Department of Computer Science
Faculty of Mathematics, Physics and Informatics
Comenius University in Bratislava

Distance oracles for timetable graphs

(Master thesis)

bc. František Hajnovič

Study program: Computer science
Branch of study: 2508 Informatics
Supervisor: doc. RNDr. Rastislav Královič, PhD.

Bratislava 2013

65787891

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname:	Bc. František Hajnovič
Study programme:	Computer Science (Single degree study, master II. deg., full time form)
Field of Study:	9.2.1. Computer Science, Informatics
Type of Thesis:	Diploma Thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Distance oracles for timetable graphs

Aim: The aim of the thesis is to explore the applicability of results about distance oracles to timetable graphs. It is known that for general graphs no efficient distance oracles exist, however, they can be constructed for many classes of graphs. Graphs defined by timetables of regular transport carriers form a specific class which it is not known to admit efficient distance oracles. The thesis should investigate to which extent the known desirable properties (e.g. small highway dimension) are present int these graphs, and/or identify new ones. Analytical study of graph operations and/or experimental verification on real data form two possible approaches to the topic.

Supervisor:	doc. RNDr. Rastislav Královič, PhD.
Department:	FMFI.KI - Department of Computer Science
Vedúci katedry:	doc. RNDr. Daniel Olejár, PhD.
Assigned:	08.11.2011

Approved: 15.11.2011 prof. RNDr. Branislav Rovan, PhD.
Guarantor of Study Programme

....

Student Supervisor

65787891

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Bc. František Hajnovič
Študijný program:	informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor:	9.2.1. informatika
Typ záverečnej práce:	diplomová
Jazyk záverečnej práce:	anglický
Sekundárny jazyk:	slovenský

Názov: Efektívny výpočet vzdialeností v grafoch spojení lineik.

Cieľ: Cieľom práce je preštudovať možnosti aplikácie výsledkov o distance oracles v grafoch reprezentujúcich dopravné siete na grafy spojení liniek. Otázka, či a aké dôležité vlastnosti ostávajú zachované sa dá riešiť teoreticky pre rôzne triedy grafov a/alebo experimentálne pre reálne dáta.

Vedúci:	doc. RNDr. Rastislav Královič, PhD.
Katedra:	FMFI.KI - Katedra informatiky
Vedúci katedry:	doc. RNDr. Daniel Olejár, PhD.
Dátum zadania:	08.11.2011
Dátum schválenia:	15.11.2011

prof. RNDr. Branislav Rovan, PhD.
garant študijného programu

....

študent vedúci práce

I hereby declare that I wrote this thesis by myself, only with the help of the referenced literature, under the careful supervision of my thesis advisor.

....

Acknowledgements

I would like to thank very much to my supervisor Rastislav Královič for valuable remarks, useful advices and consultations that helped me stay on the right path during my work on this thesis.

I am also grateful for the support of my family during my studies and the work on this thesis.

František Hajnovič

Abstract

Queries for optimal connection in timetables can be answered by running Dijkstra's algorithm on an appropriate graph. However, in certain scenarios this approach is not fast enough. In this thesis we introduce methods with much better query time than that of the efficiently implemented Dijkstra's algorithm. We analyse these methods from both theoretical and practical point of view, performing experiments on various real-world timetables of country-wide scale.

Our first method called *USP-OR* is based on pre-computing paths, that are worth to follow (the so called underlying shortest paths). This method achieves speed-ups of up to 70 (against Dijkstra's algorithm), although at the cost of high amount of preprocessed data. Our second algorithm computes a small set of important stations and additional information for optimal travelling between these stations. Named *USP-OR-A*, this method is much less space consuming but still more than 8 times faster than the Dijkstra's algorithm on some of the real-world datasets.

Key words: **optimal connection**, **timetable**, **Dijkstra's algorithm**, **distance oracles**, **underlying shortest paths**

Abstrakt

Optimálne spojenia v cestovnom poriadku vieme hľadať pomocou Dijkstrovho algoritmu na vhodnom grafe, avšak v niektorých situáciách tento prístup výkonnostne nepostačuje. V tejto práci uvádzame metódy, ktoré na dotaz na optimálne spojenie odpovedajú podstatne rýchlejšie ako efektívna implementácia Dijkstrovho algoritmu. Tieto metódy analyzujeme ako z teoretického, tak aj z praktického hľadiska pomocou experimentov na viacerých cestovných poriadkoch celonárodnej škály.

Naša prvá metóda nazvaná *USP-OR* je založená na predpočítaní trás, ktoré sa oplatí následovať (tzv. podkladové najkratšie cesty). Táto metóda dosahuje faktor zrýchlenia až 70 (oproti Dijkstrovmu algoritmu), avšak za cenu veľkej pamäťovej náročnosti. Náš druhý algoritmus predrátava malú množinu dôležitých staníc a dodatočné informácie pre optimálne cestovanie medzi nimi. Táto metóda s názvom *USP-OR-A* je už oveľa menej pamäťovo náročná, stále vyše 8 krát rýchlejšia ako Dijkstrov algoritmus na niektorých reálnych cestovných poriadkoch.

Kľúčové slová: **optimálne spojenie**, **cestovný poriadok**, **Dijkstrov algoritmus**, **dištančné orákulá**, **podkladové najkratšie cesty**

Contents

1 Introduction **1**
- 1.1 Motivation . . . 1
- 1.2 Approach . . . 2
- 1.3 Goals . . . 2
- 1.4 Theory and practice . . . 2
- 1.5 Organization & conventions . . . 3

2 Preliminaries **4**
- 2.1 Objects . . . 4
- 2.2 Earliest arrival and optimal connection . . . 7
- 2.3 (Distance) Oracles . . . 8
- 2.4 Dijkstra's algorithm . . . 9

3 Related work **11**
- 3.1 Distance oracles and route-planning . . . 11
- 3.2 Time-dependent scenario . . . 13

4 Data & analysis **15**
- 4.1 Data . . . 15
- 4.2 Analysis of properties . . . 17

5 Underlying shortest paths **20**
- 5.1 *USP-OR* . . . 21
  - 5.1.1 Analysis of *USP-OR* . . . 23
- 5.2 *USP-OR-A* . . . 25
  - 5.2.1 Analysis of *USP-OR-A* . . . 27
  - 5.2.2 Correctness of *USP-OR-A* . . . 30
  - 5.2.3 Modifications of *USP-OR-A* . . . 30
- 5.3 Selection of access node set . . . 32
  - 5.3.1 Choosing the optimal access node set . . . 32
  - 5.3.2 Choosing ANs based on node properties . . . 34
  - 5.3.3 Choosing ANs heuristically - the *locsep* algorithm . . . 34
- 5.4 Performance and comparisons . . . 38
  - 5.4.1 Performance of *USP-OR* . . . 39
  - 5.4.2 *USP-OR-A* with *locsep* . . . 42
  - 5.4.3 *USP-OR-A* with *locsep Max* . . . 45

6 Neural network approach **47**
- 6.1 Results . . . 48
- 6.2 Conclusion . . . 51

7 Application TTBlazer **52**

8 Conclusion **53**

Appendix A File formats **54**

1 Introduction

World is getting smaller every day, as new technologies constantly make communication and travelling faster and more effective then yesterday. Road network, Internet and many other networks are becoming more evolved and denser which also brings along new problems. In order to fully take advantage of such huge networks, we must have efficient algorithms that operate on these networks and give us answers to many questions. Among many others, one that we take particular interest in is the question: "What is the shortest path from place x to place y"?

In different networks, this question can make different sense. In the road network, we would like to obtain a sequence of intersections we have to go through in order to reach our destination, driving the shortest possible time (or the smallest possible distance). GPS devices and the likes of Google maps have to deal with this problem. In case of the Internet network, we might be interested in the shortest path to a destination computer in terms of router hops. In a network of social acquaintances, the smallest number of persons connecting us e.g. with guitarist Mark Knopfler or Liona Boyd could be expressed as a shortest path problem. Many problems in artificial intelligence (e.g. planning of actions) can be expressed, or include, looking for shortest paths.

The tremendous amount of work done in this area signifies the importance of quick distance or shortest path retrieval in graphs. A simple Dijkstra's or A* algorithm no longer comply to the requirements of today's applications, in which a server often has to answer hundreds of shortest path queries per second in a large-scale networks. To speed up the mentioned algorithms we usually sacrifice generality and concentrate on a particular type of network, or even on one concrete network.

In this thesis, the type of network we deal with is the one representing timetable connections, where nodes are the stations and arcs represent a direct connection between the two stations. We will talk in more details about this in following sections. However, this network has one substantial difference that we would like to point out - it is time-dependent. That means that the shortest path from station x to station y may have different solutions depending on the time when we start at station x. Therefore, we will not talk about shortest paths and distances, but rather about optimal connections and earliest arrivals and each query will now bear a third parameter - the departure time from x.

To informally develop the discussion about optimal connections in timetables, we will now clarify the motivation, approach and the goals of this thesis. We also sketch out the difference between the theory and practice when it comes to timetable search engines.

1.1 Motivation

We have already approached the motivation in the introductory text. We consider that a server (hosting e.g. journey-planning application) has to answer many queries per given time unit. What does it mean many? British National Rails Enquiries website that hosts journey planner supports over 1 million queries per day [1Te]. Even if these queries were distributed evenly throughout the whole day, there would still be more than 11 queries per second. That is why the search engine run for each query has to be fast enough to provide an answer.

11 queries per second is probably not a big issue. There is about 2500 railway stations in Great Britain and a current state of the art computer with basic implementation of a time-dependent Dijkstra's algorithm (to be talked about later) would be able handle the mentioned load without any problems. However, things get more difficult on a bigger scale, in rush hours and when additional

requirements are posed on the search results (transfers, cost of travel or simply outputting more results the user can choose from).

In shortest path routing on road networks very much has been done to speed-up the query times using pre-processing on the input graph (for a good review such methods, see [DSSW09]). Some developed methods answer distance queries more than 1 000 000 faster than the Dijkstra's algorithm on large road networks. In timetable scenario, the achieved speed-ups are much more modest. We will talk about the related work and achieved speed-ups in this area in the section 3.

1.2 Approach

We have mentioned that to get more effective algorithms with better query times, we need to focus on a special type of network and take advantage of its properties. In addition to this, what we can do is to pre-compute some information on the particular timetable and to use this information later to speed-up the answering of the queries. This is not a new technique and in the shortest path routing it is commonly referred to as creating distance oracle [TZ05]. Our approach is analogical - instead of static graphs we deal with graphs representing the timetables [1] and look for optimal connections. We will go more into the details about this approach in the preliminaries section 2.

1.3 Goals

We have set two main goals for this thesis:

- **Analyse real-world timetables** and their properties. More specifically, given the graph representing the timetable, we were interested in its sparsity, connectivity, average and maximal degrees, average optimal connection sizes... We will talk about these properties mostly in the section 4.
- **Develop methods with fast query times** for optimal connections, based on pre-computing, as outlined in the previous subsection. For this purpose, we use also the outcomes of our analysis.

1.4 Theory and practice

This thesis is more theoretically oriented - we consider the optimal connection problem in probably its purest form, which does not account for the many requirements posed by travellers using timetable search engines. Those include number of transfers, preferred route, cost of travel and others. These multi-criteria queries are discussed e.g. in [MHSWZ07]. In practice, we also usually want to output multiple connections, so that the user has a chance to choose. Needless to say, all of this makes the problem much more complicated and challenging than a pure search for an optimal connection.

On the other hand, the real-world timetable search engines concentrate usually on one given dataset, which enables them to exploit its properties [2] and tailor the search engine specifically for it. There is also a choice of a suitable timetable model based on the characteristics of the given timetable.

The aim of the theoretical works (like this thesis) is not therefore to develop an algorithm immediately deployable into practice but rather to investigate techniques which might be useful to consider when designing practical timetable search engines.

[1] Hence the name of this thesis - Distance oracles for timetable graphs, the “distance” being part of the title mostly because the term “distance oracle” is generally recognized.

[2] E.g. An the city of Bratislava has only four (functioning) bridges, which could be taken into account when designing a public transportation search engine.

1.5 Organization & conventions

This thesis is organized as follows:

- **Preliminaries**: We provide the necessary definitions (most notably timetable and its graph representations) and formally define the problem we deal with, as well as the approach we use.
- **Related work**: This section summarizes the main related work in distance oracles, static route-planning and time-dependent scenarios.
- **Data & analysis**: We introduce real-world timetables we worked with and analyse many of their properties.
- **Underlying shortest paths**: In the main part of the thesis, we present the two methods we developed to speed-up optimal connection queries in timetables. These methods are also analysed from both theoretical and practical point of view.
- **Neural network approach**: We summarize a little experiment in which we tried to train neural network to answer optimal connection queries.
- **Application TTBlazer**: This section shortly describes the application we used to analyse our datasets and test the methods.
- **Conclusion**: Finally, we conclude pointing out main results and contribution, drawbacks and possibilities for future work.

In this thesis, we also use some conventions:

- With a few exceptions, we will use **bold** font to mark currently defined term (or its notation).
- Names of our algorithms are in *italics*.

2 Preliminaries

In this section, we provide most of the definitions and terminology used throughout the thesis.

2.1 Objects

First, we will formalize the notion of a timetable and its derived graph forms, the underlying graph and terms related to these objects.

Definition 2.1. ***Timetable (TT)***
A timetable is a set $\boldsymbol{T} = \{(x, y, p, q) |\ p, q \in \mathbb{N},\ p < q\}$.

- *Elements of T (the 4-tuples) are called* ***elementary connections****. For an elementary connection* $e = (x, y, p, q)$*:*
  - $\boldsymbol{from(e)} = x$ *is the* ***departure city***
  - $\boldsymbol{to(e)} = y$ *is the* ***arrival/destination city***
  - $\boldsymbol{dep(e)} = p$ *is the* ***departure time***
  - $\boldsymbol{arr(e)} = q$ *is the* ***arrival time***
- *The set of all* ***cities*** *(stations) will be denoted as* $\boldsymbol{ct_T} = \{x |\ (x, y, p, q) \in T \text{ or } (y, x, p, q) \in T\}$ *and the number of cities as* n_T.
- *Pairs* (x, p) *or* (y, q) *such that* $(x, y, p, q) \in T$ *form the set of* ***events*** $\boldsymbol{ev_T}$. *The set of events in a specific city x is* $\boldsymbol{ev_T(x)} = \{(x, t) |\ (x, y, t, q) \in T \text{ or } (y, x, p, t) \in T\}$
- *Let* $\boldsymbol{tlow_T} = \min_{e \in T} dep(e)$ *and* $\boldsymbol{thigh_T} = \max_{e \in T} arr(e)$. *The value* $\boldsymbol{tr_T} = thigh_T - tlow_T$ *is called the* ***time range*** *of the timetable.*
- ***Height*** *of the timetable is the average number of events in a city:*

$$\boldsymbol{h_T} = \frac{|ev_T|}{n_T}$$

Let us describe some the defined terms more informally. An elementary connection corresponds to moving from one stop to the next one, e.g. with a bus (thus we disregard the notion of *lines*, i.e. getting on and off). Note that we express time as an integer - throughout this paper, this integer will represent the minutes elapsed from the time 00:00 of the first day. Thus we may take the liberty of talking about time in integer or *days hh:mm* format, as convenient at the moment. Lastly, an event simply represent an arrival or departure of a e.g. train at some station. The remaining terms should be clear enough.

Place		Time	
From	**To**	**Departure**	**Arrival**
A	B	10:00	10:45
B	C	11:00	11:30
B	C	11:30	12:10
B	A	11:20	12:30
C	A	11:45	12:15

Table 2.1: An example of a timetable - the set of elementary connections (between pairs of cities). An example of an event is a pair (A, 10:00), when some el. connection departs from A.

Following is a definition of a connection.

Definition 2.2. ***Connection***

A connection from a to b is a sequence of elementary connections $\boldsymbol{c} = (e_1, e_2, ..., e_k), k \geq 1$*, such that* $from(e_1) = a$*,* $to(e_k) = b$ *and* $\forall i \in \{2, ..., k\} : (to(e_i) = from(e_{i-1}),\ arr(e_i) \geq dep(e_{i-1})$*.*

- *Connection* ***starts*** *at the departure time* $\boldsymbol{start(c)} = dep(e_1)$ *and* ***ends*** *at the arrival time* $\boldsymbol{end(c)} = arr(e_k)$*.*
- *We also extend* $\boldsymbol{from(c)} = from(e_1)$ *and* $\boldsymbol{to(c)} = to(e_k)$
- ***Length*** *of the connection is* $\boldsymbol{len(c)} = end(c) - start(c)$
- ***Size*** *of the connection is* $\boldsymbol{size(c)} = k$ [3]
- *We will denote the set of* ***all connections*** *from a to b in a timetable T as* $\boldsymbol{C_T(a, b)}$*. We also define* $\boldsymbol{C_T} = \cup_{a,b} C_T(a, b)$

So we understand connection as a (valid) sequence of elementary connections.

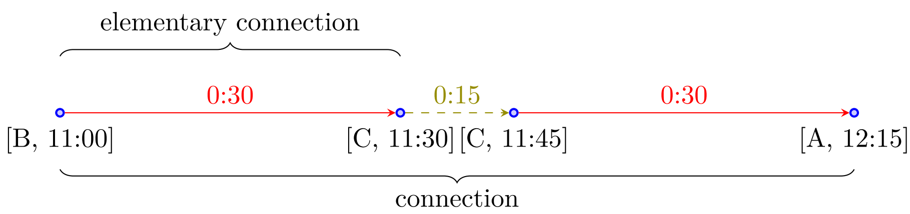


Figure 2.1: A valid connection made out of elementary connections (and waiting, which is implicit).

Next, we continue with the underlying graph - a graph representing basically the map on top of which the timetable operates.

Definition 2.3. ***Underlying graph (UG graph)***

The underlying graph of a timetable T, denoted $\boldsymbol{ug_T}$*, is an oriented graph* (V, E)*, where V is the set of all timetable cities and* $E = \{(x, y)|\ \exists(x, y, p, q) \in T\}$

- *By* $\boldsymbol{m_T}$ *we will denote the number of arcs in the UG*

Note, that we do not specify the weights of the edges in the underlying graph - they will be specified based on the current usage of the UG. Most of the time, however, if we work with a weighted UG, the weight of an arc will be the length of the shortest elementary connection on that arc. More specifically, $w(x, y) = \min_{(x,y,p,q)\in T}(q - p)\ \forall(x, y) \in E(ug_T)$. Such weighted UG will be called **optimistic** (denoted $\boldsymbol{ug_T^{opt}}$).
If we want to represent the timetable by a graph, there are two most common options [MHSWZ07] - the time-expanded and time-dependent graph.

Definition 2.4. ***Time-expanded graph (TE graph)***

Let T be a timetable. Time-expanded graph from T, denoted $\boldsymbol{te_T}$*, is an oriented graph* (V, E) *whose vertices correspond to events of T, that is* $V = \{[x, t]|\ (x, t) \in ev_T\}$*. The edges of G are of two types*

1. $([x, p], [y, q])\ \forall(x, y, p, q) \in T$ *- the so called* ***connection edges***
2. $([x, p], [x, q])\ [x, p], [x, q] \in V,\ p < q$ *and* $\nexists[x, r] \in V : p < r < q$*. - the so called* ***waiting edges***

Weight of the edge $([x, p], [y, q])$ *is* $w([x, p], [y, q]) = q - p$*.*

Informally, an edge in TE graph represent either the travelling with an elementary connection or waiting for the next event in the same city. Also, the time range and height of a timetable could be easily illustrated on the TE graph (see picture 2.3).

[3] We will use similar terminology when talking about paths - the *size* is the number of vertices (hops) in the path while the *length* refers to the actual distance (sum of weights of the edges in the path).

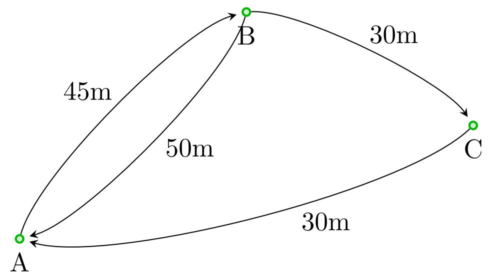


Figure 2.2: An optimistic underlying graph of the timetable 2.1. The nodes are the cities of the timetable.

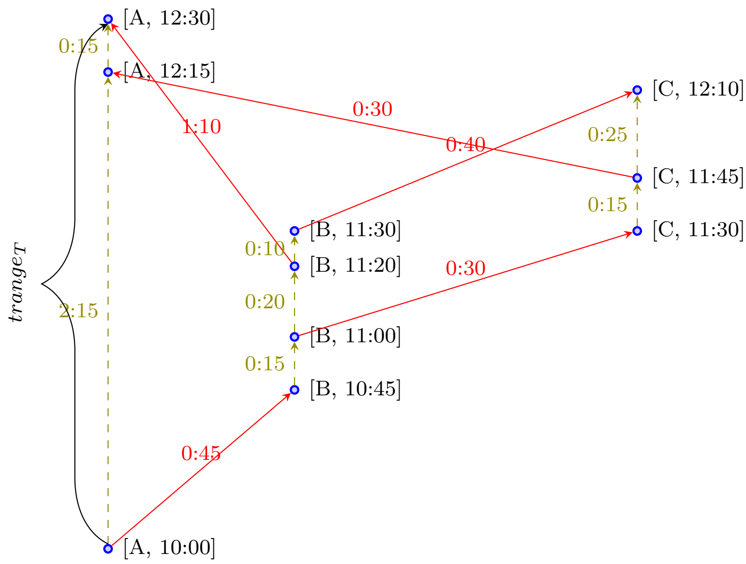


Figure 2.3: Time-expanded graph of the timetable 2.1. Nodes represent the events. There are connection and waiting edges (dashed). The time range is 2h:30m and the height is 4 (as there are 4 events in city B).

Definition 2.5. ***Time-dependent graph (TD graph)***
Let T be a timetable. Time-dependent graph from T, denoted $\boldsymbol{td_T}$, is an oriented graph (V, E) whose vertices are the timetable cities and $E = \{(x, y)| \ \exists (x, y, p, q) \in T\}$. Furthermore, the weight of an edge $(x, y) \in E$ is a piece-wise linear function $w(x, y) = f_{x,y}(t) = q - t$ where q is:

- $\min\{arr(e)|e \in T,\ dep(e) \geq t\}$
- ∞*, if* $dep(e) < t\ \forall e \in T$

Intuitively, the TD graph is simply the UG graph where each arc carries a function specifying the traversal time of that arc at any time. For an example, see picture 2.5: The latest point of every linear segment is called the **interpolation point** and it corresponds to an elementary connection (its coordinates are $dep(e), len(e)$ for corresponding el. connection e). Note that a list of all interpolation points fully defines the piece-wise linear function.

The algorithms in this thesis use almost exclusively the TD graphs, mainly because they are less space consuming. Also, time-dependent Dijkstra searches are a bit faster on TD graphs, because the search space that has to be explored is smaller. On the other hand, TE graphs are more flexible when we need to take additional search parameters into consideration (like transfers, travel costs). Since we will not talk about these, TD graphs are more suitable.

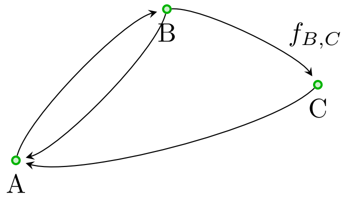


Figure 2.4: Time-dependent graph of the timetable 2.1. The nodes are the cities.

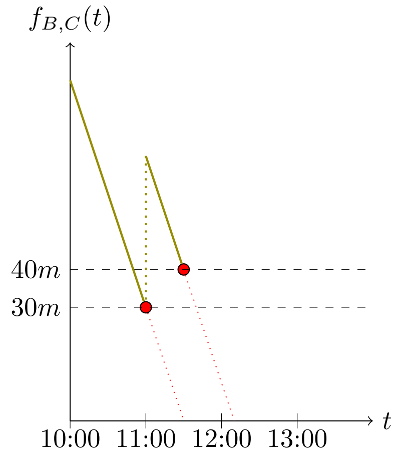


Figure 2.5: Piece-wise linear function - traversal times for the arc (B, C). The highlighted points are the interpolation points.

To sum up, there are four main types of objects we will be working with:

- Timetable (TT)
- Underlying graph (UG)
- Time-expanded graph (TE)
- Time-dependent graph (TD)

Note: Throughout this paper, we will relax a bit the notation and leave out subscripts (e.g. $ug_T \rightarrow ug$, $n_T \rightarrow n$, etc.) in situations, where the context is clear enough.

2.2 Earliest arrival and optimal connection

Now we would like to formulate the main problems this thesis deals with.

Definition 2.6. ***Earliest arrival problem (EAP)***
Given a timetable T, departure city x, destination city y and a departure time t, the task is to determine $\boldsymbol{t^*_{(x,t,y)}} = \min_{c \in C_T(x,y)}\{t + len(c) |\ start(c) \geq t\}$.

- *We will refer to the tuple* $\boldsymbol{(x, t, y)}$ *as an* ***EAP instance****, or an* ***EAP query***
- *The time* $\boldsymbol{t^*_{(x,t,y)}}$ *is called the* ***earliest arrival (EA)*** *for the given EAP instance*

A bit more difficult version of this problem is one, where we require to actually output the connection ending at time given by EA.

Definition 2.7. ***Optimal connection problem (OCP)***
Given a timetable T, departure city x, destination city y and a departure time t, the task is to find the ***optimal connection (OC)*** $\boldsymbol{c^*_{(a,t,b)}} = argmin_{c \in C_T(a,b)}\{t + len(c) |\ start(c) \geq t\}$.

The instance/query in case of the optimal connection problem has the same form as EAP query. Also, note that the OCP is at least as hard to solve as EAP since having the optimal connection implies the optimal (earliest) arrival time. In order to avoid technical issues (e.g. in the definition), we may assume that the optimal connection is unique (i.e., there is not a different connection with the same end time). However, we consider any connection which arrives at time $t^*_{(x,t,y)}$ to be optimal for the given query.

Example 2.1. *Consider our timetable from table 2.1. For the EAP instance (B, 10:45, A), the earliest arrival (EA) is 12:15 and the optimal connection (OC) is ((B, C, 11:00, 11:30), (C, A, 11:45, 12:15)), as could be easily seen from picture 2.6 of the TE graph.*

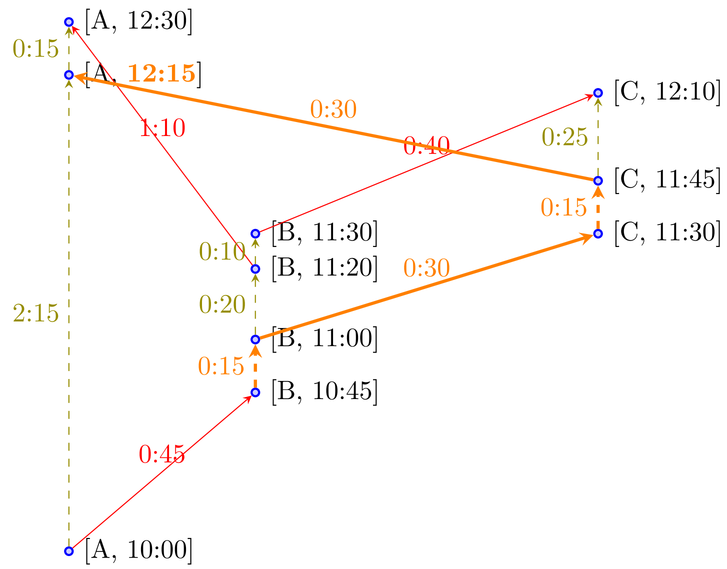


Figure 2.6: Optimal connection and earliest arrival time are marked in **bold**.

2.3 (Distance) Oracles

The term *distance oracle* was first coined in 2001 by Thorup and Zwick [TZ05], when talking about quick shortest path (or distance) computations on graphs. One approach to this problem is to pre-compute some information on the graph to speed-up answering of the queries. The paper of Thorup and Zwick was dealing with trade-offs among the time complexity of the pre-computation, the amount of pre-computed information, the speed-up in query times and the accuracy of the answers. Since the pre-computed data structure is something that helps us answer the queries more efficiently, it resembles an oracle, thus the term distance oracle.

In this thesis, we will discuss methods that behave the same way, but deal with the earliest arrival problem (or optimal connection problem) - there is some pre-processing of the timetable with a resulting data structure that speeds up answering subsequent queries. To formalize this a little more, we will refer to this kind of methods as **oracle based methods**. For such a method m, we are interested mainly in its four parameters:

- **Preprocessing time** ($\boldsymbol{prep(m)}$) - the time complexity of the pre-computation
- **Preprocessed space** ($\boldsymbol{size(m)}$) - the space complexity of the pre-computed data structure (the so called **oracle**)
- **Query time** ($\boldsymbol{qtime(m)}$) - the time complexity of answering a single query

- **Stretch** (***stretch*(*m*)**) - the worst-case ratio against the optimal value of earliest arrival (the lower, the better)

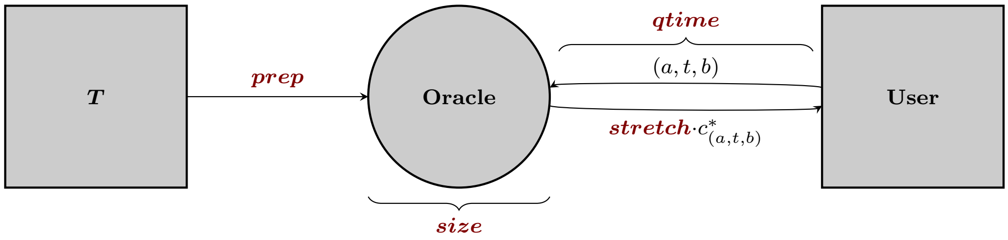


Figure 2.7: Principle of oracle based methods - we preprocess the timetable, creating a structure that helps us speed-up the answers to queries for optimal connection.

The preprocessing time is probably the least critical resource. A reasonable polynomial should bind its time complexity, depending on the computational power of the user and the scale of the timetable. The size of the preprocessed oracle is much more important - in the optimal case, it should be bound by the space complexity of the timetable itself. Optimality of the query time depends on which problem we are solving. If we query for the whole optimal connection, we have to count with a time complexity at least proportional to the diameter of the underlying graph (as connections could be that long, or even longer). If we require only the EA value as an output, much better speed-ups could be expected. The stretch should be of course as low as possible.

2.4 Dijkstra's algorithm

Throughout this thesis, we will often use Dijkstra's algorithm and its modifications both as a part of our algorithms and as a reference point against which we will compare the performance of our methods. This is a common practice. Researchers working on methods answering distance or shortest path queries in road networks commonly use the term *speed-up*, i.e. *how many times faster* is their algorithm against the Dijkstra's algorithm.

Dijkstra's algorithm is originally an algorithm that looks for shortest paths in weighted oriented graphs. It was published by E. W. Dijkstra in 1959 [Dij59] and we will not explain it at this place, as the algorithm is very well explained at many other places (e.g. [KP]). For a good summary of Dijkstra's algorithm related implementations and publications see [Som10].

As our task is to compute earliest arrivals or optimal connections instead of distances and shortest paths, our "reference point" will be a slightly modified Dijkstra's algorithm called **time-dependent Dijkstra's algorithm** [DW09] (or **TD Dijkstra** for short). The algorithm is run on a time-dependent graph and works just like the ordinary Dijkstra's algorithm, except that the weight of each arc (x, y) is determined for the time t at which we had settled vertex x.

If we assume that the evaluation of an arc by the cost function of the TD graph is implemented in constant time, the running time of the TD Dijkstra is $\mathcal{O}(n^2)$, just like the normal Dijkstra's algorithm. On sparse graphs, this bound can be improved using a quick data structure to determine the next node we settle. A good option is a priority queue implemented as a *Fibonacci heap*, which implements deletion in $\mathcal{O}(\log n)$ and all other operations in constant amortized time [Som10]. This yields the running time of TD Dijkstra $\mathcal{O}(n \log n + m)$.

We may therefore introduce a fifth parameter of our oracle based methods, the speed-up:

Definition 2.8. ***Speed-up (spd(m))***
A speed-up of an oracle based method m is the ratio $\dfrac{qtime_{avg}(TDDijkstra)}{qtime_{avg}(m)}$ *where* $qtime_{avg}(m')$ *is the average query time of the respective oracle based method* m' [4].

The definition is rather loose in the sense that we may refer to a concrete speed-up of the method on a concrete dataset or a general, theoretical speed-up expressed as a function of the size of input.

[4] Note that we may also consider the TD Dijkstra algorithm to be an oracle based method - it just happens that it does not require any preprocessing.

3 Related work

In this section, we summarize the work related to the subject of this thesis. Apart from the papers discussing searching for optimal connections and earliest arrivals in time-dependent scenarios, we also briefly summarize the research done on route planning in road networks and on distance oracles in general.

3.1 Distance oracles and route-planning

We have already mentioned in section 2 the paper of **Thorup and Zwick** [TZ05] where the term "distance oracle" originated. The authors have shown that given an undirected weighted graph of n vertices and m edges and a chosen integer $k \geq 1$, we can build a distance oracle such that:

- preprocessing takes $O(kmn^{1/k})$ expected time
- resulting distance oracle is of size $O(kn^{1+1/k})$
- answering queries takes $O(k)$ time
- stretch is at most $2k - 1$

Moreover, the authors have reasoned that their construction is essentially optimal with respect to space - i.e., if we want to have exact and constant-time answers, we will in general be forced to pre-compute $\Omega(n^2)$ information. The parameter k however provides a nice option to make trade-offs between the four parameters, as depicted on figure 3.1.

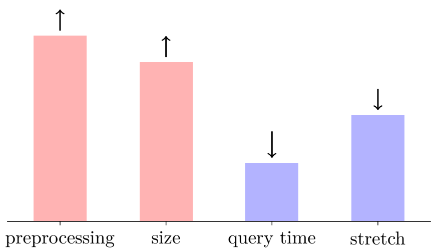
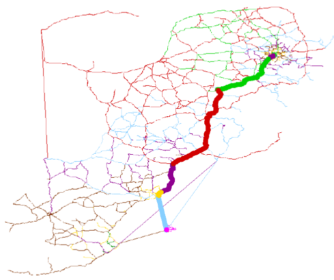


Figure 3.1: By moving k (decreasing on the picture), we can achieve compromises between the four parameters of the distance oracle.

Another work by **Gavoille** et al. [GPPR04] concerned distance labelling - a somewhat restricted version of a distance oracle where we assign each node in the graph its distance label. This is again only some pre-computed information and upon a query from x to y, we should be able to figure out their distance only using the corresponding distance labels. In the paper it is shown that for all n, there exist infinitely many graphs of n vertices for which we have an exact distance labelling scheme of a small overall size ($\mathcal{O}(n \log n)$), but for which the process of figuring out the distance from the labels takes too long from practical point of view.

Even though these results imply that we cannot create a sufficiently small efficient distance oracle in general, it may still be possible for sub-classes of general graphs, or even better, for a single particular graph. In that respect, the road network is the point of interest and fortunately it has a few "nice" properties (it is sparse, almost planar, the maximum node degree is small...) which made it possible to design exact and efficient algorithms with extremely fast query times. To name a few of these:

- **Highway hierarchies** (2005, [SS05]). The preprocessing of the algorithm works in iterations - in each of them the edges of little importance are pruned, the remaining graph is contracted (long chains of edges are replaced with shortcuts) and the result forms the new layer, connected to the previous one, and used as an input for the next iteration. On such hierarchy of layers, bidirectional Dikstra's algorithm is run, climbing up the hierarchy in each direction.
  - Speed-up: about 2500
  - Techniques: hierarchy, shortcuts, bidirectional Dijkstra
- **Transit node routing** (2006, [BFM06]). The algorithm completely replaces searching with table look-ups. There is a small set of transit nodes between which the exact distance is stored in a table. Also each node remembers its nearest transit nodes (called access nodes [5]) and the their distance. A search is necessary only in case of a local query. A disadvantage is a bigger space consumption.
  - Speed-up: more than 1 000 000
  - Techniques: landmarks



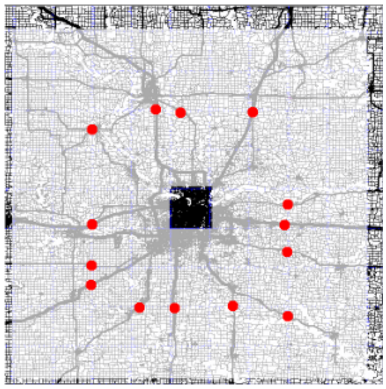


Figure 3.2: Highway hierarchies (left) - the bidirectional Dijkstra search climbs up the hierarchy to reach the most sparse level. Transit node routing (right) - access nodes (in red) that cannot be avoided when going "out of town".

- **Contraction hierarchies** (2008, [GSSD08]). The preprocessing creates additional shortcut edges in the graph. This is done by deleting one by one the vertices of the original graph and adding shortcuts where necessary - to preserve original distances. The quality depends mostly on the order in which we delete the vertices. Upon a query, a bidirectional Dijkstra search is run on the original graph enriched with the added shortcuts. The algorithm is less memory demanding than Transit node routing.
  - Speed-up: more than 30 000
  - Techniques: shortcuts (contractions), bidirectional Dijkstra

One thing these methods have in common is that their query time is very low in practice, but it is not guaranteed theoretically. This was the point of interest in the work [AFGW10], which introduces a parameter called *highway dimension*. The authors show that a low highway dimension guarantees good query times of many route-planning algorithms, including the three we have mentioned.

[5]This served partly as an inspiration for our algorithm *USP-OR-A* discussed in section 5.

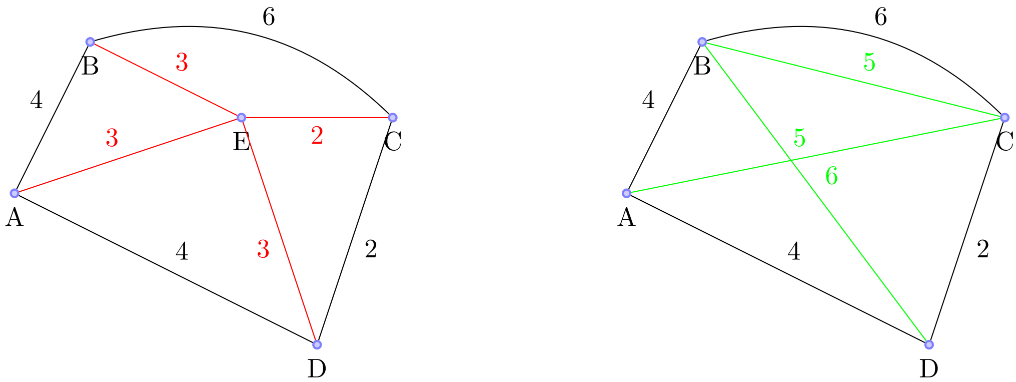


Figure 3.3: Deleting vertex E in Contraction hierarchies. Before (left) and after (right).

A very good summary of the techniques devised for road network route-planning up to the year 2009 can be found in [DSSW09]. Efficient distance oracles are also known for graphs with small recursive separators [GPPR04]. The work [Som10] suggests efficient distance oracle for power-law graphs and another distance oracle method for general graphs, offering trade-offs between stretch and query times. It also gives an exhaustive and comprehensive discussion regarding shortest path queries in general, which we point out to interested readers.

3.2 Time-dependent scenario

The time-dependent scenario has so far seen much smaller speed-ups than static routing in road networks, one reason for this being that the adaptation of the many techniques used for road networks to the time-dependent scenario is not so straightforward. This is mainly due to the fact that running bidirectional Dijkstra's algorithm (commonly used with static route-planning techniques) in time-dependent networks requires the knowledge of the destination time [DPW09]. All the same, for some methods this adaptation was carried out with good results:

- **Time-dependent contraction hierarchies** (2009, [BDSV09]). The focus in this case was on road networks having time-dependent edge weights (e.g. due to morning congestions) and on computing earliest arrival value for a given query. The main difference between the static Contraction hierarchies is that the backward search is run from more arrival times of the destination node. Each such run may then contribute a new lower or upper bound for the actual earliest arrival value, based on if the forward and backward search met (a solution was found).
  - Speed-up (TD road-network, 18 million nodes): up to 2000
- **Time-dependent SHARC** (2008, [Del08]). Static SHARC is an algorithm using unidirectional Dijkstra, it was therefore a good candidate to be adjusted for time-dependent scenario. It combines several techniques, perhaps the most important being pre-computing arc flags ([KMS06]) for a multi-partitioned graph, which is basically information stating if the given arc should/should not be considered when travelling to the destination cell.
  - Speed-up (TD road-network, 5 million nodes): up to 800
  - Speed-up (timetable, 30 000 stations): up to 27
- **Engineering TE graphs...** (2009, [DPW09]). While the previous papers used the time-dependent model of the timetable, this work concentrates on the time-expanded model. On a high level, this model is further refined by bypassing some low degree nodes, remodelling

unimportant stations and introducing time-dependent shortcuts - still in the phase of preprocessing. During the query, additional speed-up techniques are deployed to reduce the search space explored by Dijkstra's algorithm, as this can get quite huge in time-expanded graphs.

  – Speed-up (timetable, 30 000 stations): up to 57

A summary of some time-dependent route planning techniques (up to year 2009) can be found in [DW09]. The paper [MHSWZ07] discusses also multi-criteria queries and gives an overview of comparisons between time-dependent and time-expanded timetable model.

4 Data & analysis

In this section we would like to introduce the timetable datasets we were working with and provide the analysis of their properties. The main reason for this analysis is that it gives some insight into the characteristics of the timetables and so may contribute to develop an oracle based method with better qualities.

4.1 Data

We have obtained timetable datasets from numerous sources, in varying formats and of different types. Some of them were freely available on the Internet while others were provided by companies upon demand. Let us provide their brief description.

The dataset *air01* contains schedules of **domestic flights in United States** for the January of 2008. It is not comprehensive in the sense that it contains entries only for flights of some of the major airports in US. However it is large enough for our purposes (almost 300 airports). This dataset is just a fraction of the data that are freely available at the pages of American Statistical Association [6] in CSV format.

Timetable *cpsk* represent the **regional bus** schedules from the areas of **Ružomberok and Žilina, Slovakia**. The data were provided by the company in charge of the *cp.sk* portal - Inprop s.r.o. . The timetable contains about 1900 bus stops and came in a JDF 1.9 format [7]. Apart from the actual schedules, the data in JDF contain numerous other information which were not relevant for our purposes. From both timetables we have extracted subsets with a time range of one day.

The *gb-coach* and *gb-train* timetables are freely available from National Public Transport Data Repository (NPTDR) [8] in an ATCO-CIF format. These are not actually timetables but rather weekly snapshots of national public transport journeys made by **coach and train in Great Britain** (during certain week in year 2011). The datasets contain about 2500 stations each.

The *montr* dataset is part of a public feed for **Greater Montreal public transportation**, available at Google Transit Feeds [9]. The data are in a GTFS format (defines relations between CSV files listing stations, routes, stop-times...) and were made available by Montreal's Agence métropolitaine de transport. Our timetable *montr* corresponds to daily schedules of the Chambly-Richelieu-Carignan bus services (more than 200 bus stops).

Also in GTFS format come the data of **French railways** operated by company SNCF, publicly available at their website [10]. The schedules are weekly and there were two of them: one for intercity trains and one for TER trains (regional trains). Thus the three timetables *sncf-inter* (366 stations), *sncf-ter* (2637 stations) and their union *sncf* (2646 stations).

Finally, one more country-wide railway timetable was provided by ŽSR, the company in charge of the **Slovak national railways**. This timetable was exported in a MERITS format and its time range is for one year. The number of stations in *zsr* dataset is 233.

With the help of Python and Bash scripts, we converted each of these datasets to our timetable format (described in appendix A). This timetables were then loaded by our application TTBlazer,

[6] `http://stat-computing.org/dataexpo/2009/the-data.html`
[7] Jednotný dátový formát (JDF).
[8] `http://data.gov.uk`
[9] `http://code.google.com/p/googletransitdatafeed/wiki/PublicFeeds`
[10] `http://test.data-sncf.com/index.php/ter.html`

which can further generate sub-timetables (with less stations or smaller time range), underlying graphs and TE and TD graphs.

For a summary of the used timetables' descriptions, see table 4.1 and for their main properties, refer to table 4.2.

Name	Description	Format	Provided by	Publicly available
air01	domestic flights (US)	CSV	American Stat. Assoc.	✔
cpsk	regional bus (Ružomberok & Žilina, SVK)	JDF 1.9	Inprop s.r.o.	✗
gb-coach	country-wide buses (GB)	ATCO-CIF	NPTDR	✔
gb-train	country-wide rails (GB)	ATCO-CIF	NPTDR	✔
montr	public transport (Montreal, CA)	GTFS	Montreal AMT	✔
sncf	country-wide rails (FRA)	GTFS	SNCF	✔
zsr	country-wide rails (SVK)	MERITS	ŽSR	✗

Table 4.1: Datasets descriptions.

Name	El. conns.	Cities	UG arcs	Time range	Height
air01	601489	287	4668	1 month	24374
cpsk	97916	1905	5093	1 day	370
gb-coach	260710	2448	5793	1 week	3140
gb-train	1714535	2555	8335	1 week	7978
montr	7153	217	349	1 day	363
sncf	416302	2646	7994	1 week	2679
sncf-inter	22750	366	901	1 week	1052
sncf-ter	393587	2637	7647	1 week	2646
zsr	932052	233	588	1 year	60308

Table 4.2: Main properties of the timetables. The value of time range is approximate.

To see better the differences in the properties of different timetable types (train, flight, bus...), we made sub-timetables with 200 cities and with the upper bound on time range being 1 day and 6 hours [11] ($thigh_T <$ 1 day and 6 hours) from each of our dataset. We name these datasets by appending to the original name "*-200d*" [12]. See table 4.3 for details.

Name	El. conns.	Cities	UG arcs	Height
air01-200d	19010	200	3973	772
cpsk-200d	14747	200	592	370
gb-coach-200d	2760	200	564	498
gb-train-200d	24323	200	792	957
montr-200d	6841	200	320	355
sncf-200d	4192	200	611	269
sncf-inter-200d	2172	200	493	128
sncf-ter-200d	8469	200	600	419
zsr-200d	2031	200	454	133

Table 4.3: 200-station sub-timetables with the time range of one day.

Also, to further justify our choice of using TD graphs instead of TE graphs in this thesis, we provide

[11] We took all elementary connections that were within our time range. From this timetable, we made an UG and its (random) sub-graph of 200 cities. Finally we selected only those elementary connections, that were on top of this sub-graph to form a timetable with 200 cities and the desired (maximal) time range.

[12] Similarly, throughout this thesis, suffix "*-d*" would mean "with daily time range", "*-w*" "weekly time range" and suffix "*-#*" would mean sub-timetable with # stations.

their space consumption comparison in table 4.4.

	TD graph			**TE graph**		
Name	**Nodes**	**Arcs**	**Size (MB)**	**Nodes**	**Arcs**	**Size (MB)**
air01	287	4668	27	715211	1307432	72
cpsk	1905	5093	5	95601	189205	11
gb-coach	2448	5793	12	259589	512862	32
gb-train	2555	8335	79	2042316	3745751	263
montr	217	349	0.4	7182	13992	0.9
sncf	2646	7994	19	758867	1166646	85
sncf-inter	366	901	1.1	39765	60602	4.6
sncf-ter	2637	7647	18	720651	1107301	81
zsr	233	588	42	1706077	2637896	173

Table 4.4: Space consumption of time-dependent vs. time-expanded model. The number of nodes and arcs for TD graph is the same as for the corresponding underlying graph.

4.2 Analysis of properties

First we will take a look at the optimal connection *sizes* (size is the number of elementary connections) in the timetables. For a given timetable T, we will denote the average optimal connection size as $\boldsymbol{\gamma_T}$ and will call it the **optimal connection diameter** (OC diameter). We computed an approximate OC diameter for each of our datasets by measuring an average connection size of sufficiently many OCs. The results in table 4.5 indicate that the average OC size generally falls under $\sqrt{n}$.

Next we would like to get an idea of the sparsity of the underlying graphs. We see from the table 4.2 that the graphs are pretty sparse (with the exception of *air01*), but we would like to make sure that the sparsity is uniform. More specifically, we will be interested in the δ-density:

Definition 4.1. *δ-density*
A graph G of n vertices and m arcs is δ-dense $\iff \forall G' \subseteq G, n' \geq \sqrt[4]{n} : \frac{m'}{n'} \leq \delta$

- *For a timetable T, we will denote its **density** parameter [13] as $\boldsymbol{\delta_T} = min\{\delta | \; ug_T \text{ is } \delta\text{-dense}\}$*

To find out at least approximate δ_T values for our timetables, we have randomly sampled their UGs for (connected) sub-graphs of various sizes (starting from $\sqrt[4]{n}$ [14]). In table 4.6 you can see the maximal density found during the sampling.

The density is related to the **average degree** deg_{avg} in the UG, since in oriented graphs:

$$deg_{avg} = \frac{m}{n}$$

So the average degree is a lower bound on the graph's density. Table 4.7 lists the average and maximal degrees in the underlying graphs.

We would also assume, that the underlying graphs of each timetable will be **connected** (and even strongly connected), or at least that the largest connected component spans almost the whole graph.

[13] Note that this has nothing to do with the frequency of elementary connections, only with the density of the underlying graph.
[14] The choice of $\sqrt[4]{n}$ will be justified later, during the analysis of the algorithms.

Name	γ_T	Max. OC size found	$\sqrt{n}$
air01	2.4	8	16.9
cpsk	40.8	162	43.6
gb-coach	25.2	128	49.5
gb-train	25.6	111	50.5
montr	21.1	63	14.7
sncf	36.8	111	51.4
sncf-inter	17.1	58	19.1
sncf-ter	48.0	167	51.3
zsr	15.0	57	15.3

Table 4.5: With one exception, OC diameter is less then $\sqrt{n}$ (this was expected, as *montr* is the only timetable with "geographically one dimension long" - all other timetables span areas with more uniform shape). Note extremely low value for airline timetable - this is due to the fact that UGs of airline timetables have small-world characteristics [Som10]. Another thing we may notice is that regional timetables (*cpsk*, *sncf-ter*) have higher OC diameter then country-wide and inter-city timetables. We also point out that the inter-city trains in French railways decrease the average optimal connection size by one about third.

Name	Maximal δ_T found
air01	34.5
cpsk	4.1
gb-coach	5.0
gb-train	5.8
montr	1.9
sncf	5.0
sncf-inter	3.0
sncf-ter	4.8
zsr	3.2

Table 4.6: Approximate density of the underlying graphs.

Name	Avg. degree	Max. degree
air01	16.3	166
cpsk	2.7	27
gb-coach	2.4	103
gb-train	3.3	30
montr	1.6	5
sncf	3.0	27
sncf-inter	2.5	12
sncf-ter	2.9	27
zsr	2.5	12

Table 4.7: Average and maximal degree in the underlying graphs.

From the table 4.8 we may see that this assumption holds.

		Connectivity		Strong connectivity	
Name	n	**Connected**	**Largest comp.**	**Connected**	**Largest comp.**
air01	287	✔	287	✗	286
cpsk	1905	✔	1905	✗	1903
gb-coach	2448	✗	2374	✗	2332
gb-train	2555	✔	2555	✔	2555
montr	217	✗	211	✗	209
sncf	2646	✔	2646	✗	2594
sncf-inter	366	✗	328	✗	316
sncf-ter	2637	✔	2637	✗	2583
zsr	233	✔	233	✗	225

Table 4.8: Connectivity of underlying graphs.

In the previous section 3 we have mentioned the highway dimension [AFGW10] as a parameter which, when being low, guarantees low query times for certain route-planning methods. Here we were interested in the highway dimension of our underlying graphs.

Definition 4.2. ***Highway dimension***
Highway dimension $\boldsymbol{HD(G)}$ *for a directed, edge-weighted graph* $G = (V, E)$ *is the smallest integer* h*, such that:*

$$\forall r \in R^+,\ \forall u \in V,\ \exists S \subseteq B_{u,2r},\ |S| \leq h,\ \forall v, w \in B_{u,2r}:$$
$$\textit{if } r < |P(v,w)| \leq 2r \textit{ and } P(v,w) \subseteq B_{u,2r} \textit{ then } P(v,w) \cap S \neq \emptyset$$

where:

- $\boldsymbol{P(v,w)}$ *is the* ***shortest path*** *between* v *and* w
- $\boldsymbol{B_{u,r}} = \{v \in V|\ |P(u,v)| \leq r \text{ or } |P(v,u)| \leq r\}$ *and is called* ***ball*** *of radius* r *centred at* u*.*

Intuitively, a graph has a low HD, if for any r we have a *sparse* set of vertices S_r, such that every shortest path longer then r includes a vertex from S_r. By the set being sparse, we mean that every ball of radius $\mathcal{O}(r)$ contains just a few elements of S_r.

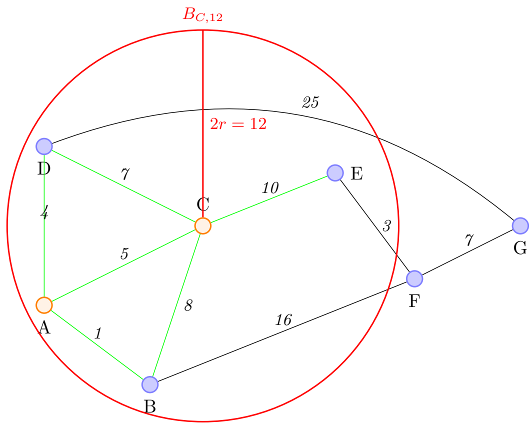


Figure 4.1: Demonstration of a definition of HD. We chose some r ($r = 6$) and some vertex v ($v = C$) to root the ball $B_{v,2r}$. All the shortest paths *longer* than r *inside* the ball have to contain a vertex from S (orange vertices C and A in our case). The upper bound on $|S|$, considering any ball with any radius, is the required highway dimension. Note: in our case, we had to choose also A to set S, since a shortest path from B to D does not include C.

Highway dimension is difficult to measure due to the way it is defined. We compute at least approximate value for sub-graphs of 200 nodes of the underlying graphs. Results could be seen in table **??**.

Name	apx. HD
svk-200	**32**
air01-200	82
cpsk-200	127
gb-coach-200	107
gb-train-200	147
montr-200	
sncf-200	
sncf-inter-200	
sncf-ter-200	
zsr-200	

Table 4.9: Highway dimension of 200 vertex sub-graphs of the underlying graphs.

5 Underlying shortest paths

In section 2 we have defined a timetable as a set of elementary connections. While do not pose any other restrictions on this set or on the elementary connections themselves, the real world timetables usually have a specific nature. Quite often are the connections repetitive, that is, the same sequence of elementary connections is repeated in several different moments throughout the day.

Another thing we may notice is that if we talk about *optimal* connections between a pair of distant cities u and v, we are often left with a few possibilities as to *which way should we go*. This is not only because the underlying graph is usually quite sparse [15], but also because for longer distances we generally need to make use of some express connection that stops only in (small number of) bigger cities.

Thus the main idea common to the methods presented in this section: *when carrying out an optimal connection between a pair of cities, one often goes along the same path regardless of the starting time* [16].

To formalize this idea, we will introduce the definition of an *underlying shortest path* - a path in the underlying graph that corresponds to some optimal connection in the timetable. To do this, we will first define a function *path* that extracts the **underlying path** (trajectory in the UG) from a given connection. Let c be a connection $c = (e_1, e_2, ..., e_k)$.

$$\boldsymbol{path(c)} = shrink(from(e_1), from(e_2), ..., from(e_k), to(e_k))$$

Note, that if the connection involves waiting in a city (as e.g. in figure 5.1), $e_x^i = e_x^{i+1}$ for some i. That is why we apply the *shrink* function, which replaces any sub-sequences of the type $(z, z, ..., z)$ by (z) in a sequence. This is rather technical way of expressing a simple intuition - for a given connection, the *path* function outputs a sequence of visited cities. Now we can formalize the notion of underlying shortest path.

Definition 5.1. ***Underlying shortest path (USP)***
*A path $p = (v_1, v_2, ..., v_k)$ in UG_T is an **underlying shortest path** if and only if $\exists t \in \mathbb{N} : p = path(c^*_{(v_1,t,v_k)}), c^*_{(v_1,t,v_k)} \in C_T$*

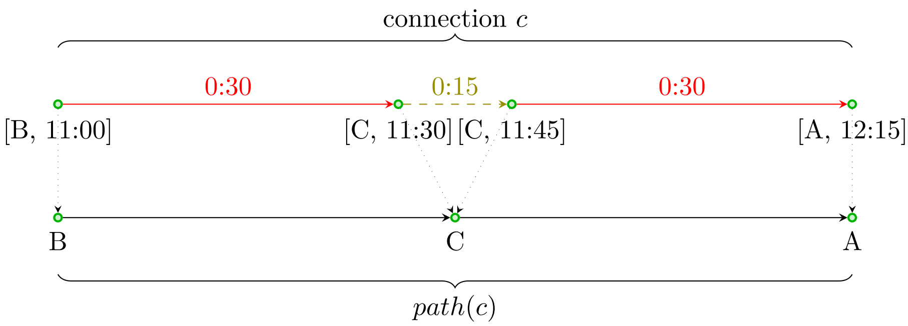


Figure 5.1: The *path* function applied on a connection to get the underlying path.

Please note that the terminology might be a bit misleading - an USP is not necessarily a shortest

[15] Maybe with exception of the airline timetables, which tend to be more dense.
[16] Or similarly, there are only few paths that are worth to follow.

path in the given UG. For example, the connections on a shortest path may require too much waiting and thus it might be that travelling along the paths with greater distance proofs to be faster.

5.1 *USP-OR*

We can easily extract the underlying path from a given connection. Now let us look at this from the other way - if, for a given EA query, we know the underlying shortest path, can we reconstruct the optimal connection? One thing we could do is to blindly follow the USP and in each city take the first elementary connection to the next one on the USP. This simple method called *expand* is described in algorithm 5.1.

Algorithm 5.1 expand

Input

- timetable T
- path $p = (v_1, v_2, ..., v_k)$, $v_i \in ct_T$
- departure time t

Algorithm

c = empty connection
$t' = t$
for all $i \in \{1, ..., k-1\}$ **do**
  $e = argmin_{e' \in C_T(v_i, v_{i+1})}\{dep(e') |\ dep(e') \geq t'\}$ *# take first available el. conn.*
  $t' = arr(e)$
  $c .= e$ *# add the el.conn to the resulting connection*
end for

Output

- connection c

A question is - will we get an optimal connection if we expanded all possible USPs between a pair of cities? We show that we will, provided the timetable has no *overtaking* [DW09] of elementary connections.

Definition 5.2. ***Overtaking***
*An elementary connection e_1 **overtakes** e_2 if, and only if $dep(e_1) > dep(e_2)$ and $arr(e_1) < arr(e_2)$.*

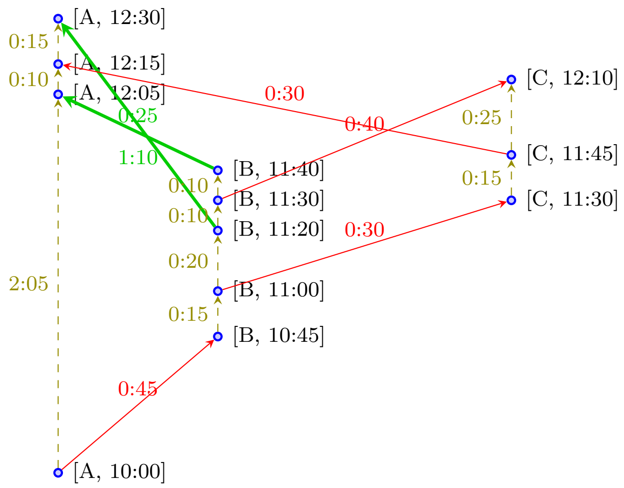


Figure 5.2: An example of **overtaking** (in thick), depicted in a TE graph.

Name	Overtaken
air01	1%
cpsk	2%
gb-coach	1%
gb-train	0%
montr	1%
sncf	1%
sncf-inter	6%
sncf-ter	1%
zsr	0%

Figure 5.3: Percentage of overtaken elementary connections in the timetables.

Lemma 5.1. *Let T be a timetable without overtaking, (x, t, y) an EA query in this timetable and* $\boldsymbol{usps(x, y)} = \{p_1, p_2, ..., p_k\}$ *a set of all USPs from x to y. Define c_i =expand(T, p_i, t) to be the connection returned by the algorithm expand 5.1. Then $\exists j : c_j = c^*_{x,t,y}$.*

Proof. The optimal connection $c^*_{x,t,y}$ has an USP p which must be present in the set *usps*(x, y), as it is the set of all USPs from x to y. So $p = p_j = (v_1, v_2, ..., v_l)$ from some j. We want to show that c_j is the optimal connection. This may be shown inductively:

1. *Base: expand* reaches city $v_1 = x$ as soon as possible (since the connection just starts there)
2. *Induction: expand* reached city v_i as soon as possible, it then takes the first available el. connection to the next city v_{i+1}. Since the el. connections do not overtake, *expand* reached the city v_{i+1} as soon as possible.

□

We would like to stress that overtaking is understood as a situation when e.g. one train overtakes another between *two subsequent stations*. This situation is not that common, however it is still present in the real world timetables [17], as shown in table 5.3. All the same, we can simply remove the overtaken elementary connections from the timetables, as they can be substituted by the quicker connection plus some waiting, thus we will not change the earliest arrival time for any query.

The basic idea of the algorithm *USP-OR* (**USP or**acle) is therefore simply to pre-compute all the USPs for each pair of cities. Upon a query, the algorithm expands all the USPs for a given pair of cities, reconstructs respective connections and chooses the best one.

[17]In Slovak rails, no overtaking has been detected. This is not surprising as (to my knowledge) there are no inter-station tracks with multiple rails going in one direction. French railways, on the other hand have designated high-speed tracks and thus overtaking is not impossible.

Algorithm 5.2 *USP-OR* query

Input
- timetable T
- query (x, t, y)

Pre-computed
- $\forall x, y:\ usps(x, y)$

Algorithm

$c^* = null$
for all $p \in usps(x, y)$ **do**
  $c = expand(T, p, t)$
  $c^* =$ better out of c^* and c
end for

Output
- connection c

5.1.1 Analysis of *USP-OR*

We will now have a look at the four parameters of this oracle based method. As for the preprocessing time, we need to find the optimal connections from each *event* in the timetable to each *city* (or in other words - solve all possible OC queries). On these connections we apply the *path* function to obtain the USPs. There is hn events and one search from a single event to all cities can be done in time $\mathcal{O}(n \log n + m)$ with TD Dijkstra (in this section we use exclusively the time-dependent graphs). In worst case, m could be as much as n^2 but we may bound it as $m \leq \delta_T n$ (where δ_T is the density of the timetable, defined in section 4). We therefore get the **preprocessing time $\mathcal{O}(hn^2(\log n + \delta))$**.

As for the preprocessed space, we need to store USPs for each pair of the cities (n^2 pairs) and each USP might be long at most $\mathcal{O}(n)$ hops. What is more, there might be multiple USPs for a single pair of cities. Therefore we have two questions with respect to the space complexity of the preprocessing:

1. What is the average size of the USPs?
2. How many are there USPs between pairs of cities on average?

As for the first question, we will call the average size of USPs in a timetable T the **USP diameter** and denote it $\boldsymbol{\omega_T}$. This value is generally higher then the OC diameter [18], but can still be very well approximated by $\sqrt{n}$ (see table 5.1 and plot 5.4).

To answer the second question, we will introduce the following definition:

Definition 5.3. ***USP coefficient***
Given a timetable T and a pair of cities x, y, we define the USP coefficient $\boldsymbol{\tau_T(x, y)} = |usps_T(x, y)|$. By $\boldsymbol{\tau_T}$ we will denote the average USP coefficient in timetable T.

From the table 5.1 we may see that τ is quite small (≈ 10). Important thing however is whether or not it is constant with respect to:

- n - we found τ to be slightly increasing, sometimes almost constant (see plot 5.5)
- time range - again the value of τ was slightly increasing (plot 5.6)

[18] If, for example, we have 8 optimal connections with size 1 and 1 optimal connection with size 10, the OC diameter will be 2 but the average USP size will be 5.5.

From the answers to our two questions we see that the **size of the preprocessed oracle** is $\mathcal{O}(\tau n^2 \omega)$.

Name	τ	max $\tau(x, y)$	ω
air01-200d	5.6	29	3.7
cpsk-200d	7.7	37	19.4
gb-coach-200d	3.5	29	7.2
gb-train-200d	6.8	40	10.3
montr-200d	2.7	18	26.1
sncf-200d	3.8	16	10.5
sncf-inter-200d	1.8	13	14.8
sncf-ter-200d	4.1	14	15.1
zsr-200d	2.3	13	16.2

Table 5.1: Average and maximal USP coefficients and USP diameter for daily timetables with 200 stations ($\sqrt{200} \approx 14$).

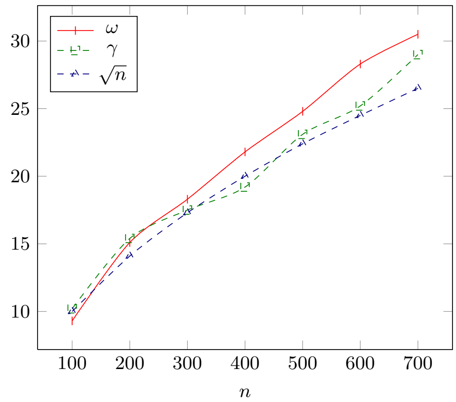


Figure 5.4: Changing of ω with increased number of stations in *cpsk* dataset. Compared to the OC diameter and $\sqrt{n}$.

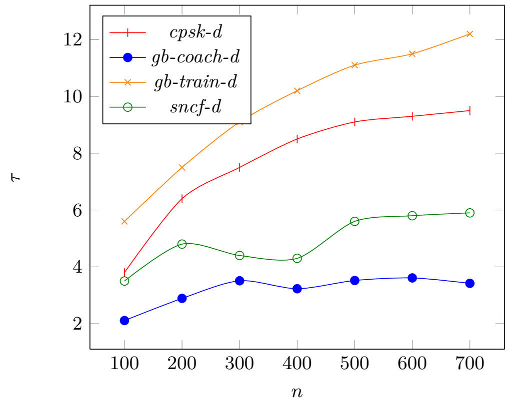


Figure 5.5: Changing of τ with increased number of stations.

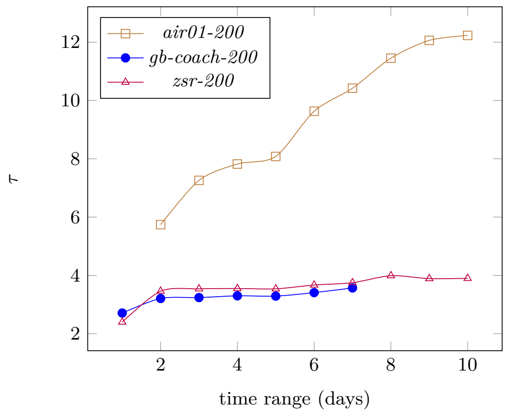


Figure 5.6: Changing of τ with increased time range.

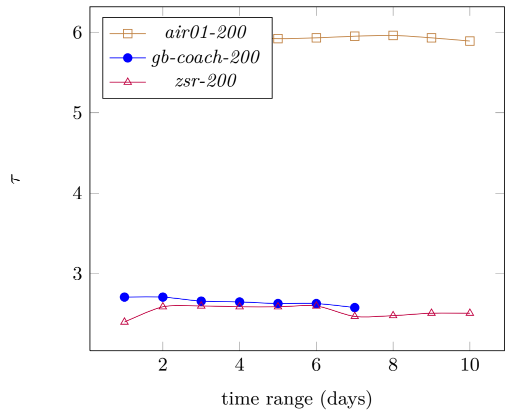


Figure 5.7: Changing of τ with increased time range when using segmentation.

Also the query time depends on the USP coefficient of a given pair of cities x, y, as we have to try out all USPs in $usps(x, y)$. The expansion of a USP by *expand* function takes time linear in the size of the USP [19], leading to **query time $\mathcal{O}(\tau\omega)$** on average. Note, that this is pretty much optimal, as τ is basically constant and we need to output the connection itself, which takes linear time in its size.

To alleviate the problem of increased τ in timetables with e.g. weekly time range, we did a simple trick called **segmentation**. First, we normally computed the USPs. Then we segmented the timetable to individual days and for each of them we stored the pointers to necessary USPs. This does not require additional memory but it makes the value of τ constant, or even decreasing (see plot 5.7) with increasing time range. Note that this would be reflected only in an improved query time of *USP-OR*, the size of preprocessed data will be left unaffected. However, from this point on we assume the use of segmentation for multi-day timetables (also in *USP-OR-A* algorithm, explained in the next section).

Finally, the **stretch** of *USP-OR* is **1**, since it returns exact answers.

USP-OR	***prep***	***size***	***qtime***	***stretch***
guaranteed	$\mathcal{O}(hn^2(\log n + \delta))$	$\mathcal{O}(\tau n^2 \omega)$	avg. $\mathcal{O}(\tau\omega)$	1
$\boldsymbol{\omega \leq \sqrt{n}, \delta \leq \log n}$	$\mathcal{O}(hn^2 \log n)$	$\mathcal{O}(\tau n^{2.5})$	avg. $\mathcal{O}(\tau\sqrt{n})$	1

Table 5.2: The summary of the *USP-OR* algorithm parameters.

5.2 *USP-OR-A*

With *USP-OR* the main disadvantage is its space consumption. We may decrease this space complexity by pre-computing USPs only between *some* cities. The nodes that we select for this purpose will be called **access nodes** (AN for short), as for each city they would be the crucial nodes we need to pass in order to access most of the cities of T. It would be suitable for this access node set to have several desirable properties. In order to formulate them, we need to define a few terms first.

Definition 5.4. ***Front neighbourhood***
Given a timetable T and access node set $\mathcal{A}$, a front neighbourhood of city x is the set of all cities (including x) that are reachable from x without the need to pass a city from $\mathcal{A}$. Formally $\boldsymbol{neigh_{\mathcal{A}}(x)} = \{y \in ct_T |\ \exists\ path\ p = (p_1, p_2, ..., p_k)\ from\ x\ to\ y\ in\ ug_T : p_i \neq a\ \forall a \in \mathcal{A},\ i \in \{2, ..., k-1\}\}$ [20]

We define analogically **back neighbourhood** (denoted $\boldsymbol{bneigh_{\mathcal{A}}(x)}$), as nodes that could be reached in UG with reversed orientation ($\overleftarrow{ug_T}$). Note that the access nodes that are on the boundary of x's neighbourhoods are also part of these neighbourhoods. These access nodes form some sort of separator between the x's neighbourhood and the rest of the graph and we will call them **local access nodes (LAN)** ($\boldsymbol{lan_{\mathcal{A}}(x)} = \mathcal{A} \cap neigh_{\mathcal{A}}(x)$), or analogically **back local access nodes** ($\boldsymbol{blan_{\mathcal{A}}(x)}$).

Now we may formulate the three desirable properties of the access node set. Given a timetable T, we would like to find access node set $\mathcal{A}$ such that for some small constants r_1, r_2 and r_3:

[19]In time-dependent graphs, this requires a constant-time retrieval of the correct interpolation point of the cost function (the piece-wise linear function that tells us the traversal time of an arc at a given time) for some time t. More specifically, we need to obtain an interpolation point $argmin_{(t',l)}\{t' |\ t' > t\}$. If we assume uniform distribution of departures throughout the time range of the timetable, this can be implemented in constant time. Otherwise, binary search lookup is possible in time $\mathcal{O}(\log h)$.

[20]In $neigh_{\mathcal{A}}(x)$ we leave out subscript identifying the timetable T. In situation with clear context, we may also leave out the $\mathcal{A}$ subscript.

1. The access node set is sufficiently small

$$|\mathcal{A}| \leq r_1 \cdot \sqrt{n} \tag{5.1}$$

2. The average square of neighbourhood [21] size for cities not in $\mathcal{A}$ is at most $r_2 \cdot n$

$$\frac{\sum_{x \in ct_T \setminus \mathcal{A}} |neigh_{\mathcal{A}}(x)|^2}{|ct_T \setminus \mathcal{A}|} \leq r_2 \cdot n \tag{5.2}$$

3. The average square of the number of local access nodes [22] for cities not in $\mathcal{A}$ is at most r_3

$$\frac{\sum_{x \in ct_T \setminus \mathcal{A}} |lan_{\mathcal{A}}(x)|^2}{|ct_T \setminus \mathcal{A}|} \leq r_3 \tag{5.3}$$

An access node set $\mathcal{A}$ with the above mentioned properties will be called **(r_1, r_2, r_3) access node set** (AN set). We will now explain how the *USP-OR-A* (**USP or**acle with **a**ccess nodes) algorithm works and return to its analysis later.

During preprocessing, we need to find a good AN set and compute the USPs between every pair of access nodes. For every city $x \notin \mathcal{A}$, we also store its $neigh_{\mathcal{A}}(x)$, $bneigh_{\mathcal{A}}(x)$, $lan_{\mathcal{A}}(x)$ and $blan_{\mathcal{A}}(x)$. On a query from x to y at time t, we will first make a local search in the neighbourhood of x up to x's local access nodes. Subsequently, we want to find out the earliest arrival times to each of y's *back* local access nodes. To do this, we take advantage of the pre-computed USPs between access nodes - try out all the pairs $u \in lan(x)$ and $v \in blan(y)$ and expand the stored USPs. Finally, we make a local search from each of y's back LANs to y, but we run the search *restricted* to y's back neighbourhood. For more details, see algorithms 5.3 and 5.4 and figure 5.8, where we have split the algorithm to 3 distinct phases.

Algorithm 5.3 *USP-OR-A* preprocessing

Input
- timetable T

Algorithm

find a good AN set $\mathcal{A}$
$\forall x, y \in \mathcal{A}$ compute $usps(x, y)$
$\forall x \in ct_T \setminus \mathcal{A}$ compute $neigh_{\mathcal{A}}(x)$, $bneigh_{\mathcal{A}}(x)$, $lan_{\mathcal{A}}(x)$ and $blan_{\mathcal{A}}(x)$

[21] We required the same for back neighbourhoods.
[22] We required the same for back LANs.

Algorithm 5.4 *USP-OR-A* query

Input
- timetable T
- query (x, t, y)

Algorithm

let $lan(x) = x$ if $x \in \mathcal{A}$
let $blan(y) = y$ if $y \in \mathcal{A}$
Local front search
perform TD Dijkstra from x at time t up to $lan(x)$
if $y \in neigh(x)$ **then**
  let c^*_{loc} be the connection to y obtained by TD Dijkstra   *# the optimal connection may still go via ANs (though it is unlikely)*
end if
$\forall u \in lan(x)$ let $ea(u)$ be the arrival time and $oc(u)$ the connection to u obtained by TD Dijkstra
Inter-AN search
for all $v \in blan(y)$ **do**
  $oc(v) = null$
  for all $u \in lan(x)$ **do**
    for all $p \in usps(u, v)$ **do**
      $c = expand(T, p, ea(u))$
      $oc(v) =$ better out of $oc(v)$ and c
    end for
  end for
end for
$\forall v \in blan(y)$ let $ea(v) = end(oc(v))$
Local back search
for all $v \in blan(y)$ **do**
  perform TD Dijkstra from v at time $ea(v)$ to y restricted to $bneigh(y)$
  let $fin(v)$ be the connection returned by TD Dijkstra
end for
$v^* = argmin_{v \in blan(y)}\{end(fin(v))\}$
$u^* = from(oc(v^*))$
let $c^* = oc(u^*).oc(v^*).fin(v^*)$ *# the dot (.) symbol is concatenation of connections*
output better out of c^*_{loc} and c^*

Output
- optimal connection $c^*_{(x,t,y)}$

5.2.1 Analysis of *USP-OR-A*

Let us now analyse the properties of this oracle-based method. Clearly, much depends on the way we look for the access node set. We will address this issue in next subsections but for now, we will assume we can find (r_1, r_2, r_3) AN set $\mathcal{A}$ in time $f(n)$. Then, in the preprocessing, we have to find USPs among the access nodes, which requires running Dijkstra's algorithm from each event in an access node (city from $\mathcal{A}$). There is $\mathcal{O}(r_1 h \sqrt{n})$ such events which leads to the time complexity $\mathcal{O}(r_1 h n^{1.5}(\log n + \delta))$. We also have to find local access nodes and neighbourhoods for each city, which can be accomplished with e.g. depth first search exploring the neighbourhood. This search algorithm (run from non-access city) has complexity linear in the number of arcs and so we could bound the total complexity as:

$$\sum_{x \in ct_T \setminus \mathcal{A}} |E(neigh_{\mathcal{A}}(x))| \leq \sum_{x \in ct_T \setminus \mathcal{A}} |neigh_{\mathcal{A}}(x)|^2 \leq r_2 n^2$$

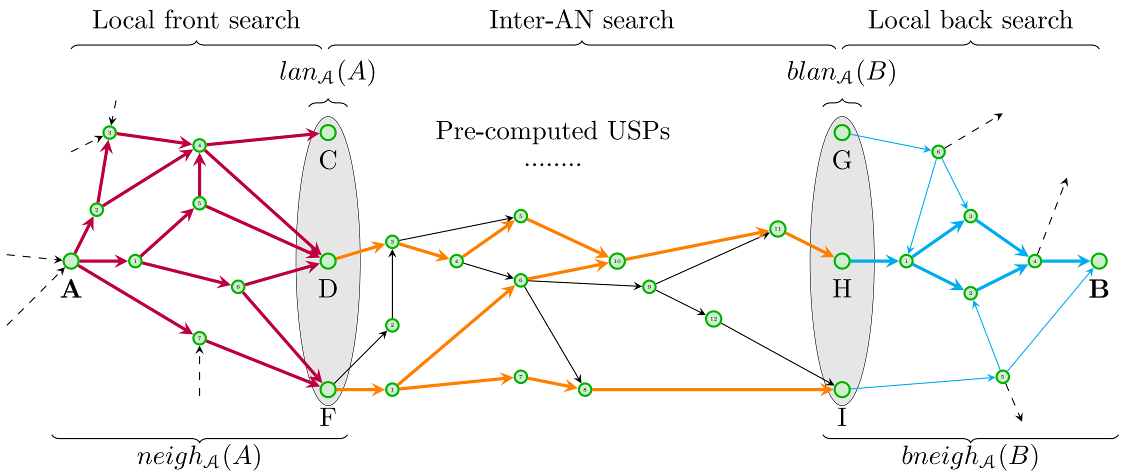


Figure 5.8: Principle of *USP-OR-A* algorithm. The arcs in **bold** mark areas that will be explored: all nodes in $neigh_{\mathcal{A}}(x)$, USPs between LANs of x and back LANs of y and the back neighbourhood of y (possibly only part of it will be explored, since the local back search goes against the direction of the back neighbourhood).

where $E(V)$ is the set of arcs among vertices of V. However this is very loose upper bound, as our UGs are actually very sparse. Therefore we can improve it. We know from the equation 5.2 that the average square of neighbourhood size is at most $r_2 \cdot n$. As a consequence of the Cauchy-Schwarz Inequality [ops] the following holds for positive real numbers x_i:

$$\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Applying this to our neighbourhood sizes, we get that the average size of the neighbourhood is at most $\sqrt{r_2 n}$. We now split the vertices of $ct_T \setminus \mathcal{A}$ to two categories: those with neighbourhoods of size at most $\sqrt[4]{n}$ will be part of the set $S_{\leq}$ and those with neighbourhoods of size bigger then $\sqrt[4]{n}$ will be in $S_{>}$. A neighbourhood in the first category cannot possibly contain more than $\sqrt{n}$ arcs while those in the second category can have at most $\delta_T |neigh_{\mathcal{A}}(x)|$ arcs (thus depending on the timetable's density).

$$\begin{aligned}
\sum_{x \in ct_T \setminus \mathcal{A}} |E(neigh_{\mathcal{A}}(x))| &\leq \\
\sum_{x \in S_{\leq}} \overbrace{|E(neigh_{\mathcal{A}}(x))|}^{\leq \sqrt{n}} + \sum_{x \in S_{>}} \overbrace{|E(neigh_{\mathcal{A}}(x))|}^{\leq \delta |neigh_{\mathcal{A}}(x)|} &\leq \\
n\sqrt{n} + \delta n \sqrt{r_2 n} &\leq \\
\delta r_2 n^{1.5} &
\end{aligned}$$

Therefore, the total **time complexity of the preprocessing** is $\mathcal{O}(f(n) + r_1 h n^{1.5}(\log n + \delta)) + \mathcal{O}(\delta r_2 n^{1.5}) = \boldsymbol{\mathcal{O}(f(n) + (r_1 + r_2)(\delta + \log n) h n^{1.5})}$.

As for the size of the preprocessed data - we need to store all the neighbourhoods, LANs and USPs between access nodes. We already know that the average size of the neighbourhood is at most $\sqrt{r_2 n}$, thus the total size of the (front and back) neighbourhoods is $\mathcal{O}(r_2 n^{1.5})$ [23]. This term bounds also the size of the pre-computed local access nodes for each node.

Finally we have the preprocessed USPs. There is at most $r_1^2 n$ pairs of access nodes and for each of them we have possibly several USPs. We will denote by $\boldsymbol{\tau_{\mathcal{A}}}$ the average USP coefficient between pairs of cities from $\mathcal{A}$ and by $\boldsymbol{\omega_{\mathcal{A}}}$ the average USP size between cities in $\mathcal{A}$. This amounts to $\mathcal{O}(r_1^2 \tau_{\mathcal{A}} \omega_{\mathcal{A}} n)$ for storage of USPs and to a total **preprocessing size** $\boldsymbol{\mathcal{O}(r_2 n^{1.5} + r_1^2 \tau_{\mathcal{A}} \omega_{\mathcal{A}} n)}$.

Name	n	$\tau_{\mathcal{A}}$	$\omega_{\mathcal{A}}$	$\sqrt{n}$
air01-d	284	10.4	3.4	16.9
cpsk-d	1905	15.9	42.6	43.6
gb-coach-d	2427	6.3	20.2	49.3
gb-train-d	2550	21.8	23.0	50.5
montr-d	217	4.0	24.3	14.7
sncf-d	2608	9.8	34.0	51.1
sncf-inter-d	344	2.8	12.2	18.5
sncf-ter-d	2600	8.9	45.5	51.1
zsr-d	225	4.7	16.5	15

Table 5.3: USP coefficient and diameter for access node sets, daily timetables.

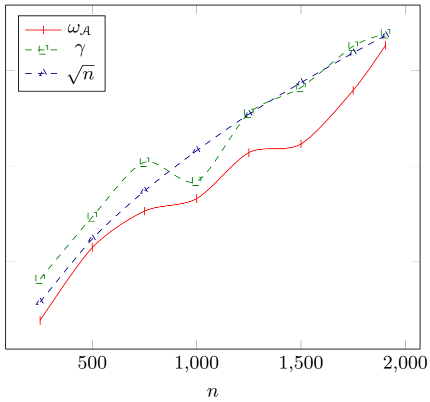


Figure 5.9: Changing of $\omega_{\mathcal{A}}$ with increased number of stations in *cpsk* dataset. Compared to the OC diameter and $\sqrt{n}$.

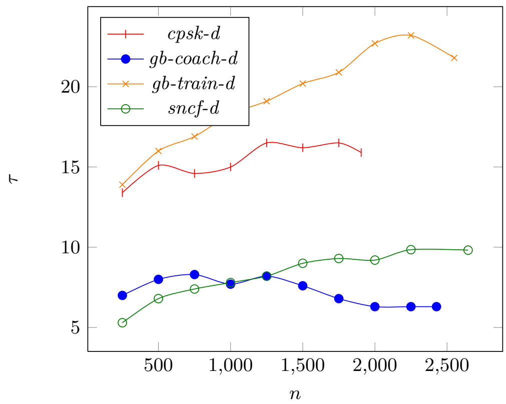


Figure 5.10: Changing of $\tau_{\mathcal{A}}$ with increased number of stations.

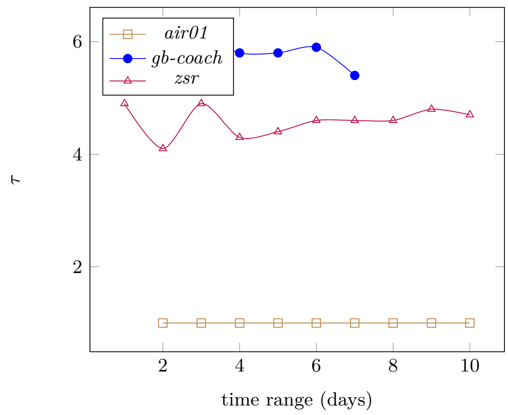


Figure 5.11: Changing of $\tau_{\mathcal{A}}$ with increased time range (using segmentation).

On a query from x at time t to y, we first perform the *local front search* (see algorithm 5.4). In this step we explore the neighbourhood of x with a time-dependent Dijkstra's algorithm, which takes on average time $\mathcal{O}(\sqrt{r_2 n}(\log(\sqrt{r_2 n}) + \delta))$. We then expand all the USPs between u and v such that $u \in lan(x)$ and $v \in blan(y)$, which takes on average $\mathcal{O}(r_3 \tau_{\mathcal{A}} \omega_{\mathcal{A}})$. Finally, from each $v \in blan(y)$ we do a TD Dijkstra, restricted to $bneigh(y)$, leading to time complexity $\mathcal{O}(\sqrt{r_3 r_2 n}(\log(\sqrt{r_2 n}) + \delta))$.

Summing up the three terms we obtain the **query time** of $\boldsymbol{\mathcal{O}(r_2 \sqrt{r_3} \sqrt{n}(\log(r_2 n) + \delta) + r_3 \tau_{\mathcal{A}} \omega_{\mathcal{A}})}$.

Stretch of the *USP-OR-A* algorithm is **1**, as it is exact algorithm.

[23] As r_2 should be a very small constant, we may disregard the square root.

The resulting bounds do not look very appealing. This is because we wanted to preserve the generality - the concrete bounds will depend on what kind of properties the timetables have and what algorithm for finding the AN set is plugged in. In table 5.4, we summarize the parameters of *USP-OR-A* method and provide the bounds for a case when the properties of the timetables correspond to those we have measured in our datasets and when we have an algorithm that finds good AN set.

USP-OR-A	**guaranteed**	$\boldsymbol{r_1, r_2, r_3 = \mathcal{O}(1), \omega \leq \sqrt{n}, \delta \leq \log n}$
prep	$\mathcal{O}(f(n) + (r_1 + r_2)(\delta + \log n)hn^{1.5})$	$\mathcal{O}(f(n) + hn^{1.5}\log n)$
size	$\mathcal{O}(r_2 n^{1.5} + r_1^2 \tau\omega n)$	$\mathcal{O}(\tau n^{1.5})$
qtime	avg. $\mathcal{O}(r_2\sqrt{r_3}\sqrt{n}(\log(r_2 n) + \delta) + r_3\tau\omega)$	avg. $\mathcal{O}(\tau\sqrt{n}\log n)$
stretch	1	1

Table 5.4: The summary of the *USP-OR-A* algorithm parameters. Here we left out subscripts identifying AN set for τ and ω.

5.2.2 Correctness of *USP-OR-A*

Finally, we will proof the correctness of the algorithm, i.e. that it always returns the optimal connection.

Theorem 5.1. *The algorithm USP-OR-A (5.3, 5.4) always returns the optimal connection.*

Proof. Let $\mathcal{A}$ be the set of access nodes and consider a query from city x to city y at any time t. If $x \in \mathcal{A}$ and $y \in \mathcal{A}$, an optimum is returned due to lemma 5.1 (in such a case, we basically run *USP-OR* algorithm).

In the following we will assume that $\boldsymbol{y \notin neigh(x)}$, which means that the optimal connection goes through some access node $u \in lan(x)$ and $v \in blan(y)$. Note that it may be that $u = v$.

What we would like to prove as a next step is that we reach the back LANs of y (or y itself if it is an access node) at the earliest arrival time. After the *local front search*, we have reached the x's local ANs at times $ea(u)$ $\forall u \in lan(x)$. For some local access node this value is the true earliest arrival. Let us denote the set of such local ANs as $lan^*(x)$. The crucial thing to realize is that the optimal connection to any city out of the x's neighbourhood will lead via some $u \in lan^*(x)$ (see figure 5.12). And because the *inter-AN search* phase finds *optimal* connections between pairs $u \in lan(x)$ and $v \in blan(y)$, it follows that for each $v \in blan(y)$ the $ea(v)$ is the earliest arrival to this city after the *inter-AN search* phase.

In the *local back search* we run a TD Dijkstra search from all back LANs of y. And since this algorithm is exact and starts from each back LAN as early as possible, we get the optimal connection to y.

It remains to show that if $\boldsymbol{y \in neigh(x)}$, we also get the optimal connection. In such case, we simply compare the connection that goes via access nodes and the one that was obtained solely within the neighbourhood and output the shorter one. As there are no other options, the proof is complete. □

5.2.3 Modifications of *USP-OR-A*

Our implementation of the *USP-OR-A* algorithm uses one slight improvement, which we did not mention in its description, since it is more of an optimization technique without any theoretical

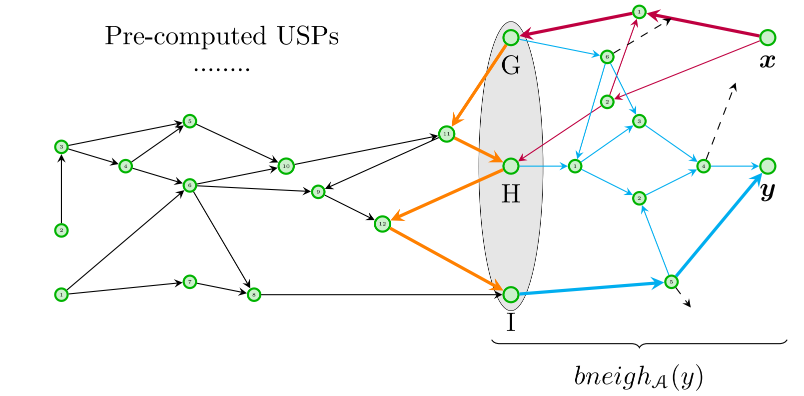


Figure 5.12: On the picture $lan(x) = \{G, H\}$ and $blan(y) = \{G, H, I\}$. In **thick** we have highlighted the optimal connection. The connection to H is sub-optimal after the *local front search* phase, however the optimal connection to y (and to H and I as well) leads through $lan^*(x)$ (some of x's local access nodes to which we have an optimal connection after the *local front search*. Particularly, in this case it goes through G).

guarantees on actual improvement of the running time. However, we consider it an interesting idea so we mention it at this place.

Definition 5.5. *USP tree*
Given a pair of cities x and y in a timetable T, we will call a USP tree the graph made out of edges of all USPs in $usp_T(x, y)$: $\boldsymbol{usp^3_T(x, y)} = (V^3, E^3)$ *where* $V^3 = \{v |\ v \text{ lays on some } p \in usp_T(x, y)\}$ *and* $E^3 = \{(a, b)|\ (a, b) \text{ is part of some } p \in usp_T(x, y)\}$.

We could take advantage of these USP trees to speed up the *local front search* phase of the algorithm, where we unnecessarily explore the whole neighbourhood when we could just go along the arcs of the USP trees. The picture 5.13 depicts this.

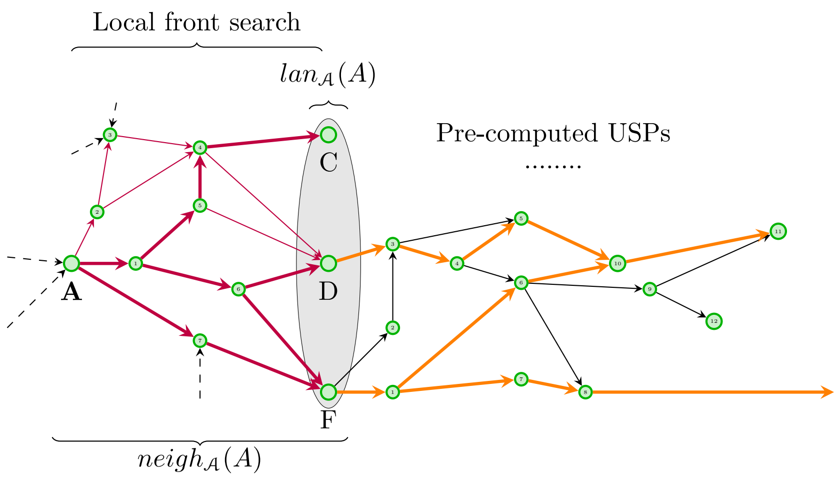


Figure 5.13: Using USP trees (**thick** non-dashed arcs in $\boldsymbol{neigh_{\mathcal{A}}(x)}$) to decrease the explored area in *local front search*. A full neighbourhood search is done only when $y \in neigh(x)$.

The interesting thing about this is the exploitation of both - timetable and its underlying graph.

While the neighbourhood of a node is something static, related only to the structure of the UG and generally time-independent, the USP trees reflect to some extent the properties of the timetable (e.g. which ways are frequently serviced and thus provide optimal connections). By intersecting these two things, we get the area that is *worth* to be explored and that is *small* at the same time (provided, of course, that the neighbourhoods are small).

5.3 Selection of access node set

The challenge in the *USP-OR-A* algorithm comes down to the selection of a good access node set - a (r_1, r_2, r_3) AN set with both three parameters as low as possible. However, intuitively (and experimentally verified), decreasing e.g. r_1 (the AN set size) increases r_2 (the size of the neighbourhoods). We therefore have to do some compromises.

In the following we first show the problem of choosing an optimal access node set to be NP-hard. We then present our methods for heuristic selection of access nodes and show their performance on real data.

5.3.1 Choosing the optimal access node set

A question stands - what is an optimal access node set? To keep the query time as low as possible, we need to avoid large neighbourhood sizes, because that would mean spending too much time doing local searches. A pretty good upper bound for neighbourhood sizes seems to be $\sqrt{n}$ (i.e. $r_1 = 1$) - the idea is that in such case the local searches cannot possibly last longer then $\mathcal{O}(n)$ while the *inter-AN search* is linear in the size of the connection and can also be at most $\mathcal{O}(n)$. In practice, both of these steps will be faster because the neighbourhoods are sparse and because the connections are on average much shorter then n. However, it gives an idea of why $\sqrt{n}$ should be considered for a target neighbourhood size.

Therefore, the question stands: What is the smallest set of ANs, such that the neighbourhood sizes are all under $\sqrt{n}$? More formally, for a timetable T, the task is to minimize $|\mathcal{A}|$ where $\mathcal{A} \subseteq ct_T$ and $\forall x \in ct_T \setminus \mathcal{A} : |neigh_{\mathcal{A}}(x)| \leq \sqrt{n}$. We will call this the **problem of the optimal access node set** and in what follows we will show that it is NP-complete.

Theorem 5.2. *The problem of the optimal access node set is NP-complete*

Proof. We will make a reduction of the *min-set cover* problem (a NP-complete problem) to the problem of optimal AN set.

Consider an instance of the min-set cover problem:

- A universe $U = \{1, 2, ..., m\}$
- k subsets of U: $S_i \subseteq U$ $i = \{1, 2, ..., k\}$ whose union is U: $\bigcup\limits_{1 \leq i \leq k} S_i = U$

Denote $\mathcal{S} = \{S_i |\ 1 \leq i \leq k\}$. The task is to choose the smallest subset $\mathcal{S}^*$ of $\mathcal{S}$ that still covers the universe ($\bigcup\limits_{S_i \in \mathcal{S}^*} S_i = U$). We will now do a simple conversion (in polynomial time) of the instance of min-set cover to the instance of the optimal AN set problem (which is represented by the underlying graph of T).

For each $j \in U$, we will make a complete graph of β_j vertices (the value of β_j will be discussed later) named m_j and for each set S_i we make a vertex s_i and vertex s'_i. We now connect all vertices of m_j to s_i for each $j \in S_i$. Finally, for we connect s_i to s'_i, $1 \leq i \leq k$.

Example. Let $m = 10$ (thus $U = \{1, 2, ..., 10\}$) and $k = 13$:

- $S_1 = \{1, 3, 10\}$
- $S_2 = \{1, 2\}$
- ...
- $S_{13} = \{2, 3, 10\}$

For this instance of min set-cover, we construct the graph as depicted on picture 5.14.

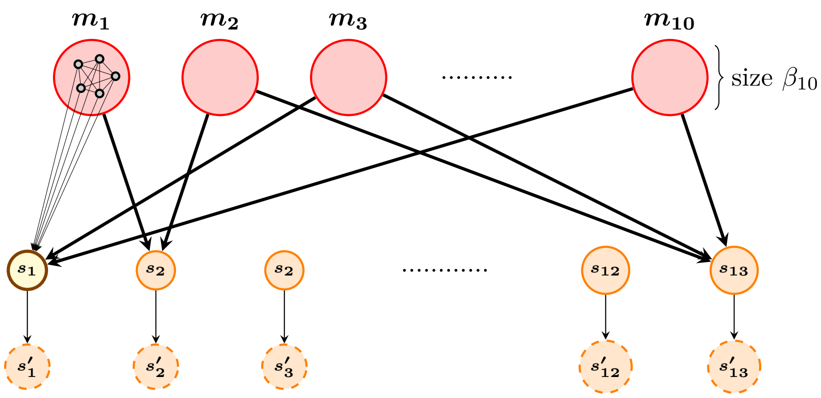


Figure 5.14: The principle of the reduction. In m_i, there are actually complete graphs of β_i vertices (as shown for m_1). **Thick** arcs represent arcs from all the vertices of respective m_i. The s_i vertices are connected to their s'_i versions. If e.g. s_1 is selected as an access node, s'_1 is no longer part of any neighbourhood (except for its own).

Now we would like to clarify the sizes of m_i. Define α_i to be the number of sets S_j that contain i: $\alpha_i = |\{S_j \in \mathcal{S} |\ i \in S_j\}|$ and assume the constructed graph has n vertices. We want the β_i to satisfy $\beta_i \geq 2$ and $\beta_i + 2\alpha_i - 1 \leq \sqrt{n}$ but $\beta_i + 2\alpha_i > \sqrt{n}$. The last two inequalities would mean that if at least one s_j connected to m_i is chosen as an access node, the neighbourhood for nodes in m_i will be still large at most $\sqrt{n}$, but if none of them is chosen, the neighbourhood size will be just over $\sqrt{n}$. We leave out the details of the construction at this place.

Now consider an optimal AN set which contains a vertex from within some m_i. If this is the case, **either** some s_j to which m_i is connected is selected as AN, **or** *all* vertices from m_i are access nodes **or** the neighbourhood is too large. Keep in mind that the local access nodes are also part of neighbourhoods, so unless we select for AN some of the s_j that m_i is connected to, the neighbourhood of any non-access node in m_i will be too large. As there are at least two nodes in every m_i, it is more efficient to select some s_j rather then select all nodes in m_i. Thus when it comes to selecting ANs *it is worth to consider only vertices* s_j.

From this point on, it is easy to see that it is optimal to select those s_j that correspond to the optimal solution of min-set cover. The reason is that each of the m_i will be connected to at least one access node s_j and will thus have neighbourhood size at most $\sqrt{n}$, while the number of selected access nodes will be optimal.

It remains to show how to choose values β_i. Due to the condition $\beta_i \leq \sqrt{n} - 2\alpha_i + 1$ we need to have sufficiently big n to fulfil $\beta_i \geq 1$. We will accomplish this by adding dummy isolated vertices to the graph. Define function *nextSquare(x)* to output the smallest $y^2 > x$ where y is a natural number. We then compute $w = (max\{2\alpha_i\} + 2)^2$ and select the starting value of n to be $n' =$ *nextSquare(*$max\{w - 1,\ \sqrt{2k + m}\}$*)*. We create the s_j and s'_j vertices and complete graphs m_i

containing so far only one vertex each. We connect everything according to the rules stated earlier in this proof and we create dummy vertices up to the capacity defined by n. Now we repeat the following:

- We compute $\sqrt{n}$ which is a natural number
- For i from 1 to m we add vertices to m_i till it does not contain $\sqrt{n} - 2\alpha_i + 1$ vertices. For each added vertex we delete one dummy vertex.
- If we run out of dummy vertices, $n = nextSquare(n)$
- Break out of the loop if $|m_i| = \sqrt{n} - 2\alpha_i + 1 \; \forall i$

With each iteration of this little algorithm we will be forced to add one more vertex to all m_i (since $\sqrt{n}$ increased by one), a so called *inefficient increase*. At the beginning, we need to make at most $m\sqrt{n'}$ efficient increases to meet the breaking condition. And since m is constant and the capacity of new dummy vertices increases linearly, after t steps we create $\mathcal{O}(t^2)$ dummy vertices that may be used for efficient increases. Therefore, the algorithm will stop after $\mathcal{O}(\sqrt{mn'})$ steps. □

5.3.2 Choosing ANs based on node properties

In the previous sub-subsection, we have shown the problem of choosing the optimal AN set to be NP-hard. In this sub-subsection we perform a simple experiment of choosing for the access nodes the cities that seem to be the most important. More specifically, in the optimistic underlying graph (see section 2) ug_T^{opt} we were looking for cities with:

1. High **degree**. We consider the sum of in-degree and out-degree [24] of the respective node x: $\boldsymbol{deg(x)} = deg_{in}(x) + deg_{out}(x)$.
2. High **betweenness centrality** (BC). Betweenness centrality for a node v is defined as

$$g(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

   where $\sigma_{st}(v)$ is the number of shortest paths from s to t passing through v and σ_{st} is the total number of shortest paths from s to t [Bra01]. We then scale the values to the range $< 0, 1 >$ to obtain for each city x its scaled betweenness centrality $\boldsymbol{bc(x)}$.

We will denote by $\mathcal{A}^{deg}(k)$ the set of k cities with highest $deg(x)$ value. We were interested in the smallest k such that $\mathcal{A}^{deg}(k)$ is (r_1, r_2, r_3) AN set with $r_2 \leq 1$ (the average square of neighbourhoods is at most n). Denote such set as $\boldsymbol{\mathcal{A}^{deg}}$ and the triplet (r_1, r_2, r_3) as $\boldsymbol{(r_1^{deg}, r_2^{deg}, r_3^{deg})}$ [25].

We define similarly $\boldsymbol{\mathcal{A}^{bc}}$ and $\boldsymbol{r_i^{bc}}$. Plots 5.15 summarize the properties of access node sets obtained this way on daily datasets *sncf* and *cpsk*.

5.3.3 Choosing ANs heuristically - the *locsep* algorithm

Clearly, selecting the cities for access nodes solely by high degree or BC value is not the best way. Probably the few nodes with highest degrees and BC will indeed be part of the AN set, as they are intuitively some sort of central hubs without which the network would not work. However, after we select these most important nodes to the AN set, we need some better measure of node's importance, or suitability to be an access node. In the following we present a simple heuristic approach run on

[24] In-degree is the number of arcs going into the node and out-degree the number of outgoing arcs.

[25] Intuitively - r_1^{deg} is the smallest r_1 such that $r_1\sqrt{n}$ highest-degree cities selected as ANs are enough to satisfy that the average square of neighbourhoods is at most n.

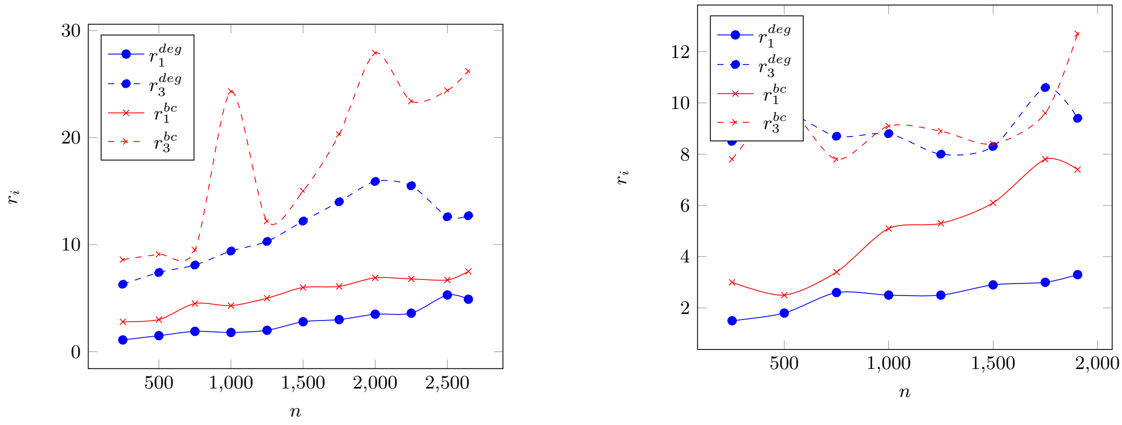


Figure 5.15: Parameters of the access node sets $\mathcal{A}^{deg}$ and $\mathcal{A}^{bc}$ with increasing n. Datasets *sncf* (left) and *cpsk* (right). $r_2 \leq 1$. The occasional "roller coaster" bumps (for value of r_3) are due to our stopping criterion which does not consider r_3 at all.

underlying graph ug_T of given timetable T that evaluates its vertices based on how good local separators they are.

The algorithm that we call *locsep* (as it looks for good local separators) will work in iterations, each of them resulting in a selection of the city with the highest score to the access node set $\mathcal{A}$ [26]. We continue to select access nodes until we meet the following stopping criterion: $\mathcal{A}$ is (r_1, r_2, r_3) AN set with $r_2 \leq 1$ (the average square of neighbourhoods is at most n) [27]. We will denote the resulting set $\boldsymbol{\mathcal{A}^{loc}}$ and its parameters as $\boldsymbol{r_i^{loc}}$.

The important thing that remains to be shown is how we compute the score for a particular city. The following text explains this.

In each iteration, we first compute the neighbourhoods and back neighbourhoods (given the current access node set $\mathcal{A}$) for each city. We need this to evaluate the stopping criteria, but the information is also used in the computation of the **potential** (the score) of the cities.

For a city x, we compute its potential $\boldsymbol{p_x}$ in the following way: we explore an area $\boldsymbol{A_x}$ of $\sqrt{n}$ nearest cities around x, ignoring branches of the search that start with an access node (x is an exception to this, since we start the search from it, although $x \notin A_x$ holds). We do this exploration in an underlying graph with no orientation and no weights. Next we get the front and back neighbourhoods of x within A_x ($\boldsymbol{fn(x)} = neigh(x) \cap A_x$, $\boldsymbol{bn(x)} = bneigh(x) \cap A_x$).

For a set of access nodes $\mathcal{A}$, let us call a path p in ug_T **access-free** if it does not contain a node from $\mathcal{A}$. Now as long as x is not in $\mathcal{A}$, we have a guarantee that for every pair $u \in bn(x)$ and $v \in fn(x)$ there is an access-free path from u to v within A_x. Our interest is how this will change after the selection of x.

Consider now a node $y \in bn(x)$. We will call $\boldsymbol{sur(y)} = \max\{0, |neigh(y)| - \sqrt{n}\}$ the **surplus** of y's neighbourhood, i.e., by how much we wish to reduce it so that it is at most $\sqrt{n}$. If the surplus is zero, y will not add anything to the x's potential. Otherwise, we run a restricted

[26] Actually, in our implementation, we allow an occasional deselection of an already selected node with the *lowest* score, to avoid having in the resulting set cities that had high score when selected but were not very useful access nodes at the end.

[27] In our implementation, we perform some further adjustments of the resulting set, such as removing unnecessary access nodes and optimising for the r_3 value.

(to A_x) search from y during which we explore j vertices in $fn(x)$. We increase the potential of x by $\min\{sur(y), |fn(x) - j|\}$ - i.e. by how much we can decrease the surplus of y's neighbourhood. We do the same for all $y \in bn(x)$ and a similar thing for all $y \in fn(x)$ (we use $\overleftarrow{ug_T}$ instead of ug_T, $bneigh(y)$ instead of $neigh(y)$ etc...). For an illustration of potential computing, see figure 5.16.

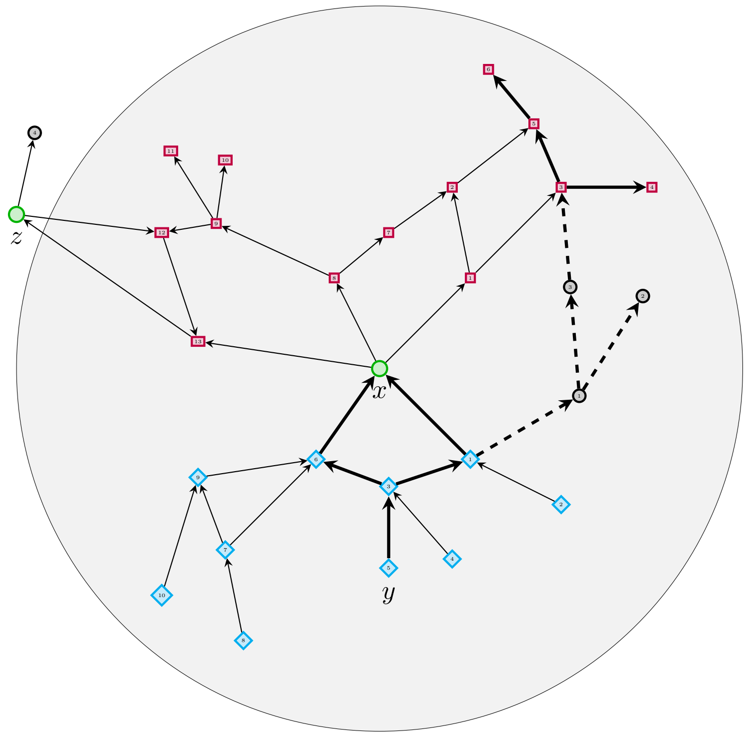


Figure 5.16: The principle of computing potentials in *locsep* algorithm. We explored an area of $\sqrt{n}$ nearest cities (in terms of hops) around x. Access nodes (like z) and cities behind them are ignored. Little squares are nodes from $fn(x)$ and diamonds are part of $bn(x)$. From y we run a forward search (the **thick** arcs). Nodes from the $fn(x)$ that were not explored in this search can only be reached via x itself. Such nodes contribute to x's potential assuming y has large neighbourhood size.

Finally, we simply get the city $x \notin \mathcal{A}$ with the highest potential and select it as an access node. We check the stopping criterion and in case it is not satisfied yet, we move on to next iteration. However, note that when a new node x' is selected to $\mathcal{A}$, we do not have to re-compute neighbourhoods and potentials of all cities - it is only necessary for those cities that could reach/be reached access-free from x' (i.e. nodes from $neigh_{\mathcal{A}}(x') \cup bneigh_{\mathcal{A}}(x')$). Algorithm 5.5 provides a high-level overview of the locsep method.

Algorithm 5.5 *locsep*

Input

- ug_T

Algorithm

$\mathcal{A} = \emptyset$
$ct' = ct_T$
while $r_2 > 1$ **do**
  $\forall x \in ct'$: compute $neigh_{\mathcal{A}}(x)$, $bneigh_{\mathcal{A}}(x)$
  $\forall x \in ct'$: compute p_x
  $x' = argmax_{x \notin \mathcal{A}}\{p_x\}$
  $\mathcal{A} = \mathcal{A} \cup \{x'\}$
  $ct' = neigh_{\mathcal{A}}(x') \cup bneigh_{\mathcal{A}}(x')$
end while
Remove unnecessary ANs
Optimise r_3

Output

- AN set $\mathcal{A}^{loc}$ ($|\mathcal{A}^{loc}| = \sqrt{n}r_1^{loc}$)

Now we would like to estimate the **time complexity** of locsep algorithm. As mentioned, one iteration consists of three parts:

1. Computing neighbourhoods. Unfortunately, at the beginning when $\mathcal{A} = \emptyset$, the neighbourhood sizes may be as large as $\mathcal{O}(n)$. Therefore, we may bound the complexity of this phase only as $\mathcal{O}(nm) = \mathcal{O}(\delta n^2)$.
2. Computing potentials. For a city x we explore area of the size $\sqrt{n}$ and from each node in that area we do a restricted search. Therefore the total complexity of this step is $\mathcal{O}(n \cdot \sqrt{n} \cdot \delta\sqrt{n}) = \mathcal{O}(\delta n^2)$.
3. Selecting node with the highest potential. This can be done in $\mathcal{O}(n)$.

Adding up the individual terms, we get the complexity of one iteration to be at most $\mathcal{O}(\delta n^2)$. As we aim for the resulting access node set of size $\mathcal{O}(\sqrt{n})$, we would get the total running time of $\boldsymbol{\mathcal{O}(\delta n^{2.5})}$. However, we remind that the algorithm is only a heuristics with no guarantees on the resulting access node set size [28].

The resulting running time is still quite impractical for bigger timetables. For example, the computation on the dataset *sncf* took more than an hour. This is due to the initial iterations, during which average neighbourhood is still very large (spanning almost the whole graph) and thus we have to do a lot of re-computations (potentials, neighbourhoods). We therefore embrace a simple trick: we do not start with $\mathcal{A} = \emptyset$ but with some access nodes already selected based on high degree. We chose to start with $\frac{2\sqrt{n}}{3}$ nodes with the highest degree ($\mathcal{A}_{deg}(\frac{2\sqrt{n}}{3})$) - enough to speed-up the computation but not influencing the resulting AN set too much.

The access node sets chosen with the *locsep* algorithm had much better properties compared to those selected by the previous approaches. The summary of their properties for each of our datasets can be seen in table 5.5, while plots 5.17 illustrate the evolution r_1 and r_3 with increasing n.

[28] The algorithm basically selects access nodes on a greedy basis. However, even that is done only heuristically, using local scope to reduce the time complexity.

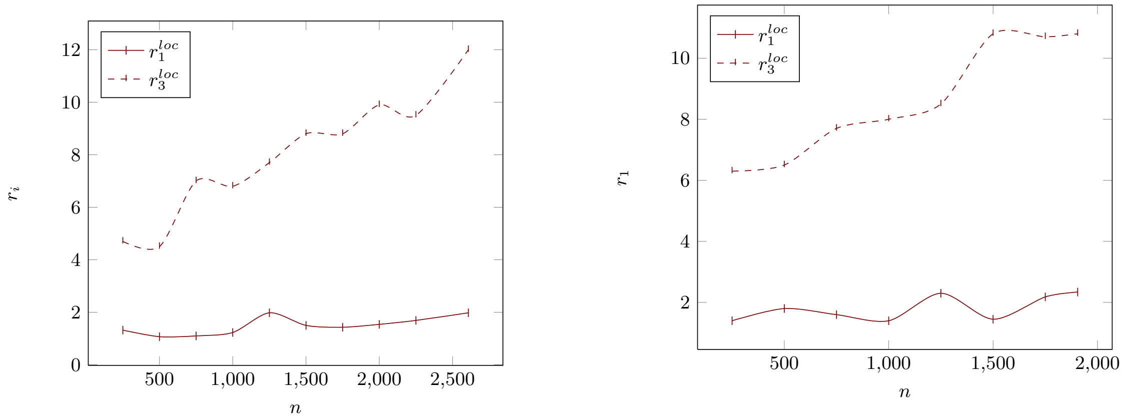


Figure 5.17: Parameters of the access node set $\mathcal{A}^{loc}$ with increasing n. Datasets *sncf* (left) and *cpsk* (right). $r_2 \leq 1$. An ideal situation would be constant (or non-increasing) functions.

Name	r_1^{loc}	r_3^{loc}
air01-d	3.9	57.4
cpsk-d	2.3	10.8
gb-coach-d	3.4	20.7
gb-train-d	3.2	22.7
montr-d	2.1	3.3
sncf-d	2.0	12.0
sncf-ter-d	1.8	9.7
sncf-inter-d	1.3	3.9
zsr-d	1.7	4.5

Table 5.5: Parameters of $\mathcal{A}^{loc}$ for all of our datasets in their maximum size. $r_2 \leq 1$. The timetable of airlines is the most challenging to find a good AN set on, since it forms a highly interconnected network and thus makes it difficult to separate the nodes' neighbourhoods with few local access nodes.

To sum up, in all of our datasets (except for *air01*), we were always able to find (r_1, r_2, r_3) access node set with the *locsep* algorithm, such that:

- $r_1 \leq 3.5$
- $r_2 \leq 1$
- $r_3 \leq 25$ (i.e. up to 5 local access nodes on average)

In what follows we try to analyse the parameters of *USP-OR-A* combined with *locsep*. We however ask the reader to consider the following more like an informal discussion providing a better insight on performance of the mentioned combination, rather than a rigorous analysis.

Suppose the following conditions to be true:

1. We have a timetable with $\delta \leq \log n$ on which we found $\mathcal{A}$ (thus $r_2^{loc} \leq 1$)
2. $\omega_{\mathcal{A}} \leq \sqrt{n}$
3. r_1^{loc} is bound by a constant
4. $r_3^{loc} \leq \log n$
5. $\sqrt{r_3^{loc}} \leq \tau_{\mathcal{A}}$

The first two conditions were generally satisfied by our timetables (again, not including *air01*). As for the third one, we found the value of r_1^{loc} almost constant and only very slightly increasing with n [29]. Finally, if the last two conditions hold, we may estimate the average query time of *USP-OR-A* with *locsep* as:

$$\begin{aligned}\mathcal{O}(r_2\sqrt{r_3}\sqrt{n}(\log(r_2 n)+\delta)+r_3\tau\omega) &=\\ \mathcal{O}(\sqrt{r_3}\sqrt{n}(\log n+\delta)+r_3\tau\omega) &=\\ \mathcal{O}(\sqrt{r_3}\sqrt{n}\log n+r_3\tau\sqrt{n}) &=\\ \mathcal{O}(\tau\sqrt{n}\log n+\log n\ \tau\sqrt{n}) &=\\ \mathcal{O}(\tau\sqrt{n}\log n)\end{aligned}$$

Most of our datasets came very close to satisfy the mentioned conditions, under which *USP-OR-A* with *locsep* reaches parameters as described in the following table.

USP-OR-A + locsep	***prep***	***size***	***qtime***	***stretch***
Under certain conditions	$\mathcal{O}(n^{2.5}\log n)$	$\mathcal{O}(\tau n^{1.5})$	avg. $\mathcal{O}(\tau\sqrt{n}\log n)$	1

Table 5.6: Parameters for *USP-OR-A* with *locsep* under certain conditions listed above.

Using *USP-OR-A* with *locsep* therefore seems to be a solution which (theoretically) should work pretty well. To justify this statement we performed experiments, the results of which we provide in the next subsection.

5.4 Performance and comparisons

In this subsection we show the performance of our algorithms on our datasets. We focus on query time and space complexity of the preprocessed oracles [30]. We have already introduced the speed-up as the ratio of average query time for the TD Dijkstra and the average query time for the given algorithm. We will have a similar measure for the size of the preprocessed data, which we compare against the amount of memory needed to store the actual timetable.

Definition 5.6. ***Size-up (szp(m))***
A size-up of an oracle based method m is the ratio $\frac{size(TD)}{size(m)}$ *where* $size(TD)$ *is the size of the memory necessary to store the time-dependent graph.*

5.4.1 Performance of *USP-OR*

Query time-wise, *USP-OR* outperforms time-dependent Dijkstra's algorithm almost 80 times (on the subset of *sncf-ter* dataset). However, this was at the cost of high space consumption of the method, in some cases requiring almost 400 times more memory then necessary for storage of the time-dependent graph. Therefore, we were not even able to preprocess the method for some of our bigger datasets. Table 5.7 gives a good overview of achieved speed-ups and size-ups.

[29] Although this is nowhere near the *proof* that r_1^{loc} could be bound by a constant.
[30] The methods are exact (i.e. *stretch* = 1) and we consider the preprocessing time as the least important parameter out of the four.

Name	*n*	*spd*	*szp*
air01-d	284		
*cpsk-d**	700	3.1	301.8
*gb-coach-d**	700	64.7	79.1
*gb-train-d**	700	24.4	134.6
montr-d	217		
*sncf-d**	700	28.7	263.7
sncf-inter-d	344		
*sncf-ter-d**	700		
zsr-d	225		

Name	*n*	*spd*	*szp*
air01-w	287		
*gb-coach-w**	700	69.5	50.6
*gb-train-w**	700		
*sncf-w**	700		
sncf-inter-w	366		
*sncf-ter-w**	700		
zsr-d	233		

Table 5.7: Speed-ups and size-ups of the *USP-OR* algorithm for the whole timetables (for those marked with asterisk we took only a subset of n stations, as we were limited by the space). Daily time range on the left, weekly on the right.

In both timetables from *cp.sk* we obtained quite similar results. The query time of *USP-OR* mildly rises with increasing n. This is due to two (not surprising) facts:

- The USP coefficient gets slightly bigger with increasing n
- The OC diameter increases too with increasing n

The speed-up was up to 15, however, at the cost of very high space complexity, which made it possible to try out only timetables of the size up to 700. With the *montr* dataset we got similar results, but on a smaller scale.

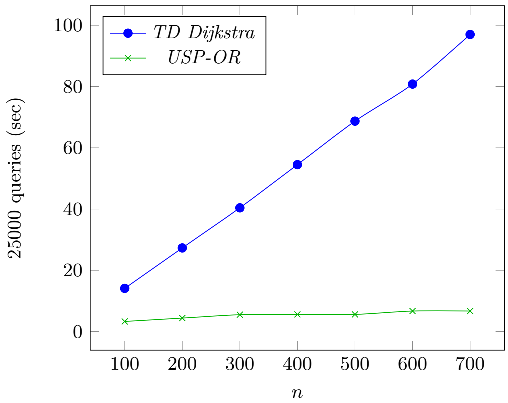


Figure 5.18: **Query time** of *USP-OR* algorithm compared to TD Dijkstra on the ***cpru*** dataset. **Changing *n*.**

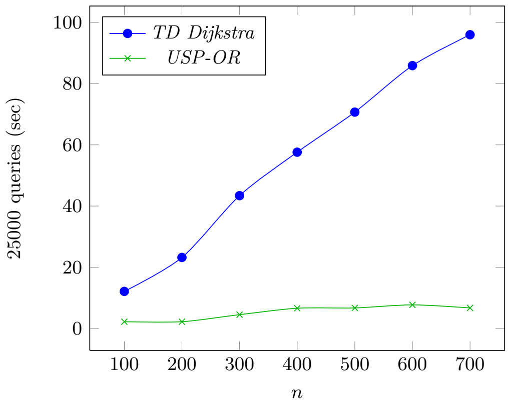


Figure 5.19: **Query time** of *USP-OR* algorithm compared to TD Dijkstra on the ***cpza*** dataset. **Changing *n*.**

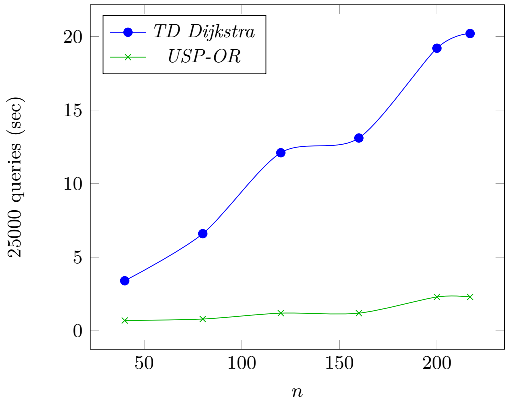


Figure 5.20: **Query time** of *USP-OR* algorithm compared to TD Dijkstra on the ***montr*** dataset. **Changing *n*.**

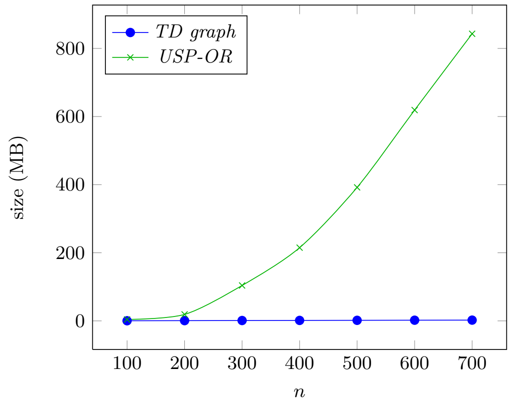


Figure 5.21: **Size** (in MB) of the oracle for *USP-OR* vs. size of TD graph on ***cpza*** dataset. **Changing *n*.**

In the datasets from SNCF, an interesting thing was that the the query-time actually decreased with increased size. This was due to the average USP coefficient getting smaller in bigger datasets while OC diameter not increasing too much. We measured here the speed-ups of up to 80 for *sncf-ter*, with smaller size-ups then in case of *cp.sk* timetables.

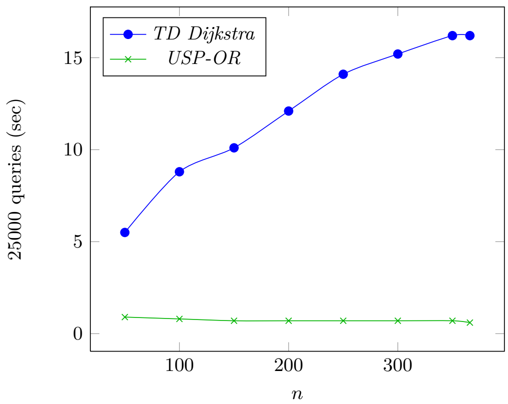


Figure 5.22: **Query time** of *USP-OR* algorithm compared to TD Dijkstra on the ***sncf-inter*** dataset. **Changing *n*.**

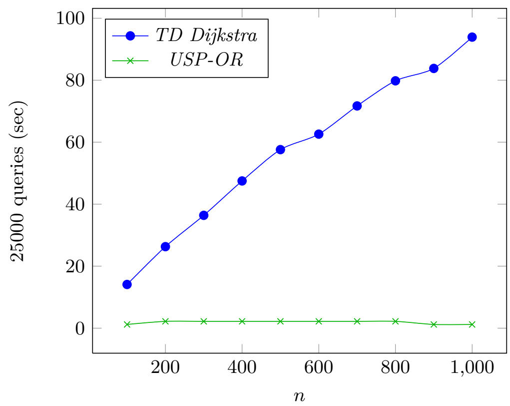


Figure 5.23: **Query time** of *USP-OR* algorithm compared to TD Dijkstra on the ***sncf-ter*** dataset. **Changing *n*.**

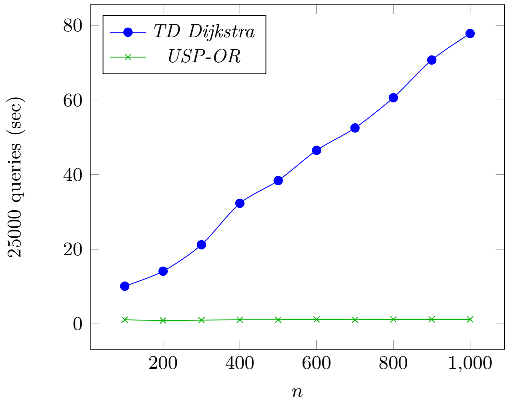


Figure 5.24: **Query time** of *USP-OR* algorithm compared to TD Dijkstra on the ***sncf*** dataset. **Changing *n*.**

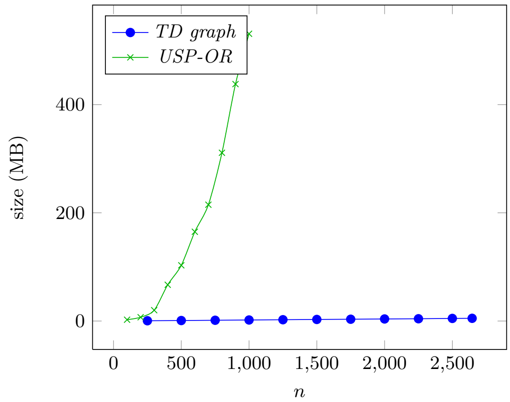


Figure 5.25: **Size** (in MB) of the oracle for *USP-OR* vs. size of TD graph on ***sncf*** dataset. **Changing *n*.**

On the *zsr* dataset we measured how increased time range influences the query time. You may see that for both algorithms the query time almost stops increasing at some point - this is because (informally) adding time range no longer brings along new optimal connections (or underlying shortest paths in case of *USP-OR*).

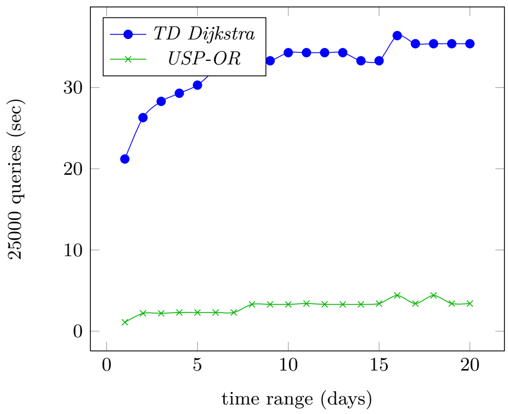


Figure 5.26: **Query time** of *USP-OR* algorithm compared to TD Dijkstra on the ***zsr*** dataset. **Changing *r*.**

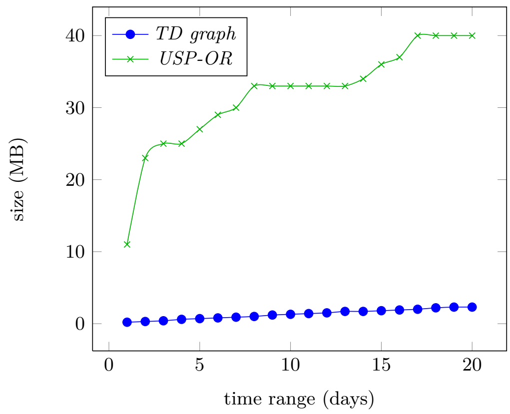


Figure 5.27: **Size** (in MB) of the oracle for *USP-OR* vs. size of TD graph on ***zsr*** dataset. **Changing *r*.**

5.4.2 *USP-OR-A*

We tested *USP-OR-A* exclusively paired with *locsep*. The speed-ups were no longer so high as in case of *USP-OR*, but neither were the size-ups, so we could fully try out all of our datasets. The maximum speed-up was achieved in *gb-coach*, where the *USP-OR-A* outperformed *TD Dijkstra* about 8.5 times. In our datasets, the preprocessed oracle of *USP-OR-A* did not need more than 3 times the size of the TD graph. Table 5.8 provides an overview of achieved speed-ups and size-ups.

Name	***n***	***spd***	***szp***
air01-d	284	2.4	1.7
cpsk-d	1905	2.9	6.6
gb-coach-d	2427	8.2	7.4
gb-train-d	2550	2.9	4.3
montr-d	217	2.8	1.7
sncf-d	2608	5.4	5.1
sncf-inter-d	344	5.2	1.9
sncf-ter-d	2600	5.5	5.2
zsr-d	225	3.0	2.7

Name	***n***	***spd***	***szp***
air01-w	287		
gb-coach-w	2448	8.5	3.8
gb-train-w	2555	2.9	2.3
sncf-w	2646	6.3	4.2
sncf-inter-w	366		
sncf-ter-w	2637		
zsr-d	233		

Table 5.8: Speed-ups and size-ups of the *USP-OR-A* for the whole timetables. Daily time range on the left, weekly on the right.

The bus timetables proved to be a bigger challenge for *USP-OR-A*, achieving milder speed-ups and requiring more memory then railways timetables. However, bigger timetables would be necessary to obtain more relevant results.

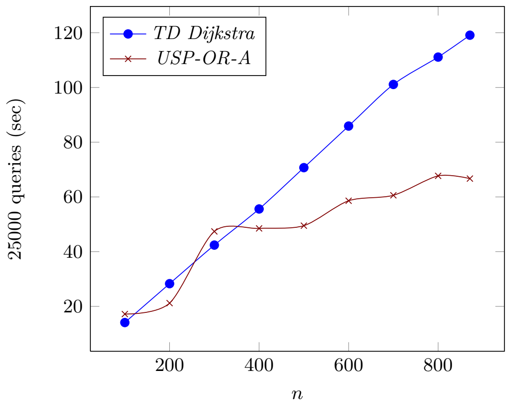


Figure 5.28: **Query time** of *USP-OR-A* with *locsep* compared to TD Dijkstra on the ***cpru*** dataset. **Changing *n*.**

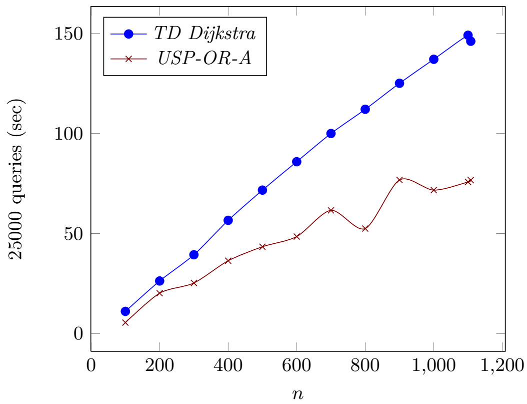


Figure 5.29: **Query time** of *USP-OR-A* with *locsep* compared to TD Dijkstra on the ***cpza*** dataset. **Changing *n*.**

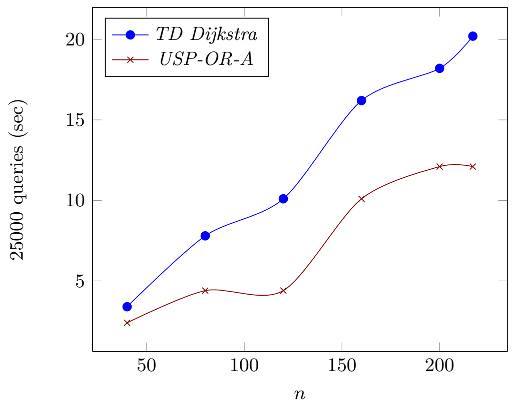


Figure 5.30: **Query time** of *USP-OR-A* with *locsep* compared to TD Dijkstra on the ***montr*** dataset. **Changing *n*.**

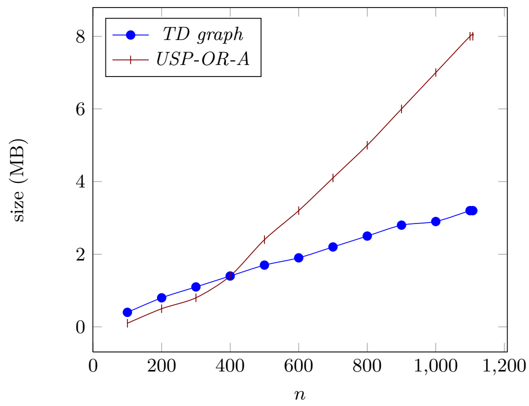


Figure 5.31: **Size** (in MB) of the oracle for *USP-OR-A* with *locsep* vs. size of TD graph on ***cpza*** dataset. **Changing *n*.**

In our biggest datasets, we achieved the best speed-ups while the size-up still stayed relatively small (though here we better see its tendency to increase as $n^{1.5}$). It would be interesting to try out even bigger datasets as the speed-up was gradually increasing with increasing n.

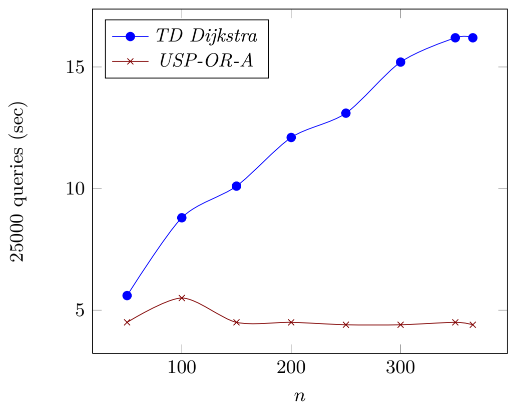


Figure 5.32: **Query time** of *USP-OR-A* with *locsep* compared to TD Dijkstra on the ***sncf-inter*** dataset. **Changing *n*.**

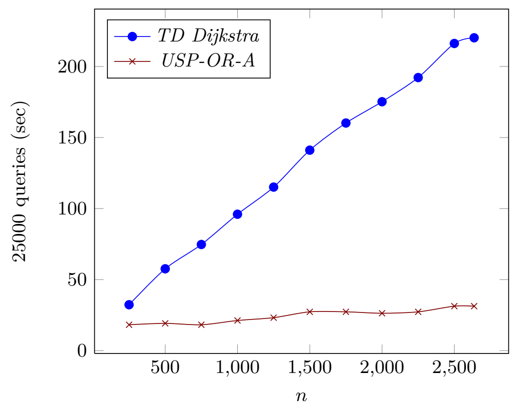


Figure 5.33: **Query time** of *USP-OR-A* with *locsep* compared to TD Dijkstra on the ***sncf-ter*** dataset. **Changing *n*.**

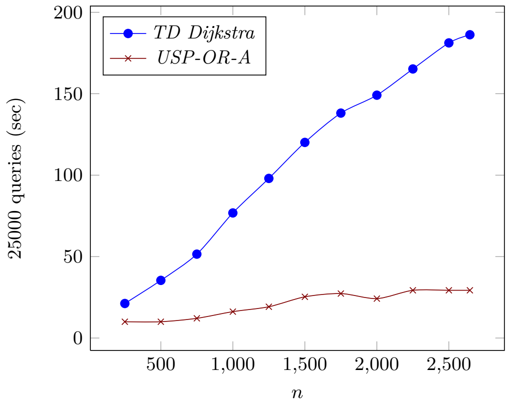


Figure 5.34: **Query time** of *USP-OR-A* with *locsep* compared to TD Dijkstra on the ***sncf*** dataset. **Changing n.**

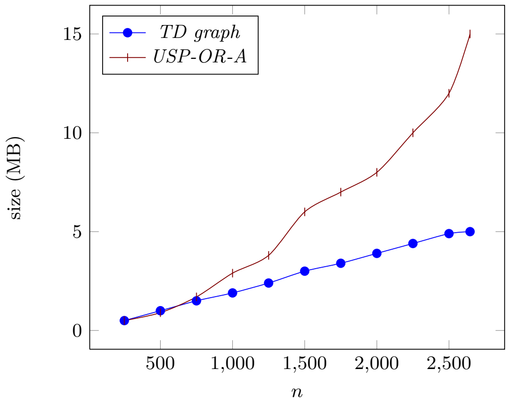


Figure 5.35: **Size** (in MB) of the oracle for *USP-OR-A* with *locsep* vs. size of TD graph on ***sncf*** dataset. **Changing n.**

Finally, on the *zsr* timetable, we see two things:

- The space-complexity of *USP-OR-A* is left pretty much unaffected with increased time range.
- The speed-up decreases (since with increased time range, there are generally more USPs between pairs of cities which we have to try out during the query)

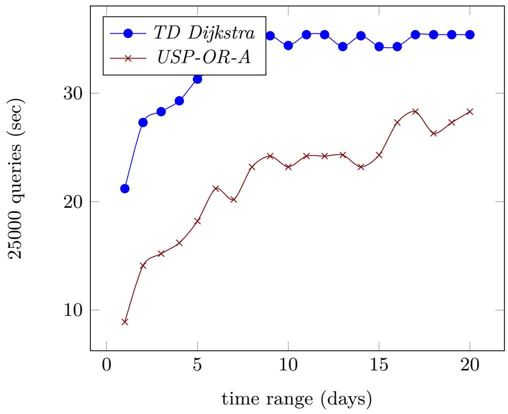


Figure 5.36: **Query time** of *USP-OR-A* with *locsep* compared to TD Dijkstra on the ***zsr*** dataset. **Changing r.**

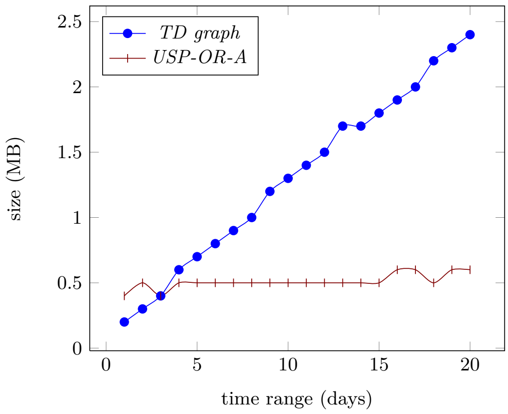


Figure 5.37: **Size** (in MB) of the oracle for *USP-OR-A* with *locsep* vs. size of TD graph on ***zsr*** dataset. **Changing r.**

6 Neural network approach

In this section we describe our experiment approaching the optimal connection problem in timetables with the help of an artificial neural network, which is the "oracle" in this case. More specifically, we consider multi-layer perceptron with back-propagation training algorithm. The training time corresponds to the preprocessing time and the size can be parametrized by the number of hidden layers (and the size of each layer). Our main interest was if the network was able to answer with reasonable connections for given queries.

The input layer of the perceptron has one neuron for each event of the timetable and one neuron for each city of the timetable. Thus each instance of the earliest arrival query can be represented by exactly two neurons on the input layer. The output layer has one neuron for each arc of the underlying graph. The idea is that the network will activate those output neurons (arcs), that correspond to the underlying path of the connection. As we have pruned the overtaking elementary connections, we can easily reconstruct the actual connection by expanding the underlying path (algorithm 5.1). See figure 6.1 for an illustration.

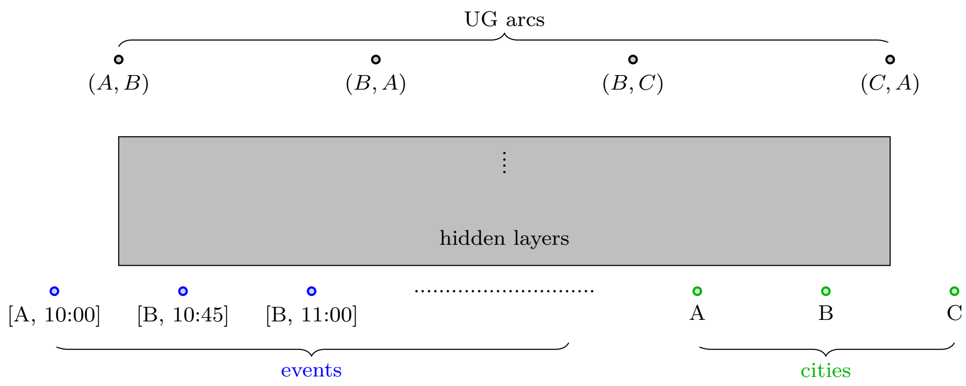


Figure 6.1:

For example, consider the following query in the timetable 2.1: "From A at *10:00* to C". The optimal connection is *[A, 10:00]*, *[B, 10:45]*, *[B, 11:00]*, *[C, 11:30]*. The query will activate two input neurons - the one corresponding to *[A, 10:00]* and the one corresponding to destination C. On the output we expect two activated neurons, since we really moved only two times - from A to B and from B to C.

After the training, we feed the network random inputs (queries for optimal connection). It might be that even though the network was trained, the desired neurons at the output will not be sufficiently activated and rounding them would yield all-zero output. That is why we employed the following method to recover the sequence of traversed cities:

- Start at departing city
- Consider all leaving arcs from this city
- Choose the one with strongest activation (leading to yet unvisited city)
- Terminate when the target city is reached or we cannot continue

From the method mentioned above we can see that the network may fail to find the answer in some cases (sometimes there is simply no solution, e.g. when "the last train already left"). In such situation, it outputs at least the partial answer, so to speak, “how to begin the journey”.

The perceptron was trained with training data divided into two groups: **validation** and **estimation** data with 80% being part of the latter group. To eliminate overfitting, early stopping was used as a stopping criterion in some cases (that is, we stopped once the validation error [31] started to increase). To disregard noise on the validation error curve (and thus too early halt of the training when validation error accidentally increased) we compare the validation error against its value from 10 iterations back. Sigmoid activation function is used in all layers.

For the testing, we used several subsets of our datasets, described in table 6.1.

Name	n	m
tt	4	5
air30-10	30	187
air50-20	50	185
cpsk30-10	30	97
cpsk50-20	50	120
montr30	30	40
montr50	50	71
zsr30-10	30	71
zsr50-20	50	137

Table 6.1: Datasets used for testing.

6.1 Results

For each dataset, we have trained 7 types of networks:

- 1 hidden layer with 30 neurons
  - 1.) 100 training examples (t = 100), α [32] = 0.1, early stopping
  - 2.) 300 training examples (t = 300), $\alpha = 0.1$, early stopping
  - 3.) 100 training examples (t = 100), $\alpha = 0.1$, minimum of 300 iterations
- 5 hidden layers, each with 30 neurons
  - 4.) 100 training examples (t = 100), $\alpha = 1$, early stopping
  - 5.) 300 training examples (t = 300), $\alpha = 1$, early stopping
  - 6.) 100 training examples (t = 100), $\alpha = 1$, minimum of 300 iterations
- 1 hidden layer with 100 neurons
  - 7.) 100 training examples (t = 100), $\alpha = 0.1$, minimum of 500 iterations

For the first six types of network, the *maximum* number of iterations was set to 300, for the last one it was 500 (thus it was trained with exactly 500 iterations). Also, each network was trained on a different randomly generated training set.

Once the networks were trained, we have compared the first triple (table 6.2) on 1000 randomly generated earliest arrival queries (on a given dataset) and the second triple (table 6.3) on another 1000 generated queries. In each case, Dijkstra’s algorithm was part of the comparison so that

[31] Validation error was computed as $\sum_{v\in\mathcal{V}} \frac{\sum_i^m (d_i^v - y_i^v)^2}{2}$ where $\mathcal{V}$ is the validation set, m the number of output neurons, d_i^v the i-th neuron’s target value for training example v and y_i^v the actual value of i-th neuron for that training example.

[32] α is the parameter used in adjusting weights in the back-propagation algorithm

we obtained the optimal values and could evaluate the accuracy of the networks (the seventh network was compared simply against the Dijkstra's algorithm, table 6.4). Even though the generated queries were different in each case, the comparison among them all can still be made based on the percentage of the times when the network was able to answer with the optimum (table 6.5).

Name	Conn ∃	1.) t = 100, α = 0.1, E.S.				2.) t = 300, α = 0.1, E.S.				3.) t = 100, α = 0.1, 300 it.			
		E.E.	V.E.	Found	Opt	E.E.	V.E.	Found	Opt	E.E.	V.E.	Found	Opt
tt	462	0.442	0.77	462	435	0.31	0.91	462	462	0.67	1.02	450	450
air30-10	945	105.29	28.93	184	21	331.61	88.84	187	51	46.20	26.21	464	82
air50-20	931	133.90	36.57	112	33	385.32	94.89	166	104	48.30	31.52	426	204
cpsk30-10	504	63.83	12.42	146	57	124.55	32.96	287	160	16.74	13.96	270	147
cpsk50-20	620	140.59	47.67	214	182	429.81	119.36	197	181	20.63	43.59	221	189
montr30	545	69.28	35.02	268	234	52.24	60.05	418	376	4.89	18.51	366	310
montr50	572	81.19	54.28	290	253	161.01	124.99	351	314	6.61	33.43	341	300
zsr30-10	643	57.08	29.24	278	204	298.64	69.62	179	144	13.59	23.02	306	229
zsr50-20	892	130.74	46.05	152	53	448.37	134.80	185	60	34.02	41.85	347	122

Table 6.2: The first triple of trained neural network. *Conn* ∃ - number of test cases (out of 1000) when there existed a connection (found by Dijkstra's algorithm) for the query. *E.S.* - early stopping. *E.E. and V.E.* - estimation and validation error at the end of training. *Found* - found a connection for the query. *Opt* - found optimal connection

Name	Conn ∃	4.) t = 100, α = 1, E.S.				5.) t = 300, α = 1, E.S.				6.) t = 100, α = 1, 300 it.			
		E.E.	V.E.	Found	Opt	E.E.	V.E.	Found	Opt	E.E.	V.E.	Found	Opt
tt	449	12.16	3.13	292	292	2.79	1.02	403	403	0.00	1.35	449	449
air30-10	939	119.56	30.77	178	21	339.35	79.04	121	26	34.21	37.8	505	94
air50-20	922	132.05	28.36	30	16	371.12	106.18	50	36	33.86	54.17	387	171
cpsk30-10	518	61.83	13.91	133	94	195.12	35.84	156	101	30.25	20.52	174	80
cpsk50-20	601	133.25	46.01	164	119	363	115.19	179	159	64.85	53.07	162	143
montr30	495	103.71	33.63	349	248	191.92	107.10	304	273	8.67	36.86	250	214
montr50	565	124.11	44.92	240	226	411.93	111.62	249	243	68.37	50.77	238	229
zsr30-10	658	115.50	26.28	140	117	331.12	82.83	147	124	16.42	37.08	211	165
zsr50-20	900	156.06	32	110	57	475.49	104.63	135	45	43.78	50.12	321	134

Table 6.3: The second triple of trained neural network. *Conn* ∃ - number of test cases (out of 1000) when there existed a connection (found by Dijkstra's algorithm) for the query. *E.S.* - early stopping. *E.E. and V.E.* - estimation and validation error at the end of training. *Found* - found a connection for the query. *Opt* - found optimal connection

Name	Conn ∃	7.) t = 100, α = 0.1, E.S.			
		E.E.	V.E.	Found	Opt
tt	471	0.1	2.24	471	471
air30-10	931	23.47	30.76	573	107
air50-20	938	37.66	38.44	405	203
cpsk30-10	481	11.67	17.22	281	135
cpsk50-20	605	8.51	46.87	252	195
montr30	527	2.85	43.44	346	300
montr50	615	4.13	24.97	386	346
zsr30-10	672	7.7	26.92	307	234
zsr50-20	887	22.68	42.68	426	178

Table 6.4: The 7th trained neural network. *Conn* ∃ - number of test cases (out of 1000) when there existed a connection (found by Dijkstra's algorithm) for the query. *E.S.* - early stopping. *E.E. and V.E.* - estimation and validation error at the end of training. *Found* - found a connection for the query. *Opt* - found optimal connection

There are several things to notice about these results:

- Interestingly, validation error is smaller (in most cases) when we did not use the early stopping criterion than in the case when we used it for networks with one hidden layer (first triple). For the second triple, this is no longer the case. Upon closer look, we have found out that this phenomenon is simply due to different training sets used with each network (e.g. on the montr50 dataset, type *3.)*, the network has already started with validation error being lower than validation error on a trained network of type *1.)* on this dataset).

- Unfortunately, already we can see that the neural networks performed very poorly, especially on the datasets *zsr* and *air*, finding optimal connections in barely 5% of the queries. Better summary will be visible in table 6.5.

We have taken a closer look at the evolution of validation errors in the training of type *7.)* networks and have found that in general, the validation error went steeply down at the beginning, followed by an interval of fluctuations (which probably caused early stopping when this criterion was in use) and finally found a pretty stable value. The estimation error was decreasing all this time, which is why the networks of type *7.)* obtained most of the time the best results in the final comparison.

Name	1.)		2.)		3.)		4.)		5.)		6.)		7.)	
	O.f.F.	F.O.	O.f.F.	F.O.	O.f.F.	F.O.	O.f.F.	F.O.	O.f.F.	F.O.	O.f.F.	F.O.	O.f.F.	F.O.
tt	94.2	94.2	100.0	100.0	100.0	97.4	100.0	65.0	100.0	89.8	100.0	100.0	100.0	**100.0**
air30-10	11.4	2.2	27.3	5.4	17.7	8.7	11.8	2.2	21.5	2.8	18.6	10.0	18.7	**11.5**
air50-20	29.5	3.5	62.7	11.2	47.9	**21.9**	53.3	1.7	72.0	3.9	44.2	18.5	50.1	21.6
cpsk30-10	39.0	11.3	55.7	**31.7**	54.4	29.2	70.7	18.1	64.7	19.5	46.0	15.4	48.0	28.1
cpsk50-20	85.0	29.4	91.9	29.2	85.5	30.5	72.6	19.8	88.8	26.5	88.3	23.8	77.4	**32.2**
montr30	87.3	42.9	90.0	**69.0**	84.7	56.9	71.1	50.1	89.8	55.2	85.6	43.2	86.7	56.9
montr50	87.2	44.2	89.5	54.9	88.0	52.4	94.2	40.0	97.6	43.0	96.2	40.5	89.6	**56.3**
zsr30-10	73.4	31.7	80.4	22.4	74.8	**35.6**	83.6	17.8	84.4	18.8	78.2	25.1	76.2	34.8
zsr50-20	34.9	5.9	32.4	6.7	35.2	13.7	51.8	6.3	33.3	5.0	41.7	14.9	41.8	**20.1**

Table 6.5: Table summarizes in how many cases (in percents) the network found the optimum value (*F.O. - found optimum*) and in how many cases (percents) when the network found some connection sequence, it was the optimal sequence (*O.f.F - optimum from found*)

In table 6.5 we note down two measures of the network's quality:

- **F.O.**: The network's capability to output the optimum connection (when it, in fact, exists)
- **O.f.F**: The network's reliability - that once the network has found a connection, it is really the best one.

Network with one big hidden layer, trained without the early stopping criterion was the most successful one, when it comes to the F.O. measure, though the best value (69 %) was achieved by a network of type *2.)* trained on a bigger training set. Perceptrons trained on these bigger training sets (*2.)* and *5.)*) were also more reliable.

6.2 Conclusion of the experiment

First of all, we would like to point out some positive aspects. The networks were able to learn optimal answers to queries that were *not* part of the training. Also, it looks like enhanced training of the network could produce much better results.

On the other hand, there are other parameters that could possibly help more then just increasing the number of iterations: different activation functions, error functions, weights adjustments (using momentum, or weight decay) or even using different type of network (some publications have used Hopfield network to search for shortest paths in graphs).

However, we conclude that we failed to train neural network to answer optimal connection queries in timetables. We suppose the main reason for this is that the problem is simply too challenging for a neural network.

7 Application TTBlazer

8 Conclusion

In this thesis we have

We have developed exact methods to considerably speed-up the query time for optimal connections in timetables compared to the time-dependent Dijkstra's algorithm (running in $\mathcal{O}(m + n \log n)$). Our first algorithm - *USP-OR* - achieves speed-ups of up to 70 in the sub-timetable of country-wide coaches in Great Britain. However, it does so at the cost of high space consumption, requiring more that 50 times the space that is needed to represent the timetable itself. Theoretically, for real-world timetables with certain properties, this algorithm has the space complexity $\mathcal{O}(n^{2.5})$ and the average query time $\mathcal{O}(\sqrt{n})$.

Our second algorithm called *USP-OR-A* is still 8.5 times faster then the time-dependent Dijkstra's algorithm on the dataset of British country-wide coaches (2500 stations) and at the same time, it requires about 4 times the space needed to represent the timetable. We believe the speed-up of *USP-OR-A* against Dijkstra's algorithm can be even higher for bigger timetables, since its query time is under certain conditions theoretically determined as $\mathcal{O}(\sqrt{n} \log n)$, while the algorithm can handle much bigger datasets for its space complexity is essentially $\mathcal{O}(n^{1.5})$.

Finally, it would be interesting to measure the query times of *USP-OR-A* if we used random sampling of queries with a distribution according to the reality. Such distribution strongly favours queries concerning the most important cities which are generally part of the access node set in *USP-OR-A*. As computing optimal connections between these cities is very fast (just like in *USP-OR*), we could expect much better speed-ups in real-world situations.

Appendices

A File formats

Timetable is simply a set of elementary connections, thus the format is:

- number of el. connections
- the list of all el. connections (one per line, format "*FROM TO DEP-DAY DEP-TIME ARR-DAY ARR-TIME*")

```
7                       //number of elementary connections
A B 0 10:00 0 10:45     //el. connection
A B 0 11:00 0 11:45
A B 0 12:00 0 12:45
A C 0 09:30 0 10:00
A C 0 10:15 0 10:45
C D 0 11:00 0 11:30
C D 0 13:00 0 13:30
```

Listing 1: TT file format.

Underlying graph is basically an oriented graph, with some optional parameters. The format is the following:

- number of cities
- number of arcs
- the list of all cities (one per line)
  - optional coordinates (otherwise null)
- the list of all arcs (one per line, format "*FROM TO*")
  - optional length (otherwise null)
  - optional list of lines operating on that arc (otherwise null)

```
4                                    //number of cities
5                                    //number of arcs
A 45 32                              //name of the city, optional coordinates
B null
C 56 34
D null
A B 57 Northern                      //arc, optional length and list of lines
A C null Picadilly Victoria
C B 45 Circle Jubilee Picadilly
C D 32 null
D A null null
```

Listing 2: UG file format.

Time-expanded graph is simply an oriented weighted graph, with nodes being the events and arcs being the elementary connections or waiting edges:

- number of nodes (i.e. events)
- number of arcs (el. connections + waiting)
- the list of all events (in the format "CITY DAY TIME")
- the list of all arcs (in the format "FROM-EVENT TO-EVENT")

```
5                        //number of events
15                       //number of arcs
A 0 13:30                //event
A 0 14:00
B 0 13:45
B 0 15:00
C 0 14:15
A 0 13:30 A 0 14:00      //waiting arc
A 0 13:30 B 0 13:45      //el. connection arc
A 0 14:00 B 0 15:00
A 0 13:30 B 0 15:00
C 0 14:15 B 0 15:00
...
```

Listing 3: TE file format.

Time-dependent graph is an oriented graph with a function on the arc specifying the arc's traversal time at any moment. In timetable networks this function is piece-wise linear and it is fully represented by the list of its interpolation points. Thus the TD file format:

- number of cities
- number of arcs
- the list of all cities (one per line)
  - optional coordinates (otherwise null)
- the list of all arcs (one per line). Arc has the format "*FROM TO INT-POINTS*" where *INT-POINTS* is a list of interpolation points [33], see the listing 4 for an example.

```
4                                             //number of stations
5                                             //number of arcs
A 0 0                                         //name of the city, optional coordinates
B 4 4
C null
D 12 0
A B (0 13:30 45) (0 14:00 40)                 //arc and the list of interpolation
    points
A C (1 14:15 10)
C B (0 15:00 20)
C D (2 10:00 70)
D A (1 17:20 35) (1 18:00 40) (1 18:50 35)
...
```

Listing 4: TD file format.

[33] An interpolation point is described by a triple "*DAY TIME MINUTES*", where *MINUTES* are the traversal time

References

[1Te] 1Tech. National rail case study.

[AFGW10] Ittai Abraham, Amos Fiat, Andrew V. Goldberg, and Renato Fonseca F. Werneck. Highway dimension, shortest paths, and provably efficient algorithms. In Moses Charikar, editor, *SODA*, pages 782–793. SIAM, 2010.

[BDSV09] Gernot Veit Batz, Daniel Delling, Peter Sanders, and Christian Vetter. Time-dependent contraction hierarchies. In Irene Finocchi and John Hershberger, editors, *ALENEX*, pages 97–105. SIAM, 2009.

[BFM06] Holger Bast, Stefan Funke, and Domagoj Matijevic. Transit ultrafast shortest-path queries with linear-time preprocessing, 2006.

[Bra01] Ulrik Brandes. A faster algorithm for betweenness centrality. *Journal of Mathematical Sociology*, 25:163–177, 2001.

[Del08] Daniel Delling. Time-dependent sharc-routing. In Dan Halperin and Kurt Mehlhorn, editors, *ESA*, volume 5193 of *Lecture Notes in Computer Science*, pages 332–343. Springer, 2008. ISBN 978-3-540-87743-1.

[Dij59] E. W. Dijkstra. A note on two problems in connexion with graphs. *NUMERISCHE MATHEMATIK*, 1(1):269–271, 1959.

[DPW09] Daniel Delling, Thomas Pajor, and Dorothea Wagner. Engineering time-expanded graphs for faster timetable information. In Ravindra Ahuja, Rolf Mohring, and Christos Zaroliagis, editors, *Robust and Online Large-Scale Optimization*, volume 5868 of *Lecture Notes in Computer Science*, pages 182–206. Springer Berlin / Heidelberg, 2009. ISBN 978-3-642-05464-8.

[DSSW09] Daniel Delling, Peter Sanders, Dominik Schultes, and Dorothea Wagner. Engineering route planning algorithms. In *ALGORITHMICS OF LARGE AND COMPLEX NETWORKS. LECTURE NOTES IN COMPUTER SCIENCE*. Springer, 2009.

[DW09] Daniel Delling and Dorothea Wagner. Time-dependent route planning. In *Robust and Online Large-Scale Optimization, LNCS*. Springer, 2009.

[GPPR04] Cyril Gavoille, David Peleg, Stephane Perennes, and Ran Raz. Distance labeling in graphs. *Journal of Algorithms*, 53(1):85 – 112, 2004. ISSN 0196-6774. URL `http://www.sciencedirect.com/science/article/pii/S0196677404000884`.

[GSSD08] Robert Geisberger, Peter Sanders, Dominik Schultes, and Daniel Delling. Contraction hierarchies: Faster and simpler hierarchical routing in road networks. In Catherine C. McGeoch, editor, *WEA*, volume 5038 of *Lecture Notes in Computer Science*, pages 319–333. Springer, 2008. ISBN 978-3-540-68548-7.

[KMS06] Ekkehard Köhler, Rolf H. Möhring, and Heiko Schilling. Fast point-to-point shortest path computations with arc-flags. In *IN: 9TH DIMACS IMPLEMENTATION CHALLENGE [29*, 2006.

[KP] Kelly Kinahan and Jennifer Pryor. Dijkstra's algorithm for shortest route problems.

[MHSWZ07] Matthias Müller-Hannemann, Frank Schulz, Dorothea Wagner, and Christos Zaroliagis. *Algorithmic Methods for Railway Optimization*, volume 4359 of *Lecture Notes in Computer Science*, chapter Timetable Information: Models and Algorithms, pages 67 – 90. Springer, 2007.

[ops] Art of problem solving. Root-mean square-arithmetic mean-geometric mean-harmonic mean inequality.

[Som10] Christian Sommer. *Approximate Shortest Path and Distance Queries in Networks*. PhD thesis, Graduate School of Information Science and Technology, The University of Tokyo, 2010.

[SS05] Peter Sanders and Dominik Schultes. Highway hierarchies hasten exact shortest path queries. In Gerth Stølting Brodal and Stefano Leonardi, editors, *ESA*, volume 3669 of *Lecture Notes in Computer Science*, pages 568–579. Springer, 2005. ISBN 3-540-29118-0.

[TZ05] Mikkel Thorup and Uri Zwick. Approximate distance oracles. *J. ACM*, 52(1):1–24, 2005.