The Arc Index of Theta-Curve and Handcuff Graph

Eunchan Cho[1] Jeongwon Shin[1] Boyeon Seo[1] Minho Choi[1]

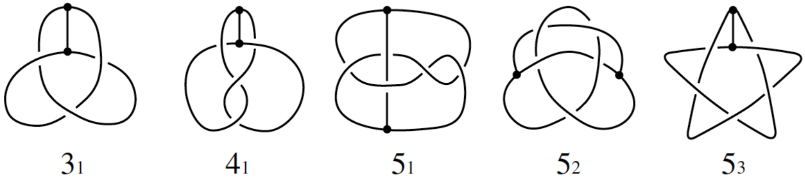
SEP 6, 2025

[1]Korea Science Academy of KAIST

Introduction

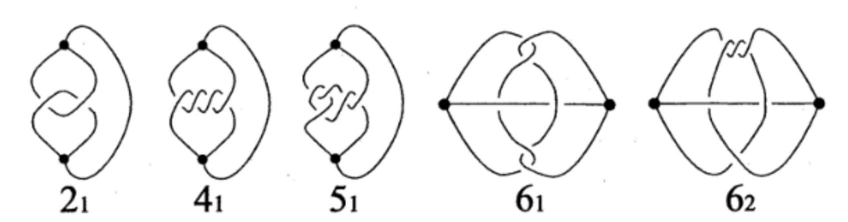
Theta-Curves

- A **theta-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.
- Theta-curves are roughly classified by comparing the triples of constituent knots.
- A theta-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3.

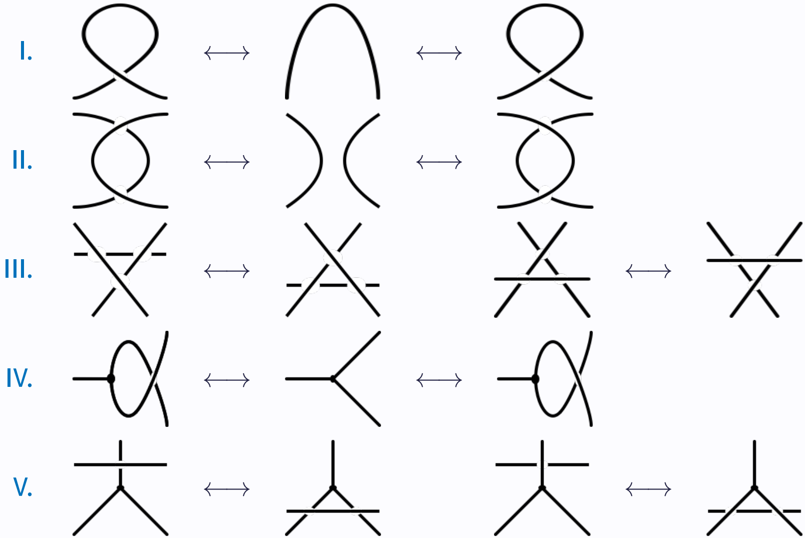


Handcuff Graphs

- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.
- A **constituent link H_{12}**, is a subgraph of H that consists of two vertices v_1, v_2 and two edges e_1, e_2.



Reidemaister Moves for Theta-Curves and Handcuff Graphs

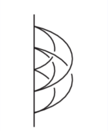


Arc Presentation

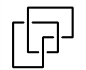
- **Arc presentation** of a theta-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called **pages**).
- The embedding is with the common boundary line (called **axis**).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- **Arc index**, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is **minimal arc presentation**.

Arc Presentation

Trefoil

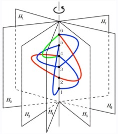


Open Book



Grid Diagram

$\theta_{5,2}$



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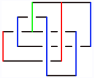
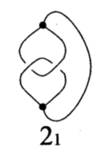
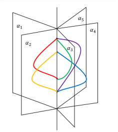


Grid Diagram



$\Phi_{2,1}$



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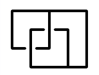


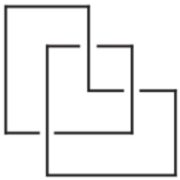
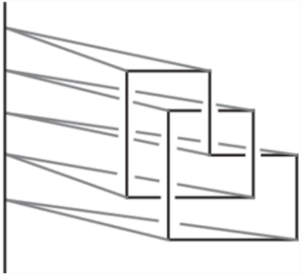
Grid Diagram

Grid Diagram

- The **grid diagram** of theta-curve or handcuff graph is a diagram with only vertical strand and horizontal strands.
- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Grid Diagram

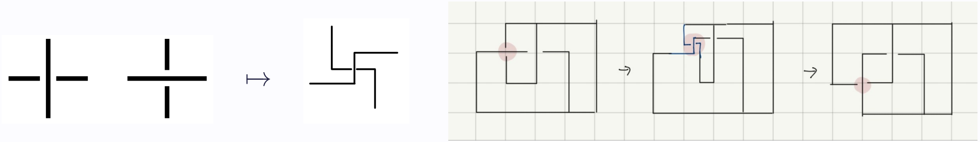
- A grid diagram gives rise to an arc presentation and vice versa.


Theorem

Every theta-curve and handcuff graph admit a grid diagram.

Proof



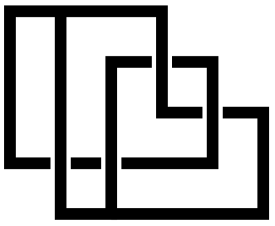
Corollary

Every theta-curve and handcuff graph admit a arc presentation.

Classifying by Determinant

THC-cromwell matrix

- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ-curves and handcuff graphs



$$\longmapsto \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

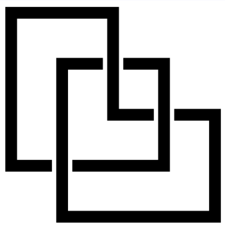
Determinant of the cromwell matrices of Knot

Theorem

Let K be any knot then its determinant of the cromwell matrix is 0 or ± 2.

Proof

$\xmapsto{\text{grid diagram}}$



$\xmapsto{THC-cromwell}$

$$\xmapsto{THC-cromwell} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

CASE 1. When n is an even number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the determinant of K is 0.

CASE 2. When n is an odd number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

So the determinant of K is ± 2. $\square$

H-deletion of THC-cromwell matricies

- The **H-deletion** Matrix of the THC-cromwell matrix G is $(n-1)\times(n-1)$ matrix which deleted vertex-row and its two side-rows from the matrix G.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Determinant of the THC-cromwell matrices

Theorem

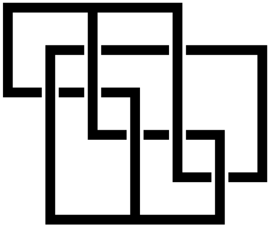
Let M be any THC-cromwell matrice of θ-curve or handcuff graph.

- *$det(M) = \pm 1 \iff M$ represents θ-curve*
- *$det(M) = 0$ or $\pm 2 \iff M$ represents handcuff graph*

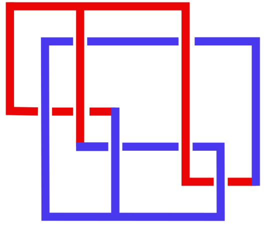
**det(M) = determinant of H-deletion matrix of M*

Proof

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longmapsto$$



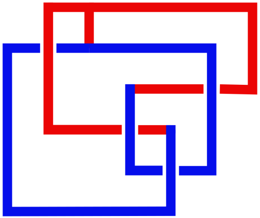
$\longmapsto$



CASE 1. When M represents θ-curve

ⓘ Line-shape

$$
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\longmapsto
$$



$$
\xmapsto{H-\text{deletion}}
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

and simplify with

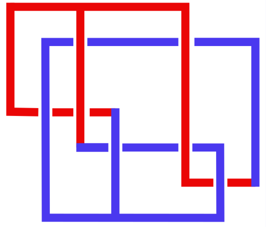
$$
\xmapsto[\text{operations}]{\text{row/column}}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\xmapsto{\text{subtracting}}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

So $\det(M) = \pm 1$

CASE 1. When M represents θ-curve

ⓘⓘ T-shape

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

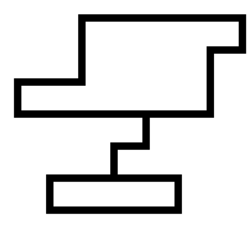
and simplify with

$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xmapsto{\text{regioning}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

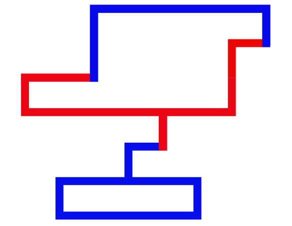
So $\det(M) = \pm 1$

CASE 2. When M represents handcuff graph

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



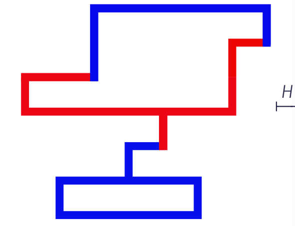
$$\longmapsto$$



CASE 2. When M represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and simplify with

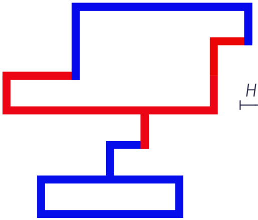
$$\xmapsto[operations]{row/column} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xmapsto{seperating} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So det(M) = 0 or ± 2

CASE 2. When M represents handcuff graph

ⓘ T-loop & Line-shape

$$\begin{bmatrix} 0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&1&1\\ 1&0&1&0&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 0&0&0&1&1&0&0&0\\ 0&1&0&1&0&1&0&0\\ 0&1&0&0&0&1&0&0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 0&1&0&0&0&1\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&1&0&0\\ 1&0&1&0&1&0\\ 1&0&0&0&1&0 \end{bmatrix}$$

and simplify with

$$\xmapsto{operations} \begin{bmatrix} T-loop & *\\ 0 & Line-shape \end{bmatrix} \xmapsto{seperating} \begin{bmatrix} Knot & * & *\\ 0 & Line-shape & *\\ 0 & 0 & Line-shape \end{bmatrix}$$

So $\det(M)$ = 0 or ± 2

CASE 2. When M represents handcuff graph

(ii) Knot & Line-shape

cromwell matrix $\xmapsto{H-deletion}$ H-deletion matrix $\xmapsto[operations]{seperating}$ $\begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$

So det(M) = 0 or ± 2