

The Determinant and Arc Indices of θ -Curves and Handcuff-Graphs

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Lower and Upper Bounds of Arc Index

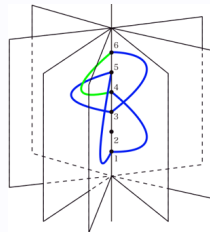
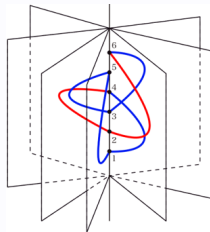
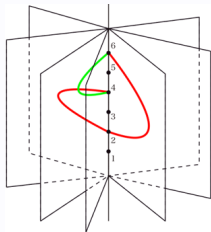
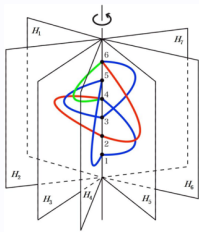
Lower Bounds from Constituent Knots/Links

Theorem

Let T be any θ -curve and K_1, K_2, K_3 be three constituent knots of T . Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

PROOF



Theorem

Let H be any handcuff graph and L be constituent link of H . Then

$$\alpha(H) \geq \alpha(L) + 1$$

PROOF

It is similar to the previous proof.



Lower Bounds from Constituent Knots/Links

Theorem

Let T be any θ -curve and K_1, K_2, K_3 be three constituent knots of T . Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^3 \alpha(K_i)$$

PROOF

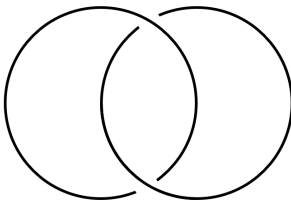
- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i .
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i .

$$\alpha(K_i) \leq |S_i| + |S_{i+1}| \implies \sum_{i=1}^3 \alpha(K_i) \leq 2 \sum_{i=1}^3 |S_i| = 2\alpha(T)$$

Theorem

If L is an alternating and non-split link, then

$$\alpha(L) = c(L) + 2.$$



Theorem

Let H be any handcuff graph, and L be the constituent link of H . If L is an alternating and non-split link, then

$$\alpha(H) \geq c(L) + 3.$$

PROOF

- We know $\alpha(H) \geq \alpha(L) + 1$.
- By the previous theorem, $\alpha(L) = c(L) + 2$.
- Thus, $\alpha(H) \geq c(L) + 3$.

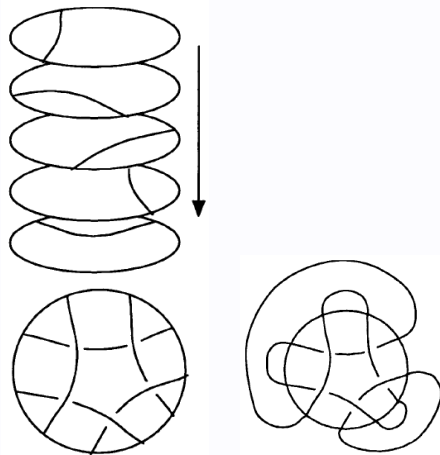


Corollary

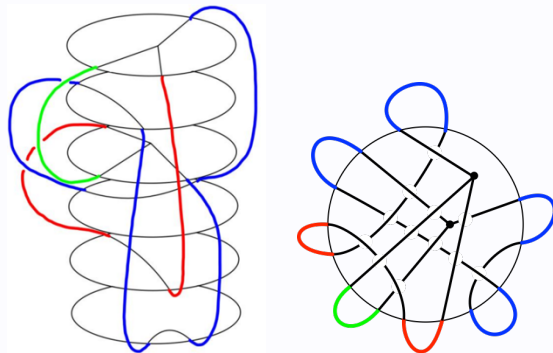
Let H be any handcuff graph, and L be a constituent link of T . If L is alternating and non-split,

$$\alpha(H) = c(L) + 3.$$

Stacked Tangle of an θ -Curve



Stacked Tangle of a Link

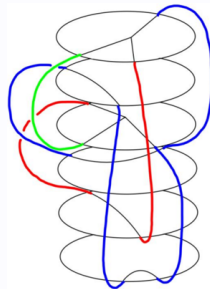


Stacked Tangle of a θ -Curve

Stacked Tangle of an θ -Curve

Stacked tangle of an θ -curve is stacked disks each with the frame as boundary with following properties:

- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
 - two arcs in different simple disks intersect at most one point(by RII)
 - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)

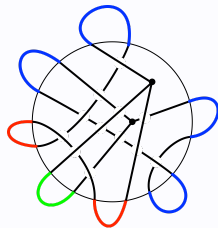


Stacked Tangle of an θ -Curve

Simple closure of stacked tangle is a **stacked tangle** with **caps** satisfying following properties:

- A **cap** is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.

Then a simple closure of a stacked tangle **without any nested caps** is corresponding to an arc presentation.



A **reduced simple closure of a stacked tangle** is

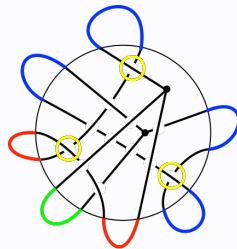
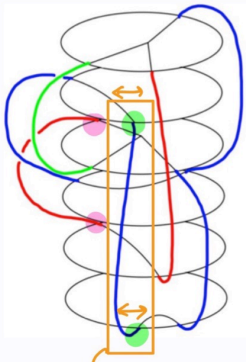
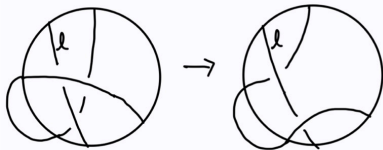
- a simple closure of a stacked tangle **without any nested caps**
- any two arcs(including line segment) joining by caps have **no intersection** when view from above

Stacked Tangle of an θ -Curve

Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

PROOF



Yamada Polynomials

Let D_T be a diagram of an θ -curve T . Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}[x^{\pm 1}]$ is calculated by the following properties:

- **Y6:** $R(\bigoplus) = -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R(\bigcirc \text{---} \bigcirc) = 0$
- **Y8:** $R(T' \cup \bigcirc) = (x + 1 + x^{-1})R(T')$ for an arbitrary θ -curve diagram T'
- **Y9:** $R(\bigwedge) - R(\bigvee) = (x - x^{-1}) [R(\bigcirc) - R(\bigcirc)]$
- **Y10:** $R(\bigcirc) = x^2 R(\bigcap), \quad R(\bigcirc) = x^{-2} R(\bigcap)$
- **Y11:** $R(\bigcirc) = R(\bigcirc)$ **Y12:** $R(\bigcirc) = R(\bigcirc)$
- **Y13:** $R(\bigcirc) = R(\bigcirc), \quad R(\bigcirc) = R(\bigcirc)$
- **Y14:** $R(\bigcirc) = -xR(\bigcirc), \quad R(\bigcirc) = -x^{-1}R(\bigcirc)$

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n .

Theorem

Let T be any θ -curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T .

Lower Bounds from Yamada Polynomial

Proposition

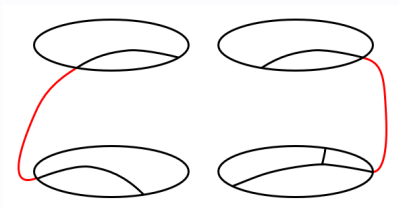
Let S_T be a simple closure of stacked tangle of a θ -curve or handcuff graph T **without nested caps**. Then

$$\max \deg_x R(S_T) \leq c + n, \quad \min \deg_x R(S_T) \geq -(c + n),$$

where **c, n** is the number of caps and crossings in S_T , respectively.

PROOF

- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Basis Step:

- If $c_S + c_{SS} = 0$, then S_T has no simple disks and is equivalent to the result of applying Y14 to \bigoplus .
 $\therefore R(S_T) = -x^{\pm 3} [-x^2 - x - 2 - x^{-1} - x^{-2}] \implies 5 \leq c + n.$
- If $n = 0$, then S_T is equivalent to $\bigcirc - \bigcirc \cup \bigcirc \cup \dots \cup \bigcirc$.
 $\therefore R(S_T) = 0 \implies 0 < 2 \leq c + n.$

All base cases satisfy the inequality.

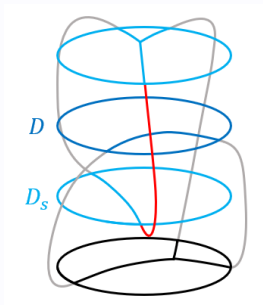
Proof of Theorem

Inductive Step:

Assume that it holds for any $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$, and $c_s + c_{ss} > 0$.

Let S_T be a **simple closure of stacked tangle** of a θ -curve or handcuff graph T such that the number of simple caps, semi-simple caps, and crossings are c_s, c_{ss}, n , respectively.

Take the topmost **simple disk** D_s connected to the top disk, and a **disk** D directly above D_s .



Proof of Theorem

CASE 1. Suppose that there is no cap between D_S and D .

① Suppose that there is no intersection between D_S and D in S_T .

- D_S and D do not affect each other.
- We can swap the position of D_S and D without affecting the rest of the diagram.

② Suppose that there is an intersection between D_S and D in S_T .

- Let S_T^- , S_T^0 and S_T^∞ be the simple closure of stacked tangle which is obtained by replacing \times with \times , $\rangle\langle$ and \asymp , respectively.
- The simple caps, semi-simple caps, and crossings of the both are $c_s, c_{ss}, n - 1$.
- Applying Y9

$$R(\times) - R(\times) = (x - x^{-1}) [R(\rangle\langle) - R(\asymp)],$$

then

$$R(S_T) - R(S_T^-) = (x - x^{-1})(R(S_T^0) - R(S_T^\infty)).$$

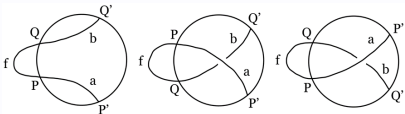
- Then, it is sufficient to show that the interchanged one holds.

Proof of Theorem

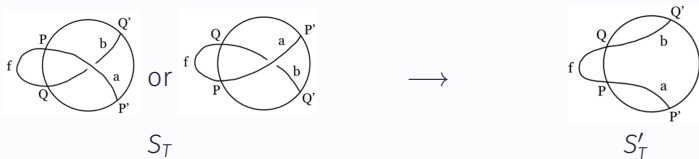
CASE 2. Suppose that there is a cap between D_s and D .

① Suppose that D is a simple disk.

- When view from above, there are three cases:



- After applying **Y10**, the second and third cases can be regarded as the first case, and the cap can be reduced.



- S'_T has $c - 1$ caps, $c_s - 1$ simple caps, c_{ss} semi-simple caps and $n - 1$ crossings.

- By induction hypothesis,

$$\begin{aligned}\max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 2 \\ &\leq [(c-1) + (n-1)] \pm 2 \\ &\leq c + n, \\ \min \deg_x R(S_T) &= \min \deg_x R(S'_T) \pm 2 \\ &\geq -[(c-1) + (n-1)] \pm 2 \\ &\geq -(c+n).\end{aligned}$$

② D is not a simple disk.

- When viewed from above, all the cases can be reduced as follows.



- $R(S_T) = -x^{\pm 1}R(S'_T)$ and $R(S_T) = x^{\pm 2}R(S''_T)$ by **Y14** and **Y10**, respectively.
- Both of S'_T and S''_T have $c - 1$ caps, c_s simple caps, $c_{ss} - 1$ semi-simple caps, and $n - 1$ crossing.

Proof of Theorem

- By induction hypothesis, in the first case,

$$\begin{aligned}\max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 1 \\ &\leq [(c-1) + (n-1)] \pm 1 \\ &\leq c + n.\end{aligned}$$

- Similarly, in the second case,

$$\begin{aligned}\max \deg_x R(S_T) &= \max \deg_x R(S''_T) \pm 2 \\ &\leq [(c-1) + (n-1)] \pm 2 \\ &\leq c + n.\end{aligned}$$

- It holds for $\min \deg_x R(S_T)$ in the same way.



Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ -curve or handcuff graph T corresponding to minimal arc presentation of T . Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

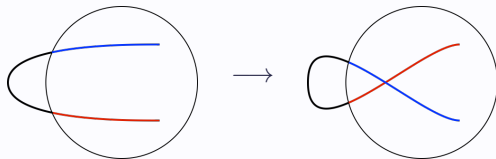
where n is the number of crossings in S_T .

PROOF

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T , $\alpha(T)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the p negative($c - p$ positive) curls.

Proof of Theorem

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n + p$	$n + (c - p)$

$$\bullet R(S_T^{neg}) = x^{-2p}R(S_T) \text{ and } R(S_T^{pos}) = x^{2(c-p)}R(S_T)$$

$$\min \deg_x R(S_T) - 2p = \min \deg_x R(S_T^{neg})$$

$$\geq -c + -(n + p)$$

$$\max \deg_x R(S_T) + 2(c - p) = \max \deg_x R(S_T^{pos})$$

$$\leq c + [n + (c - p)]$$

$$\min \deg_x R(S_T) \geq -c - n + p$$

$$\max \deg_x R(S_T) \leq n + p$$

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq c + 2n$$

□

Proof of Theorem

Theorem

Let T be any θ -curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T .

PROOF

Let S_T be a reduce simple closure of stacked tangle of a θ -curve or handcuff graph T corresponding to minimal arc presentation of T .

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

Proof of Theorem

① Let T be any θ -curve.

Consider the maximum number of crossings in S_T .

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4)$
- number of crossings by a simple disk and non-simple disk : $2 (\alpha(T) - 3)$
- number of crossings by two non-simple disks : 2
- number of crossings counted by disks joined by cap : $\alpha(T) - 2$

Thus

$$\begin{aligned} n &\leq \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4) + 2 (\alpha(T) - 3) + 2 - (\alpha(T) - 2) \\ &= \frac{1}{2} [(\alpha(T))^2 - 5\alpha(T) + 8] \end{aligned}$$

By Lemma,

$$\begin{aligned} \max \deg_x R(S_T) - \min \deg_x R(S_T) &\leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8 \\ 2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} &\leq \alpha(T) \end{aligned}$$

Proof of Theorem

② Let T be any handcuff graph.

Consider the maximum number of crossings in S_T .

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4)$
- number of crossings by a simple disk and non-simple disk : $2 (\alpha(T) - 3)$
- number of crossings by two non-simple disks : 1
- number of crossings counted by disks joined by cap : $\alpha(T) - 1 - 2 = \alpha(T) - 3$

Thus

$$\begin{aligned} n &\leq \frac{1}{2} (\alpha(T) - 3) (\alpha(T) - 4) + 2 (\alpha(T) - 3) + 1 - (\alpha(T) - 3) \\ &= \frac{1}{2} [(\alpha(T))^2 - 5\alpha(T) + 8] \end{aligned}$$

By Lemma,

$$\begin{aligned} \max \deg_x R(S_T) - \min \deg_x R(S_T) &\leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8 \\ 2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} &\leq \alpha(T) \end{aligned}$$

□