

# The Determinant and Arc Indices of Theta Curves and Handcuff-Graphs

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NOV 22, 2025

R&E 2025

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## Lower Bounds of Arc Index

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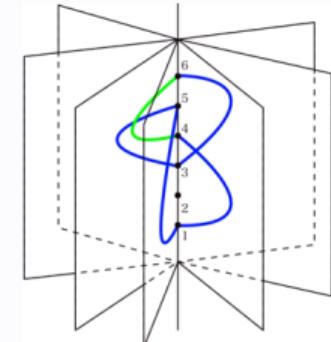
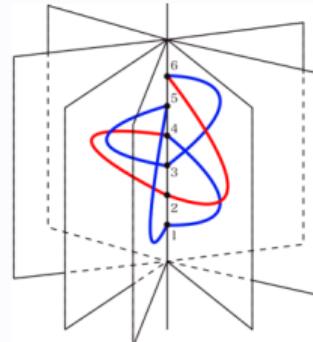
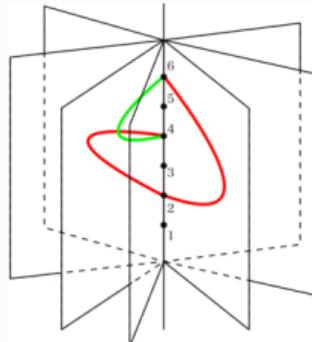
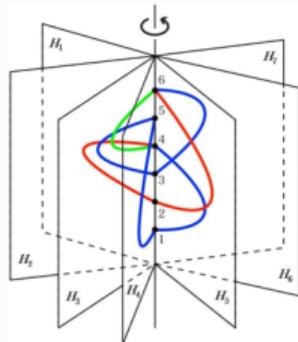
# Lower Bounds from Constituent Knots

## Theorem

Let  $T$  be any  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

## PROOF



□

## Theorem

Let  $T$  be any  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^3 \alpha(K_i)$$

## PROOF

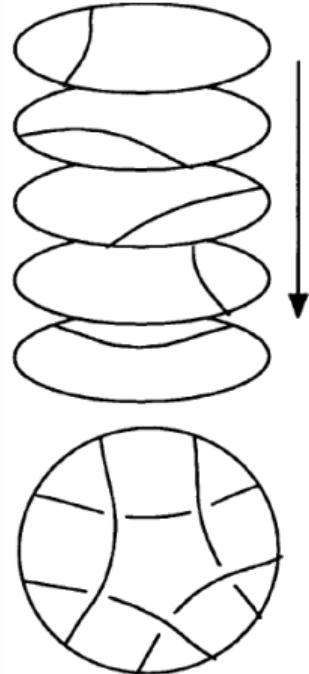
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- A minimal arc presentation of  $T$  is given.
- $K_1 = e_1 \cup e_2$ ,  $K_2 = e_2 \cup e_3$ , and  $K_3 = e_3 \cup e_1$ .
- $S_i$  be the set of half plane corresponding the edge  $e_i$ .
- $S_i \cup S_{i+1}$  form an arc presentation of the knot  $K_i$ .
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

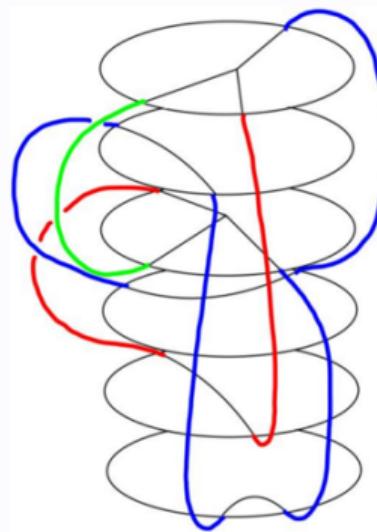
$$\sum_{i=1}^3 \alpha(K_i) \leq 2 \sum_{i=1}^3 |S_i| = 2\alpha(T)$$

□

## Stacked Tangle of an $\theta$ -Curve



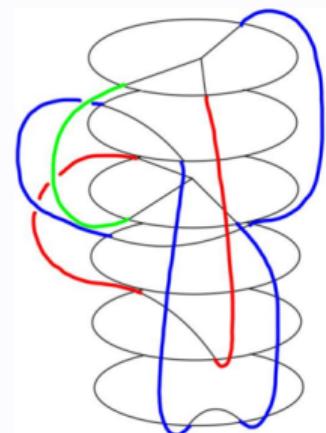
Stacked Tangle of a Link



Stacked Tangle of a  $\theta$ -Curve

**Stacked tangle** of an  $\theta$ -curve is stacked disks each with the frame as boundary with following properties:

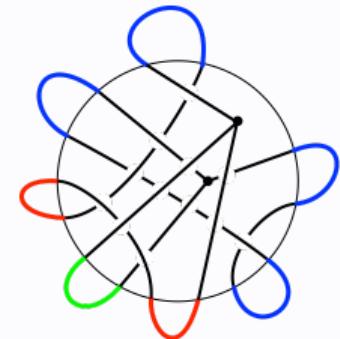
- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point(by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



**Simple closure** of stacked tangle is a **stacked tangle** with **caps** satisfying following properties:

- A **cap** is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any two caps have no intersection.

Then a simple closure of a stacked tangle **without any nested caps** is corresponding to an arc presentation.



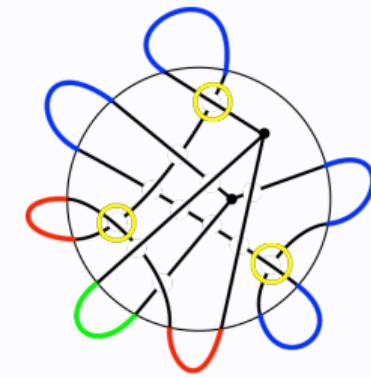
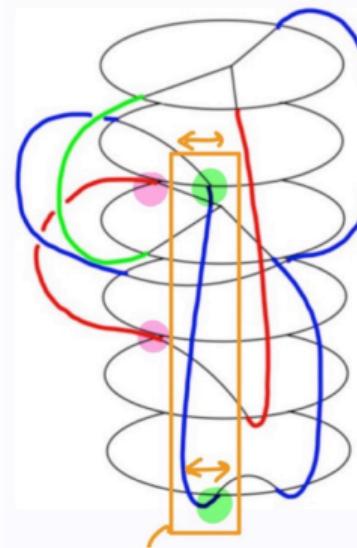
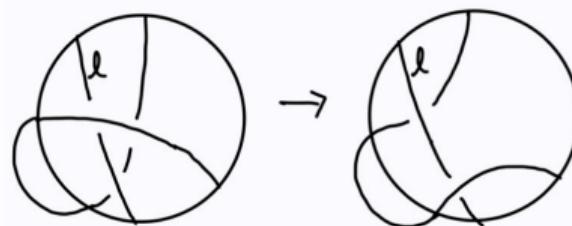
A **reduced simple closure of a stacked tangle** is

- a simple closure of a stacked tangle **without any nested caps**
- any two arcs(including line segment) joining by caps have **no intersection** when view from above

## Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemeister Moves.

## PROOF



□

# Yamada Polynomials

Let  $D_T$  be a diagram of an  $\theta$ -curve  $T$ . Then, the **Yamada Polynomial**  $R(D_T) \in \mathbb{Z} [x^{\pm 1}]$  is calculated by the following properties:

- **Y6:**  $R(\bigoplus) = -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$       **Y7:**  $R(\bigcirc\bigcirc) = 0$
- **Y8:**  $R(T' \cup \bigcirc) = (x + 1 + x^{-1})R(T')$  for an arbitrary  $\theta$ -curve diagram  $T'$
- **Y9:**  $R(\bigotimes) - R(\bigotimes) = (x - x^{-1}) [R(\bigcirc\bigcirc) - R(\bigcirc\bigcirc)]$
- **Y10:**  $R(\bigcirclearrowleft) = x^2 R(\bigcap), \quad R(\bigcirclearrowright) = x^{-2} R(\bigcap)$
- **Y11:**  $R(\bigotimes) = R(\bigcirc\bigcirc)$       **Y12:**  $R(\bigotimes) = R(\bigotimes)$
- **Y13:**  $R(\bigtriangleup) = R(\bigtriangleup), \quad R(\bigtriangleup) = R(\bigtriangleup)$
- **Y14:**  $R(\neg\bigcirclearrowleft) = -x R(\neg\bigtriangleleft), \quad R(\neg\bigcirclearrowright) = -x^{-1} R(\neg\bigtriangleleft)$

## Proposition ([?])

$R(D_T)$  is an ambient isotopy invariant of  $T$  up to multiplying  $(-x)^n$  for some integer  $n$ .

# Lower Bounds from Yamada Polynomial

## Theorem

Let  $T$  be any  $\theta$ -curve or handcuff graph. Then

$$2 + \sqrt{\max_{x \in T} \deg_x R(S_T) - \min_{x \in T} \deg_x R(S_T) - 4} \leq \alpha(T)$$

where  $R(T)$  is a Yamada Polynomial of  $T$ .

## Proposition

Let  $S_T$  be a simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $T$  **without nested caps**. Then

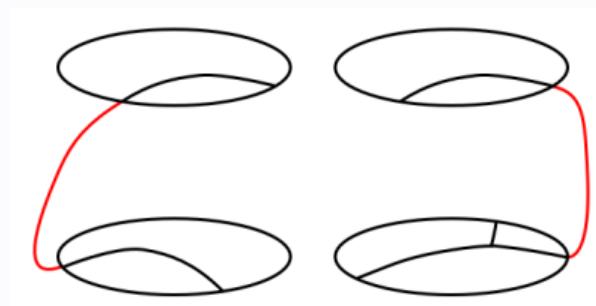
$$\max \deg_x R(S_T) \leq c + n, \quad \min \deg_x R(S_T) \geq -(c + n),$$

where  $c, n$  is the number of caps and crossings in  $S_T$ , respectively.

## PROOF

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- Use double mathematical induction of  $(c_s + c_{ss}, n)$ .



### Basis Step:

- If  $c_s + c_{ss} = 0$ , then  $S_T$  has no simple disks and is equivalent to the result of applying Y14 to  $\ominus$ .  
 $\therefore R(S_T) = -x^{\pm 3} [-x^2 - x - 2 - x^{-1} - x^{-2}] \implies 5 \leq c + n.$
- If  $n = 0$ , then  $S_T$  is equivalent to  $\bigcirc \ominus \bigcirc \cup \bigcirc \cup \dots \cup \bigcirc$ .  
 $\therefore R(S_T) = 0 \implies 0 < 2 \leq c + n.$

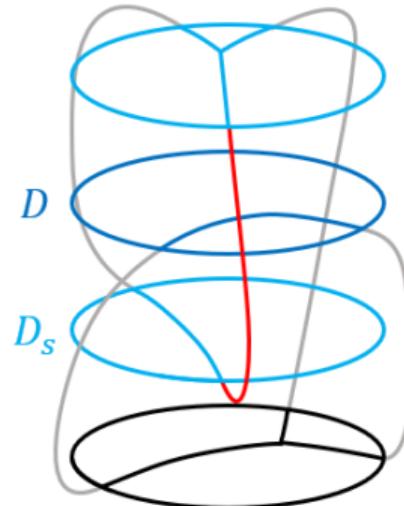
All base cases satisfy the inequality.

## Inductive Step:

Assume that it holds for any  $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$ , and  $c_s + c_{ss} > 0$ .

Let  $S_T$  be a **simple closure of stacked tangle** of a  $\theta$ -curve or handcuff graph  $T$  such that the number of simple caps, semi-simple caps, and crossings are  $c_s, c_{ss}, n$ , respectively.

Take the topmost **simple disk**  $D_s$  connected to the top disk, and a **disk**  $D$  directly above  $D_s$ .



CASE 1. Suppose that there is no cap between  $D_s$  and  $D$ .

① Suppose that there is no intersection between  $D_s$  and  $D$  in  $S_T$ .

- $D_s$  and  $D$  do not affect each other.
- We can swap the position of  $D_s$  and  $D$  without affecting the rest of the diagram.

② Suppose that there is an intersection between  $D_s$  and  $D$  in  $S_T$ .

- Let  $S_T^-$ ,  $S_T^0$  and  $S_T^\infty$  be the simple closure of stacked tangle which is obtained by replacing  with ,  and , respectively.
- The simple caps, semi-simple caps, and crossings of the both are  $c_s$ ,  $c_{ss}$ ,  $n - 1$ .
- Applying Y9

$$R(\text{X}) - R(\text{X}) = (x - x^{-1}) [R(\text{O}) - R(\text{O})],$$

then

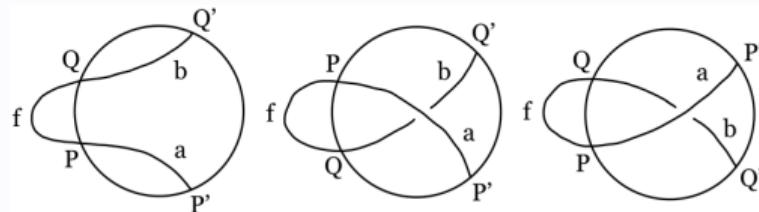
$$R(S_T) - R(S_T^-) = (x - x^{-1})(R(S_T^0) - R(S_T^\infty)).$$

- Then, it is sufficient to show that the interchanged one holds.

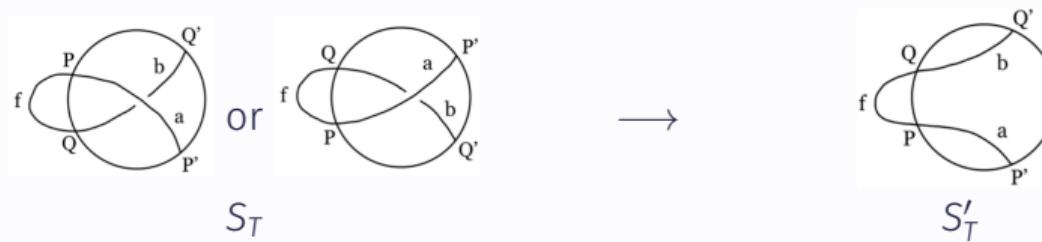
CASE 2. Suppose that there is a cap between  $D_s$  and  $D$ .

① Suppose that  $D$  is a simple disk.

- When view from above, there are three cases:



- After applying Y10, the second and third cases can be regarded as the first case, and the cap can be reduced.



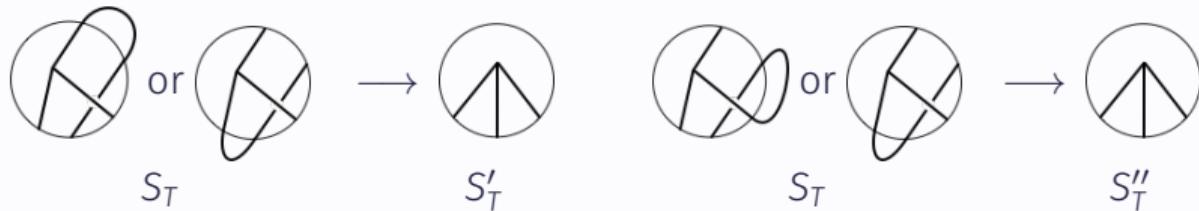
- $S'_T$  has  $c - 1$  caps,  $c_s - 1$  simple caps,  $c_{ss}$  semi-simple caps and  $n - 1$  crossings.

- By induction hypothesis,

$$\begin{aligned}
 \max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 2 \\
 &\leq [(c - 1) + (n - 1)] \pm 2 \\
 &\leq c + n, \\
 \min \deg_x R(S_T) &= \min \deg_x R(S'_T) \pm 2 \\
 &\geq -[(c - 1) + (n - 1)] \pm 2 \\
 &\geq -(c + n).
 \end{aligned}$$

## ② $D$ is not a simple disk.

- When viewed from above, all the cases can be reduced as follows.



- $R(S_T) = -x^{\pm 1}R(S'_T)$  and  $R(S_T) = x^{\pm 2}R(S''_T)$  by **Y14** and **Y10**, respectively.
- Both of  $S'_T$  and  $S''_T$  have  $c - 1$  caps,  $c_s$  simple caps,  $c_{ss} - 1$  semi-simple caps, and  $n - 1$  crossing.

- By induction hypothesis, in the first case,

$$\begin{aligned}
 \max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 1 \\
 &\leq [(c - 1) + (n - 1)] \pm 1 \\
 &\leq c + n.
 \end{aligned}$$

- Similarly, in the second case,

$$\begin{aligned}
 \max \deg_x R(S_T) &= \max \deg_x R(S''_T) \pm 2 \\
 &\leq [(c - 1) + (n - 1)] \pm 2 \\
 &\leq c + n.
 \end{aligned}$$

- It holds for  $\min \deg_x R(S_T)$  in the same way.

□

## Proposition

Let  $S_T$  be a reduced simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $T$  corresponding to minimal arc presentation of  $T$ . Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

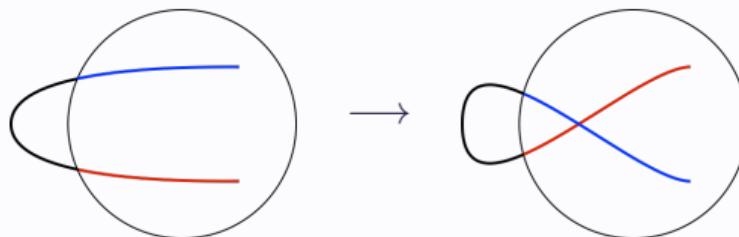
where  $n$  is the number of crossings in  $S_T$ .

## PROOF

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- $S_T$  is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps  $c$  in  $S_T$  is exactly arc index of  $T$ ,  $\alpha(T)$ .

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- $p$  of the caps yield a negative curl, and the remaining  $c - p$  yield a positive curl.
- $S_T^{neg}(S_T^{pos})$  is the diagram obtained by inserting the  $p$  negative( $c - p$  positive) curls.

	$S_T^{neg}$	$S_T^{pos}$
Number of Caps	$c$	$c$
Number of Crossings	$n + p$	$n + (c - p)$

- $R(S_T^{neg}) = x^{-2p}R(S_T)$  and  $R(S_T^{pos}) = x^{2(c-p)}R(S_T)$

$$\begin{aligned} \min \deg_x R(S_T) - 2p &= \min \deg_x R(S_T^{neg}) \\ &\geq -c + -(n + p) \end{aligned}$$

$$\begin{aligned} \max \deg_x R(S_T) + 2(c - p) &= \max \deg_x R(S_T^{pos}) \\ &\leq c + [n + (c - p)] \end{aligned}$$

$$\min \deg_x R(S_T) \geq -c - n + p$$

$$\max \deg_x R(S_T) \leq n + p$$

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq c + 2n$$

□

# Proof of Theorem

## Theorem

Let  $T$  be any  $\theta$ -curve or handcuff graph. Then

$$2 + \sqrt{\max_{x \in T} \deg_x R(S_T) - \min_{x \in T} \deg_x R(S_T) - 4} \leq \alpha(T)$$

where  $R(T)$  is a Yamada Polynomial of  $T$ .

## PROOF

Let  $S_T$  be a reduce simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $T$  corresponding to minimal arc presentation of  $T$ .

- The number of caps :  $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks :  $\alpha(T) - 3$

① Let  $T$  be any  $\theta$ -curve.

Consider the maximum number of crossings in  $S_T$ .

- number of crossings by two simple disks :  $\binom{\alpha(T)-3}{2} = \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4)$
- number of crossings by a simple disk and non-simple disk :  $2 (\alpha(T) - 3)$
- number of crossings by two non-simple disks :  $2$
- number of crossings counted by disks joined by cap :  $\alpha(T) - 2$

Thus

$$\begin{aligned} n &\leq \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4) + 2 (\alpha(T) - 3) + 2 - (\alpha(T) - 2) \\ &= \frac{1}{2} [(\alpha(T))^2 - 5\alpha(T) + 8] \end{aligned}$$

By Lemma,

$$\begin{aligned} \max \deg_x R(S_T) - \min \deg_x R(S_T) &\leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8 \\ 2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} &\leq \alpha(T) \end{aligned}$$

② Let  $T$  be any handcuff graph.

Consider the maximum number of crossings in  $S_T$ .

- number of crossings by two simple disks :  $\binom{\alpha(T)-3}{2} = \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)$
- number of crossings by a simple disk and non-simple disk :  $2(\alpha(T)-3)$
- number of crossings by two non-simple disks : 1
- number of crossings counted by disks joined by cap :  $\alpha(T) - 1 - 2 = \alpha(T) - 3$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4) + 2(\alpha(T)-3) + 1 - (\alpha(T)-3) \\ &= \frac{1}{2}[(\alpha(T))^2 - 5\alpha(T) + 8] \end{aligned}$$

By Lemma,

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8$$

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

□

## Result

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# Grid Diagrams of $\theta$ -Curves



# Grid Diagrams of Handcuff-Graphs

