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The Determinant and an Arc Index of Theta Curve and Handcuff-Graph

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Lower Bounds of Arc Index

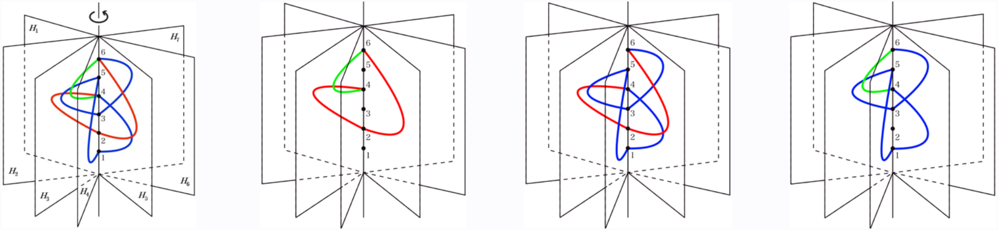
Lower Bounds from Constituent Knots

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Proof



$\square$

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

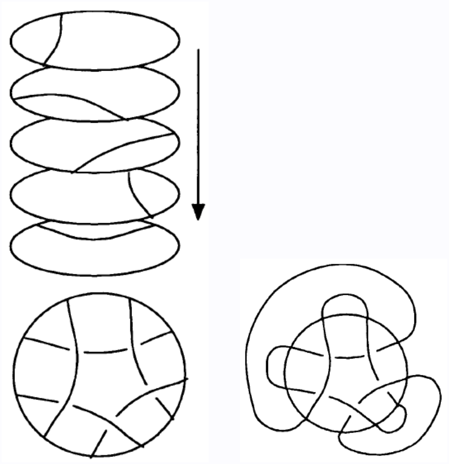
Proof

- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i.
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i.
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

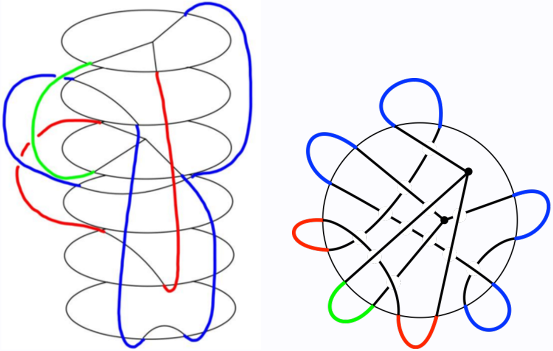
$$\sum_{i=1}^{3} \alpha(K_i) \leq 2 \sum_{i=1}^{3} |S_i| = 2\alpha(T)$$

$\square$

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link

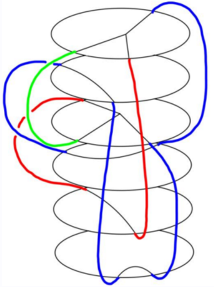


Stacked Tangle of a θ-Curve

Figure from [?]

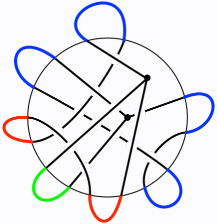
Stacked tangle of an θ-curve is stacked disks each with the frame as boundary with following properties:

- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point(by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Simple closure of stacked tangle is a stacked tangle with caps satisfying following properties:

- A cap is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.



Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

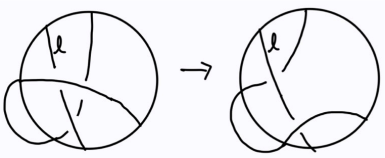
A reduced simple closure of a stacked tangle is

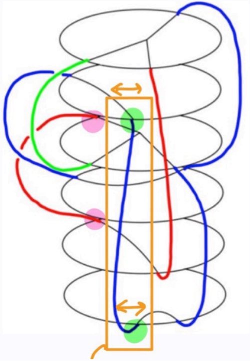
- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

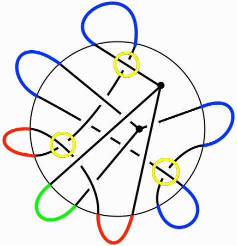
Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

Proof

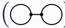
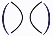
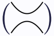
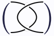
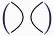
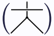
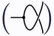
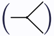
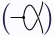
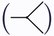






□

Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** R () $= -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** R () $= 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** R () $- R$ () $= (x - x^{-1})$ [R () $- R$ ()]
- **Y10:** R () $= x^2 R$ (), R () $= x^{-2} R$ ()
- **Y11:** R () $= R$ () **Y12:** R () $= R$ ()
- **Y13:** R () $= R$ (), R () $= R$ ()
- **Y14:** R () $= -xR$ (), R () $= -x^{-1}R$ ()

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of the θ-curve T.

Proposition

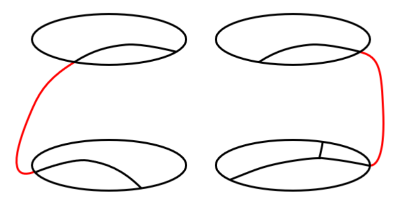
*Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T **without nested caps**. Then*

$$\max\deg_x R(S_T) \le c+n, \quad \min\deg_x R(S_T) \ge -(c+n),$$

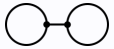
*where **c, n** is the number of caps and crossings in S_T, respectively.*

Proof

- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Basis Step:

- If $c_s + c_{ss} = 0$, then S_T has no simple disks and is equivalent to the result of applying Y14 to .
  $\therefore R(S_T) = -x^{\pm 3}\left[-x^2 - x - 2 - x^{-1} - x^{-2}\right] \implies 5 \leq c + n.$
- If $n = 0$, then S_T is equivalent to  $\cup \bigcirc \cup \cdots \cup \bigcirc$.
  $\therefore R(S_T) = 0 \implies 0 < 2 \leq c + n.$

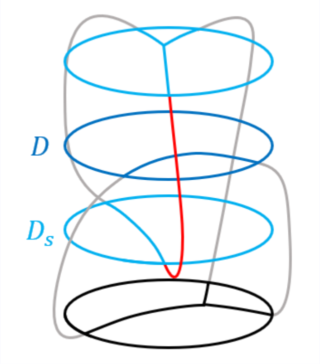
All base cases satisfy the inequality.

Inductive Step:

Assume that it holds for any $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$, and $c_s + c_{ss} > 0$.

Let S_T be a **simple closure of stacked tangle** of a θ-curve or handcuff graph T such that the number of simple caps, semi-simple caps, and crossings are c_s, c_{ss}, n, respectively.

Take the topmost **simple disk D_s** connected to the top disk, and a **disk D** directly above D_s.



CASE 1. Suppose that there is no cap between D_s and D.

① Suppose that there is no intersection between D_s and D in S_T.

- D_s and D do not affect each other.
- We can swap the position of D_s and D without affecting the rest of the diagram.

② Suppose that there is an intersection between D_s and D in S_T.

- Let S_T^-, S_T^0 and S_T^∞ be the simple closure of stacked tangle which is obtained by replacing $\times$ with $\times$, $)\,($ and $\asymp$, respectively.
- The simple caps, semi-simple caps, and crossings of the both are $\mathbf{c_s, c_{ss}, n-1}$.
- Applying Y9

$$R(\times) - R(\times) = (x - x^{-1})\left[R()\,() - R(\asymp)\right],$$

then

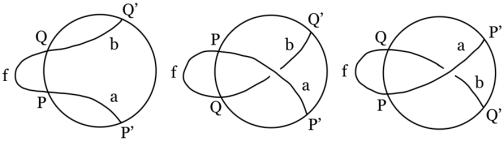
$$R(S_T) - R(S_T^-) = (x - x^{-1})(R(S_T^0) - R(S_T^\infty)).$$

- Then, it is sufficient to show that the interchanged one holds.

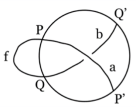
CASE 2. Suppose that there is a cap between D_s and D.

① Suppose that D is a simple disk.

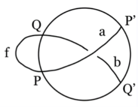
- When view from above, there are three cases:



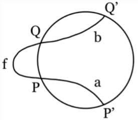
- After applying Y10, the second and third cases can be regarded as the first case, and the cap can be reduced.



or



$\longrightarrow$



S_T $\qquad$ S'_T

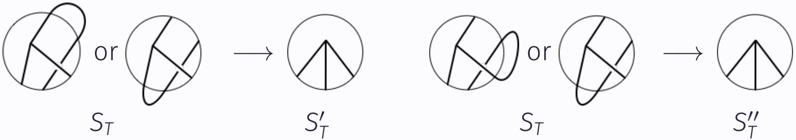
- S'_T has $c-1$ caps, c_s-1 simple caps, c_{ss} semi-simple caps and $n-1$ crossings.

- By induction hypothesis,

$$\begin{aligned}
\max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 2 \\
&\leq [(c-1)+(n-1)] \pm 2 \\
&\leq c+n, \\
\min \deg_x R(S_T) &= \min \deg_x R(S'_T) \pm 2 \\
&\geq -[(c-1)+(n-1)] \pm 2 \\
&\geq -(c+n).
\end{aligned}$$

② D is not a simple disk.

- When viewed from above, all the cases can be reduced as follows.



- $R(S_T) = -x^{\pm 1}R(S'_T)$ and $R(S_T) = x^{\pm 2}R(S''_T)$ by Y14 and Y10, respectively.
- Both of S'_T and S''_T have $c-1$ caps, c_s simple caps, $c_{ss}-1$ semi-simple caps, and $n-1$ crossing.

- By induction hypothesis, in the first case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S_T') \pm 1\\ &\le [(c-1)+(n-1)] \pm 1\\ &\le c+n.\end{aligned}$$

- Similarly, in the second case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S_T'') \pm 2\\ &\le [(c-1)+(n-1)] \pm 2\\ &\le c+n.\end{aligned}$$

- It holds for $\min\deg_x R(S_T)$ in the same way.

$\square$

Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve T corresponding to minimal arc presentation of T. Then

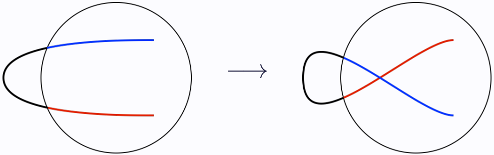
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the p negative($c - p$ positive) curls.

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

- $R\left(S_T^{neg}\right) = x^{-2p}R(S_T)$ and $R\left(S_T^{pos}\right) = x^{2(c-p)}R(S_T)$

$$
\begin{aligned}
\min\deg_x R(S_T) - 2p &= \min\deg_x R\left(S_T^{neg}\right) \\
&\geq -c + -(n+p) \\
\max\deg_x R(S_T) + 2(c-p) &= \max\deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n + (c-p)] \\
\min\deg_x R(S_T) &\geq -c - n + p \\
\max\deg_x R(S_T) &\leq n + p \\
\max\deg_x R(S_T) - \min\deg_x R(S_T) &\leq c + 2n
\end{aligned}
$$

□

Proof of Theorem

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proof

Let S_T be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

① Let T be any θ-curve.

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings by two non-simple disks : 2
- number of crossings counted by disks joined by cap : $\alpha(T)-2$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)+2\left(\alpha(T)-3\right)+2-\left(\alpha(T)-2\right) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

$$\max\deg_x R(S_T)-\min\deg_x R(S_T) \leq 2n+\alpha(T) \leq \alpha(T)^2-4\alpha(T)+8$$

$$2+\sqrt{\max\deg_x R(S_T)-\min\deg_x R(S_T)-4} \leq \alpha(T)$$

② Let T be any handcuff graph.

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings by two non-simple disks : 1
- number of crossings counted by disks joined by cap : $\alpha(T)-1-2 = \alpha(T)-3$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right) + 2\left(\alpha(T)-3\right) + 1 - \left(\alpha(T)-3\right) \\ &= \frac{1}{2}\left[(\alpha(T))^2 - 5\alpha(T) + 8\right] \end{aligned}$$

By Lemma,

$$\max\deg_x R(S_T) - \min\deg_x R(S_T) \leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8$$

$$2 + \sqrt{\max\deg_x R(S_T) - \min\deg_x R(S_T) - 4} \leq \alpha(T)$$

$\square$