

The Determinant and Arc Indices of θ -Curves and Handcuff-Graphs

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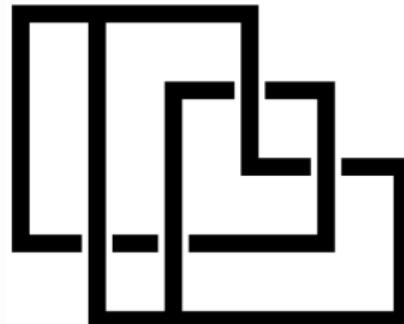
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Determinant of θ -curve and Handcuff graph

THC-cromwell matrix

- The **Cromwell Matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ -curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Determinants of the cromwell matrices of Knot

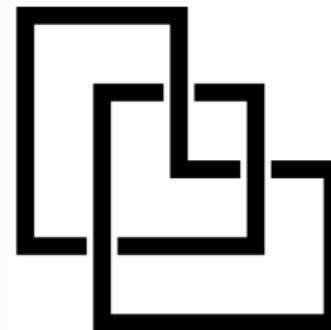
Theorem

Let K be a knot. Then the determinant of a cromwell matrix of K is 0 or ± 2 .

PROOF



grid diagram



Cromwell

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

row/column
operations

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof of Theorem

CASE 1. When n is an even number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the determinant of K is 0.

CASE 2. When n is an odd number.

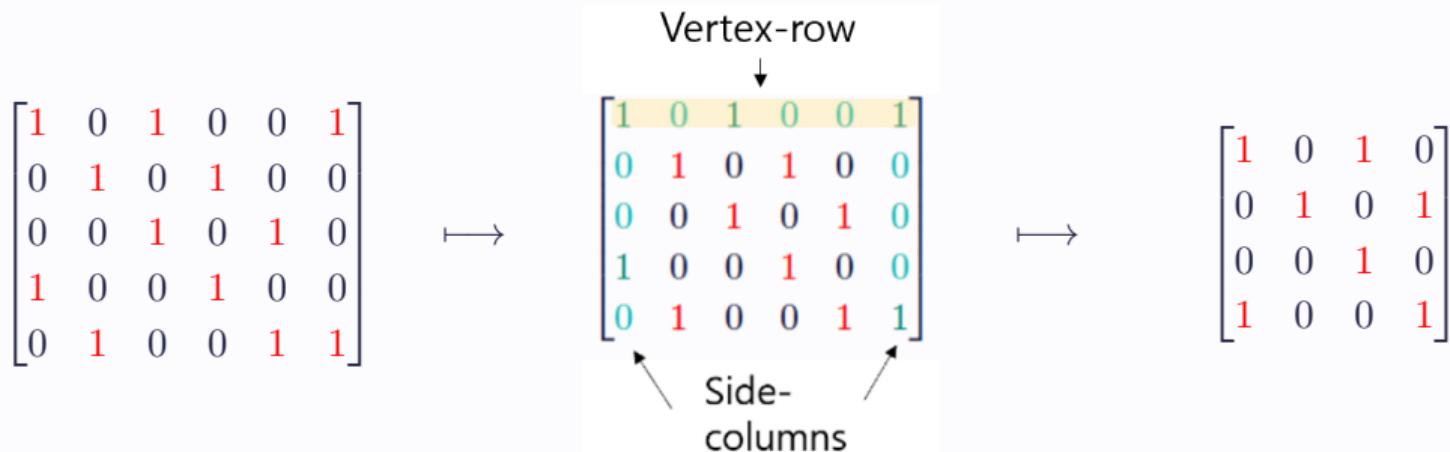
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

So the determinant of K is ± 2 .

□

H-deletion of THC-cromwell matrices

- The **vertex-row** of THC-cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic} , where $a < b < c$, as its elements.
- The **side-column** of THC-cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-cromwell matrix G is $(n - 1) \times (n - 1)$ matrix which deleted vertex-row and its two side-columns from the matrix G .



Determinants of the THC-cromwell matrices

Theorem

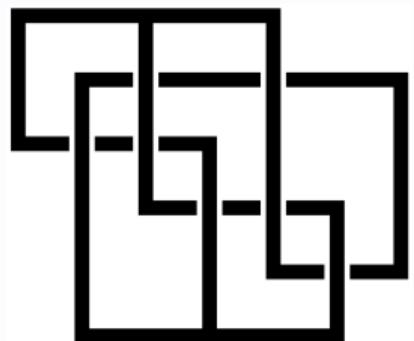
Let M be a THC-cromwell matrix of θ -curve or handcuff graph.

- $\det^*(M) = \pm 1 \iff M \text{ represents } \theta\text{-curve}$
- $\det^*(M) = 0 \text{ or } \pm 2 \iff M \text{ represents handcuff graph}$

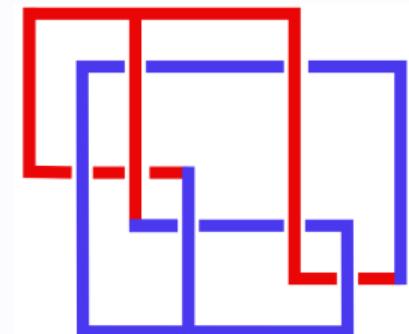
where $\det^*(M)$ = determinant of H -deletion matrix of M

PROOF

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$

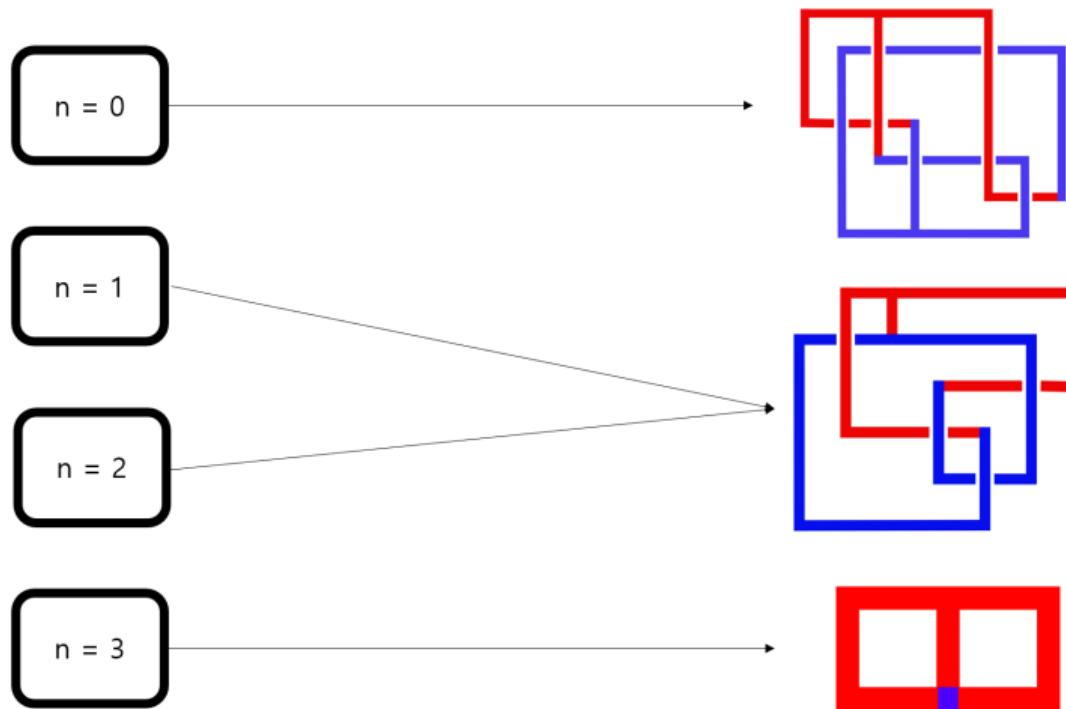


H -deletion



Proof of Theorem

CASE 1. When M represents θ -curve

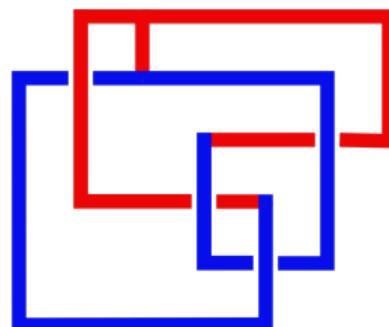


Proof of Theorem

CASE 1. When M represents θ -curve

i) Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow$$



$$\xrightarrow{H\text{-deletion}} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\xrightarrow{\text{row/column operations}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\xrightarrow{\text{subtracting}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

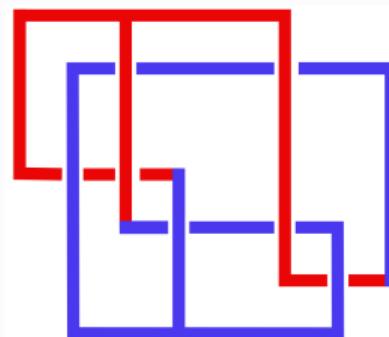
So $\det^*(M) = \pm 1$

Proof of Theorem

CASE 1. When M represents θ -curve

(ii) T-shape

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$



H -deletion

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$\xrightarrow{\text{row/column operations}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{\text{regioning}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det^*(M) = \pm 1$