

# The Determinants and Arc Indices of $\theta$ -Curves and Handcuff Graphs

---

Eunchan Cho<sup>1</sup> Jeongwon Shin<sup>1</sup> Boyeon Seo<sup>1</sup> Minho Choi<sup>1</sup>  
Supervisor: Hun Kim<sup>1</sup> GyoTaek Jin<sup>2</sup>

NOV 22, 2025  
R&E 2025

<sup>1</sup>Korea Science Academy of KAIST

<sup>2</sup>Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology

# Introduction

---

# $\theta$ -Curves and Handcuff Graphs

- A  **$\theta$ -curve**  $T$  is a graph embedded in  $S^3$ , which consists of two vertices  $v_1, v_2$  and three edges  $e_1, e_2, e_3$ , such that each edge joins the vertices.
- A **handcuff graph**  $H$  is a graph embedded in  $S^3$  consisting of two vertices ( $v_1, v_2$ ) and three edges ( $e_1, e_2, e_3$ ), where  $e_3$  has distinct endpoints  $v_1$  and  $v_2$ , and  $e_1$  and  $e_2$  are loops based at  $v_1$  and  $v_2$ .



3<sub>1</sub>



4<sub>1</sub>



5<sub>1</sub>



5<sub>2</sub>



5<sub>3</sub>

$\theta$ -Curves



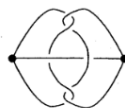
2<sub>1</sub>



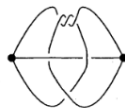
4<sub>1</sub>



5<sub>1</sub>



6<sub>1</sub>



6<sub>2</sub>

Handcuff Graphs

# Arc Presentation



Trefoil



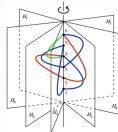
Arc Presentation



Grid Diagram



$\theta_{5,2}$



Arc Presentation



Grid Diagram



$2_1$



Arc Presentation



Grid Diagram

$\Phi_{2,1}$

**Arc presentation** of a  $\theta$ -curve or handcuff graph is an embedding of them.

- It is contained in the union of finitely many half planes (called **pages**).
- The embedding is with the common boundary line (called **axis**).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.

**Arc index** is the minimal number of pages among all possible arc presentations of graph.

- Arc presentation with the minimal number of pages is called **minimal arc presentation**.

# Grid Diagram



Trefoil



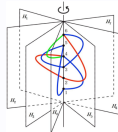
Arc Presentation



Grid Diagram



$\theta_{5,2}$



Arc Presentation



Grid Diagram



$2_1$



Arc Presentation



Grid Diagram

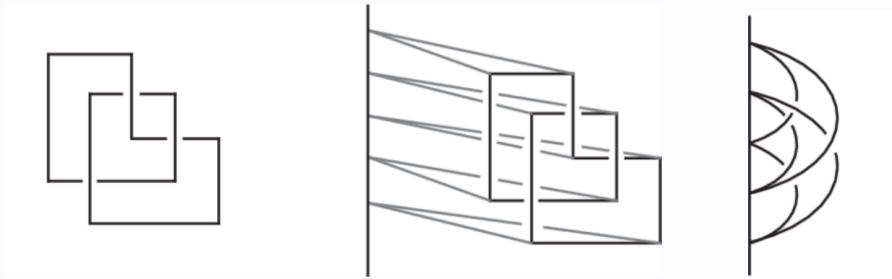
$\Phi_{2,1}$

A **grid diagram** of  $\theta$ -curve or handcuff graph is a diagram with only vertical and horizontal strands.

- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

# Arc Presentation of the $\theta$ -Curve and Handcuff Graph

- A grid diagram gives rise to an arc presentation and vice versa.



## Theorem

*Every  $\theta$ -curve and handcuff graph admit a grid diagram.*

## Corollary

*Every  $\theta$ -curve and handcuff graph admit an arc presentation.*

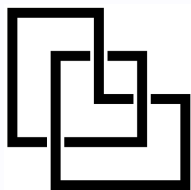


# Determinants of $\theta$ -Curves and Handcuff Graphs

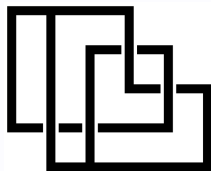
---

# THC-Cromwell matrix

- The **Cromwell Matrix** of a knot is an  $n \times n$  binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-Cromwell matrix** is an expansion of Cromwell matrix into  $\theta$ -curves and handcuff graphs.



$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



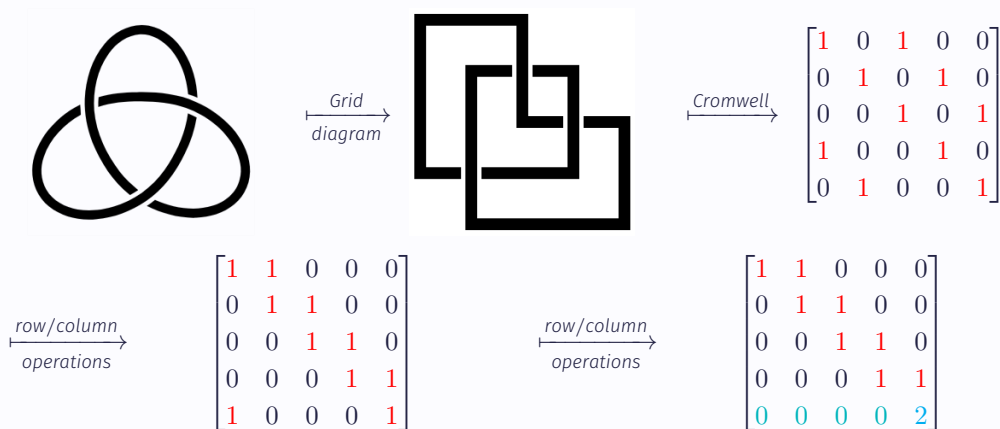
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

# Determinants of the Cromwell matrices of Knot

## Theorem

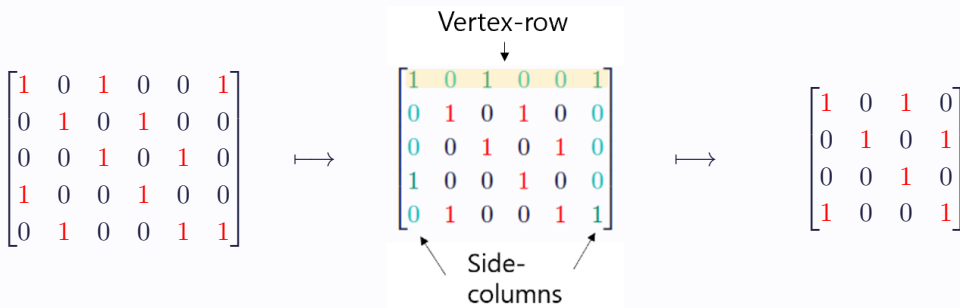
Let  $K$  be a knot. Then the determinant of a Cromwell matrix of  $K$  is 0 or  $\pm 2$ .

## PROOF



## H-deletion of THC-Cromwell matrices

- The **vertex-row** of THC-Cromwell matrix  $M$  is a row which contains three 1s,  $M_{ia}, M_{ib}, M_{ic}$ , where  $a < b < c$ , as its elements.
- The **side-column** of THC-Cromwell matrix  $M$  is a column which contains the leftmost 1 of vertex-row ( $M_{ia}$ ) or the rightmost 1 of vertex row ( $M_{ic}$ ).
- The **H-deletion** Matrix of the THC-Cromwell matrix  $G$  is  $(n - 1) \times (n - 1)$  matrix which deleted vertex-row and its two side-columns from the matrix  $G$ .



# Determinants of the THC-Cromwell matrices

## Theorem

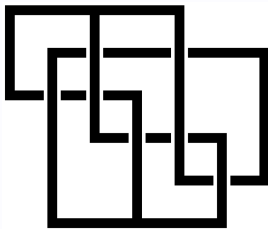
Let  $M_G$  be a THC-Cromwell matrix of  $\theta$ -curve or handcuff graph  $G$ .

- $\det^*(M_G) = \pm 1 \iff M_G$  represents  $\theta$ -curve
- $\det^*(M_G) = 0$  or  $\pm 2 \iff M_G$  represents handcuff graph

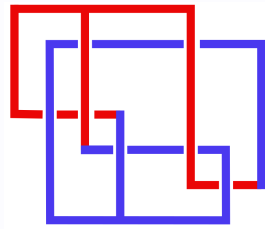
where  $\det^*(M_G)$  = determinant of  $H$ -deletion matrix of  $M_G$

## PROOF

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$



$H$ -deletion  $\rightarrow$

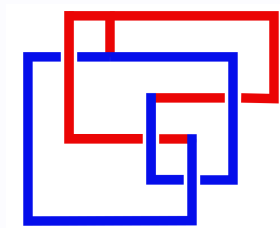


# Proof of Theorem

## CASE 1. When $M_G$ represents $\theta$ -curve

Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



$H$ -deletion

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

row/column  
operations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

subtracting

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So  $\det^*(M_G) = \pm 1$

# Proof of Theorem

CASE 2. When  $M_G$  represents handcuff graph

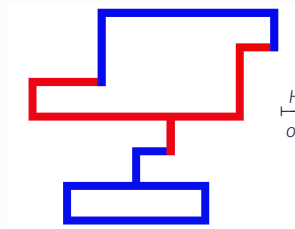
T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

row/column operations  $\rightarrow$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So  $\det^*(M_G) = 0$  or  $\pm 2$



$H$ -deletion  
only T-loop

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

seperating  $\rightarrow$

$$\begin{bmatrix} \text{Knot} & * \\ 0 & \text{Line - shape} \end{bmatrix}$$

# Proof of Theorem

## CASE 2. When $G$ represents handcuff graph

### ① T-loop & Line-shape

cromwell matrix

$\xrightarrow{H\text{-deletion}}$

H-deletion matrix

$\xrightarrow{\text{seperating operations}}$

$$\begin{bmatrix} T\text{-loop} & * \\ 0 & \text{Line-shape} \end{bmatrix}$$

$\xrightarrow{\text{seperating operations}}$

$$\begin{bmatrix} \text{Knot} & * & * \\ 0 & \text{Line-shape} & * \\ 0 & 0 & \text{Line-shape} \end{bmatrix}$$

### ② Knot & Line-shape

cromwell matrix

$\xrightarrow{H\text{-deletion}}$

H-deletion matrix

$\xrightarrow{\text{seperating operations}}$

$$\begin{bmatrix} \text{Knot} & * \\ 0 & \text{Line-shape} \end{bmatrix}$$

So  $\det^*(M_G) = 0$  or  $\pm 2$



## Upper Bounds of Arc Index

---

# Upper Bounds of Arc Index

For a spatial graph  $G$ ,  $c(G)$  denotes the crossing number of  $G$ .

## Theorem (Lee et al., 2018)

*Let  $G$  be a spatial graph. Then,*

$$\alpha(G) \leq c(G) + e + b$$

*where  $e$  and  $b$  are the number of edges and bouquet cut points, respectively.*

## Corollary

*Let  $G$  be a non-trivial prime  $\theta$ -curve or handcuff graph. Then,*

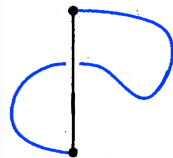
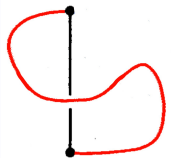
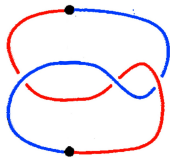
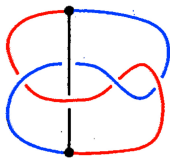
$$\alpha(G) \leq c(G) + 3.$$

## Lower Bounds of Arc Index

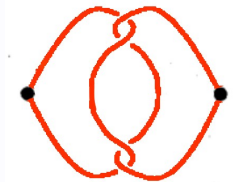
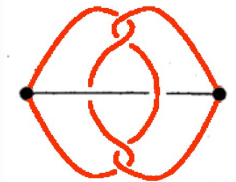
---

## Lower Bound from Constituent Knots/Links

- A **constituent knot**  $T_{ij}$ ,  $1 \leq i < j \leq 3$ , is a subgraph of  $\theta$ -curve  $T$  that consists of two vertices  $v_1, v_2$  and two edges  $e_i, e_j$ .



- A **constituent link**  $H_{12}$ , is a subgraph of handcuff graph  $H$  that consists of two vertices  $v_1, v_2$  and two edges  $e_1, e_2$ .



## Lower Bound from Constituent Knots/Links

### Theorem (Lee, 2023)

*Let  $T$  be a  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then*

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

### Theorem (Lee, 2023)

*Let  $T$  be a  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then*

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^3 \alpha(K_i)$$

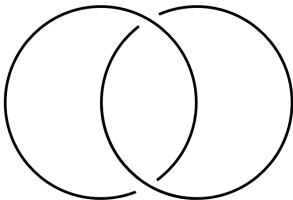
## Lower Bound from Constituent Knots/Links

### Theorem

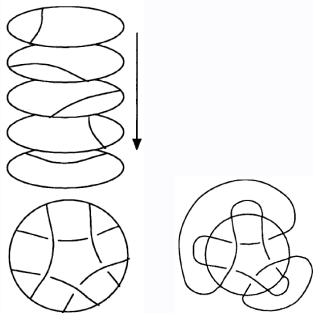
*Let  $H$  be a handcuff graph, and  $L$  be the constituent link of  $H$ . If  $L$  is an alternating and non-split link, then*

$$\alpha(H) \geq c(L) + 3,$$

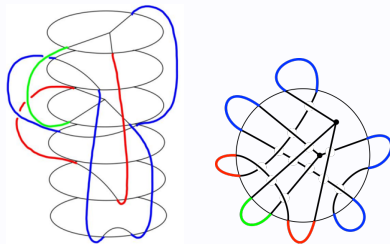
*where  $c(L)$  is the crossing number of  $L$ .*



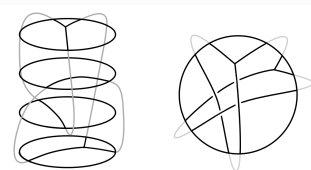
# Stacked Tangle of an $\theta$ -Curve and Handcuff Graph



Stacked Tangle of a Link



$\theta$ -Curve



Handcuff Graph

# Yamada Polynomials

Let  $D_G$  be a diagram of an  $\theta$ -curve or handcuff graph  $G$ . Then, the **Yamada Polynomial**  $R(D_G) \in \mathbb{Z}[x^{\pm 1}]$  is calculated by the following properties:

- **Y6:**  $R(\bigoplus) = -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$       **Y7:**  $R(\bigcirc \bigcirc) = 0$
- **Y8:**  $R(G \cup \bigcirc) = (x + 1 + x^{-1})R(G)$  for an arbitrary  $\theta$ -curve or handcuff graph diagram  $G$
- **Y9:**  $R(\bigwedge) - R(\bigvee) = (x - x^{-1}) [R(\bigcirc \bigcirc) - R(\bigcirc \bigcirc)]$
- **Y10:**  $R(\bigcirc) = x^2 R(\bigcap), \quad R(\bigcirc) = x^{-2} R(\bigcap)$
- **Y11:**  $R(\bigcirc \bigcirc) = R(\bigcirc \bigcirc)$       **Y12:**  $R(\bigcirc \bigcirc) = R(\bigcirc \bigcirc)$
- **Y13:**  $R(\bigwedge) = R(\bigwedge), \quad R(\bigwedge) = R(\bigwedge)$
- **Y14:**  $R(\bigcirc \bigcirc) = -x R(\bigcirc \bigcirc), \quad R(\bigcirc \bigcirc) = -x^{-1} R(\bigcirc \bigcirc)$

## Proposition (Yamada, 1989)

$R(D_G)$  is an ambient isotopy invariant of  $G$  up to multiplying  $(-x)^n$  for some integer  $n$ .



### Theorem

*Let  $G$  be a  $\theta$ -curve or handcuff graph. Then*

$$2 + \sqrt{\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4} \leq \alpha(G).$$

# Sketch of Proof

## Proposition

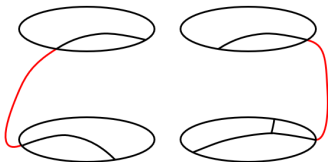
Let  $S_G$  be a simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $G$  **without nested caps**. Then

$$\max \deg_x R(S_G) \leq c + n, \quad \min \deg_x R(S_G) \geq -(c + n),$$

where  $c$  and  $n$  is the number of caps and crossings in  $S_G$ , respectively.

## PROOF

- Let  $c_s$  and  $c_{ss}$  be the number of **simple caps** and **semi-simple caps**, respectively.
- Use double mathematical induction of  $(c_s + c_{ss}, n)$ .



# Sketch of Proof

## Proposition

*Let  $S_G$  be a reduced simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $G$  corresponding to minimal arc presentation of  $G$ . Then*

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) - 2n \leq \alpha(G)$$

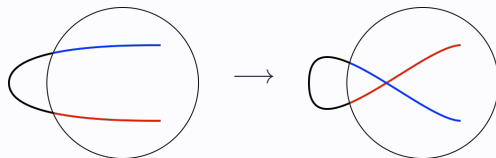
*where  $n$  is the number of crossings in  $S_G$ .*

## PROOF

- $S_G$  is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps  $c$  in  $S_G$  is exactly arc index of  $G$ ,  $\alpha(G)$ .

## Sketch of Proof

- Take a cap and add a positive or negative curl



	$S_G^{neg}$	$S_G^{pos}$
Number of Caps	$c$	$c$
Number of Crossings	$n + p$	$n + (c - p)$

$$\begin{aligned}\min \deg_x R(S_G) - 2p &= \min \deg_x R(S_G^{neg}) \geq -c + -(n + p) \\ \max \deg_x R(S_G) + 2(c - p) &= \max \deg_x R(S_G^{pos}) \leq c + [n + (c - p)] \\ \max \deg_x R(S_G) - \min \deg_x R(S_G) &\leq c + 2n\end{aligned}$$

□

# Sketch of Proof

## Theorem

Let  $G$  be a  $\theta$ -curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4} \leq \alpha(G).$$

## PROOF

Let  $S_G$  be a reduce simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $G$  corresponding to minimal arc presentation of  $G$ .

- The number of caps :  $\alpha(G)$
- The number of non-simple disks : 2
- The number of simple disks :  $\alpha(G) - 3$

The theorem is followed by considering the maximum of crossings in  $S_G$  and applying previous theorems.

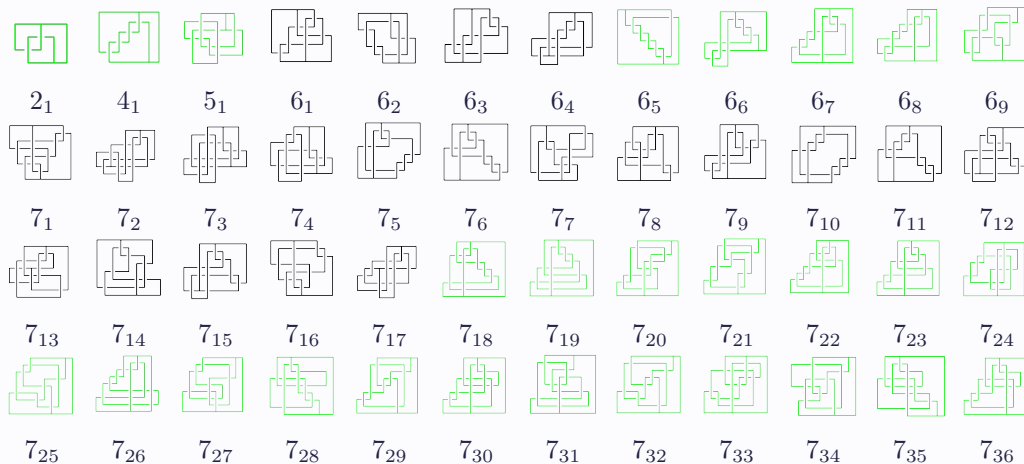
# Minimal Grid Diagrams

---

# Grid Diagrams of $\theta$ -Curves



# Grid Diagrams of Handcuff Graphs





## Further Studies

---

## Further Studies

- We tried to use the Python for determining arc indices. For the Python program, we used the Topoly package, but the package had an error. We would like to use another tool to completely determine the arc indices of  $\theta$ -curves and handcuff graphs, such as Yamada package.
- Applying the result of our research to  $\theta$ -curves or handcuff graphs of higher crossings or to other spatial graphs can be researched further.

## References

---

- 1 Moriuchi, H. (2019). A table of  $\theta$ -curves and handcuff graphs with up to seven crossings. *Advanced Studies in Pure Mathematics*, 281–290.
- 2 Yoonsang Lee. (2023). A Study on Arc Index of Theta-Curves. Korea Science Academy of KAIST
- 3 Minjung Lee, Sungjong No, and Seungsang Oh. (2018). Arc index of spatial graphs. *Journal of Graph Theory*, 90(3), 406–415.
- 4 Yamada, S. (1989). An invariant of spatial graphs. *Journal of Graph Theory*, 13(5), 537–551.