세타 커브와 수갑 그래프의 행렬식과 호 지수

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초록. 본 연구에서는 공간 그래프(spatial graph), 특히 엇갈림 수(crossing number) 7까지의 세타 커브(theta-curve)와 수갑 그래프(handcuff graph)에서의 최소 그물 그림(minimal grid diagram)과 그 후보군을 구해 정리함으로써 그들의 호 지수(arc index)를 확정하고자 하였다. 이 과정에서 세타 커브와 수갑 그래프의 행렬식을 정의하고 이를 이용하여 두 커브를 구분하는 결과를 얻었다. 또한 세타 커브와 수갑 그래프에 대한 호 지수의 상한과 하한의 결과를 정립하고, 하한에 대한 정리를 쌓인 얽힘 표현(stacked tangle representation)을 통해 완전히 증명하여 호 지수를 추정할 수 있었다. 더 나아가 야마다 다항식(Yamada polynomial)에 기반한 파이선 프로그램을 작성하여 이를 확인해 보았다.

중심어 : 세타 커브, 수갑 그래프, 호 지수, 최소 그물 그림

The Determinants and Arc Indices of θ-Curves and Handcuff Graphs

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Abstract. Our purpose of this research is classifying theta-curves and handcuff graphs by their arc indices, which their crossing is up to 7. We tried to find those arc indices by proving lower bounds and upper bounds of the theta-curves and handcuff graphs. Also we programmed Python code that returns the minimal grid diagram of the theta curves and handcuff graphs which have crossing up to 7. Yamada polynomial was used to prove bounds and programming the Python code.

Keywords : theta-curve, handcuff graph, arc index, minimal grid diagram

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1 Introduction

Knot theory is a field of mathematics that studies simple closed curves embedded in three-dimensional space. Study of theta-curves and handcuff graphs is included in knot theory. In this research, we tried to get the minimal grid diagrams of those graphs, which are up to 7 crossings.

We first decided to use the Python code to examine the minimal grid diagrams of theta-curves and handcuff graphs. For this, we found the method to classify theta-curve and handcuff graphs by their Cromwell matrices, which are called THC matrices. Finding all possible matrices for those graphs and finding out the Yamada polynomial of each graphs could give the minimal grid diagram of the graph.

Also, we used some mathematical methods to found out the minimal grid diagrams by using the bounds of the arc index, and used the Python code to examine if we found the right grid diagrams. Then we tried to decrease the gap of upper and lower bounds, and found out arc indices of some graphs.

2 Theoretical Background

Definition 1. *θ-curve* is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.

Definition 2. *Handcuff graph* is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.

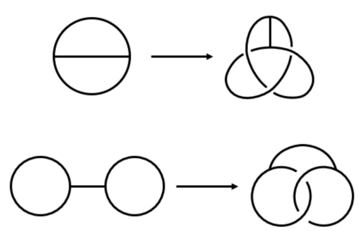


Figure 1: Examples of theta-curves and handcuff graphs.

In the projection of the handcuff graph and theta-curve, the section where they meet themselves is named *crossing*. For each handcuff graphs and theta-curves, the minimal number of crossings are called the *crossing number*. If the one graph and other graph's continuous transform of the graph is same, these graphs are *equivalent*. The *Generalized Reidemeister Moves* are used to transfrom a projection of handcuff graphs and theta-curves.

A theta-curve is said to be trivial if it can be embedded in a 2-sphere in S^3. In the similiar way, a handcuff graph is said to be trivial if it can be embedded in a 2-sphere in S^3.

Definition 3. *Arc presentation* is an open-book decomposition of $\mathbb{R}^3$ which has open half-planes as pages and the standard z-axis as the binding axis.

Every spatial graph G can be embedded in an open-book decomposition with finitely many pages so that it meets each page in exactly one simple arc with two different end-points on the binding axis. In knot theory, *arc index*, is the minimal number of pages among all possible arc presentations of graph. This arc presentation with the minimal number of pages is *minimal arc presentation. Prime knots* are knots that is not constructed by combining simpler knots. We are finding the arc index for the prime theta-curve and handcuff graph up to seven crossings.

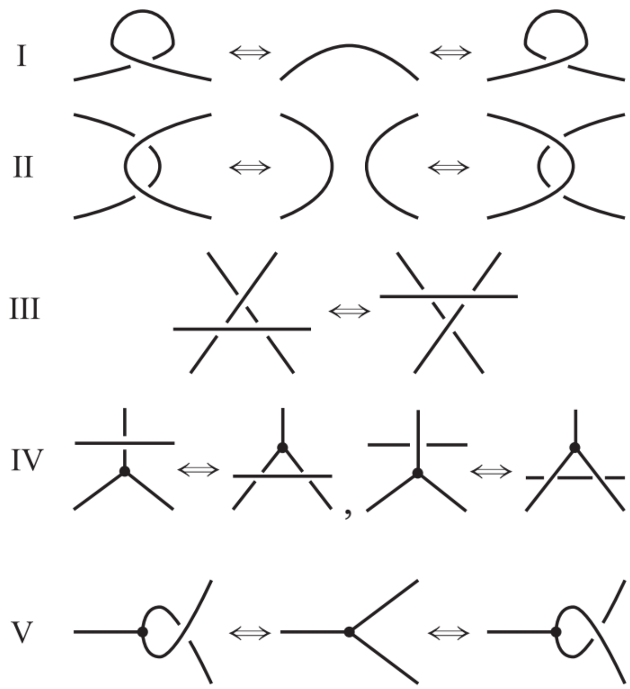


Figure 2: Generalized Reidemeister moves.

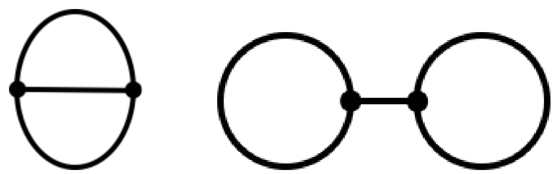


Figure 3: Trivial theta-curve and handcuff graph.

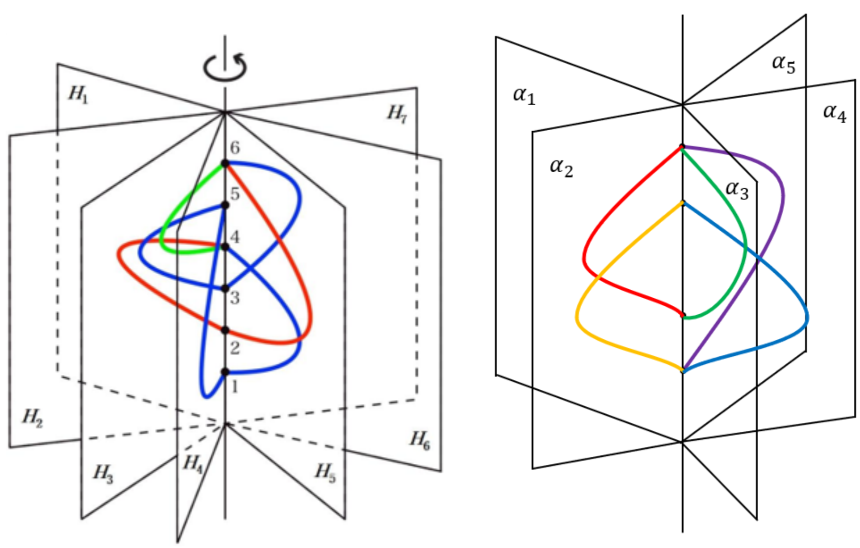


Figure 4: Arc presentation of theta-curve 3_1 and a trivial handcuff graph

Definition 4. The *grid diagram* is a handcuff graph or theta-curve diagram of vertical strands and one less number of horizontal strands with the properties that at every crossing the vertical strand crosses over the horizontal strand

and no two horizontal segments are co-linear and no two vertical segments are co-linear.

Definition 5. The *Cromwell matrix* is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s. For theta-curve and handcuff graph, its Cromwell matrix is called the the *THC-Cromwell matrix* is the matrix that satisfies the following conditions.

1. It is a binary $n \times (n+1)$ matrix.
2. It has exactly two 1s in every column.
3. There are only three 1s in two distinct rows (which are called the *Three-row*) and every other rows has exactly two 1s.

If the 1s of the Cromwell matrix are connected by horizontal and vertical lines with vertical lines are always on the horizontal lines, it leads to the grid diagram. The arc presentation can be expressed by grid diagram and vice versa. They are in one-to-one correspondence. Also, if the number of half planes in arc presentation is α, then the size of corresponding grid diagram is $(\alpha - 1) \times \alpha$.

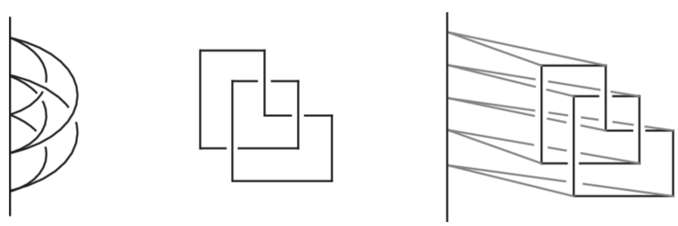


Figure 5: An arc presentation and a grid diagram of the knot

Theorem 6. *Every θ-curve and handcuff graph admit an arc presentation.*

Proof. For any theta-curve and handcuff graph, we can put it on a grid by using some suitable planar isotropy. Then for every crossing it has, there are only two following cases.

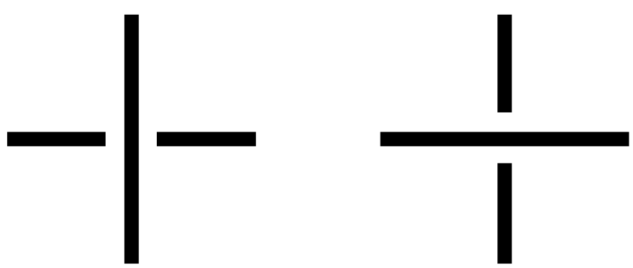


Figure 6: Cases for crossings

For each crossings, if it is a first case, then it is over. However, if it is a second case, then we can make that crossing to first case using some suitable movement.

Therefore, it becomes the grid diagram and since the grid diagrams and arc presentations are in one-to-one correspondence, there always exists arc presentations for every theta-curve and handcuff graph. □

Definition 7. The *link with n-components* is an embedding of the disjoint union of n circles $S^1 \cup \cdots \cup S^1$ in $\mathbb{R}^3$. 1-component link is called a *knot*.

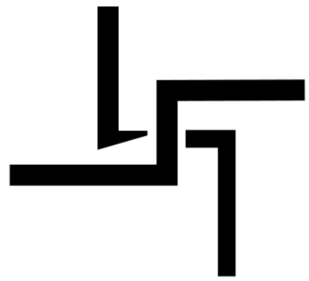


Figure 7: Suitable movement for case 2

The polynomial of the knot is invariant. *Yamada polynomial* is invariant when it is mulitplied $(-x)^n$ for some integer n. The *spread* of a knot, is substraction of maximal degree of minimal degree of the polynomial. It is a invariant since Yamada polynomial is invariant when mulitplied by $(-x)^n$.

Let 3 edges of theta-curve is e_1, e_2, e_3, then $e_1 \cup e_2$, $e_2 \cup e_3$, $e_3 \cup e_1$ becomes a knot on 3-sphere. This knots are *consituent knot* of the theta-curve.

Definition 8. *Stacked tangle* of an theta-curve and handcuff graph is stacked disks each with the frame as boundary with following properties:

- Only two disk called *non-simple disks* contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple disks is at the top.
- Other disks called *simple disks* contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point (by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point (by RV)

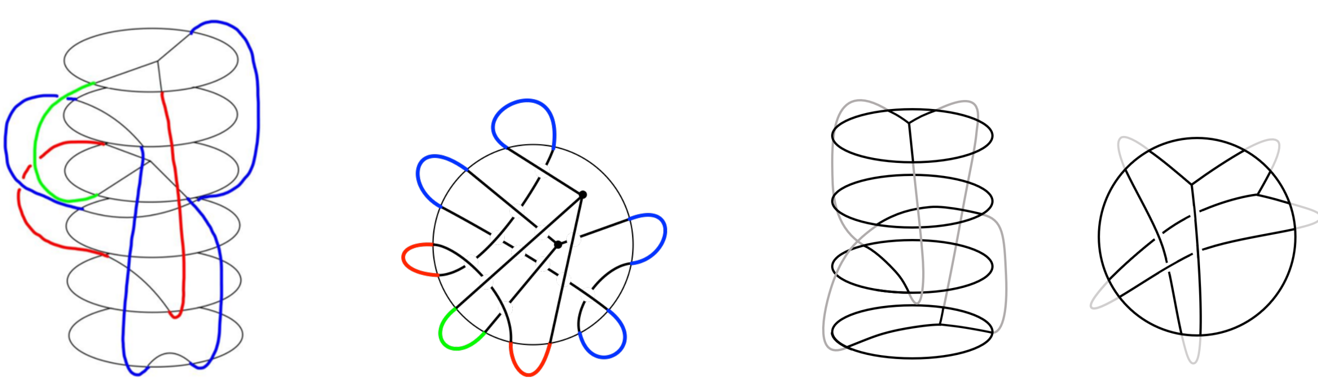


Figure 8: Stacked tangle representations.

Simple closure of stacked tangle is a stacked tangle with *caps* satisfying following properties:

- A *cap* is a simple arc in outside of stacked tangle joining end points of arcs or line segments

- When view from above any tow caps have no intersection.

Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

A *reduced simple closure of a stacked tangle* is

- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

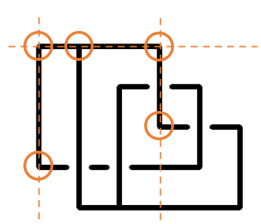
3 Research Methods and Procedure

First, we made a Python Code to find the arc index and the cromwell matrix of the knots. Then we used the upper and lower bounds, and also found the smaller upper bounds and larger lower bounds. Finally we used the direct method to find the arc index by drawing the binding circle.

3.1 The Python Code

The Python code generates the possible Cromwell matrices of theta-curves and handcuff graphs, and compares with the existing ones in the table which is less or equal to 7 crossings. It eventually returns the Cromwell matrix of the compared theta-curve and handcuff graph. To distinguish the Cromwell matrices from theta-curves and handcuff graphs, which is generated and removed by the rules from the above section, we should use the determinant of the Cromwell matrices.

Definition 9. Let any THC-cromwell matrix G. Consider any Three-row i(called *deleted-row*) and its three 1s, a_{ij}, a_{ik}, a_{il}, which means that $a_{ij} = a_{ik} = a_{il} = 1$ and $j < k < l$. The *H-deletion Matrix* of the THC-cromwell matrix is $(n-1) \times (n-1)$ matrix which deleted row i and column j, l (which are called the *edge-columns*) from the matrix G. The 1s contained in both ith row and jth, lth columns are called the *edge-one*, and the 1 contained in both ith row and kth column is called the *middle-one*.



$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure 9: H deletion of THC-matrix of theta-curve 3_1.

3.1.1 Main Theorem

Theorem 10. *In case the given THC-cromwell matrix is θ-curve or handcuff graph, then the given THC-Cromwell matrix is theta-curve if and only if determinant of the H-deletion matrix is ± 1. Also, the given THC-Cromwell matrix is handcuff graph if and only if determinant of the H-deletion matrix is 0 or ± 2.*

Proof. 1. **Determinant of the cromwell matrices of Knot**

The cromwell matrix of a knot is a $n \times n$ binary matrix which has exactly two 1s for each rows and columns We are interested of the determinant of the cromwell matrix of knots, so we are going to use the row and column operations to get simpler matrix. In this way, determinant of the Cromwell matrix would change only by multiplying ± 1.

Now we follow these *H to D* steps :

(a) Since there are always exactly two 1s in every row, there exists two columns that containing the 1s of the top row. Using column operations, interchange those two columns with the left most and secondary left most columns of the matrix.

(b) Choose secondary column from the left. It should contain another 1 below the 1 which is from the top row, so interchange a row containing that below 1 with secondary top row by using row operations.

(c) Cover the left most column and top row and repeat (a), (b) (What you need to careful is that when you interchange the row, you should also change the covered elements together.) until we get the following matrix.

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Next, we apply the row operation. We add each row above the bottom row to the bottom row.

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \\ 2 & 2 & \cdots & 2 & 2 \end{pmatrix}$$

If n is even, by subtracting bottom row with $2R_1, 2R_3, ..., 2R_{n-1}$ We can get a matrix like

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Since the last row is zero row, the determinant is 0.

If n is odd, by subtracting bottom row with $2R_1, 2R_3, ...2R_{n-2}$, we can get a following matrix.

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}$$

Since it is upper triangular matrix, we can obtain determinant by trace. Hence the other entry is all 1, the determinant is ± 2.

2. **Proof in the case of theta-curve**

   Since the THC-cromwell matrix is not a $n \times n$ matrix, we consider H-deletion matrix of THC-cromwell matrix to get its determinant. Actually, in the THC-cromwell matrix, the 1s in same column represents the vertical line of the grid diagram, and the 1s in same row represents the horizontal line of the grid diagram. Therefore, we can think H-delation matrix represents a grid diagram (here, we consider the more general version of grid diagram. It is a diagram consisted of vertical strands and horizontal strands with the properties that at every crossing, the vertical strand crosses over the horizontal strand, and no two horizontal segments and vertical segments are co-linear) without three horizontal lines and three vertical lines. Now, we are going to count n, which is the number of 1s in the middle column and edge-columns(deleted horizontal lines) of Three-row i, which is also included in another Three-row j.

   (a) **When $n = 0$**

   As we consider the non-deleted row's middle-one, the horizontal line and vertical lines that contain it is not deleted since $n = 0$. Therefore, the grid diagram represents a graph consisted a vertex (which is corressponding to middle-one) and three edges that connect this vertex with another three vertices. We'll call it as the *T-shape*.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

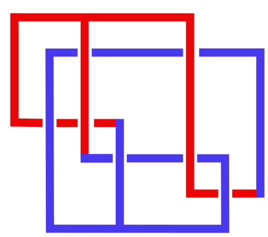


Figure 10: Examples of T-shape (Red line is a deleted line and blue line is a T-shape)

   (b) **When $n = 1$ (middle-one)**

   As we consider the non-deleted row's middle-one, a horixontal line that contains it is deleted since $n = 1$ at the middle-one. Therefore, the grid diagram represents a graph consisted a vertex (which is corressponding to middle-one) and two edges that connect this vertex with another two vertices. Since there are only two lines connected to middle-one, we can say that this grid diagram represents a graph consisted of two vertices and an edge connecting them. We'll call it as the *Line-shape*.

   (c) **When $n = 1$ (edge-one)**

   As we consider the non-deleted row's middle-one, a vertical line that consisted of the middle-one and deleted edge-one is deleted. Therefore, the grid diagram represents a graph consisted a vertex (which is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

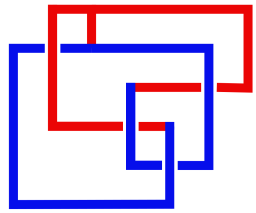


Figure 11: Examples of Line-shape (Red line is a deleted line and blue line is a Line-shape)

corressponding to middle-one) and two edges that connect this vertex with another two vertices. Since there are only two lines connected to middle-one, the grid diagram represents a Line-shape.

(d) **When $n = 2$ (edge-ones)**
As we consider the non-deleted row's middle-one, the horizontal line connected to it is not deleted, but other lines are all deleted. Therefore, the grid diagram represents a Line-shape.

(e) **When $n = 2$ (middle-one and edge-one)**
As we consider the non-deleted row's middle-one, the horizontal line and the vertical line(consisted of middle-one and deleted edge-one) are deleted. Hence, there is only one vertical line is connected to middle one. Therefore, the grid diagram represents a Line-shape.

(f) **When $n = 3$ (middle-one and edge-ones)**
Trivial θ-curve is the only way satisfying this condition. Therefore, its H-deletion matrix is (1) hence the determinant of H-deletion matrix is 1.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(i) **Line-shape**
The Line-shape is consisted of two vertices and an edge connecting them, and the grid diagram of the Line-shape is consisted of horizontal and vertical segments. This means that the H-deletion matrix, which is a Line-shape, is a $n \times n$ matrix with following properties :

- It is a binary matrix.
- One and only one of the following is satisfied
  - There exists only one row which contains exactly one 1, and another row which contains exactly three 1s. Every other row contains exactly two 1s. Also, there exists only one column which contains exactly one 1. Every other column contains exactly two 1s. (Type-(b))
  - There exists only one row which contains exactly one 1. Every other row contains exactly two 1s. Also, there exists only one column which contains exactly one 1. Every other column contains exactly two 1s. (Type-(c),(d),(e))

In case the Line-shape is a type-(b), we use the *H to D* steps, including following step instead of (a) step of first cycle:

i. Interchange a row and the column with exactly one 1 with the first row, and the last column. Interchange the rest row with exactly one 1 with the last row.

Therefore, after applying *H to D* steps, we can get a following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & & & 1 \\ \vdots & & & \ddots & 1 & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Also, if we subtract each row with its upper row starting from the top, we can get a upper triangular matrix, which diagonal entries are 1 or -1. Therefore, the determinant of the type-(b) H-deletion matrix is ± 1.

In case the Line-shape is a type-(c),(d),(e), we use the *H to D* steps, including following step instead of (a) step of first cycle:

i. Interchange a row with exactly 1 one with the top row.

Therefore, after applying *H to D steps*, we can get a following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & & & \\ 0 & 1 & 1 & & & \vdots \\ \vdots & & & \ddots & 1 & \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

Then, if we subtract each row with its upper row starting from the top, we can get a diagonal matrix, which diagonal entries are all 1. Therefore, the determinant of the type-(c),(d),(e) H-deletion matrix is ± 1.

As a result, the determinant of Line-shape H-deletion matrix is ± 1.

(ii) **T-shape**

The T-shape can be understood as the combination of two Line-shapes. To be specific, the H-deletion matrix of T-shape is $n \times n$ matrix with following properties :

- It is a bianry matrix.
- There exists only one row with exactly three 1s, and two rows which contains exactly one 1. Also, there exists only one column which contains exactly one 1, and other columns contain exactly two 1s.

Using row and column operations, we can make two regions in the matrix. One is the intersection of the first to kth row and first to $(k-1)$th column. Another is the intersection of kth to nth row and kth to nth column. Each region represents a Line-shape.

The red square region is a $(n+1) \times n$ matrix such that :

- It is a binary matrix.
- There are only two rows contains exactly one 1, and every other rows and columns contain exactly two 1s.

The blue square region is a H-deletion matrix, which is a type-(c),(d),(e) Line-shape. Using *H to D* steps in the red square including 'Interchange a row with exactly one 1 with the top row.' instead of step (a) at the first time, and using *H to D* steps in the blue square including 'Interchange the columns with

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Figure 12: Dividing Regions.

exactly one 1 with the fist column and last column.' instead of step (a) at the first time, we can get a following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & & \vdots & 0 & 0 \\ 0 & 1 & 1 & & 1 & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & \vdots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

For a red square, we subtract each row with its upper row starting from the top row. For a blue square, we subtract each column with its right column starting from the last column. As a result, we can get a upper triangular matrix, which diagnoal entries are all 1. Hence the determinant of T-shape is also ± 1.

Therefore, ± 1 will be the determinant of the theta-curve's H-deletion matrix.

3. **Proof in the case of handcuff graph**

   Similiar with θ-curve, we are going to count n of the THC-cromwell matrix, which is the number of 1s in the middle column and edge-columns of Three-row i, which is also included in another Three-row j.

   (a) **When $n = 0$**

   Considering non-deleted row's middle one, there exists three lines connected to it. We'll call it as the *T-loop*. Therefore, the H-deletion matrix will be union of a T-loop and a Line-shape.

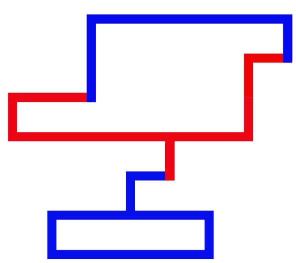


Figure 13: Examples of union of T-loop and Line-shape.

   (b) **When $n = 1$ (middle-one)**

   Considering non-deleted row's middle one, the horizontal line connected to this middle-one is deleted. Therefore, the H-deletion matrix will be union of a knot and a Line-shape.

(c) **When $n = 1$ (edge-one)**

Considering non-deleted row's middle one, a vertical line that consisted of the middle-one and deleted edge-one is deleted. Therefore, the H-deletion matrix will be union of a knot and a Line-shape.

(d) **When $n = 2$ (edge-ones)**

This is not possible since if two different row's edge-ones are connected, then it becomes the θ-curve.

(e) **When $n = 2$ (middle-one and edge-one)**

Similiar with (d), This is not possible.

(f) **When $n = 3$**

Similiar with (d), This is not possible.

(i) **T-loop**

T-loop can be understood as the combination of a knot and a Line-shape. Therefore, the H-deletion matrix is $n(n+1)$ matrix. If we delete the last column(we will call this matrix as *deleted-T-loop*), we can get a form of $\left(\begin{array}{c|c} Knot & * \\ \hline 0 & Line - shape \end{array}\right)$, where knot and Line-shape is a $n \times n$ and $k \times k$ matrices. Hence the determinant is det(knot)det(Line-shape) = 0 or ± 2.

(ii) **Union of T-loop and Line-shape**

We knot that T-loop and Line-shape is independent, which means that there are no 1s in same rows or columns. Therefore, the H-deletion matrix is a form of $\left(\begin{array}{c|c} deleted - T - loop & * \\ \hline 0 & Line - shape \end{array}\right)$ hence the determinant is det(deleted-T-loop)det(Line-shape) = 0 or ± 2.

(iii) **Union of Knot and Line-shape**

Knot and Line-shape is also independent. Therefore, the H-deletion matrix is a form of $\left(\begin{array}{c|c} Knot & * \\ \hline 0 & Line - shape \end{array}\right)$ hence the determinant is 0 or ± 2

Therefore, 0 or ± 2 will be the determinant of the handcuff graph's H-deletion matrix.

□

3.2 Bounds of Arc Index

We found the bounds of arc index by the theoretical background, or we found the new bounds inspired by the theoretical background. For the θ-curve, it is known that the following theorems hold.

Theorem 11 (2). *Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then*

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1.$$

Theorem 12 (2). *Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then*

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i).$$

Theorem 13 (2)**.** *For any θ-curve θ,*

$$\alpha(\theta) \geq \frac{1+\sqrt{f(\alpha(\theta))+36c(\theta)}}{3} \left(f(x) = \begin{cases} 73 & x \equiv 0 \pmod 6 \\ 4 & x \equiv 1 \pmod 6 \\ 25 & x \equiv 2 \pmod 6 \\ -8 & x \equiv 3 \pmod 6 \\ 49 & x \equiv 4 \pmod 6 \\ 20 & x \equiv 5 \pmod 6 \end{cases} \right).$$

Proposition 14 (2)**.** *For θ-curve T,*

$$\alpha(T) \leq c(T) + 3$$

where $c(T)$ is the crossing number of T.

Now, for the handcuff graph, the followings hold.

Theorem 15 (5)**.** *If L is an alternating and non-split link, then*

$$\alpha(L) = c(L) + 2.$$

Theorem 16 (3)**.** *For any spatial graph H,*

$$\alpha(H) \leq c(H) + e + b,$$

where e is the number of the edge and b is the number of the bouquet.

Corollary 17. *If H is a handcuff curve,*

$$\alpha(H) \leq c(H) + 5.$$

Especially, if the constituent link of H is non-split,

$$\alpha(H) \leq c(H) + 3.$$

Proof. If H is a handcuff curve, the number of the edge is 3, and the number of the bouquet is at most 2. Thus the first inequality holds since $e = 3, b \leq 2$. If there is a bouquet, one of the loops can be pulled out without affecting the rest. (See figure 13.) Then, if we remove the vertex edge of H, the remaining constituent link is a split link. Thus there are no bouquet if the constituent link of H is non-split, and we have second inequality since $e = 3, b = 0$. □

Proposition 18. *For a handcuff curve H, let L be the constituent link of H. Then,*

$$\alpha(H) \geq \alpha(L) + 1.$$

Proof. In the arc presentation of H, let v_1, v_2 be the vertices of H. Then, there are half-planes that contain the vertex edge of H. If we remove them, the remainder is the arc presentation of L, the constituent link of H. Since the number of half-planes that contain the vertex edge is at least 1, we obtain

$$\alpha(H) \geq \alpha(L) + (\text{the number of half plane that contain vertex edge}) \geq \alpha(L) + 1.$$

□

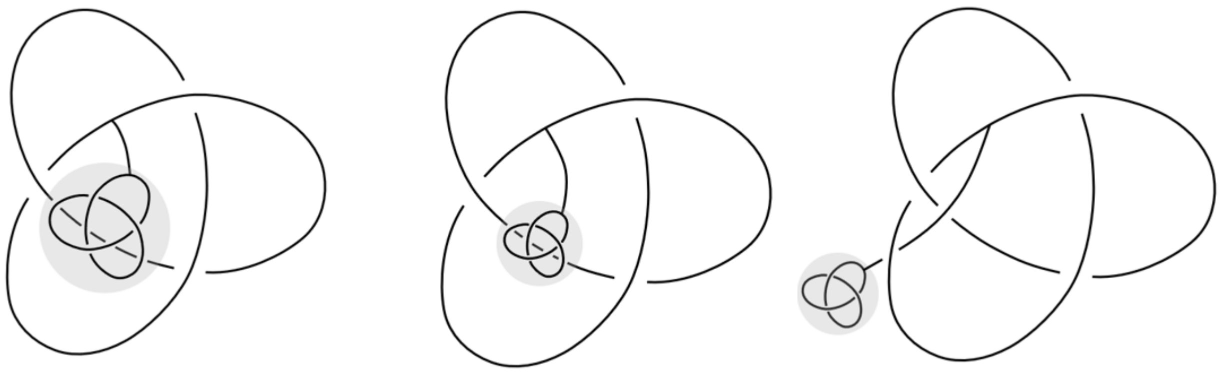


Figure 14: Pulling the bouquet

Using the Theorem 15, we obtain the following corollary.

Corollary 19. *In the handcuff curve, if its constituent link L is alternating and non-split, then*

$$\alpha(H) \geq c(L) + 3.$$

Proof. Since L is alternating and non-split link, then $\alpha(L) = c(L) + 2$ by Theorem 15. Thus,

$$\alpha(H) \geq \alpha(L) + 1 = (c(L) + 2) + 1 = c(L) + 3$$

according to Proposition 18. □

We now consider the Yamada polynomial of the handcuff curve and investigate the relationship between the difference of its maximum and minimum degrees and the arc index.

Definition 20. For a graph $G = (V, E)$, where V is vertex set of G and E is edge set of G, let us define 2-variable Laurent polynomial

$$h(G)(x, y) = \sum_{F \subset E} (-x)^{-|F|} x^{\mu(G-F)} y^{\beta(G-F)}$$

where $\mu(G)$ and $\beta(G)$ is the number of connected components of G and the first Betti number of G.
Then, the Yamada polynomial of a graph G, $R(G)$ is defined by

$$R(G)(x) = h(G)(-1, -x - 2 - x^{-1}).$$

It is known that, for some integer n, the product $(-x)^n R(G)$ (where $R(G)$ is the Yamada polynomial of the spatial graph G) is an ambient isotopy invariant. Moreover, the Yamada polynomial satisfies the following properties.

Theorem 21. *For the Yamada polynomial, the following properties hold.*

1. $R(\cdot) = -1$

2. *Let e be a non-loop edge of a graph G. Then, $R(G) = R(G/e) + R(G - e)$, where G/e, $G - e$ denote the graphs obtained by contracting and deleting the edge e, respectively.*

3. *Let e be a loop edge of a graph G. Then, $R(G) = -(x + 1 + x^{-1})R(G - e)$.*

4. *Let* $G_1 \cup G_2$ *be a disjoint union of graphs* G_1 *and* G_2*. Then,* $R(G_1 \cup G_2) = R(G_1)R(G_2)$*.*
5. *Let* $G_1 \cdot G_2$ *be a union of graphs* G_1 *and* G_2 *having one common point. Then,* $R(G_1 \cdot G_2) = -R(G_1)R(G_2)$*.*
6. *If* G *has an isthmus, then* $R(G) = 0$*.*

Theorem 22. *For the Yamada polynomial, the following properties hold.*

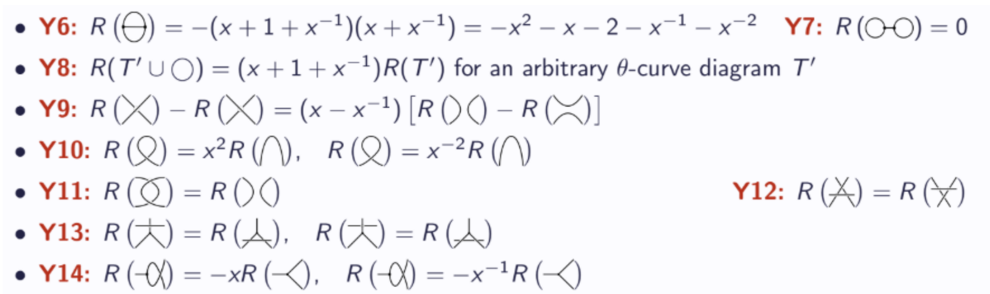


Figure 15: Properties of Yamada polynomial

Now, using the stacked tangle representation and the Yamada polynomial, we establish lower bounds for the arc index of θ-curves and handcuff graphs.

Proposition 23. *Let* S_G *be the simple closure of a stacked tangle of a* θ*-curve or handcuff graph* G*. Then*

$$\max \deg_x R(G) \le c + n, \quad \min \deg_x R(G) \ge -(c+n),$$

where c *and* n *are the number of caps and crossings in* S_G*, respectively.*

Proof. Let a *simple cap* be a cap connecting two simple disks, and a *semi-simple cap* be a cap connecting a simple disk and a non-simple disk. We use double induction on $(c_s + c_{ss}, n)$, where c_s and c_{ss} are the numbers of simple caps and semi-simple caps, respectively.

Basis Step

1. When $c_s + c_{ss} = 0$
   Since $c_s = c_{ss} = 0$, S_G has no simple disks. Applying the Yamada relation Y14, we obtain

   $$R(S_G) = -x^{\pm 3}(-x^2 - x - 2 - x^{-1} - x^{-2}),$$

   and it holds the inequalities $\max \deg_x R(S_G) \le c + n$ and $\min \deg_x R(S_G) \ge -(c+n)$.
2. When $n = 0$
   There are no crossings, so S_G consists of trivial handcuff graph and the disjoint disks. Then $R(S_G) = 0$ and clearly $0 \le c + n$, so the inequalities hold.

Inductive Step

Assume the theorem holds for all $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$ with $c_s + c_{ss} > 0$. Consider S_G with c_s simple caps, c_{ss} semi-simple caps, and n crossings.
Let D_s be the topmost simple disk connected to the top disk, and D the disk directly above D_s. We distinguish two cases:

Case 1. No cap between D_s and D

1. If D_s and D do not intersect, swapping their positions does not change S_G.

2. If D_s and D intersect, apply Yamada relation Y9:

$$R(S_G) - R(S_G^-) = (x - x^{-1})\big(R(S_G^0) - R(S_G^\infty)\big),$$

where S_G^-, S_G^0, S_G^∞ are obtained by replacing with , , , respectively. S_G^0 and S_G^∞ has fewer crossings than S_G, so the induction hypothesis ensures the following inequalities:

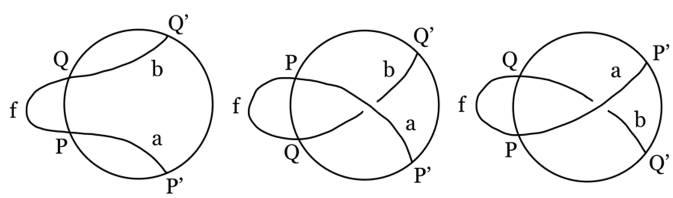
$$\max\deg_x R(S_G^0) \le c + n - 1, \quad \min\deg_x R(S_G^0) \ge -(c + n - 1)$$
$$\max\deg_x R(S_G^\infty) \le c + n - 1, \quad \min\deg_x R(S_G^\infty) \ge -(c + n - 1)$$

Therefore, if S_G^- satisfies the inequality, then S_G does also.

From the two cases above, when there is no cap between D_s and D, it suffices to consider the diagram obtained by swapping D_s and D. Repeating this process, we reduce to the case where a cap exists between D_s and D.

Case 2. There is a cap between D_s and D

1. If D is a simple disk
   When view from above, there are three cases:



When the first case, we can easily reduce the caps, and the inequality holds by inductive hypothesis. When the second and third cases, we can reduce the cap and crossing using Y10. Let S'_G be a reduced diagram with $c-1$ caps, $c_s - 1$ simple caps, c_{ss} semi-simple caps, and $n-1$ crossings. By inductive hypothesis,

$$\max\deg_x R(S_G) = \max\deg_x R(S'_G) \pm 2 \le c + n, \quad \min\deg_x R(S_G) = \min\deg_x R(S'_G) \pm 2 \ge -(c + n).$$

2. If D is non-simple disk
   When there is no intersection between D_s and D, we can easily reduce the caps, and the inequality holds by inductive hypothesis. Suppose there is an intersection.
   When viewed from above, all the cases can be reduced as follows:

Applying Y10 and Y14, we can reduce diagrams S'_G and S''_G with $c-1$ caps, c_s simple caps, $c_{ss}-1$ semi-simple caps, and $n-1$ crossings. Then by induction,

$$\max\deg_x R(S_G) = \max\deg_x R(S'_G) \pm 1 \leq c+n, \quad \min\deg_x R(S_G) = \min\deg_x R(S'_G) \pm 1 \geq -(c+n)$$
$$\max\deg_x R(S_G) = \max\deg_x R(S''_G) \pm 2 \leq c+n, \quad \min\deg_x R(S_G) = \min\deg_x R(S''_G) \pm 1 \geq -(c+n)$$

and the inequalities hold for S_G.

Thus, by induction, the inequalities hold for all simple closures of stacked tangles of θ-curves and handcuff graphs.

□

Theorem 24. *Let S_G be a reduced simple closure of a stacked tangle corresponding to a minimal arc presentation of G. Then*

$$\max\deg_x R(S_G) - \min\deg_x R(S_G) - 2n \leq \alpha(G),$$

where n is the number of crossings and $\alpha(G)$ is the arc index of G.

Proof. In a minimal arc presentation, the number of caps c equals the arc index $\alpha(G)$. Since S_G is reduced simple closure, we can insert curls at each caps. Depending on the direction of the curl, we classify them into positive and negative curls, and let p denote the number of maximum negative curls.
Let S_G^{neg} and S_G^{pos} denote the diagrams with negative and positive curls inserted, respectively. Then,

$$R(S_G^{neg}) = x^{-2p}R(S_G), \quad R(S_G^{pos}) = x^{2(c-p)}R(S_G).$$

By analyzing the maximum and minimum degrees,

$$\begin{aligned}
\min\deg_x R(S_T) - 2p &= \min\deg_x R\left(S_T^{neg}\right) \\
&\geq -c-(n+p) \\
\max\deg_x R(S_T) + 2(c-p) &= \max\deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n+(c-p)] \\
\min\deg_x R(S_T) &\geq -c-n+p \\
\max\deg_x R(S_T) &n+p \\
\max\deg_x R(S_T) - \min\deg_x R(S_T) &\leq c+2n
\end{aligned}$$

and we obtain

$$\max\deg_x R(S_G) - \min\deg_x R(S_G) \leq c+2n = \alpha(G)+2n.$$

□

Theorem 25. *Let G be any θ-curve or handcuff graph. Then*

$$2+\sqrt{\max\deg_x R(S_G) - \min\deg_x R(S_G) - 4} \leq \alpha(G),$$

where $R(G)$ is the Yamada polynomial of G.

Proof. Let S_G be a reduced simple closure of a stacked tangle corresponding to a minimal arc presentation of G. Then, the number of caps is $\alpha(G)$, and S_G has 2 non-simple disks and $\alpha(G)-3$ simple disks. The maximal number of crossings n in S_G can be estimated as follows.

Case 1. θ-curve

1. The number of crossings between simple disks is at most $\binom{\alpha(G)-3}{2}$.
2. The number of crossings between simple and non-simple disks is at most $2 \cdot (\alpha(G) - 3)$.
3. The number of crossings between the two non-simple disks is at most 2, except for the trivial theta curve.

However, for each cap, the two arcs that it connects have no crossing since RS_T has no nested caps. Moreover, any pair of simple arcs is connected by at most one cap. Note that between the two non-simple disks, there are at most two caps, except in the trivial theta curve case.
Therefore, we obtain the following upper bound on n in terms of $\alpha(G)$:

$$n \leq \binom{\alpha(G)-3}{2} + 2(\alpha(G)-3) + 2 - (\alpha(G)-2) = \frac{1}{2}(\alpha(G)^2 - 5\alpha(G) + 8).$$

Case 2. Handcuff graph

1. The number of crossings between simple disks is at most $\binom{\alpha(G)-3}{2}$.
2. The number of crossings between simple and non-simple disks is at most $2 \cdot (\alpha(G) - 3)$.
3. The number of crossings between the two non-simple disks is at most 1.

In contrast to the θ-curve, a simple arc in a handcuff graph can intersect a non-simple arc connected to two caps, due to the two loops. Moreover, as there is at most one cap between the non-simple disks since there is only one edge between the two vertices.
Thus, the number of crossings satisfies the same upper bound:

$$n \leq \binom{\alpha(G)-3}{2} + 2(\alpha(G)-3) + 1 - (\alpha(G)-3) = \frac{1}{2}(\alpha(G)^2 - 5\alpha(G) + 8).$$

By Theorem 24,

$$\max\deg_x R(S_G) - \min\deg_x R(S_G) \leq 2n + \alpha(G) \leq \alpha(G)^2 - 4\alpha(G) + 8.$$

Solving for $\alpha(G)$ yields

$$2 + \sqrt{|\max\deg_x R(S_G) - \min\deg_x R(S_G) - 4|} \leq \alpha(G),$$

which completes the proof. $\square$

4 Research Result

The Python code did not work since the Topoly package had error of the Yamada polynomial function. However, we were able to examine some of our answers by the Python code. We used the bounds from theoretical background, and used the Python code to find the arc index. If we could not find the arc index by the computer, we directly found the arc index by drawing the binding circle of the theta-curve. Also, we were able to found the bounds of the arc index of the handcuff graphs. The following result of the graphs with arc index is at below. The graphs with confirmed arc index is colored in green.

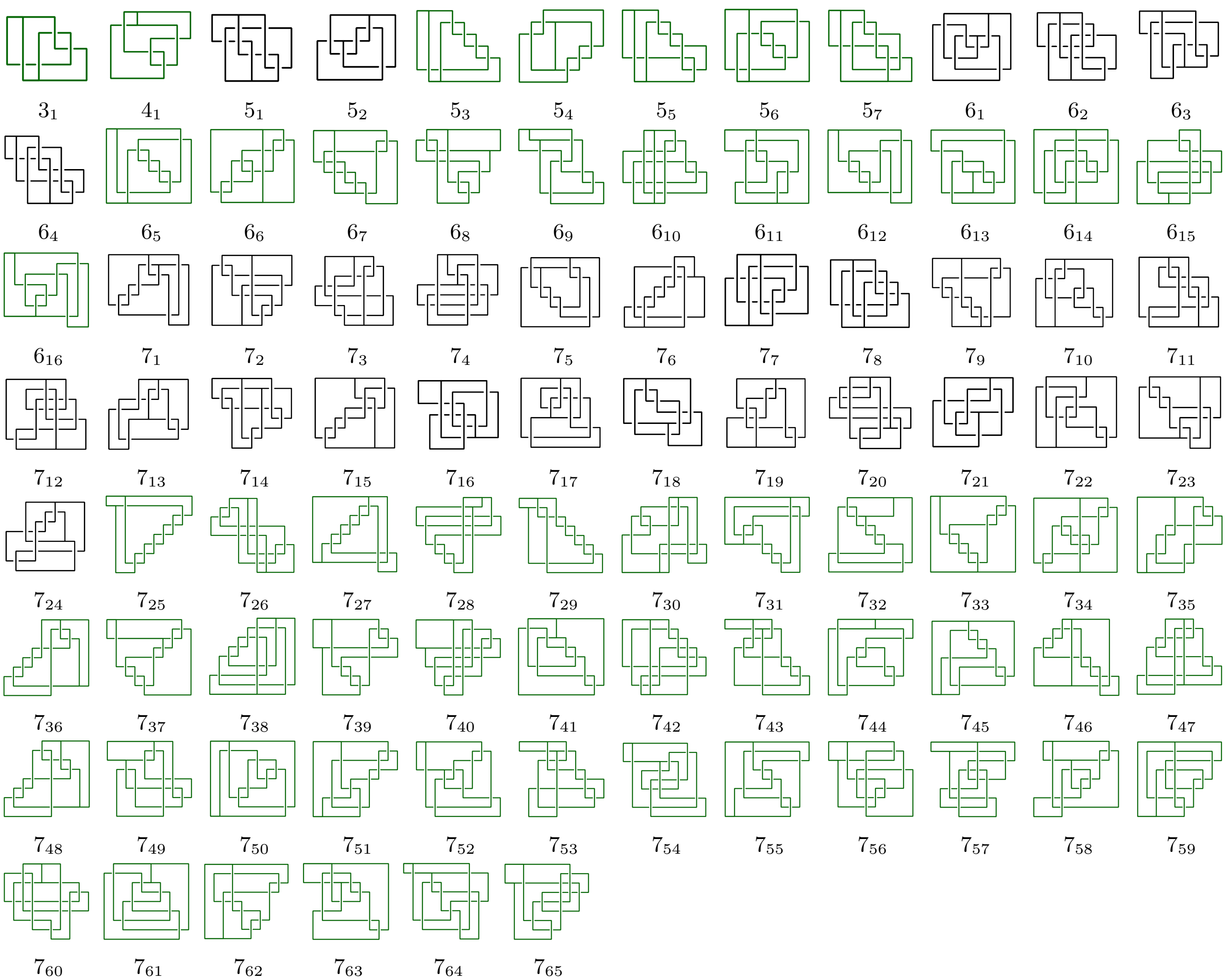


Figure 16: Grid Diagram of the Theta-Curves Up to 7 Crossings

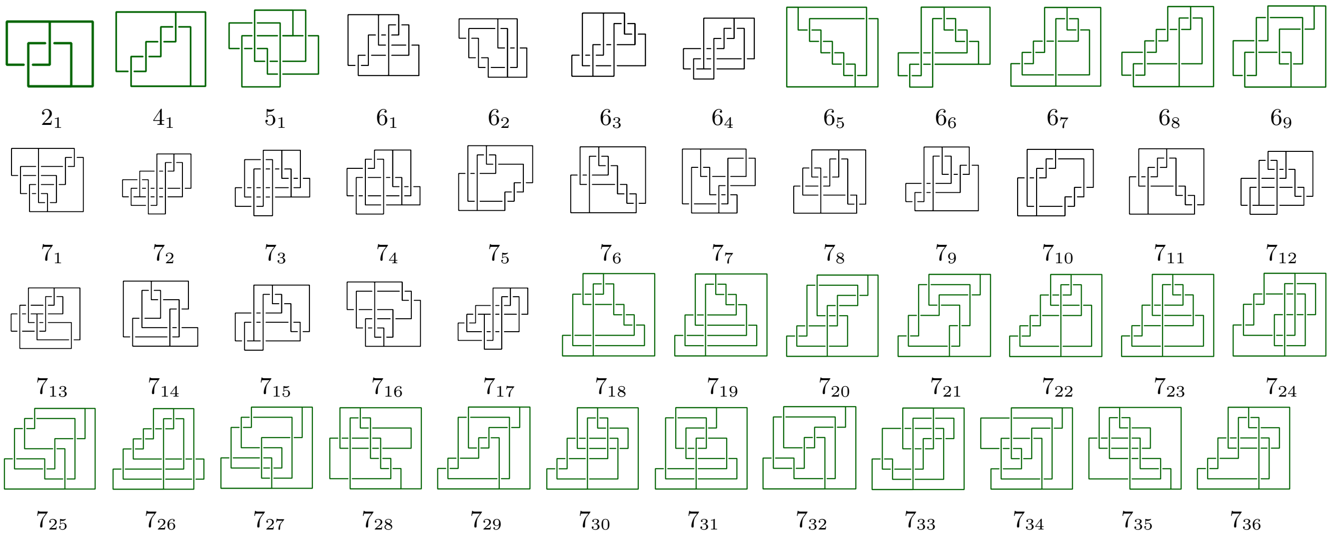


Figure 17: Grid Diagram of the Handcuff Graphs Up to 7 Crossings

5 Further Researches

We want to fix every arc index of handcuff graph and theta-curve graph up to 7 crossings. Also, we want to apply bounds of spatial graphs to knots. Since we found the minimal diagram of some theta-curves and handcuff graphs up to 7 crossings, it can be used as reference for other researches. Moreover, since we could not use the Topoly package, we would like to use another tool for examining the arc indicies.

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