

# The Arc Index of Theta-Curve and Handcuff Graph

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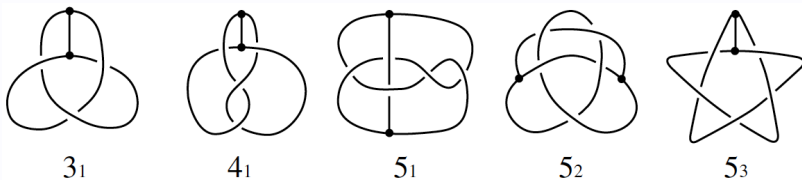
SEP 6, 2025

# Introduction

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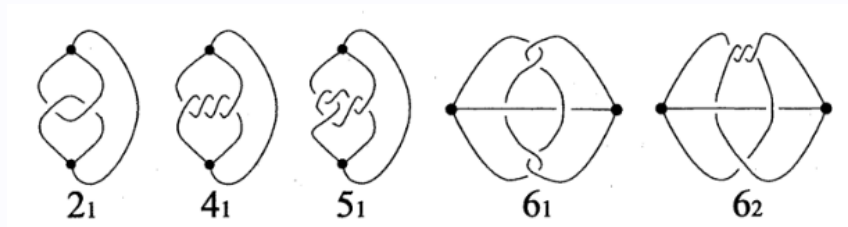
# Theta-Curves

- A **theta-curve**  $T$  is a graph embedded in  $S^3$ , which consists of two vertices  $v_1, v_2$  and three edges  $e_1, e_2, e_3$ , such that each edge joins the vertices.
- A **constituent knot**  $T_{ij}$ ,  $1 \leq i < j \leq 3$ , is a subgraph of  $T$  that consists of two vertices  $v_1, v_2$  and two edges  $e_i, e_j$ .
- Theta-curves are roughly classified by comparing the triples of constituent knots.
- A theta-curve is said to be **trivial** if it can be embedded in a 2-sphere in  $S^3$ .

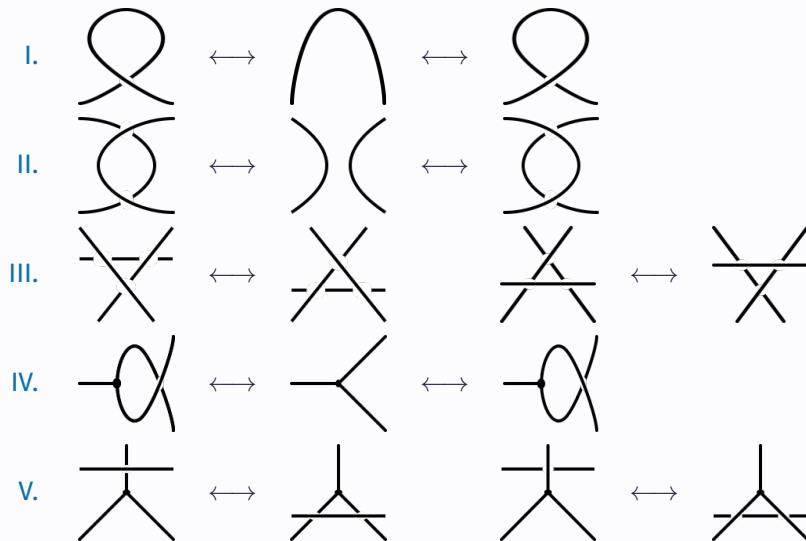


# Handcuff Graphs

- **Handcuff graph**  $H$  is the graph which consists of two loops and an edge joining the vertices of each loop.
- A **constituent link**  $H_{12}$ , is a subgraph of  $H$  that consists of two vertices  $v_1, v_2$  and two edges  $e_1, e_2$ .



# Reidemeister Moves for Theta-Curves and Handcuff Graphs



- **Arc presentation** of a theta-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called **pages**).
- The embedding is with the common boundary line (called **axis**).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- **Arc index**, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is **minimal arc presentation**.

# Arc Presentation



Trefoil



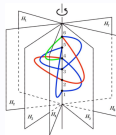
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Grid Diagram



$\theta_{5,2}$



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Grid Diagram



$2_1$

$\Phi_{2,1}$



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Grid Diagram

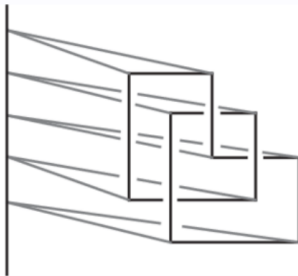
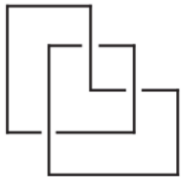
# Grid Diagram

- The **grid diagram** of theta-curve or handcuff graph is a diagram with only vertical strand and horizontal strands.
- $(\text{number of vertical strands}) + 1 = (\text{number of horizontal strands})$
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same strand.



# Grid Diagram

- A grid diagram gives rise to an arc presentation and vice versa.

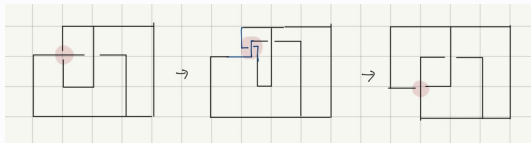
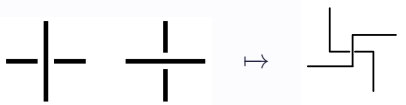


# Arc Presentation of the Theta-Curve and Handcuff Graph

## Theorem

*Every theta-curve and handcuff graph admit a grid diagram.*

## PROOF



## Corollary

*Every theta-curve and handcuff graph admit a arc presentation.*

