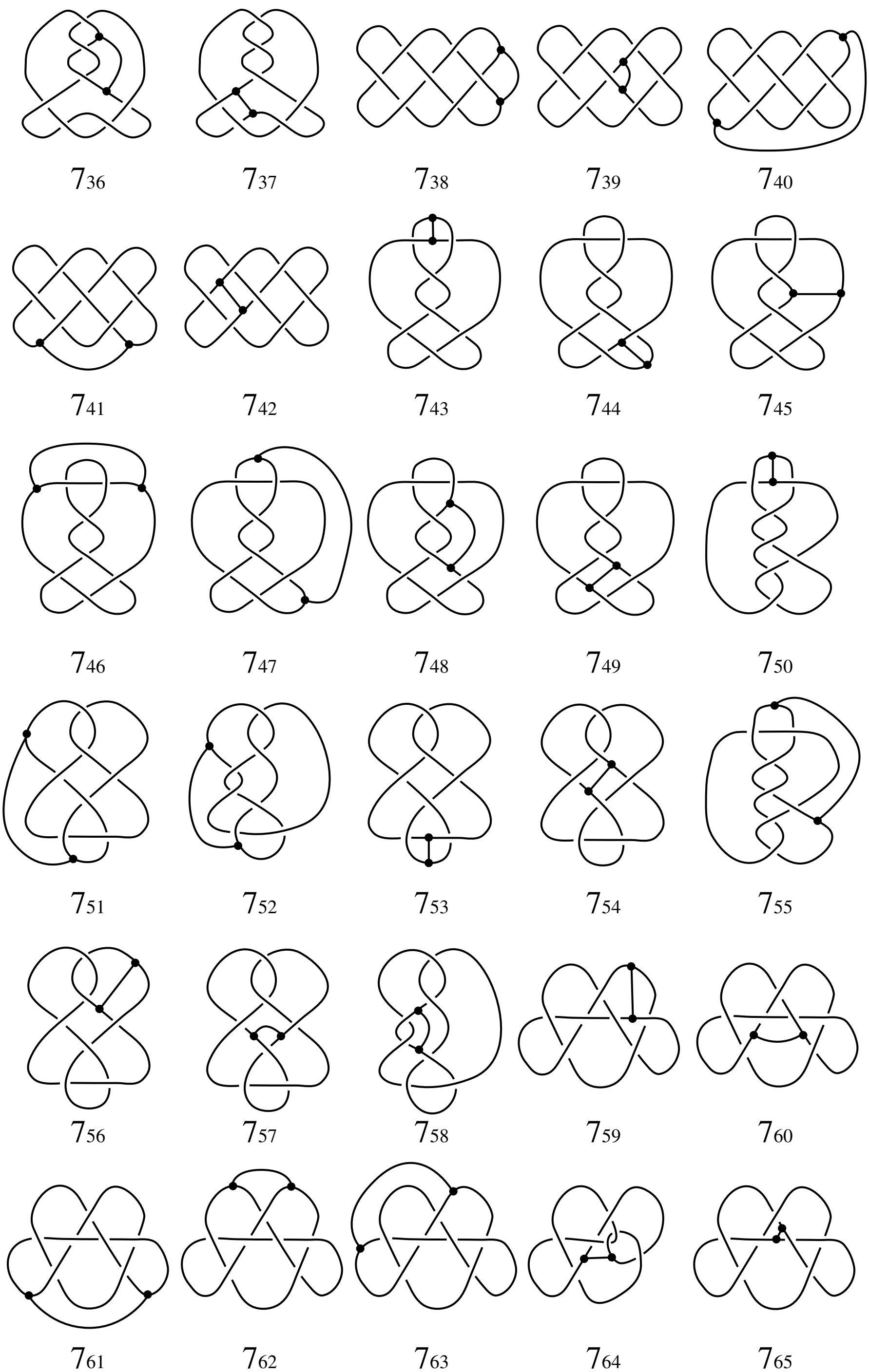


Fig. 1. (*Continued*)

2. θ-Curve

A *θ-curve* Θ is a graph embedded in S^3, which consists of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), such that each edge joins the vertices. A *constituent knot* Θ_{ij}, $1 \leq i < j \leq 3$, is a subgraph of Θ that consists of two vertices (v_1, v_2) and two edges (e_i, e_j). θ-curves are roughly classified by comparing the triples of constituent knots. A θ-curve is said to be *trivial* if it can be embedded in a 2-sphere in S^3.

Definition 2.1 (Litherland [12]). A θ-curve is said to be *prime* if it satisfies the following conditions:

(C1) it is nontrivial;
(C2) it is not the order-2 vertex connected sum of nontrivial knot and (possibly trivial) θ-curve;
(C3) it is not the order-3 vertex connected sum of two nontrivial θ-curves.

Here, an *order-n vertex connected sum* of spatial graphs is defined as in Fig. 2 (see [26]). A nontrivial knot K, such as in Fig. 2(a), is called a *local knot* of a θ-curve.

Remark 2.2. We may consider that a θ-curve Θ exists in $\mathbf{R}^3$, and present Θ by a *θ-curve diagram* θ in an obvious manner, which consists of two vertices and three

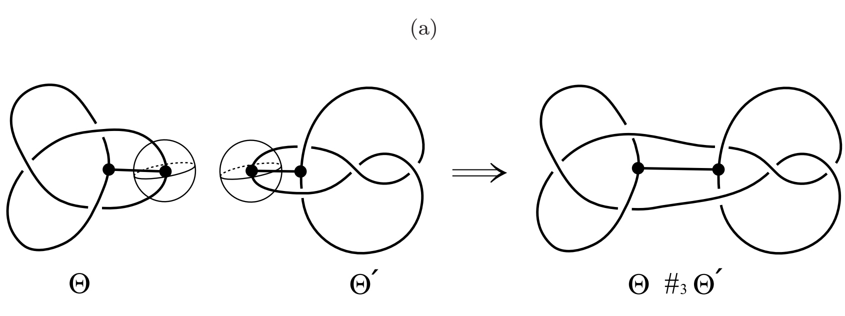


Fig. 2. (a) An order-2 vertex connected sum of spatial graphs; (b) an order-3 vertex connected sum of spatial graphs.

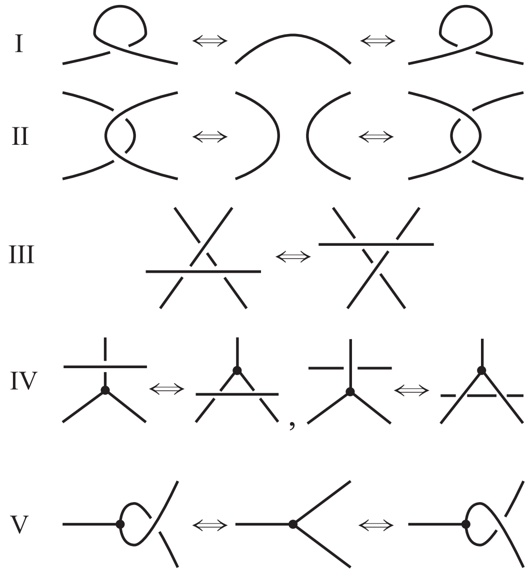


Fig. 3. Reidemeister moves for θ-curves.

edges immersed in $\mathbf{R}^2$ such that

(D1) each intersection is a transverse double point, and
(D2) in a neighborhood of each double point one of the two segments is removed.
We also present a constituent knot Θ_{ij} by a diagram θ_{ij}.

We often use the following lemma to conclude that two θ-curves are equivalent.

Lemma 2.3 (Kauffman [6]). *If two θ-curves are ambient isotopic, then any diagrams of them are related by a finite sequence of the moves of Fig.* 3.

3. Conway's Method

We review Conway's method [2]. We define a *tangle* as a pair (B^3, t), where t is a 1-manifold properly embedded in a unit 3-ball

$B^3 = \{(x, y, z) \in R^3 \mid x^2 + y^2 + z^2 \leq 1\}$ with four boundary components

$$\mathrm{NE} = \big(1/\sqrt{2}, 1/\sqrt{2}, 0\big), \qquad \mathrm{SE} = \big(1/\sqrt{2}, -1/\sqrt{2}, 0\big),$$
$$\mathrm{SW} = \big(-1/\sqrt{2}, -1/\sqrt{2}, 0\big), \quad \mathrm{NW} = \big(-1/\sqrt{2}, 1/\sqrt{2}, 0\big);$$

see Fig. 4.

Let $T = (B^3, t)$ be a tangle such that t consists of two arcs and n circles. We say T is a V^n*-tangle* (respectively, an H^n*-tangle*, an X^n*-tangle*) if T has an arc connecting NE and SE (respectively, NW, SW).

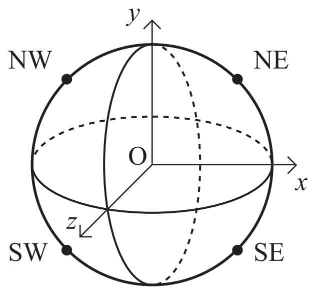


Fig. 4. A unit 3-ball B^3 and 4 boundary components of t.

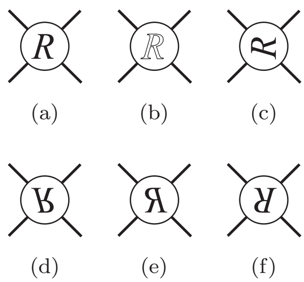


Fig. 5. (a) A tangle R; (b) the tangle μR; (c) the tangle νR; (d) the tangle $\rho_x R$; (e) the tangle $\rho_y R$; (f) the tangle $\rho_z R$.

We present a tangle by a regular diagram as in Fig. 5(a), where we use the projection $(x, y, z) \mapsto (x, y)$. Let R be a tangle. We denote by μR, νR, $\rho_x R$, $\rho_y R$, $\rho_z R$ the tangles obtained from R by reflecting with regard to the xy-plane; $\mu(x, y, z) = (x, y, -z)$, by turning it counter-clockwise by $\pi/2$; $\nu(x, y, z) = (-y, x, z)$, by rotating it through angle π; $\rho_x(x, y, z) = (x, -y, -z)$, $\rho_y(x, y, z) = (-x, y, -z)$, and $\rho_z(x, y, z) = (-x, -y, z)$, respectively. We present these tangles diagrammatically as shown in Fig. 5. We call μR the *mirror image* of R.

We say that two tangles are *isotopic* if there is an isotopy of the 3-ball B^3 that takes one tangle to the other while fixing each point of the boundary, that is, their diagrams are related by a finite sequence of *Reidemeister moves* I–III as shown in Fig. 3 inside the circle defining the tangle while the endpoints of the strings remain fixed.

Definition 3.1. We say that two tangles T and T' are *equivalent* if T is isotopic to one of the following eight tangles:

$$T', \rho_x T', \rho_y T', \rho_z T', \nu T', \nu\rho_x T', \nu\rho_y T', \nu\rho_z T'.$$

For a tangle diagram D, we denote by $c(D)$ the number of crossings of D. The *crossing number* of a tangle T, denoted by $c(T)$, is the minimal number of $c(D)$'s for all the diagrams D which present the equivalence class of T.

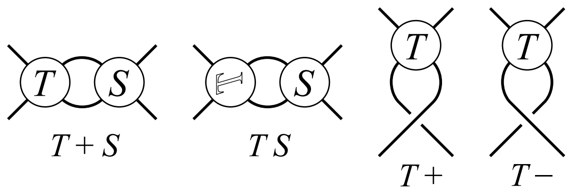


Fig. 6. The operations.

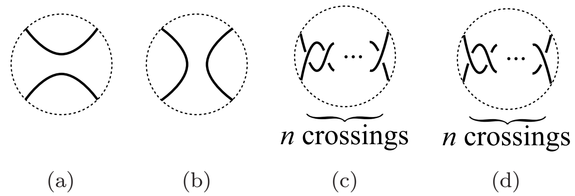


Fig. 7. (a) The 0 tangle; (b) the ∞ tangle; (c) the n tangle; (d) the $-n$ tangle.

Given two tangles T and S, we define new tangles $T+S$, TS, $T+$ and $T-$ as shown in Fig. 6; $T+S$ and TS are the *sum* and *product* of T and S, respectively. Notice that $TS = \rho_x\mu\nu(T) + S$, where $\rho_x\mu\nu(T)$ is the tangle obtained from T by reflecting across the NW and SE diagonal line.

The simplest tangles are the 0 and ∞ tangles as shown in Figs. 7(a) and 7(b). Further, for a positive integer n, we define the n tangle and the $-n$ tangle as shown in Figs. 7(c) and 7(d), which are called *integral tangles*.

A tangle T is said to be *algebraic* if T is obtained from the 0 and ∞ tangles by a finite sequence of the operations given in Fig. 6. Thus, an algebraic tangle is obtained from the 0, ∞ and integral tangles by the operations of addition and multiplication. We denote the n tangle simply by n, and the $-n$ tangle by $\bar{n}$. For integral tangles $a_1, a_2, a_3, \ldots, a_{i-1}, a_i$, the tangle $a_1a_2a_3\cdots a_{i-1}a_i$, abbreviating $((\cdots(a_1\ a_2)a_3\cdots a_{i-1})a_i)$, is called a *rational tangle*. Two rational tangles $a_1a_2\ldots a_{i-1}a_i$ and $b_1b_2\ldots b_{j-1}b_j$ are isotopic if and only if the corresponding rational numbers (including $1/0 = \infty$)

$$a_i + \cfrac{1}{a_{i-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}} \quad \text{and} \quad b_j + \cfrac{1}{b_{j-1} + \cfrac{1}{\ddots + \cfrac{1}{b_2 + \cfrac{1}{b_1}}}}$$

are the same.

Remark 3.2. For the above continued fraction, we can assume that each a_m $(1 \leq m \leq i)$ has the same sign.

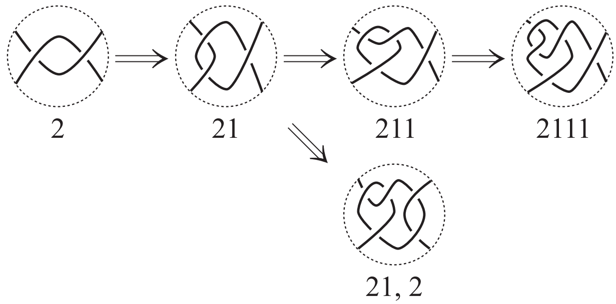


Fig. 8. Algebraic tangles.

The comma notation $(a_1, a_2, \ldots, a_i) = (a_1\, 0) + (a_2\, 0) + \cdots + (a_i\, 0)$ is preferred to the sum notation, but is only used with two or more terms in the bracket. Figure 8 shows the step-by-step formation of two algebraic tangles 2 1 1 1 and 2 1, 2 as examples.

In [14], we listed a table of algebraic tangles with up to seven crossings, where we identified a tangle T with the tangles which are equivalent to T or $\mu(T)$.

Lemma 3.3. *Unsplittable algebraic tangle diagrams with up to seven crossings are as follows*:

1

2

3   2 1

4   3 1   2 2   2 1 1

2, 2   2, $\bar{2}$

5   4 1   3 2   3 1 1   2 3   2 2 1   2 1 2   2 1 1 1

2, 2+   (2, 2)1   (2, 2)$\bar{1}$ 3, 2   3, $\bar{2}$ 2 1, 2

6   5 1   4 2   4 1 1   3 3   3 2 1   3 1 2   3 1 1 1   2 4   2 3 1   2 2 2   2 2 1 1

2 1 3   2 1 2 1   2 1 1 2   2 1 1 1 1

2, 2 + +   (2, 2)2   (2, 2)$\bar{2}$ (2, $\bar{2}$)2   3, 2+   2 1, 2+

(3, 2)1   (3, 2)$\bar{1}$ (3, $\bar{2}$)$\bar{1}$ (2 1, 2)1   (2, 2+)1   (2, 2)1 1

4, 2   4, $\bar{2}$ 3 1, 2   2 2, 2   2 2, $\bar{2}$ 2 1 1, 2   (2, 2), 2   (2, 2), $\bar{2}$ (2, $\bar{2}$), 2

3, 3   3, $\bar{3}$ 3, 2 1   3, $\bar{2}$ $\bar{1}$ 2 1, 2 1

2, 2, 2   2, 2, $\bar{2}$

7   6 1   5 2   5 1 1   4 3   4 2 1   4 1 2   4 1 1 1   3 4   3 3 1   3 2 2   3 2 1 1

3 1 3   3 1 2 1   3 1 1 2   3 1 1 1 1   2 5   2 3 1   2 3 2   2 3 1 1   2 2 3   2 2 2 1

2 2 1 2   2 2 1 1 1   2 1 4   2 1 3 1   2 1 2 2   2 1 2 1 1   2 1 1 3   2 1 1 2 1

2 1 1 1 2   2 1 1 1 1 1

2, 2 + ++   (2, 2)3   (2, 2)$\bar{3}$ (2, $\bar{2}$)3   3, 2 + +   2 1, 2 + +   (3, 2)2   (3, 2)$\bar{2}$

(3, $\bar{2}$)2   (3, $\bar{2}$)$\bar{2}$ (2 1, 2)2   (2 1, 2)$\bar{2}$ (2, 2+)2   (2, 2)1 2

4, 2+   3 1, 2+   2 2, 2+   2 1 1, 2+   (2, 2), 2+   3, 3+   3, 2 1+   2 1, 2 1+   2, 2, 2+

(2, 2 + +)1   (2, 2)2 1   (2, 2)$\bar{2}$ $\bar{1}$ (2, $\bar{2}$)2 1   (3, 2+)1   (2 1, 2+)1

$(3,2)1\ 1 \quad (2\ 1,2)1\ 1 \quad (2,2+)1\ 1 \quad (2,2)1\ 1\ 1$
$(4,2)1 \quad (4,2)\bar{1} \quad (4,\bar{2})1 \quad (4,\bar{2})\bar{1} \quad (3\ 1,2)1 \quad (2\ 2,2)1 \quad (2\ 2,2)\bar{1} \quad (2\ 2,\bar{2})1 \quad (2\ 2,\bar{2})\bar{1}$
$(2\ 1\ 1,2)1 \quad ((2,2),2)1 \quad ((2,2),2)\bar{1} \quad ((2,2),\bar{2})\bar{1} \quad ((2,\bar{2}),2)1 \quad ((2,\bar{2}),2)\bar{1}$
$(3,3)1 \quad (3,3)\bar{1} \quad (3,\bar{3})1 \quad (3,2\ 1)1 \quad (3,\bar{2}\ \bar{1})\bar{1} \quad (2\ 1,2\ 1)1$
$(2,2,2)1 \quad (2,2,2)\bar{1} \quad (2,2,\bar{2})\bar{1}$
$5,2 \quad 5,\bar{2} \quad 4\ 1,2 \quad 3\ 2,2 \quad 3\ 2,\bar{2} \quad 3\ 1\ 1,2 \quad 2\ 3,2 \quad 2\ 3,\bar{2} \quad 2\ 2\ 1,2 \quad 2\ 1\ 2,2 \quad 2\ 1\ 2,\bar{2}$
$2\ 1\ 1\ 1,2 \quad (3,2),2 \quad (3,2),\bar{2} \quad (3,\bar{2}),2 \quad (3,\bar{2}),\bar{2} \quad (2\ 1,2),2 \quad (2\ 1,2),\bar{2}$
$(2,2+),2 \quad (2,2+),\bar{2} \quad ((2,2)1),2 \quad ((2,2)\bar{1}),\bar{2} \quad ((2,\bar{2})1),2$
$4,3 \quad 4,\bar{3} \quad 3\ 1,3 \quad 3\ 1,\bar{3} \quad 2\ 2,3 \quad 2\ 2,\bar{3} \quad 2\ 1\ 1,3 \quad 2\ 1\ 1,\bar{3} \quad (2,2),3 \quad (2,2),\bar{3} \quad (2,\bar{2}),3$
$4,2\ 1 \quad 3\ 1,2\ 1 \quad 2\ 2,2\ 1 \quad 2\ 1\ 1,2\ 1 \quad (2,2),2\ 1 \quad (2,2),\bar{2}\ \bar{1} \quad (2,\bar{2}),2\ 1$
$3,2,2 \quad 3,2,\bar{2} \quad 3,\bar{2},\bar{2} \quad 2\ 1,2,2.$

A connected 4-valent graph embedded in 2-sphere we call a *polyhedron*, and a polyhedron is *basic* if there exists no bigon. We can obtain link diagrams from basic polyhedra by substituting tangles for their vertices. By using Conway's method, we can enumerate all links in order of crossing numbers thoroughly.

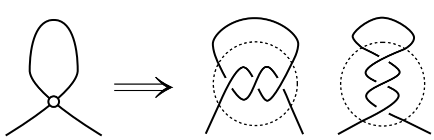
4. Prime Basic θ-Polyhedron

A *θ-polyhedron* is a connected graph embedded in 2-sphere, whose two vertices are 3-valent, and the others are 4-valent. We can obtain a θ-curve diagram from a θ-polyhedron by substituting algebraic tangles for their 4-valent vertices. A θ-polyhedron P_Θ is said to be *basic* if it does not contain a loop nor a bigon whose vertices are 4-valent.

Remark 4.1. If P_Θ contains a loop, then we have a θ-curve diagram with a local knot or nugatory crossings from P_Θ (see Fig. 9).

Remark 4.2. If P_Θ contains a bigon, then we have a θ-curve diagram obtained from P_Θ is also obtained from another polyhedron P'_Θ with fewer 4-valent vertices than P_Θ. In fact, adding two algebraic tangles, we have an algebraic tangle (see Fig. 10).

Let C be a circle which decomposes S^2 into 2-disks D_1, D_2. A circle C is called a *cutting circle* for P_Θ if C meets P_Θ in less than or equal to three points, and both $P_\Theta \cap D_1$ and $P_\Theta \cap D_2$ contain 4-valent vertices.



(a) (b)

Fig. 9. (a) A local knot, or (b) nugatory crossings obtained from a loop in a θ-polyhedron.

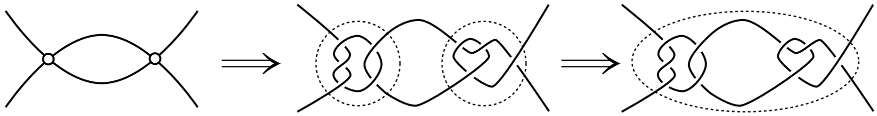


Fig. 10. The sum of algebraic tangles obtained from a bigon in a θ-polyhedron.

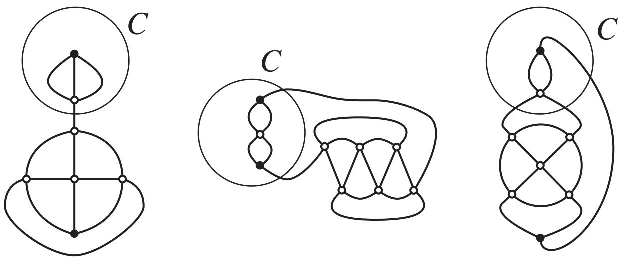


Fig. 11. Nonprime θ-polyhedra.

A θ-polyhedron P_Θ is said to be *prime* if it does not have a cutting circle (see Fig. 11).

Remark 4.3. Nonprime θ-polyhedra may produce a nonprime θ-curve.

A graph is said to be 3-*connected* if we must remove at least three vertices to disconnect it. Using the following lemma, the uniqueness of planar embedding of a prime basic θ-polyhedron with more than three vertices are guaranteed.

Lemma 4.4 (Whitney's Theorem [24, 25], cf. [1]). *A* 3-*connected planar graph has a unique planar embedding.*

Prime basic θ-polyhedra are classified into two types, according as if their 3-valent vertices are adjacent or not. We call the former *type-$\times$ prime basic θ-polyhedra*, and the latter *type-$*$ prime basic θ-polyhedra*. First, we give type-$\times$ prime basic θ-polyhedra with up to seven 4-valent vertices.

Theorem 4.5. *There exist seven type-$\times$ prime basic θ-polyhedra with up to seven* 4-*valent vertices as in Fig.* 12.

Remark 4.6. From now on, 4-valent vertices in a θ-polyhedron will be denoted by T or $\mathbb{T}$ in the figures, because we substitute 4-valent vertices by tangles later. The vertex T corresponds to a tangle T, and $\mathbb{T}$ corresponds to a tangle μT.

To prove Theorem 4.5, we use the following result. In [29], Yamano classified tangles of seven crossings or less, where he used the concept of *prime basic* 4-*regular disk graphs*. Let Q be a connected 4-regular planar graph, and v be a vertex of Q.

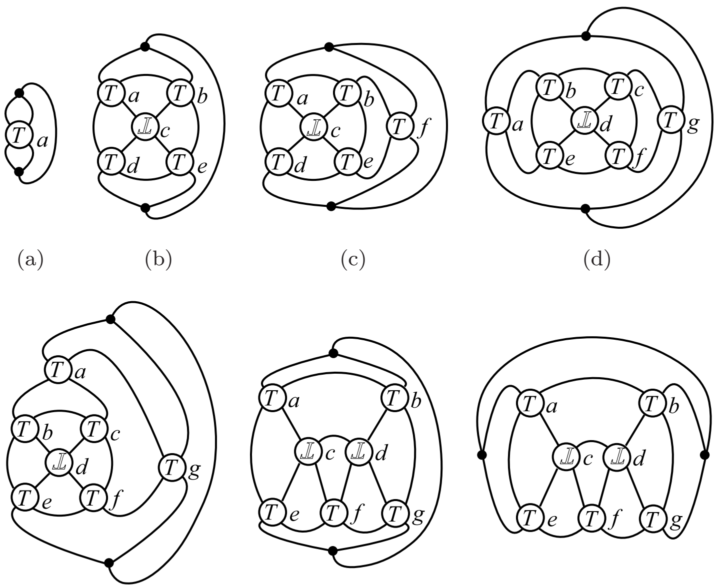


Fig. 12. (a) $1^1_\times a$; (b) $5^1_\times a.b.c.d.e$; (c) $6^1_\times a.b.c.d.e.f$; (d) $7^1_\times a.b.c.d.e.f.g$; (e) $7^2_\times a.b.c.d.e.f.g$; (f) $7^3_\times a.b.c.d.e.f.g$; (g) $7^4_\times a.b.c.d.e.f.g$.

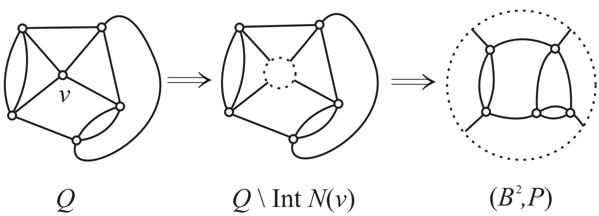


Fig. 13. The making of a 4-regular disk graph.

We denote the neighborhood of v by $N(v)$, and the interior of $N(v)$ by $\mathrm{Int}N(v)$. A pair (B^2, P) is called a 4-*regular disk graph* if (B^2, P) and $(S^2\backslash\mathrm{Int}N(v), Q\backslash\mathrm{Int}N(v))$ are homeomorphic (see Fig. 13).

A 4-regular disk graph (B^2, P) is said to be *basic* if P contains no loop and no bigon, and be *prime* if for any disk D in B^2, such that ∂D meets P transversely in two points, D contains no vertex.

Lemma 4.7 [29]. *There exist six prime basic* 4*-regular disk graphs with up to seven vertices as in Fig.* 14.

Yamano enumerated a tangle diagram from a prime basic 4-regular disk graph by substituting algebraic tangles for their vertices.

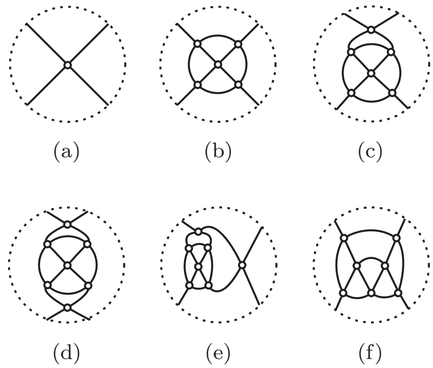


Fig. 14. (a) P_1; (b) P_5; (c) P_6; (d) P_{7-1}; (e) P_{7-2}; (f) P_{7-3}.

Proof of Theorem 4.5. By using Lemma 4.7, we construct type-$\times$ prime basic θ-polyhedra. First, we construct the "numerator" and the "denominator" of a prime basic 4-regular disk graph. Second, we add a vertex on each of the new edges. Third, we join these two vertices by an edge (see Fig. 15). Finally, we check whether it is prime and 3-connected. From Lemma 4.4, we obtain Theorem 4.5. □

Remark 4.8. Since the θ-polyhedron $1^1_\times a$ has only three vertices, it is not 3-connected. However, there does not exist another planar embedding.

Next, we give type-$*$ prime basic θ-polyhedra with up to seven 4-valent vertices.

Theorem 4.9. *There exist seventeen type-$*$ prime basic θ-polyhedra with up to seven* 4*-valent vertices as in Fig.* 16.

We need the following well-known theorem to prove Theorem 4.9.

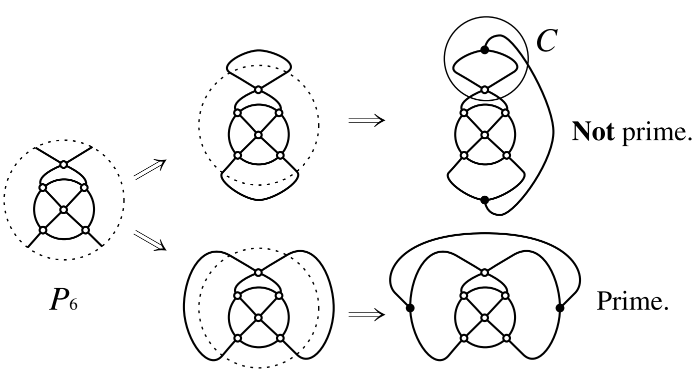


Fig. 15. The making of a θ-polyhedron.

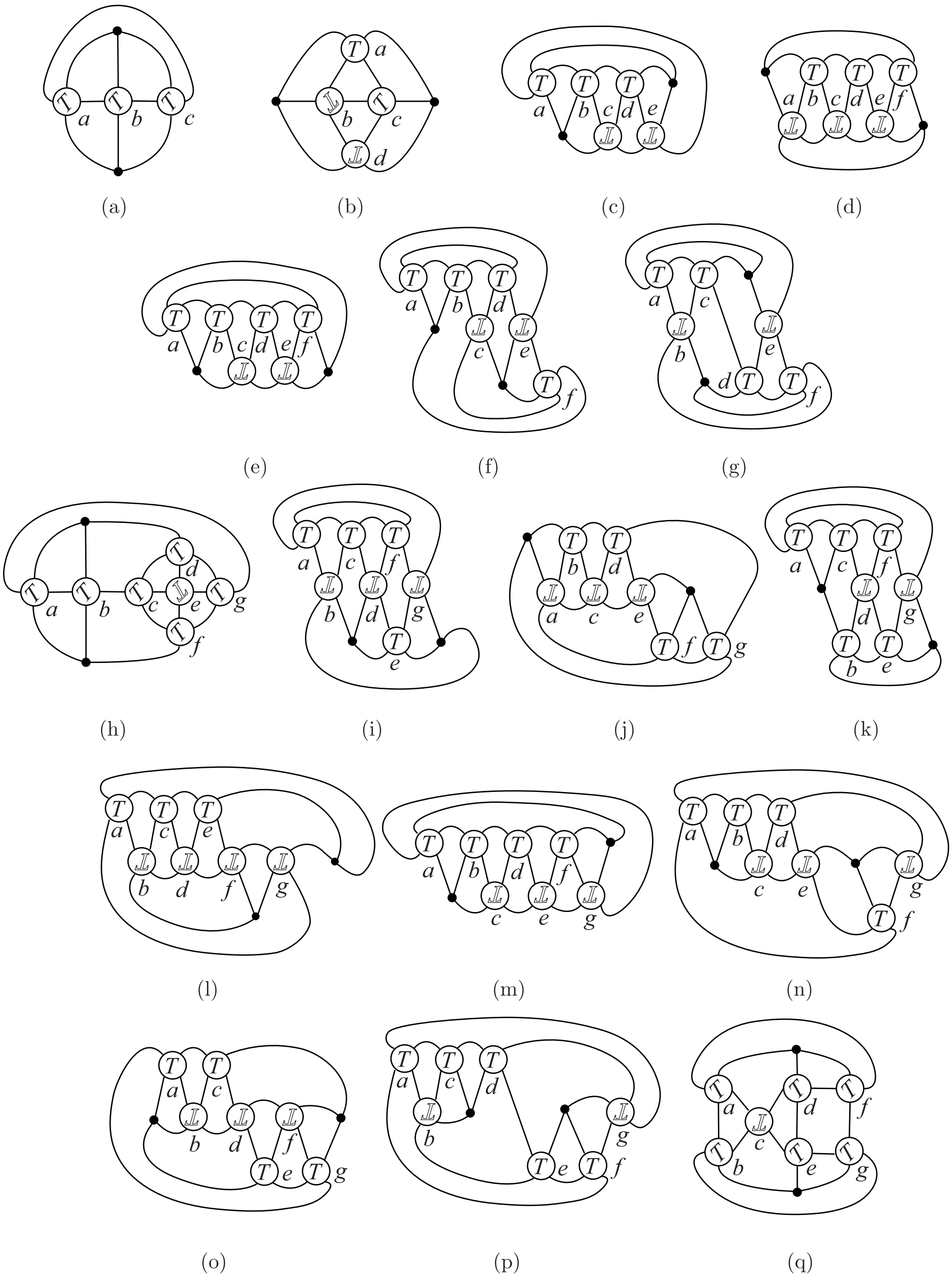


Fig. 16. (a) $3^1_*a.b.c$; (b) $4^1_*a.b.c.d$; (c) $5^1_*a.b.c.d.e$; (d) $6^1_*a.b.c.d.e.f$; (e) $6^2_*a.b.c.d.e.f$; (f) $6^3_*a.b.c.d.e.f$; (g) $6^4_*a.b.c.d.e.f$; (h) $7^1_*a.b.c.d.e.f.g$; (i) $7^2_*a.b.c.d.e.f.g$; (j) $7^3_*a.b.c.d.e.f.g$; (k) $7^4_*a.b.c.d.e.f.g$; (l) $7^5_*a.b.c.d.e.f.g$; (m) $7^6_*a.b.c.d.e.f.g$; (n) $7^7_*a.b.c.d.e.f.g$; (o) $7^8_*a.b.c.d.e.f.g$; (p) $7^9_*a.b.c.d.e.f.g$; (q) $7^{10}_*a.b.c.d.e.f.g$.

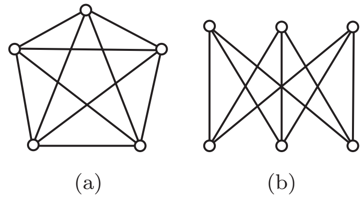


Fig. 17. (a) K_5; (b) $K_{3,3}$.

Lemma 4.10 (Kuratowski's Theorem [9]). *A graph G is planar if and only if G does not have a subgraph which is subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$ (see Fig. 17).*

Outline of a proof of Theorem 4.9. We enumerate type-$*$ prime basic θ-polyhedra according to the number of 4-valent vertices t, $3 \leq t \leq 7$. Such a prime basic θ-polyhedron satisfies the following conditions:

(P1) It contains no loop.
(P2) It contains no bigon.
(P3) Its 3-valent vertex and 4-valent vertex are not connected with two edges.
(P4) Its 3-valent vertex and 4-valent vertex are not connected with three edges.
(P5) It is a planar graph.

In fact, these conditions follow from the definition of a basic prime θ-polyhedron; see Fig. 18. If 4-valent vertices X and Y are connected by an edge, we write $X \sim Y$. From now on, we call small arcs which join a 4-valent vertex "hands". The conditions (P3) and (P4) induce the following four patterns according to the number of 4-valent vertices which are adjacent to both the 3-valent vertices: (a) three, (b) two, (c) one,

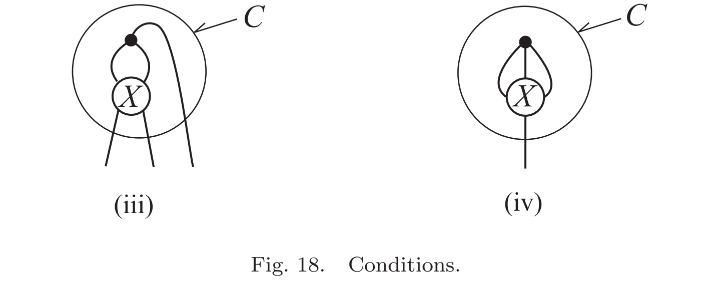
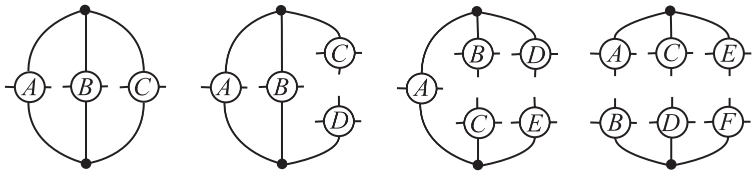


Fig. 18. Conditions.



(a) (b) (c) (d)

Fig. 19. Patterns.

(d) zero; see Fig. 19. Conversely, all prime basic θ-polyhedra are constructed from one of these patterns.

We explain how to enumerate prime basic θ-polyhedra. First, we prepare above patterns and 4-valent vertices with hands. Second, we connect remaining hands that satisfy (P1) and (P2). Third, we will use Lemma 4.10 to decide whether a θ-polyhedron satisfies (P5). Fourth, we check their primeness. Finally, we check their 3-connectedness.

For example, if $t = 4$, we consider the cases (a) and (b). See Fig. 20. When we consider the case (a), we need one more 4-valent vertex with four hands. We call the vertex D. We consider how the hands of D connect. By the condition (P2), $D \sim A$, B, C, which gives a graph containing $K_{3,3}$, and it does not satisfy the condition (P5).

See Fig. 21. When we consider the case (b), we need no more 4-valent vertex. Each of the vertices C and D has three remaining hands, and so we consider how

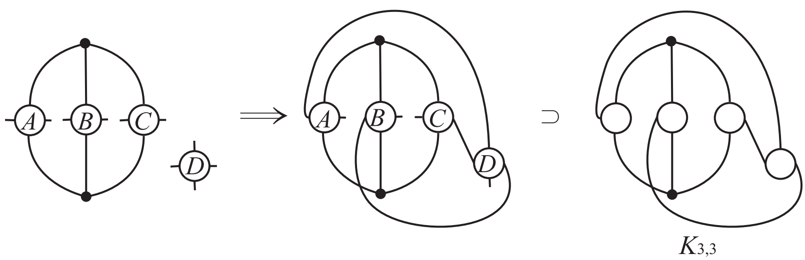


Fig. 20. $t = 4$ and the case (a).

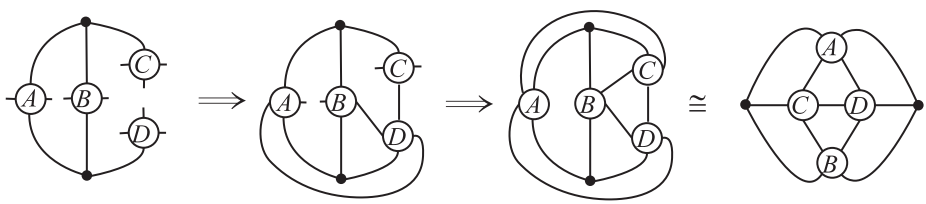


Fig. 21. $t = 4$ and the case (b).

the hands of D connect. By the condition (P2), $D \sim A, B, C$. Then $C \sim A, B$. This θ-polyhedron has no cutting circle, so it is prime. Moreover, each graph obtained by removing any two vertices of it is connected, so it is 3-connected. Therefore, we obtain 4^1_*.

We obtain the other prime basic θ-polyhedra in similar way (cf. [13]). □

5. Conclusion

From Lemma 3.3, Theorems 4.5 and 4.9, we can obtain all the prime θ-curves with up to seven crossings, which are exactly the same ones as in Litherland's. Litherland showed that these θ-curves are mutually distinct by investigating the *Alexander polynomial* [11] and constituent knots.

We show this using the *Yamada polynomial* [27], which will be explained in Sec. 6. Furthermore, we show the primeness of these θ-curves by using the following proposition originally due to Thurston. We can apply this since each θ-curve in the table has a trivial constituent knot.

Proposition 5.1 [12]. *Suppose a θ-curve Θ contains a trivial constituent knot. Take the 2-fold branched cover over this trivial knot; the two lifts of the remaining edge give a knot $\tilde{K}$ in S^3. Then, Θ is prime if and only if $\tilde{K}$ is prime.*

Example 5.2. For the θ-curve 7_{41}, we obtain the knot 8_{18} (see Fig. 22).

6. Yamada Polynomial

In [27], Yamada introduced a polynomial invariant for spatial graph diagrams. Let γ be a spatial graph diagram. Then, the Yamada polynomial $R(\gamma) \in \mathbf{Z}[x^{\pm 1}]$ is

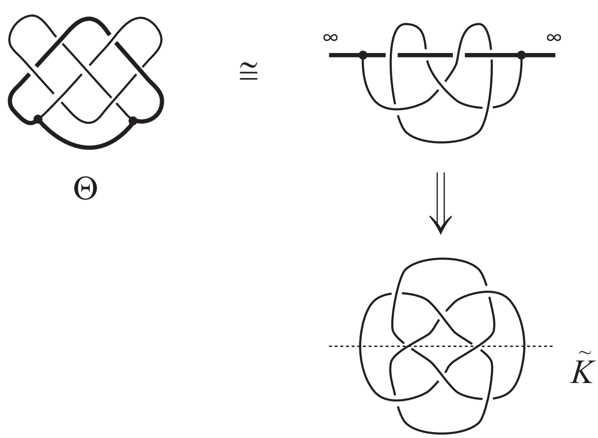
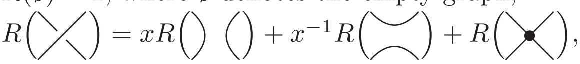


Fig. 22. The θ-curve 7_{41} is prime.

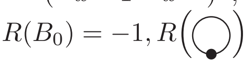
defined by the following recursive formulas:

(Y1) $R(\emptyset) = 1$, where $\emptyset$ denotes the empty graph,

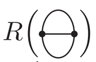
(Y2) 

(Y3) , where e is a nonloop edge,

(Y4) $R(\gamma_1 \sqcup \gamma_2) = R(\gamma_1)R(\gamma_2)$, where $\gamma_1 \sqcup \gamma_2$ denotes the disjoint union of spatial graph diagrams γ_1 and γ_2,

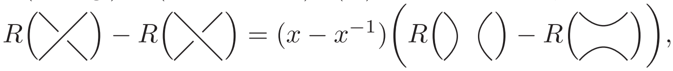
(Y5) $R(B_n) = -(-x-1-x^{-1})^n$, where B_n is the n-leafed bouquet. In particular, $R(\bullet) = R(B_0) = -1,$  $= R(B_1) = x + 1 + x^{-1}$.

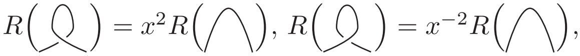
For a θ-curve diagram θ, we easily calculate $R(\theta)$ by using the following properties:

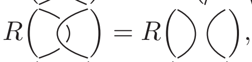
(Y6)  $= -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$,

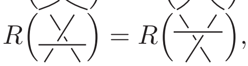
(Y7)  $= 0$,

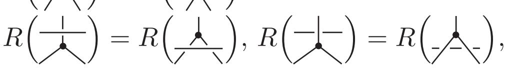
(Y8) $R(\theta' \sqcup \bigcirc) = (x+1+x^{-1})R(\theta')$ for an arbitrary θ-curve diagram θ',

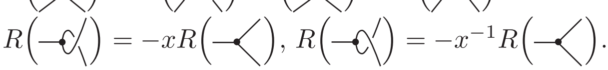
(Y9) 

(Y10) 

(Y11) 

(Y12) 

(Y13) 

(Y14) 

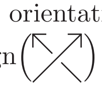
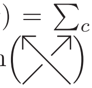
From Lemma 2.3, Yamada obtained the following proposition:

Proposition 6.1 [27, Theorem 4]. *Let Θ be a θ-curve, and θ be a diagram of Θ. Then, $R(\theta)$ is an ambient isotopy invariant of Θ up to multiplying $(-x)^n$ for some integer n.*

Here, we mention the influence of the order-2 and order-3 vertex connected sum.

Proposition 6.2 [27, Theorem 5]. *Let γ_1 and γ_2 be spatial graph diagrams. Then*

(1) $R(\gamma_1 \#_2 \gamma_2) = R(\gamma_1)R(\gamma_2)/(x + 1 + x^{-1})$,
(2) $R(\gamma_1 \#_3 \gamma_2) = R(\gamma_1)R(\gamma_2)/(-x^2 - x - 2 - x^{-1} - x^{-2})$.

Moreover, Yamada defined another polynomial invariant of a θ-curve. First, we review the *twisting number* of a knot diagram. For a knot diagram k, we fix an orientation. Put $t(k) = \sum_c \mathrm{sign}(c)$, where c ranges over all crossings of k and sign $= +1$, sign $= -1$. Here, we note that $t(k)$ does not depend on the choice of the orientation of k. Second, let θ be a θ-curve diagram, and θ_{ij} the constituent knot diagram of θ. Then, the *twisting number* of θ is defined by $t(\theta) = \sum_{i<j} t(\theta_{ij})/2$. Finally, we put $S(\Theta) = (-x)^{-2t(\theta)}R(\theta)$.

Proposition 6.3 [27, Theorem 7]. *Let Θ be a θ-curve. Then, $S(\Theta)$ is an ambient isotopy invariant of Θ.*

In [28], Yamada also introduced a polynomial invariant for 3-regular spatial graph Γ with some good weight ω, which is the linear combination of the bracket polynomial. First, we mention the *Temperley–Lieb algebra*. The mth Temperley–Lieb algebra V_m is an algebra over $\mathbf{C}$ generated by the elements $1_m, \varepsilon_1, \ldots, \varepsilon_{m-1}$ and the following relations.

(TL1) $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i \quad \text{if} \quad |i-j| \geq 2$,
(TL2) $\varepsilon_i^2 = \delta\varepsilon_i \quad$ where $\delta = -q - q^{-1}$ and $q \in \mathbf{C}$,
(TL3) $\varepsilon_i\varepsilon_{i\pm1}\varepsilon_i = \varepsilon_i$.

The geometrical meaning of ε_i is shown in Fig. 23.

For any nonnegative integer m, we set

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

Then, we define elements g_n and f_n of V_m by

$$g_n = \frac{1}{[n]}\sum_{j=1}^{n}[j]\varepsilon_j \ldots \varepsilon_{n-1},$$

$$f_n = g_n g_{n-1} \ldots g_1.$$

Elements g_n and f_n hold the following recursive formulas,

$$g_{n+1} = \frac{[n]}{[n+1]}g_n\varepsilon_n + 1,$$

$$f_{n+1} = g_{n+1}f_n.$$

By induction, it follows easily that $g_n - 1$ and $f_n - 1$ are elements in the proper subalgebra $\mathfrak{A}(\varepsilon_1, \ldots, \varepsilon_{n-1})$ generated by $\varepsilon_1, \ldots, \varepsilon_{n-1}$. So that, g_n and f_n are commute with $\varepsilon_{n+1}, \ldots, \varepsilon_{m-1}$.

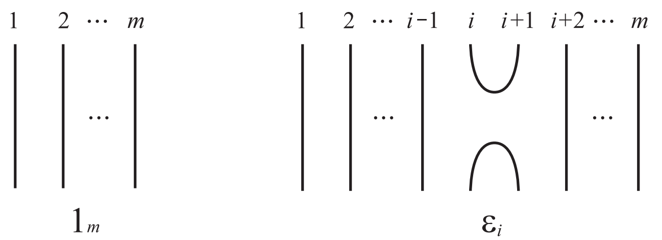


Fig. 23. The elements 1_m and ε_i.

The following lemma is given by Wenzl [23], Lickorish [10] and Yamada [28].

Lemma 6.4 [23, 10, 28]. *f_n is the unique element in V_m which is generated by $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$ and holds the following*:

(E1) $\varepsilon_i f_n = f_n \varepsilon_i = 0$, *for all* $i \le n-1$,
(E2) $f_n^2 = f_n$.

We call f_n the magic knitting of degree n.

Let Γ be a 3-regular graph and $\omega : E(\Gamma) \to \mathbf{Z}_+$ be a positive integer valued map. We say ω is a *good weight* on Γ if the following conditions are satisfied. For each vertex of Γ, if e_1, e_2 and e_3 are the edges incident with the vertex, then

(V1) $\omega(e_1) + \omega(e_2) + \omega(e_3) \in \mathbf{Z}_2$,
(V2) $|\omega(e_1) - \omega(e_2)| \le \omega(e_3) \le \omega(e_1) + \omega(e_2)$.

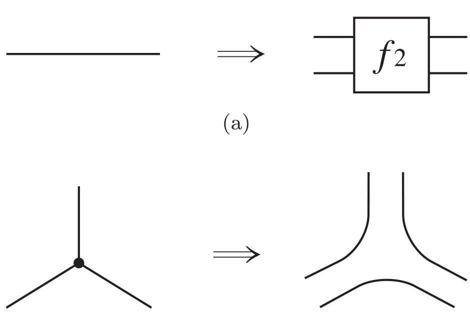
From now on, we consider the case $\omega(e) = 2$ for each edge e. Let ω be a good weight on a spatial 3-regular graph Γ and γ be a diagram of Γ. Then, γ^ω is defined as the linear combination of link diagrams derived from γ by the following:

(W1) Parallelize each edge e by the weight $\omega(e)$ and immerse the magic knitting of degree $\omega(e)$ as in Fig. 24(a).
(W2) Connect those parallelized edges at each vertex as in Fig. 24(b).

The bracket polynomial $\langle l \rangle \in \mathbf{Z}[A^{\pm 1}]$ of a nonoriented link diagram l is defined by the following formulas (originally introduced in [5]):

(B1) $\langle \bigcirc \rangle = -A^2 - A^{-2}$,
(B2) $\langle l \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle l \rangle$,
(B3) $\langle \times \rangle = A \langle)\,(\rangle + A^{-1} \langle \asymp \rangle$.

He defined $\langle \gamma^\omega \rangle$ by the linear extension of the bracket polynomial, and gave a relation between $\langle \gamma^\omega \rangle$ and $R(\gamma)$.



(a)

(b)

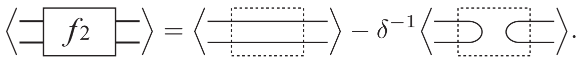
Fig. 24. How to make γ^ω.

Lemma 6.5 [28]. *Let γ be a diagram of a spatial 3-regular graph Γ with the good weight defined $\omega(e) = 2$ for each edge e. Then*

$$\langle \gamma^{\omega} \rangle (A) = (-\delta)^{\chi(\Gamma)} R(\gamma)(A^4),$$

where $\delta = -A^2 - A^{-2}$ and $\chi(\Gamma)$ the Euler characteristic of Γ.

We assume Γ is a θ-curve Θ. Then, we can describe $\langle g^{\omega} \rangle$ concretely. Here, the magic knitting of degree 2 is decomposed as follows:



(6.1)

Since $\chi(\Theta) = -1$, we obtain the following proposition.

Proposition 6.6. *For a θ-curve diagram θ and its constituent knot diagram θ_{ij}, let $\theta^{(2)}$ be the link diagram obtained from θ by parallelizing each edge, and $\theta_{ij}^{(2)}$ be the link diagram obtained from θ_{ij} by $(2, 0)$-cabling. Then*

$$R(\theta)(x) = -\delta \, \langle \theta^{(2)} \rangle (x^{1/4}) + \sum_{i<j} \langle \theta_{ij}^{(2)} \rangle (x^{1/4}) - 2,$$

where $\delta = -x^{1/2} - x^{-1/2}$.

Proof. From Lemma 6.5, we obtain the following equation:

$$R(\theta)(x) = (-\delta) \langle \theta^{\omega} \rangle (x^{1/4}), \tag{6.2}$$

where $\delta = -x^{1/2} - x^{-1/2}$. We can describe $\langle \theta^{\omega} \rangle$ concretely by using Eq. (6.1):

$$\begin{aligned}
\langle \theta^{\omega} \rangle &= \langle \theta^{(2)} \rangle - \delta^{-1} \sum_{i<j} \langle \theta_{ij}^{(2)} \rangle + 3\delta^{-2} \langle \bigcirc \rangle - \delta^{-3} \langle \bigcirc \sqcup \bigcirc \rangle \\
&= \langle \theta^{(2)} \rangle - \delta^{-1} \sum_{i<j} \langle \theta_{ij}^{(2)} \rangle + 3\delta^{-1} - \delta^{-1} \\
&= \langle \theta^{(2)} \rangle - \delta^{-1} \sum_{i<j} \langle \theta_{ij}^{(2)} \rangle + 2\delta^{-1}.
\end{aligned}$$

Then, we replace $\langle \theta^{\omega} \rangle$ in the equation (6.2) with the above. □

By using Proposition 6.6 and KNOT program [8], we compute the Yamada polynomial of all the prime θ-curve with up to seven crossings immediately (see Tables 1–3). For example, the entry for 3_1 appears as follows.

3_1 $\{-10\}(1\ 0\ 0\ 0\ -1\ 0\ -1\ 0\ -1\ -1\ -1\ -1\ -1)$

This means that $S(3_1) = x^{-10} - x^{-6} - x^{-4} - x^{-2} - x^{-1} - 1 - x - x^2$.

Table 1. Yamada polynomial of θ-curve with up to seven crossings.

Θ	$S(\Theta)$
3_1	{-10} (1 0 0 0 -1 0 -1 0 -1 -1 -1 -1 -1)
4_1	{-9} (1 0 -1 0 -1 -1 0 -1 0 -1 0 0 -1 0 0 -1)
5_1	{-12} (-1 1 2 -1 1 0 -2 -1 -2 0 -1 0 1 -1 -1 1 -1 -1)
5_2	{-10} (1 1 0 0 0 -2 -2 -1 -2 0 -1 1 0 0 1 -1 -1)
5_3	{-2} (-1 -1 -1 -1 -1 0 0 0 0 0 -1 0 -1 0 -1 0 1 0 1)
5_4	{0} (-1 -1 -1 -1 -2 -1 -1 -1 0 1 0 1 0 1 0 0 1 -1)
5_5	{-2} (-1 0 -1 -2 0 -1 -2 0 -1 1 0 1 1 -1 0 0 -1 0 1)
5_6	{-4} (-1 0 0 -2 0 0 -1 1 -1 0 -2 -1 0 -1 0 1 0 0 1)
5_7	{0} (-2 -1 0 -2 -1 0 -2 0 -1 1 0 0 2 0 0 1 0 -1)
6_1	{-11} (-1 0 2 -1 -1 2 -1 -1 1 -1 0 -2 0 -1 -3 1 0 -2 1 1)
6_2	{-14} (1 -1 -2 2 0 -2 3 1 -1 1 -1 0 -3 -1 0 -4 0 2 -2 0 1)
6_3	{-6} (-2 0 2 -2 0 1 -2 0 -1 1 -1 -1 1 -2 -2 1 0 -1 1 1)
6_4	{-6} (-1 -1 0 0 -1 1 1 -1 0 0 -2 -1 -2 0 -1 0 2 -1 0 1)
6_5	{-11} (1 0 0 1 -2 -1 0 -2 0 -1 0 -1 -1 1 0 0 1 0 -1 0 0 -1)
6_6	{-6} (-1 0 -1 -1 1 -1 -1 1 -1 0 -1 0 0 -1 1 0 -1 0 0 -1 0 1)
6_7	{-7} (1 -1 -1 1 -2 -2 0 -2 0 0 2 1 -1 1 -1 -2 0 0 -1 0 1)
6_8	{-8} (-1 1 0 -2 1 0 -2 0 -1 0 -1 0 1 -2 0 1 -1 0 1 0 -1)
6_9	{-9} (1 0 -1 1 0 -2 0 -1 -3 -1 0 -1 1 0 2 -1 -1 2 -2 0 1 -1)
6_{10}	{-6} (-1 0 1 -1 -1 1 -2 -2 0 -1 -1 1 0 1 -1 1 1 -3 1 0 -1 1)
6_{11}	{-7} (1 0 -2 0 1 -3 -1 1 -2 -1 1 -1 0 -1 2 0 -2 3 -1 -2 1)
6_{12}	{-4} (-1 0 0 -2 -1 1 -2 -1 1 -1 -1 1 0 1 -1 1 0 -3 2 0 -1 1)
6_{13}	{-8} (-1 0 1 -1 -1 2 -1 -2 1 -1 -2 0 -1 0 -2 1 2 -2 2 1 -2)
6_{14}	{-10} (1 -1 -1 2 -2 -1 3 -1 -1 1 -2 -1 -3 0 0 -3 2 1 -2 2 1 -1)
6_{15}	{-8} (2 0 -3 2 0 -4 1 0 -3 0 -1 1 -2 0 3 -3 0 3 -2 -1 1)
6_{16}	{-13} (-1 1 1 -2 2 2 -3 1 1 -2 0 -2 0 -3 -2 2 -2 -1 3 -1 -1 1)
7_1	{-7} (-1 0 1 -3 1 3 -4 0 2 -3 0 -1 1 -2 -2 3 -2 -2 3 0 -2 2 1 -1)
7_2	{-12} (-1 1 3 -2 -1 4 -3 -4 3 -2 -3 1 -1 0 -3 3 2 -5 3 2 -4 1 1 -1)
7_3	{-14} (-1 0 3 -2 -3 6 0 -5 4 0 -4 0 -1 0 -4 2 4 -6 1 4 -5 -1 2)
7_4	{-7} (1 0 -3 1 3 -5 -1 3 -4 -2 2 -1 0 -1 3 -1 -4 5 -1 -4 3 1 -1)
7_5	{-18} (-1 1 0 -1 4 -1 -1 2 -1 -1 -2 0 -1 -2 1 1 -2 1 1 -2 -1 1 -1 -1)
7_6	{-12} (1 0 -1 3 0 -4 2 -1 -3 1 -1 0 -3 0 1 -3 1 3 -2 0 2 -1 -1)
7_7	{-18} (-1 0 2 0 -1 2 1 -3 1 0 -2 0 -1 1 -2 1 2 -3 -1 1 -2 -1)
7_8	{-9} (-1 1 0 -1 1 0 1 -1 0 -1 -2 -1 -1 -2 -1 0 0 0 1 1)
7_9	{-16} (-1 -1 3 0 -3 4 2 -4 2 1 -3 0 -1 1 -4 -1 3 -5 -1 4 -2 -1 1)
7_{10}	{-10} (1 -1 0 4 -3 -2 5 -4 -4 1 -3 -1 -2 3 1 -3 4 1 -5 2 2 -3 0 1)
7_{11}	{-13} (1 -1 0 3 -2 -1 1 -2 -1 -1 1 -1 -2 1 -1 -2 2 1 -1 0 1 -1 -1)
7_{12}	{-14} (1 -2 -2 6 -2 -4 7 -2 -4 2 -1 -1 -3 2 1 -6 3 4 -6 1 3 -3 -1 1)
7_{13}	{-9} (-1 0 2 -3 -1 4 -2 -1 2 -2 -1 -2 1 -1 -3 3 0 -3 2 1 -2 0 1)
7_{14}	{-10} (-1 1 2 -3 -1 4 -4 -3 4 -2 -2 2 0 0 -3 2 0 -6 4 2 -4 2 1 -1)
7_{15}	{-16} (1 0 0 2 0 -2 2 -1 -3 0 -2 0 -2 1 1 -2 1 1 -2 0 1 -1 -1)
7_{16}	{-4} (-1 -1 0 0 -1 0 1 -1 -1 0 -2 0 -1 1 -1 -1 1 -1 0 1 1)
7_{17}	{-20} (1 -1 -1 3 -1 -3 3 -1 -2 2 1 2 -1 1 0 -5 0 1 -4 0 1 -1 -1)
7_{18}	{0} (-1 -1 0 0 -3 -1 0 -3 -1 1 -1 1 0 2 0 0 3 -1 -1 1 0 -1)
7_{19}	{-8} (-1 0 2 -2 -2 4 -1 -4 3 0 -3 1 -1 0 -4 1 2 -5 2 3 -3 0 2)
7_{20}	{-8} (-1 0 0 -2 2 1 -1 1 -1 0 -2 0 0 -2 -1 0 -2 -1 1 0 0 1 1)
7_{21}	{-4} (-1 -1 1 0 -3 1 1 -3 0 1 -2 0 -1 1 -2 -1 3 -2 0 3 0 -1)
7_{22}	{-2} (-1 0 1 -2 -2 2 -3 -4 1 -1 -2 2 1 2 -1 2 2 -4 2 1 -3 0 1)
7_{23}	{-6} (-2 1 1 -4 2 1 -3 1 0 1 -2 0 1 -3 -1 2 -2 -2 2 0 -1 1 1)
7_{24}	{-6} (-1 0 1 -1 -2 1 -1 -3 2 1 -1 1 -1 0 -3 0 1 -3 1 2 -1 0 1)
7_{25}	{-2} (-1 -1 -1 -1 -1 0 0 0 0 0 0 0 0 0 -1 0 -1 0 -1 0 0 0 1 0 1)
7_{26}	{0} (-1 -1 -1 -1 -2 -1 0 0 0 1 0 -1 0 0 -1 1 0 1 -1 0 1 -1 1)
7_{27}	{2} (-1 -1 -1 -1 -2 -1 -1 -1 -1 0 0 0 1 1 0 1 0 1 0 1 1 -1 0 -1)

Table 1. (*Continued*)

Θ	$S(\Theta)$
7_{28}	{-22} (1 0 -1 0 0 -1 0 1 0 0 1 0 0 1 0 0 -2 0 -1 -2 0 -1 -1 0 -1)
7_{29}	{-18} (1 0 0 1 0 -1 0 0 -1 0 0 -1 -1 0 -1 0 -1 1 -1 -1 1 -1 0 0 -1)
7_{30}	{-20} (1 0 -1 0 1 -1 -1 2 0 -1 2 0 -2 0 -1 0 -2 1 1 -3 0 0 -2)
7_{31}	{-21} (-1 0 1 0 0 2 1 -1 0 0 -2 0 1 -1 0 -1 1 -2 -1 1 -2 -1 0 -1)
7_{32}	{-23} (-1 0 1 0 -1 1 1 -1 1 2 -1 0 1 -1 0 -1 1 -2 -2 1 -2 -2 0 -1)
7_{33}	{-22} (1 0 0 1 -1 -2 2 -1 -1 2 -1 1 -1 1 0 -2 1 0 -2 0 0 -2 -1 0 -1)
7_{34}	{-20} (1 0 1 1 -1 0 1 -3 -1 1 -2 1 -1 1 -1 -1 2 -1 -1 1 -1 -2 0 -1)
7_{35}	{-21} (-1 1 1 -1 1 3 -2 0 1 -3 0 -1 2 -1 -1 2 -2 -2 1 -1 -2 0 0 -1)
7_{36}	{-25} (-1 0 0 -1 1 2 -2 2 2 -1 2 0 1 -2 -1 1 -2 -1 1 -2 -2 0 -1 -2)
7_{37}	{-23} (-1 1 0 -2 1 2 -2 2 3 -1 1 -1 1 -2 -1 2 -3 -2 1 -2 -2 0 0 -1)
7_{38}	{-20} (1 0 -1 1 1 -2 1 1 -2 0 1 -1 -1 1 -1 0 -2 2 -1 -3 2 -2 -1 1 -1)
7_{39}	{-18} (1 0 0 2 1 -2 1 0 -4 -1 0 -2 -1 2 0 1 -1 2 -2 -3 3 -2 -1 1 -1)
7_{40}	{-22} (1 0 -2 0 2 -3 -1 4 0 -1 4 1 -2 1 -1 -1 -5 1 1 -5 2 1 -3)
7_{41}	{-21} (-1 0 2 0 -2 3 1 -3 1 2 -2 0 2 -2 -1 -2 2 -3 -2 4 -3 -2 1 -1)
7_{42}	{-23} (-1 0 1 -1 -1 3 0 -1 3 1 -2 1 1 -2 0 -1 1 -4 -1 2 -4 0 1 -2)
7_{43}	{-20} (1 -1 0 3 -2 -1 4 -3 -1 2 -2 0 -2 1 -1 -3 3 1 -3 2 0 -3 0 0 -1)
7_{44}	{-22} (1 -1 0 2 -3 0 4 -3 0 2 -2 0 -1 2 0 -2 3 -1 -4 2 -1 -3 0 0 -1)
7_{45}	{-23} (-1 1 1 -3 2 3 -4 3 2 -3 0 -1 1 -2 0 4 -2 -2 3 -3 -3 1 -1 -2)
7_{46}	{-24} (1 -1 -1 3 -2 -3 4 -2 -1 3 0 1 -1 2 1 -3 2 1 -5 0 0 -3 -1 0 -1)
7_{47}	{-21} (-1 2 1 -4 2 3 -5 2 3 -2 1 0 2 -3 -2 3 -4 -3 4 -2 -3 1 0 -1)
7_{48}	{-25} (-1 1 0 -3 3 2 -5 3 2 -2 2 1 2 -2 0 3 -4 -2 3 -4 -3 1 -1 -2)
7_{49}	{-23} (-2 1 3 -3 0 4 -4 0 3 -1 1 0 3 -1 -3 3 -2 -5 2 -1 -3 0 0 -1)
7_{50}	{-18} (1 -1 -1 3 -3 -2 5 -3 0 4 -1 0 -2 1 -1 -4 2 0 -5 2 1 -3 1 1 -1)
7_{51}	{-16} (1 -2 -1 5 -3 -3 7 -2 -3 3 -1 -1 -3 2 0 -6 3 2 -6 2 3 -3 -1 1)
7_{52}	{-15} (-1 2 2 -5 2 5 -6 1 3 -4 -1 -1 2 -3 -2 5 -3 -4 5 -1 -4 2 1 -1)
7_{53}	{-14} (1 -1 0 3 -3 0 4 -5 -1 2 -3 0 -1 2 -1 -3 3 -1 -4 3 0 -3 2 1 -1)
7_{54}	{-14} (2 -1 -3 4 -1 -5 5 0 -3 3 0 0 -4 0 1 -6 1 4 -4 0 3 -2 -1 1)
7_{55}	{-19} (-1 1 1 -4 3 4 -6 3 3 -3 1 0 1 -4 -2 3 -4 -3 5 -2 -3 3 0 -2)
7_{56}	{-17} (-1 1 0 -2 4 0 -4 5 0 -3 2 -1 0 -3 0 1 -5 0 3 -4 0 3 -2 -1 1)
7_{57}	{-16} (1 -1 0 3 -4 -1 5 -4 -1 3 -2 0 -1 2 -1 -4 3 -1 -5 3 1 -3 1 1 -1)
7_{58}	{-19} (-2 1 3 -4 1 6 -5 0 4 -2 0 -1 2 -3 -4 4 -2 -5 4 0 -4 1 1 -1)
7_{59}	{-13} (1 -1 -2 3 -1 -4 5 0 -4 4 0 -3 0 -1 1 -4 0 4 -6 0 4 -4 1 2 -1)
7_{60}	{-10} (-2 0 4 -3 -3 7 -2 -5 5 -1 -4 1 0 0 -5 2 3 -8 3 5 -5 1 2 -1)
7_{61}	{-14} (-1 1 3 -4 -2 7 -4 -5 6 -2 -3 3 1 0 -4 3 1 -9 3 3 -7 2 3 -1)
7_{62}	{-10} (-1 1 1 -4 1 4 -5 1 5 -4 -1 1 -3 -2 -3 3 -1 -4 7 0 -4 4 -1 -2 1)
7_{63}	{-15} (1 -1 -2 4 0 -6 5 2 -7 3 2 -3 0 0 2 -5 -1 6 -6 -2 6 -4 -2 2)
7_{64}	{-15} (2 0 -4 3 2 -7 2 4 -6 1 4 -1 0 -1 3 -5 -5 6 -4 -4 6 -1 -2 1)
7_{65}	{-12} (-1 1 1 -3 2 3 -6 1 4 -5 0 2 -2 -1 -2 3 -2 -4 6 -2 -4 5 -1 -2 1)

Appendix A

We give an enumeration of θ-curve with up to seven crossings by using our notation. Knots in the second column correspond to Rolfsen's knot table [20], and θ-curves in the last column correspond to Litherland's table [11]. $\overline{K}$ and $\overline{\Theta}$ denote mirror images of K and Θ, respectively.

Example A.1. The θ-curve diagram as in Fig. 25(a) is denoted by $4^1_* 2\ 1\ 0.1.1.\bar{2}\ 0$. Its constituent knots are $5_2, 3_1, 0_1$. This θ-curve and 7_{22} are ambient isotopic (see Fig. 25).

Table 2. θ-curves with our notation.

Notation	Constituent knot	θ-curve
$1^1_\times\ 3$	3_1, 0, 0	$\overline{3_1}$
$1^1_\times\ 2\ 2$	4_1, 0, 0	4_1
$1^1_\times\ 5$	5_1, 0, 0	5_3
$1^1_\times\ 3\ 2$	5_2, 0, 0	5_6
$1^1_\times\ 2\ 3$	$\overline{5_2}$, 0, 0	$\overline{5_5}$
$1^1_\times\ 3, 2$	$\overline{5_1}$, $\overline{3_1}$, 0	$\overline{5_4}$
$1^1_\times\ 2\ 1, 2$	5_2, 3_1, 0	5_7
$3^1_*\ 2.2.\bar{1}$	$\overline{3_1}$, 0, 0	5_2
$4^1_*\ 2.1.1.1$	$\overline{3_1}$, 0, 0	5_2
$4^1_*\ 2\ 0.1.1.1$	0, 0, 0	5_1
$1^1_\times\ 4\ 2$	6_1, 0, 0	6_5
$1^1_\times\ 2\ 4$	$\overline{6_1}$, 0, 0	$\overline{6_6}$
$1^1_\times\ 3\ 1\ 2$	6_2, 0, 0	6_9
$1^1_\times\ 2\ 1\ 3$	6_2, 0, 0	6_{10}
$1^1_\times\ 2\ 1\ 1\ 2$	6_3, 0, 0	$\overline{6_{14}}$
$1^1_\times\ 2\ 2, 2$	6_1, 4_1, 0	6_8
$1^1_\times\ 2\ 1\ 1, 2$	$\overline{6_2}$, 4_1, 0	$\overline{6_{13}}$
$1^1_\times\ 3, 2\ 1$	$\overline{6_1}$, 0, 0	$\overline{6_7}$
$1^1_\times\ 3, 2+$	$\overline{6_2}$, $\overline{3_1}$, 0	$\overline{6_{12}}$
$1^1_\times\ 2\ 1, 2+$	6_3, 3_1, 0	$\overline{6_{16}}$
$3^1_*\ 3.2.\bar{1}$	4_1, 3_1, 0	6_4
$3^1_*\ 2.2.2$	0, 0, 0	5_1
$3^1_*\ 2.2.\bar{2}$	0, 0, 0	6_1
$4^1_*\ 2\ 1.1.1.1$	0, 0, 0	6_1
$4^1_*\ 2\ 1\ 0.1.1.1$	$\overline{3_1}$, 0, 0	6_2
$4^1_*\ \bar{2}\ \bar{1}\ 0.1.1.1$	4_1, $\overline{3_1}$, 0	$\overline{6_4}$
$4^1_*\ 2.2\ 0.1.1$	4_1, 3_1, 0	6_4
$4^1_*\ 2.1.1.2\ 0$	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
$4^1_*\ 2.1.1.\bar{2}\ 0$	4_1, 3_1, 0	6_4
$4^1_*\ 2\ 0.2\ 0.1.1$	$\overline{3_1}$, 0, 0	6_2
$4^1_*\ \bar{2}\ 0.2\ 0.1.1$	0, 0, 0	$\overline{6_1}$
$5^1_\times\ 2.1.1.1.1$	6_3, 0, 0	6_{15}
$5^1_\times\ 2\ 0.1.1.1.1$	$\overline{6_2}$, 0, 0	$\overline{6_{11}}$
$5^1_*\ 2\ 0.1.1.1.1$	0, 0, 0	$\overline{6_1}$
$5^1_*\ 1.1.1.1.2\ 0$	$\overline{5_2}$, $\overline{3_1}$, 0	$\overline{5_7}$
$6^2_*\ 1.1.1.1.1.1$	0, 0, 0	$\overline{6_1}$
$6^4_*\ 1.1.1.1.1.1$	0, 0, 0	$\overline{5_1}$
$1^1_\times\ 7$	7_1, 0, 0	7_{25}
$1^1_\times\ 5\ 2$	7_2, 0, 0	$\overline{7_{29}}$
$1^1_\times\ 4\ 3$	7_3, 0, 0	7_{33}
$1^1_\times\ 3\ 4$	$\overline{7_3}$, 0, 0	$\overline{7_{34}}$

Table 2. (*Continued*)

Notation	Constituent knot	θ-curve
$1^1_\times$ 2 5	$\overline{7_2}$, 0, 0	7_{28}
$1^1_\times$ 3 2 2	7_5, 0, 0	$\overline{7_{43}}$
$1^1_\times$ 3 1 3	7_4, 0, 0	7_{38}
$1^1_\times$ 2 2 3	7_5, 0, 0	$\overline{7_{44}}$
$1^1_\times$ 2 2 1 2	7_6, 0, 0	$\overline{7_{53}}$
$1^1_\times$ 2 1 2 2	$\overline{7_6}$, 0, 0	7_{50}
$1^1_\times$ 2 1 1 1 2	7_7, 0, 0	7_{59}
$1^1_\times$ 5, 2	$\overline{7_1}$, $\overline{5_1}$, 0	$\overline{7_{27}}$
$1^1_\times$ 4 1, 2	$\overline{7_3}$, 5_1, 0	$\bar{7}_{36}$
$1^1_\times$ 3 2, 2	7_3, $\overline{5_2}$, 0	7_{37}
$1^1_\times$ 2 3, 2	7_2, 5_2, 0	$\overline{7_{32}}$
$1^1_\times$ 3 1 1, 2	$\overline{7_4}$, 5_2, 0	$\overline{7_{42}}$
$1^1_\times$ 2 2 1, 2	$\overline{7_5}$, $\overline{5_2}$,0	7_{49}
$1^1_\times$ (3, 2), 2	7_5, 5_1, 0	$\overline{7_{48}}$
$1^1_\times$ (3, 2), $\bar{2}$	5_1, 5_2, 0	7_{18}
$1^1_\times$ (3, $\bar{2}$), 2	6_2, 0, 0	6_{11}
$1^1_\times$ (3, $\bar{2}$), $\bar{2}$	6_3, 0, 0	6_{15}
$1^1_\times$ (2 1, 2), 2	$\overline{7_6}$, $\overline{5_2}$, 0	7_{58}
$1^1_\times$ (2 1, 2), $\bar{2}$	$\overline{5_1}$, $\overline{5_2}$, 0	$\overline{7_{18}}$
$1^1_\times$ (2 1, $\bar{2}$), 2	6_3, 0, 0	$\overline{6_{15}}$
$1^1_\times$ (2 1, $\bar{2}$), $\bar{2}$	$\overline{6_2}$, 0, 0	$\overline{6_{11}}$
$1^1_\times$ 4, 3	$\overline{7_1}$, $\overline{3_1}$, 0	$\overline{7_{26}}$
$1^1_\times$ 3 1, 3	7_3, $\overline{3_1}$, 0	7_{35}
$1^1_\times$ 2 2, 3	7_2, 0, 0	$\overline{7_{30}}$
$1^1_\times$ 4, 2 1	7_2, 3_1, 0	$\overline{7_{31}}$
$1^1_\times$ 3 1, 2 1	7_5, 3_1, 0	$\overline{7_{45}}$
$1^1_\times$ 2 1 1, 2 1	$\overline{7_6}$, 0, 0	7_{54}
$1^1_\times$ 2 2, 2+	$\overline{7_6}$, 4_1, 0	7_{57}
$1^1_\times$ 2 2, $\bar{2}-$	6_2, 4_1, 0	6_{13}
$1^1_\times$ 2 1 1, 2+	$\overline{7_7}$, 4_1, 0	$\overline{7_{65}}$
$1^1_\times$ 2 1 1, $\bar{2}-$	$\overline{6_1}$, 4_1, 0	$\overline{6_8}$
$1^1_\times$ 3, 3+	$\overline{7_4}$, 0, 0	$\overline{7_{39}}$
$1^1_\times$ 3, $\bar{3}-$	$\overline{6_1}$, 0, 0	$\overline{6_7}$
$1^1_\times$ 2 1, 2 1+	7_7, 0, 0	7_{62}
$1^1_\times$ 2 1, 2 1−	$\overline{3_1}$, 0, 0	5_2
$1^1_\times$ 2 1, $\bar{2}$ $\bar{1}+$	6_1, 0, 0	6_7
$1^1_\times$ 2 1, $\bar{2}$ $\bar{1}-$	$\overline{6_1}$, 0, 0	$\overline{6_7}$
$1^1_\times$ 3, 2 + +	$\overline{7_5}$, $\overline{3_1}$, 0	7_{46}
$1^1_\times$ 3, $\bar{2}$ − −	6_3, $\overline{3_1}$, 0	6_{16}
$1^1_\times$ 2 1, 2 + +	7_6, 3_1, 0	$\overline{7_{56}}$

Table 2. (*Continued*)

Notation	Constituent knot	θ-curve
$1^1_\times$ $2\ 1,\bar{2}--$	6_2, 3_1, 0	6_{12}
3^1_* $(3,2).1.\bar{1}$	$\overline{6_2}$, 0, 0	$\overline{6_{11}}$
3^1_* $(3,\bar{2}).1.\bar{1}$	$\overline{5_1}$, $\overline{5_2}$, 0	$\overline{7_{18}}$
3^1_* $(2\ 1,2).1.\bar{1}$	6_3, 0, 0	6_{15}
3^1_* $(2\ 1,\bar{2}).1.\bar{1}$	5_1, 5_2, 0	7_{18}
3^1_* $4.2.\bar{1}$	$\overline{5_2}$, 0, 0	$\overline{7_{20}}$
3^1_* $3\ 1.2.\bar{1}$	$\overline{5_1}$, 0, 0	7_{15}
3^1_* $2\ 1\ 1.2.\bar{1}$	5_2, 3_1, 0	7_{22}
3^1_* $2\ 1\ 1\ 0.\bar{2}.1$	$\overline{6_2}$, $\overline{3_1}$, 0	$\overline{6_{12}}$
3^1_* $3.2\ 1.\bar{1}$	5_2, 4_1, 0	$\overline{7_{24}}$
3^1_* $2\ 1.2\ 1.\bar{1}$	$\overline{5_1}$, 0, 0	7_{17}
3^1_* $3.2.2$	$\overline{3_1}$, 0, 0	6_2
3^1_* $3.2.\bar{2}$	4_1, 0, 0	$\overline{7_{13}}$
3^1_* $3.\bar{2}.2$	4_1, 0, 0	$\overline{7_{13}}$
3^1_* $3.\bar{2}.\bar{2}$	3_1, 0, 0	$\overline{7_6}$
3^1_* $3\ 0.\bar{2}.2$	4_1, $\overline{3_1}$, 0	$\overline{6_4}$
3^1_* $3\ 0.\bar{2}.\bar{2}$	3_1, $\overline{3_1}$, 0	7_8
3^1_* $2\ 1.\bar{2}\ 0.2\ 0$	$\overline{3_1}$, 0, 0	6_2
3^1_* $2\ 1.\bar{2}\ 0.\bar{2}\ 0$	$\overline{3_1}$, 0, 0	7_7
4^1_* $4.1.1.1$	3_1, $\overline{3_1}$, 0	$\overline{7_8}$
4^1_* $4\ 0.1.1.1$	$\overline{3_1}$, 0, 0	7_5
4^1_* $\bar{4}\ 0.1.1.1$	5_1, 0, 0	$\overline{7_{15}}$
4^1_* $3\ 1.1.1.1$	$\overline{3_1}$, 0, 0	7_6
4^1_* $3\ 1\ 0.1.1.1$	0, 0, 0	$\overline{7_1}$
4^1_* $\bar{3}\ \bar{1}\ 0.1.1.1$	5_2, 0, 0	7_{20}
4^1_* $2\ 2.1.1.1$	$\overline{3_1}$, 0, 0	7_7
4^1_* $2\ 2\ 0.1.1.1$	3_1, $\overline{3_1}$, 0	7_{10}
4^1_* $\bar{2}\ \bar{2}\ 0.1.1.1$	$\overline{5_2}$, $\overline{3_1}$, 0	$\overline{7_{23}}$
4^1_* $2\ 1.2\ 0.1.1$	4_1, 0, 0	$\overline{7_{13}}$
4^1_* $2\ 1\ 0.2\ 0.1.1$	4_1, 4_1, 0	7_{14}
4^1_* $\bar{2}\ \bar{1}\ 0.2\ 0.1.1$	4_1, 0, 0	7_{13}
4^1_* $3.1.1.2$	5_1, 0, 0	7_{16}
4^1_* $3\ 0.1.1.2$	5_2, 4_1, 0	7_{23}
4^1_* $\bar{3}\ 0.1.1.2$	$\overline{5_2}$, 4_1, 0	$\overline{7_{24}}$
4^1_* $2\ 1.1.1.2$	$\overline{5_2}$, 0, 0	$\overline{7_{21}}$
4^1_* $3.1.1.2\ 0$	4_1, 0, 0	7_{11}
4^1_* $3.1.1.\bar{2}\ 0$	$\overline{5_2}$, 0, 0	$\overline{7_{20}}$
4^1_* $3\ 0.1.1.2\ 0$	0, 0, 0	$\overline{7_2}$
4^1_* $3\ 0.1.1.\bar{2}\ 0$	$\overline{5_1}$, 0, 0	7_{15}
4^1_* $\bar{3}\ 0.1.1.2\ 0$	5_1, 0, 0	$\overline{7_{17}}$

Table 2. (*Continued*)

Notation	Constituent knot	θ-curve
$4^1_*\ 2\ 1\ 0.1.1.2\ 0$	$4_1, 0, 0$	7_{12}
$4^1_*\ 2\ 1\ 0.1.1.\bar{2}\ 0$	$5_2, 3_1, 0$	7_{22}
$4^1_*\ \bar{2}\ \bar{1}\ 0.1.1.2\ 0$	$\overline{5_2}, 4_1, 0$	$\overline{7_{24}}$
$4^1_*\ 2.3.1.1$	$\overline{5_1}, 0, 0$	7_{15}
$4^1_*\ 2.3\ 0.1.1$	$\overline{5_2}, 0, 0$	$\overline{7_{20}}$
$4^1_*\ 2.2\ 1.1.1$	$5_2, 3_1, 0$	7_{22}
$4^1_*\ 2\ 0.3.1.1$	$\overline{3_1}, 0, 0$	7_5
$4^1_*\ \bar{2}\ 0.3.1.1$	$3_1, \overline{3_1}, 0$	7_8
$4^1_*\ 2\ 0.3\ 0.1.1$	$0, 0, 0$	$\overline{7_1}$
$4^1_*\ \bar{2}\ 0.3\ 0.1.1$	$3_1, 0, 0$	$\overline{7_6}$
$4^1_*\ 2\ 0.2\ 1.1.1$	$3_1, \overline{3_1}, 0$	7_{10}
$4^1_*\ \bar{2}\ 0.2\ 1.1.1$	$3_1, 0, 0$	$\overline{7_7}$
$4^1_*\ 2.2.2.1$	$5_2, 4_1, 0$	7_{24}
$4^1_*\ 2.2.2\ 0.1$	$\overline{5_1}, 0, 0$	7_{17}
$4^1_*\ \bar{2}.2.2\ 0.1$	$\overline{5_1}, 0, 0$	7_{17}
$4^1_*\ 2.2.1.2$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$4^1_*\ 2.2.1.2\ 0$	$5_2, 0, 0$	7_{19}
$4^1_*\ 2.2.1.\bar{2}\ 0$	$4_1, 0, 0$	7_{13}
$4^1_*\ 2\ 0.2\ 0.2.1$	$4_1, 4_1, 0$	7_{14}
$4^1_*\ \bar{2}\ 0.2\ 0.2.1$	$\overline{3_1}, \overline{3_1}, 0$	7_9
$5^1_\times\ 2\ 1.1.1.1.1$	$\overline{7_7}, 0, 0$	$\overline{7_{63}}$
$5^1_\times\ \bar{2}\ \bar{1}.1.1.1.1$	$\overline{6_1}, 4_1, 0$	$\overline{6_8}$
$5^1_\times\ 2\ 1\ 0.1.1.1.1$	$\overline{7_6}, 0, 0$	$\overline{7_{51}}$
$5^1_\times\ 2.1.1.2.1$	$7_7, 0, 0$	7_{60}
$5^1_\times\ 2.1.1.\bar{2}.1$	$6_2, 0, 0$	6_{11}
$5^1_\times\ \bar{2}.1.1.\bar{2}.1$	$5_1, 0, 0$	7_{16}
$5^1_\times\ 2.1.1.1.2$	$7_7, \overline{3_1}, 0$	7_{64}
$5^1_\times\ 2.1.1.1.\bar{2}$	$6_2, 4_1, 0$	6_{13}
$5^1_\times\ 2.2\ 0.1.1.1$	$\overline{7_6}, 0, 0$	7_{52}
$5^1_\times\ 2.\bar{2}\ 0.1.1.1$	$\overline{5_1}, 0, 0$	$\overline{7_{16}}$
$5^1_\times\ \bar{2}.2\ 0.1.1.1$	$6_3, 0, 0$	$\overline{6_{15}}$
$5^1_\times\ \bar{2}.\bar{2}\ 0.1.1.1$	$\overline{6_2}, 0, 0$	$\overline{6_{11}}$
$5^1_\times\ 2.1.2\ 0.1.1$	$7_6, 3_1, 0$	$\overline{7_{55}}$
$5^1_\times\ 2\ 0.1.2\ 0.1.1$	$\overline{7_5}, \overline{3_1}, 0$	7_{47}
$5^1_\times\ 2\ 0.1.1.2\ 0.1$	$\overline{7_4}, 3_1, 0$	$\overline{7_{41}}$
$5^1_\times\ 2\ 0.1.1.1.2\ 0$	$\overline{7_4}, 3_1, 0$	$\overline{7_{40}}$
$5^1_*\ \bar{1}.\bar{3}.1.1.1$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$5^1_*\ 1.1.1.1.\bar{3}$	$5_1, 5_2, 0$	7_{18}
$5^1_*\ 3\ 0.1.1.1.1$	$4_1, 0, 0$	7_{13}
$5^1_*\ \bar{3}\ 0.1.1.1.1$	$4_1, 0, 0$	$\overline{7_{13}}$
$5^1_*\ \bar{1}.3\ 0.1.1.1$	$6_1, 4_1, 0$	6_8

Table 2. (*Continued*)

Notation	Constituent knot	θ-curve
5^1_* $1.1.1.1.3\ 0$	$\overline{6_1}$, 4_1, 0	$\overline{6_8}$
5^1_* $\bar{1}.2\ 1.1.1.1$	6_3, 0, 0	$\overline{6_{15}}$
5^1_* $1.1.1.1.\bar{2}\bar{1}$	$\overline{5_1}$, $\overline{5_2}$, 0	$\overline{7_{18}}$
5^1_* $\bar{2}.\bar{2}.1.1.1$	$\overline{5_1}$, 0, 0	$\overline{7_{16}}$
5^1_* $\bar{2}.\bar{1}.\bar{1}.\bar{1}.2$	$\overline{5_1}$, 0, 0	$\overline{7_{16}}$
5^1_* $\bar{1}.\bar{2}.1.2.1$	5_1, 0, 0	7_{16}
5^1_* $\bar{1}.\bar{2}.1.1.2$	$\overline{5_1}$, $\overline{5_2}$, 0	$\overline{7_{18}}$
5^1_* $\bar{1}.\bar{2}.\bar{1}.\bar{1}.2$	$\overline{5_1}$, $\overline{5_2}$, 0	$\overline{7_{18}}$
5^1_* $\bar{1}.\bar{1}.\bar{2}.\bar{1}.2$	5_1, 0, 0	7_{16}
5^1_* $\bar{2}.2\ 0.1.1.1$	4_1, 3_1, 0	6_3
5^1_* $2.1.1.1.2\ 0$	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
5^1_* $\bar{1}.\bar{2}.1.2\ 0.1$	$\overline{5_2}$, 0, 0	$\overline{7_{21}}$
5^1_* $\bar{1}.\bar{2}.1.\bar{2}\ 0.1$	4_1, 3_1, 0	6_3
5^1_* $1.2.1.1.2\ 0$	6_2, 4_1, 0	6_{13}
5^1_* $\bar{1}.\bar{2}.1.1.2\ 0$	5_1, 5_2, 0	7_{18}
5^1_* $1.1.1.2.2\ 0$	$\overline{6_1}$, 0, 0	$\overline{6_7}$
5^1_* $2\ 0.2.1.1.1$	$\overline{3_1}$, $\overline{3_1}$, 0	7_9
5^1_* $2\ 0.\bar{2}.1.1.1$	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
5^1_* $\bar{2}\ 0.2.1.1.1$	0, 0, 0	7_4
5^1_* $\bar{2}\ 0.\bar{2}.1.1.1$	5_2, 0, 0	7_{21}
5^1_* $2\ 0.1.2.1.1$	4_1, 0, 0	7_{13}
5^1_* $\bar{2}\ 0.1.2.1.1$	$\overline{3_1}$, 0, 0	7_6
5^1_* $2\ 0.1.1.2.1$	3_1, 0, 0	$\overline{7_6}$
5^1_* $\bar{2}\ 0.1.1.2.1$	4_1, 0, 0	$\overline{7_{13}}$
5^1_* $2\ 0.1.1.1.2$	0, 0, 0	$\overline{7_4}$
5^1_* $2\ 0.1.1.1.\bar{2}$	$\overline{5_2}$, 0, 0	$\overline{7_{21}}$
5^1_* $\bar{2}\ 0.1.1.1.2$	3_1, 3_1, 0	$\overline{7_9}$
5^1_* $\bar{2}\ 0.1.1.1.\bar{2}$	4_1, 3_1, 0	6_3
5^1_* $\bar{1}.2\ 0.2.1.1$	6_1, 0, 0	6_7
5^1_* $\bar{1}.2\ 0.1.1.2$	$\overline{6_2}$, 4_1, 0	$\overline{6_{13}}$
5^1_* $\bar{1}.\bar{2}\ 0.\bar{1}.\bar{1}.2$	5_1, 5_2, 0	7_{18}
5^1_* $\bar{1}.\bar{1}.2\ 0.\bar{1}.2$	4_1, 3_1, 0	6_3
5^1_* $\bar{1}.\bar{1}.\bar{2}\ 0.\bar{1}.2$	$\overline{5_2}$, 0, 0	$\overline{7_{21}}$
5^1_* $\bar{1}.2\ 0.1.2\ 0.1$	$\overline{6_2}$, 0, 0	$\overline{6_{11}}$
5^1_* $\bar{1}.2\ 0.1.\bar{2}\ 0.1$	$\overline{5_1}$, 0, 0	$\overline{7_{16}}$
5^1_* $\bar{1}.\bar{1}.2\ 0.2\ 0.1$	0, 0, 0	6_1
5^1_* $\bar{1}.\bar{1}.2\ 0.\bar{2}\ 0.1$	3_1, 0, 0	$\overline{6_2}$
5^1_* $\bar{1}.\bar{1}.\bar{2}\ 0.2\ 0.1$	$\overline{5_1}$, 0, 0	$\overline{7_{16}}$
5^1_* $\bar{1}.\bar{1}.\bar{2}\ 0.\bar{2}\ 0.1$	0, 0, 0	6_1
5^1_* $1.1.2\ 0.1.2\ 0$	6_2, 0, 0	6_{11}
5^1_* $1.1.\bar{2}\ 0.1.2\ 0$	5_1, 0, 0	7_{16}

Table 2. (*Continued*)

Notation	Constituent knot	θ-curve
6^1_* 1.2.1.1.1.1	0, 0, 0	7_2
6^1_* 1.2 0.1.1.1.1	4_1, 0, 0	7_{12}
6^1_* 1.$\bar{2}$ 0.1.1.1.1	4_1, 3_1, 0	6_3
6^2_* 1.1.1.2.1.1	3_1, 0, 0	$\overline{7_6}$
6^2_* 2 0.1.1.1.1.1	0, 0, 0	$\overline{7_3}$
6^2_* $\bar{2}$ 0.1.1.1.1.1	0, 0, 0	$\overline{7_4}$
6^2_* 1.2 0.1.1.1.1	4_1, 0, 0	7_{13}
6^2_* 1.1.2 0.1.1.1	$\overline{3_1}$, $\overline{3_1}$, 0	7_9
6^2_* 1.1.$\bar{2}$ 0.1.1.1	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
6^3_* 2.1.$\bar{1}$.1.1.1	3_1, 3_1, 0	$\overline{3_1}\#_3\overline{3_1}$
6^3_* 2.1.1.1.$\bar{1}$.$\bar{1}$	4_1, 3_1, 0	6_4
6^3_* 1.2.$\bar{1}$.1.1.1	4_1, 3_1, 0	6_4
6^3_* 1.2.1.1.$\bar{1}$.$\bar{1}$	3_1, 3_1, 0	$\overline{3_1}\#_3\overline{3_1}$
6^3_* 1.1.2.1.1.1	6_3, 0, 0	6_{15}
6^3_* 1.1.$\bar{2}$.1.1.1	5_2, 0, 0	7_{21}
6^3_* 1.1.2.1.$\bar{1}$.$\bar{1}$	0, 0, 0	$\overline{6_1}$
6^3_* 1.1.$\bar{2}$.1.$\bar{1}$.$\bar{1}$	5_1, 0, 0	7_{16}
6^3_* 1.1.1.1.2.$\bar{1}$	6_3, 0, 0	6_{15}
6^3_* 1.1.1.1.$\bar{2}$.$\bar{1}$	5_2, 0, 0	7_{21}
6^3_* 1.1.$\bar{1}$.1.2.1	0, 0, 0	$\overline{6_1}$
6^3_* 1.1.$\bar{1}$.1.$\bar{2}$.1	5_1, 0, 0	7_{16}
6^3_* 2 0.1.1.1.$\bar{1}$.$\bar{1}$	$\overline{3_1}$, 0, 0	6_2
6^3_* 2 0.1.$\bar{1}$.1.1.1	3_1, $\overline{3_1}$, 0	$3_1\#_3\overline{3_1}$
6^3_* 2.$\bar{1}$.1.$\bar{1}$.$\bar{1}$.$\bar{1}$	$\overline{3_1}$, 0, 0	3_1
6^3_* 2.$\bar{1}$.$\bar{1}$.$\bar{1}$.1.1	0, 0, 0	6_1
6^3_* 1.2 0.1.1.$\bar{1}$.$\bar{1}$	3_1, $\overline{3_1}$, 0	$3_1\#_3\overline{3_1}$
6^3_* 1.2 0.$\bar{1}$.1.1.1	$\overline{3_1}$, 0, 0	6_2
6^3_* 1.$\bar{2}$ 0.1.1.$\bar{1}$.$\bar{1}$	3_1, 0, 0	$\overline{3_1}$
6^3_* 1.$\bar{2}$ 0.$\bar{1}$.1.1.1	0, 0, 0	$\overline{6_1}$
6^3_* 1.1.2 0.1.$\bar{1}$.$\bar{1}$	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
6^3_* 1.1.$\bar{2}$ 0.1.1.1	0, 0, 0	6_1
6^3_* 1.1.1.1.$\bar{2}$ 0.$\bar{1}$	0, 0, 0	6_1
6^3_* 1.1.$\bar{1}$.1.$\bar{2}$ 0.1	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
6^4_* 2 0.1.1.1.1.1	3_1, 0, 0	$\overline{6_2}$
6^4_* 1.2 0.1.1.1.1	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
6^4_* 1.1.2 0.1.1.1	0, 0, 0	$\overline{6_1}$
$7^4_\times$ 1.1.1.1.1.1.1	7_7, 0, 0	7_{61}
7^3_* 1.1.1.1.1.1.1	6_3, 0, 0	$\overline{6_{15}}$
7^4_* 1.1.1.1.1.1.1	0, 0, 0	7_4
7^4_* 1.$\bar{1}$.1.$\bar{1}$.$\bar{1}$.1.$\bar{1}$	4_1, $\overline{3_1}$, 0	$\overline{6_3}$
7^9_* 1.1.1.1.1.1.1	$\overline{3_1}$, 0, 0	6_2

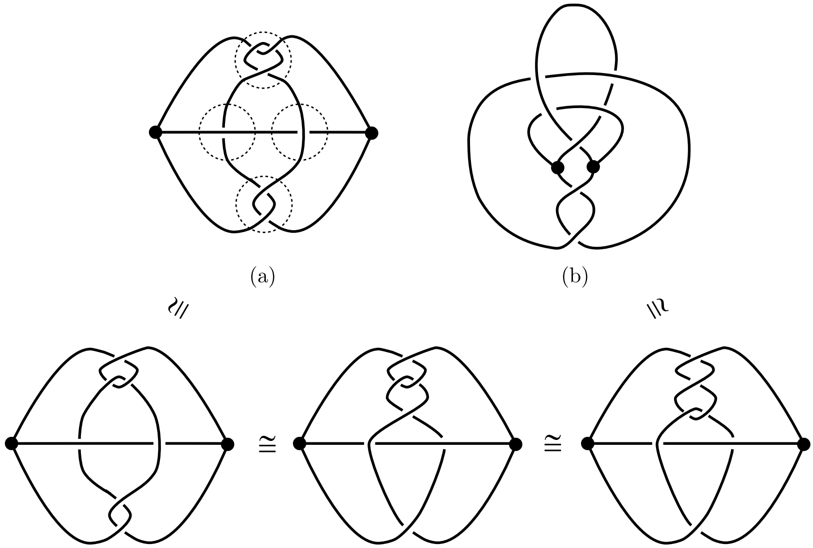


Fig. 25. (a) $4^1_*2\ 1\ 0.1.1.\bar{2}\ 0$; (b) 7_{22}.

Acknowledgments

The author would like to thank Professor Taizo Kanenobu and Professor Akio Kawauchi for their valuable advice and encouragement. I gratefully acknowledge the financial support from Japan Society for the Promotion of Science (JSPS) and Osaka City University Advanced Mathematical Institute (OCAMI).

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