The Determinant and Arc Indices of θ-Curves and Handcuff Graphs

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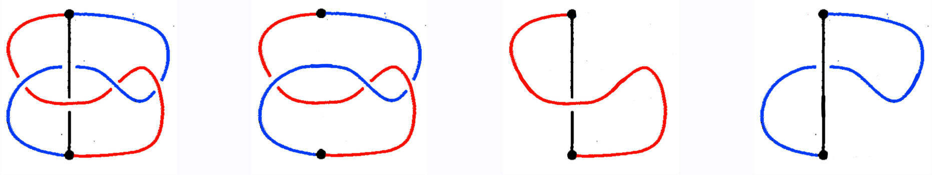
Lower Bounds of Arc Index

- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of θ-curve T that consists of two vertices v_1, v_2 and two edges e_i, e_j.



- A **constituent link** H_{12}, is a subgraph of handcuff graph H that consists of two vertices v_1, v_2 and two edges e_1, e_2.

Lower Bound from Constituent Knots/Links

Theorem (Lee, 2023)

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Theorem (Lee, 2023)

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

Lower Bound from Constituent Knots/Links

Theorem

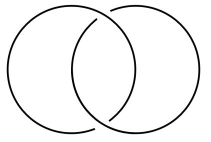
Let H be a handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then

$$\alpha(H) \geq c(L) + 3.$$

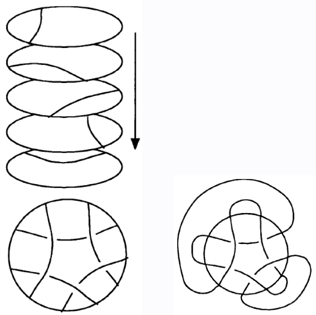
Corollary

Let H be a handcuff graph, and L be the constituent link of H. If L is alternating and non-split, then

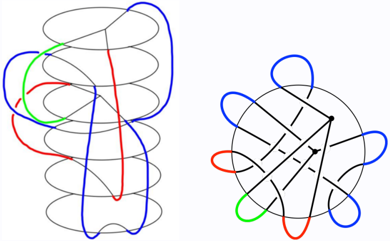
$$\alpha(H) = c(L) + 3.$$



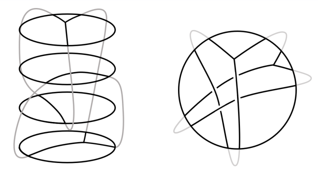
Stacked Tangle of an θ-Curve and Handcuff Graph



Stacked Tangle of a Link



θ-Curve



Handcuff Graph

Yamada Polynomials

Let D_G be a diagram of an θ-curve or handcuff graph G. Then, the **Yamada Polynomial** $R(D_G) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- Y6: $R\left(\ \right) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ Y7: $R\left(\ \right) = 0$
- Y8: $R(G \cup \bigcirc) = (x+1+x^{-1})R(G)$ for an arbitrary θ-curve or handcuff graph diagram G
- Y9: $R\left(\ \right) - R\left(\ \right) = (x - x^{-1})\left[R\left(\ \right) - R\left(\ \right)\right]$
- Y10: $R\left(\ \right) = x^2 R\left(\ \right)$, $R\left(\ \right) = x^{-2} R\left(\ \right)$
- Y11: $R\left(\ \right) = R\left(\ \right)$ Y12: $R\left(\ \right) = R\left(\ \right)$
- Y13: $R\left(\ \right) = R\left(\ \right)$, $R\left(\ \right) = R\left(\ \right)$
- Y14: $R\left(\ \right) = -xR\left(\ \right)$, $R\left(\ \right) = -x^{-1}R\left(\ \right)$

Proposition (Yamada, 1989)

$R(D_G)$ is an ambient isotopy invariant of G up to multiplying $(-x)^n$ for some integer n.

Lower Bound from Yamada Polynomial

Theorem

Let G be a θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4} \leq \alpha(G).$$

Lower Bound from Yamada Polynomial

Proposition

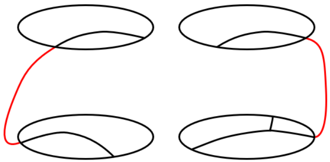
*Let S_G be a simple closure of stacked tangle of a θ-curve or handcuff graph G **without nested caps**. Then*

$$\max \deg_x R(S_G) \leq c + n, \quad \min \deg_x R(S_G) \geq -(c+n),$$

where c and n is the number of caps and crossings in S_G, respectively.

Proof

- Let c_s and c_{ss} be the number of **simlpe caps** and **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Proposition

Let S_G be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to minimal arc presentation of G. Then

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) - 2n \leq \alpha(G)$$

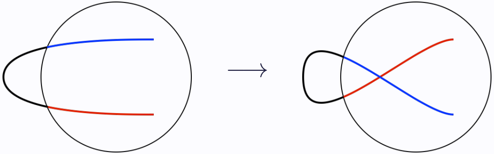
where n is the number of crossings in S_G.

Proof

- S_G is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_G is exactly arc index of G, $\alpha(G)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



	S_G^{neg}	S_G^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

$$\min \deg_x R(S_G) - 2p = \min \deg_x R\left(S_G^{neg}\right) \geq -c + -(n+p)$$
$$\max \deg_x R(S_G) + 2(c-p) = \max \deg_x R\left(S_G^{pos}\right) \leq c + [n+(c-p)]$$
$$\max \deg_x R(S_G) - \min \deg_x R(S_G) \leq c + 2n \qquad \square$$

Proof of Theorem

Theorem

Let G be a θ-curve or handcuff graph. Then

$$2+\sqrt{\max\deg_x R(S_G)-\min\deg_x R(S_G)-4}\leq\alpha(G).$$

PROOF

Let S_G be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to minimal arc presentation of G.

- The number of caps : $\alpha(G)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(G)-3$

The theorem is followed by considering the maximum of crossings in S_G and applying previous theorems.