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Embedding knots and links in an open book I: Basic properties

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Abstract

Birman and Menasco recently introduced a new way of presenting knots and links together with a corresponding link invariant. This paper examines the fundamental properties of this *arc-presentation*. In particular, a set of moves is described which relate two different arc-presentations of the same knot, and the behaviour under the knot operations of distant union and connected sum is established.

Keywords: Knot; Link; Knot invariant; Connected sum

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In their study of the braid index of satellite links [4], Birman and Menasco introduce a new way of presenting knots and links and a corresponding knot invariant. This presentation of a link involves embedding it in a sheaf of half-planes. More specifically, we note that there is an open-book decomposition of the 3-sphere which has open discs as pages and an unknotted circle as the binding. We can think of the 3-sphere as $\mathbb{R}^3 \cup \{\infty\}$ and of the circle as the z-axis $\cup\{\infty\}$. If we use polar coordinates in the x-y plane then the pages are half-planes H_θ at angle θ. A link L can be embedded in finitely many such half-planes so that it meets each half-plane in a single simple arc. Such an embedding is called an *arc-presentation* of L. The minimum number of planes required to present a given link in this manner is a knot invariant which we shall call the *arc index* of L and denote by $\alpha(L)$.

Birman and Menasco were studying the braid index of satellite links. The binding circle of an arc-presentation of the companion knot can be used as a braid axis for the satellite. They conjecture that the braid index of the double of a knot K is determined by $\alpha(K)$ and the framing. More generally, we might expect that

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the braid index of a satellite depends on the arc index of the companion and some properties of the pattern.

This paper establishes some fundamental properties of this new presentation. In particular, we discuss the equivalence of such embeddings via a finite sequence of moves (analogous to the Reidemeister or Markov moves), and study the behaviour of arc index under the operations of distant union and connected sum. This leads to the following result.

Theorem.
(a) $\alpha(L_1 \sqcup L_2) = \alpha(L_1) + \alpha(L_2)$.
(b) $\alpha(L_1 \# L_2) = \alpha(L_1) + \alpha(L_2) - 2$.

In a follow-up paper [6], the arc index of many links is related to the crossing number of a link, $c(L)$. The following upper bound on arc index is derived: $\alpha(L) \leqslant c(L) + 2$. This result is established by describing an algorithm for drawing a binding circle on a link diagram. We also show a lower bound which grows as the square root of $c(L)$.

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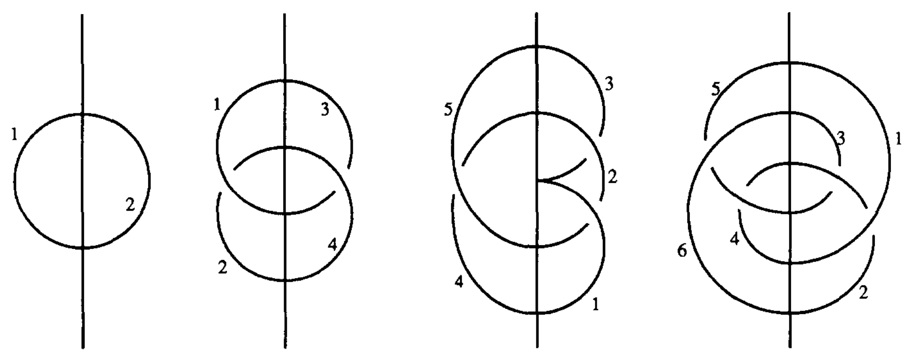
1. Introduction

We begin by giving some examples of arc-presentations and then show that arc index is well defined. Fig. 1 shows arc-presentations of (a) the unknot, (b) the Hopf link, (c) the trefoil, (d) the figure-8 knot. These are presented with two, four, five and six arcs respectively. The straight line represents the binding circle and the semicircles represent the arcs. The order of the arcs around the binding circle is indicated by the numbering.



(a) (b) (c) (d)

Fig. 1.

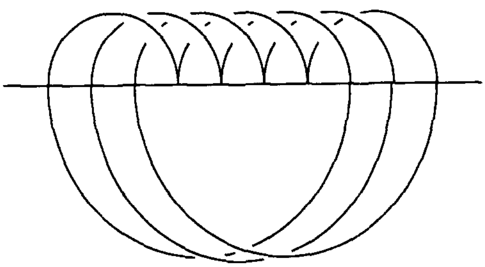


Fig. 2.

The knot in Fig. 2 is the (3, 7) torus knot. Its construction can be generalised to show that the arc index of a (p, q) torus link is at most $p+q$.

Different ways of drawing arc-presentations can provide other information. Two alternatives are described below.

Firstly, we can draw a circle to represent the binding circle and mark n points on it to be the endpoints of n arcs. The arcs are then chords across the circle — two meeting at each of the n points. With this method it is easy to enumerate the links of small arc index. The possibilities when $n=2$, 3 and 4 are illustrated in

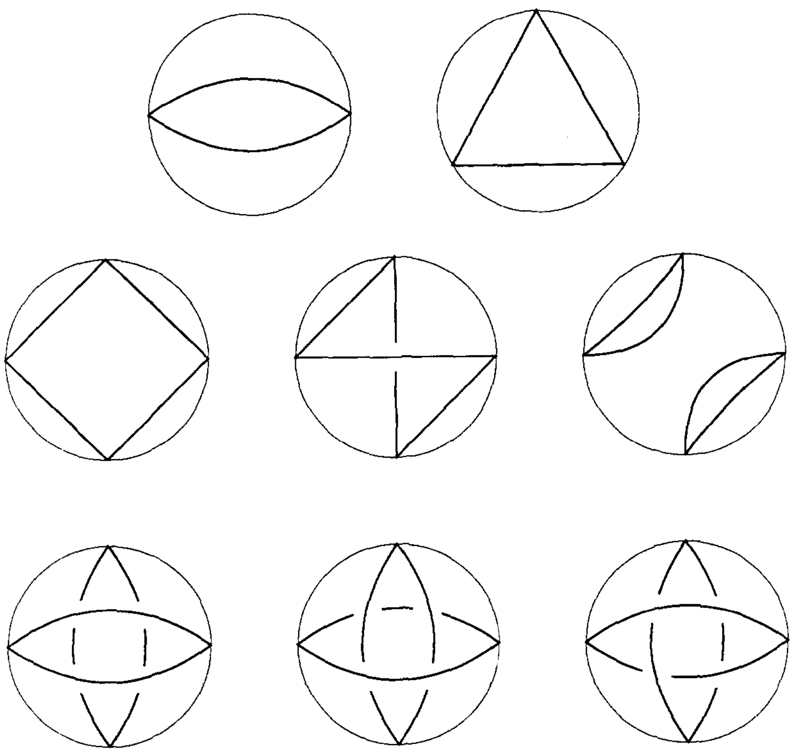


Fig. 3.

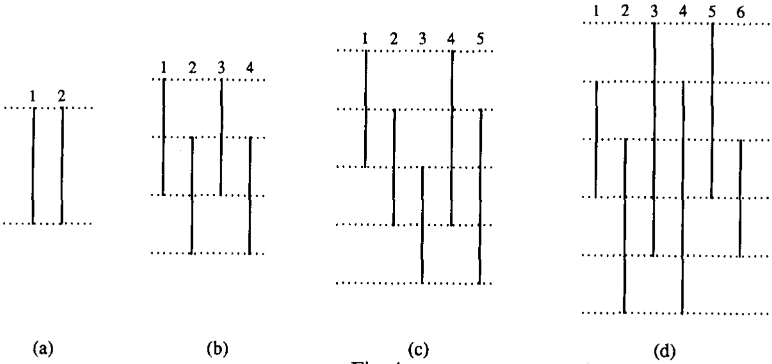


Fig. 4.

Fig. 3. This shows that the unknot has arc index 2, the Hopf link and the 2-component unlink have arc index 4. There is no link with arc index 3. Since the arc index of any other link must be at least 5 and Fig. 1(c) shows an arc-presentation of the trefoil with five arcs, we have $\alpha(3_1) = 5$. We can note in passing that a knot and its mirror image have the same arc index.

The second method involves "unwrapping" an arc-presentation to lie flat in a plane. Given an embedding of a link L in a sheaf of n half-planes with binding circle A, we take a small solid tubular neighbourhood $N(A)$ of A such that $\partial N(A)$ intersects all the half-planes transversely in a single line and each component of $L \cap N(A)$ meets A in a single point. Let $T(A) = \partial N(A)$ and let $P = \{p_1, \ldots, p_n\}$ be the set of points of $L \cap A$. We can define a height function h in $N(A)$ by projection onto A. Isotop L so that h is constant on each component of $L \cap N(A)$. Replace the component of $L \cap N(A)$ containing p_i with a loop λ_i in $T(A)$ such that $h(\lambda_i) = h(p_i)$. In the figures these loops are drawn as dotted lines.

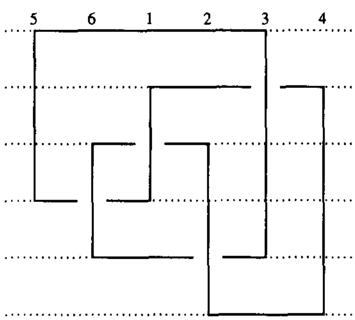


Fig. 5.

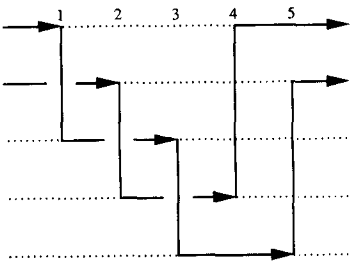


Fig. 6.

Suppose that the arc in half-plane H_θ connects p_i to p_j. Draw an arc in $H_\theta \cap T(A)$ connecting the loops λ_i and λ_j.

The examples of Fig. 1 are redrawn in this format in Fig. 4. The horizontal lines are the loops λ_i and the vertical lines correspond to the arcs. The numbering in Figs. 1 and 4 is the same.

Notice that each loop is divided naturally into two pieces. Making a choice of one of these for each loop will convert the "loops and lines" diagram into a more conventional link diagram. We can achieve useful results by making suitable choices. For example, we can nominate a line in $T(A)$ parallel to A and take the parts of the loops which avoid this line. This produces a diagram of L in a rectangle. Fig. 5 shows the figure-8 knot of Fig. 4(d) after it has been cut between half-planes 4 and 5.

We could also choose pieces of the loops so that when following round the link the angle θ increases monotonically. The link then winds around A, and L is presented as a braid with axis A. The braiding of the trefoil of Fig. 4(c) is shown in Fig. 6. This construction can be used to give a lower bound on the arc index.

Proposition. *Let $s(L)$ denote the braid index of L. Then $\alpha(L) \geqslant 2s(L)$.*

Proof. To convert a "loops and lines" diagram into a braid we must choose an orientation around A. (In the figures this will either be left-to-right or right-to-left.) Let β be an m-string braid constructed from an arc-presentation with n arcs which comprises arcs in half-planes and semi-loops in $T(A)$. At any angle θ, the half-plane H_θ meets β in m components. These will either be m points, or $m-1$ points and an arc. If we choose the other orientation to construct the braid then we get the complementary set of semi-loops and the braid has $n-m$ strings. One of these two braids has at most $n/2$ strings. □

Corollary. *Let* breadth(P_L) *denote the difference in the maximum and minimum degrees in v in the knot polynomial $P_L(v, z)$. Then $\alpha(L) \geqslant$* breadth$(P_L) + 2$.

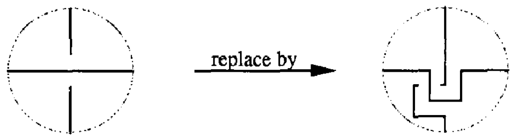


Fig. 7.

Proof. It is known [7,9] that the braid index of a link satisfies

$$2(s(L)-1) \geqslant \operatorname{breadth}(P_L). \qquad \square$$

In general this bound will be very crude since there are only finitely many links with a given arc index but an unlimited number of braids with a given braid index. However, it is sufficient to show that the arc index of the figure-8 knot is 6.

We now show that arc index is well defined.

Proposition. *Every link has an arc-presentation.*

Proof. Let $\pi(L)$ be a projection of L and let D be the associated diagram obtained by indicating the under/over-crossing information at the double points of $\pi(L)$. We can assume that L is piecewise linear and that all the lines in $\pi(L)$ are parallel to one of the two coordinate axes in the plane. (Think of these two directions as vertical and horizontal.) We can further assume that $\pi(L)$ is a regular projection so that all double points are transverse intersections and do not occur at corners. The diagram D has crossings at the double points of $\pi(L)$. If the over-crossing line of a crossing is horizontal then we can make a local adjustment as shown in Fig. 7. Repeating this wherever possible we reach a diagram in which all the over-crossings are vertical.

Since there are finitely many line segments in D we can assume that no two are collinear. This produces a diagram of L similar to the "loop and lines" diagrams described above. $\square$

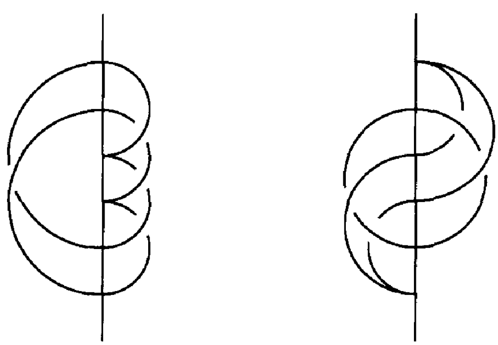


Fig. 8.

Jozef Przytycki noted that "one of the oldest theorems in knot theory" due to Brunn is an easy corollary: every knot has a projection with a unique singular point [5]. His proof is to construct an arc-presentation.

2. Equivalence of arc-presentations by moves

Clearly, a link has many arc-presentations. Even if we consider only those presentations which use the minimum number of arcs, there may not be a unique presentation. Fig. 8 shows two presentations of the (2, 4) torus link each using six arcs. They are not "equivalent" under any natural interpretation of the term since in the left-hand diagram, both the components meet the binding circle three times, while in the other diagram this is not the case.

For presentations of links as diagrams or braids there are sets of combinatorial moves whereby any two presentations of the same link can be interchanged by applying a finite sequence of these moves. They are due to Reidemeister and Markov respectively. There is an analogous result for arc-presentations of links. To describe the moves we need the following notation.

Say that two arcs *interleave* if the endpoints of one alternate with the endpoints of the other around the binding circle. Say that two points of $L \cap A$ *interleave* if the two half-planes containing the arcs meeting at one point alternate with the two half-planes containing the arcs which meet at the other point. The examples in Fig. 9 should make these definitions clear.

The four moves described below are indicated schematically in Fig. 10.

Move I: If two adjacent points of $L \cap A$ do not interleave then they can be interchanged.

Move II: If two arcs in adjacent half-planes do not interleave then they can be interchanged.

Move III: If both endpoints of an arc are adjacent points of $L \cap A$ then the arc can be removed and its endpoints amalgamated. Conversely, an arc can be inserted between two arcs with a common endpoint by separating the two arcs (thereby

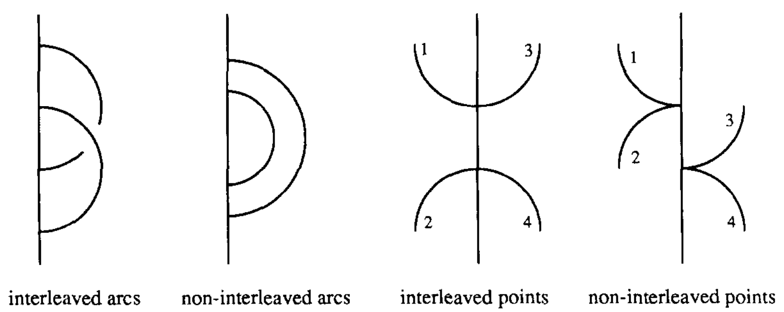


Fig. 9.

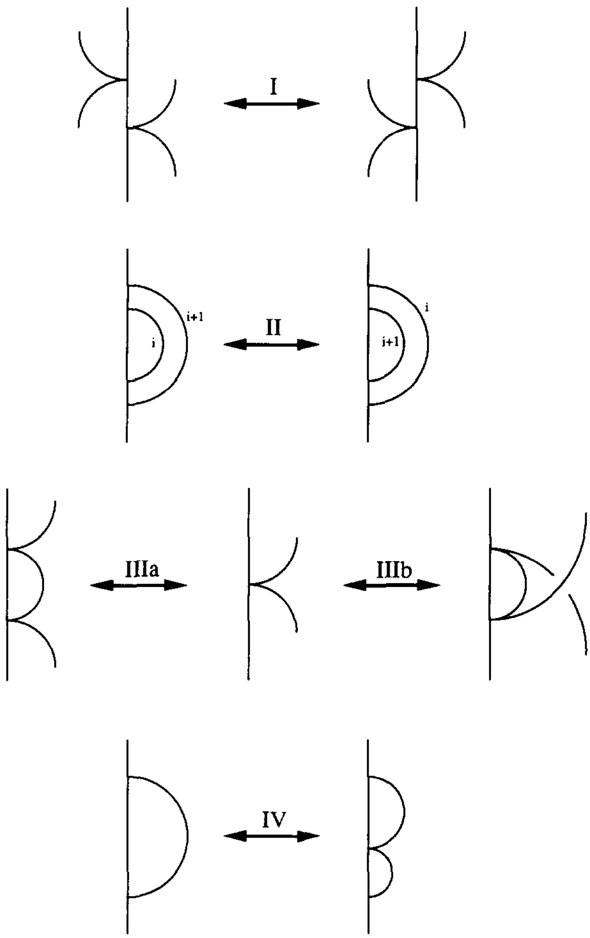


Fig. 10.

adding an extra point to $L \cap A$) and adding a new half-plane containing a new arc. Note that this can be done in two ways, one of which preserves the framing.

Move IV: If the two arcs meeting at a point p are adjacent then they can be amalgamated and p can be removed. Conversely, an arc can be split into two adjacent arcs with the addition of an extra intersection point in $L \cap A$.

These four moves can be thought of as two dual pairs. To see this more clearly notice that an arc-presentation of a link L can be completely described by a square matrix of 0's and 1's. Suppose that L meets the binding circle in n points. Number these points from 1 to n in order round the binding circle. There are also n half-planes containing an arc of L. Number these from 1 to n in order of increasing polar angle. Now take an $n \times n$ matrix and index the columns by arcs, and the rows by points of $L \cap A$. Set the (i, j) entry to 1 if arc i ends at point j, and to zero otherwise. This produces a matrix having exactly two 1's in each row

Fig. 11.

and each column. (The transpose matrix describes the mirror image of L.) Move I says that adjacent columns can be interchanged if their entries are separated. Move II is the same statement applied to the rows. Move III says that two 1's in the same row and adjacent columns can be inserted or deleted by expanding or reducing the matrix. Move IV is the same statement for 1's in the same column and adjacent rows.

Although four moves have been described to exhibit the symmetry involved, one of them is redundant. Move IV can be obtained by repeating move II until move III can be applied.

Theorem. *Any two arc-presentations of L are related by a finite sequence of the moves* I, II, III *described above.*

Proof. Let E_1 and E_2 be two arc-presentations of an oriented link L. From each of E_1 and E_2 we can form a braid presentation as shown in Fig. 6 so that in both cases the braid winds around the binding circle in the same direction. By using move III if necessary we can assume that each crossing looks locally like one of those in Fig. 11. Denote the resulting diagrams by B_1 and B_2.

Since B_1 and B_2 are braid presentations of the same link L, they are related by a finite sequence of Markov moves. (Proofs of Markov's theorem can be found in [1,8,10,12].) We shall show that the Markov moves can be accomplished using moves I to III. The Markov moves can be summarised as follows:

(1) braid-preserving isotopy.

(2) If β is an n-string braid then β can be replaced by the $(n+1)$-string braid $\beta \cdot \sigma_n^{\pm 1}$. Conversely, $\beta \cdot \sigma_n^{\pm 1}$ can be reduced to β.

Braid isotopy is completely described by the braid relations:

(3) $\sigma_i \cdot \sigma_i^{-1} = 1 = \sigma_i^{-1} \cdot \sigma_i$ for all i.

(4) $\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i$ when $|i-j| \geqslant 2$.

(5) $\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$.

Fig. 12 shows how each of (2) $\cdots$ (5) can be achieved by applying a finite sequence of moves I, II and III. □

3. Effects of knot operations on arc index

In this section we investigate how arc index behaves under the knot operations of distant union and connected sum. When taken together, the propositions in this section constitute the theorem stated at the beginning of the paper.

Proposition. *Suppose that a link L is a distant union of two sublinks so that $L = L_1 \sqcup L_2$. Then $\alpha(L) \leqslant \alpha(L_1) + \alpha(L_2)$.*

Proof. Arc-presentations of L_1 and L_2 can be placed together to form an arc-presentation for L. The example in Fig. 13 shows the distant union of a trefoil with a Hopf link. □

Proposition. *Suppose that a link L is a distant union of two sublinks so that $L = L_1 \sqcup L_2$. Then $\alpha(L) \geqslant \alpha(L_1) + \alpha(L_2)$.*

Proof. Suppose that we have an arc-presentation for L. There is a 2-sphere S in $S^3 - L$ which separates L_1 and L_2. Since S does not meet L, every arc in the arc-presentation is either inside or outside S and hence belongs to either L_1 or L_2. Removing the half-planes containing the arcs of L_2 leaves an arc-presentation of L_1, and conversely. □

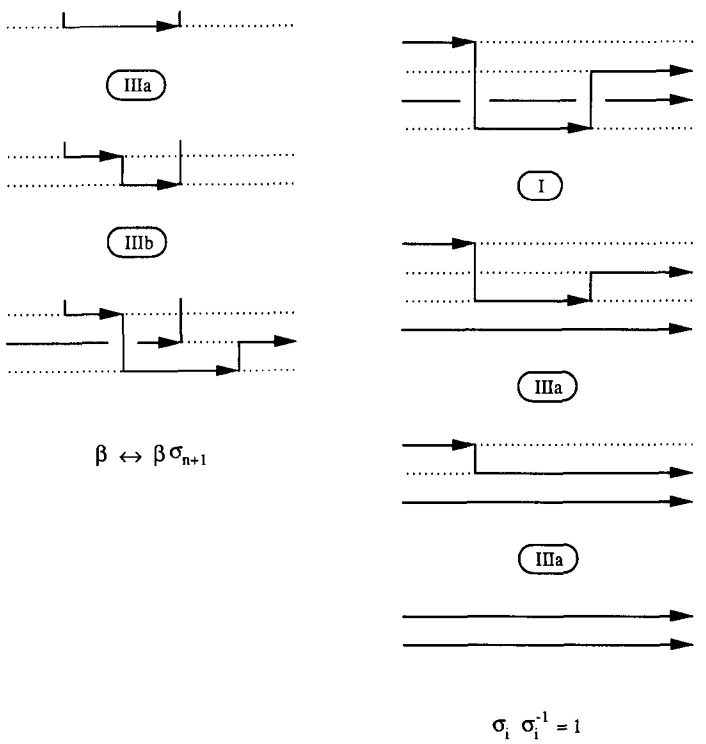


Fig. 12.

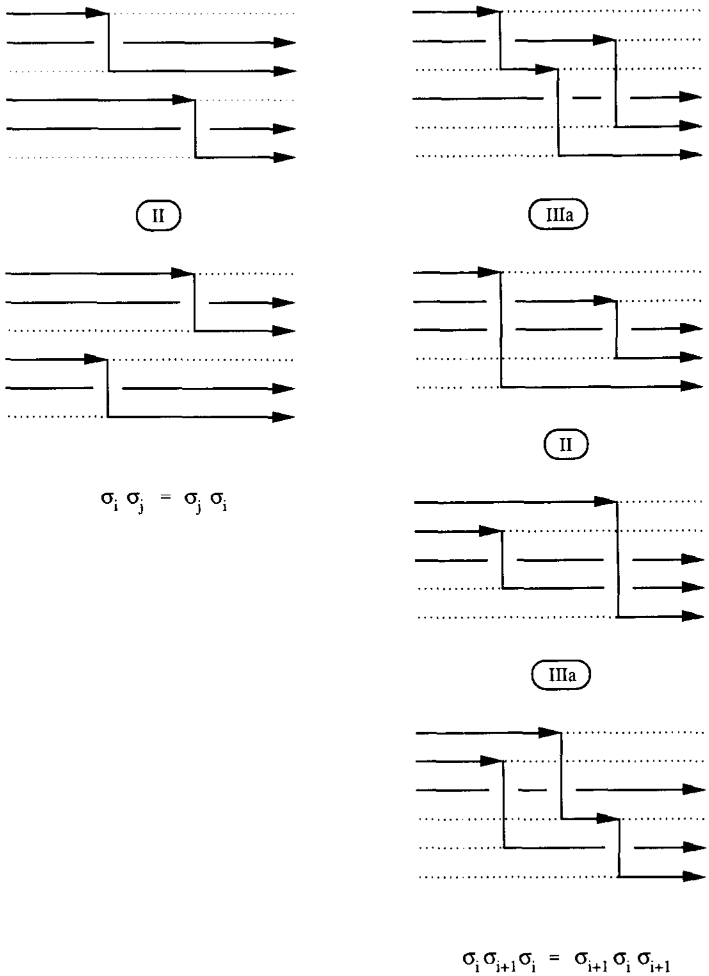


Fig. 12 (continued).

Proposition. *Suppose that a nonsplit link L is a connected sum of two links so that $L = L_1 \# L_2$. Then $\alpha(L) \leqslant \alpha(L_1) + \alpha(L_2) - 2$.*

Proof. Suppose we have arc-presentations of L_1 and L_2 with axes A_1 and A_2. We choose an arc in each factor and remove them. The half-planes containing the remaining arcs of each link L_i can be swivelled around the axis A_i so that they all lie in a half-space. Assume that the angles of the fibres containing L_1 lie in $(0, \pi)$ and that the angles of the fibres of L_2 lie in $(\pi, 2\pi)$. The spacing of the points of $L_i \cap A_i$ can also be adjusted. With a suitable choice of spacing of these points, the axes A_1 and A_2 can be made to coincide so that the only points where L_1 meets

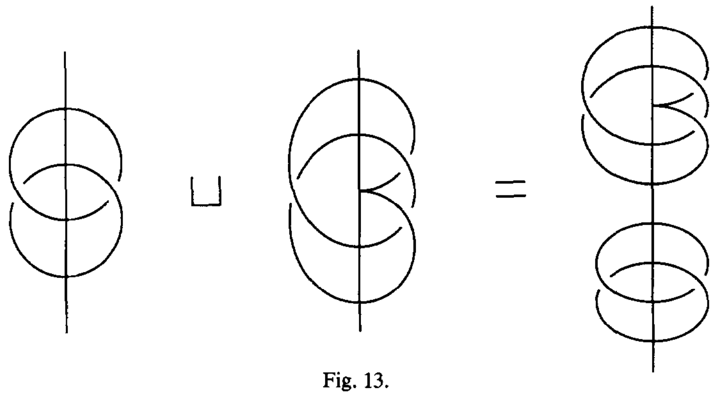


Fig. 13.

L_2 are precisely the free ends produced by the removal of the two arcs. We now have an arc-presentation of the connected sum $L_1 \# L_2$. The process is illustrated in Fig. 14 where the factors are a trefoil and a figure-8 knot. □

Proposition. *Suppose that a nonsplit link L is a connected sum of two links so that* $L = L_1 \# L_2$. *Then* $\alpha(L) \geqslant \alpha(L_1) + \alpha(L_2) - 2$.

Proof. The proof is based on the arguments used by Birman and Menasco in their theorem on the additivity of braid index [2]. My presentation follows theirs and the following discussion concentrates on those points where their method needs modification to fit the current situation. Lemmas 1–4 correspond in both papers.

Suppose that we have an arc-presentation of L with the fewest possible arcs. The aim is to construct arc-presentations for the two factors L_1 and L_2. Let A denote the binding circle. Let S be a 2-sphere which meets L in two points factorising it as $L_1 \# L_2$. To define the sign of some singularities we will need to endow S with an orientation. In the general case A will intersect L many times. To reduce the number of these intersections as much as possible we shall modify the arc-presentation of L by a "generalised type I" move. This is illustrated in

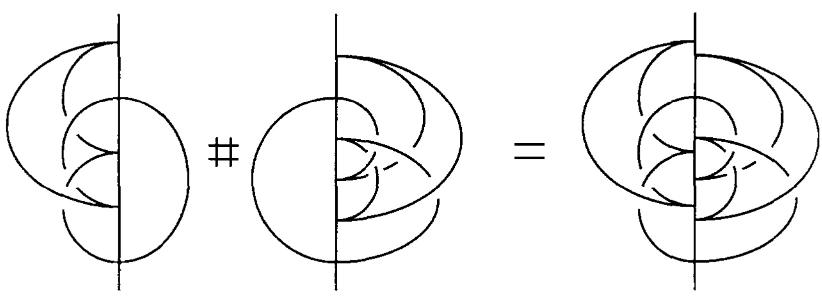


Fig. 14.

Fig. 15.

Fig. 15. It is the same as an ordinary type I move except that several arcs are moved simultaneously. We can think of this as allowing several arcs to lie in the same half-plane with small bubbles at the crossings or as part of a conventional knot diagram in the half-plane. The whole tangle is then moved up or down instead of just one point of $L \cap A$. It is this move which plays the role of the exchange move in [2].

If we can reduce $|S \cap A|$ to 2 then we can recover arc-presentations of the factors of L and obtain bounds on their arc indices.

Lemma 0. *The proposition holds when the sphere meets the binding circle in two points.*

Proof. Suppose that $|S \cap A| = 2$. Let U_1 and U_2 be the two 3-balls bounded by S. Let m_i be the number of arcs lying entirely in the interior of U_i. There are two further arcs both of which meet S. Thus $\alpha(L) = m_1 + m_2 + 2$.

We can form an arc-presentation for L_i as follows. Start with $L \cap U_i$. This is a set of m_i arcs and two semi-arcs which terminate at the points of $S \cap L$. Let p be one of the two points of $S \cap A$. Extend each semi-arc in its own half-plane to connect with p. It is straightforward to see that this closing path can be deformed to lie in S and hence we have an arc-presentation for L_i which uses $m_i + 2$ arcs. Hence we have

$$\begin{aligned} \alpha(L_1) + \alpha(L_2) &\leqslant m_1 + m_2 + 4 \\ &= \alpha(L) + 2. \qquad \square \end{aligned}$$

We need to show how $|S \cap A|$ can be reduced to a useful level. As in [2] we study the singular foliation of S induced by the fibration of $\mathbb{R}^3 - A$ by the half-planes $\{H_\theta \mid \theta \in [0, 2\pi]\}$. The basic set-up is summarised below. The leaves of the foliation are the components of $S \cap H_\theta$ for $\theta \in [0, 2\pi]$. We assign an orientation to S so that there is a well-defined normal vector at each point.

We can place S in general position with respect to A and H so that

(1) S meets A transversally in a finite number of points.

(2) There is a solid torus neighbourhood $N(A)$ of A in $S^3 - L$ such that each component of $N(A) \cap S$ is a disc.

(3) Each component of $N(A) \cap S$ has a radial foliation.

(4) S meets all but finitely many half-planes transversally. Each singular fibre H_θ is tangent to S at exactly one point in the interior of H_θ.

(5) All these tangencies are saddles or local extrema.

(6) The singular fibres do not contain arcs of L.

A leaf of the foliation of S is *singular* if it contains a point of tangency. The other leaves are *nonsingular*. Each nonsingular leaf is either a simple loop or an arc whose endpoints are in $S \cap A$.

The complexity of the pair (S, H) is measured by the complexity function c. A value of c is a pair $(|S \cap A|, |H \cdot S|)$ where the second term denotes the number of saddle singularities in the foliation of S. The values of c are ordered lexicographically.

Say that the pair (S, H) has loops if there exists a nonsingular fibre H_t such that some component of $S \cap H_t$ is a simple loop.

Lemma 1. *Suppose that* (S, H) *satisfies* (1)–(6) *and has loops. Then there is a 2-sphere* S' *which factorises* L *such that* (S', H) *also satisfies* (1)–(6), *has no loops, and moreover,* $c(S', H) \leqslant c(S, H)$.

Proof. The technique for removing simple closed curves by surgery is well known and the details can be found in [2,3]. The idea is sketched below.

Suppose that S intersects a nonsingular fibre in a loop, λ_t. We follow this loop as it evolves in the fibration as we increase or decrease the angle parameter, t. Eventually we must get to a singularity and, since we cannot reach a centre by going both backwards and forwards in the fibration, we can assume that we reach a saddle. Let λ_θ be the loop in the singular fibre H_θ into which λ_t has evolved.

Suppose that λ_θ bounds a disc Δ in H_θ whose interior does not meet S. Since H_θ is a singular fibre, its interior does not meet L and hence $\Delta \cap L = \emptyset$. The loop λ_θ also bounds two discs in S. Because Δ does not meet L one of these discs also avoids L, and the other is punctured twice. Performing a surgery at Δ we replace S by two spheres one of which does not meet L. Since L is nonsplit, the sphere which avoids L is inessential and can be removed. The other is the desired sphere S'.

If the interior of Δ does meet S then a surgery on an innermost loop can be found which achieves the same effect. □

We now have that each component of the intersection of a nonsingular fibre H_t with S is an arc whose endpoints lie on A. These endpoints separate A into two segments. If either of these segments does not meet L then we can isotop S to remove two points of $S \cap A$. This reduces the complexity, and any loops formed by the process can be removed by applying Lemma 1.

We call a leaf β in $S \cap H_t$ *essential* if L meets both segments of $A - \partial\beta$. We can strengthen (1)–(6) above as follows

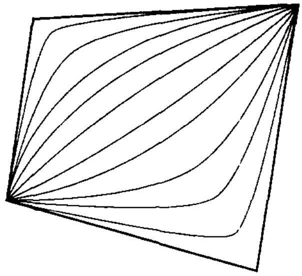


Fig. 16.

(5′) All tangencies are saddles.

(7) If H_t is a nonsingular fibre then each component of $S \cap H_t$ is essential.

We now consider how we can reduce the number of times A pierces S. Assume that $|S \cap A| = n \geqslant 4$. Each nonsingular fibre meets S in $n/2$ arcs connecting the n points of $S \cap A$. Since the foliation is radial near these points, there must be singularities in the foliation each of which is a saddle point tangency of a singular fibre with S. In a singular fibre there will be four leaves emanating from the singular point and ending in $S \cap A$. The complement of these singular leaves in S is a union of regions foliated without singularities, each of which looks like the one in Fig. 16.

We form a tiling of S whose vertices are the points $S \cap A$, and whose edges are obtained by nominating one leaf from each nonsingular region. The 2-cells of this tiling are all 4-gons and will be called tiles. Each tile can be given a sign ($\pm$) according to whether the normal vectors of S and the singular fibre at the point of tangency have the same orientation or not.

Lemma 2. *There is always a 2-valent or 3-valent vertex in the tiling of S.*

Proof. This is an easy consequence of Euler's formula. □

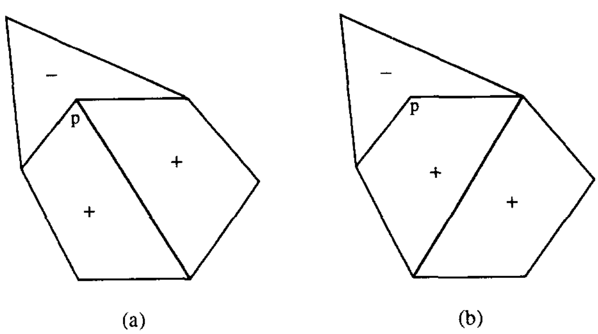
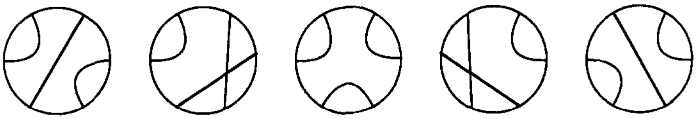


Fig. 17.



In this sequence of fibres an arc of L between the two singular fibres can be moved back or forwards without meeting S until it lies in the leftmost or rightmost picture.

Fig. 18.

Lemma 3. *If there is no 2-valent vertex in the tiling of S then we can isotop S to produce one without increasing the complexity.*

Proof. If there is no 2-valent vertex, there must be a 3-valent vertex, p. Two of the three tiles meeting at this vertex must have the same sign, and we consider these. The aim is to replace the tiling shown in Fig. 17(a) by that in (b) by changing the order in which the singularities appear in the fibration. This proceeds as in [2] to which the reader is referred for the details.

We assume that neither tile meets L. (We shall see later that this is not a limitation.) Fig. 18 shows a sequence of five fibres in increasing order. The bounding circles represent the axis A. The only obstruction to the isotopy will be arcs of L which lie in fibres between the two singular fibres. However, if such an

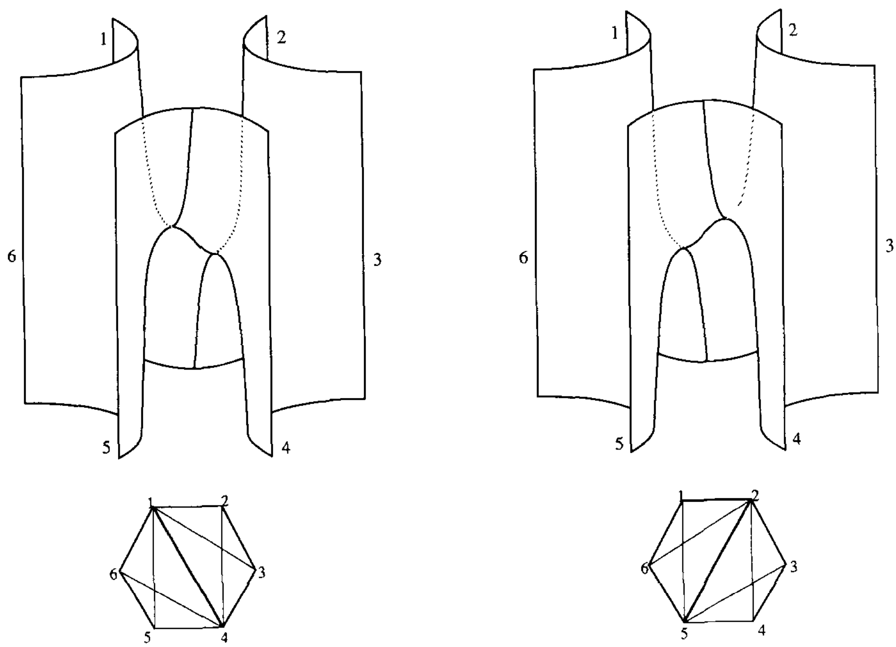


Fig. 19.

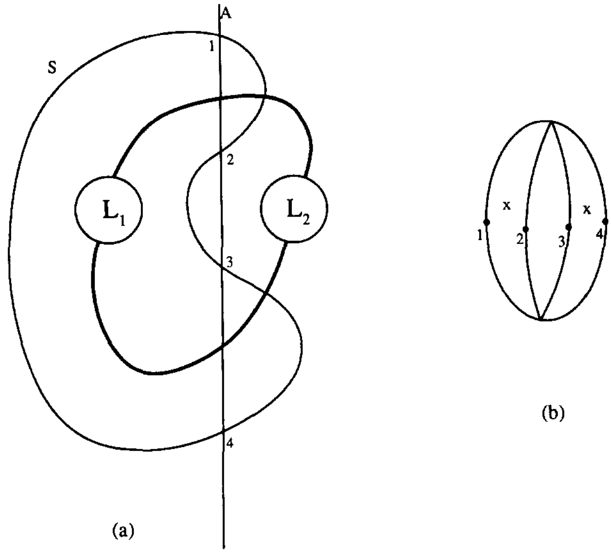


Fig. 20.

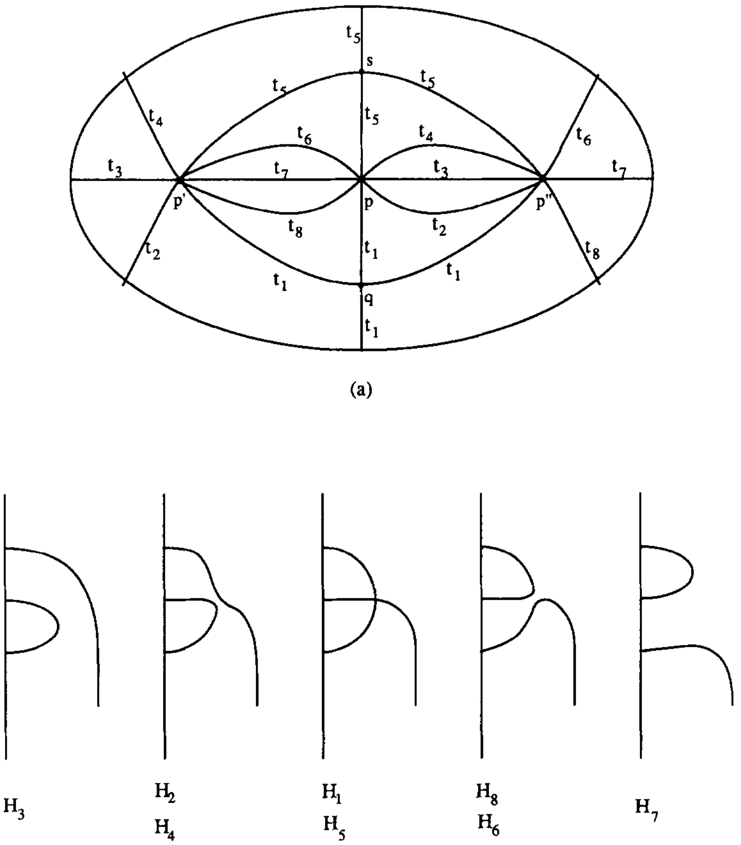
arc exists it can be moved as follows: keeping the arc planar, it can swivel around A keeping its endpoints fixed. It can move forward or backward in the fibration until it lies outside the interval between the two singular fibres. In the figure any arc drawn in the central disc can be transferred to one of the two end-discs. If several arcs obstruct the isotopy they can be moved in sequence preserving the arc-presentation at all times.

We can now isotop S as indicated in Fig. 19 to interchange the two saddle points and thus produce a 2-valent vertex. The vertical lines in the figure are loops in $\partial N(A)$ which encircle the binding circle.

We now return to the problem of why we can assume both tiles avoid L. Say that a region R in S (like that in Fig. 16) bounded by singular leaves and foliated by nonsingular leaves is *bad* if it is pierced by L. There are at most two bad regions and these are adjacent to at most four points of $S \cap A$ — the vertices of the tiling. Therefore if the tiling has more than four vertices, at least one of the vertices is not near a puncture point. On the other hand, the tiling may have four vertices which are all adjacent to bad regions. In this case Birman and Menasco show that the sphere S must be situated as shown in Fig. 20(a) with $|S \cap A| = 4$. The tiling of S is shown in Fig. 20(b): the two points where L punctures S are marked "x".

We now apply a similar construction to that in Lemma 0. Let U_1 and U_2 be the two 3-balls bounded by S. We will form an arc-presentation for L_1 as follows.

Start with $L \cap U_1$. This is a set of m_1 arcs and two semi-arcs which terminate at the points of $S \cap L$. Let p_1 and p_2 be two points of $S \cap A$ which are adjacent on A. Extend each semi-arc in its own half-plane to connect with p_1 or p_2 and so that



(a)

(b)
Fig. 21.

the extensions lie in S. There is a leaf in the foliation of S which connects p_1 to p_2 and which lies in a half-plane that does not contain an arc of L. We use this leaf as an arc to join the two free ends thus forming an arc-presentation having $m_1 + 3$ arcs. The three parts of this closing path (two semi-arcs and a complete arc) lie in S and hence we have constructed L_1. The presentation uses $m_1 + 3$ arcs but we can apply type III move, removing the inserted leaf, to reduce this to $m_1 + 2$. The rest of the argument is the same as that proving Lemma 0. □

Lemma 4. *If the tiling contains a good 2-valent vertex then a generalised type* I *move can be performed on the arc-presentation of L. After the move the complexity of (S, H) can be reduced by removing points of $S \cap A$.*

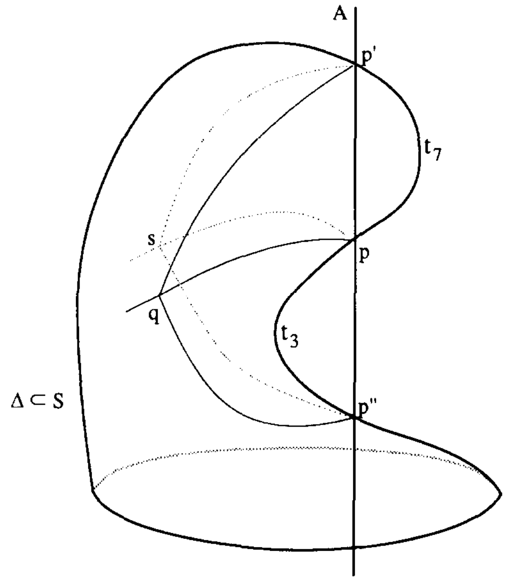


Fig. 22.

Proof. The proof follows that in [2] substituting the generalised type I move for the exchange move. First we outline the discussion which describes how S must be embedded near the 2-valent vertex in order to establish some notation.

Let p be the 2-valent vertex in question, let q and s be the singular points of the two tiles which meet at p, and let p' and p'' be the two vertices adjacent to p in the tiling. Suppose the singular fibres containing q and s occur at angles t_1 and t_5 respectively. Parts of the singular leaves through q and s bound a disc on S which contains p, and a neighbourhood Δ of this region is shown in Fig. 21(a). Other leaves are labelled in the figure occurring at angles $t_1, \ldots, t_8$. In the fibres $H_1, \ldots, H_8$ these leaves are as shown in Fig. 21(b). The arguments in [2] show that Δ must be embedded in $\mathbb{R}^3$ as shown in Fig. 22.

The nonsingular leaves at angles t_3 and t_7 are essential by hypothesis. Therefore, there are points of $L \cap A$ between p and p', and also between p and p''. We shall perform a generalised type I move to transfer the points of intersection between p and p' to a new position just below p''. This will be achieved by an isotopy which takes the given arc-presentation to a new arc-presentation for L and which preserves the tiling of S except for relabelling the vertex set: $S \cap A$. To perform the isotopy in a controlled manner, a region of 3-space is created which will contain all the moving parts. The region is constructed as follows.

Each nonsingular fibre H_t meets S in a unique arc b_t connecting q to p' or p''. This arc, together with A, bounds two discs in H_t. If b_t connects q to p' (respectively p'') choose the disc which does not contain p'' (respectively p') and denote it by μ_t. The closure of the intersection of all these discs $t \in [0, 2\pi]$ is two

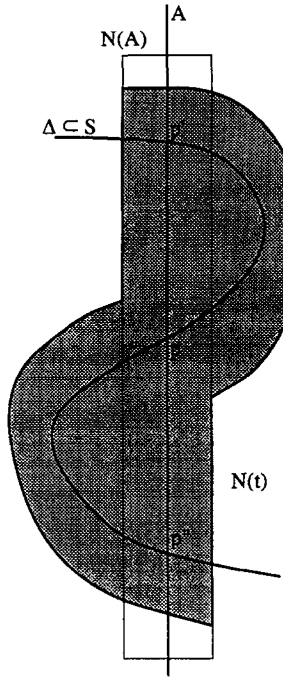


Fig. 23.

3-balls $B_1 \cup B_2$ which intersect along an arc connecting the two singular points q and s via p. The sphere S may meet $B_1 \cup B_2$ in a number of sheets which are locally parallel to Δ and which may intersect A inside $B_1 \cup B_2$.

Reparametrise the interval $[0, 2\pi]$ so that $t_1 = 0$ and $t_5 = \pi$. Then discs μ_t and $\mu_{t+\pi}$ can be paired up so that $\mu_t \subset B_1$ and $\mu_{t+\pi} \subset B_2$ for $t \in [0, \pi)$. Let M_t denote the set $\mu_t \cup \mu_{t+\pi}$.

If H_t is a nonsingular fibre, $t \in (0, \pi)$, then it contains two leaves of the foliation of Δ: b_t connecting p to p'' and a short arc a_t running from p'' to $\partial\Delta$. Similarly, in $H_{t+\pi}$ we have arcs $b_{t+\pi}$ connecting p to p' and $a_{t+\pi}$ connecting p' to $\partial\Delta$. We shall form a neighbourhood of $B_1 \cup B_2$ as follows. Let N_t be a neighbourhood in the plane $H_t \cup H_{t+\pi}$ of $M_t \cup a_t \cup a_{t+\pi}$ such that

(i) $N_t \cap L = M_t \cap L$,

(ii) $N_t \cap S = (M_t \cap S) \cup ((H_t \cup H_{t+\pi}) \cap N(A) \cap S)$ where $N(A)$ is a solid tubular neighbourhood of A.

A typical $N(t)$ is shown in Fig. 23. Choose the neighbourhoods N_t so that they vary smoothly as t goes from 0 to π. Let N be the closure of the set $\{N_t \mid t \in [0, \pi)\}$. The isotopy will be supported inside N.

The isotopy will be carried out in several stages. First we isotop the arcs of L inside B_1 within their own half-planes so that they lie in $N(A)$ — carrying sheets of S along as necessary. Second, we regard the arcs of $L \cap B_1$ as pivoted where

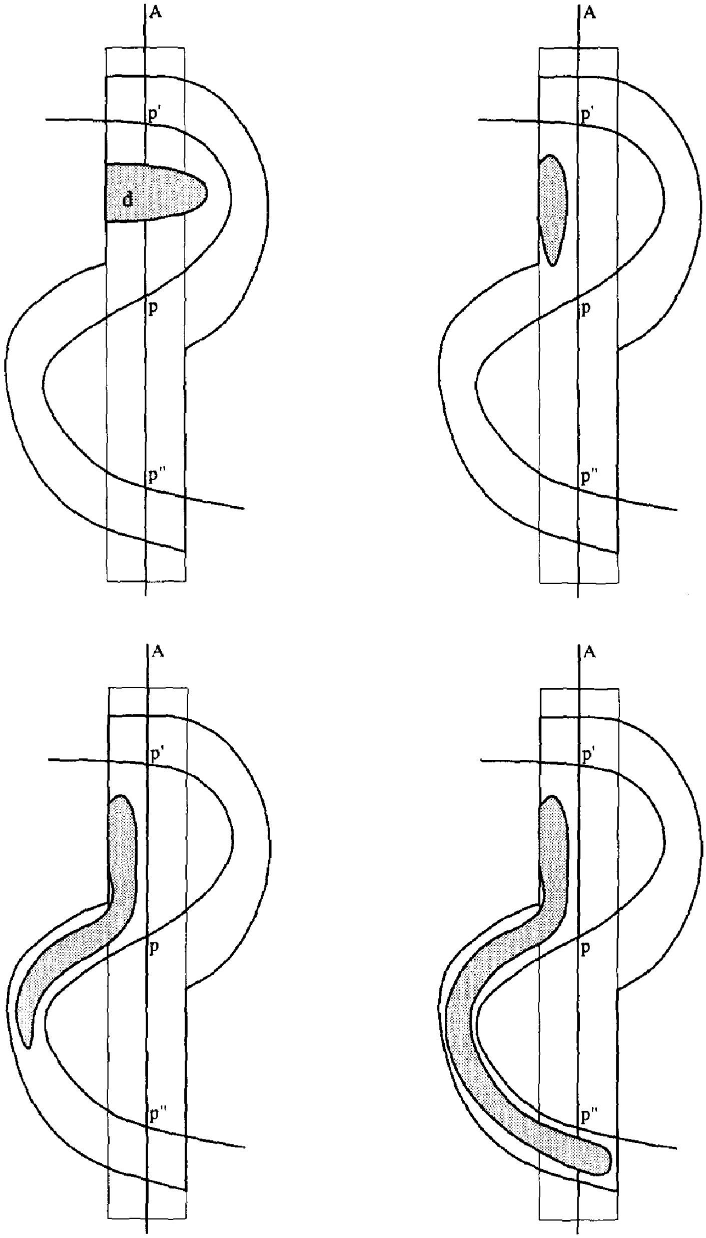


Fig. 24.

they join A and we swivel them round so that each lies in a nonsingular fibre H_t for some $t \in (\pi, 2\pi)$. (This is a kind of generalised type II move.) Thirdly, we detach L from the segment of A between p and p'. To do this take a solid tubular neighbourhood $Z(A)$ of A of small radius so that $Z(A) \subset N(A)$. Then deform the arcs of $Z(A) \cap L$ so that they lie on $\partial Z(A)$ outside B_1.

The critical part of the isotopy is carried out in the same way as the exchange move in [2]. Choose a disc d in $(N(A) \cap \mu_{t+\pi}) \cup \mu_t$ such that

(i) $d \cap L = ((N(A) \cap \mu_{t+\pi}) \cup \mu_t) \cap L$,

(ii) $d \cap S = ((N(A) \cap \mu_{t+\pi}) \cup \mu_t) \cap S - b_{t+\pi}$.

The isotopy pushes the disc d (and the points of L and S meeting it) across A and down into the region below p''. The isotopy is to vary continuously as t varies. At the end, the sheets of S which intersected A between p and p' are to intersect A below p''. This part of the isotopy is indicated by the series of pictures in Fig. 24. It can be formalised as in [2].

The isotopies in the first three stages can now be reversed: the arcs in $\partial Z(A)$ can be reattached to A, then they can be swivelled round A to lie in their original fibres.

The sphere S can be repositioned so that the tiling is the same before and after the isotopy. The only change will be in the order of the points of $S \cap A$ along A. We can now modify S to reduce the complexity because b_{t_7} is no longer an essential arc. $\square$

Note added in proof

Further work on arc index is reported in I.J. Nutt, The braid index of satellite links, Ph.D thesis, Liverpool (1995). Two papers are in preparation.

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