KOREA SCIENCE ACADEMY OF KAIST

The Arc Index of Theta-Curve and Handcuff Graph

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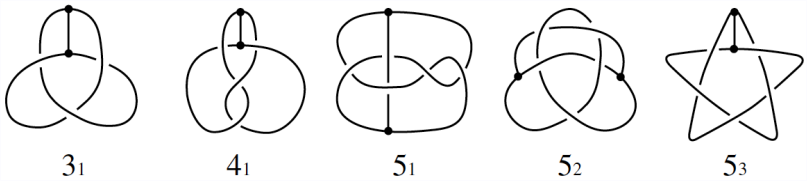
Introduction

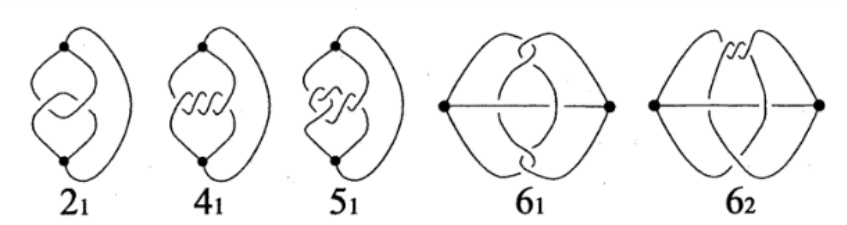
Theta-Curves

- A **theta-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.
- Theta-curves are roughly classified by comparing the triples of constituent knots.
- A theta-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3.

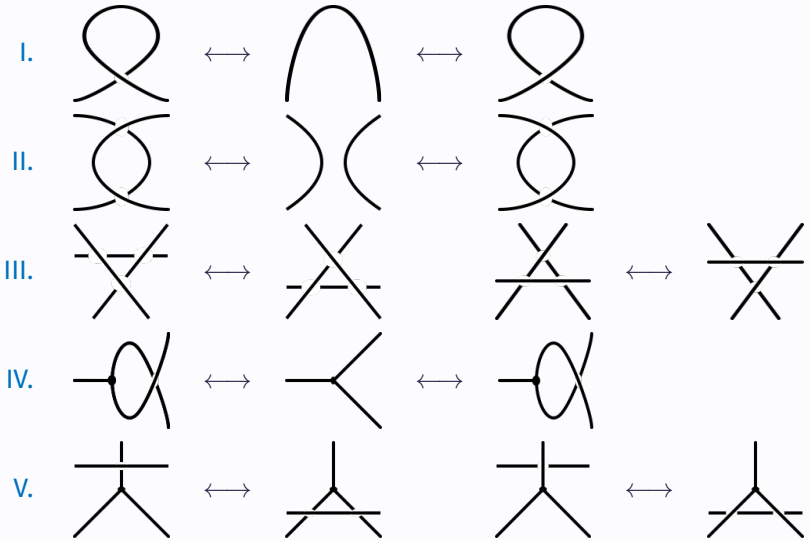


Handcuff Graphs

- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.
- A **constituent link H_{12}**, is a subgraph of H that consists of two vertices v_1, v_2 and two edges e_1, e_2.



Reidemaister Moves for Theta-Curves and Handcuff Graphs



Arc Presentation

- Arc presentation of a theta-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called pages).
- The embedding is with the common boundary line (called axis).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- Arc index, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is minimal arc presentation.

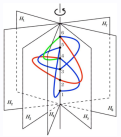
Arc Presentation

Trefoil

Open Book

Grid Diagram

$\theta_{5,2}$



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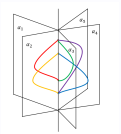
Grid Diagram

$\Phi_{2,1}$



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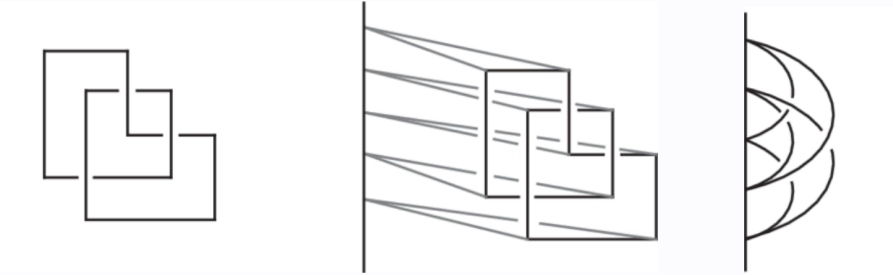
Grid Diagram

Grid Diagram

- The **grid diagram** of theta-curve or handcuff graph is a diagram with only vertical strand and horizontal strands.
- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Grid Diagram

- A grid diagram gives rise to an arc presentation and vice versa.

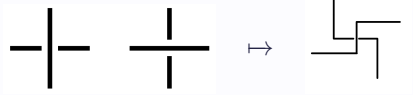


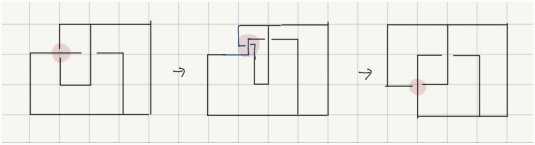
Arc Presentation of the Theta-Curve and Handcuff Graph

Theorem

Every theta-curve and handcuff graph admit a grid diagram.

Proof





Corollary

Every theta-curve and handcuff graph admit a arc presentation.