The Determinants and Arc Indices of θ-Curves and Handcuff Graphs

Eunchan Cho[1] Jeongwon Shin[1] Boyeon Seo[1] Minho Choi[1]
Supervisor: Hun Kim[1] GyoTaek Jin[2]

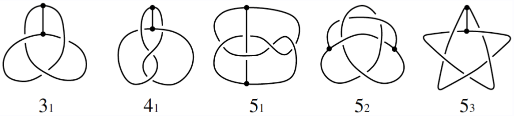
NOV 22, 2025
R&E 2025

[1]Korea Science Academy of KAIST
[2]Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology

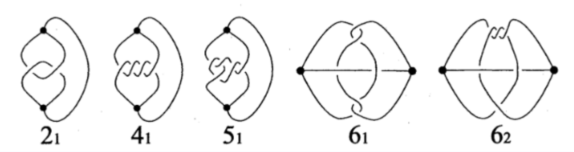
Introduction

θ-Curves and Handcuff Graphs

- A ***θ*-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.



θ-Curves

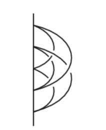


Handcuff Graphs

Arc Presentation

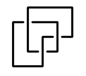
Trefoil

$\Longleftrightarrow$



Arc Presentation

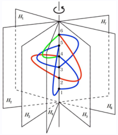
$\Longleftrightarrow$



Grid Diagram

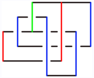
$\theta_{5,2}$

$\Longleftrightarrow$

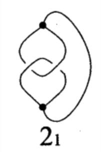


Arc Presentation

$\Longleftrightarrow$

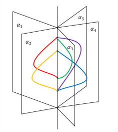


Grid Diagram



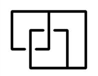
$\Phi_{2,1}$

$\Longleftrightarrow$



Arc Presentation

$\Longleftrightarrow$



Grid Diagram

Arc Presentation

Arc presentation of a θ-curve or handcuff graph is an embedding of them.

- It is contained in the union of finitely many half planes (called **pages**).
- The embedding is with the common boundary line (called **axis**).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.

Arc index is the minimal number of pages among all possible arc presentations of graph.

- Arc presentation with the minimal number of pages is called **minimal arc presentation**.

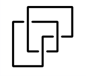
Grid Diagram

Trefoil

$\Longleftrightarrow$

Arc Presentation

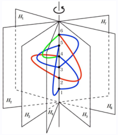
$\Longleftrightarrow$



Grid Diagram

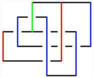
$\theta_{5,2}$

$\Longleftrightarrow$



Arc Presentation

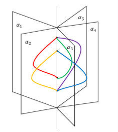
$\Longleftrightarrow$



Grid Diagram

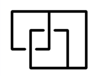
$\Phi_{2,1}$

$\Longleftrightarrow$



Arc Presentation

$\Longleftrightarrow$



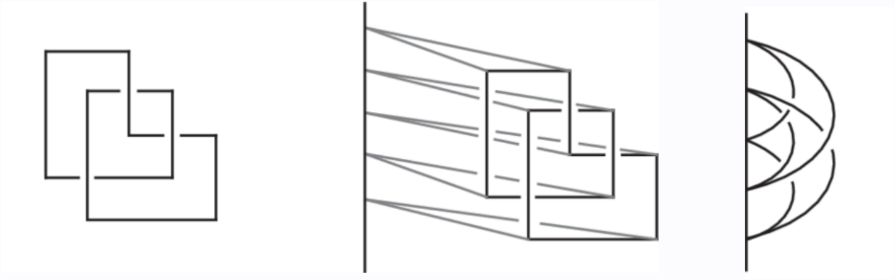
Grid Diagram

Grid Diagram

A grid diagram of θ-curve or handcuff graph is a diagram with only vertical and horizontal strands.

- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

- A grid diagram gives rise to an arc presentation and vice versa.



Theorem

Every θ-curve and handcuff graph admits a grid diagram.

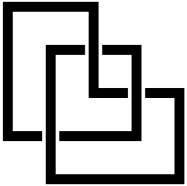
Corollary

Every θ-curve and handcuff graph admits an arc presentation.

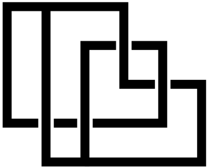
Determinants of θ-Curves and Handcuff Graphs

THC-Cromwell matrix

- The **Cromwell matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-Cromwell matrix** is an expansion of Cromwell matrix into θ-curves and handcuff graphs.



$$\longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\longmapsto \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

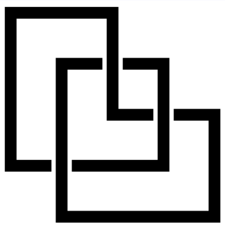
Determinants of the Cromwell matrices of Knot

Theorem

Let K be a knot. Then the determinant of a Cromwell matrix of K is 0 or ± 2.

Proof

$\xmapsto[\text{diagram}]{\text{Grid}}$



$$\xmapsto{\text{Cromwell}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

H-deletion of THC-Cromwell matrices

- The **vertex-row** of THC-Cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic}, where $a < b < c$, as its elements.
- The **side-column** of THC-Cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-Cromwell matrix G is $(n-1) \times (n-1)$ matrix which deleted vertex-row and its two side-columns from the matrix G.

Vertex-row

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Side-columns

Determinants of the THC-Cromwell matrices

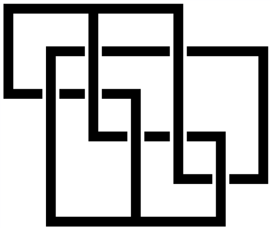
Let M_G be a THC-Cromwell matrix of θ-curve or handcuff graph G.

- $\det^*(M_G) = \pm 1 \iff M_G$ represents θ-curve
- $\det^*(M_G) = 0$ or $\pm 2 \iff M_G$ represents handcuff graph

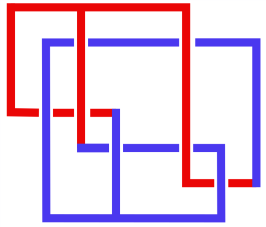
where $\det^*(M_G)$ = determinant of H-deletion matrix of M_G

PROOF

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



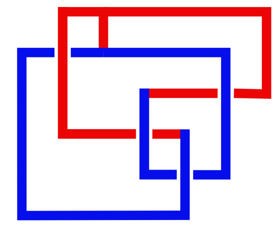
$\xmapsto{H-deletion}$



CASE 1. When M_G represents θ-curve

Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

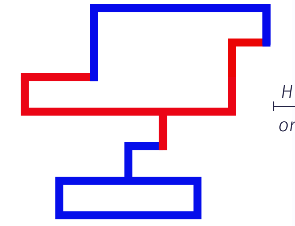
$$\xmapsto[operations]{row/column} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xmapsto{subtracting} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det^*(M_G) = \pm 1$

CASE 2. When M_G represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto[only\ T-loop]{H-deletion} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\xmapsto[operations]{row/column} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xmapsto{seperating} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So det*(M_G) = 0 or ± 2

CASE 2. When G represents handcuff graph

ⓘ T-loop & Line-shape

THC-Cromwell matrix $\xmapsto{H-deletion}$ H-deletion matrix $\xmapsto[operations]{seperating}$

$$\begin{bmatrix} T-loop & * \\ O & Line-shape \end{bmatrix} \xmapsto[operations]{seperating} \begin{bmatrix} Knot & * & * \\ 0 & Line-shape & * \\ 0 & 0 & Line-shape \end{bmatrix}$$

ⓘⓘ Knot & Line-shape

THC-cromwell matrix $\xmapsto{H-deletion}$ H-deletion matrix $\xmapsto[operations]{seperating}$ $\begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$

So $\det^*(M_G) = 0$ or ± 2

Upper Bounds of Arc Index

Upper Bounds of Arc Index

For a spatial graph G, $c(G)$ denotes the crossing number of G.

Theorem (Lee* et al., 2018)

Let G be a spatial graph. Then,

$$\alpha(G) \leq c(G) + e + b$$

where e and b are the number of edges and bouquet cut points, respectively.

Corollary

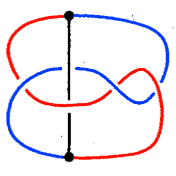
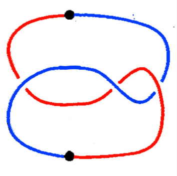
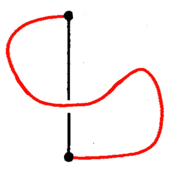
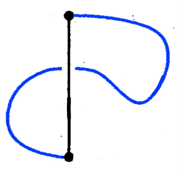
Let G be a non-trivial prime θ-curve or handcuff graph. Then,

$$\alpha(G) \leq c(G) + 3.$$

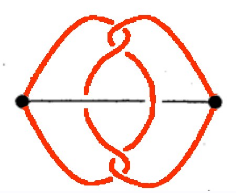
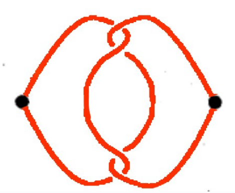
Lower Bounds of Arc Index

Lower Bound from Constituent Knots/Links

- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of θ-curve T that consists of two vertices v_1, v_2 and two edges e_i, e_j.

- A **constituent link** H_{12}, is
  a subgraph of handcuff graph H that consists of two vertices v_1, v_2 and two edges e_1, e_2.

Lower Bound from Constituent Knots/Links

Theorem (Lee[†], 2023)

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Theorem (Lee, 2023)

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

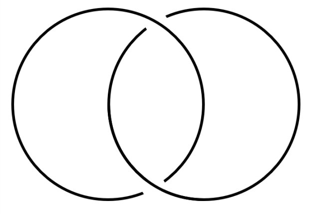
$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

Theorem

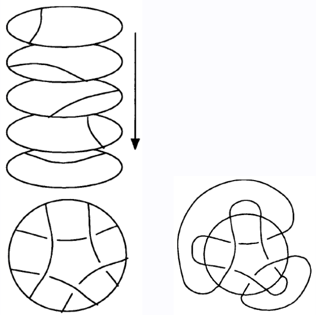
Let H be a handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then

$$\alpha(H) \geq c(L) + 3,$$

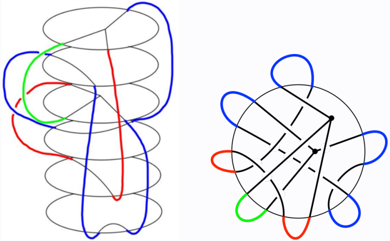
where $c(L)$ is the crossing number of L.



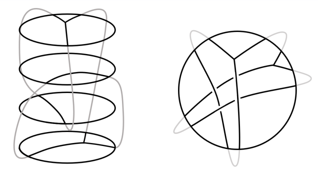
Stacked Tangle of an θ-Curve and Handcuff Graph



Stacked Tangle of a Link



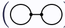
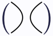
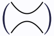
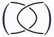
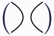
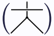
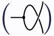
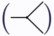
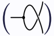
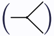
θ-Curve



Handcuff Graph

Yamada Polynomials

Let D_G be a diagram of an θ-curve or handcuff graph G. Then, the **Yamada Polynomial** $R(D_G) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** R () $= -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** R () $= 0$
- **Y8:** $R(G \cup$ $) = (x+1+x^{-1})R(G)$ for an arbitrary θ-curve or handcuff graph diagram G
- **Y9:** R () $- R$ () $= (x - x^{-1})\left[R\right.$ () $- R$ ()$\left.\right]$
- **Y10:** R () $= x^2 R$ (), R () $= x^{-2} R$ ()
- **Y11:** R () $= R$ () **Y12:** R () $= R$ ()
- **Y13:** R () $= R$ (), R () $= R$ ()
- **Y14:** R () $= -xR$ (), R () $= -x^{-1}R$ ()

Proposition (Yamada, 1989)

$R(D_G)$ is an ambient isotopy invariant of G up to multiplying $(-x)^n$ for some integer n.

Lower Bound from Yamada Polynomial

Theorem

Let G be a θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4} \leq \alpha(G).$$

Sketch of Proof

Proposition

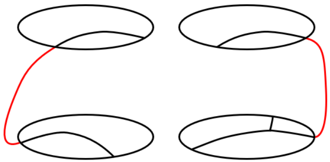
*Let S_G be a simple closure of stacked tangle of a θ-curve or handcuff graph G **without nested caps**. Then*

$$\max\deg_x R(S_G) \leq c+n, \quad \min\deg_x R(S_G) \geq -(c+n),$$

where c and n is the number of caps and crossings in S_G, respectively.

Proof

- Let c_s and c_{ss} be the number of **simple caps** and **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Sketch of Proof

Proposition

Let S_G be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to minimal arc presentation of G. Then

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) - 2n \leq \alpha(G)$$

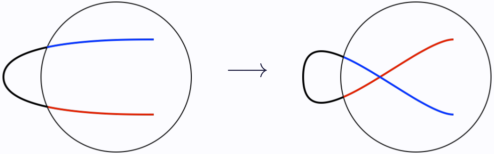
where n is the number of crossings in S_G.

Proof

- S_G is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_G is exactly arc index of G, $\alpha(G)$.

Sketch of Proof

- Take a cap and add a positive or negative curl



	S_G^{neg}	S_G^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

$$\min\deg_x R(S_G) - 2p = \min\deg_x R\left(S_G^{neg}\right) \geq -c + -(n+p)$$
$$\max\deg_x R(S_G) + 2(c-p) = \max\deg_x R\left(S_G^{pos}\right) \leq c + [n+(c-p)]$$
$$\max\deg_x R(S_G) - \min\deg_x R(S_G) \leq c + 2n \qquad \square$$

Sketch of Proof

Theorem

Let G be a θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4} \leq \alpha(G).$$

Proof

Let S_G be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to minimal arc presentation of G.

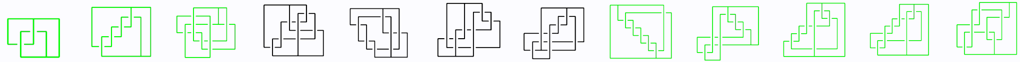
- The number of caps : $\alpha(G)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(G) - 3$

The theorem is followed by considering the maximum of crossings in $S_G gsin S_G$ and applying previous theorems.

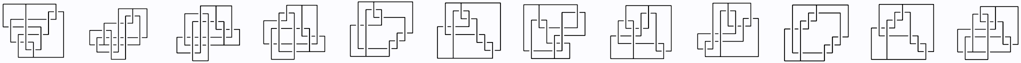
Minimal Grid Diagrams

Grid Diagrams of θ-Curves

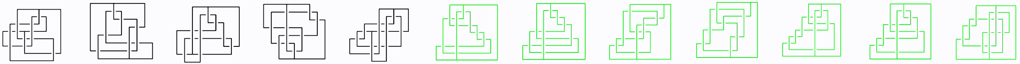
Grid Diagrams of Handcuff Graphs



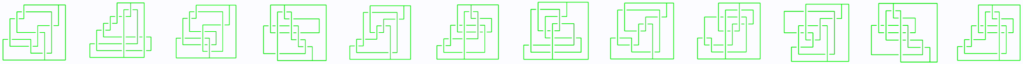
2_1 4_1 5_1 6_1 6_2 6_3 6_4 6_5 6_6 6_7 6_8 6_9



7_1 7_2 7_3 7_4 7_5 7_6 7_7 7_8 7_9 7_{10} 7_{11} 7_{12}



7_{13} 7_{14} 7_{15} 7_{16} 7_{17} 7_{18} 7_{19} 7_{20} 7_{21} 7_{22} 7_{23} 7_{24}



7_{25} 7_{26} 7_{27} 7_{28} 7_{29} 7_{30} 7_{31} 7_{32} 7_{33} 7_{34} 7_{35} 7_{36}

Further Studies

Further Studies

- Using another tool rather than Topoly package of Python to completely determine the arc indices.
- Applying the result of our research to graph of higher crossings or other spatial graphs.

References

References

1 Moriuchi, H. (2019). A table of θ-curves and handcuff graphs with up to seven crossings. *Advanced Studies in Pure Mathematics*, 281–290.

2 Yoonsang Lee. (2023). A Study on Arc Index of Theta-Curves. Korea Science Academy of KAIST

3 Minjung Lee, Sungjong No, and Seungsang Oh. (2018). Arc index of spatial graphs. *Journal of Graph Theory, 90*(3), 406–415.

4 Yamada, S. (1989). An invariant of spatial graphs. *Journal of Graph Theory, 13*(5), 537–551.