

Arc Index of a Theta Curve

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Introduction

θ -Curves

- A **θ -curve** T is a graph embedded in S^3 , which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3 , such that each edge joins the vertices.
- A **constituent knot** T_{ij} , $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j .
- θ -curves are roughly classified by comparing the triples of constituent knots.
- A θ -curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3 .

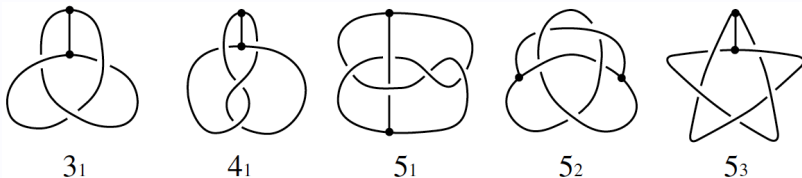
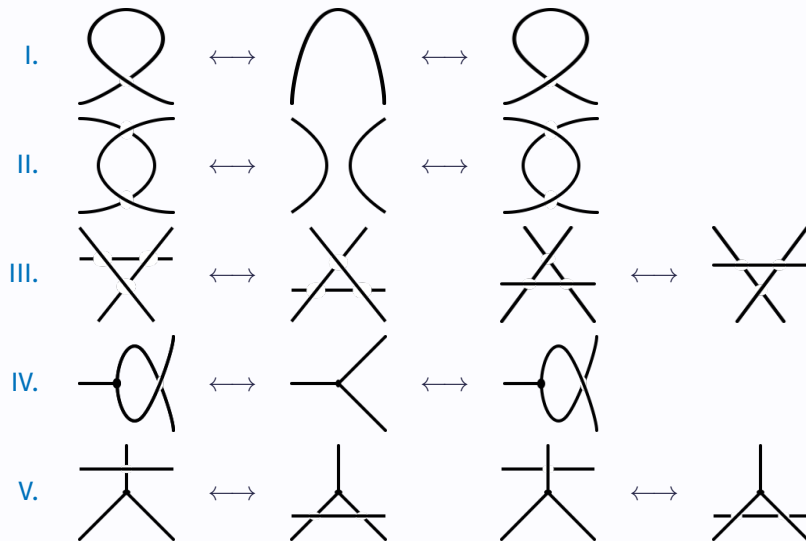


Figure from [Moriuchi, 2007]

Reidemeister Moves for θ -Curves

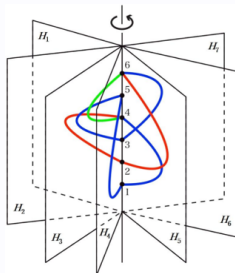


Arc Index

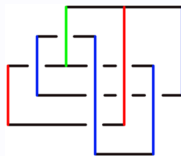
- An **arc presentation** of a θ -curve is defined in the same manner as an arc presentation of a knot.
- The binding axis contains all **vertices** of θ -curve.
- **Minimal arc presentation** and **arc index** are defined in the same manner.



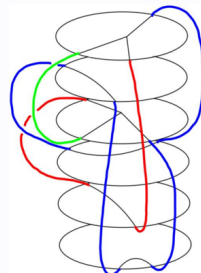
$\theta 5_2$



Open Book



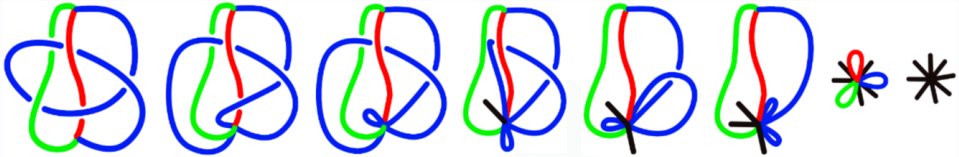
Grid Diagram



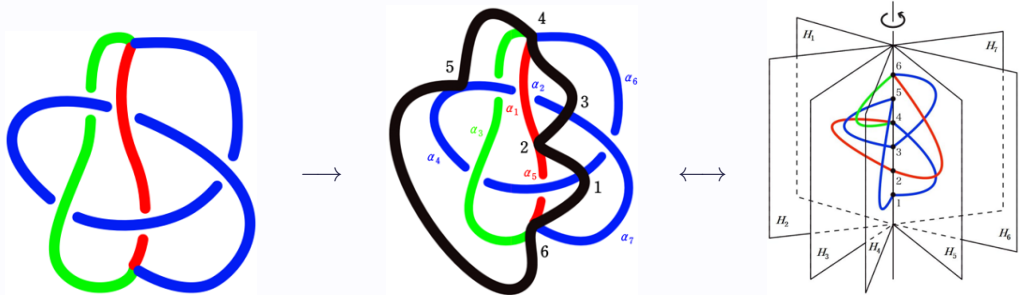
Stacked Tangle

Spoke Algorithm

- Gathering all crossings and vertices to one point to obtain a **wheel diagram**, which is corresponding to an arc presentation.



Binding Circle Method



Theorem ([Lee et al., 2019])

Let G be any spatial graph with e edges and b bouquet cut components. Then

$$\alpha(G) \leq c(G) + e + b$$

Corollary

Let T be any θ -curve. Then

$$\alpha(T) \leq c(T) + 3$$

Classifying by Determinant

THC-cromwell matrix

- The **cromwell matrix** of a knot is an $n \times n$ binary matrix such that each row and column has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ -curves and handcuff graphs that satisfies the following conditions :
 1. It is a $(n + 1) \times n$ binary matrix.
 2. It contains exactly two 1s in every column.
 3. There exists two distinct rows which contains exactly three 1s, which is called the **three-row**. Every other row contains exactly two 1s.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Cromwell matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

THC-cromwell matrix

Lower Bounds of Arc Index

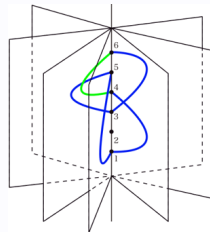
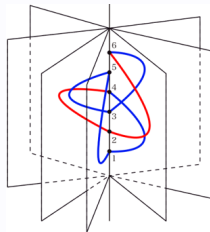
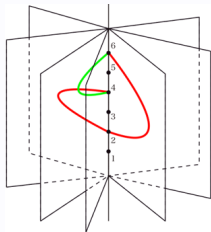
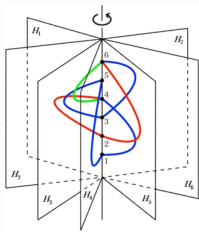
Lower Bounds from Constituent Knots

Theorem

Let T be any θ -curve and K_1, K_2, K_3 be three constituent knots of T . Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

PROOF



Theorem

Let T be any θ -curve and K_1, K_2, K_3 be three constituent knots of T . Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^3 \alpha(K_i)$$

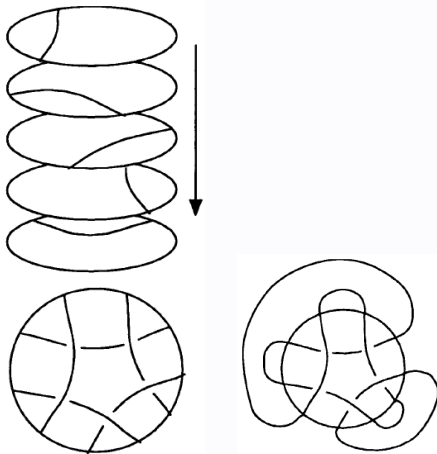
PROOF

- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i .
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i .
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

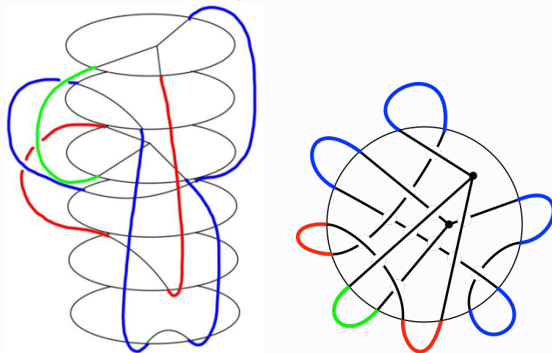
$$\sum_{i=1}^3 \alpha(K_i) \leq 2 \sum_{i=1}^3 |S_i| = 2\alpha(T)$$



Stacked Tangle of an θ -Curve



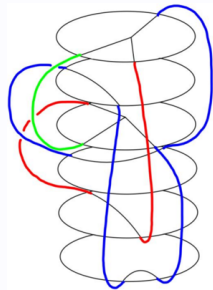
Stacked Tangle of a Link



Stacked Tangle of a θ -Curve

Stacked tangle of an θ -curve is stacked disks each with the frame as boundary with following properties:

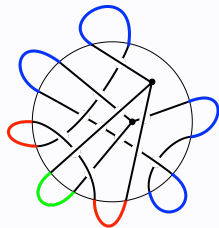
- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
 - two arcs in different simple disks intersect at most one point(by RII)
 - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Simple closure of stacked tangle is a **stacked tangle** with **caps** satisfying following properties:

- A **cap** is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.

Then a simple closure of a stacked tangle **without any nested caps** is corresponding to an arc presentation.



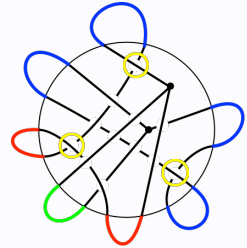
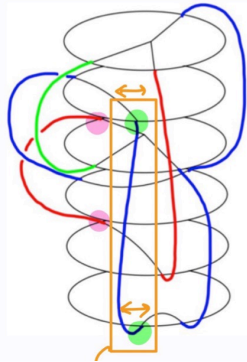
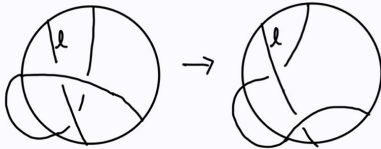
A **reduced simple closure of a stacked tangle** is

- a simple closure of a stacked tangle **without any nested caps**
- any two arcs(including line segment) joining by caps have **no intersection** when view from above

Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

PROOF



Yamada Polynomials

Let D_T be a diagram of an θ -curve T . Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}[x^{\pm 1}]$ is calculated by the following properties:

- **Y6:** $R(\bigoplus) = -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R(\bigcirc \text{---} \bigcirc) = 0$
- **Y8:** $R(T' \cup \bigcirc) = (x + 1 + x^{-1})R(T')$ for an arbitrary θ -curve diagram T'
- **Y9:** $R(\bigwedge) - R(\bigvee) = (x - x^{-1}) [R(\bigcirc \bigcirc) - R(\text{---})]$
- **Y10:** $R(\bigcirc) = x^2 R(\bigcap), \quad R(\bigcirc) = x^{-2} R(\bigcup)$
- **Y11:** $R(\bigotimes) = R(\bigcirc \bigcirc)$ **Y12:** $R(\bigotimes) = R(\bigvee \bigwedge)$
- **Y13:** $R(\bigstar) = R(\bigwedge), \quad R(\bigstar) = R(\bigvee)$
- **Y14:** $R(\text{---}) = -x R(\text{---}), \quad R(\text{---}) = -x^{-1} R(\text{---})$

Proposition ([Yamada, 1989])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n .

Theorem

Let T be any θ -curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of the θ -curve T .

Proposition

Let S_T be a simple closure of stacked tangle of a θ -curve T **without any nested caps**.
Then

$$\max \deg_x R(S_T) \leq c + n - 2 \quad \text{and} \quad \min \deg_x R(S_T) \geq -c - n + 2$$

where c is the **number of caps** and n is the **number of crossings** in S_T .

PROOF

- A **simple cap** is a cap joining simple disks.
- Let s be the **number of simple caps** in S_T .
- Use double mathematical induction of (s, n) .

Basis Step:

If $s = 0$, then S_T is either equivalent to \ominus or $\bigcirc\text{---}\bigcirc$.

- If $S_T \equiv \ominus$, then $R(S_T) = -x^2 - x - 2 - x^{-1} - x^{-2}$ and $4 \leq s + n$.
- If $S_T \equiv \bigcirc\text{---}\bigcirc$, then $R(S_T) = 0$ and $3 \leq c + n$.

If $n = 0$, then S_T is equivalent to $\bigcirc\text{---}\bigcirc \cup \bigcirc \cup \cdots \cup \bigcirc$.

- $R(S_T) = 0$ and $2 \leq c + n$.

All of the cases satisfy the inequalities.

Inductive Step:

Assume that the inequalities hold for any (s', n') where $0 \leq s' < s$ or $0 \leq n' < n$.

Let S_T be a simple closure of stacked tangle of a θ -curve T such that the number of simple caps is s and the number of crossings is n .

Take a **simple cap f** in S_T , joining boundary points P and Q .

CASE 1. Suppose that P and Q are boundary points of a single disk.

- $S_T = S'_T \cup \bigcirc$
- $R(S_T) = (x + 1 + x^{-1})R(S'_T)$
- The number of caps is $c - 1$ and the number of crossings n' is less than or equal to n in S'_T .

$$\max \deg_x R(S_T) = \max \deg_x R(S'_T) + 1 \leq [(c - 1) + n' - 2] + 1 \leq c + n - 2$$

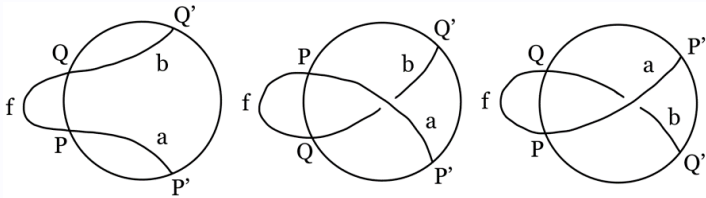
$$\min \deg_x R(S_T) = \min \deg_x R(S'_T) - 1 \geq [-(c - 1) - n' + 2] - 1 \geq -c - n + 2$$

- S'_T satisfy the inequalities implies S_T satisfy the inequalities.

CASE 2. Suppose that P and Q are boundary points of different disks D_P and D_Q , respectively.

① Suppose that D_P and D_Q are adjacent disks.

- When view from above, there are three cases:



- At first case, we can reduce the simple cap f .
- After applying **Y10**, other cases can be regarded as first case.

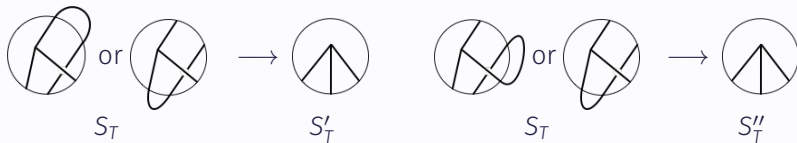
$$R(\text{cap}) = x^2 R(\cap), \quad R(\text{cap}) = x^{-2} R(\cap) \tag{Y10}$$

② Suppose that D_P and D_Q are not adjacent disks and D_P is above D_Q .

- Let D be the disk just above D_Q .
- If arcs or line segment in D and D_Q have no intersection, then we can change the position of D and D_Q without any quantities.
- We can assume that the arc in D_Q intersect arc or line segment in D , when view from above.

① There is a cap joining D_Q and D .

- D_Q and D are adjacent disks.
- If D is a simple disk, then we can reduce a simple cap as Case 2-①.
- If D is a non-simple disk, then



- $R(S_T) = -x^{\pm 1}R(S'_T)$ and $R(S_T) = x^{\pm 2}R(S''_T)$ by **Y14** and **Y10**, respectively.
- Both of S'_T and S''_T have $s - 1$ simple caps, $c - 1$ caps, and $n - 1$ crossing.
- By induction hypothesis,

$$\begin{aligned}\max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 1 \\ &\leq [(c - 1) + (n - 1) - 2] \pm 1 \\ &< c + n - 2\end{aligned}$$

$$\begin{aligned}\min \deg_x R(S_T) &= \min \deg_x R(S'_T) \pm 2 \\ &\geq [-(c - 1) - (n - 1) + 2] \pm 2 \\ &\geq -c - n + 2\end{aligned}$$

② There is no cap joining D_Q and D .

- Applying Y9

$$R(\text{X}) = R(\text{X}) + (x - x^{-1}) [R(\text{O}) - R(\text{X})]$$

then

$$R(S_T) = R(S_T^-) + (x - x^{-1}) [R(S_T^0) - R(S_T^\infty)]$$

- S_T^0 and S_T^∞ have c caps and $n - 1$ crossings.
- $(x - x^{-1}) [R(S_T^0) - R(S_T^\infty)]$ satisfy the inequalities.
- If S_T^- satisfy the inequalities, then S_T also satisfy the inequalities.
- The gap between D_P and D_Q is reduced in S_T^- .
- For S_T^- , investigate above cases.

This process will terminate after a finite number of investigations. It is the end of **CASE 2**.

□

Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ -curve T corresponding to minimal arc presentation of T . Then

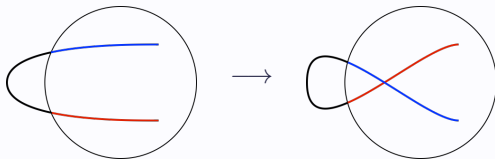
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n + 4 \leq \alpha(T)$$

where n is the number of crossings in S_T .

PROOF

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T , $\alpha(T)$.

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the p negative($c - p$ positive) curls.

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n + p$	$n + (c - p)$

$$\bullet R(S_T^{neg}) = x^{-2p}R(S_T) \text{ and } R(S_T^{pos}) = x^{2(c-p)}R(S_T)$$

$$\begin{aligned} \min \deg_x R(S_T) - 2p &= \min \deg_x R(S_T^{neg}) \\ &\geq -c + -(n + p) + 2 \end{aligned}$$

$$\begin{aligned} \max \deg_x R(S_T) + 2(c - p) &= \max \deg_x R(S_T^{pos}) \\ &\leq c + [n + (c - p)] - 2 \end{aligned}$$

$$\min \deg_x R(S_T) \geq -c - n + p + 2$$

$$\max \deg_x R(S_T) \leq n + p - 2$$

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq c + 2n - 4$$

□

Proof of Theorem

Theorem

Let T be any θ -curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of the θ -curve T .

PROOF

Let S_T be a reduce simple closure of stacked tangle of a θ -curve T corresponding to minimal arc presentation of T .

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

Consider the maximum number of crossings in S_T .

- number of crossings by two simple disks : $\frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4)$
- number of crossings by a simple disk and non-simple disk : $2 (\alpha(T) - 3)$
- number of crossings counted by disks joined by cap : $\alpha(T)$
- number of crossings by two non-simple disks : 2

Thus

$$\begin{aligned} n &\leq \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4) + 2 (\alpha(T) - 3) - \alpha(T) + 2 \\ &= \frac{1}{2} [(\alpha(T))^2 - 5\alpha(T) + 4] \end{aligned}$$

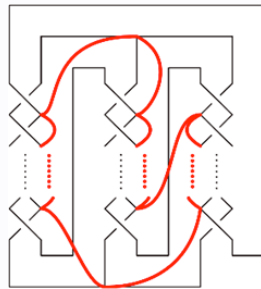
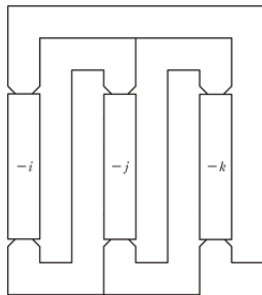
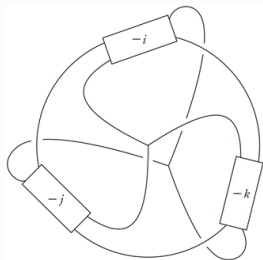
By Lemma,

$$\begin{aligned} \max \deg_x R(S_T) - \min \deg_x R(S_T) &\leq 2n - 4 + \alpha(T) \leq [\alpha(T)]^2 - 4\alpha(T) \\ 2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} &\leq \alpha(T) \end{aligned}$$

□

Further Studies

Kinoshita-Wolcott θ -Curve

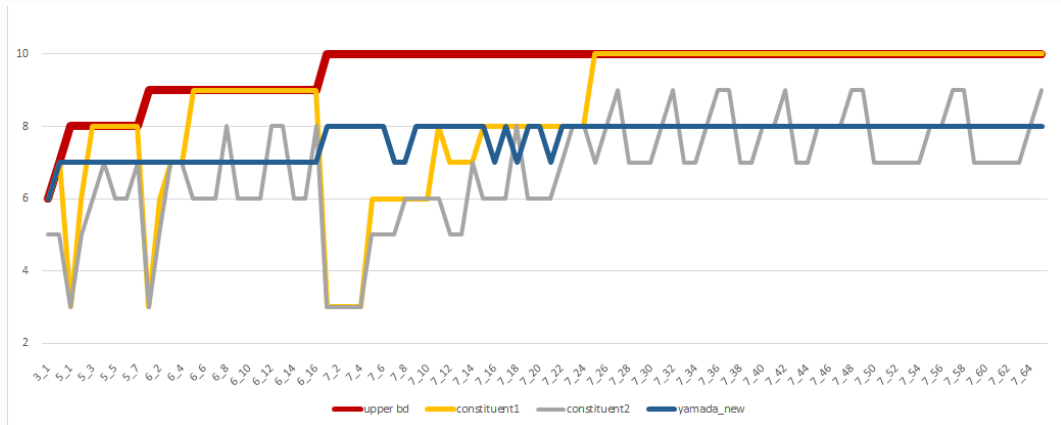


Theorem

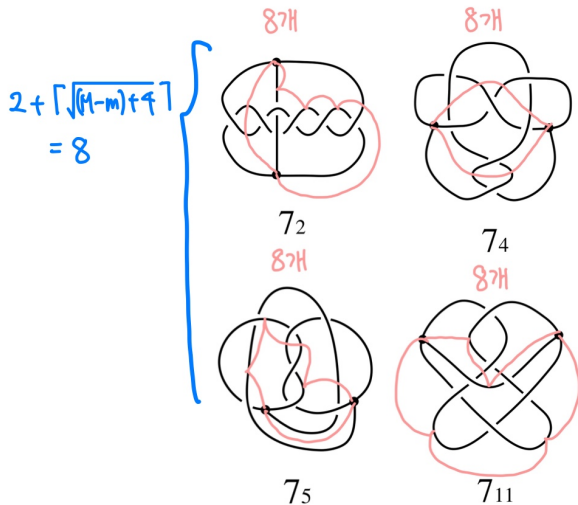
Let $K(-i, -j, -k)$ be the Kinoshita-Wolcott θ -curve. Then

$$\alpha(K(-i, -j, -k)) \leq i + j + k + 2$$







Bounds of Arc Index



Arc Index of Some θ -Curves



Thank You for Your Attention.

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