



# The Arc Index of Theta-Curve and Handcuff Graph

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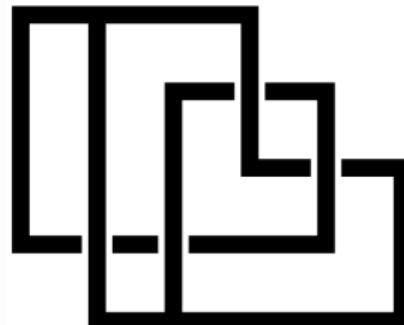
## Determinant of $\theta$ -curve and Handcuff graph

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## THC-cromwell matrix

- The **Cromwell Matrix** of a knot is an  $n \times n$  binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into  $\theta$ -curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

# Determinant of the cromwell matrices of Knot

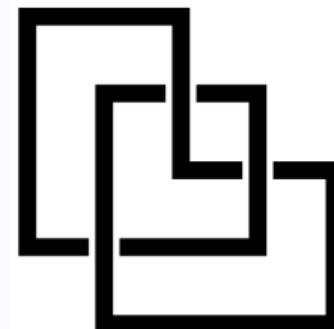
## Theorem

Let  $K$  be a knot. Then the determinant of a cromwell matrix of  $K$  is 0 or  $\pm 2$ .

## PROOF



grid diagram



Cromwell

Cromwell

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

row/column  
operations

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

### CASE 1. When $n$ is an even number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the determinant of  $K$  is 0.

### CASE 2. When $n$ is an odd number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

So the determinant of  $K$  is  $\pm 2$ .

□

## H-deletion of THC-cromwell matrices

- The **vertex-row** of THC-cromwell matrix  $M$  is a row which contains three 1s,  $M_{ia}, M_{ib}, M_{ic}$ , where  $a < b < c$ , as its elements.
- The **side-column** of THC-cromwell matrix  $M$  is a column which contains the leftmost 1 of vertex-row ( $M_{ia}$ ) or the rightmost 1 of vertex row ( $M_{ic}$ ).
- The **H-deletion** Matrix of the THC-cromwell matrix  $G$  is  $(n - 1) \times (n - 1)$  matrix which deleted vertex-row and its two side-columns from the matrix  $G$ .

The diagram illustrates the H-deletion process of a THC-cromwell matrix  $G$ . It shows three matrices connected by arrows:

- Matrix  $G$ :**  $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$
- Matrix  $M$ :**  $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$  (Note: The first row of  $M$  is highlighted in red, while the first row of  $G$  is highlighted in green.)
- Matrix  $H$ :**  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Annotations indicate the components of the middle matrix  $M$ :

- An arrow labeled "Vertex-row" points to the first row of  $M$ .
- Two arrows labeled "Side-columns" point to the first and last columns of  $M$ .

# Determinant of the THC-cromwell matrices

## Theorem

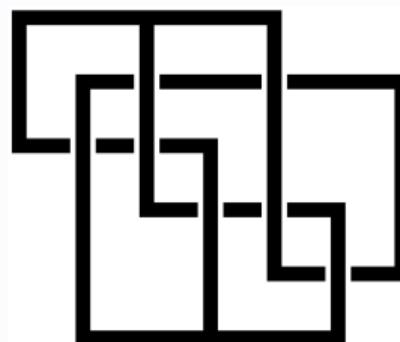
Let  $M$  be any THC-cromwell matrix of  $\theta$ -curve or handcuff graph.

- $\det^*(M) = \pm 1 \iff M \text{ represents } \theta\text{-curve}$
- $\det^*(M) = 0 \text{ or } \pm 2 \iff M \text{ represents handcuff graph}$

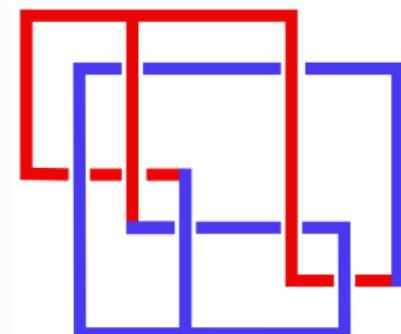
where  $\det^*(M)$  = determinant of  $H$ -deletion matrix of  $M$

## PROOF

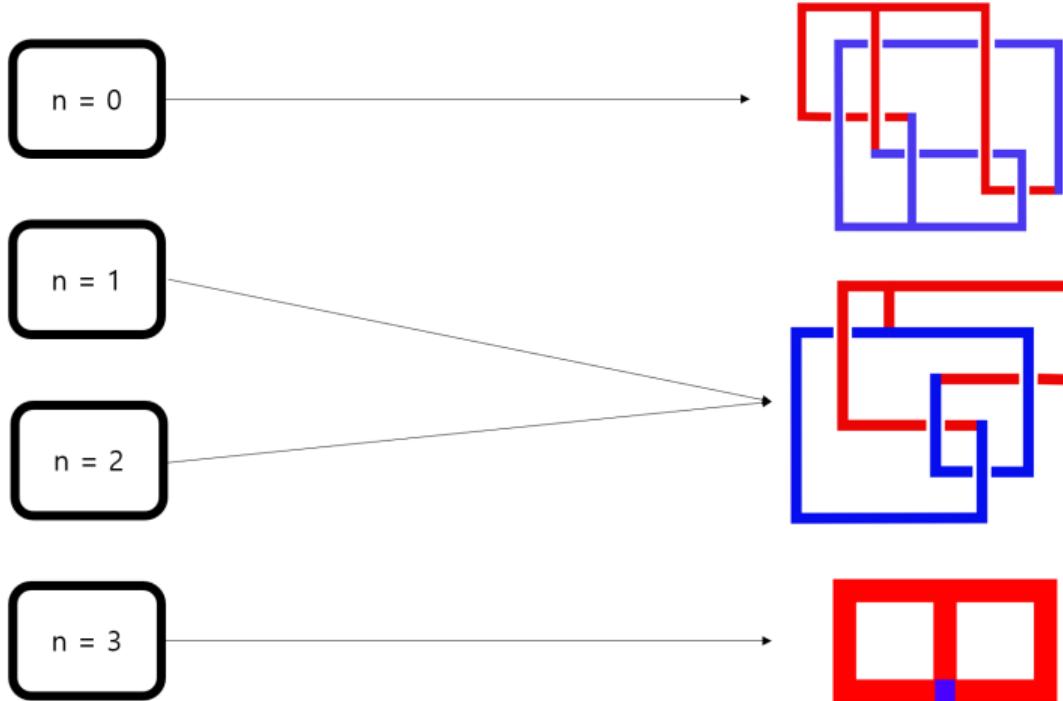
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



$H$ -deletion



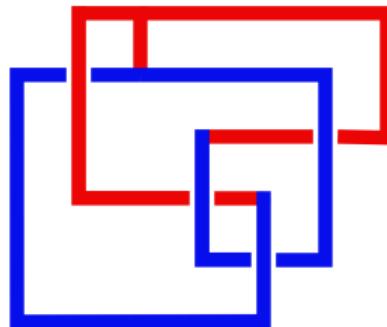
## CASE 1. When $M$ represents $\theta$ -curve



## CASE 1. When $M$ represents $\theta$ -curve

### i Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow$$



$$\xrightarrow{H\text{-deletion}} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and simplify with

$$\xrightarrow[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{subtracting}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So  $\det(M) = \pm 1$

## CASE 1. When $M$ represents $\theta$ -curve

ii) T-shape

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \text{red T-shape} \\ \text{blue T-shape} \end{array} \xrightarrow{H\text{-deletion}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and simplify with

$$\xrightarrow{\substack{\text{row/column} \\ \text{operations}}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{regioning}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So  $\det(M) = \pm 1$

