Arc Index of a Theta Curve

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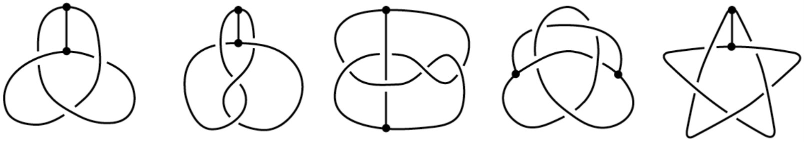
Introduction

θ-Curves

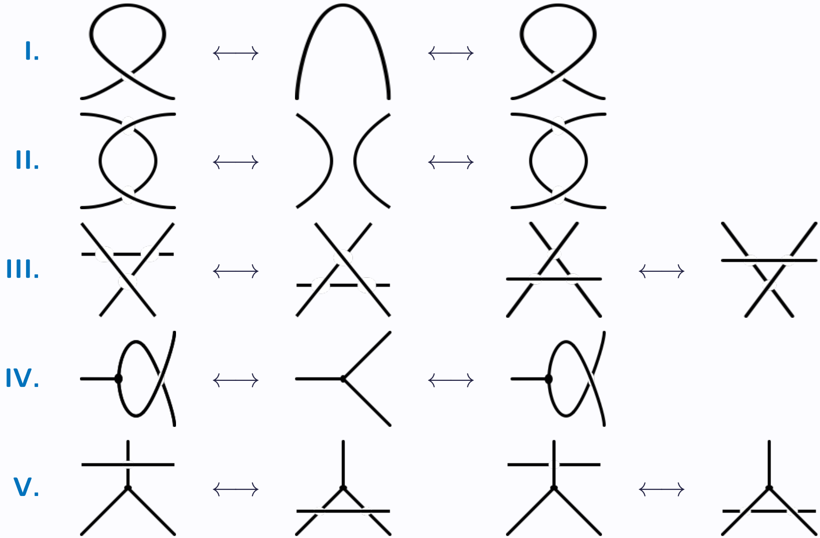
- A **θ-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **constituent knot T_{ij}**, $1 \leq i < j \leq 3$, is a subgraph of T that consists of two vertices v_1, v_2 and two edges e_i, e_j.
- θ-curves are roughly classified by comparing the triples of constituent knots.
- A θ-curve is said to be **trivial** if it can be embedded in a 2-sphere in S^3.



3_1 4_1 5_1 5_2 5_3

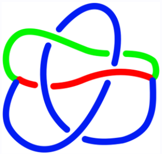
Figure from [Moriuchi, 2007]

Reidemaister Moves for θ-Curves

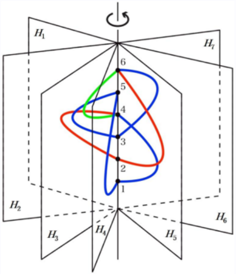


Arc Index

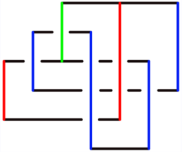
- An **arc presentation** of a θ-curve is defined in the same manner as an arc presentation of a knot.
- The binding axis contains all **vertices** of θ-curve.
- **Minimal arc presentation** and **arc index** are defined in the same manner.



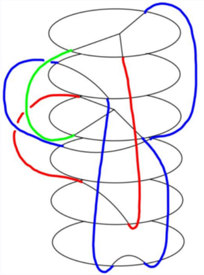
$\theta 5_2$



Open Book

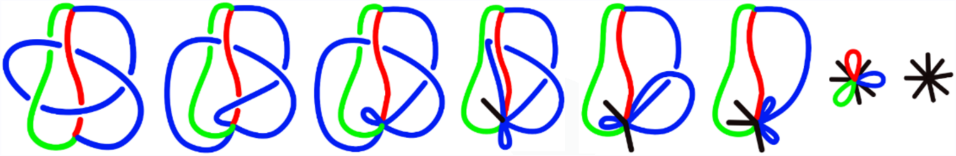


Grid Diagram

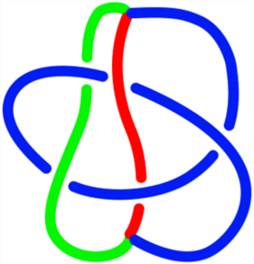


Stacked Tangle

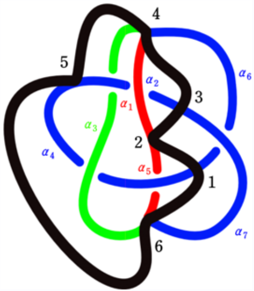
- Gathering all crossings and vertices to one point to obtain a **wheel diagram**, which is corresponding to an arc presentation.



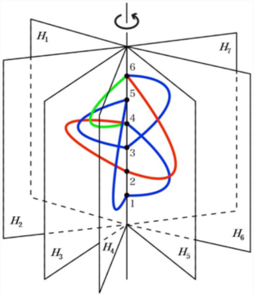
Binding Circle Method



$\longrightarrow$



$\longleftrightarrow$



Upper Bound of Arc Index

Theorem ([Lee et al., 2019])

Let G be any spatial graph with e edges and b bouquet cut components. Then

$$\alpha(G) \leq c(G) + e + b$$

Corollary

Let T be any θ-curve. Then

$$\alpha(T) \leq c(T) + 3$$

Lower Bounds of Arc Index

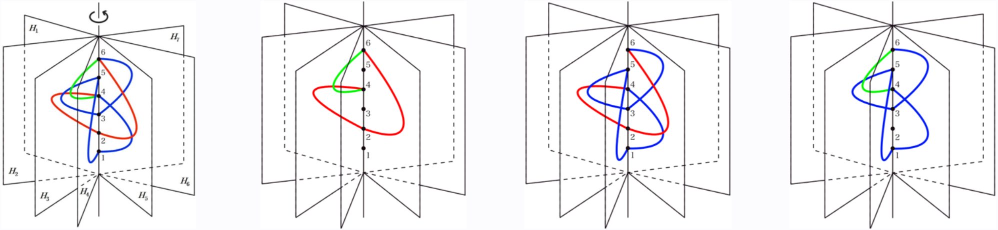
Lower Bounds from Constituent Knots

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Proof



$\square$

Theorem

Let T be any θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2}\sum_{i=1}^{3}\alpha(K_i)$$

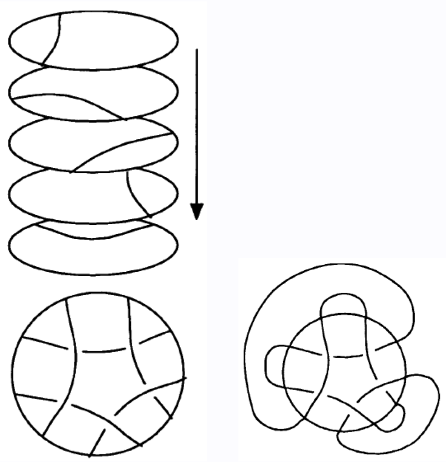
Proof

- A minimal arc presentation of T is given.
- $K_1 = e_1 \cup e_2$, $K_2 = e_2 \cup e_3$, and $K_3 = e_3 \cup e_1$.
- S_i be the set of half plane corresponding the edge e_i.
- $S_i \cup S_{i+1}$ form an arc presentation of the knot K_i.
- $\alpha(K_i) \leq |S_i| + |S_{i+1}|$

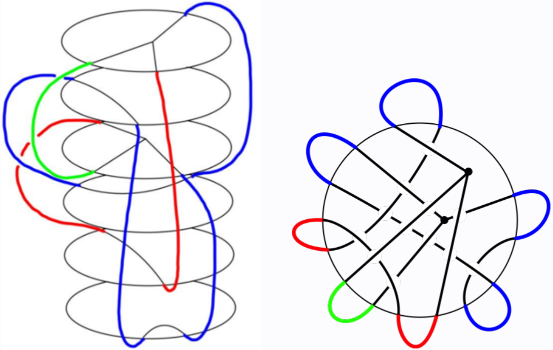
$$\sum_{i=1}^{3}\alpha(K_i) \leq 2\sum_{i=1}^{3}|S_i| = 2\alpha(T)$$

$\square$

Stacked Tangle of an θ-Curve



Stacked Tangle of a Link

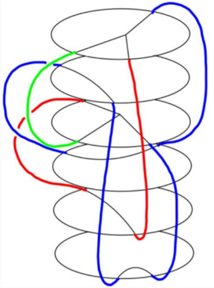


Stacked Tangle of a θ-Curve

Figure from [Morton and Beltrami, 1998]

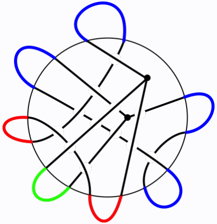
Stacked tangle of an θ-curve is stacked disks each with the frame as boundary with following properties:

- Only two disk called **non-simple disks** contain one vertex and three line segments which joins the vertex and boundary point.
- One of the non-simple discs is at the top.
- Other disks called **simple disks** contain simple arc which joins two points on the boundary.
- When view from above
    - two arcs in different simple disks intersect at most one point(by RII)
    - arc in simple disk and tree in non-simple disk intersect at most one point(by RV)



Simple closure of stacked tangle is a **stacked tangle** with **caps** satisfying following properties:

- A **cap** is a simple arc in outside of stacked tangle joining end points of arcs or line segments.
- When view from above any tow caps have no intersection.



Then a simple closure of a stacked tangle **without any nested caps** is corresponding to an arc presentation.

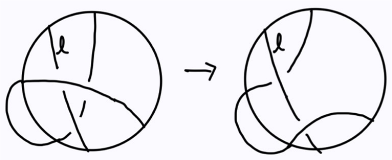
A **reduced simple closure of a stacked tangle** is

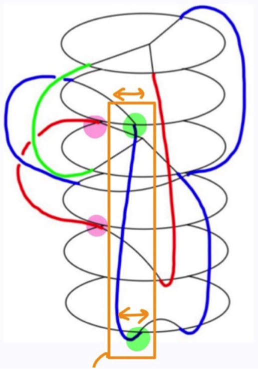
- a simple closure of a stacked tangle **without any nested caps**
- any two arcs(including line segment) joining by caps have **no intersection** when view from above

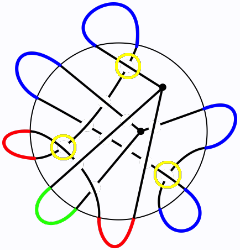
Proposition

A reduced simple closure of a stacked tangle can be obtained a simple closure of a stacked tangle without any nested caps by applying Reidemaister Moves.

Proof







□

Yamada Polynomials

Let D_T be a diagram of an θ-curve T. Then, the **Yamada Polynomial $R(D_T) \in \mathbb{Z}\left[x^{\pm 1}\right]$** is calculated by the following properties:

- **Y6:** $R\left(\;\right) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R\left(\;\right) = 0$
- **Y8:** $R(T' \cup \bigcirc) = (x+1+x^{-1})R(T')$ for an arbitrary θ-curve diagram T'
- **Y9:** $R\left(\;\right) - R\left(\;\right) = (x - x^{-1})\left[R\left(\;\right) - R\left(\;\right)\right]$
- **Y10:** $R\left(\;\right) = x^2 R\left(\;\right)$, $R\left(\;\right) = x^{-2} R\left(\;\right)$
- **Y11:** $R\left(\;\right) = R\left(\;\right)$ **Y12:** $R\left(\;\right) = R\left(\;\right)$
- **Y13:** $R\left(\;\right) = R\left(\;\right)$, $R\left(\;\right) = R\left(\;\right)$
- **Y14:** $R\left(\;\right) = -xR\left(\;\right)$, $R\left(\;\right) = -x^{-1}R\left(\;\right)$

Proposition ([Yamada, 1989])

$R(D_T)$ is an ambient isotopy invariant of T up to multiplying $(-x)^n$ for some integer n.

Lower Bounds from Yamada Polynomial

Theorem

Let T be any θ-curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of the θ-curve T.

Proposition

Let S_T be a simple closure of stacked tangle of a θ-curve T ***without any nested caps****. Then*

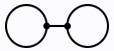
$$\max \deg_x R(S_T) \leq c + n - 2 \quad \textit{and} \quad \min \deg_x R(S_T) \geq -c - n + 2$$

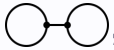
where ***c*** *is the* ***number of caps*** *and* ***n*** *is the* ***number of crossings*** *in S_T.*

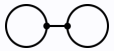
Proof

- A **simple cap** is a cap joining simple disks.
- Let ***s*** be the **number of simple caps** in S_T.
- Use double mathematical induction of (s, n).

Basis Step:

If $s = 0$, then S_T is either equivalent to  or .

- If $S_T \equiv$ , then $R(S_T) = -x^2 - x - 2 - x^{-1} - x^{-2}$ and $4 \leq s + n$.
- If $S_T \equiv$ , then $R(S_T) = 0$ and $3 \leq c + n$.

If $n = 0$, then S_T is equivalent to  $\cup$  $\cup \cdots \cup$ .

- $R(S_T) = 0$ and $2 \leq c + n$.

All of the cases satisfy the inequalities.

Inductive Step:

Assume that the inequalities hold for any (s', n') where $0 \leq s' < s$ or $0 \leq n' < n$.

Let S_T be a simple closure of stacked tangle of a θ-curve T such that the number of simple caps is s and the number of crossings is n.

Take a **simple cap f** in S_T, joining boundary points P and Q.

CASE 1. Suppose that P and Q are boundary points of a single disk.

- $S_T = S'_T \cup \bigcirc$
- $R(S_T) = (x + 1 + x^{-1})R(S'_T)$
- The number of caps is $c - 1$ and the number of crossings n' is less than or equal to n in S'_T.

$$\max \deg_x R(S_T) = \max \deg_x R(S'_T) + 1 \leq [(c-1) + n' - 2] + 1 \leq c + n - 2$$

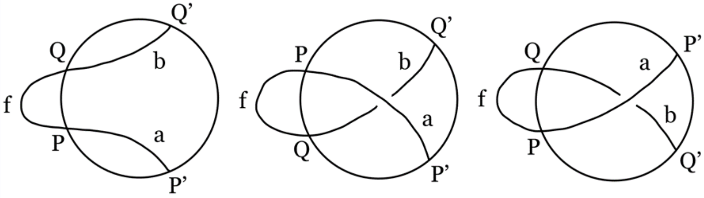
$$\min \deg_x R(S_T) = \min \deg_x R(S'_T) - 1 \geq [-(c-1) - n' + 2] - 1 \geq -c - n + 2$$

- S'_T satisfy the inequalities implies S_T satisfy the inequalities.

CASE 2. Suppose that P and Q are boundary points of different disks D_P and D_Q, respectively.

① Suppose that D_P and D_Q are adjacent disks.

- When view from above, there are three cases:



- At first case, we can reduce the simple cap f.
- After applying **Y10**, other cases can be regarded as first case.

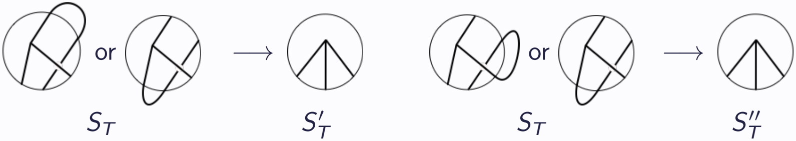
$$R\left(\ \right) = x^2 R\left(\ \right), \quad R\left(\ \right) = x^{-2} R\left(\ \right) \qquad (\textbf{Y10})$$

② Suppose that D_P and D_Q are not adjacent disks and D_P is above D_Q.

- Let D be the disk just above D_Q.
- If arcs or line segment in D and D_Q have no intersection, then we can change the position of D and D_Q without any quantities.
- We can assume that the arc in D_Q intersect arc or line segment in D, when view from above.

ⓘ There is a cap joining D_Q and D.

- D_Q and D are adjacent disks.
- If D is a simple disk, then we can reduce a simple cap as Case 2–①.
- If D is a non-simple disk, then



- $R(S_T) = -x^{\pm 1} R(S'_T)$ and $R(S_T) = x^{\pm 2} R(S''_T)$ by **Y14** and **Y10**, respectively.
- Both of S'_T and S''_T have $s-1$ simple caps, $c-1$ caps, and $n-1$ crossing.
- By induction hypothesis,

$$\begin{aligned}
\max \deg_x R(S_T) &= \max \deg_x R(S'_T) \pm 1 \\
&\leq [(c-1)+(n-1)-2] \pm 1 \\
&< c+n-2 \\
\min \deg_x R(S_T) &= \min \deg_x R(S'_T) \pm 2 \\
&\geq [-(c-1)-(n-1)+2] \pm 2 \\
&\geq -c-n+2
\end{aligned}$$

ⓘⓘ There is no cap joining D_Q and D.

- Applying **Y9**

  then

$$R(S_T) = R(S_T^-) + (x - x^{-1})\left[R(S_T^0) - R(S_T^\infty)\right]$$

- S_T^0 and S_T^∞ have c caps and $n-1$ crossings.
- $(x - x^{-1})\left[R(S_T^0) - R(S_T^\infty)\right]$ satisfy the inequalities.
- If S_T^- satisfy the inequalities, then S_T also satisfy the inequalities.
- The gap between D_P and D_Q is reduced in S_T^-.
- For S_T^-, investigate above cases.

This process will terminate after a finite number of investigations. It is the end of **CASE 2**. $\square$

Proposition

Let S_T be a reduced simple closure of stacked tangle of a θ-curve T corresponding to minimal arc presentation of T. Then

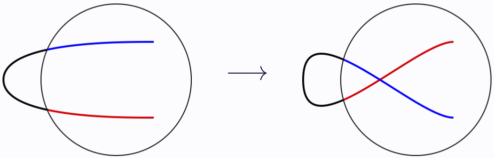
$$\max \deg_x R(S_T) - \min \deg_x R(S_T) - 2n + 4 \leq \alpha(T)$$

where n is the number of crossings in S_T.

Proof

- S_T is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_T is exactly arc index of T, $\alpha(T)$.

- Take a cap and add a positive or negative curl



- After modification of diagram as above, resulting diagram is also a simple closure of stacked tangle.
- The number of crossings is increased by 1.
- p of the caps yield a negative curl, and the remaining $c - p$ yield a positive curl.
- $S_T^{neg}(S_T^{pos})$ is the diagram obtained by inserting the p negative($c - p$ positive) curls.

	S_T^{neg}	S_T^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

- $R\left(S_T^{neg}\right) = x^{-2p} R(S_T)$ and $R\left(S_T^{pos}\right) = x^{2(c-p)} R(S_T)$

$$
\begin{aligned}
\min \deg_x R(S_T) - 2p &= \min \deg_x R\left(S_T^{neg}\right) \\
&\geq -c + -(n+p) + 2 \\
\max \deg_x R(S_T) + 2(c-p) &= \max \deg_x R\left(S_T^{pos}\right) \\
&\leq c + [n + (c-p)] - 2 \\
\min \deg_x R(S_T) &\geq -c - n + p + 2 \\
\max \deg_x R(S_T) &\leq n + p - 2 \\
\max \deg_x R(S_T) - \min \deg_x R(S_T) &\leq c + 2n - 4
\end{aligned}
$$

$\square$

Theorem

Let T be any θ-curve. Then

$$2 + \sqrt{\max \deg_x R(S_T) - \min \deg_x R(S_T) + 4} \leq \alpha(T)$$

where $R(T)$ is a Yamada Polynomial of the θ-curve T.

Proof

Let S_T be a reduce simple closure of stacked tangle of a θ-curve T corresponding to minimal arc presentation of T.

- The number of caps : $\alpha(T)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(T) - 3$

Consider the maximum number of crossings in S_T.

- number of crossings by two simple disks : $\frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)$
- number of crossings by a simple disk and non-simple disk : $2\left(\alpha(T)-3\right)$
- number of crossings counted by disks joined by cap : $\alpha(T)$
- number of crossings by two non-simple disks : 2

Thus

$$\begin{aligned} n &\leq \frac{1}{2}\left(\alpha(T)-3\right)\left(\alpha(T)-4\right)+2\left(\alpha(T)-3\right)-\alpha(T)+2 \\ &= \frac{1}{2}\left[(\alpha(T))^2-5\alpha(T)+4\right] \end{aligned}$$

By Lemma,

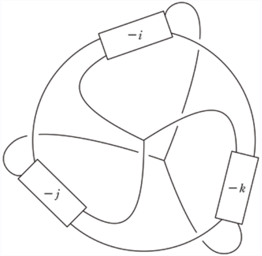
$$\max\deg_x R(S_T)-\min\deg_x R(S_T) \leq 2n-4+\alpha(T) \leq [\alpha(T)]^2-4\alpha(T)$$

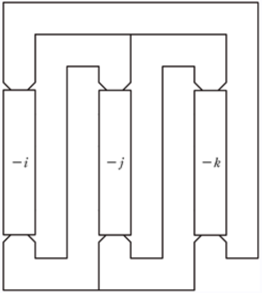
$$2+\sqrt{\max\deg_x R(S_T)-\min\deg_x R(S_T)+4} \leq \alpha(T)$$

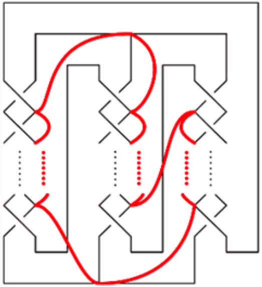
$\square$

Further Studies

Kinoshita-Wolcott θ-Curve





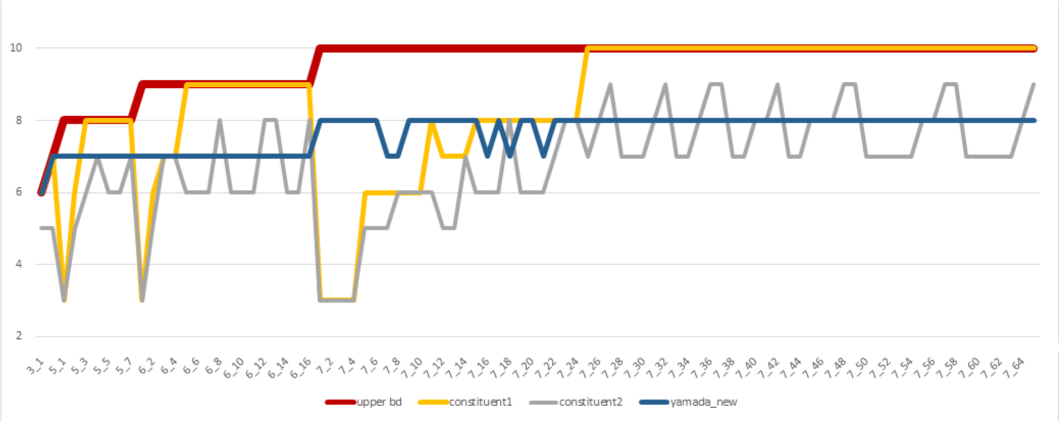


Theorem

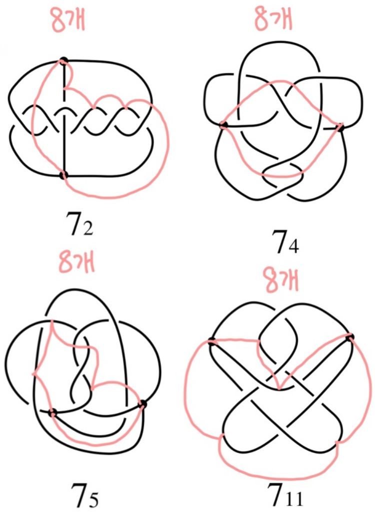
Let $K(-i,-j,-k)$ be the Kinoshita-Wolcott θ-curve. Then

$$\alpha(K(-i,-j,-k)) \leq i+j+k+2$$

Bounds of Arc Index



$$2+\left\lceil\sqrt{(M-m)+4}\right\rceil = 8$$



Thank You for Your Attention.

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