The Determinant and Arc Indices of θ-Curves and Handcuff-Graphs

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Lower and Upper Bounds of Arc Index

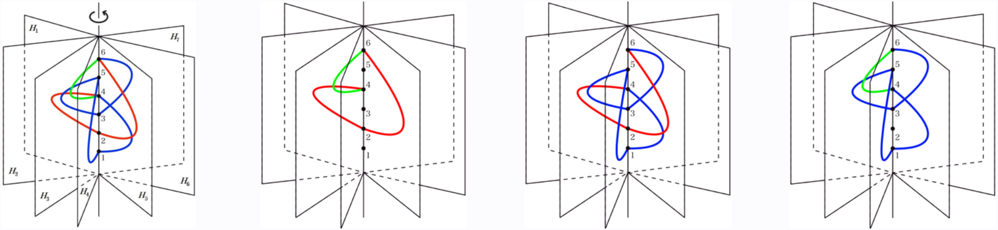
Lower Bounds from Constituent Knots/Links

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Proof



$\square$

Lower Bounds from Constituent Knots/Links

Theorem

Let H be any handcuff graph and L be constituent link of H. Then

$$\alpha(H) \geq \alpha(L) + 1$$

Proof

It is similar to the previous proof. $\square$

Lower Bounds from Constituent Knots/Links

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2}\sum_{i=1}^{3}\alpha(K_i)$$

Proof

- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i.
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i.

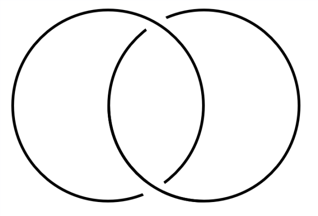
$$\alpha(K_i) \leq |S_i| + |S_{i+1}| \implies \sum_{i=1}^{3}\alpha(K_i) \leq 2\sum_{i=1}^{3}|S_i| = 2\alpha(T)$$

$\square$

Theorem

If L is an alternating and non-split link, then

$$\alpha(L) = c(L) + 2.$$



Lower Bounds from Constituent Knots/Links

Theorem

Let H be any handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then

$$\alpha(H) \geq c(L) + 3.$$

Proof

- We know $\alpha(H) \geq \alpha(L) + 1$.
- By the previous theorem, $\alpha(L) = c(L) + 2$.
- Thus, $\alpha(H) \geq c(L) + 3$.

$\square$

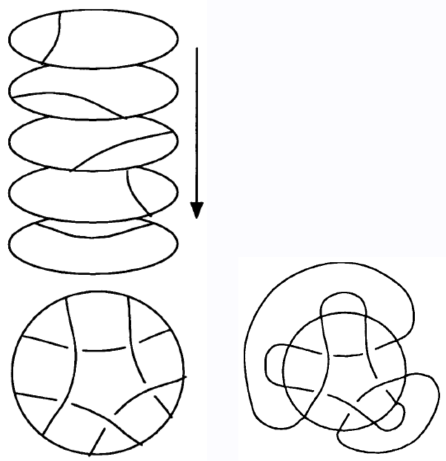
Lower Bounds from Constituent Knots/Links

Corollary

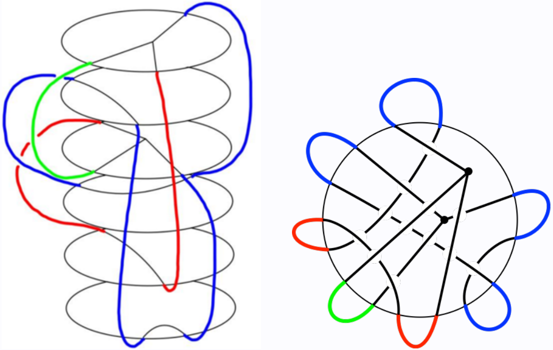
Let H be any handcuff graph, and L be a constituent link of T. If L is alternating and non-split,

$$\alpha(H) = c(L) + 3.$$

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link



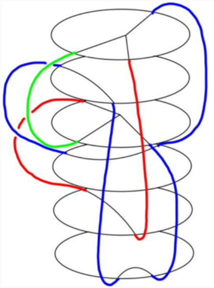
Stacked Tangle of a θ-Curve

Figure from [?]

Stacked Tangle of an θ-Curve

Stacked tangle of an θ-curve is stacked disks each with the frame as boundary with following properties:

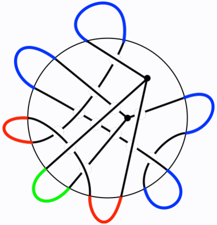
- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
  - two arcs in different simple disks intersect at most one point(by RII)
  - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Stacked Tangle of an θ-Curve

Simple closure of stacked tangle is a stacked tangle with caps satisfying following properties:

- A cap is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.



Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

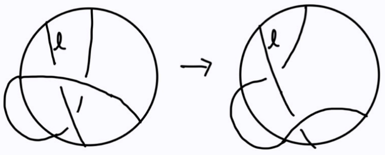
A reduced simple closure of a stacked tangle is

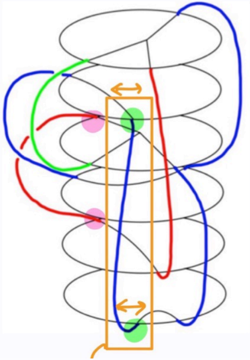
- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

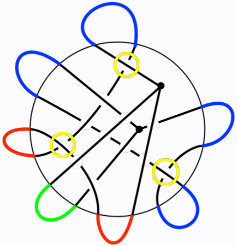
Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

Proof







Yamada Polynomials

Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** $R\left(\;\right) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R\left(\;\right) = 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** $R\left(\;\right) - R\left(\;\right) = (x - x^{-1})\left[R\left(\;\right) - R\left(\;\right)\right]$
- **Y10:** $R\left(\;\right) = x^2 R\left(\;\right)$, $R\left(\;\right) = x^{-2} R\left(\;\right)$
- **Y11:** $R\left(\;\right) = R\left(\;\right)$ **Y12:** $R\left(\;\right) = R\left(\;\right)$
- **Y13:** $R\left(\;\right) = R\left(\;\right)$, $R\left(\;\right) = R\left(\;\right)$
- **Y14:** $R\left(\;\right) = -xR\left(\;\right)$, $R\left(\;\right) = -x^{-1}R\left(\;\right)$

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proposition

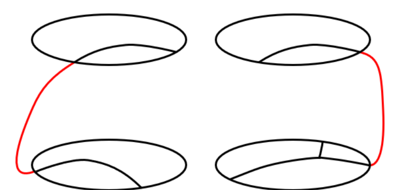
Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T ***without nested caps****. Then*

$$\max\deg_x R(S_T) \le c+n, \quad \min\deg_x R(S_T) \ge -(c+n),$$

where c, n *is the number of caps and crossings in S_T, respectively.*

Proof

- Let c_s, c_{ss} be the number of **simlpe caps** and **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.

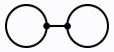


Basis Step:

When $c_s + c_{ss} = 0$,

- S_T has no simple disks.
- S_T is equivalent to the result of applying Y14 to .
- Thus, $R(S_T) = -x^{\pm 3}\left[-x^2 - x - 2 - x^{-1} - x^{-2}\right] \implies 5 \leq c + n$.

When $n = 0$,

- S_T is equivalent to  $\cup \bigcirc \cup \cdots \cup \bigcirc$.
- Thus, $R(S_T) = 0 \implies 0 < 2 \leq c + n$.

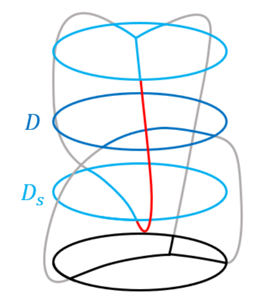
All base cases satisfy the inequality.

Inductive Step:

Assume that it holds for any $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$, and $c_s + c_{ss} > 0$.

Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T such that the number of simple caps, semi-simple caps, and crossings are c_s, c_{ss}, n, respectively.

Take the topmost simple disk D_s connected to the top disk, and a disk D directly above D_s.



Proof of Theorem

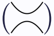
CASE 1. Suppose that there is no cap between D_s and D.

① Suppose that there is no intersection between D_s and D in S_T.

- D_s and D do not affect each other.
- We can swap the position of D_s and D without affecting the rest of the diagram.

② Suppose that there is an intersection between D_s and D in S_T.

- Let S_T^-, S_T^0 and S_T^∞ be the simple closure of stacked tangle which is obtained by replacing  with ,)(and , respectively.
- The simple caps, semi-simple caps, and crossings of the both are $c_s, c_{ss}, n-1$.
- Applying Y9

$R($  $) - R($  $) = (x - x^{-1})\,[R()() - R($  $)]$,

then

$$R(S_T) - R(S_T^-) = (x - x^{-1})(R(S_T^0) - R(S_T^\infty)).$$

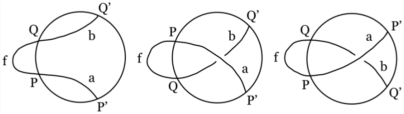
- Then, it is sufficient to show that the interchanged one holds.

Proof of Theorem

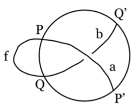
CASE 2. Suppose that there is a cap between D_s and D.

① Suppose that D is a simple disk.

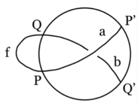
- When view from above, there are three cases:



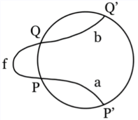
- After applying Y10, the second and third cases can be regarded as the first case, and the cap can be reduced.



or



$\longrightarrow$



S_T S'_T

- S'_T has $c-1$ caps, c_s-1 simple caps, c_{ss} semi-simple caps and $n-1$ crossings.

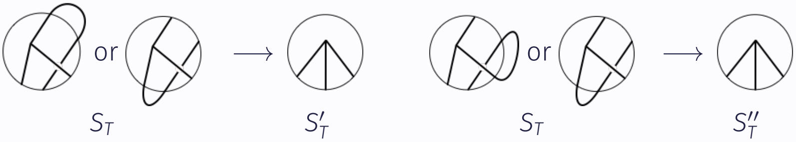
Proof of Theorem

- By induction hypothesis,

$$
\begin{aligned}
\max\deg_x R(S_T) &= \max\deg_x R(S'_T) \pm 2 \\
&\leq [(c-1)+(n-1)] \pm 2 \\
&\leq c+n, \\
\min\deg_x R(S_T) &= \min\deg_x R(S'_T) \pm 2 \\
&\geq -[(c-1)+(n-1)] \pm 2 \\
&\geq -(c+n).
\end{aligned}
$$

② D is not a simple disk.

- When viewed from above, all the cases can be reduced as follows.



- $R(S_T) = -x^{\pm 1} R(S'_T)$ and $R(S_T) = x^{\pm 2} R(S''_T)$ by **Y14** and **Y10**, respectively.
- Both of S'_T and S''_T have $c-1$ caps, c_s simple caps, $c_{ss}-1$ semi-simple caps, and $n-1$ crossing.

Proof of Theorem

- By induction hypothesis, in the first case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S'_T) \pm 1\\ &\leq [(c-1)+(n-1)] \pm 1\\ &\leq c+n.\end{aligned}$$

- Similarly, in the second case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S''_T) \pm 2\\ &\leq [(c-1)+(n-1)] \pm 2\\ &\leq c+n.\end{aligned}$$

- It holds for $\min\deg_x R(S_T)$ in the same way.

$\square$

Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T. Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

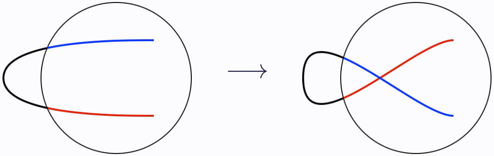
where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the p negative($c - p$ positive) curls.

Proof of Theorem

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

- $R\left(S_T^{neg}\right) = x^{-2p}R(S_T)$ and $R\left(S_T^{pos}\right) = x^{2(c-p)}R(S_T)$

$$\begin{aligned}
\min\deg_x R(S_T) - 2p &= \min\deg_x R\left(S_T^{neg}\right) \\
&\geq -c + -(n+p) \\
\max\deg_x R(S_T) + 2(c-p) &= \max\deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n + (c-p)] \\
\min\deg_x R(S_T) &\geq -c - n + p \\
\max\deg_x R(S_T) &\leq n + p \\
\max\deg_x R(S_T) - \min\deg_x R(S_T) &\leq c + 2n
\end{aligned}$$

$\square$

Proof of Theorem

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proof

Let S_T be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

Proof of Theorem

① Let T be any θ-curve.

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)$
- number of crossings by a simple disk and non-simple disk : $2(\alpha(T)-3)$
- number of crossings by two non-simple disks : 2
- number of crossings counted by disks joined by cap : $\alpha(T)-2$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4) + 2(\alpha(T)-3) + 2 - (\alpha(T)-2) \\ &= \frac{1}{2}\left[(\alpha(T))^2 - 5\alpha(T) + 8\right] \end{aligned}$$

By Lemma,

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8$$

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

② Let T be any handcuff graph.

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)$
- number of crossings by a simple disk and non-simple disk : $2(\alpha(T)-3)$
- number of crossings by two non-simple disks : 1
- number of crossings counted by disks joined by cap : $\alpha(T)-1-2=\alpha(T)-3$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}(\alpha(T)-3)(\alpha(T)-4)+2(\alpha(T)-3)+1-(\alpha(T)-3) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

$$\max\deg_x R(S_T)-\min\deg_x R(S_T) \leq 2n+\alpha(T) \leq \alpha(T)^2-4\alpha(T)+8$$

$$2+\sqrt{\max\deg_x R(S_T)-\min\deg_x R(S_T)-4} \leq \alpha(T) \qquad \square$$