



The Arc Index of Theta-Curve and Handcuff Graph

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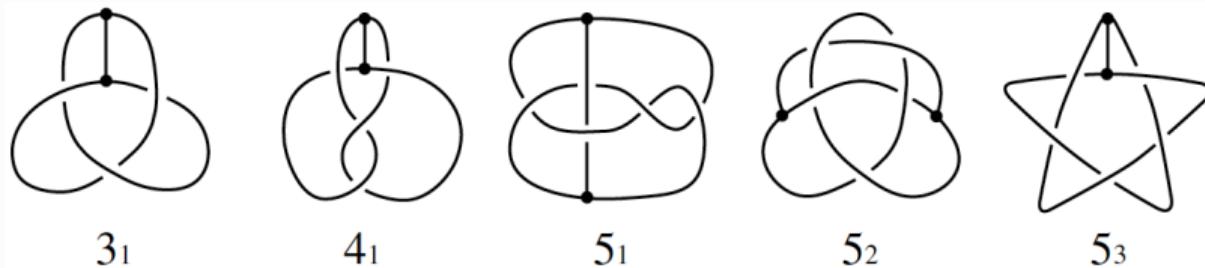
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Introduction

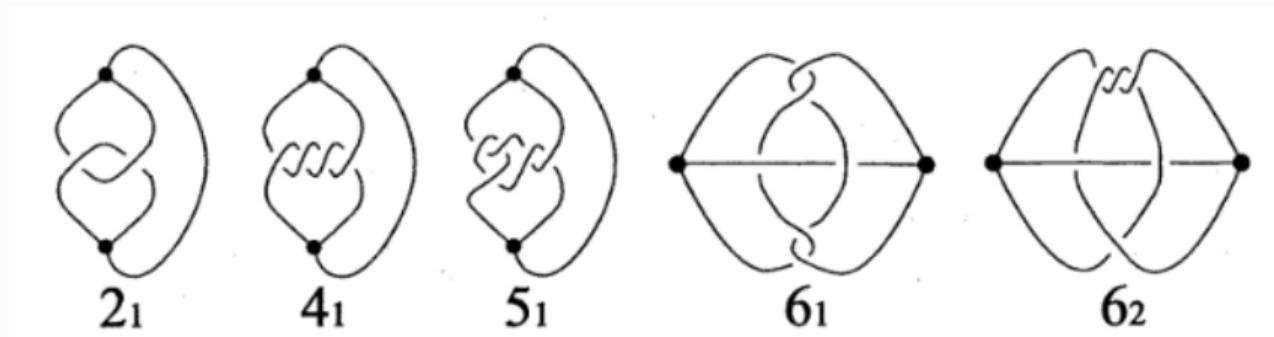
Theta-Curves

- A **theta-curve** T is a graph embedded in S^3 , which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3 , such that each edge joins the vertices.
- A **constituent knot** T_{ij} , $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j .
- Theta-curves are roughly classified by comparing the triples of constituent knots.
- A theta-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3 .

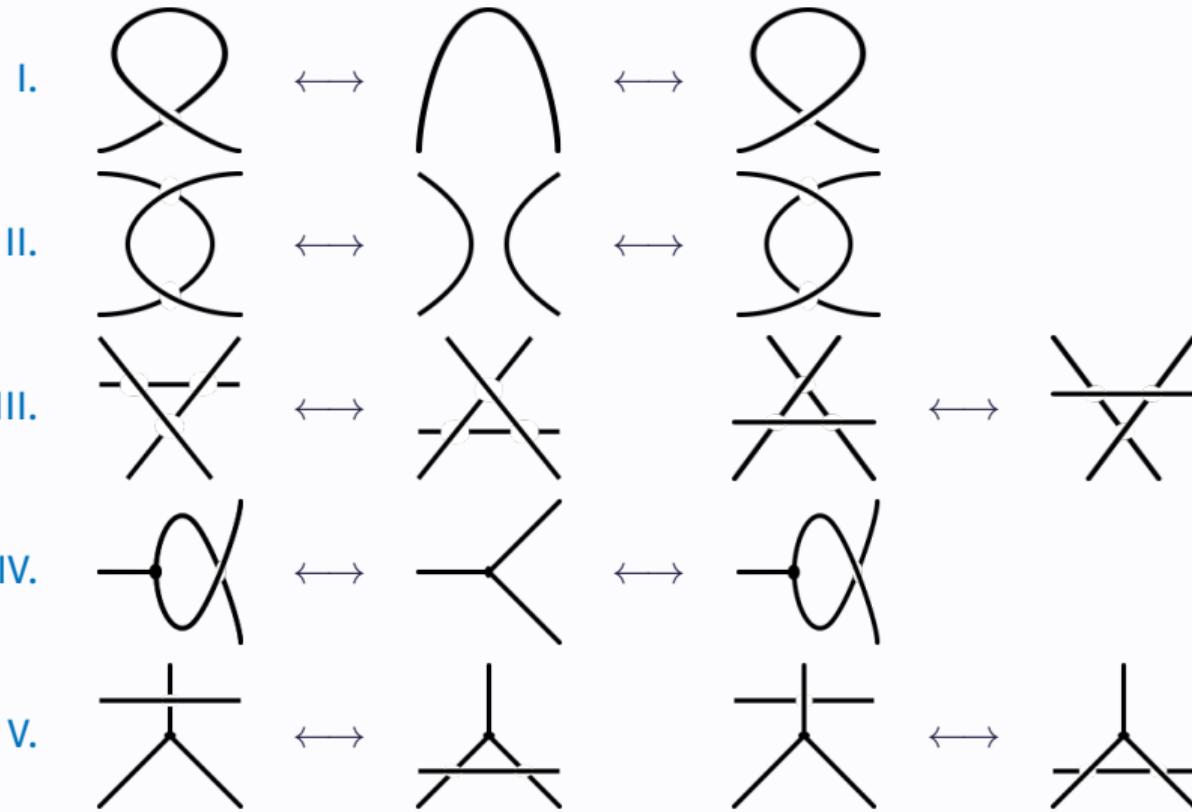


Handcuff Graphs

- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3) , where e_3 has distinct endpoints v_1 and v_2 , and e_1 and e_2 are loops based at v_1 and v_2 .
- A **constituent link H_{12}** , is a subgraph of H that consists of two vertices v_1, v_2 and two edges e_1, e_2 .



Reidemeister Moves for Theta-Curves and Handcuff Graphs



Arc Presentation

- **Arc presentation** of a theta-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called **pages**).
- The embedding is with the common boundary line (called **axis**).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- **Arc index**, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is **minimal arc presentation**.

Arc Presentation



Trefoil



θ_{5,2}



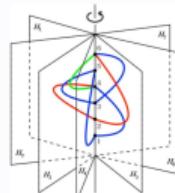
Φ_{2,1}



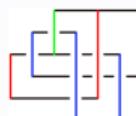
Open Book



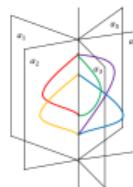
Grid Diagram



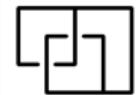
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Grid Diagram



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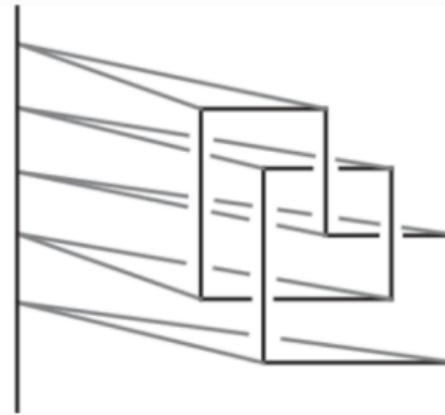
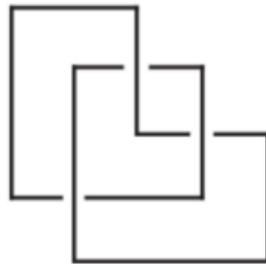
Grid Diagram

Grid Diagram

- The **grid diagram** of theta-curve or handcuff graph is a diagram with only vertical and horizontal strands.
- $(\text{number of vertical strands}) + 1 = (\text{number of horizontal strands})$
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Grid Diagram

- A grid diagram gives rise to an arc presentation and vice versa.

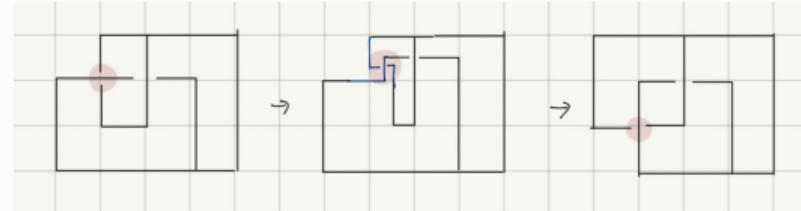
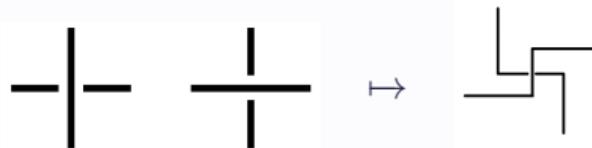


Arc Presentation of the Theta-Curve and Handcuff Graph

Theorem

Every theta-curve and handcuff graph admit a grid diagram.

PROOF



Corollary

Every theta-curve and handcuff graph admit a arc presentation.

