

# The Determinant and an Arc Index of Theta Curve and Handcuff-Graph

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# Introduction

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# $\theta$ -Curves

- A  **$\theta$ -curve**  $T$  is a graph embedded in  $S^3$ , which consists of two vertices  $v_1, v_2$  and three edges  $e_1, e_2, e_3$ , such that each edge joins the vertices.
- A **constituent knot**  $T_{ij}$ ,  $1 \leq i < j \leq 3$ , is a subgraph of  $T$  that consists of two vertices  $v_1, v_2$  and two edges  $e_i, e_j$ .
- $\theta$ -curves are roughly classified by comparing the triples of constituent knots.
- A  $\theta$ -curve is said to be **trivial** if it can be embedded in a 2-sphere in  $S^3$ .

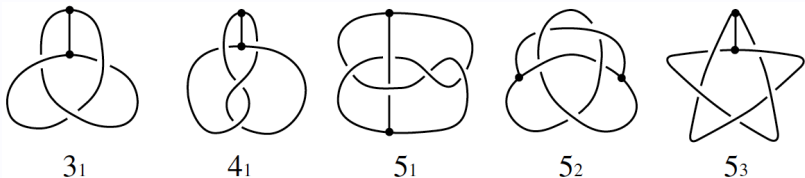


Figure from [Moriuchi, 2007]

# Handcuff-Graphs

- **Handcuff graph** consists of 2 loops and 1 edge joining the loops.

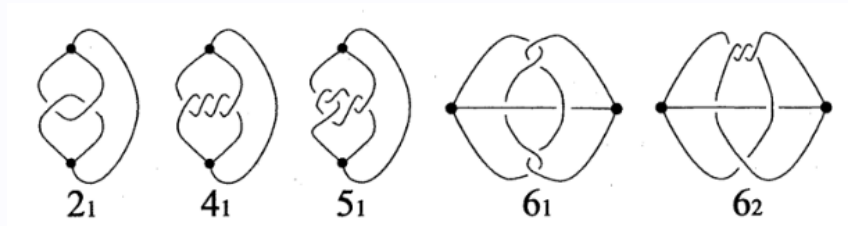
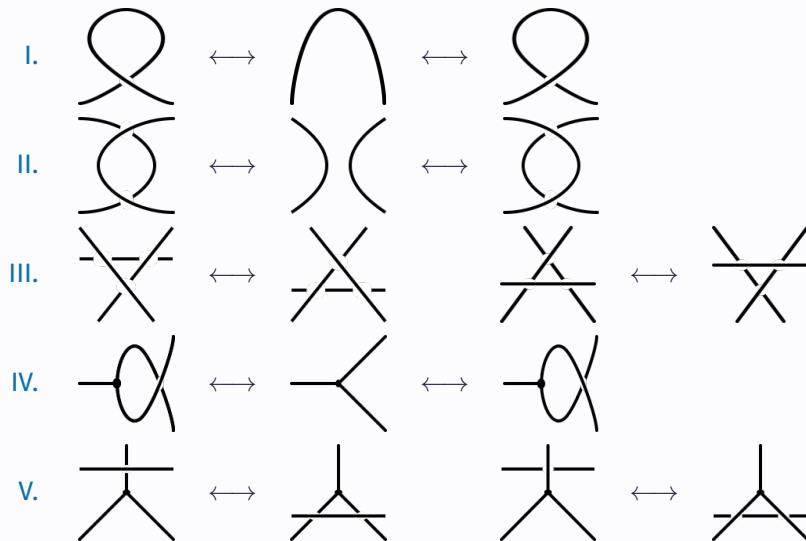


Figure from [?] with up to seven crossings

# Reidemeister Moves for $\theta$ -Curves and Handcuff Graphs

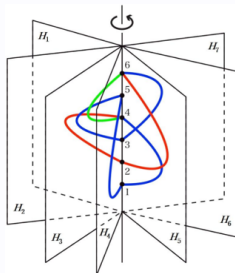


# Arc Index

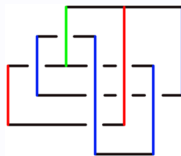
- An **arc presentation** of a  $\theta$ -curve is defined in the same manner as an arc presentation of a knot.
- The binding axis contains all **vertices** of  $\theta$ -curve.
- **Minimal arc presentation** and **arc index** are defined in the same manner.



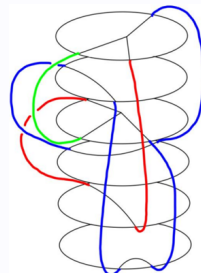
$\theta 5_2$



Open Book



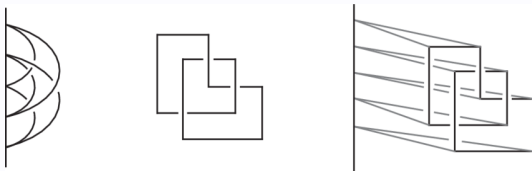
Grid Diagram



Stacked Tangle

# Grid Diagram

- The **grid diagram** is a handcuff graph or theta-curve diagram of vertical strands and one less number of horizontal strands with the properties that at every crossing the vertical strand crosses over the horizontal strand and no two horizontal segments are co-linear and no two vertical segments are co-linear.

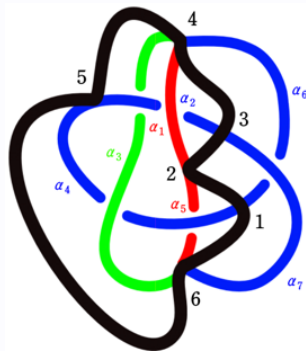


# Binding Circle Method

Let  $D_T$  be a diagram of an  $\theta$ -curve  $T$ . A **binding circle** of  $D_T$  is a simple closed curve  $C$  meeting  $D_T$  in  $n$  distinct points with the following properties:

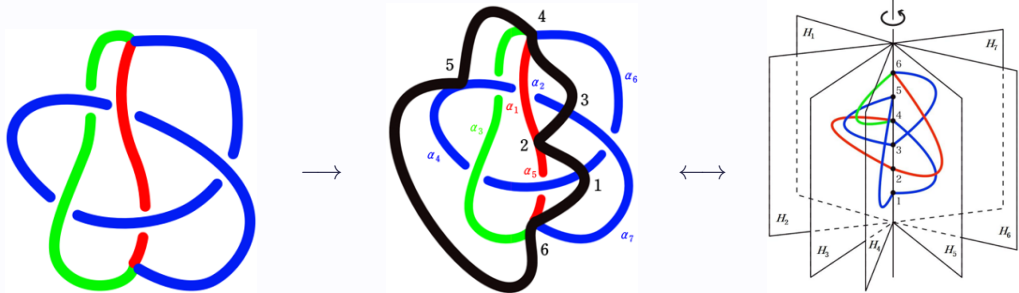
- $C$  must meet  $D_T$  at **two vertices**.
- $C$  divide  $D_T$  into  $n$  arcs  $\alpha_1, \alpha_2, \dots, \alpha_n$ .
- Each  $\alpha_i$  has no self-crossings.
- If  $\alpha_i$  crosses over  $\alpha_j$  at a crossing in inside(resp. outside)  $C$ , then  $i < j$ (resp.  $i > j$ ) and it crosses over  $\alpha_i$  at any other crossings with  $\alpha_j$ , respectively.
- For each  $i$ , there exists an embedded disk  $d_i$  such that  $\partial d_i = C$  and  $\alpha_i \subset d_i$ .
- $d_i \cap d_j = C$ , for distinct  $i$  and  $j$ .

Then the pair  $(D_T, C)$  is also corresponding to an arc presentation.





# Binding Circle Method



## Theorem ([Lee et al., 2019])

*Let  $G$  be any spatial graph with  $e$  edges and  $b$  bouquet cut components. Then*

$$\alpha(G) \leq c(G) + e + b$$

## Corollary

*Let  $T$  be any  $\theta$ -curve. Then*

$$\alpha(T) \leq c(T) + 3$$

## Classifying by Determinant

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## THC-cromwell matrix

- The **cromwell matrix** of a knot is an  $n \times n$  binary matrix such that each row and column has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into  $\theta$ -curves and handcuff graphs that satisfies the following conditions :
  1. It is a  $(n + 1) \times n$  binary matrix.
  2. It contains exactly two 1s in every column.
  3. There exists two distinct rows which contains exactly three 1s, which is called the **three-row**. Every other row contains exactly two 1s.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

# Determinant of the cromwell matrices of Knot

## Theorem

*Let  $K$  be any knot then its determinant of the cromwell matrix is 0 or 2.*

## PROOF



$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

CASE 1. When  $n$  is an even number.

## Lower Bounds of Arc Index

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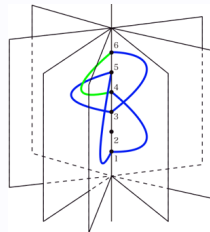
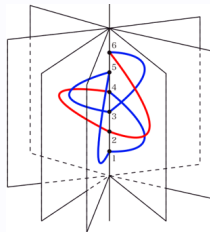
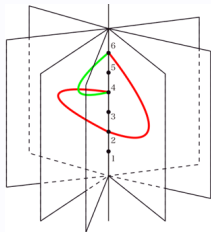
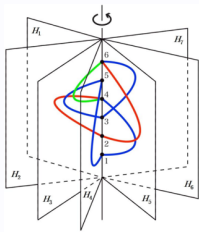
# Lower Bounds from Constituent Knots

## Theorem

Let  $T$  be any  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

## PROOF





## Theorem

Let  $T$  be any  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^3 \alpha(K_i)$$

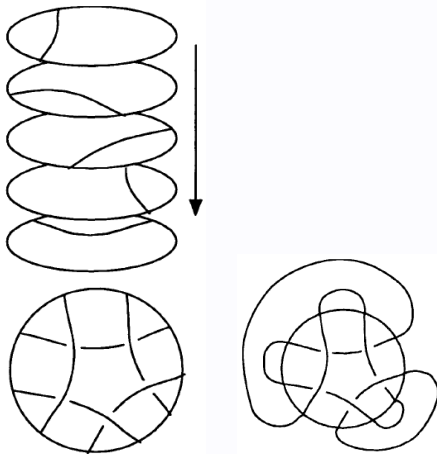
## PROOF

- A minimal arc presentation of  $T$  is given.
- $K_1 = e_1 \cup e_2$ ,  $K_2 = e_2 \cup e_3$ , and  $K_3 = e_3 \cup e_1$ .
- $S_i$  be the set of half plane corresponding the edge  $e_i$ .
- $S_i \cup S_{i+1}$  form an arc presentation of the knot  $K_i$ .
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

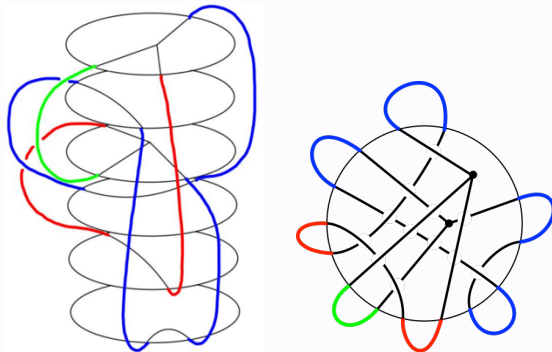
$$\sum_{i=1}^3 \alpha(K_i) \leq 2 \sum_{i=1}^3 |S_i| = 2\alpha(T)$$



## Stacked Tangle of an $\theta$ -Curve



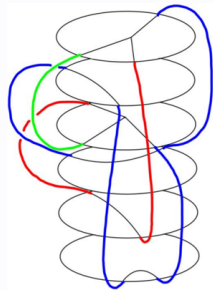
Stacked Tangle of a Link



Stacked Tangle of a  $\theta$ -Curve

**Stacked tangle** of an  $\theta$ -curve is stacked disks each with the frame as boundary with following properties:

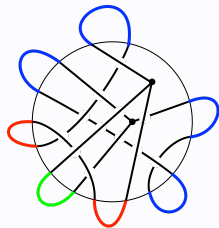
- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point(by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



**Simple closure** of stacked tangle is a **stacked tangle** with **caps** satisfying following properties:

- A **cap** is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.

Then a simple closure of a stacked tangle **without any nested caps** is corresponding to an arc presentation.



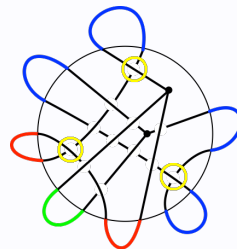
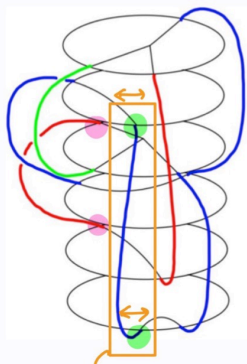
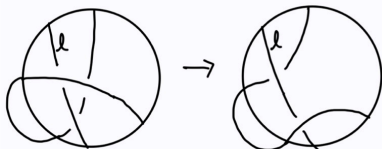
A **reduced simple closure of a stacked tangle** is

- a simple closure of a stacked tangle **without any nested caps**
- any two arcs(including line segment) joining by caps have **no intersection** when view from above

## Proposition

*A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.*

## PROOF



# Yamada Polynomials

Let  $D_T$  be a diagram of an  $\theta$ -curve  $T$ . Then, the **Yamada Polynomial**  $R(D_T) \in \mathbb{Z}[x^{\pm 1}]$  is calculated by the following properties:

- **Y6:**  $R(\bigoplus) = -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$       **Y7:**  $R(\bigcirc \bigcirc) = 0$
- **Y8:**  $R(T' \cup \bigcirc) = (x + 1 + x^{-1})R(T')$  for an arbitrary  $\theta$ -curve diagram  $T'$
- **Y9:**  $R(\bigwedge) - R(\bigvee) = (x - x^{-1}) [R(\bigcirc) - R(\bigcirc)]$
- **Y10:**  $R(\bigcirc) = x^2 R(\bigcap), \quad R(\bigcirc) = x^{-2} R(\bigcap)$
- **Y11:**  $R(\bigcirc) = R(\bigcirc)$       **Y12:**  $R(\bigcirc) = R(\bigcirc)$
- **Y13:**  $R(\bigcirc) = R(\bigcirc), \quad R(\bigcirc) = R(\bigcirc)$
- **Y14:**  $R(\bigcirc) = -xR(\bigcirc), \quad R(\bigcirc) = -x^{-1}R(\bigcirc)$

## Proposition ([Yamada, 1989])

$R(D_T)$  is an ambient isotopy invariant of  $T$  up to multiplying  $(-x)^n$  for some integer  $n$ .

### Theorem

*Let  $T$  be any  $\theta$ -curve. Then*

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

*where  $R(T)$  is a Yamada Polynomial of the  $\theta$ -curve  $T$ .*

## Proposition

Let  $S_T$  be a simple closure of stacked tangle of a  $\theta$ -curve  $T$  **without any nested caps**. Then

$$\max \deg_x R(S_T) \leq c + n - 2 \quad \text{and} \quad \min \deg_x R(S_T) \geq -c - n + 2$$

where  $c$  is the **number of caps** and  $n$  is the **number of crossings** in  $S_T$ .

## PROOF

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- A **simple cap** is a cap joining simple disks.
- Let  $s$  be the **number of simple caps** in  $S_T$ .
- Use double mathematical induction of  $(s, n)$ .



### Basis Step:

If  $s = 0$ , then  $S_T$  is either equivalent to  $\ominus$  or  $\bigcirc\text{---}\bigcirc$ .

- If  $S_T \equiv \ominus$ , then  $R(S_T) = -x^2 - x - 2 - x^{-1} - x^{-2}$  and  $4 \leq s + n$ .
- If  $S_T \equiv \bigcirc\text{---}\bigcirc$ , then  $R(S_T) = 0$  and  $3 \leq c + n$ .

If  $n = 0$ , then  $S_T$  is equivalent to  $\bigcirc\text{---}\bigcirc \cup \bigcirc \cup \cdots \cup \bigcirc$ .

- $R(S_T) = 0$  and  $2 \leq c + n$ .

All of the cases satisfy the inequalities.

## Inductive Step:

Assume that the inequalities hold for any  $(s', n')$  where  $0 \leq s' < s$  or  $0 \leq n' < n$ .

Let  $S_T$  be a simple closure of stacked tangle of a  $\theta$ -curve  $T$  such that the number of simple caps is  $s$  and the number of crossings is  $n$ .

Take a **simple cap  $f$**  in  $S_T$ , joining boundary points  $P$  and  $Q$ .

### CASE 1. Suppose that $P$ and $Q$ are boundary points of a single disk.

- $S_T = S'_T \cup \bigcirc$
- $R(S_T) = (x + 1 + x^{-1})R(S'_T)$
- The number of caps is  $c - 1$  and the number of crossings  $n'$  is less than or equal to  $n$  in  $S'_T$ .

$$\max \deg_x R(S_T) = \max \deg_x R(S'_T) + 1 \leq [(c - 1) + n' - 2] + 1 \leq c + n - 2$$

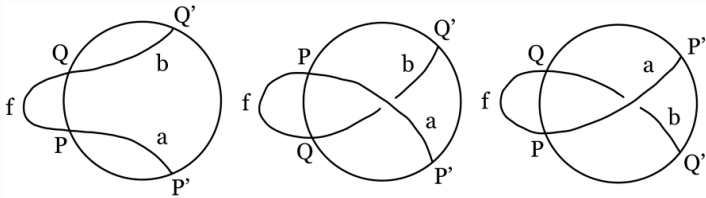
$$\min \deg_x R(S_T) = \min \deg_x R(S'_T) - 1 \geq [-(c - 1) - n' + 2] - 1 \geq -c - n + 2$$

- $S'_T$  satisfy the inequalities implies  $S_T$  satisfy the inequalities.

CASE 2. Suppose that  $P$  and  $Q$  are boundary points of different disks  $D_P$  and  $D_Q$ , respectively.

① Suppose that  $D_P$  and  $D_Q$  are adjacent disks.

- When view from above, there are three cases:



- At first case, we can reduce the simple cap  $f$ .
- After applying **Y10**, other cases can be regarded as first case.

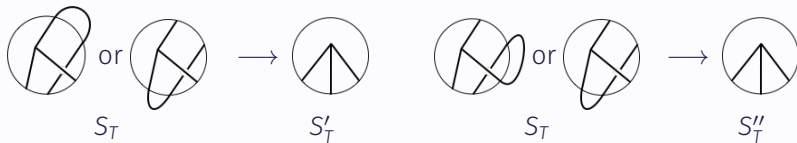
$$R(\text{cap}) = x^2 R(\cap), \quad R(\text{cap}) = x^{-2} R(\cap) \tag{Y10}$$

② Suppose that  $D_P$  and  $D_Q$  are not adjacent disks and  $D_P$  is above  $D_Q$ .

- Let  $D$  be the disk just above  $D_Q$ .
- If arcs or line segment in  $D$  and  $D_Q$  have no intersection, then we can change the position of  $D$  and  $D_Q$  without any quantities.
- We can assume that the arc in  $D_Q$  intersect arc or line segment in  $D$ , when view from above.

① There is a cap joining  $D_Q$  and  $D$ .

- $D_Q$  and  $D$  are adjacent disks.
- If  $D$  is a simple disk, then we can reduce a simple cap as Case 2-①.
- If  $D$  is a non-simple disk, then



- $R(S_T) = -x^{\pm 1}R(S'_T)$  and  $R(S_T) = x^{\pm 2}R(S''_T)$  by **Y14** and **Y10**, respectively.
- Both of  $S'_T$  and  $S''_T$  have  $s - 1$  simple caps,  $c - 1$  caps, and  $n - 1$  crossing.
- By induction hypothesis,

$$\begin{aligned}\max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 1 \\ &\leq [(c - 1) + (n - 1) - 2] \pm 1 \\ &< c + n - 2\end{aligned}$$

$$\begin{aligned}\min \deg_x R(S_T) &= \min \deg_x R(S'_T) \pm 2 \\ &\geq [-(c - 1) - (n - 1) + 2] \pm 2 \\ &\geq -c - n + 2\end{aligned}$$

② There is no cap joining  $D_Q$  and  $D$ .

- Applying Y9

$$R(\text{X}) = R(\text{X}) + (x - x^{-1}) [R(\text{O}) - R(\text{X})]$$

then

$$R(S_T) = R(S_T^-) + (x - x^{-1}) [R(S_T^0) - R(S_T^\infty)]$$

- $S_T^0$  and  $S_T^\infty$  have  $c$  caps and  $n - 1$  crossings.
- $(x - x^{-1}) [R(S_T^0) - R(S_T^\infty)]$  satisfy the inequalities.
- If  $S_T^-$  satisfy the inequalities, then  $S_T$  also satisfy the inequalities.
- The gap between  $D_P$  and  $D_Q$  is reduced in  $S_T^-$ .
- For  $S_T^-$ , investigate above cases.

This process will terminate after a finite number of investigations. It is the end of **CASE 2**.

□

### Proposition

*Let  $S_T$  be a reduced simple closure of stacked tangle of a  $\theta$ -curve  $T$  corresponding to minimal arc presentation of  $T$ . Then*

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n + 4 \leq \alpha(T)$$

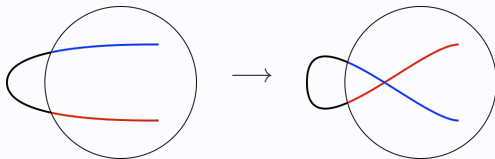
*where  $n$  is the number of crossings in  $S_T$ .*

### PROOF

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- $S_T$  is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps  $c$  in  $S_T$  is exactly arc index of  $T$ ,  $\alpha(T)$ .

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- $p$  of the caps yield a negative curl, and the remaining  $c - p$  yield a positive curl.
- $S_T^{neg}(S_T^{pos})$  is the diagram obtained by inserting the  $p$  negative( $c - p$  positive) curls.



	$S_T^{neg}$	$S_T^{pos}$
Number of Caps	$c$	$c$
Number of Crossings	$n + p$	$n + (c - p)$

$$\bullet R(S_T^{neg}) = x^{-2p}R(S_T) \text{ and } R(S_T^{pos}) = x^{2(c-p)}R(S_T)$$

$$\begin{aligned} \min \deg_x R(S_T) - 2p &= \min \deg_x R(S_T^{neg}) \\ &\geq -c + -(n + p) + 2 \end{aligned}$$

$$\begin{aligned} \max \deg_x R(S_T) + 2(c - p) &= \max \deg_x R(S_T^{pos}) \\ &\leq c + [n + (c - p)] - 2 \end{aligned}$$

$$\min \deg_x R(S_T) \geq -c - n + p + 2$$

$$\max \deg_x R(S_T) \leq n + p - 2$$

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq c + 2n - 4$$

□

# Proof of Theorem

## Theorem

Let  $T$  be any  $\theta$ -curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where  $R(T)$  is a Yamada Polynomial of the  $\theta$ -curve  $T$ .

## PROOF

Let  $S_T$  be a reduce simple closure of stacked tangle of a  $\theta$ -curve  $T$  corresponding to minimal arc presentation of  $T$ .

- The number of caps :  $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks :  $\alpha(T) - 3$

Consider the maximum number of crossings in  $S_T$ .

- number of crossings by two simple disks :  $\frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4)$
- number of crossings by a simple disk and non-simple disk :  $2 (\alpha(T) - 3)$
- number of crossings counted by disks joined by cap :  $\alpha(T)$
- number of crossings by two non-simple disks : 2

Thus

$$\begin{aligned} n &\leq \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4) + 2 (\alpha(T) - 3) - \alpha(T) + 2 \\ &= \frac{1}{2} [(\alpha(T))^2 - 5\alpha(T) + 4] \end{aligned}$$

By Lemma,

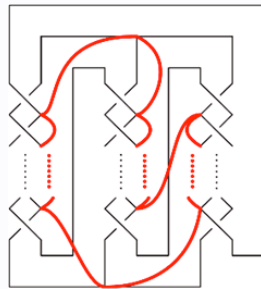
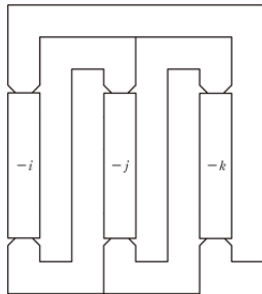
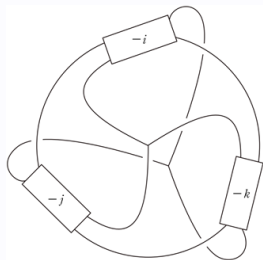
$$\begin{aligned} \max \deg_x R(S_T) - \min \deg_x R(S_T) &\leq 2n - 4 + \alpha(T) \leq [\alpha(T)]^2 - 4\alpha(T) \\ 2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} &\leq \alpha(T) \end{aligned}$$

□

## Further Studies

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# Kinoshita-Wolcott $\theta$ -Curve

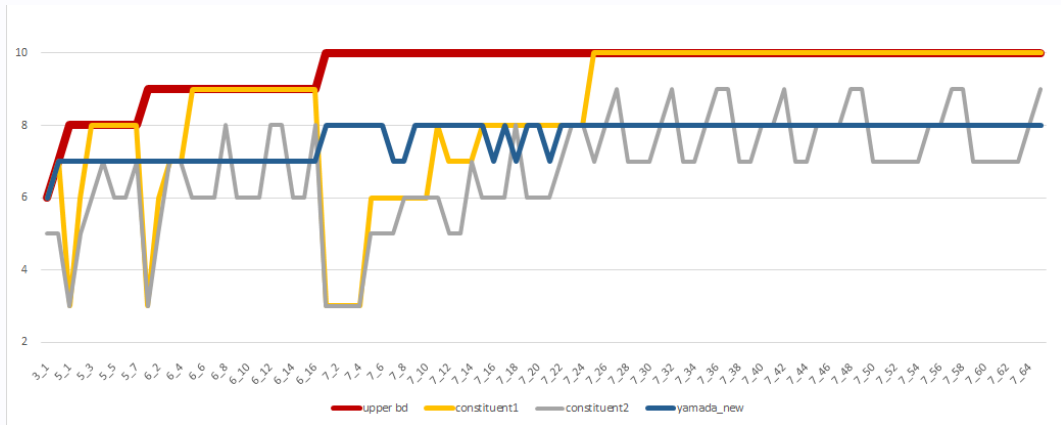


## Theorem

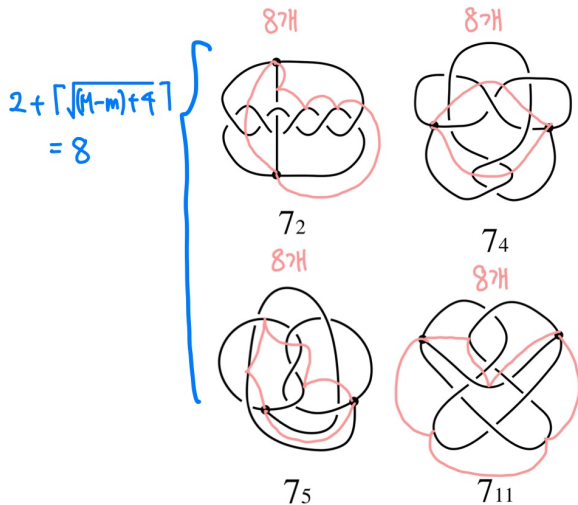
Let  $K(-i, -j, -k)$  be the Kinoshita-Wolcott  $\theta$ -curve. Then

$$\alpha(K(-i, -j, -k)) \leq i + j + k + 2$$

## Bounds of Arc Index









# Arc Index of Some $\theta$ -Curves



Thank You for Your Attention.



-  Bae, Y. and Park, C. Y. (2000).  
**An upper bound of arc index of links.**  
*Mathematical Proceedings of the Cambridge Philosophical Society*, 129(3):491–500.
-  Cromwell, P. R. (1995).  
**Embedding knots and links in an open book I: Basic properties.**  
*Topology and its Applications*, 64:37–58.
-  Cromwell, P. R. (1998).  
**Arc presentation of knots and links.**  
*BANACH CENTER PUBLICATIONS*, 42:57–64.
-  Cromwell, P. R. and Nutt, I. J. (1996).  
**Embedding knots and links in an open book II: Bounds on arc index.**  
*Mathematical Proceedings of the Cambridge Philosophical Society*, 119(2):309–319.
-  Dabrowski-Tumanski, P., Goundaroulis, D., Stasiak, A., and Sulkowska, J. I. (2024).  
 **$\theta$ -curves in proteins.**  
*Protein Science*, 33(9).
-  Jang, B., Kronaeur, A., Luitel, P., Medici, D., Taylor, S. A., and Zupan, A. (2016).  
**New examples of brunnian theta graphs.**