The Determinant and Arc Indices of θ-Curves and Handcuff-Graphs

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Lower Bounds of Arc Index

Lower Bound from Constituent Knots/Links

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

Lower Bound from Constituent Knots/Links

Theorem

Let H be any handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then

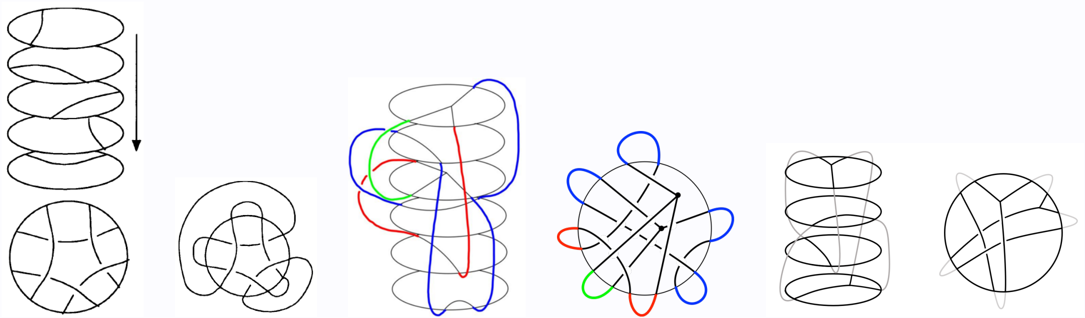
$$\alpha(H) \geq c(L) + 3.$$

Corollary

Let H be any handcuff graph, and L be a constituent link of T. If L is alternating and non-split,

$$\alpha(H) = c(L) + 3.$$

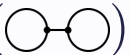
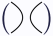
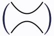
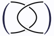
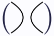
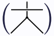
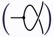
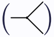
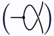
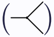
Stacked Tangle of an θ-Curve and Handcuff-Graph



Stacked Tangle of a Link $\qquad$ θ-Curve $\qquad$ Handcuff-Graph

Figure from [?]

Let D_G be a diagram of an θ-curve or handcuff-graph G. Then, the **Yamada Polynomial** $R(D_G) \in Z\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** R () $= -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** R () $= 0$
- **Y8:** $R(G \cup$ $) = (x+1+x^{-1})R(G)$ for an arbitrary θ-curve or handcuff-graph diagram G
- **Y9:** R () $- R$ () $= (x - x^{-1})\left[R\right.$ () $- R$ ()$\left.\right]$
- **Y10:** R () $= x^2 R$ (), R () $= x^{-2} R$ ()
- **Y11:** R () $= R$ () **Y12:** R () $= R$ ()
- **Y13:** R () $= R$ (), R () $= R$ ()
- **Y14:** R () $= -xR$ (), R () $= -x^{-1} R$ ()

Proposition ([?])

$R(D_G)$ is an ambient isotopy invariant of G up to multiplying $(-x)^n$ for some integer n.

Lower Bound from Yamada Polynomial

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Lower Bound from Yamada Polynomial

Proposition

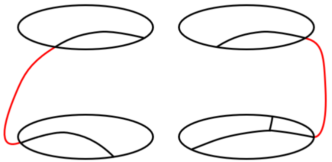
Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T ***without nested caps****. Then*

$$\max\deg_x R(S_T) \leq c + n, \quad \min\deg_x R(S_T) \geq -(c+n),$$

where $\boldsymbol{c}, \boldsymbol{n}$ *is the number of caps and crossings in S_T, respectively.*

Proof

- Let c_s, c_{ss} be the number of **simlpe caps** or **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T. Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

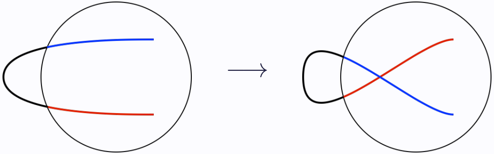
where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

$$\min \deg_x R(S_T) - 2p = \min \deg_x R\left(S_T^{neg}\right) \geq -c + -(n+p)$$

$$\max \deg_x R(S_T) + 2(c-p) = \max \deg_x R\left(S_T^{pos}\right) \leq c + [n+(c-p)]$$

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq c + 2n$$

□

Proof of Theorem

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T).$$

PROOF

Let S_T be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

Consider the maximum number of crossings in S_T, and use the previous theorem.