

# The Determinant and Arc Indices of $\theta$ -Curves and Handcuff Graphs

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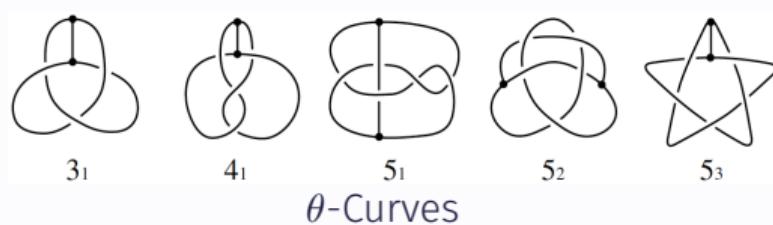
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# Introduction

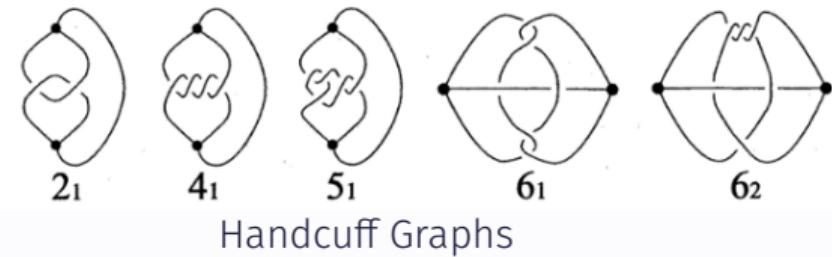
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# $\theta$ -Curves and Handcuff Graphs

- A  **$\theta$ -curve**  $T$  is a graph embedded in  $S^3$ , which consists of two vertices  $v_1, v_2$  and three edges  $e_1, e_2, e_3$ , such that each edge joins the vertices.
- A **constituent knot**  $T_{ij}$ ,  $1 \leq i < j \leq 3$ , is a subgraph of  $T$  that consists of two vertices  $v_1, v_2$  and two edges  $e_i, e_j$ .
- A **handcuff graph**  $H$  is a graph embedded in  $S^3$  consisting of two vertices  $(v_1, v_2)$  and three edges  $(e_1, e_2, e_3)$ , where  $e_3$  has distinct endpoints  $v_1$  and  $v_2$ , and  $e_1$  and  $e_2$  are loops based at  $v_1$  and  $v_2$ .
- A **constituent link**  $H_{12}$ , is a subgraph of  $H$  that consists of two vertices  $v_1, v_2$  and two edges  $e_1, e_2$ .



$\theta$ -Curves



Handcuff Graphs

## Arc Presentation

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- **Arc presentation** of a  $\theta$ -curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called **pages**).
- The embedding is with the common boundary line (called **axis**).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- **Arc index**, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is **minimal arc presentation**.

# Arc Presentation



Trefoil



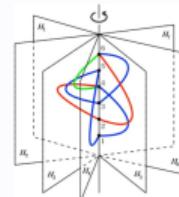
θ<sub>5,2</sub>



Φ<sub>2,1</sub>



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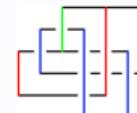
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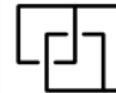
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Grid Diagram



Grid Diagram



Grid Diagram

## Grid Diagram

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- The **grid diagram** of  $\theta$ -curve or handcuff graph is a diagram with only vertical and horizontal strands.
- $(\text{number of vertical strands}) + 1 = (\text{number of horizontal strands})$
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

# Arc Presentation of the $\theta$ -Curve and Handcuff Graph

- A grid diagram gives rise to an arc presentation and vice versa.



## Theorem

*Every  $\theta$ -curve and handcuff graph admit a grid diagram.*

## Corollary

*Every  $\theta$ -curve and handcuff graph admit an arc presentation.*

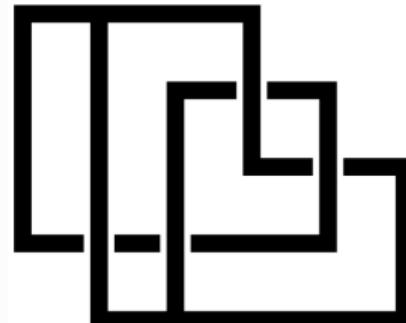
# Determinants of $\theta$ -Curves and Handcuff Graphs

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## THC-Cromwell matrix

- The **Cromwell Matrix** of a knot is an  $n \times n$  binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-Cromwell matrix** is an expansion of Cromwell matrix into  $\theta$ -curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



→

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

# Determinants of the Cromwell matrices of Knot

## Theorem

Let  $K$  be a knot. Then the determinant of a Cromwell matrix of  $K$  is 0 or  $\pm 2$ .

## PROOF



Grid  
diagram



Cromwell

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

row/column  
operations

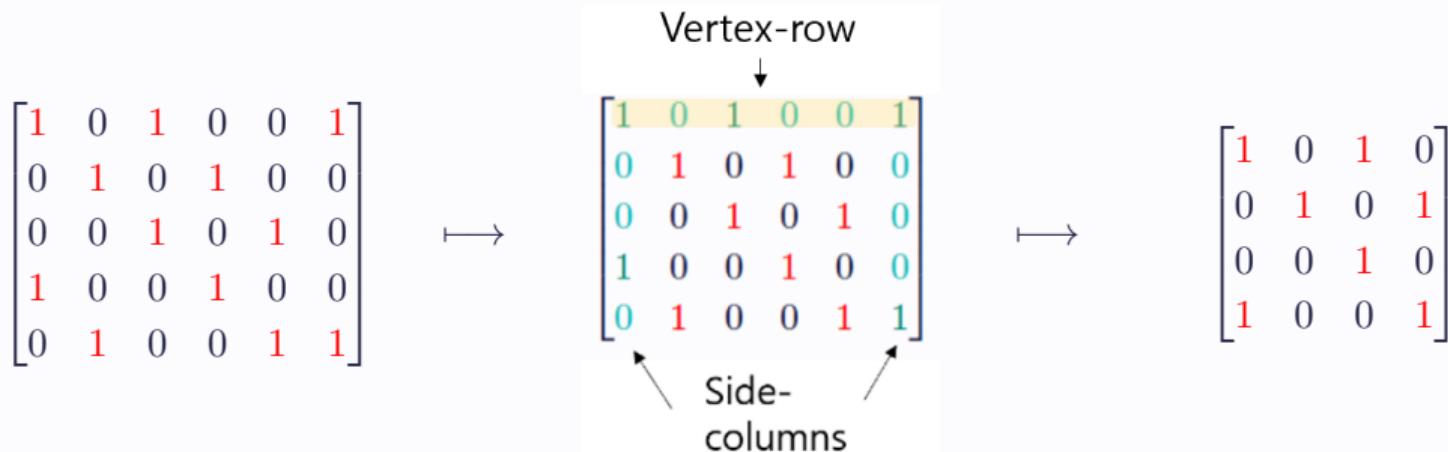
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

row/column  
operations

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

## H-deletion of THC-Cromwell matrices

- The **vertex-row** of THC-Cromwell matrix  $M$  is a row which contains three 1s,  $M_{ia}, M_{ib}, M_{ic}$ , where  $a < b < c$ , as its elements.
- The **side-column** of THC-Cromwell matrix  $M$  is a column which contains the leftmost 1 of vertex-row ( $M_{ia}$ ) or the rightmost 1 of vertex row ( $M_{ic}$ ).
- The **H-deletion** Matrix of the THC-Cromwell matrix  $G$  is  $(n - 1) \times (n - 1)$  matrix which deleted vertex-row and its two side-columns from the matrix  $G$ .



# Determinants of the THC-Cromwell matrices

## Theorem

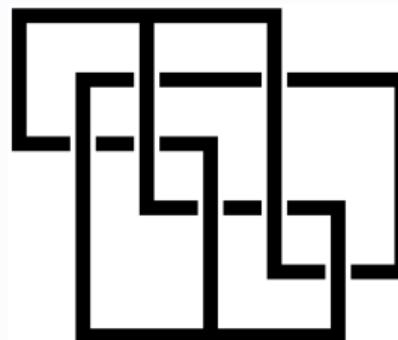
Let  $M$  be a THC-Cromwell matrix of  $\theta$ -curve or handcuff graph.

- $\det^*(M) = \pm 1 \iff M \text{ represents } \theta\text{-curve}$
- $\det^*(M) = 0 \text{ or } \pm 2 \iff M \text{ represents handcuff graph}$

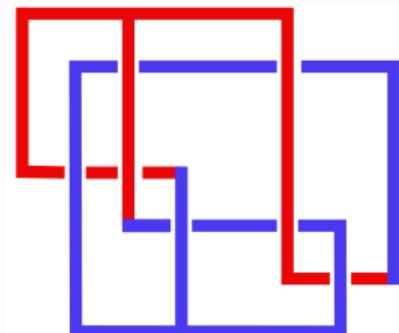
where  $\det^*(M)$  = determinant of  $H$ -deletion matrix of  $M$

## PROOF

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



$H$ -deletion

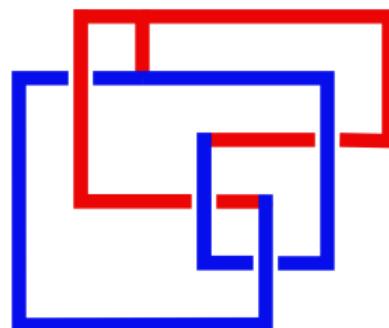


# Proof of Theorem

CASE 1. When  $M$  represents  $\theta$ -curve

Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow$$



$$\xrightarrow{H\text{-deletion}} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\xrightarrow{\text{row/column operations}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\xrightarrow{\text{subtracting}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So  $\det^*(M) = \pm 1$

# Proof of Theorem

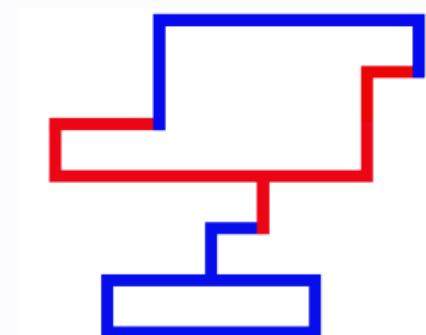
CASE 2. When  $M$  represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$\xrightarrow{\text{row/column operations}}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



$\xrightarrow{\text{H-deletion}} \text{only T-loop}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{\text{seperating}}$

$$\begin{bmatrix} \text{Knot} & * \\ 0 & \text{Line - shape} \end{bmatrix}$$

So  $\det(M) = 0$  or  $\pm 2$

# Proof of Theorem

## CASE 2. When $M$ represents handcuff graph

### i) T-loop & Line-shape

$$\begin{array}{ccc} \text{cromwell matrix} & \xrightarrow{\text{H-deletion}} & \text{H-deletion matrix} \\ \left[ \begin{array}{cc} \text{T-loop} & * \\ 0 & \text{Line - shape} \end{array} \right] & \xrightarrow{\text{seperating operations}} & \left[ \begin{array}{cccc} \text{Knot} & & * & * \\ 0 & \text{Line - shape} & 0 & \text{Line - shape} \\ 0 & & * & * \end{array} \right] \end{array}$$

### ii) Knot & Line-shape

$$\begin{array}{ccc} \text{cromwell matrix} & \xrightarrow{\text{H-deletion}} & \text{H-deletion matrix} \\ \left[ \begin{array}{cc} \text{Knot} & * \\ 0 & \text{Line - shape} \end{array} \right] & \xrightarrow{\text{seperating operations}} & \left[ \begin{array}{cc} \text{Knot} & * \\ 0 & \text{Line - shape} \end{array} \right] \end{array}$$

So  $\det(M) = 0$  or  $\pm 2$

## Upper Bounds of Arc Index

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# Upper Bounds of Arc Index

**Theorem (Lee et al., 2018)**

Let  $G$  be any spatial graph. Then,

$$\alpha(G) \leq c(G) + e + b$$

where  $e, b$  are the number of edges and bouquet cut, respectively.

**Corollary**

Let  $T$  be any non-trivial prime  $\theta$ -curve or handcuff graph. Then,

$$\alpha(T) \leq c(T) + 3.$$

## Lower Bounds of Arc Index

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# Lower Bound from Constituent Knots/Links

## Theorem (Lee, 2023)

Let  $T$  be any  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then

$$\alpha(T) \geq \max_{i \in \{1, 2, 3\}} \alpha(K_i) + 1$$

## Theorem (Lee, 2023)

Let  $T$  be any  $\theta$ -curve and  $K_1, K_2, K_3$  be three constituent knots of  $T$ . Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^3 \alpha(K_i)$$

# Lower Bound from Constituent Knots/Links

## Theorem

Let  $H$  be any handcuff graph, and  $L$  be the constituent link of  $H$ . If  $L$  is an alternating and non-split link, then

$$\alpha(H) \geq c(L) + 3.$$

## Corollary

Let  $H$  be any handcuff graph, and  $L$  be a constituent link of  $T$ . If  $L$  is alternating and non-split,

$$\alpha(H) = c(L) + 3.$$

# Stacked Tangle of an $\theta$ -Curve and Handcuff Graph

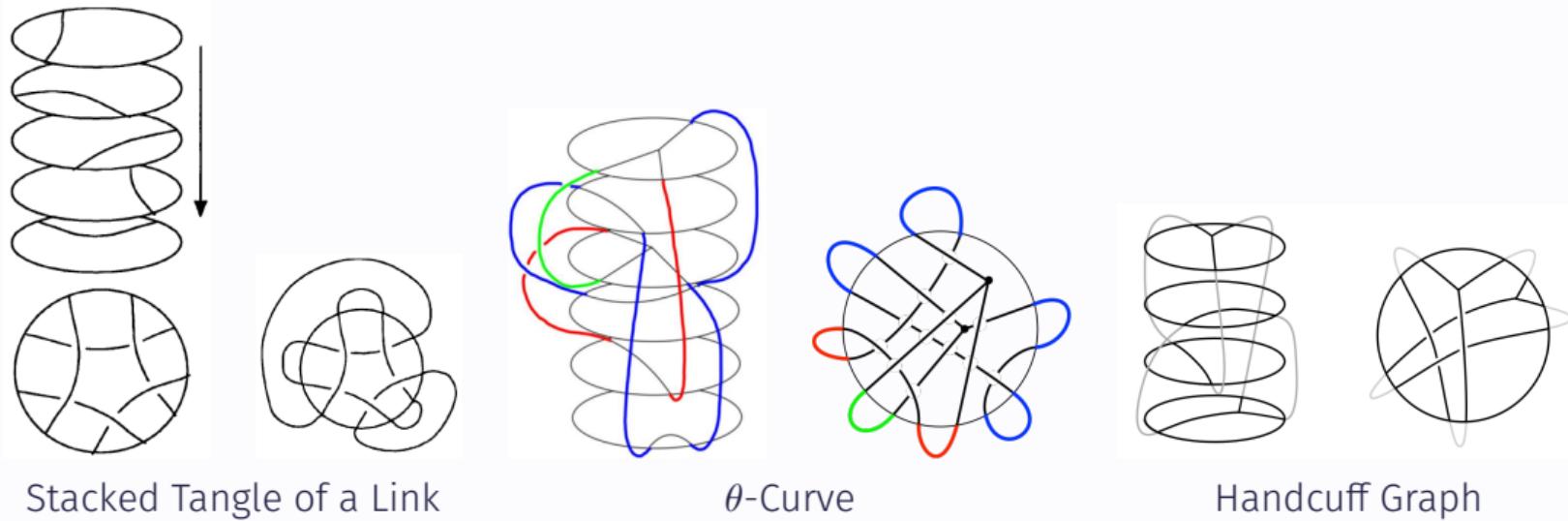


Figure from [?]

# Yamada Polynomials

Let  $D_G$  be a diagram of an  $\theta$ -curve or handcuff graph  $G$ . Then, the **Yamada Polynomial**  $R(D_G) \in \mathbb{Z}[x^{\pm 1}]$  is calculated by the following properties:

- **Y6:**  $R(\bigoplus) = -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$       **Y7:**  $R(\bigcirc\bigcirc) = 0$
- **Y8:**  $R(G \cup \bigcirc) = (x + 1 + x^{-1})R(G)$  for an arbitrary  $\theta$ -curve or handcuff graph diagram  $G$
- **Y9:**  $R(\bigotimes) - R(\bigotimes) = (x - x^{-1}) [R(\bigcirc\bigcirc) - R(\bigcirc\bigcirc)]$
- **Y10:**  $R(\bigcirc\bigcirc) = x^2 R(\bigcap), \quad R(\bigcirc\bigcirc) = x^{-2} R(\bigcap)$
- **Y11:**  $R(\bigotimes) = R(\bigcirc\bigcirc)$       **Y12:**  $R(\bigtimes) = R(\bigtimes)$
- **Y13:**  $R(\bigtriangleup) = R(\bigtriangleup), \quad R(\bigtriangleup) = R(\bigtriangleup)$
- **Y14:**  $R(\neg\bigcirc\bigcirc) = -xR(\neg\bigtriangleleft), \quad R(\neg\bigcirc\bigcirc) = -x^{-1}R(\neg\bigtriangleleft)$

**Proposition (Yamada, 1989)**

$R(D_G)$  is an ambient isotopy invariant of  $G$  up to multiplying  $(-x)^n$  for some integer  $n$ .

## Lower Bound from Yamada Polynomial

### Theorem

Let  $G$  be any  $\theta$ -curve or handcuff graph. Then

$$2 + \sqrt{\max_{x \in S_G} \deg_x R(S_G) - \min_{x \in S_G} \deg_x R(S_G) - 4} \leq \alpha(G).$$

# Lower Bound from Yamada Polynomial

## Proposition

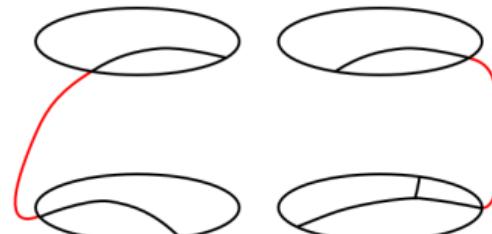
Let  $S_G$  be a simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $G$  **without nested caps**. Then

$$\max \deg_x R(S_G) \leq c + n, \quad \min \deg_x R(S_G) \geq -(c + n),$$

where  $c, n$  is the number of caps and crossings in  $S_G$ , respectively.

## PROOF

- Let  $c_s, c_{ss}$  be the number of **simple caps** or **semi-simple caps**, repectively.
- Use double mathematical induction of  $(c_s + c_{ss}, n)$ .



## Proposition

Let  $S_G$  be a reduced simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $G$  corresponding to minimal arc presentation of  $G$ . Then

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) - 2n \leq \alpha(G)$$

where  $n$  is the number of crossings in  $S_G$ .

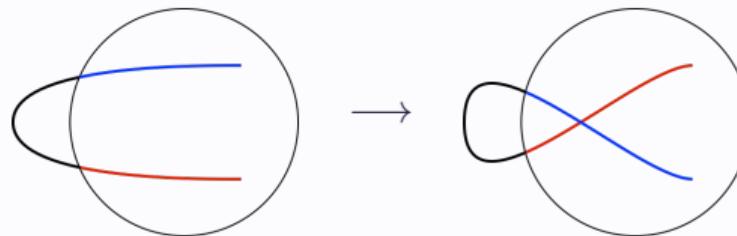
## PROOF

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- $S_G$  is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps  $c$  in  $S_G$  is exactly arc index of  $G$ ,  $\alpha(G)$ .

## Proof of Theorem

- Take a cap and add a positive or negative curl



	$S_G^{neg}$	$S_G^{pos}$
Number of Caps	$c$	$c$
Number of Crossings	$n + p$	$n + (c - p)$

$$\min \deg_x R(S_G) - 2p = \min \deg_x R(S_G^{neg}) \geq -c + -(n + p)$$

$$\max \deg_x R(S_G) + 2(c - p) = \max \deg_x R(S_G^{pos}) \leq c + [n + (c - p)]$$

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) \leq c + 2n$$

□

# Proof of Theorem

## Theorem

Let  $G$  be any  $\theta$ -curve or handcuff graph. Then

$$2 + \sqrt{\max_{x \in S_G} \deg_x R(S_G) - \min_{x \in S_G} \deg_x R(S_G) - 4} \leq \alpha(G).$$

## PROOF

Let  $S_G$  be a reduce simple closure of stacked tangle of a  $\theta$ -curve or handcuff graph  $G$  corresponding to minimal arc presentation of  $G$ .

- The number of caps :  $\alpha(G)$
- The number of non-simple disks : 2
- The number of simple disks :  $\alpha(G) - 3$

Consider the maximum number of crossings in  $S_G$ , and use the previous theorem.

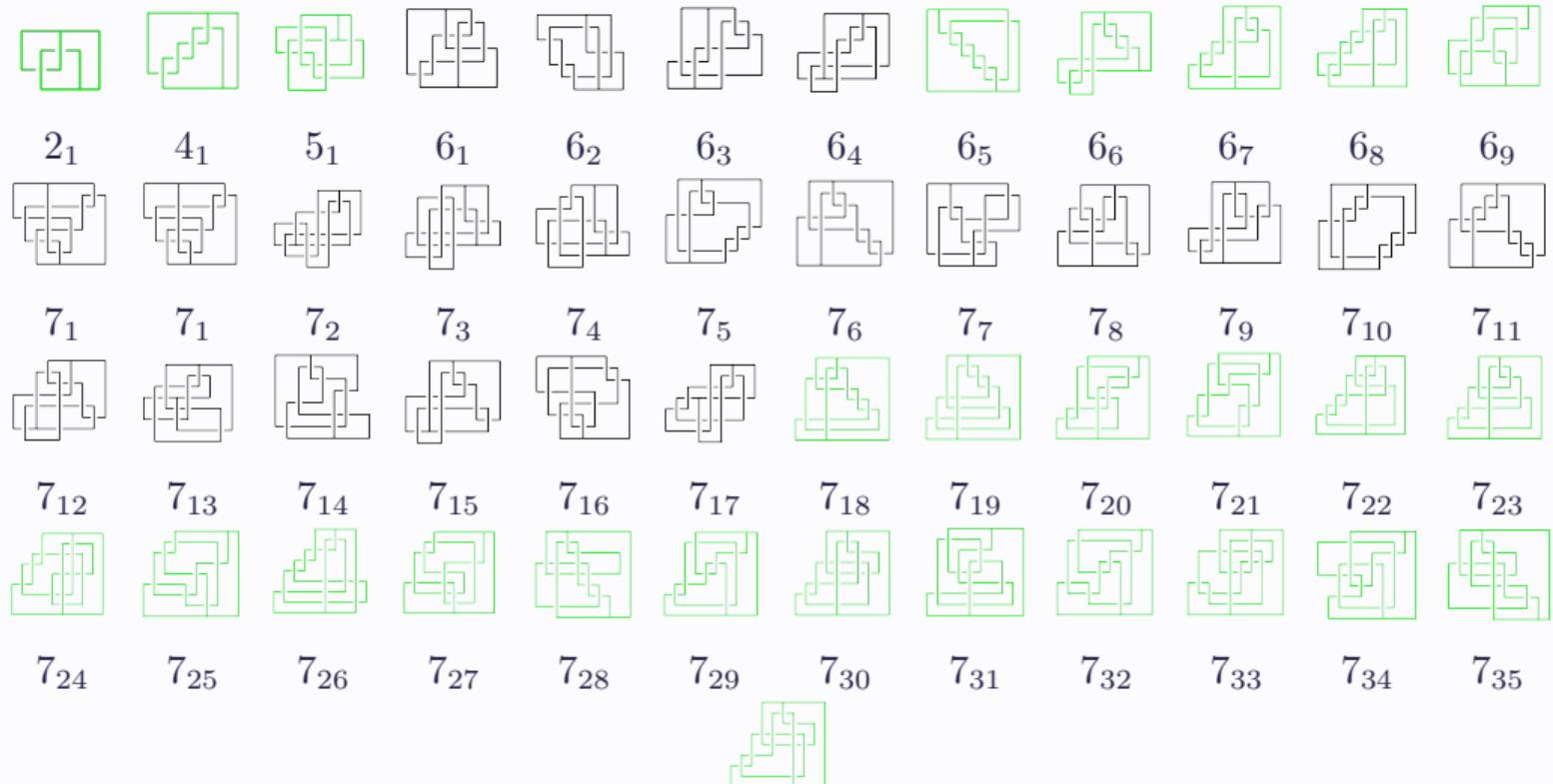
## Minimal Grid Diagrams

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# Grid Diagrams of $\theta$ -Curves



# Grid Diagrams of Handcuff Graphs



## Further Studies

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## Further Studies

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- We tried to use the Python for determining arc indices. For the Python program, we used the Topoly package, but the package had an error. We would like to use another tool to completely determine the arc indices of  $\theta$ -curves and handcuff graphs, such as Yamada package.
- Applying the result of our research to  $\theta$ -curves or handcuff graphs of higher crossings or to other spatial graphs can be researched further.