The Determinants and Arc Indices of θ-Curves and Handcuff Graphs

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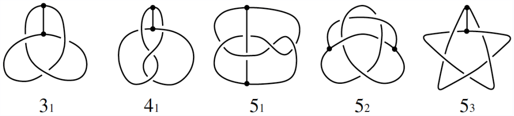
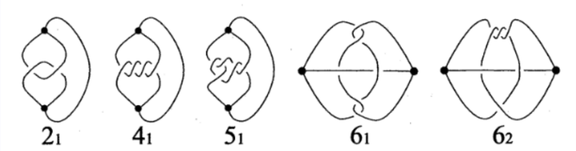
Introduction

- A **θ-curve** T is a graph embedded in S^3, which consists of two vertices v_1, v_2 and three edges e_1, e_2, e_3, such that each edge joins the vertices.
- A **handcuff graph** H is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2.



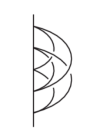
θ-Curves



Handcuff Graphs

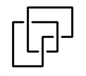
Trefoil

$\Longleftrightarrow$



Arc Presentation

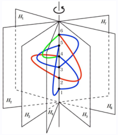
$\Longleftrightarrow$



Grid Diagram

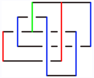
$\theta_{5,2}$

$\Longleftrightarrow$



Arc Presentation

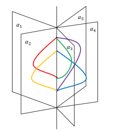
$\Longleftrightarrow$



Grid Diagram

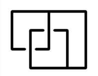
$\Phi_{2,1}$

$\Longleftrightarrow$



Arc Presentation

$\Longleftrightarrow$



Grid Diagram

Arc Presentation

Arc presentation of a θ-curve or handcuff graph is an embedding of them.

- It is contained in the union of finitely many half planes (called **pages**).
- The embedding is with the common boundary line (called **axis**).
- Each vertex lies in the axis.
- Each page contains a properly embedded single arc.

Arc index is the minimal number of pages among all possible arc presentations of graph.

- Arc presentation with the minimal number of pages is called **minimal arc presentation**.

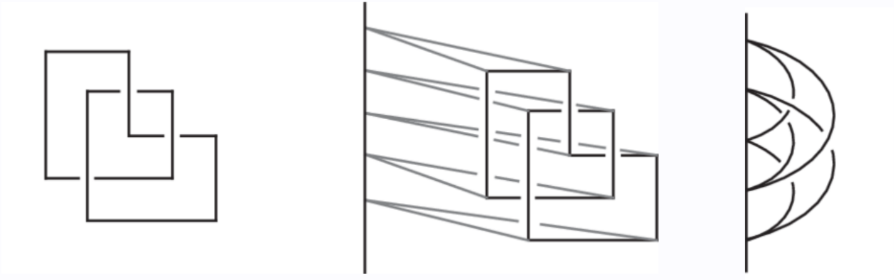
Grid Diagram

A **grid diagram** of θ-curve or handcuff graph is a diagram with only vertical and horizontal strands.

- (number of vertical strands) + 1 = (number of horizontal strands)
- At every crossing, the vertical strand crosses over the horizontal strand.
- No two horizontal strands are in the same row.
- No two vertical strands are in same column.

Arc Presentation of the θ-Curve and Handcuff Graph

- A grid diagram gives rise to an arc presentation and vice versa.



Theorem

Every θ-curve and handcuff graph admit a grid diagram.

Corollary

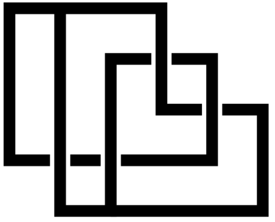
Every θ-curve and handcuff graph admit an arc presentation.

Determinants of θ-Curves and Handcuff Graphs

THC-Cromwell matrix

- The **Cromwell Matrix** of a knot is an $n \times n$ binary matrix each of whose rows and columns has exactly two 1s.
- The **THC-Cromwell matrix** is an expansion of Cromwell matrix into θ-curves and handcuff graphs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$$\longmapsto \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

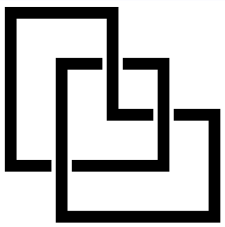
Determinants of the Cromwell matrices of Knot

Theorem

Let K be a knot. Then the determinant of a Cromwell matrix of K is 0 or ± 2.

PROOF

$\xmapsto[\text{diagram}]{\text{Grid}}$



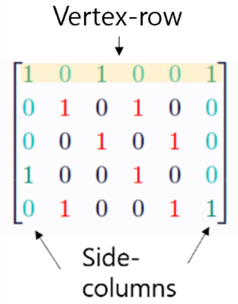
$$\xmapsto{\text{Cromwell}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xmapsto[\text{operations}]{\text{row/column}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

H-deletion of THC-Cromwell matricies

- The **vertex-row** of THC-Cromwell matrix M is a row which contains three 1s, M_{ia}, M_{ib}, M_{ic}, where $a < b < c$, as its elements.
- The **side-column** of THC-Cromwell matrix M is a column which contains the leftmost 1 of vertex-row (M_{ia}) or the rightmost 1 of vertex row (M_{ic}).
- The **H-deletion** Matrix of the THC-Cromwell matrix G is $(n-1) \times (n-1)$ matrix which deleted vertex-row and its two side-columns from the matrix G.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \longmapsto$$



$$\longmapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Determinants of the THC-Cromwell matrices

Theorem

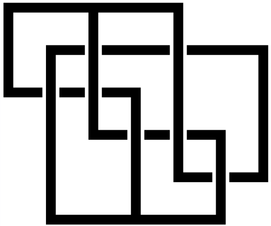
Let M_G be a THC-Cromwell matrice of θ-curve or handcuff graph G.

- *$det^*(M_G) = \pm 1 \iff M_G$ represents θ-curve*
- *$det^*(M_G) = 0$ or $\pm 2 \iff M_G$ represents handcuff graph*

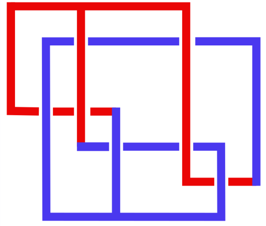
where $det^*(M_G)$ = determinant of H-deletion matrix of M_G

Proof

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



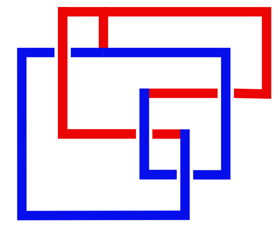
$\xmapsto{H-deletion}$



CASE 1. When M_G represents θ-curve

Line-shape

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto{H-deletion} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

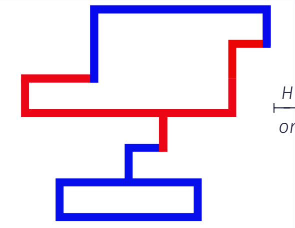
$$\xmapsto[operations]{row/column} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xmapsto{subtracting} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det^*(M_G) = \pm 1$

CASE 2. When M_G represents handcuff graph

T-loop

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto$$



$$\xmapsto[only\ T-loop]{H-deletion} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\xmapsto[operations]{row/column} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xmapsto{seperating} \begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$$

So det*(M_G) = 0 or ± 2

CASE 2. When G represents handcuff graph

ⓘ T-loop & Line-shape

cromwell matrix $\xmapsto{H-deletion}$ H-deletion matrix $\xmapsto[operations]{seperating}$

$$\begin{bmatrix} T-loop & * \\ O & Line-shape \end{bmatrix} \xmapsto[operations]{seperating} \begin{bmatrix} Knot & * & * \\ 0 & Line-shape & * \\ 0 & 0 & Line-shape \end{bmatrix}$$

ⓘⓘ Knot & Line-shape

cromwell matrix $\xmapsto{H-deletion}$ H-deletion matrix $\xmapsto[operations]{seperating}$ $\begin{bmatrix} Knot & * \\ O & Line-shape \end{bmatrix}$

So $\det^*(M_G) = 0$ or ± 2

Upper Bounds of Arc Index

Upper Bounds of Arc Index

For a spatial graph G, $c(G)$ denotes the crossing number of G.

Theorem (Lee et al., 2018)
Let G be a spatial graph. Then,

$$\alpha(G) \leq c(G) + e + b$$

where e and b are the number of edges and bouquet cut points, respectively.

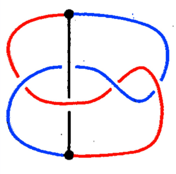
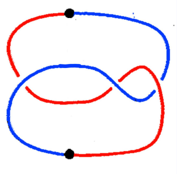
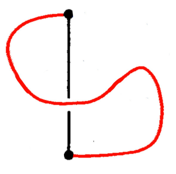
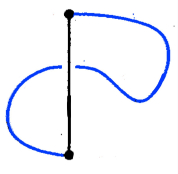
Corollary
Let G be a non-trivial prime θ-curve or handcuff graph. Then,

$$\alpha(G) \leq c(G) + 3.$$

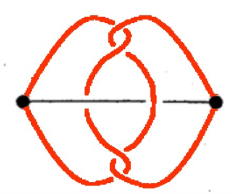
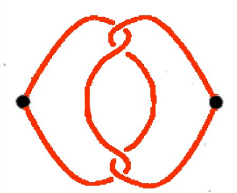
Lower Bounds of Arc Index

Lower Bound from Constituent Knots/Links

- A constituent knot T_{ij}, $1 \leq i < j \leq 3$, is a subgraph of θ-curve T that consists of two vertices v_1, v_2 and two edges e_i, e_j.

- A constituent link H_{12}, is a subgraph of handcuff graph H that consists of two vertices v_1, v_2 and two edges e_1, e_2.

Lower Bound from Constituent Knots/Links

Theorem (Lee, 2023)

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \max_{i \in \{1,2,3\}} \alpha(K_i) + 1$$

Theorem (Lee, 2023)

Let T be a θ-curve and K_1, K_2, K_3 be three constituent knots of T. Then

$$\alpha(T) \geq \frac{1}{2} \sum_{i=1}^{3} \alpha(K_i)$$

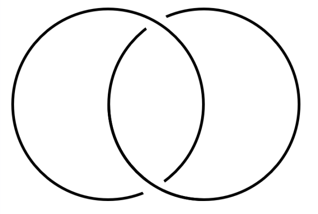
Lower Bound from Constituent Knots/Links

Theorem

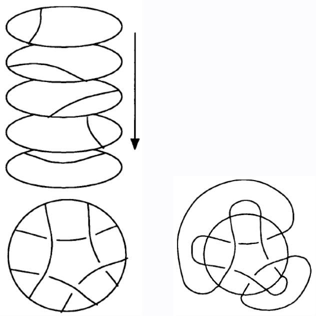
Let H be a handcuff graph, and L be the constituent link of H. If L is an alternating and non-split link, then

$$\alpha(H) \geq c(L) + 3,$$

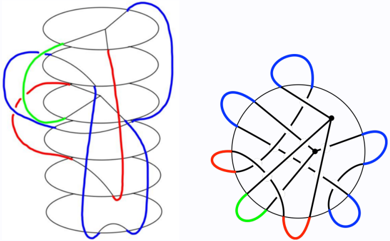
where $c(L)$ is the crossing number of L.



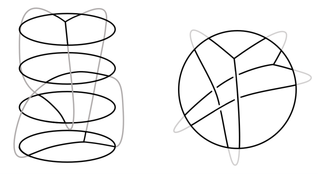
Stacked Tangle of an θ-Curve and Handcuff Graph



Stacked Tangle of a Link



θ-Curve



Handcuff Graph

Let D_G be a diagram of an θ-curve or handcuff graph G. Then, the **Yamada Polynomial** $R(D_G) \in Z\left[x^{\pm 1}\right]$ is calculated by the following properties:

- **Y6:** $R(\ ) = -(x+1+x^{-1})(x+x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$ **Y7:** $R(\ ) = 0$
- **Y8:** $R(G \cup \bigcirc) = (x+1+x^{-1})R(G)$ for an arbitrary θ-curve or handcuff graph diagram G
- **Y9:** $R(\ ) - R(\ ) = (x - x^{-1})\left[R(\ ) - R(\ )\right]$
- **Y10:** $R(\ ) = x^2 R(\ )$, $\quad R(\ ) = x^{-2} R(\ )$
- **Y11:** $R(\ ) = R(\ )$ **Y12:** $R(\ ) = R(\ )$
- **Y13:** $R(\ ) = R(\ )$, $\quad R(\ ) = R(\ )$
- **Y14:** $R(\ ) = -xR(\ )$, $\quad R(\ ) = -x^{-1}R(\ )$

Proposition (Yamada, 1989)

$R(D_G)$ is an ambient isotopy invariant of G up to multiplying $(-x)^n$ for some integer n.

Lower Bound from Yamada Polynomial

Theorem

Let G be a θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4} \leq \alpha(G).$$

Sketch of Proof

Proposition

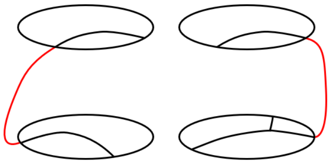
*Let S_G be a simple closure of stacked tangle of a θ-curve or handcuff graph G **without nested caps**. Then*

$$\max\deg_x R(S_G) \leq c + n, \quad \min\deg_x R(S_G) \geq -(c+n),$$

where c and n is the number of caps and crossings in S_G, respectively.

Proof

- Let c_s and c_{ss} be the number of **simlpe caps** and **semi-simple caps**, repectively.
- Use double mathematical induction of $(c_s + c_{ss}, n)$.



Sketch of Proof

Proposition

Let S_G be a reduced simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to minimal arc presentation of G. Then

$$\max \deg_x R(S_G) - \min \deg_x R(S_G) - 2n \leq \alpha(G)$$

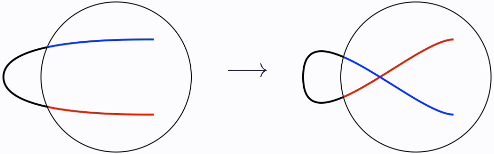
where n is the number of crossings in S_G.

Proof

- S_G is a reduced simple closure of stacked tangle corresponding to minimal arc presentation.
- The number of caps c in S_G is exactly arc index of G, $\alpha(G)$.

Sketch of Proof

- Take a cap and add a positive or negative curl



	S_G^{neg}	S_G^{pos}
Number of Caps	c	c
Number of Crossings	$n+p$	$n+(c-p)$

$$\min\deg_x R(S_G) - 2p = \min\deg_x R\left(S_G^{neg}\right) \geq -c + -(n+p)$$

$$\max\deg_x R(S_G) + 2(c-p) = \max\deg_x R\left(S_G^{pos}\right) \leq c + [n+(c-p)]$$

$$\max\deg_x R(S_G) - \min\deg_x R(S_G) \leq c + 2n \qquad \square$$

Sketch of Proof

Theorem

Let G be a θ-curve or handcuff graph. Then

$$2 + \sqrt{\max \deg_x R(S_G) - \min \deg_x R(S_G) - 4} \leq \alpha(G).$$

Proof

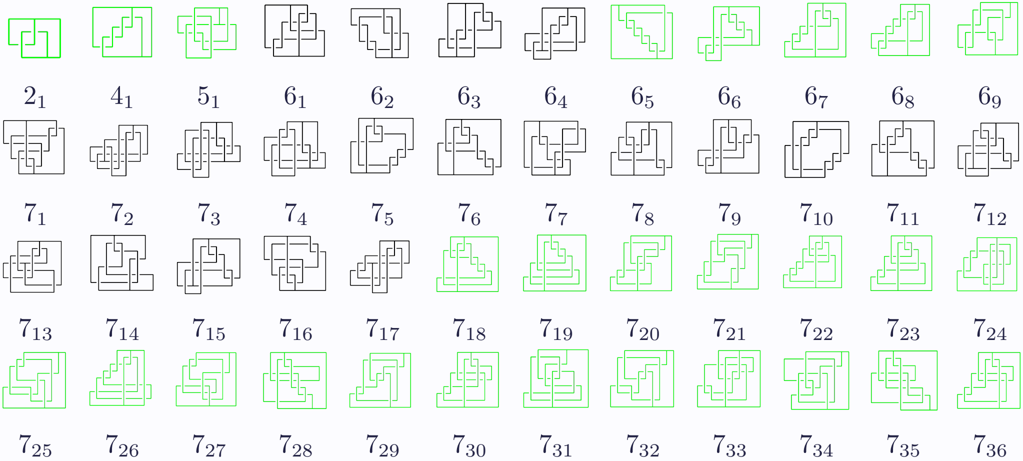
Let S_G be a reduce simple closure of stacked tangle of a θ-curve or handcuff graph G corresponding to minimal arc presentation of G.

- The number of caps : $\alpha(G)$
- The number of non-simple disks : 2
- The number of simple disks : $\alpha(G) - 3$

The theorem is followed by considering the maximum of crossings in S_G and applying previous theorems.

Minimal Grid Diagrams

Grid Diagrams of θ-Curves


2_1 4_1 5_1 6_1 6_2 6_3 6_4 6_5 6_6 6_7 6_8 6_9
7_1 7_2 7_3 7_4 7_5 7_6 7_7 7_8 7_9 7_{10} 7_{11} 7_{12}
7_{13} 7_{14} 7_{15} 7_{16} 7_{17} 7_{18} 7_{19} 7_{20} 7_{21} 7_{22} 7_{23} 7_{24}
7_{25} 7_{26} 7_{27} 7_{28} 7_{29} 7_{30} 7_{31} 7_{32} 7_{33} 7_{34} 7_{35} 7_{36}

Further Studies

Further Studies

- We tried to use the Python for determining arc indices. For the Python program, we used the Topoly package, but the package had an error. We would like to use another tool to completely determine the arc indices of θ-curves and handcuff graphs, such as Yamada package.
- Applying the result of our research to θ-curves or handcuff graphs of higher crossings or to other spatial graphs can be researched further.

References

References

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3 Minjung Lee, Sungjong No, and Seungsang Oh. (2018). Arc index of spatial graphs. *Journal of Graph Theory, 90*(3), 406–415.

4 Yamada, S. (1989). An invariant of spatial graphs. *Journal of Graph Theory, 13*(5), 537–551.