The Determinant and an Arc Index of θ-Curve and Handcuff Graph

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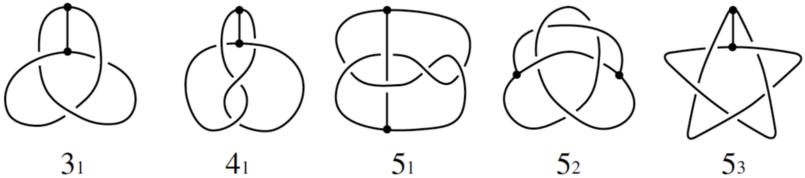
Introduction

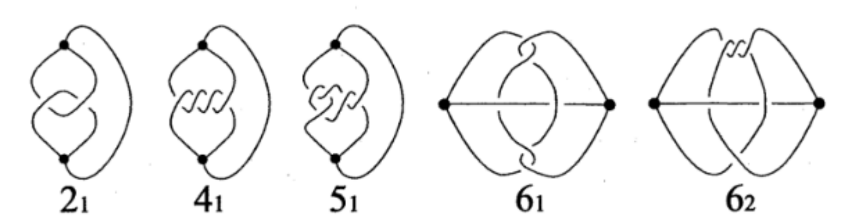
θ-Curves

- A **θ-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot T_{ij}**, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.
- θ-curves are roughly classified by comparing the triples of constituent knots.
- A θ-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3.

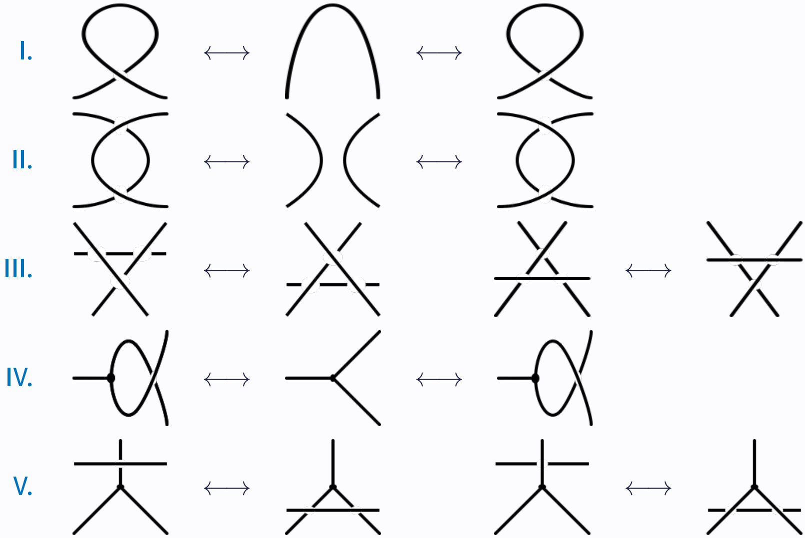


Handcuff Graphs

- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.
- A **constituent link H_{12}**, is a subgraph of H that consists of two vertices v_1, v_2 and two edges e_1, e_2.



Reidemaister Moves for θ-Curves and Handcuff Graphs

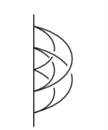


Arc Presentation

- Arc presentation of a θ-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called pages).
- The embedding is with the common boundary line (called axis).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- Arc index, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is minimal arc presentation.

Arc Presentation

Trefoil



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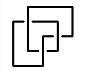
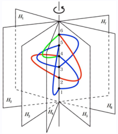


Grid Diagram

$\theta_{5,2}$



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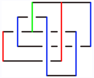
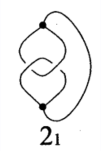
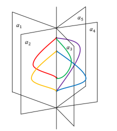


Grid Diagram



$\Phi_{2,1}$



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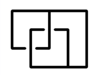
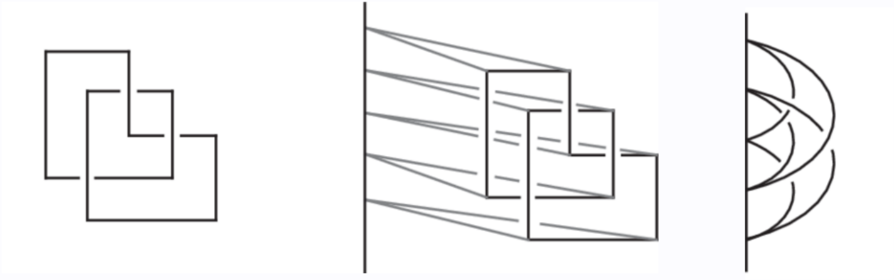


Grid Diagram

Grid Diagram

- The **grid diagram** of θ-curve or handcuff graph is a diagram with only vertical and horizontal strands.
- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

- A grid diagram gives rise to an arc presentation and vice versa.

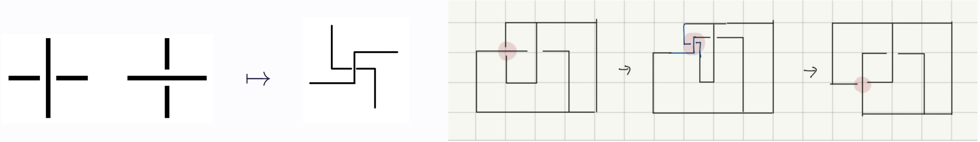


Arc Presentation of the θ-Curve and Handcuff Graph

Theorem

Every θ-curve and handcuff graph admit a grid diagram.

Proof



Corollary

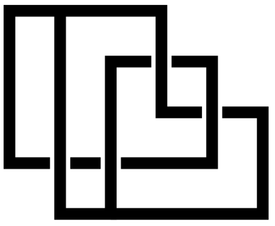
Every θ-curve and handcuff graph admit a arc presentation.

Determinant of θ-curve and Handcuff graph

THC-cromwell matrix

- The **Cromwell Matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ-curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



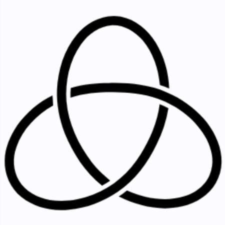
$$\longmapsto \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Determinants of the cromwell matrices of Knot

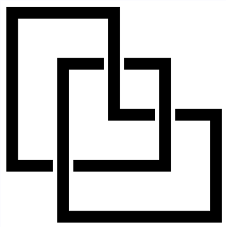
Theorem

Let K be a knot. Then the determinant of a cromwell matrix of K is 0 or ± 2.

PROOF



$\xmapsto{\text{grid diagram}}$



$$\xmapsto{\textit{Cromwell}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xmapsto[\textit{operations}]{\textit{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

CASE 1. When n is an even number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the determinant of K is 0.

CASE 2. When n is an odd number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

So the determinant of K is ± 2. $\square$

H-deletion of THC-cromwell matricies

- The **vertex-row** of THC-cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic}, where $a < b < c$, as its elements.
- The **side-column** of THC-cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-cromwell matrix G is $(n-1) \times (n-1)$ matrix which deleted vertex-row and its two side-columns from the matrix G.

Vertex-row

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Side-
columns

Determinants of the THC-cromwell matrices

Theorem

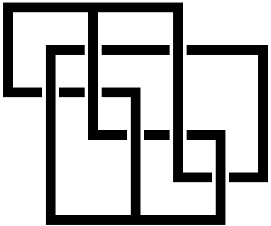
Let M be a THC-cromwell matrice of θ-curve or handcuff graph.

- *$det^*(M) = \pm 1 \iff M$ represents θ-curve*
- *$det^*(M) = 0$ or $\pm 2 \iff M$ represents handcuff graph*

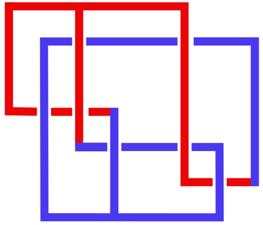
where $det^*(M)$ = determinant of H-deletion matrix of M

Proof

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$

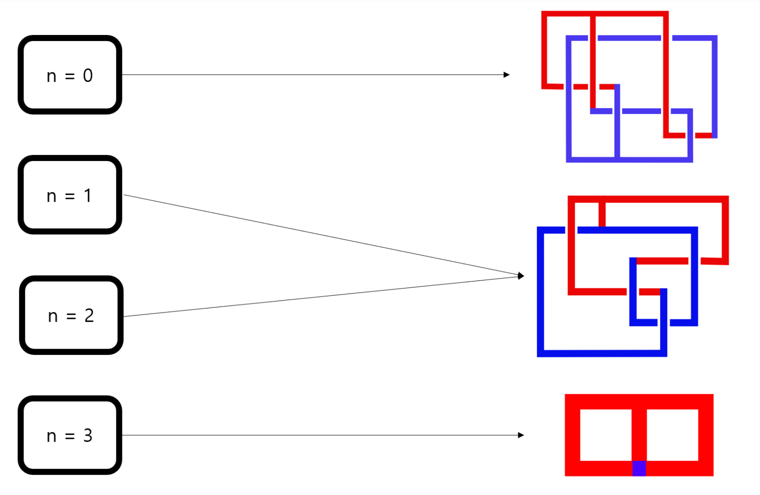


$$\xmapsto{H-deletion}$$



Proof of Theorem

CASE 1. When M represents θ-curve

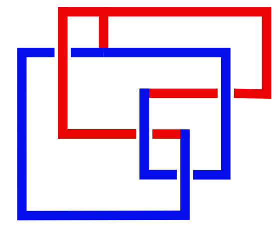


Proof of Theorem

CASE 1. When M represents θ-curve

ⓘ Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

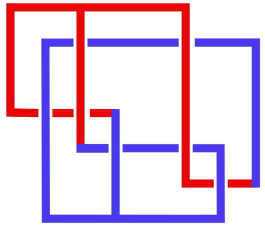
$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xmapsto{\text{subtracting}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det(M) = \pm 1$

CASE 1. When M represents θ-curve

(ii) T-shape

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longmapsto$$



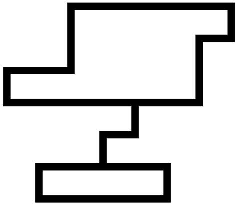
$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xmapsto{\text{regioning}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

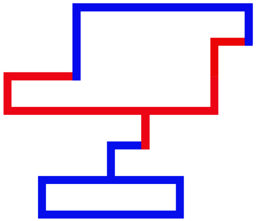
So $\det(M) = \pm 1$

CASE 2. When M represents handcuff graph

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



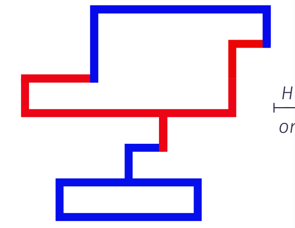
$$\longmapsto$$



CASE 2. When M represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xrightarrow[\text{only } T-loop]{H-deletion} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

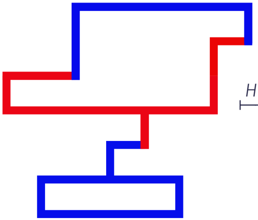
$$\xrightarrow[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{seperating}} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So det(M) = 0 or ± 2

CASE 2. When M represents handcuff graph

ⓘ T-loop & Line-shape

$$\begin{bmatrix} 0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&1&1\\ 1&0&1&0&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 0&0&0&1&1&0&0&0\\ 0&1&0&1&0&1&0&0\\ 0&1&0&0&0&1&0&0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 0&1&0&0&0&1\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&1&0&0\\ 1&0&1&0&1&0\\ 1&0&0&0&1&0 \end{bmatrix}$$

$$\xmapsto{operations} \begin{bmatrix} T-loop & * \\ 0 & Line-shape \end{bmatrix} \xmapsto{seperating} \begin{bmatrix} Knot & * & * \\ 0 & Line-shape & * \\ 0 & 0 & Line-shape \end{bmatrix}$$

So $\det(M) = 0$ or ± 2

CASE 2. When M represents handcuff graph

ii Knot & Line-shape

cromwell matrix $\xmapsto{H-deletion}$ H-deletion matrix $\xmapsto[operations]{seperating}$ $\begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$

So det(M) = 0 or ± 2

Lower Bounds of Arc Index

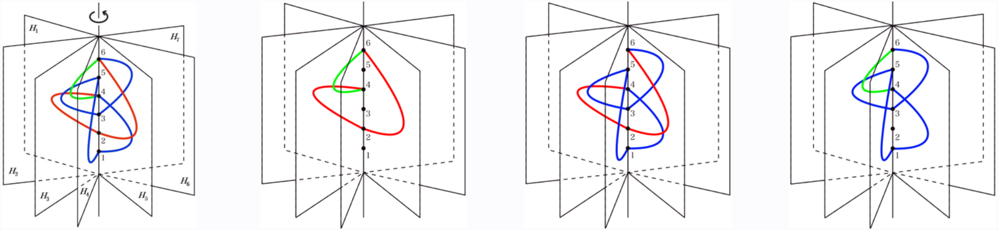
Lower Bounds from Constituent Knots

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Proof



$\square$

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

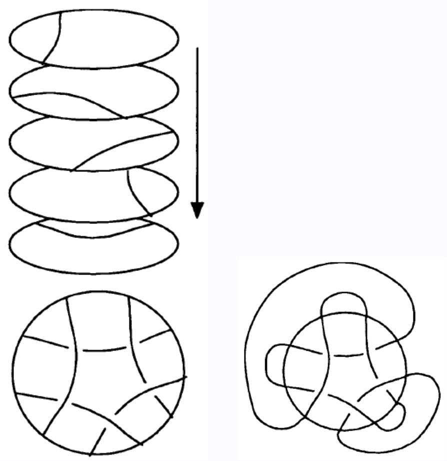
Proof

- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i.
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i.
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

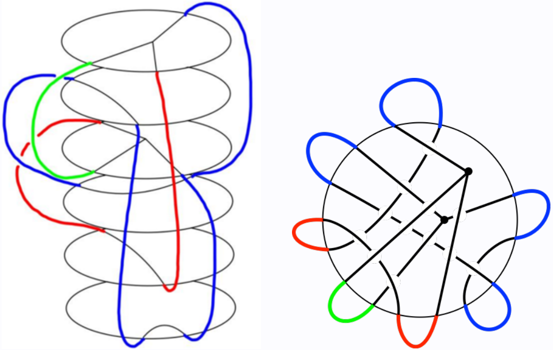
$$\sum_{i=1}^{3} \alpha(K_i) \leq 2 \sum_{i=1}^{3} |S_i| = 2\alpha(T)$$

$\square$

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link

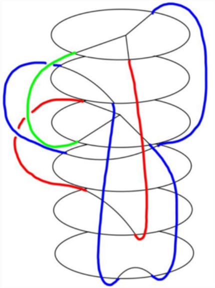


Stacked Tangle of a θ-Curve

Stacked Tangle of an θ-Curve

Stacked tangle of an θ-curve is stacked disks each with the frame as boundary with following properties:

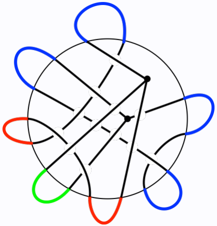
- Only two disk called non-simple disks contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called simple disks contain simple arc which joins two points on the boundary.
- When view from above
    - two arcs in different simple disks intersect at most one point(by RII)
    - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Stacked Tangle of an θ-Curve

Simple closure of stacked tangle is a stacked tangle with caps satisfying following properties:

- A cap is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.



Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

A reduced simple closure of a stacked tangle is

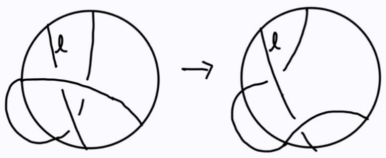
- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

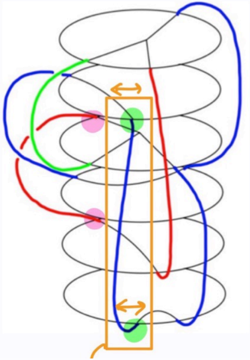
Stacked Tangle of an θ-Curve

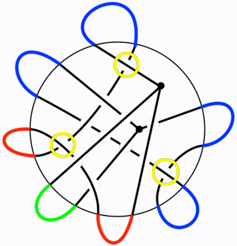
Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

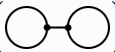
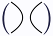
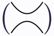
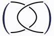
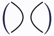
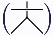
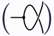
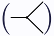
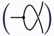
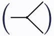
Proof







Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** R () $= -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** R () $= 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** R () $- R$ () $= (x - x^{-1})$ [R () $- R$ ()]
- **Y10:** R () $= x^2 R$ (), R () $= x^{-2} R$ ()
- **Y11:** R () $= R$ () **Y12:** R () $= R$ ()
- **Y13:** R () $= R$ (), R () $= R$ ()
- **Y14:** R () $= -xR$ (), R () $= -x^{-1} R$ ()

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Lower Bounds from Yamada Polynomial

Proposition

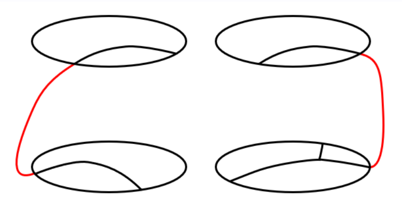
*Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T **without nested caps**. Then*

$$\max \deg_x R(S_T) \leq c + n, \quad \min \deg_x R(S_T) \geq -(c + n),$$

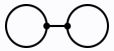
*where **c, n** is the number of caps and crossings in S_T, respectively.*

Proof

- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Basis Step:

- If $c_s + c_{ss} = 0$, then S_T has no simple disks and is equivalent to the result of applying Y14 to .
  $\therefore R(S_T) = -x^{\pm 3}\left[-x^2 - x - 2 - x^{-1} - x^{-2}\right] \implies 5 \leq c + n.$
- If $n = 0$, then S_T is equivalent to  $\cup \bigcirc \cup \cdots \cup \bigcirc$.
  $\therefore R(S_T) = 0 \implies 0 < 2 \leq c + n.$

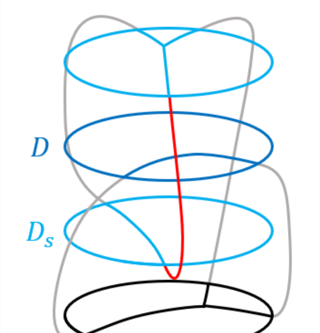
All base cases satisfy the inequality.

Inductive Step:

Assume that it holds for any $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$, and $c_s + c_{ss} > 0$.

Let S_T be a **simple closure of stacked tangle** of a θ-curve or handcuff graph T such that the number of simple caps, semi-simple caps, and crossings are c_s, c_{ss}, n, respectively.

Take the topmost **simple disk** D_s connected to the top disk, and a **disk** D directly above D_s.



Proof of Theorem

CASE 1. Suppose that there is no cap between D_s and D.

① Suppose that there is no intersection between D_s and D in S_T.

- D_s and D do not affect each other.
- We can swap the position of D_s and D without affecting the rest of the diagram.

② Suppose that there is an intersection between D_s and D in S_T.

- Let S_T^-, S_T^0 and S_T^∞ be the simple closure of stacked tangle which is obtained by replacing with ,)(and , respectively.
- The simple caps, semi-simple caps, and crossings of the both are $c_s, c_{ss}, n-1$.
- Applying Y9

$$R\left(\;\right) - R\left(\;\right) = (x - x^{-1})\left[R\left()(\right) - R\left(\;\right)\right],$$

then

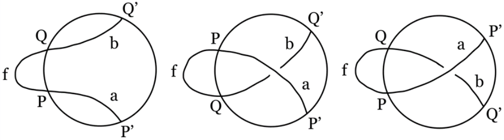
$$R(S_T) - R(S_T^-) = (x - x^{-1})(R(S_T^0) - R(S_T^\infty)).$$

- Then, it is sufficient to show that the interchanged one holds.

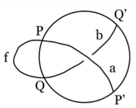
CASE 2. Suppose that there is a cap between D_s and D.

① Suppose that D is a simple disk.

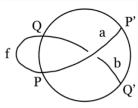
- When view from above, there are three cases:



- After applying Y10, the second and third cases can be regarded as the first case, and the cap can be reduced.

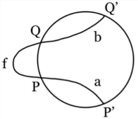


or



S_T

$\longrightarrow$



S'_T

- S'_T has $c-1$ caps, c_s-1 simple caps, c_{ss} semi-simple caps and $n-1$ crossings

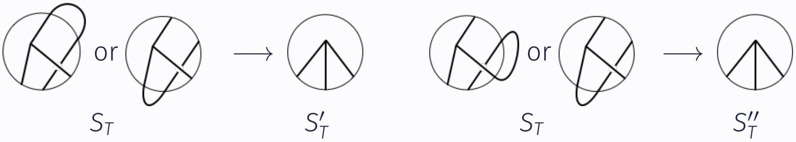
Proof of Theorem

- By induction hypothesis,

$$
\begin{aligned}
\max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 2 \\
&\leq [(c-1)+(n-1)] \pm 2 \\
&\leq c+n, \\
\min \deg_x R(S_T) &= \min \deg_x R(S'_T) \pm 2 \\
&\geq -[(c-1)+(n-1)] \pm 2 \\
&\geq -(c+n).
\end{aligned}
$$

② D is not a simple disk.

- When viewed from above, all the cases can be reduced as follows.



- $R(S_T) = -x^{\pm 1} R(S'_T)$ and $R(S_T) = x^{\pm 2} R(S''_T)$ by **Y14** and **Y10**, respectively.
- Both of S'_T and S''_T have $c-1$ caps, c_s simple caps, $c_{ss}-1$ semi-simple caps, and $n-1$ crossing.

Proof of Theorem

- By induction hypothesis, in the first case,

$$\begin{aligned} \max\deg_x R(S_T) &= \max\deg_x R(S'_T) \pm 1 \\ &\leq [(c-1)+(n-1)] \pm 1 \\ &\leq c+n. \end{aligned}$$

- Similarly, in the second case,

$$\begin{aligned} \max\deg_x R(S_T) &= \max\deg_x R(S''_T) \pm 2 \\ &\leq [(c-1)+(n-1)] \pm 2 \\ &\leq c+n. \end{aligned}$$

- It holds for $\min\deg_x R(S_T)$ in the same way.

$\square$

Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T. Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

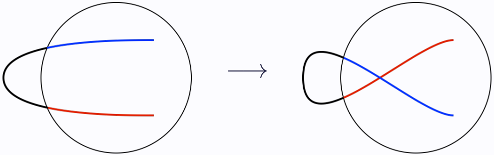
where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the p negative($c - p$ positive) curls.

Proof of Theorem

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

- $R\left(S_T^{neg}\right) = x^{-2p}R(S_T)$ and $R\left(S_T^{pos}\right) = x^{2(c-p)}R(S_T)$

$$\begin{aligned}
\min\deg_x R(S_T) - 2p &= \min\deg_x R\left(S_T^{neg}\right) \\
&\geq -c + -(n+p) \\
\max\deg_x R(S_T) + 2(c-p) &= \max\deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n + (c-p)] \\
\min\deg_x R(S_T) &\geq -c - n + p \\
\max\deg_x R(S_T) &\leq n + p \\
\max\deg_x R(S_T) - \min\deg_x R(S_T) &\leq c + 2n
\end{aligned}$$

$\square$

Proof of Theorem

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proof

Let S_T be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

① Let T be any θ-curve.

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings by two non-simple disks : 2
- number of crossings counted by disks joined by cap : $\alpha(T)-2$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)+2\left(\alpha(T)-3\right)+2-\left(\alpha(T)-2\right) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq 2n+\alpha(T) \leq \alpha(T)^2-4\alpha(T)+8$$

$$2+\sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

Proof of Theorem

② **Let T be any handcuff graph.**

Consider the maximum number of crossings in S_T.

Proof of Theorem

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings by two non-simple disks : 1
- number of crossings counted by disks joined by cap : $\alpha(T)-1-2=\alpha(T)-3$

Thus

$$\begin{aligned} n &\le \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)+2\left(\alpha(T)-3\right)+1-\left(\alpha(T)-3\right) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

$$\max\deg_x R(S_T)-\min\deg_x R(S_T) \le 2n+\alpha(T) \le \alpha(T)^2-4\alpha(T)+8$$