The Determinant and an Arc Index of Theta Curve and Handcuff-Graph

Eunchan Cho[1] Jeongwon Shin[1] Boyeon Seo[1] Minho Choi[1]

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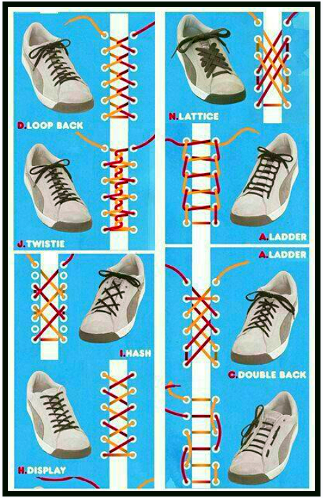
학회이름

[1]Korea Science Academy of KAIST

What is a Knot Theory?



What is a Knot Theory?

- **Knot theory** is a field of mathematics that studies simple closed curves embedded in three-dimensional space.

Introduction

θ-Curves

- A **θ-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.
- θ-curves are roughly classified by comparing the triples of constituent knots.
- A θ-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3.

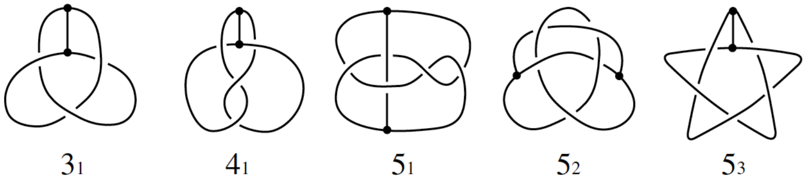


Figure from [?]

Handcuff-Graphs

- Handcuff graph consists of 2 loops and 1 edge joining the loops.

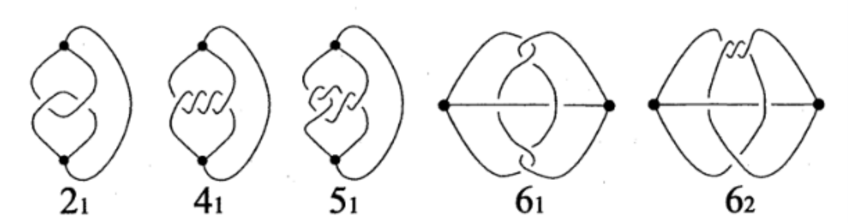
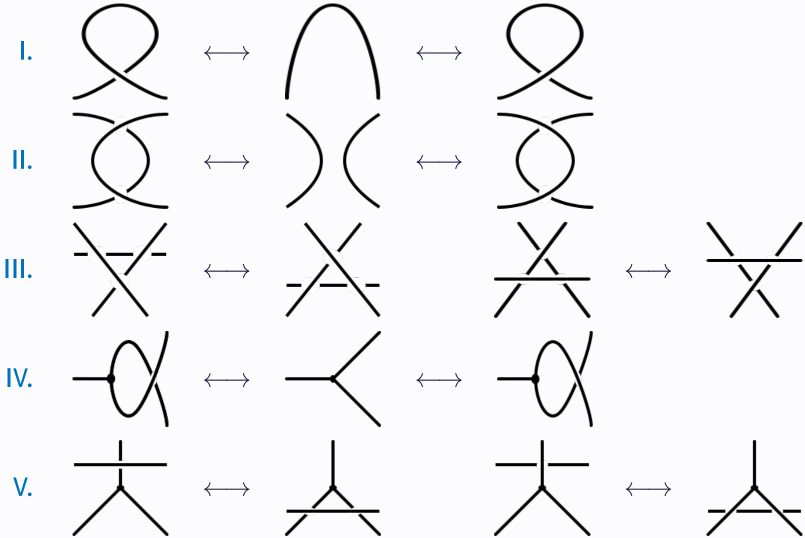


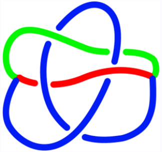
Figure from [?]

Reidemaister Moves for θ-Curves and Handcuff Graphs



Arc Index

- An **arc presentation** of a θ-curve is defined in the same manner as an arc presentation of a knot.
- The binding axis contains all **vertices** of θ-curve.
- **Minimal arc presentation** and **arc index** are defined in the same manner.



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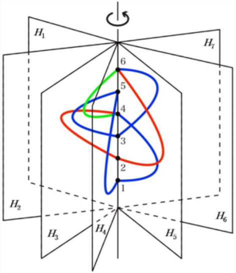
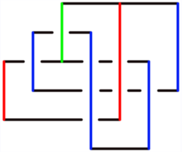
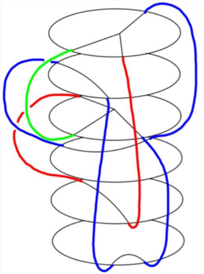


Open Book



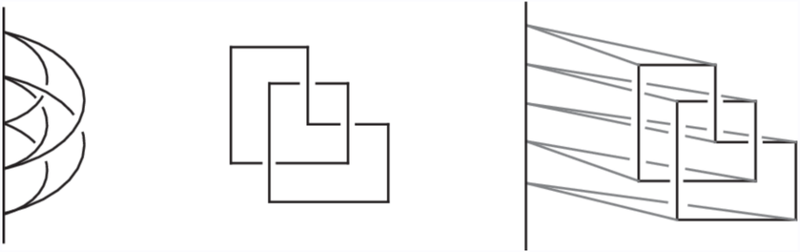
Grid Diagram



Stacked Tangle

Grid Diagram

- The **grid diagram** is a handcuff graph or theta-curve diagram of vertical strands and one less number of horizontal strands.
- At every crossing the vertical strand crosses over the horizontal strand and no two horizontal segments are co-linear and no two vertical segments are co-linear.



Cromwell Matrix

- The **Cromwell matrix** is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s. For theta-curve and handcuff graph, its Cromwell matrix is called the the *THC-Cromwell matrix* is the matrix that satisfies the following conditions.
  1. It is a binary $n \times (n+1)$ matrix.
  2. It has exactly two 1s in every column.
  3. There are only three 1s in two distinct rows (which are called the *Three-row*) and every other rows has exactly two 1s.
- If the 1s of the Cromwell matrix are connected by horizontal and vertical lines with vertical lines are always on the horizontal lines, it leads to the grid diagram. The arc presentation can be expressed by grid diagram and vice versa.
- If the number of half planes in arc presentation is α, then the size of corresponding grid diagram is $(\alpha - 1) \times \alpha$.

Cromwell Matrix

$$
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
$$

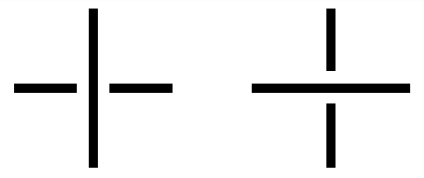
Arc Presentation of the Theta-Curve and Handcuff Graph

Theorem

Arc presentations exist for every theta-curve and handcuff graph.

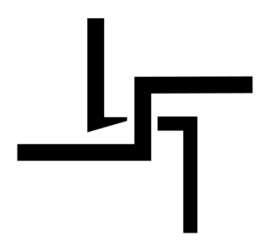
Proof

For any theta-curve and handcuff graph, we can put it on a grid by using some suitable planar isotropy. Then for every crossing it has, there are only two following cases.



Arc Presentation of the Theta-Curve and Handcuff Graph

For each crossings, if it is a first case, then it is over. However, if it is a second case, then we can make that crossing to first case using some suitable movement.

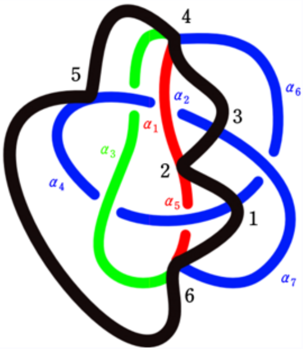


Therefore, it becomes the grid diagram and since the grid diagrams and arc presentations are in one-to-one correspondence, there always exists arc presentations for every theta-curve and handcuff graph. □

Binding Circle Method

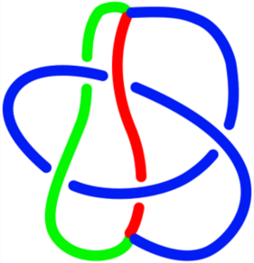
Let D_T be a diagram of an θ-curve T. A **binding circle** of D_T is a simple closed curve C meeting D_T in n distinct points with the following properties:

- C must meet D_T at **two vertices**.
- C divide D_T into n arcs $\alpha_1, \alpha_2, \ldots, \alpha_n$.
- Each α_i has no self-crossings.
- If α_i crosses over α_j at a crossing in inside(resp. outside) C, then $i < j$(resp. $i > j$) and it crosses over α_i at any other crossings with α_j, respectively.
- For each i, there exists an embedded disk d_i such that $\partial d_i = C$ and $\alpha_i \subset d_i$.
- $d_i \cap d_j = C$, for distinct i and j.

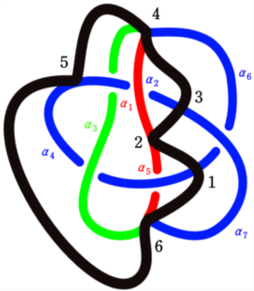


Then the pair (D_T, C) is also corresponding to an arc presentation.

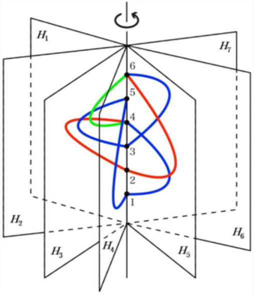
Binding Circle Method



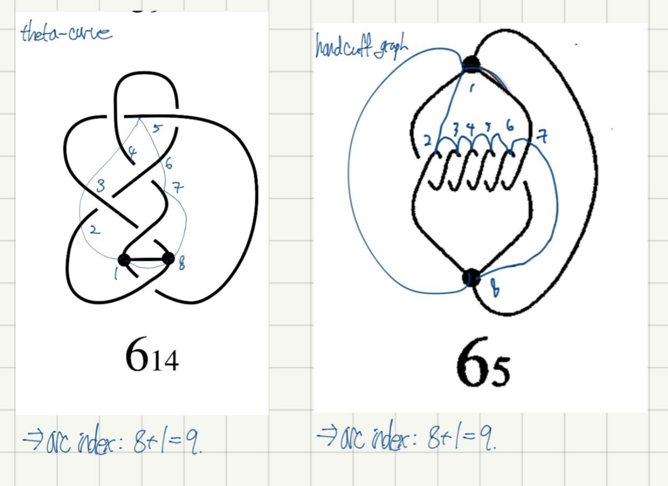
$\longrightarrow$



$\longleftrightarrow$



Arc Index of Spatial Graphs



Classifying by Determinant

THC-cromwell matrix

- The **cromwell matrix** of a knot is an $n \times n$ binary matrix such that each row and column has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ-curves and handcuff graphs that satisfies the following conditions :
  1. It is a $(n+1) \times n$ bianry matrix.
  2. It contains exactly two 1s in every column.
  3. There exists two distinct rows which contains exactly three 1s, which is called the **three-row**. Every other row contains exactly two 1s.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Determinant of the cromwell matrices of Knot

Theorem

Let K be any knot then its determinant of the cromwell matrix is 0 or ± 2.

Proof

$$\longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

CASE 1. When n is an even number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the determinant of K is 0.

CASE 2. When n is an odd number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

So the determinant of K is ± 2. $\square$

H-deletion of THC-cromwell matricies

Definition

The H-deletion Matrix of the THC-cromwell matrix G is $(n-1) \times (n-1)$ matrix which deleted vertex-row and its two side-rows from the matrix G.

Determinant of the THC-cromwell matrices

Theorem

Let K be any THC-cromwell matrice of θ-curve or handcuff graph.
Then, its determinant is ± 1 iff the THC-cromwell matrix represents θ-curve, and 0 or ± 2 iff the THC-cromwell matrix represents handcuff graph.

Proof

$$\longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Lower Bounds of Arc Index

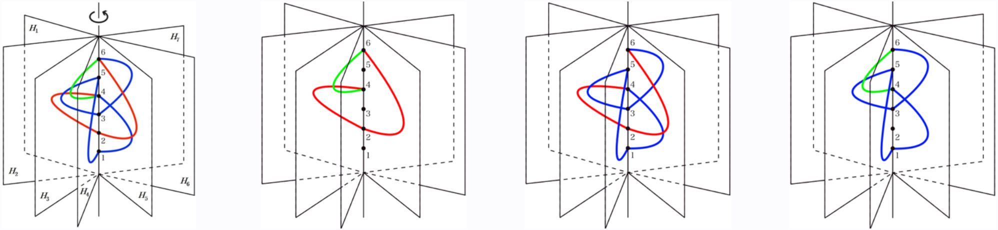
Lower Bounds from Constituent Knots

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Proof



$\square$

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

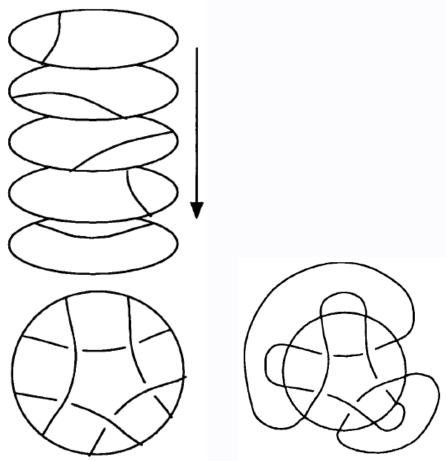
Proof

- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i.
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i.
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

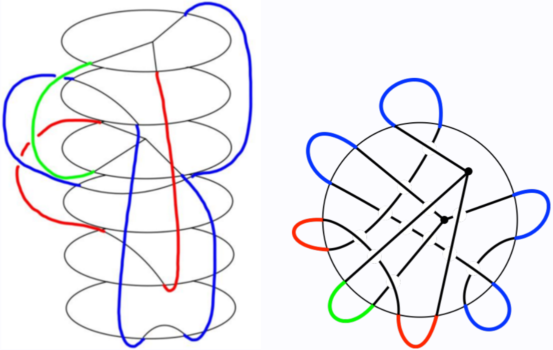
$$\sum_{i=1}^{3} \alpha(K_i) \leq 2 \sum_{i=1}^{3} |S_i| = 2\alpha(T)$$

$\square$

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link

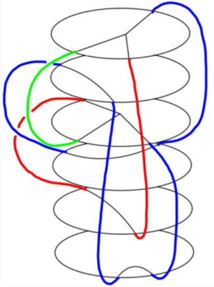


Stacked Tangle of a θ-Curve

Figure from [?]

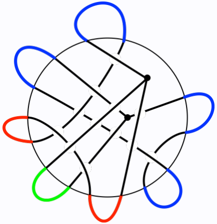
Stacked tangle of an θ-curve is stacked disks each with the frame as boundary with following properties:

- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point(by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Simple closure of stacked tangle is a stacked tangle with caps satisfying following properties:

- A cap is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.



Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

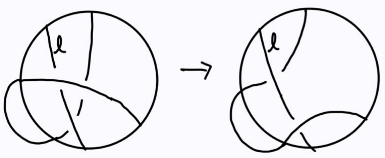
A reduced simple closure of a stacked tangle is

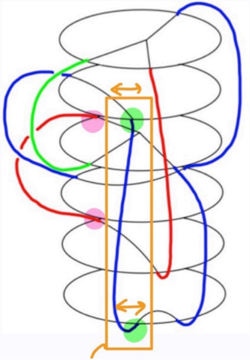
- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

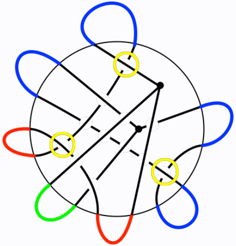
Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

Proof



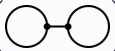
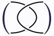
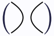
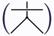
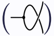
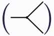
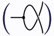
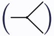




□

Yamada Polynomials

Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** R () $= -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** R () $= 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** R () $- R$ () $= (x - x^{-1})$ [R () $- R$ ()]
- **Y10:** R () $= x^2 R$ (), R () $= x^{-2} R$ ()
- **Y11:** R () $= R$ () **Y12:** R () $= R$ ()
- **Y13:** R () $= R$ (), R () $= R$ ()
- **Y14:** R () $= -xR$ (), R () $= -x^{-1}R$ ()

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of the θ-curve T.

Proposition

*Let S_T be a simple closure of stacked tangle of a θ-curve T **without any nested caps**. Then*

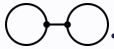
$$\max \deg_x R(S_T) \leq c + n - 2 \quad \textit{and} \quad \min \deg_x R(S_T) \geq -c - n + 2$$

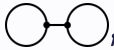
*where c is the **number of caps** and n is the **number of crossings** in S_T.*

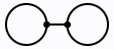
Proof

- A **simple cap** is a cap joining simple disks.
- Let s be the **number of simple caps** in S_T.
- Use double mathematical induction of (s, n).

Basis Step:

If $s = 0$, then S_T is either equivalent to  or .

- If $S_T \equiv$ , then $R(S_T) = -x^2 - x - 2 - x^{-1} - x^{-2}$ and $4 \leq s + n$.
- If $S_T \equiv$ , then $R(S_T) = 0$ and $3 \leq c + n$.

If $n = 0$, then S_T is equivalent to  $\cup$  $\cup \cdots \cup$ .

- $R(S_T) = 0$ and $2 \leq c + n$.

All of the cases satisfy the inequalities.

Inductive Step:

Assume that the inequalities hold for any (s', n') where $0 \leq s' < s$ or $0 \leq n' < n$.

Let S_T be a simple closure of stacked tangle of a θ-curve T such that the number of simple caps is s and the number of crossings is n.

Take a **simple cap f** in S_T, joining boundary points P and Q.

CASE 1. Suppose that P and Q are boundary points of a single disk.

- $S_T = S'_T \cup \bigcirc$
- $R(S_T) = (x + 1 + x^{-1})R(S'_T)$
- The number of caps is $c - 1$ and the number of crossings n' is less than or equal to n in S'_T.

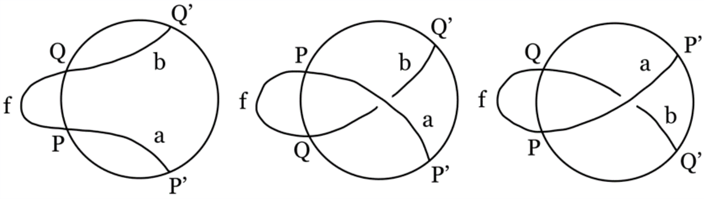
$$\max \deg_x R(S_T) = \max \deg_x R(S'_T) + 1 \leq [(c-1) + n' - 2] + 1 \leq c + n - 2$$
$$\min \deg_x R(S_T) = \min \deg_x R(S'_T) - 1 \geq [-(c-1) - n' + 2] - 1 \geq -c - n + 2$$

- S'_T satisfy the inequalities implies S_T satisfy the inequalities.

CASE 2. Suppose that P and Q are boundary points of different disks D_P and D_Q, respectively.

① Suppose that D_P and D_Q are adjacent disks.

- When view from above, there are three cases:



- At first case, we can reduce the simple cap f.
- After applying Y10, other cases can be regarded as first case.

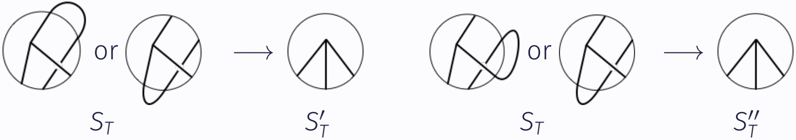
R () $= x^2 R$ () , R () $= x^{-2} R$ ()   (Y10)

② Suppose that D_P and D_Q are not adjacent disks and D_P is above D_Q.

- Let D be the disk just above D_Q.
- If arcs or line segment in D and D_Q have no intersection, then we can change the position of D and D_Q without any quantities.
- We can assume that the arc in D_Q intersect arc or line segment in D, when view from above.

ⓘ There is a cap joining D_Q and D.

- D_Q and D are adjacent disks.
- If D is a simple disk, then we can reduce a simple cap as Case 2–①.
- If D is a non-simple disk, then

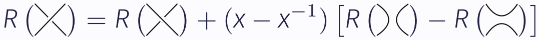


- $R(S_T) = -x^{\pm 1} R(S_T')$ and $R(S_T) = x^{\pm 2} R(S_T'')$ by Y14 and Y10, respectively.
- Both of S_T' and S_T'' have $s-1$ simple caps, $c-1$ caps, and $n-1$ crossing.
- By induction hypothesis,

$$\begin{aligned}
\max\deg_x R(S_T) &= \max\deg_x R(S_T') \pm 1 \\
&\leq [(c-1)+(n-1)-2] \pm 1 \\
&< c+n-2 \\
\min\deg_x R(S_T) &= \min\deg_x R(S_T') \pm 2 \\
&\geq [-(c-1)-(n-1)+2] \pm 2 \\
&\geq -c-n+2
\end{aligned}$$

ⓘⓘ There is no cap joining D_Q and D.

- Applying Y9



then

$$R(S_T) = R(S_T^-) + (x - x^{-1})\left[R(S_T^0) - R(S_T^\infty)\right]$$

- S_T^0 and S_T^∞ have c caps and $n-1$ crossings.
- $(x - x^{-1})\left[R(S_T^0) - R(S_T^\infty)\right]$ satisfy the inequalities.
- If S_T^- satisfy the inequalities, then S_T also satisfy the inequalities.
- The gap between D_P and D_Q is reduced in S_T^-.
- For S_T^-, investigate above cases.

This process will terminate after a finite number of investigations. It is the end of CASE 2.

$\square$

Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve T corresponding to minimal arc presentation of T. Then

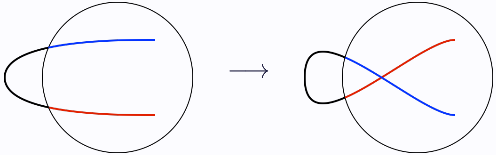
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n + 4 \leq \alpha(T)$$

where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the p negative($c - p$ positive) curls.

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

- $R\left(S_T^{neg}\right) = x^{-2p}R(S_T)$ and $R\left(S_T^{pos}\right) = x^{2(c-p)}R(S_T)$

$$\begin{aligned}
\min\deg_x R(S_T) - 2p &= \min\deg_x R\left(S_T^{neg}\right) \\
&\geq -c + -(n+p) + 2 \\
\max\deg_x R(S_T) + 2(c-p) &= \max\deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n+(c-p)] - 2 \\
\min\deg_x R(S_T) &\geq -c - n + p + 2 \\
\max\deg_x R(S_T) &\leq n + p - 2 \\
\max\deg_x R(S_T) - \min\deg_x R(S_T) &\leq c + 2n - 4
\end{aligned}$$

$\square$

Proof of Theorem

Theorem

Let T be any θ-curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of the θ-curve T.

Proof

Let S_T be a reduce simple closure of stacked tangle of a θ-curve T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)$
- number of crossings by a simple disk and non-simple disk : $2(\alpha(T)-3)$
- number of crossings counted by disks joined by cap : $\alpha(T)$
- number of crossings by two non-simple disks : 2

Thus

$$\begin{aligned} n &\leq \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)+2(\alpha(T)-3)-\alpha(T)+2 \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+4\right] \end{aligned}$$

By Lemma,

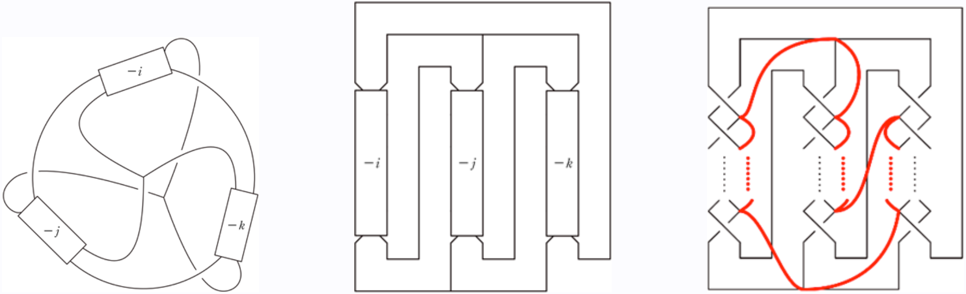
$$\max\deg_x R(S_T)-\min\deg_x R(S_T) \leq 2n-4+\alpha(T) \leq [\alpha(T)]^2-4\alpha(T)$$

$$2+\sqrt{\max\deg_x R(S_T)-\min\deg_x R(S_T)+4} \leq \alpha(T)$$

$\square$

Further Studies

Kinoshita-Wolcott θ-Curve

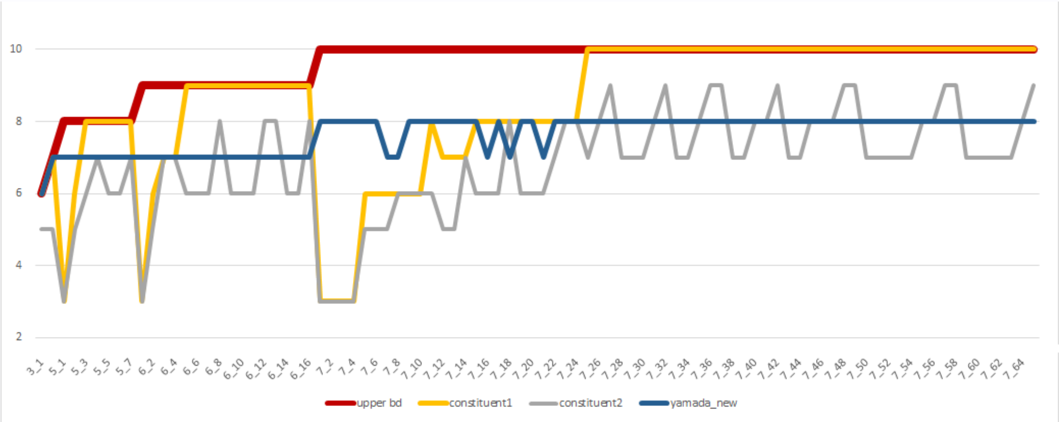


Theorem

Let $K(-i,-j,-k)$ be the Kinoshita-Wolcott θ-curve. Then

$$\alpha(K(-i,-j,-k)) \le i+j+k+2$$

Bounds of Arc Index



Thank You for Your Attention.