

# The Arc Index of Theta-Curve and Handcuff Graph

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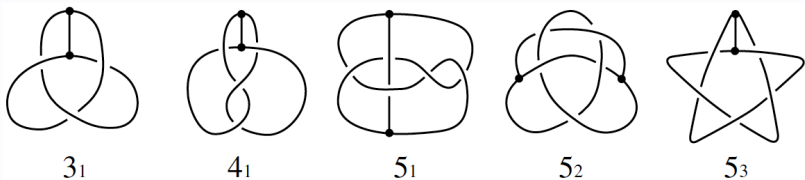
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# Introduction

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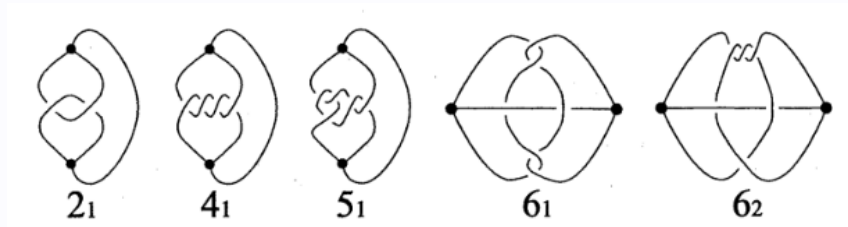
# Theta-Curves

- A **theta-curve**  $T$  is a graph embedded in  $S^3$ , which consists of two vertices  $v_1, v_2$  and three edges  $e_1, e_2, e_3$ , such that each edge joins the vertices.
- A **constituent knot**  $T_{ij}$ ,  $1 \leq i < j \leq 3$ , is a subgraph of  $T$  that consists of two vertices  $v_1, v_2$  and two edges  $e_i, e_j$ .
- Theta-curves are roughly classified by comparing the triples of constituent knots.
- A theta-curve is said to be **trivial** if it can be embedded in a 2-sphere in  $S^3$ .

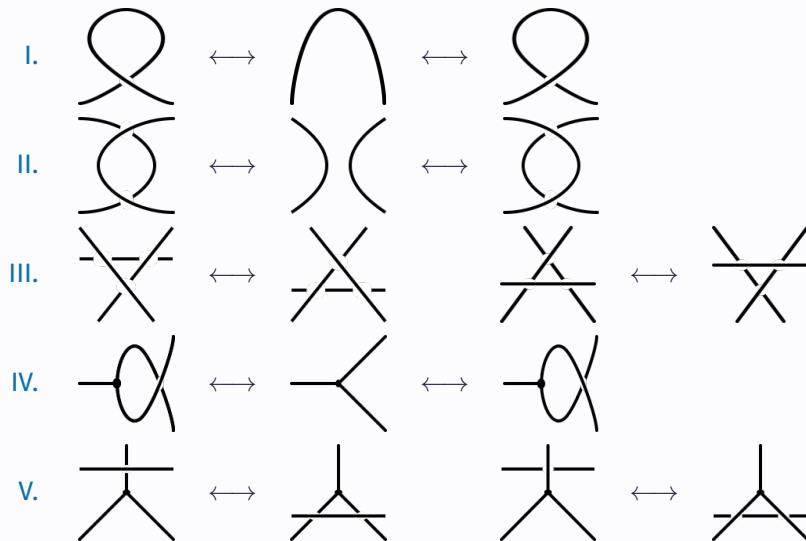


# Handcuff Graphs

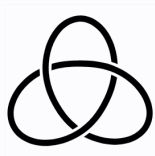
- **Handcuff graph** is the graph which consists of two loops and an edge joining the vertices of each loop.



# Reidemeister Moves for Theta-Curves and Handcuff Graphs



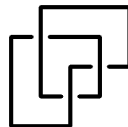
- **Arc presentation** is an open-book decomposition of  $\mathbb{R}^3$  which has open half-planes as pages and the standard z-axis as the binding axis.
- **Arc index**, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is **minimal arc presentation**.



Trefoil



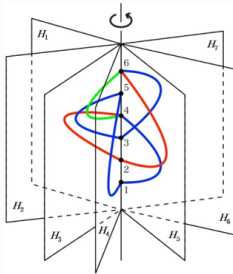
Open Book



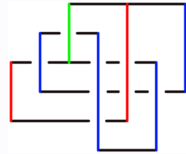
Grid Diagram



$\theta_{5,2}$



Open Book

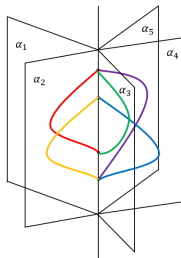


Grid Diagram

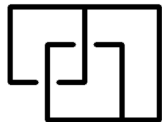




$\Phi_{2,1}$



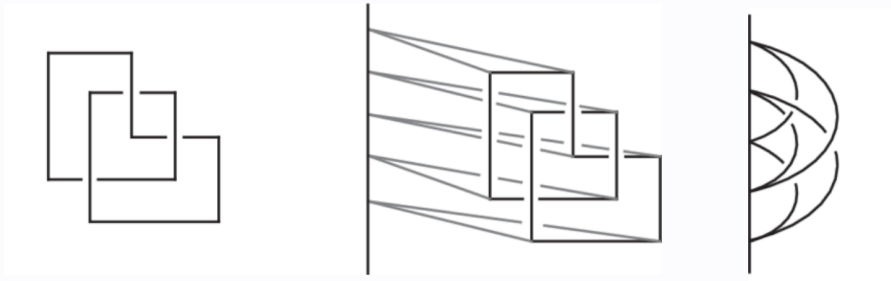
Open Book



Grid Diagram

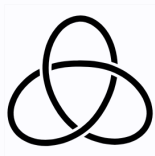
# Grid Diagram

- The **grid diagram** of a theta-curve or handcuff graph is a diagram of vertical strands and one less number of horizontal strands.
- At every crossing the vertical strand crosses over the horizontal strand and no two horizontal segments are co-linear and no two vertical segments are co-linear.

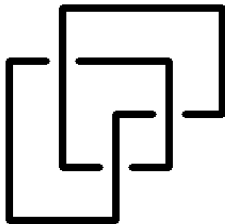


# Cromwell Matrix

- The **Cromwell matrix** of a knot is an  $n \times n$  binary matrix each of whose rows and columns has exactly two 1s.



Trefoil



Grid Diagram

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

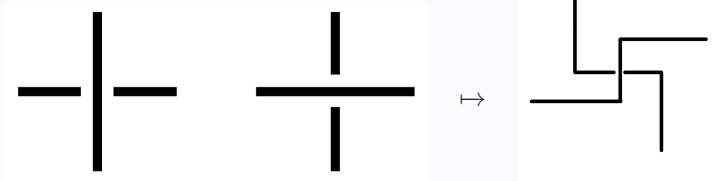
Cromwell Matrix

# Arc Presentation of the Theta-Curve and Handcuff Graph

## Theorem

*Arc presentations exist for every theta-curve and handcuff graph.*

## PROOF



# Arc Presentation of the Theta-Curve and Handcuff Graph

## PROOF

