KOREA SCIENCE ACADEMY OF KAIST

The Arc Index of Theta-Curve and Handcuff Graph

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SEP 6, 2025
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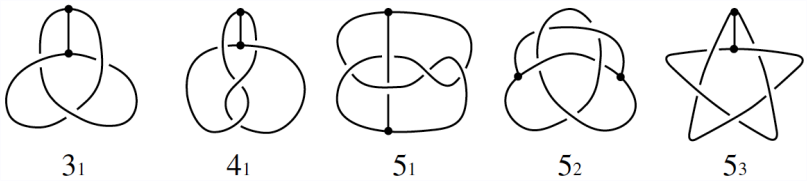
Introduction

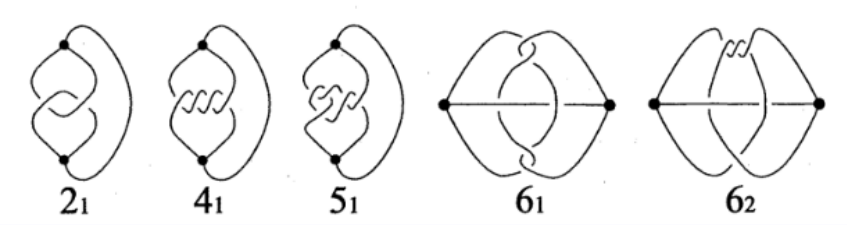
Theta-Curves

- A **theta-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.
- Theta-curves are roughly classified by comparing the triples of constituent knots.
- A theta-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3.

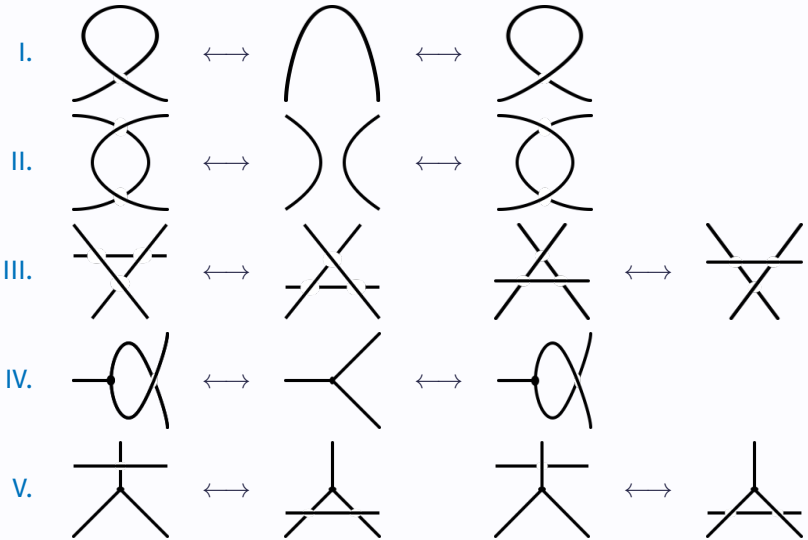


Handcuff Graphs

- **Handcuff graph** is the graph which consists of two loops and an edge jointing the vertices of each loop.



Reidemaister Moves for Theta-Curves and Handcuff Graphs

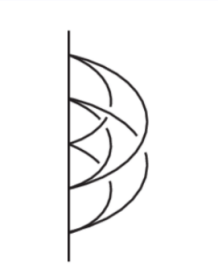


Arc Index

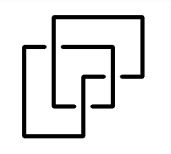
- **Arc presentation** is an open-book decomposition of $\mathbb{R}^3$ which has open half-planes as pages and the standard z-axis as the binding axis.
- **Arc index**, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is **minimal arc presentation**.

Arc Index

Trefoil



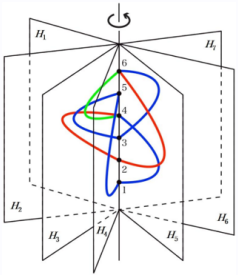
Open Book



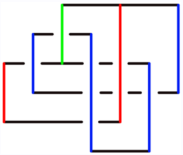
Grid Diagram

Arc Index

$\theta_{5,2}$



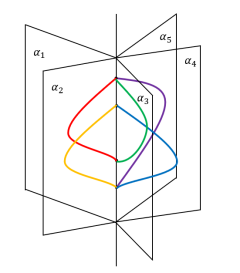
Open Book



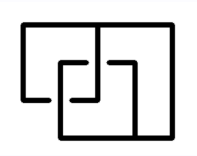
Grid Diagram

Arc Index

$\Phi_{2,1}$



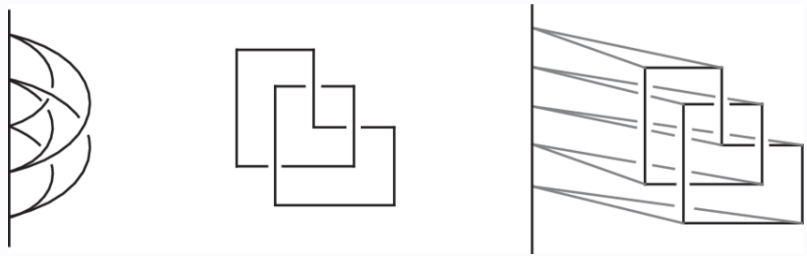
Open Book



Grid Diagram

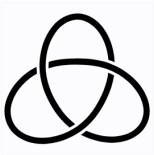
Grid Diagram

- The grid diagram of a theta-curve or handcuff graph is a diagram of vertical strands and one less number of horizontal strands.
- At every crossing the vertical strand crosses over the horizontal strand and no two horizontal segments are co-linear and no two vertical segments are co-linear.

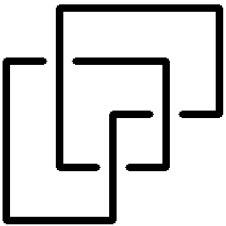


Cromwell Matrix

- The **Cromwell matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.



Trefoil



Grid Diagram

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Cromwell Matrix

Arc Presentation of the Theta-Curve and Handcuff Graph

Theorem

Arc presentations exist for every theta-curve and handcuff graph.

Proof