엇갈림수 7 이하의 세타커브와 수갑 그래프의 최소 그물 그림

조은찬[1], 신정원[1], 서보연[1], 최민호[1], 김훈[2], 진교택[3]

[1] 연구자, 한국과학영재학교

[2] 책임지도자, 한국과학영재학교

[3] 공동지도자, 한국과학기술원 수리과학과

초록. 본 연구에서는 공간 그래프(spatial graph), 특히 엇갈림수(crossing number) 7까지의 θ-커브(θ-curve)과 수갑 그래프(handcuff graph)에서의 최소 그물 그림(minimal grid diagram)을 구함으로써 그들의 호 지수(arc index)를 확정시키고자 하였다. 또한 그 과정에서 호 지수의 상한과 하한을 구할 수 있었으며, 야마다 다항식(yamada polynomial)에 기반한 파이썬 프로그램을 작성하여 이를 확인해 보았다.

중심어 : θ-커브, 수갑 그래프, 호 지수, 최소 그물 그림

Minimal Grid Diagrams of Theta-Curves and Handcuff Graphs up to 7 Crossings

Eunchan Cho[1], Jeongwon Shin[1], Boyeon Seo[1], Minho Choi[1], Hun Kim[2], Gyotaek Jin[3]

[1]Researcher, Korea Science Academy of KAIST

[2]Supervisor, Korea Science Academy of KAIST

[3]Co-Supervisor, Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology

---

Abstract. Our purpose of this research is classifying theta-curves and handcuff graphs by their arc indices, which their crossing is up to 7. We tried to find those arc indices by proving lower bounds and upper bounds of the theta-curves and handcuff graphs. Also we programmed Python code that returns the minimal grid diagram of the theta curves and handcuff graphs which have crossing up to 7. Yamada polynomial was used to prove bounds and programming the Python code.

Keywords : theta-curve, handcuff graph, arc index, minimal grid diagram

---

1 Introduction

Knot theory is a field of mathematics that studies simple closed curves embedded in three-dimensional space. Study of theta-curves and handcuff graphs is included in knot theory. In this research, we tried to get the minimal grid diagrams of those graphs, which are up to 7 crossings.

We first decided to use the Python code to examine the minimal grid diagrams of θ-curves and handcuff graphs. For this, we found the method to classify theta-curve and handcuff graphs by their Cromwell matrices, which are called THC matrices. Finding all possible matrices for those graphs and finding out the Yamada polynomial of each graphs could give the minimal grid diagram of the graph.

Also, we used some mathematical methods to found out the minimal grid diagrams by using the bounds of the arc index, and used the Python code to examine if we found the right grid diagrams. Then we tried to decrease the gap of upper and lower bounds, and found out arc indices of some graphs.

2 Theoretical Background

Definition 1. *θ-curve* is a spatial knot on 3-sphere which has 2 vertices and 3 edges.

Definition 2. *Handcuff graph* consists of 2 loops and 1 edge joining the loops.

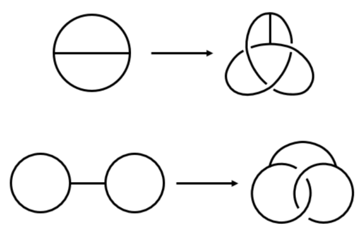

Figure 1: Examples of θ-curves and handcuff graphs.

In the projection of the handcuff graph and θ-curve, the section where they meet themselves is named *crossing*. For each handcuff graphs and θ-curves, the minimal number of crossings are called the *crossing number*. If the one graph and other graph's continuous transform of the graph is same, these graphs are *equivalent*. The *Generalized Reidemeister Moves* are used to transfrom a projection of handcuff graphs and θ-curves.

A θ-curve is said to be trivial if it can be embedded in a 2-sphere in S^3. In the similiar way, a handcuff graph is said to be trivial if it can be embedded in a 2-sphere in S^3.

Definition 3. *Arc presentation* is an open-book decomposition of $\mathbb{R}^3$ which has open half-planes as pages and the standard z-axis as the binding axis.

Every spatial graph G can be embedded in an open-book decomposition with finitely many pages so that it meets each page in exactly one simple arc with two different end-points on the binding axis. In knot theory, *arc index*, is the minimal number of pages among all possible arc presentations of graph. This arc presentation with the minimal number of pages is *minimal arc presentation. Prime knots* are knots that is not constructed by combining simpler knots. We are finding the arc index for the prime theta-curve and handcuff graph up to seven crossings.

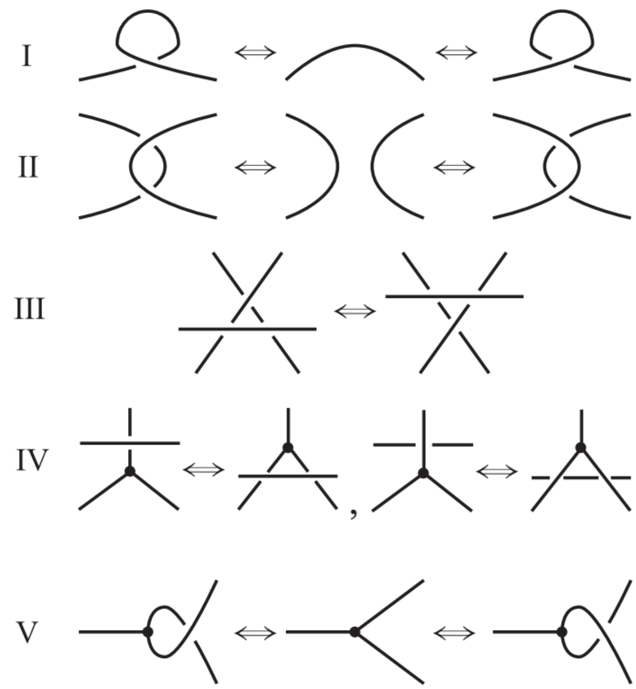

Figure 2: Generalized Reidemeister moves.

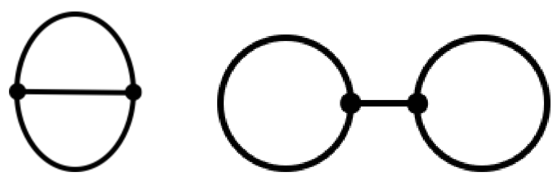

Figure 3: Trivial θ-curve and handcuff graph.

Definition 4. The *grid diagram* is a handcuff graph or theta-curve diagram of vertical strands and one less number of horizontal strands with the properties that at every crossing the vertical strand crosses over the horizontal strand and no two horizontal segments are co-linear and no two vertical segments are co-linear.

Definition 5. The *Cromwell matrix* is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s. For theta-curve and handcuff graph, its Cromwell matrix is called the the *THC-Cromwell matrix* is the matrix that satisfies the following conditions.

1. It is a binary $n \times (n+1)$ matrix.

2. It has exactly two 1s in every column.

3. There are only three 1s in two distinct rows (which are called the *Three-row*) and every other rows has exactly two 1s.

If the 1s of the Cromwell matrix are connected by horizontal and vertical lines with vertical lines are always on the horizontal lines, it leads to the grid diagram. The arc presentation can be expressed by grid diagram and vice versa. They are in one-to-one correspondence. Also, if the number of half planes in arc presentation is α, then the size of corresponding grid diagram is $(\alpha - 1) \times \alpha$.

Theorem 6. *Arc presentations exist for every θ-curve and handcuff graph.*

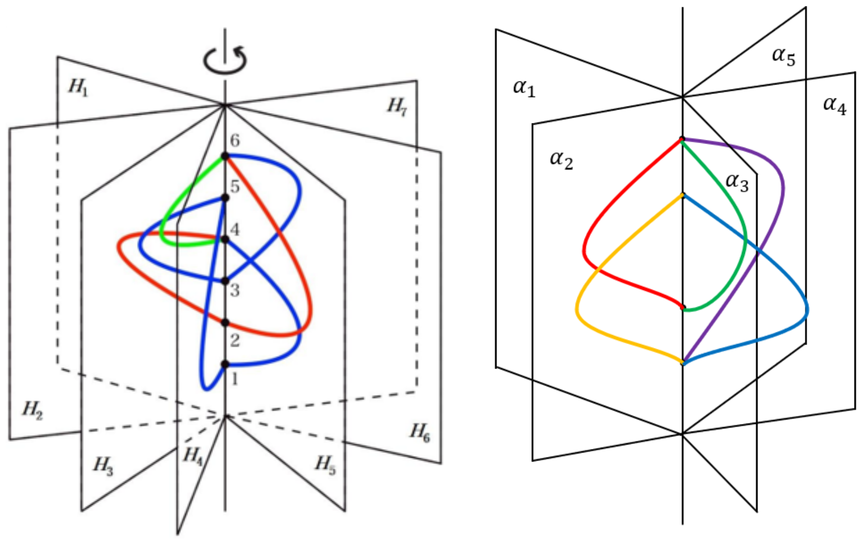

Figure 4: Arc presentation of θ-curve 3_1 and a trivial handcuff graph

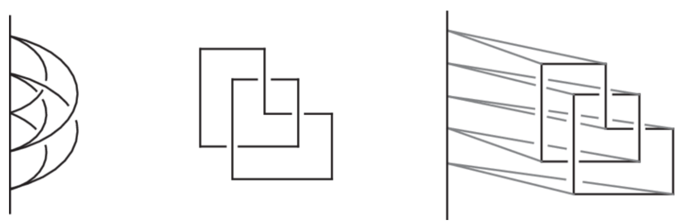

Figure 5: An arc presentation and a grid diagram of the knot

Proof. For any θ-curve and handcuff graph, we can put it on a grid by using some suitable planar isotropy. Then for every crossing it has, there are only two following cases.

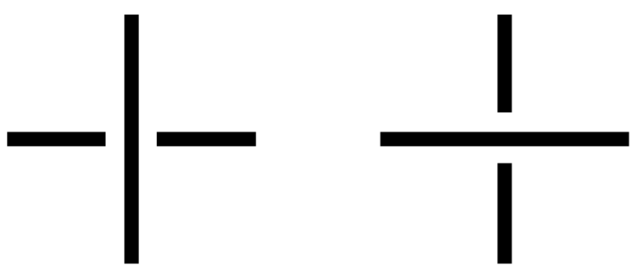

Figure 6: Cases for crossings

For each crossings, if it is a first case, then it is over. However, if it is a second case, then we can make that crossing to first case using some suitable movement.

Therefore, it becomes the grid diagram and since the grid diagrams and arc presentations are in one-to-one correspondence, there always exists arc presentations for every θ-curve and handcuff graph. $\square$

Definition 7. The *link with n-components* is an embedding of the disjoint union of n circles $S^1 \cup \cdots \cup S^1$ in $\mathbb{R}^3$. 1-component link is called a *knot*.

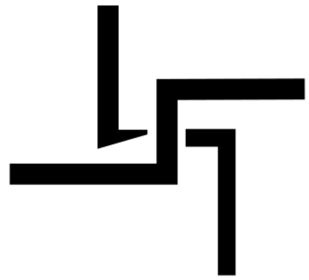

Figure 7: Suitable movement for case 2

The polynomial of the knot is invariant. *Yamada polynomial* is invariant when it is mulitplied $(-x)^n$ for some integer n. The *spread* of a knot, is substraction of maximal degree of minimal degree of the polynomial. It is a invariant since Yamada polynomial is invariant when mulitplied by $(-x)^n$.

Let 3 edges of theta-curve is e_1, e_2, e_3, then $e_1 \cup e_2$, $e_2 \cup e_3$, $e_3 \cup e_1$ becomes a knot on 3-sphere. This knots are *consituent knot* of the theta-curve.

Definition 8. *Stacked tangle* of an theta-curve and handcuff graph is stacked disks each with the frame as boundary with following properties:

- Only two disk called *non-simple disks* contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple disks is at the top.
- Other disks called *simple disks* contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point (by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point (by RV)

Simple closure of stacked tangle is a stacked tangle with *caps* satisfying following properties:

- A *cap* is a simple arc in outside of stacked tangle joining end points of arcs or line segments
- When view from above any tow caps have no intersection.

Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

A *reduced simple closure of a stacked tangle* is

- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

3 Research Methods and Procedure

First, we made a Python Code to find the arc index and the cromwell matrix of the knots. Then we used the upper and lower bounds, and also found the smaller upper bounds and larger lower bounds. Finally we used the direct method to find the arc index by drawing the binding circle.

3.1 The Python Code

The Python code generates the possible Cromwell matrices of theta-curves and handcuff graphs, and compares with the existing ones in the table which is less or equal to 7 crossings. It eventually returns the Cromwell matrix of the compared theta-curve and handcuff graph. To distinguish the Cromwell matrices from theta-curves and handcuff graphs, which is generated and removed by the rules from the above section, we should use the determinant of the Cromwell matrices.

Theorem 9. *Every theta-curve and handcuff graph has its corresponding THC-Cromwell matrix.*

Definition Let any Three-row i and its two outer '1's j, k. The *H-deletion Matrix* of THC-Cromwell matrix is $(n-1)\times(n-1)$ matrix which deleted row i and column j, k.

3.1.1 Main Theorem

Theorem 10. *The given THC-Cromwell matrix is theta-curve if and only if determinant of the H-deletion matrix is* ± 1*.The given THC-Cromwell matrix is handcuff graph if and only if determinant of the H-deletion matrix is 0 or* ± 2*.*

Proof.

1. **Method of changing grid diagram of a knot to simple matrix**

   First, we should change grid diagram of a knot into Cromwell matrix. However, we can apply row operation of interchanging the rows. In this way, determinant of the Cromwell matrix would change only by multiplying ± 1.

   By applying row operation, we should make matrix such as

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

   Let matrix is $n \times n$. Next, we apply the row operation. We add the upper rows to the most bottom row. Then,

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \\ 2 & 2 & \cdots & 2 & 2 \end{pmatrix}$$

   Lastly, by subtracting $(2i-1)$th row ($i \in \mathbb{N}$) with multiplying row by 2 and substract with most bottom row, if n is even, the most bottom row only contains 0. Since it is upper triangular matrix, we can obtain determinant by trace. Hence the last entry is 0, the determinant is 0. If n is odd, the last entry of most bottom row is 2 and other entry is all 0. Since it is upper triangular matrix, we can obtain determinant by trace too. Hence the other entry is all 1, the determinant is ± 2.

2. **Proof in the case of theta-curve**

   Do the H-deletion. Then, we can make the matrix of this grid diagram by previous section. If end vertices is erased when we delete the row, then by what vertices is erased in the other three vertices row in the grid diagram can make different matrix.

   (a) **0**

   T-shaped figure is given.

   (b) **1 (middle vertex)**

   Line-shaped figure is given.

   (c) **1 (end vertex)**

   Line-shaped figure is given.

   (d) **2 (two end vertices)**

   T-shaped figure is given.

   (e) **2 (middle and end vertices)**

   Line-shaped figure is given.

   (i) **Line-shaped figure**

   Let the matrix is given.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

   If you work on deleting rows and columns here, you can get the line-shaped figure of the following figure. If you convert this into a matrix according to the previous section,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

   is given.

   If we appropriately perform adding multiple of one row to another row and replacing the second row with the result, the determinant changes only by ± 1, and the resulting matrix will be identity matrix. If we appropriately transform other theta-curves, we can get the same result.

   (ii) **T-shaped figure**

   Let the matrix is given.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

   If you work on deleting rows and columns here, you can get the T-shaped figure of the following figure. If you convert this into a matrix according to the previous section,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   is given.

   If we appropriately perform adding multiple of one row to another row and replacing the second row with the result, the determinant changes only by ± 1, and the resulting matrix will be identity matrix. If we appropriately transform other theta-curves, we can get the same result.

Therefore, ± 1 will be the determinant of the theta-curve's H-deletion matrix.

3. **Proof in the case of handcuff graph**
   Do the H-deletion. Then, we can make the matrix of this grid diagram by upper section. If end vertices is erased when we delete the row, then by what vertices is erased in the other three vertices row in the grid diagram can make different matrix.

   (a) **0**
   Knot is given.

   (b) **1 (middle vertex)**
   Line-shaped figure is given.

   (c) **1 (end vertex)**
   It is not given since one column might have 3 vertices.

   (d) **2 (two end vertices)**
   It is not given since rows which have 3 vertices is connected with 2 line.

   (e) **2 (middle and end vertices)**
   It is not given since rows which have 3 vertices is connected with 2 line.

   (i) **Knot**
   By upper section, the determinant becomes 0 or ± 2.

   (ii) **Line-shaped figure** Let the matrix is given.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   If you work on deleting rows and columns here, you can get the line-shaped figure of the following figure. If you convert this into a matrix according to the previous section,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

   Since it is upper triangular matrix, the determinant of the matrix is trace, so the determinant of this matrix is 0.

$\square$

3.2 Bounds of Arc Index

We found the bounds of arc index by the theoretical background, or we found the new bounds inspired by the theoretical background.

Theorem 11. *Let T be any theta-curve and K_1, K_2, K_3 be three constituent knots of T. Then*

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1.$$

Theorem 12. *Let T be any theta-curve and K_1, K_2, K_3 be three constituent knots of T. Then*

$$\alpha(T) \geq \frac{1}{2}\sum_{i=1}^{3}\alpha(K_i).$$

Theorem 13. *For any theta-curve θ,*

$$\alpha(\theta) \geq \frac{1+\sqrt{f(\alpha(\theta))+36c(\theta)}}{3}\left(f(x) = \begin{cases} 73 & x \equiv 0 \pmod 6 \\ 4 & x \equiv 1 \pmod 6 \\ 25 & x \equiv 2 \pmod 6 \\ -8 & x \equiv 3 \pmod 6 \\ 49 & x \equiv 4 \pmod 6 \\ 20 & x \equiv 5 \pmod 6 \end{cases} \right).$$

Theorem 14. *For theta-curve θ, if $c(\theta)$ is the crossing number, then*

$$\alpha(\theta) \leq c(\theta)+3.$$

Theorem 15. *Let T be any theta-curve. Then*

$$2+\sqrt{\max\deg_x R(S_T) - \min\deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada polynomial of the theta-curve T, and S_T be a reduce simple closure of stacked tangle of a theta-curve T corresponding to minimal arc presentation of T.

Definition 16. The *non-simple cap* is the cap that connects simple disk and non-simple disk or two non-simple disks.

Theorem 17. *The number of the simple cap of the graph G is invariant if least one of the edges are not crossed with others when the projection has minimal crossing, except the trivial theta-curve.*

Proof. To find the number of the non-simple cap from the grid diagram, we can draw the horizontal lines which include 1 from row that has three 1s. Since non-simple cap connects 1s from rows which have three 1s, the number of horizontal lines is the number of the non-simple caps. When the edge that directly connects two vertices is not crossed, the number of edges that starts from the vertices, which is 5, is the number of the horizontal lines. It can be shown by drawing the binding circle of the knot. Since the binding circle crosses the crossed area of two other edges, the number of edges that starts from the vertices is 5, 2 from each vertices and 1 that directly connects the vertices, except the trivial theta-curve. Since $\alpha(G))$ is the number of caps, the number of simple cap is $\alpha(G)-5$. □

Definition 18. In a handcuff curve, the *vertex edge* is an edge that is connected to both vertices.

Definition 19. In a handcuff curve, the *link component* is a union of the loops from each vertex to itself.

Theorem 20 (Cromwell, P. R.)**.** *If L is an alternating and non-split link, then*

$$\alpha(L) = c(L)+2.$$

Theorem 21 (Lee, M. J., No, S., & Oh, S.)**.** *For any spatial graph H,*

$$\alpha(H) \leq c(H)+e+b,$$

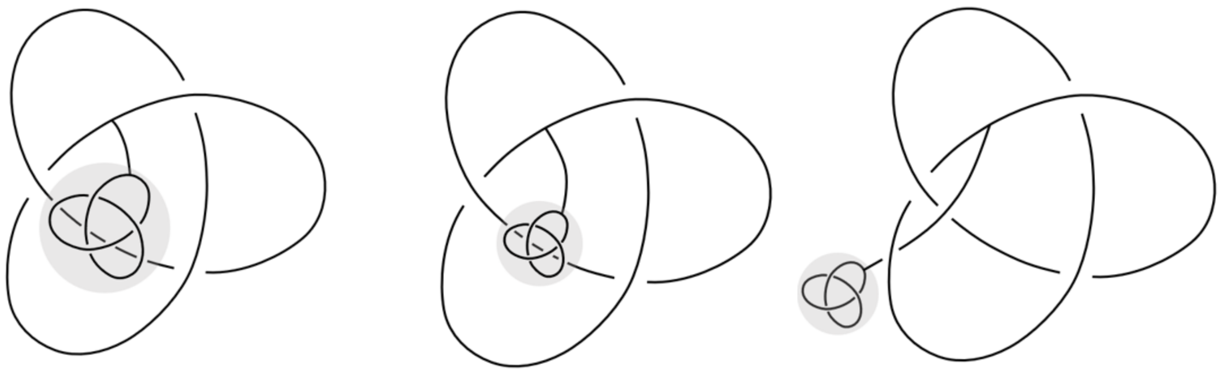

Figure 8: caption

where e is the number of the edge and b is the number of the bouquet.

Corollary 22. *If H is a handcuff curve,*

$$\alpha(H) \leq c(H) + 5.$$

Especially, if the link component of H is non-split,

$$\alpha(H) \leq c(H) + 3.$$

Proof. If H is a handcuff curve, the number of the edge is 3, and the number of the bouquet is at most 2. Thus the first inequality holds since $e = 3, b \leq 2$. If there is a bouquet, one of the loops can be pulled out without affecting the rest. (See figure 1.) Then, if we remove the vertex edge of H, the remaining link component is a split link. Thus there are no bouquet if the link component of H is non-split, and we have second inequality since $e = 3, b = 0$. □

Proposition 1. *For a handcuff curve H, let L be a link component of H. Then,*

$$\alpha(H) \geq \alpha(L) + 1.$$

Proof. In the arc presentation of H, let v_1, v_2 be the vertices of H. Then, there are half-planes that contain the vertex edge of H. If we remove them, the remainder is the arc presentation of L, the link component of H. Since the number of half-planes that contain the vertex edge is at least 1, we obtain

$$\alpha(H) \geq \alpha(L) + (\text{the number of half plane that contain vertex edge}) \geq \alpha(L) + 1.$$

□

Using the Theorem 1, we obtain the following corollary.

Corollary 23. *In the handcuff curve, if the link component L is alternating and non-split, then*

$$\alpha(H) \geq c(L) + 3.$$

Proof. Since L is alternating and non-split link, $\alpha(L) = c(L) + 2$ by Theorem 1. Thus,

$$\alpha(H) \geq \alpha(L) + 1 = (c(L) + 2) + 1 = c(L) + 3$$

according to Proposition 1. □

Combining the above corollary and the corollary of Theorem 2, we have the following theorem.

Theorem 24. *For the handcuff curve H, if the link component L is alternating and non-split, then*

$$\alpha(H) = c(L) + 3.$$

We now consider the Yamada polynomial of the handcuff curve and investigate the relationship between the difference of its maximum and minimum degrees and the arc index.

Definition 25. For a graph $G = (V, E)$, where V is vertex set of G and E is edge set of G, let us define 2-variable Laurent polynomial

$$h(G)(x, y) = \sum_{F \subset E} (-x)^{-|F|} x^{\mu(G-F)} y^{\beta(G-F)}$$

where $\mu(G)$ and $\beta(G)$ is the number of connected components of G and the first Betti number of G.
Then, the Yamada polynomial of a graph G, $R(G)$ is defined by

$$R(G)(x) = h(G)(-1, -x - 2 - x^{-1}).$$

It is known that, for some integer n, the product $(-x)^n R(G)$ (where $R(G)$ is the Yamada polynomial of the spatial graph G) is an ambient isotopy invariant. Moreover, the Yamada polynomial satisfies the following properties.

Theorem 26. *For the Yamada polynomial, the following properties hold.*

1. $R(\cdot) = -1$
2. *Let e be a non-loop edge of a graph G. Then, $R(G) = R(G/e) + R(G - e)$, where G/e, $G - e$ denote the graphs obtained by contracting and deleting the edge e, respectively.*
3. *Let e be a loop edge of a graph G. Then, $R(G) = -(x + 1 + x^{-1})R(G - e)$.*
4. *Let $G_1 \cup G_2$ be a disjoint union of graphs G_1 and G_2. Then, $R(G_1 \cup G_2) = R(G_1)R(G_2)$.*
5. *Let $G_1 \cdot G_2$ be a union of graphs G_1 and G_2 having one common point. Then, $R(G_1 \cdot G_2) = -R(G_1)R(G_2)$.*
6. *If G has an isthmus, then $R(G) = 0$.*

Theorem 27. *For the Yamada polynomial, the following properties hold.*

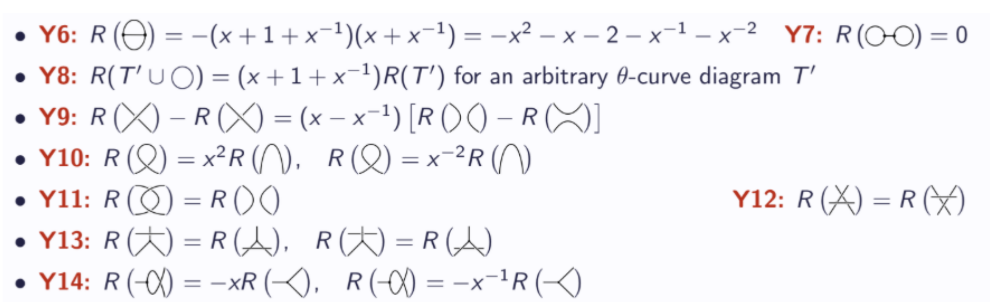

Figure 9: Properties of Yamada polynomial.

Now, using the stacked tangle representation and the Yamada polynomial, we will prove the following theorem. The following theorem gives a lower bound for the arc index in terms of the Yamada polynomial.

Theorem 28. *Let S_T be the closure of stacked tangle of theta curve or handcuff curve. Then,*

$$spr(R(S_T)) \leq 2n+2$$

where n is the number of crossings in S_T and $\operatorname{spr}(f)$ *denotes the spread of f.*

Corollary 29. *If G is a theta curve or handcuff curve, then*

$$\alpha(G) \geq \frac{5+\sqrt{4\operatorname{spr}(R(G))-15}}{2},$$

except when G is the trivial theta curve.

Proof. Let S_T be the closure of stacked tangle of theta curve or handcuff curve. Let c_s, c_{ss}, c_n, n be the number of simple cap, semi-simple cap, non-simple cap and the number of crossings in S_T, respectively. We will prove the theorem using the mathematical induction on the pair $(c_s + c_{ss}, n)$, ordered lexicographically.

1. Basis cases
   First, suppose $c_s + c_{ss} = 0$. Then, since there are no simple disks, S_T must be the trivial theta curve. In this case, the number of crossing is at least 1. Since the spread of Yamada polynomial of trivial theta curve is 4, $\operatorname{spr}(R(S_T)) = 4 \leq 2n+2$.
   Second, suppose $n = 0$. Then, S_T must be the disjoint union of trivial handcuff graph and possibly some circles(unlink). Since the Yamada polynomial of trivial handcuff is zero, $\operatorname{spr}(R(S_T)) = 0 \leq 2 = 2n+2$.
   Hence, basis step is proven.

2. Inductive step
   Assume that the theorem holds for all pairs $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$, and suppose $c_s + c_{ss} > 0$. Then the top disk must have a semi-simple cap. Let the disk connected to the top disk be denoted by D_s. (Note that if the top disk had only a non-simple cap, then there would be no simple or semi-simple cap at all.) Since D_s is not top disk, there exists a disk D directly above D_s. Now we consider two cases, depending on whether there exists a cap between D_s and D. First, suppose that there is no cap between D_s and D. Then, there are two possibilities: there is an intersection between D_s and D, or there is not.
   If there is no intersection between D_s and D, we can swap the position of D_s and D without affecting the rest of the diagram. If there is an intersection between D_s and D, we can use the relation

   $$R\left(\right) - R\left(\right) = (x - x^{-1})\left(R\left(\right) - R\left(\right)\right).$$

   In S_T, let S'_T, S^0_T and S^∞_T be the diagrams obtained by replacing the crossing with , and , respectively. Since both $R\left(S^0_T\right)$ and $R\left(S^\infty_T\right)$ have c_s simple disks, c_{ss} semi-simple disks, and $n-1$ crossings, their spread is at most $2(n-1)+2 = 2n$ by the induction hypothesis. Therefore, the spread of $(x - x^{-1})\left(R\left(S^0_T\right) - R\left(S^\infty_T\right)\right)$ is at most $2n+2$. It is thus sufficient to prove that the spread of $R\left(S'_T\right)$ is at most $2n+2$. Now, observe that S'_T can be interpreted as the diagram obtained by swapping the positions of D_s and D in S_T. Therefore, in both cases, it is enough to prove that the inequality holds after this swapping. However, after each swap, the distance between the top disk and D_s decreases. Hence, we can continue swapping D_s with the disk directly above it until either the disk above D_s is the top disk, or there is a cap between D_s and the disk above it. Then it suffices to consider the case where there is a cap between D_s and D.
   Now, consider the case where there is a cap between D_s and D. There are two cases: either D_s is simple or it is non-simple. If D_s is simple, we can reduce the number of caps using Reidemeister moves, as illustrated in Figure 2. Its spread is at most $2n+2$ by the induction hypothesis. Therefore, the spread of S_T is at most

$2n+2$. If D_s is non-simple, the number of caps can be similarly reduced using the Reidemeister moves, as shown in Figure 3.

Hence, by the base cases and the inductive step, the theorem follows by mathematical induction.

Corollary follows by bounding the number of crossings n in terms of the arc index $\alpha(G)$. Suppose G is a theta curve, let RS_T be a reduced stacked tangle with no nested caps. Since the stacked tangle has 2 non-simple disks and $\alpha(G)-3$ simple disks, the following holds.

1. The number of crossings between simple disks is at most $\binom{\alpha(G)-3}{2}$.
2. The number of crossings between simple and non-simple disks is at most $2 \cdot (\alpha(G)-3)$.
3. The number of crossings between the two non-simple disks is at most 2, except for the trivial theta curve.

However, for each cap, the two arcs that it connects have no crossing since RS_T has no nested caps. Moreover, any pair of simple arcs is connected by at most one cap. Note that between the two non-simple disks, there are at most two caps, except in the trivial theta curve case. Therefore, we obtain the following upper bound on n in terms of $\alpha(G)$:

$$n \leq \binom{\alpha(G)-3}{2} + 2(\alpha(G)-3) + 2 - (\alpha(G)-2) = \frac{1}{2}(\alpha(G)^2 - 5\alpha(G) + 8).$$

Using the above theorem, we have the following inequality:

$$\alpha(G) \geq \frac{5 + \sqrt{4\mathrm{spr}(R(G)) - 15}}{2}.$$

Now suppose G is a handcuff curve. Similarly, the following holds.

1. The number of crossings between simple disks is at most $\binom{\alpha(G)-3}{2}$.
2. The number of crossings between simple and non-simple disks is at most $2 \cdot (\alpha(G)-3)$.
3. The number of crossings between the two non-simple disks is at most 1.

Note that, in contrast to the theta case, some two arcs in a handcuff curve can be connected to two caps simultaneously when a simple arc intersects a non-simple one, because there are two loops in a handcuff curve. Moreover, since a handcuff curve has only one edge connecting the two vertices, there can be at most one cap between the two non-simple disks. Thus, the number of crossings satisfies the same upper bound:

$$n \leq \binom{\alpha(G)-3}{2} + 2(\alpha(G)-3) + 1 - (\alpha(G)-1-2) = \frac{1}{2}(\alpha(G)^2 - 5\alpha(G) + 8),$$

and we obtain the following inequality:

$$\alpha(G) \geq \frac{5 + \sqrt{4\mathrm{spr}(R(G)) - 15}}{2}.$$

□

4 Research Result

The Python code did not work since the Topoly Library had error of the Yamada polynomial function. However, we were able to examine some of our answers by the Python code. We used the bounds from theoretical background, and used the Python code to find the arc index. If we could not find the arc index by the computer, we directly found the arc index by drawing the binding circle of the theta-curve. Also, we were able to found the bounds of the arc index of the handcuff graphs. The following result of the graphs with arc index is at below. The graphs with confirmed arc index is colored in green.

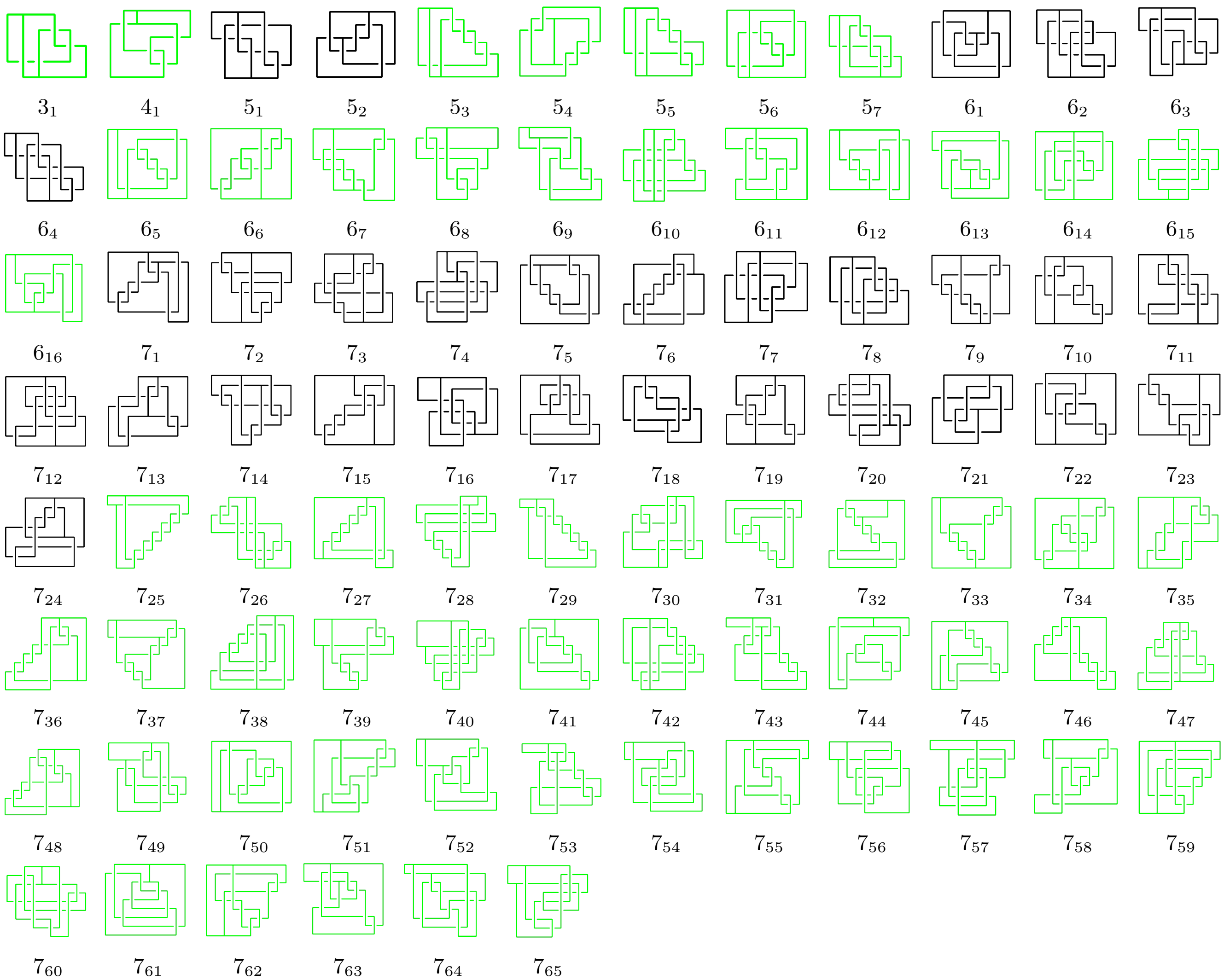

Figure 10: Grid Diagram of the Theta-Curves Up to 7 Crossings

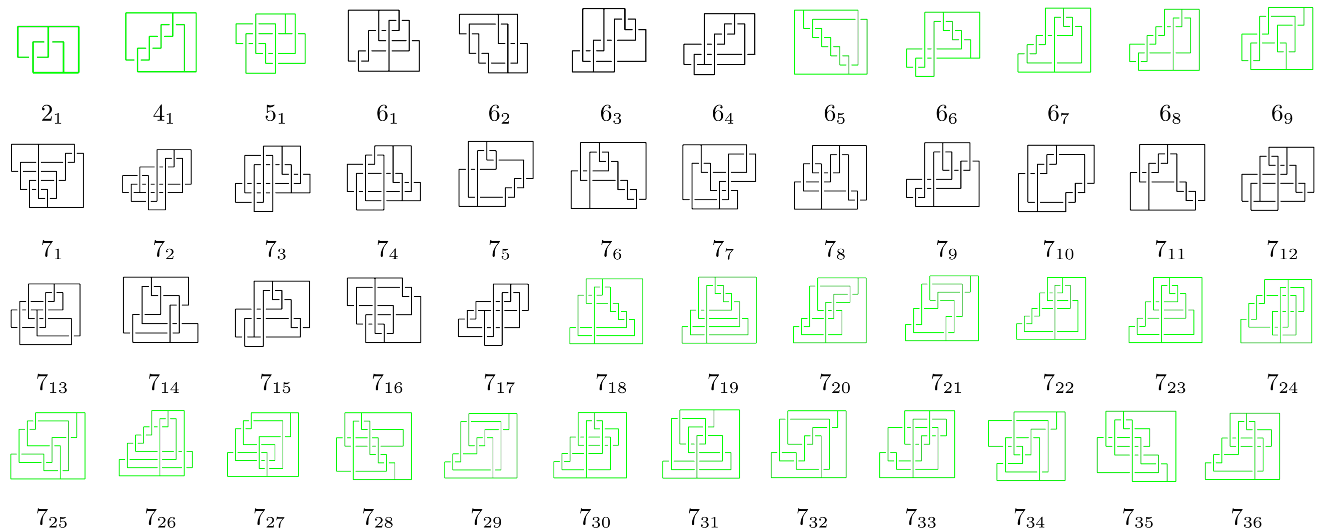

Figure 11: Grid Diagram of the Handcuff Graphs Up to 7 Crossings

5 Conclusion

요약, 더 나아가서 어떻게 써먹을 수 있을지?

References

1. Yoonsang Lee. (2023). *A Study on Arc Index of Theta-Curves*. Korea Science Academy of KAIST

2. Lee, H. J., Hong, K., Lee, H., & Oh, S. (2014). Mosaic number of knots. *Journal of Knot Theory and Its Ramifications, 23*(13), 1450069.

3. Lee, M. J., No, S., & Oh, S. (2018). Arc index of spatial graphs. *Journal of Graph Theory, 90*(3), 406–415.

4. Moriuchi, H. (2019). A table of θ-curves and handcuff graphs with up to seven crossings. *Advanced Studies in Pure Mathematics*, 281–290.

5. Yamada, S. (1989). An invariant of spatial graphs. *Journal of Graph Theory, 13*(5), 537–551.

6. Cromwell, P. R. (1995b). Embedding knots and links in an open book I: Basic properties. *Topology and Its Applications, 64*(1), 37–58.