The Determinant and Arc Indices of θ-Curves and Handcuff-Graphs

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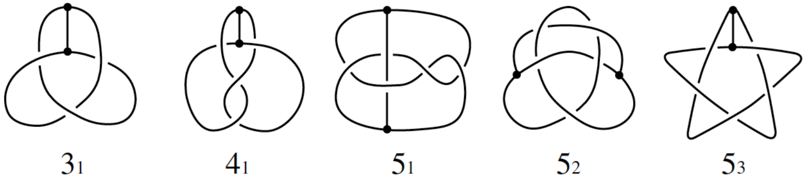
Introduction

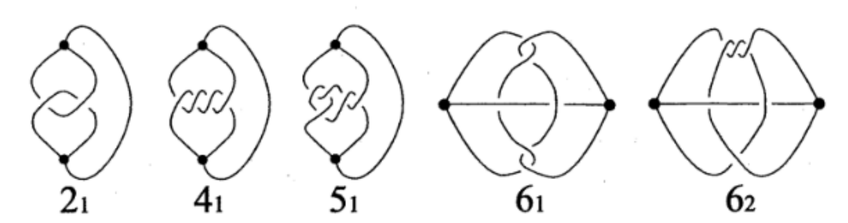
θ-Curves

- A ***θ*-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot** T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.
- θ-curves are roughly classified by comparing the triples of constituent knots.
- A θ-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3.

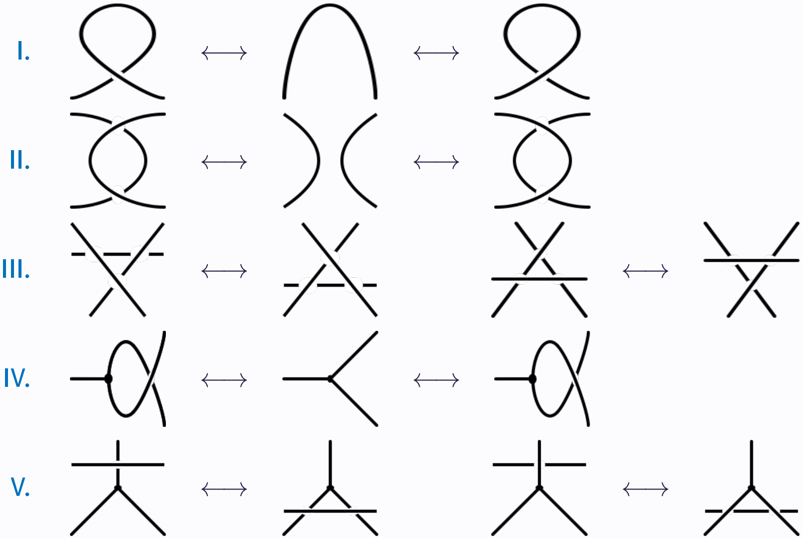


Handcuff Graphs

- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.
- A **constituent link H_{12}**, is a subgraph of H that consists of two vertices v_1, v_2 and two edges e_1, e_2.



Reidemaister Moves for θ-Curves and Handcuff Graphs



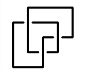
Arc Presentation

- Arc presentation of a θ-curve or handcuff graph is an embedding of them.
- It is contained in the union of finitely many half planes (called pages).
- The embedding is with the common boundary line (called axis).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.
- Arc index, is the minimal number of pages among all possible arc presentations of graph.
- This arc presentation with the minimal number of pages is minimal arc presentation.

Arc Presentation

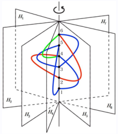
Trefoil

Open Book



Grid Diagram

$\theta_{5,2}$



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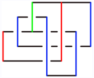
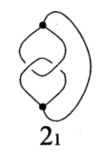
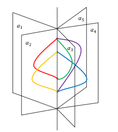


Grid Diagram



$\Phi_{2,1}$



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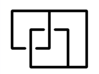


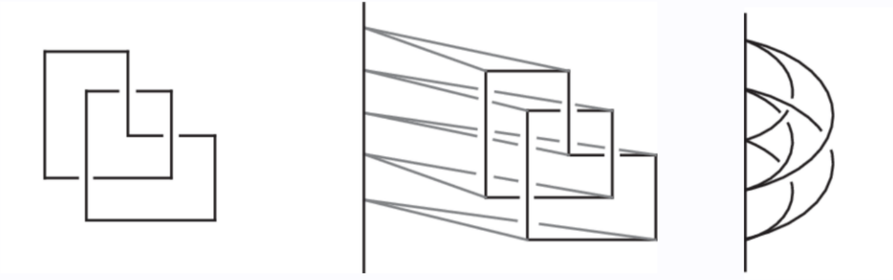
Grid Diagram

Grid Diagram

- The **grid diagram** of θ-curve or handcuff graph is a diagram with only vertical and horizontal strands.
- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Grid Diagram

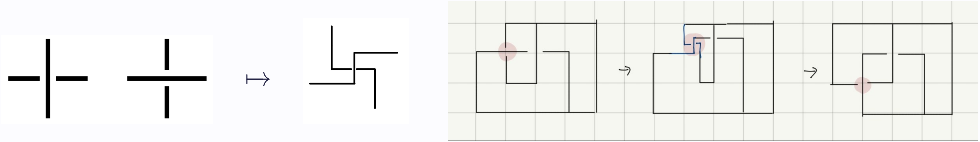
- A grid diagram gives rise to an arc presentation and vice versa.



Theorem

Every θ-curve and handcuff graph admit a grid diagram.

Proof



Corollary

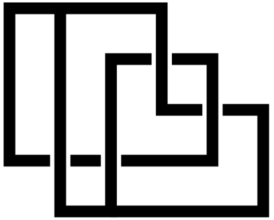
Every θ-curve and handcuff graph admit an arc presentation.

Determinant of θ-curve and Handcuff graph

THC-cromwell matrix

- The **Cromwell Matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-cromwell matrix** is an expansion of cromwell matrix into θ-curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\longmapsto \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

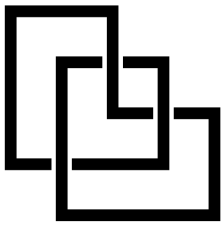
Determinants of the cromwell matrices of Knot

Theorem

Let K be a knot. Then the determinant of a cromwell matrix of K is 0 or ± 2.

Proof

$\xmapsto{\text{grid diagram}}$



$$\xmapsto{\textit{Cromwell}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xmapsto[\textit{operations}]{\textit{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

CASE 1. When n is an even number.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the determinant of K is 0.

CASE 2. When n is an odd number.

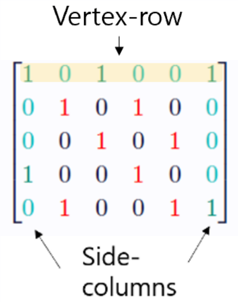
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

So the determinant of K is ± 2. $\square$

H-deletion of THC-cromwell matricies

- The **vertex-row** of THC-cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic}, where $a < b < c$, as its elements.
- The **side-column** of THC-cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-cromwell matrix G is $(n-1) \times (n-1)$ matrix which deleted vertex-row and its two side-columns from the matrix G.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto$$



$$\longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Determinants of the THC-cromwell matrices

Theorem

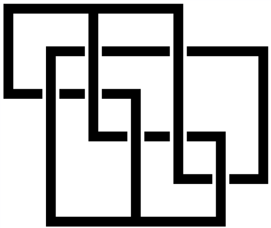
Let M be a THC-cromwell matrice of θ-curve or handcuff graph.

- *$det^*(M) = \pm 1 \iff M$ represents θ-curve*
- *$det^*(M) = 0$ or $\pm 2 \iff M$ represents handcuff graph*

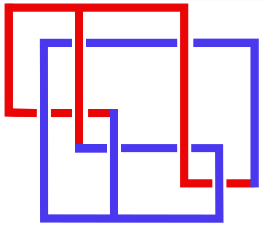
where $det^*(M)$ = determinant of H-deletion matrix of M

Proof

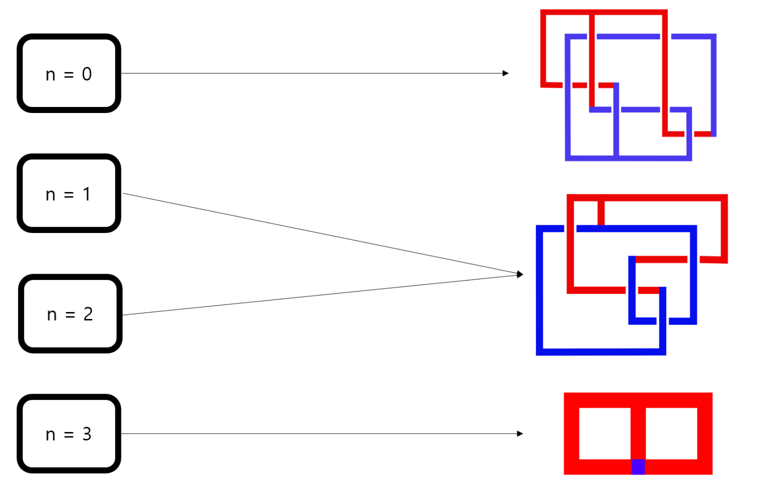
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



$$\xmapsto{H-deletion}$$



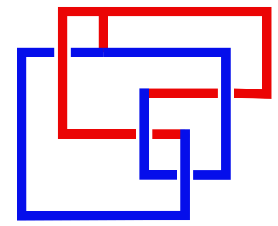
CASE 1. When M represents θ-curve



CASE 1. When M represents θ-curve

ⓘ Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H\text{—}deletion} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

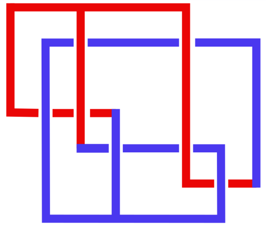
$$\xmapsto[operations]{row/column} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xmapsto{subtracting} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det^*(M) = \pm 1$

CASE 1. When M represents θ-curve

ⓘⓘ T-shape

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\xmapsto[operations]{row/column} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xmapsto{regioning} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det^*(M) = \pm 1$

Lower and Upper Bounds of Arc Index

Upper Bounds of Arc Index

Theorem

Let T be any θ-curve. Then,

$$\alpha(T) \leq c(T) + 3.$$

Theorem

Let H be any handcuff graph. Then,

$$\alpha(H) \leq c(H) + 5.$$

Especially, if the constituent link of H is non-split,

$$\alpha(H) \leq c(H) + 3.$$

Upper Bounds of Arc Index

Theorem

Let T be any non-trivial prime θ-curve or handcuff graph. Then,

$$\alpha(T) \leq c(T) + 3.$$

Lower and Upper Bounds of Arc Index

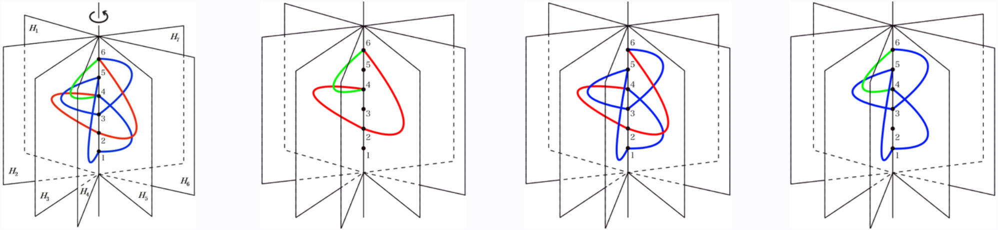
Lower Bounds from Constituent Knots/Links

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Proof



$\square$

Lower Bounds from Constituent Knots/Links

Theorem

Let H be any handcuff graph and L be constituent link of H. Then

$$\alpha(H) \geq \alpha(L) + 1$$

Proof

It is similar to the previous proof. $\square$

Lower Bounds from Constituent Knots/Links

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2}\sum_{i=1}^{3}\alpha(K_i)$$

Proof

- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i.
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i.

$$\alpha(K_i) \leq |S_i| + |S_{i+1}| \implies \sum_{i=1}^{3}\alpha(K_i) \leq 2\sum_{i=1}^{3}|S_i| = 2\alpha(T)$$

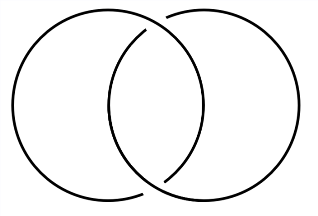
$\square$

Lower Bounds from Constituent Knots/Links

Theorem

If L is an alternating and non-split link, then

$$\alpha(L) = c(L) + 2.$$



Lower Bounds from Constituent Knots/Links

Theorem

Let H be any handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then

$$\alpha(H) \geq c(L) + 3.$$

Proof

- We know $\alpha(H) \geq \alpha(L) + 1$.
- By the previous theorem, $\alpha(L) = c(L) + 2$.
- Thus, $\alpha(H) \geq c(L) + 3$.

$\square$

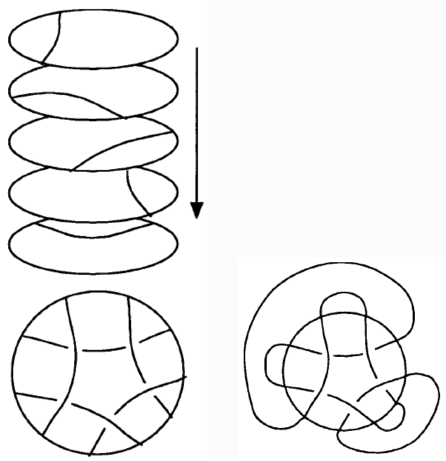
Lower Bounds from Constituent Knots/Links

Corollary

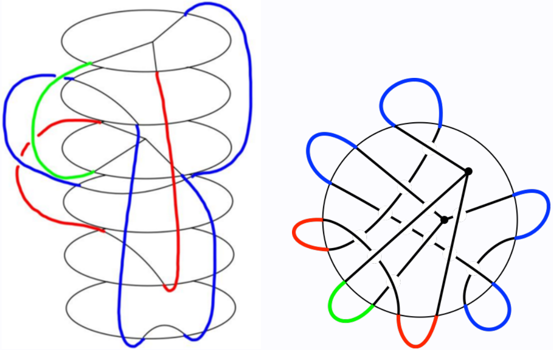
Let H be any handcuff graph, and L be a constituent link of T. If L is alternating and non-split,

$$\alpha(H) = c(L) + 3.$$

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link



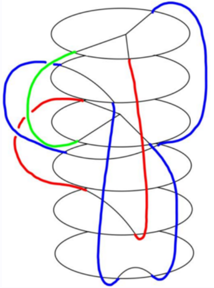
Stacked Tangle of a θ-Curve

Figure from [?]

Stacked Tangle of an θ-Curve

Stacked tangle of an θ-curve is stacked disks each with the frame as boundary with following properties:

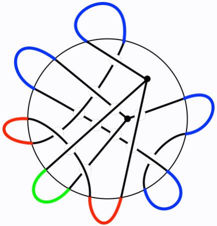
- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
    - two arcs in different simple disks intersect at most one point(by RII)
    - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Stacked Tangle of an θ-Curve

Simple closure of stacked tangle is a stacked tangle with caps satisfying following properties:

- A cap is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.



Then a simple closure of a stacked tangle without any nested caps is corresponding to an arc presentation.

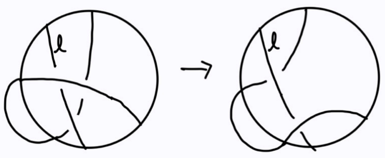
A reduced simple closure of a stacked tangle is

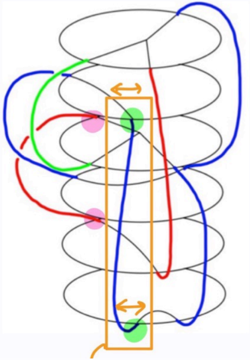
- a simple closure of a stacked tangle without any nested caps
- any two arcs(including line segment) joining by caps have no intersection when view from above

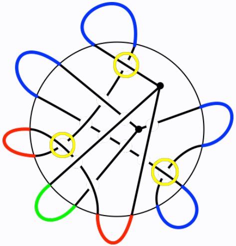
Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

Proof







Yamada Polynomials

Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial** $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** $R\left(\;\right) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R\left(\;\right) = 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** $R\left(\;\right) - R\left(\;\right) = (x - x^{-1})\left[R\left(\;\right) - R\left(\;\right)\right]$
- **Y10:** $R\left(\;\right) = x^2 R\left(\;\right)$, $R\left(\;\right) = x^{-2} R\left(\;\right)$
- **Y11:** $R\left(\;\right) = R\left(\;\right)$ **Y12:** $R\left(\;\right) = R\left(\;\right)$
- **Y13:** $R\left(\;\right) = R\left(\;\right)$, $R\left(\;\right) = R\left(\;\right)$
- **Y14:** $R\left(\;\right) = -xR\left(\;\right)$, $R\left(\;\right) = -x^{-1}R\left(\;\right)$

Proposition ([?])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proposition

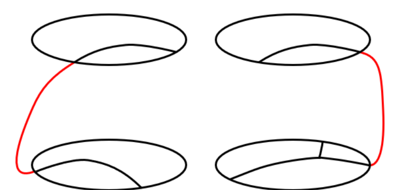
*Let S_T be a simple closure of stacked tangle of a θ-curve or handcuff graph T **without nested caps**. Then*

$$\max \deg_x R(S_T) \leq c + n, \quad \min \deg_x R(S_T) \geq -(c + n),$$

*where **c, n** is the number of caps and crossings in S_T, respectively.*

Proof

- Let c_s, c_{ss} be the number of **simlpe caps** and **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.

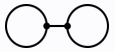


Basis Step:

When $c_s + c_{ss} = 0$,

- S_T has no simple disks.
- S_T is equivalent to the result of applying Y14 to .
- Thus, $R(S_T) = -x^{\pm 3}\left[-x^2 - x - 2 - x^{-1} - x^{-2}\right] \implies 5 \leq c + n$.

When $n = 0$,

- S_T is equivalent to  $\cup \bigcirc \cup \cdots \cup \bigcirc$.
- Thus, $R(S_T) = 0 \implies 0 < 2 \leq c + n$.

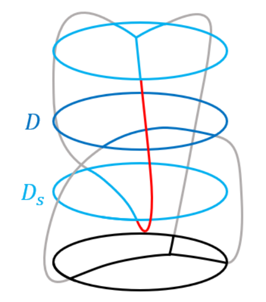
All base cases satisfy the inequality.

Inductive Step:

Assume that it holds for any $(c'_s + c'_{ss}, n') < (c_s + c_{ss}, n)$, and $c_s + c_{ss} > 0$.

Let S_T be a **simple closure of stacked tangle** of a θ-curve or handcuff graph T such that the number of simple caps, semi-simple caps, and crossings are c_s, c_{ss}, n, respectively.

Take the topmost **simple disk D_s** connected to the top disk, and a **disk D** directly above D_s.



Proof of Theorem

CASE 1. Suppose that there is no cap between D_s and D.

① Suppose that there is no intersection between D_s and D in S_T.

- D_s and D do not affect each other.
- We can swap the position of D_s and D without affecting the rest of the diagram.

② Suppose that there is an intersection between D_s and D in S_T.

- Let S_T^-, S_T^0 and S_T^∞ be the simple closure of stacked tangle which is obtained by replacing with , $)($ and , respectively.
- The simple caps, semi-simple caps, and crossings of the both are $c_s, c_{ss}, n-1$.
- Applying Y9

$$R\left(\;\right) - R\left(\;\right) = (x - x^{-1})\left[R\left()(\right) - R\left(\;\right)\right],$$

then

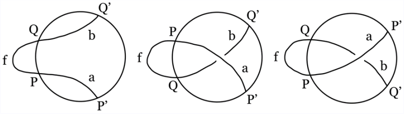
$$R(S_T) - R(S_T^-) = (x - x^{-1})(R(S_T^0) - R(S_T^\infty)).$$

- Then, it is sufficient to show that the interchanged one holds.

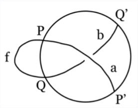
CASE 2. Suppose that there is a cap between D_s and D.

① Suppose that D is a simple disk.

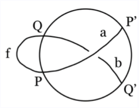
- When view from above, there are three cases:



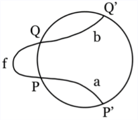
- After applying Y10, the second and third cases can be regarded as the first case, and the cap can be reduced.



or



$\longrightarrow$



S_T $\qquad$ S'_T

- S'_T has $c-1$ caps, c_s-1 simple caps, c_{ss} semi-simple caps and $n-1$ crossings.

Proof of Theorem

- By induction hypothesis,

$$\begin{aligned}
\max\deg_x R(S_T) &= \max\deg_x R(S'_T) \pm 2 \\
&\leq [(c-1)+(n-1)] \pm 2 \\
&\leq c+n, \\
\min\deg_x R(S_T) &= \min\deg_x R(S'_T) \pm 2 \\
&\geq -[(c-1)+(n-1)] \pm 2 \\
&\geq -(c+n).
\end{aligned}$$

Proof of Theorem

② *D* is not a simple disk.

- When viewed from above, all the cases can be reduced as follows.

$S_T \qquad S'_T \qquad S_T \qquad S''_T$

- $R(S_T) = -x^{\pm 1}R(S'_T)$ and $R(S_T) = x^{\pm 2}R(S''_T)$ by **Y14** and **Y10**, respectively.
- Both of S'_T and S''_T have $c-1$ caps, c_s simple caps, $c_{ss}-1$ semi-simple caps, and $n-1$ crossing.

Proof of Theorem

- By induction hypothesis, in the first case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S'_T) \pm 1 \\ &\leq [(c-1)+(n-1)] \pm 1 \\ &\leq c+n.\end{aligned}$$

- Similarly, in the second case,

$$\begin{aligned}\max\deg_x R(S_T) &= \max\deg_x R(S''_T) \pm 2 \\ &\leq [(c-1)+(n-1)] \pm 2 \\ &\leq c+n.\end{aligned}$$

- It holds for $\min\deg_x R(S_T)$ in the same way.

$\square$

Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T. Then

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n \leq \alpha(T)$$

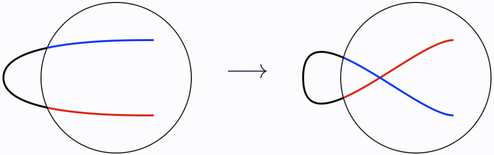
where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

Proof of Theorem

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- S_T^{neg}(S_T^{pos}) is the diagram obtained by inserting the p negative($c - p$ positive) curls.

Proof of Theorem

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

- $R\left(S_T^{neg}\right) = x^{-2p}R(S_T)$ and $R\left(S_T^{pos}\right) = x^{2(c-p)}R(S_T)$

$$\begin{aligned}
\min\deg_x R(S_T) - 2p &= \min\deg_x R\left(S_T^{neg}\right) \\
&\geq -c + -(n+p) \\
\max\deg_x R(S_T) + 2(c-p) &= \max\deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n + (c-p)] \\
\min\deg_x R(S_T) &\geq -c - n + p \\
\max\deg_x R(S_T) &\leq n + p \\
\max\deg_x R(S_T) - \min\deg_x R(S_T) &\leq c + 2n \qquad \square
\end{aligned}$$

Proof of Theorem

Theorem

Let T be any θ-curve or handcuff graph. Then

$$2+\sqrt{\max\deg_x R(S_T)-\min\deg_x R(S_T)-4}\leq\alpha(T)$$

where $R(T)$ is a Yamada Polynomial of T.

Proof

Let S_T be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T)-3$

Proof of Theorem

① Let T be any θ-curve.

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings by two non-simple disks : 2
- number of crossings counted by disks joined by cap : $\alpha(T)-2$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)+2\left(\alpha(T)-3\right)+2-\left(\alpha(T)-2\right) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8$$

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T)$$

Proof of Theorem

② Let T be any handcuff graph.

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\binom{\alpha(T)-3}{2} = \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings by two non-simple disks : 1
- number of crossings counted by disks joined by cap : $\alpha(T)-1-2=\alpha(T)-3$

Thus

$$\begin{aligned} n &\leq \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)+2\left(\alpha(T)-3\right)+1-\left(\alpha(T)-3\right) \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+8\right] \end{aligned}$$

By Lemma,

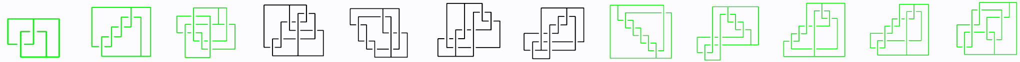
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) \leq 2n + \alpha(T) \leq \alpha(T)^2 - 4\alpha(T) + 8$$

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) - 4} \leq \alpha(T) \qquad \square$$

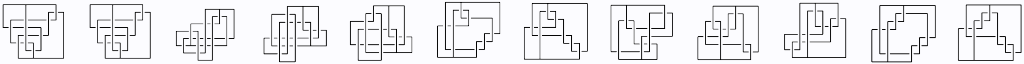
Result

Grid Diagrams of θ-Curves

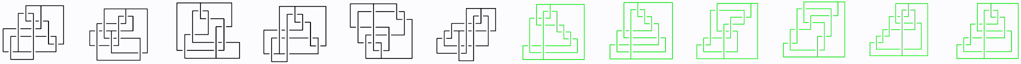
Grid Diagrams of Handcuff-Graphs



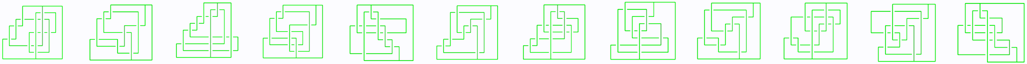
2_1 4_1 5_1 6_1 6_2 6_3 6_4 6_5 6_6 6_7 6_8 6_9



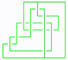
7_1 7_1 7_2 7_3 7_4 7_5 7_6 7_7 7_8 7_9 7_{10} 7_{11}



7_{12} 7_{13} 7_{14} 7_{15} 7_{16} 7_{17} 7_{18} 7_{19} 7_{20} 7_{21} 7_{22} 7_{23}



7_{24} 7_{25} 7_{26} 7_{27} 7_{28} 7_{29} 7_{30} 7_{31} 7_{32} 7_{33} 7_{34} 7_{35}



7_{36}

Further Research

Further Research

- We tried to use the Python for determining arc indices. For the Python program, we used the Topoly package, but the package had an error. We would like to use another tool to completely determine the arc indices of θ-curves and handcuff-graphs, such as Yamada package.
- Applying the result of our research to θ-curves or handcuff-graphs of higher crossings or to other spatial graphs can be researched further.