The Determinants and Arc Indices of θ-Curves and Handcuff Graphs

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NOV 22, 2025
R&E 2025

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Introduction

θ-Curves and Handcuff Graphs

- A **θ-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.

3_1 4_1 5_1 5_2 5_3

θ-Curves

2_1 4_1 5_1 6_1 6_2
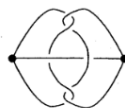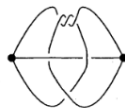
Handcuff Graphs
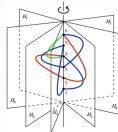
Arc Presentation

Trefoil ⟺ Arc Presentation ⟺ Grid Diagram

$\theta_{5,2}$ ⟺ Arc Presentation ⟺ Grid Diagram

$\Phi_{2,1}$ ⟺ Arc Presentation ⟺ Grid Diagram

Arc Presentation

Arc presentation of a θ-curve or handcuff graph is an embedding of them.

- It is contained in the union of finitely many half planes (called pages).
- The embedding is with the common boundary line (called axis).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.

Arc index is the minimal number of pages among all possible arc presentations of graph.

- Arc presentation with the minimal number of pages is called minimal arc presentation.
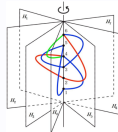
Grid Diagram

|		
Trefoil	Arc Presentation	Grid Diagram
$\theta_{5,2}$	Arc Presentation	Grid Diagram
$\Phi_{2,1}$	Arc Presentation	Grid Diagram

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Grid Diagram

A **grid diagram** of θ-curve or handcuff graph is a diagram with only vertical and horizontal strands.

- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Arc Presentation of the θ-Curve and Handcuff Graph

- A grid diagram gives rise to an arc presentation and vice versa.
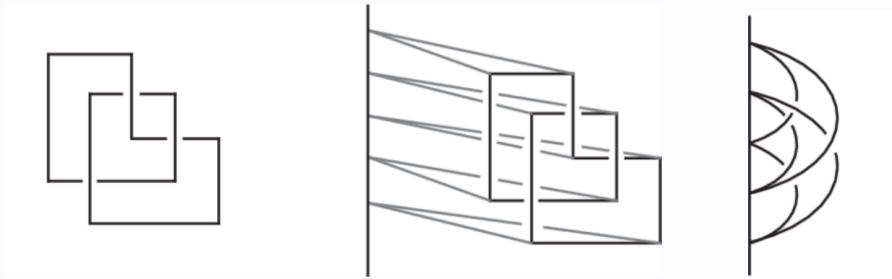
Theorem
*Every θ-curve and handcuff graph admits a grid diagram.*

Corollary
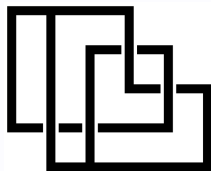*Every θ-curve and handcuff graph admits an arc presentation.*

Determinants of
θ-Curves and Handcuff Graphs

THC-Cromwell matrix

- The **Cromwell matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-Cromwell matrix** is an expansion of Cromwell matrix into θ-curves and handcuff graphs.
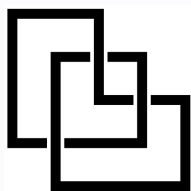
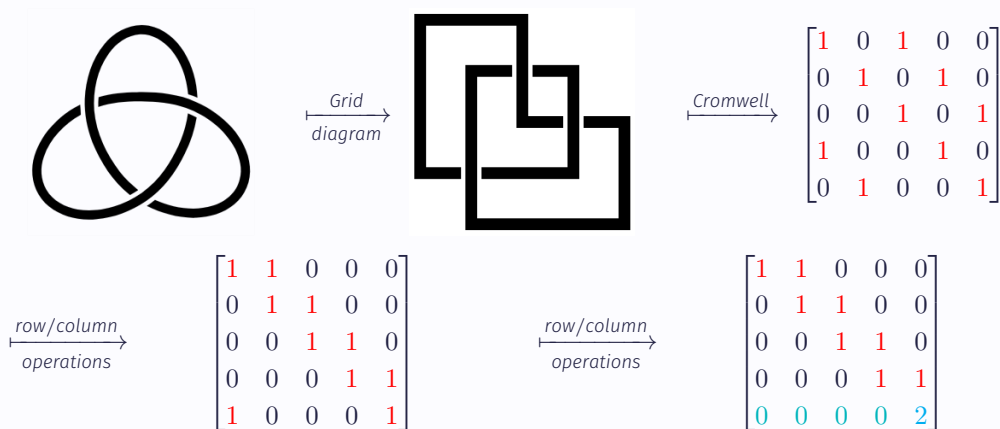 $\longmapsto$ $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ $\longmapsto$ $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$

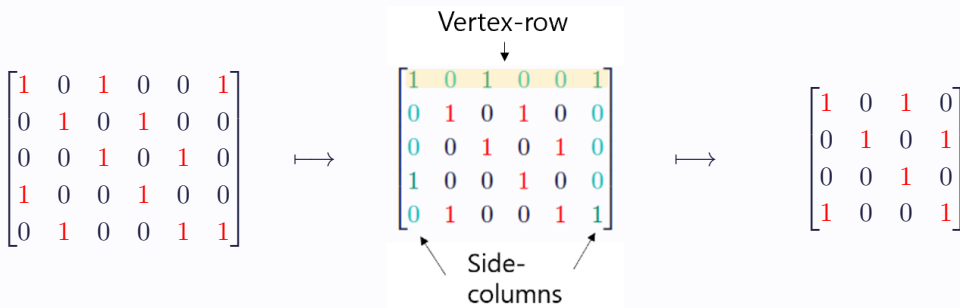# Determinants of the Cromwell matrices of Knot

Theorem

Let K be a knot. Then the determinant of a Cromwell matrix of K is 0 or ± 2.

PROOF

$$\xmapsto{\text{Grid diagram}}$$

$$\xmapsto{\text{Cromwell}}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xmapsto{\text{row/column operations}}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xmapsto{\text{row/column operations}}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

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H-deletion of THC-Cromwell matrices

- The **vertex-row** of THC-Cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic}, where $a < b < c$, as its elements.
- The **side-column** of THC-Cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-Cromwell matrix G is $(n-1) \times (n-1)$ matrix which deleted vertex-row and its two side-columns from the matrix G.

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\longmapsto
$$

Vertex-row
↓

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

Side-columns

$$
\longmapsto
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
$$

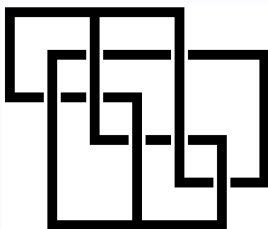# Determinants of the THC-Cromwell matrices

Let M_G be a THC-Cromwell matrix of θ-curve or handcuff graph G.

- $\det^*(M_G) = \pm 1 \iff M_G$ represents θ-curve
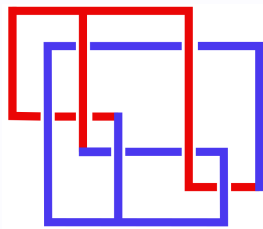- $\det^*(M_G) = 0$ or $\pm 2 \iff M_G$ represents handcuff graph

where $\det^*(M_G) = $ determinant of H-deletion matrix of M_G

PROOF

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$
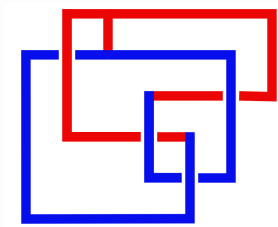
$\xmapsto{H-deletion}$

CASE 1. When M_G represents θ-curve

Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$

$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xmapsto[\text{operations}]{row/column} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xmapsto{subtracting} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So det*(M_G) = ± 1
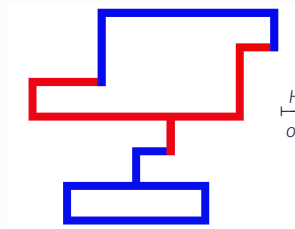
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CASE 2. When M_G represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$\longmapsto$

$\xrightarrow[only\,T-loop]{H-deletion}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$\xrightarrow[operations]{row/column}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{seperating}$

$$\begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So det*(M_G) = 0 or ± 2

CASE 2. When G represents handcuff graph

(i) T-loop & Line-shape

THC-Cromwell matrix $\xrightarrow{\;H-deletion\;}$ H-deletion matrix $\xrightarrow{\substack{seperating \\ operations}}$

$$\begin{bmatrix} T-loop & * \\ O & Line-shape \end{bmatrix} \xrightarrow{\substack{seperating \\ operations}} \begin{bmatrix} Knot & * & * \\ 0 & Line-shape & * \\ 0 & 0 & Line-shape \end{bmatrix}$$

(ii) Knot & Line-shape

THC-cromwell matrix $\xrightarrow{\;H-deletion\;}$ H-deletion matrix $\xrightarrow{\substack{seperating \\ operations}} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$

So $\det^\star(M_G) = 0$ or ± 2

Upper Bounds of Arc Index

For a spatial graph G, $c(G)$ denotes the crossing number of G.

Theorem (Lee[*] et al., 2018)

Let G be a spatial graph. Then,

$$\alpha(G) \leq c(G) + e + b$$

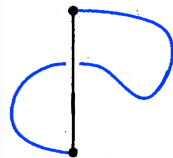*where e and b are the number of edges and bouquet cut points, respectively.*
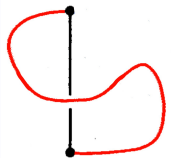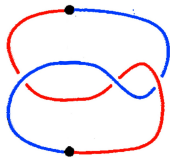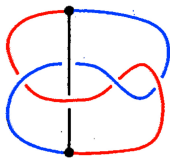
Corollary

Let G be a non-trivial prime θ-curve or handcuff graph. Then,

$$\alpha(G) \leq c(G) + 3.$$

Lower Bounds of Arc Index

- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of θ-curve T that consists of two vertices v_1, v_2 and two edges e_i, e_j.

- A **constituent link** H_{12}, is a subgraph of handcuff graph H that consists of two vertices v_1, v_2 and two edges e_1, e_2.
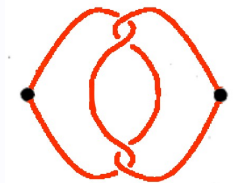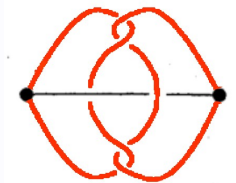
Lower Bound from Constituent Knots/Links

Theorem (Lee[†]**, 2023)**

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Theorem (Lee[†]**, 2023)**

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

Theorem
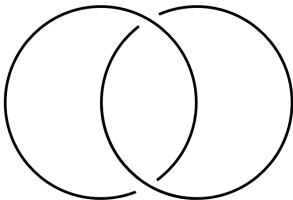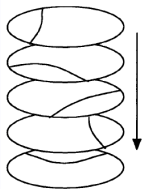
Let H be a handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then
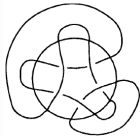
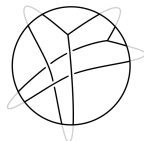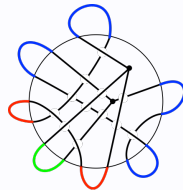$$\alpha(H) \geq c(L) + 3,$$

where c(L) is the crossing number of L.

Stacked Tangle of a Link
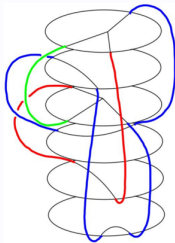
θ-Curve

Handcuff Graph

Yamada Polynomials

Let D_G be a diagram of an θ-curve or handcuff graph G. Then, the **Yamada Polynomial** $R(D_G) \in Z\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** $R\left(\ominus\right) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R\left(\bigcirc\!-\!\bigcirc\right) = 0$
- **Y8:** $R(G \cup \bigcirc) = (x+1+x^{-1})R(G)$ for an arbitrary θ-curve or handcuff graph diagram G
- **Y9:** $R\left(\diagdown\!\!\!\!\diagup\right) - R\left(\diagdown\!\!\!\!\diagup\right) = (x - x^{-1})\left[R\left(\,\right)\left(\,\right) - R\left(\asymp\right)\right]$
- **Y10:** $R\left(\mathcal{Q}\right) = x^2 R\left(\cap\right), \quad R\left(\mathcal{Q}\right) = x^{-2} R\left(\cap\right)$
- **Y11:** $R\left(\overline{\mathcal{Q}}\right) = R\left(\,\right)\left(\,\right)$ **Y12:** $R\left(\overset{\curlyvee}{\curlywedge}\right) = R\left(\overset{\curlyvee}{\curlywedge}\right)$
- **Y13:** $R\left(\bigstar\right) = R\left(\perp\right), \quad R\left(\bigstar\right) = R\left(\perp\right)$
- **Y14:** $R\left(-\!\!\!\triangleleft\!\!\right) = -xR\left(-\!\!\triangleleft\right), \quad R\left(-\!\!\!\triangleleft\!\!\right) = -x^{-1}R\left(-\!\!\triangleleft\right)$

Proposition (Yamada, 1989)
$R(D_G)$ is an ambient isotopy invariant of G up to multiplying $(-x)^n$ for some integer n.

Lower Bound from Yamada Polynomial

Theorem

Let G be a θ-curve or handcuff graph. Then

$$2 + \sqrt{|\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4|} \leq \alpha(G).$$

Sketch of Proof

Proposition

*Let S_G be a simple closure of stacked tangle of a θ-curve or handcuff graph G **without nested caps**. Then*

$$\max \deg_x R(S_G) \leq c + n, \quad \min \deg_x R(S_G) \geq -(c + n),$$

*where **c** and **n** is the number of caps and crossings in S_G, respectively.*

PROOF

- Let c_s and c_{ss} be the number of **simple caps** and **semi-simple caps**, respectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.

Sketch of Proof

Proposition

Let S_G be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to a minimal arc presentation of G. Then

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) - 2n \leq \alpha(G)$$

where n is the number of crossings in S_G.

PROOF

- S_G is a reduced simple closure of stacked tangle corresponding to a minimal arc presentation.
- The number of caps c in S_G is exactly the arc index of G, $\alpha(G)$.

Sketch of Proof

· Take a cap and add a positive or negative curl
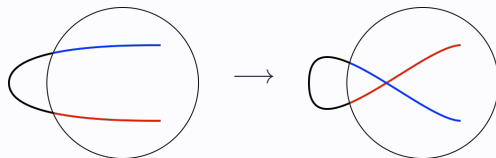
	S_G^{neg}	S_G^{pos}
Number of Caps	c	c
Number of Crossings	$n + p$	$n + (c - p)$

$$\min \deg_x R(S_G) - 2p = \min \deg_x R\left(S_G^{neg}\right) \geq -c + -(n + p)$$

$$\max \deg_x R(S_G) + 2(c - p) = \max \deg_x R\left(S_G^{pos}\right) \leq c + [n + (c - p)]$$

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) \leq c + 2n \qquad \square$$

Sketch of Proof

Theorem

Let G be a θ-curve or handcuff graph. Then

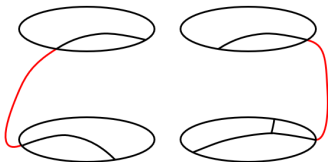$$2 + \sqrt{|\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4|} \leq \alpha(G).$$

PROOF

Let S_G be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to a minimal arc presentation of G.
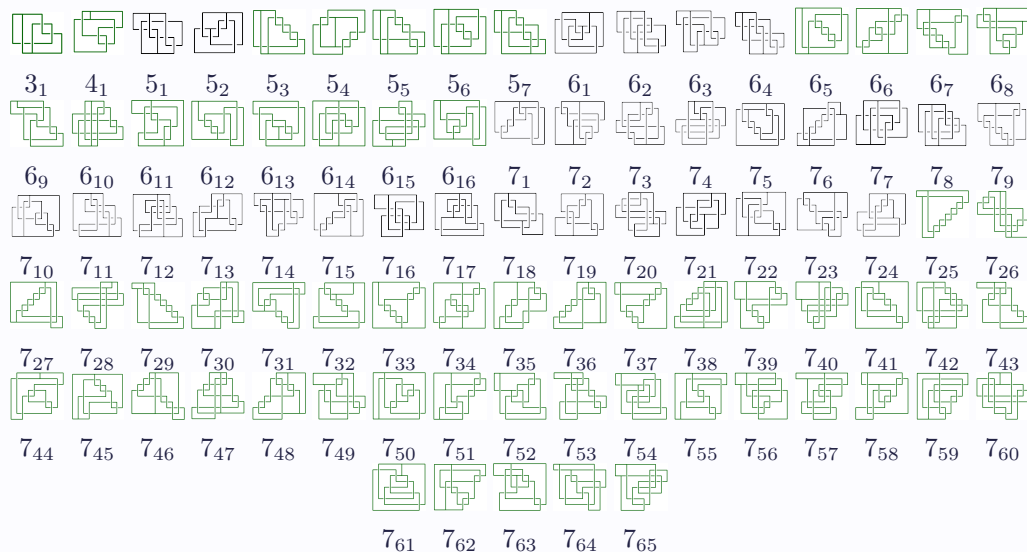
- The number of caps : $\alpha(G)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(G) - 3$

The theorem is followed by considering the maximum of crossings in S_G and applying previous theorems.

Minimal Grid Diagrams

3_1 4_1 5_1 5_2 5_3 5_4 5_5 5_6 5_7 6_1 6_2 6_3 6_4 6_5 6_6 6_7 6_8

6_9 6_{10} 6_{11} 6_{12} 6_{13} 6_{14} 6_{15} 6_{16} 7_1 7_2 7_3 7_4 7_5 7_6 7_7 7_8 7_9

7_{10} 7_{11} 7_{12} 7_{13} 7_{14} 7_{15} 7_{16} 7_{17} 7_{18} 7_{19} 7_{20} 7_{21} 7_{22} 7_{23} 7_{24} 7_{25} 7_{26}

7_{27} 7_{28} 7_{29} 7_{30} 7_{31} 7_{32} 7_{33} 7_{34} 7_{35} 7_{36} 7_{37} 7_{38} 7_{39} 7_{40} 7_{41} 7_{42} 7_{43}

7_{44} 7_{45} 7_{46} 7_{47} 7_{48} 7_{49} 7_{50} 7_{51} 7_{52} 7_{53} 7_{54} 7_{55} 7_{56} 7_{57} 7_{58} 7_{59} 7_{60}

7_{61} 7_{62} 7_{63} 7_{64} 7_{65}

| 2_1 | 4_1 | 5_1 | 6_1 | 6_2 | 6_3 | 6_4 | 6_5 | 6_6 | 6_7 | 6_8 | 6_9 |

| 7_1 | 7_2 | 7_3 | 7_4 | 7_5 | 7_6 | 7_7 | 7_8 | 7_9 | 7_{10} | 7_{11} | 7_{12} |

| 7_{13} | 7_{14} | 7_{15} | 7_{16} | 7_{17} | 7_{18} | 7_{19} | 7_{20} | 7_{21} | 7_{22} | 7_{23} | 7_{24} |

| 7_{25} | 7_{26} | 7_{27} | 7_{28} | 7_{29} | 7_{30} | 7_{31} | 7_{32} | 7_{33} | 7_{34} | 7_{35} | 7_{36} |
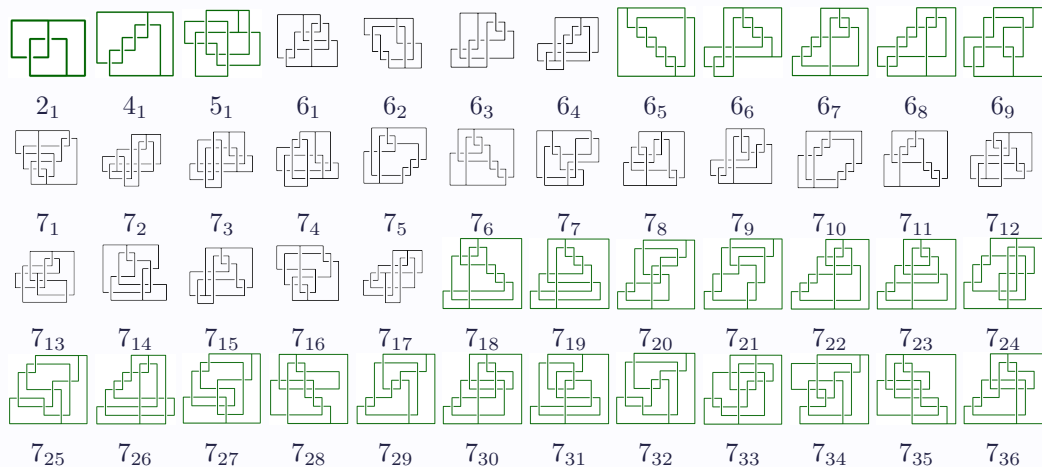
Further Studies

Further Studies

- Using another tool rather than Topoly package of Python to completely determine the arc indices.
- Applying the result of our research to graph of higher crossings or other spatial graphs.

References

References

1 Moriuchi, H. (2019). A table of θ-curves and handcuff graphs with up to seven crossings. *Advanced Studies in Pure Mathematics*, 281–290.

2 Yoonsang Lee. (2023). A Study on Arc Index of Theta-Curves. Korea Science Academy of KAIST

3 Minjung Lee, Sungjong No, and Seungsang Oh. (2018). Arc index of spatial graphs. *Journal of Graph Theory, 90*(3), 406–415.

4 Yamada, S. (1989). An invariant of spatial graphs. *Journal of Graph Theory, 13*(5), 537–551.