深蓝学院
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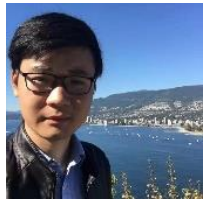
Lecture 5

MINIMUM SNAP TRAJECTORY GENERATION



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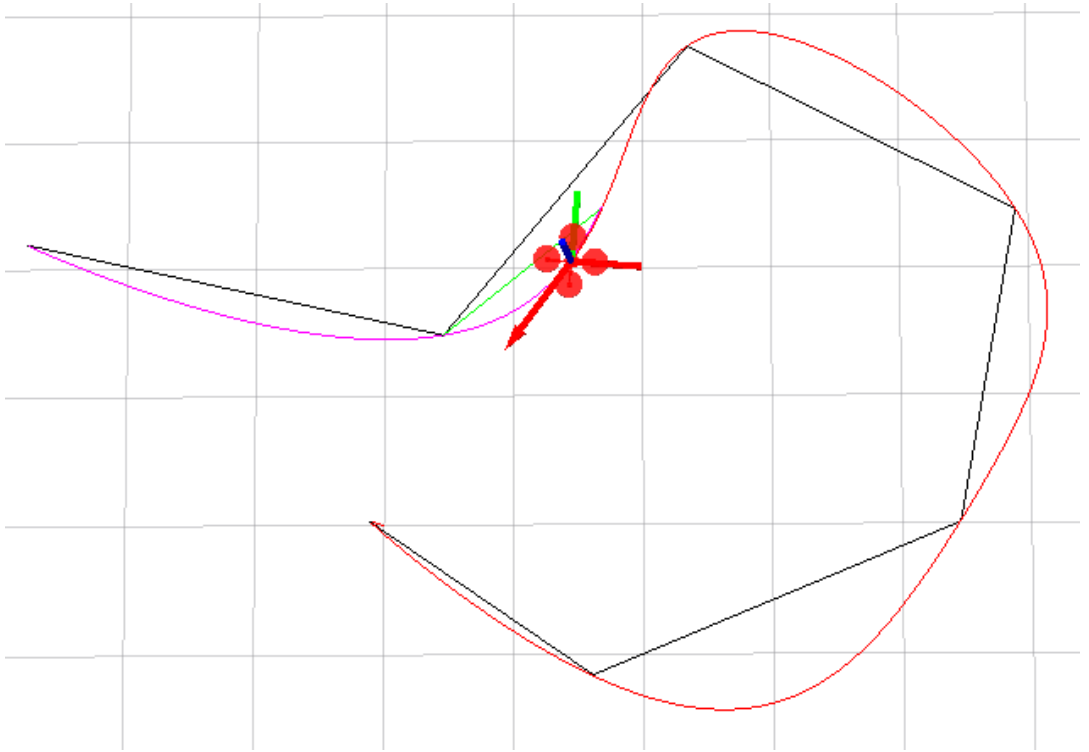
Outline

1. Introduction
2. Minimum Snap Optimization
3. Closed-form Solution to Minimum Snap
4. Implementation Details
5. Homework

Introduction

Why smooth trajectory

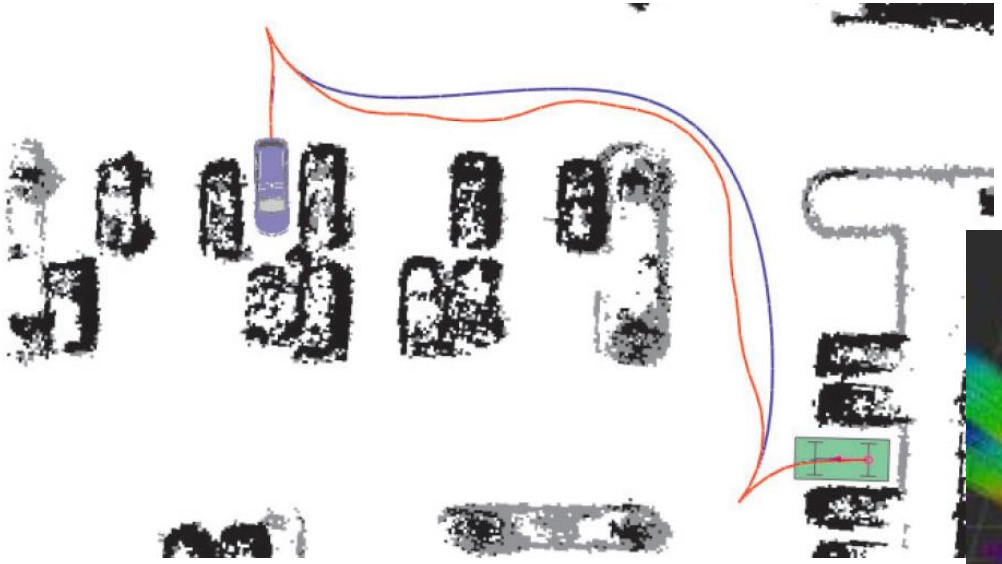
- Good for autonomous moving.
- Velocity/higher order dynamics can’t change immediately.
- The robot should not stop at turns.
- Save energy.

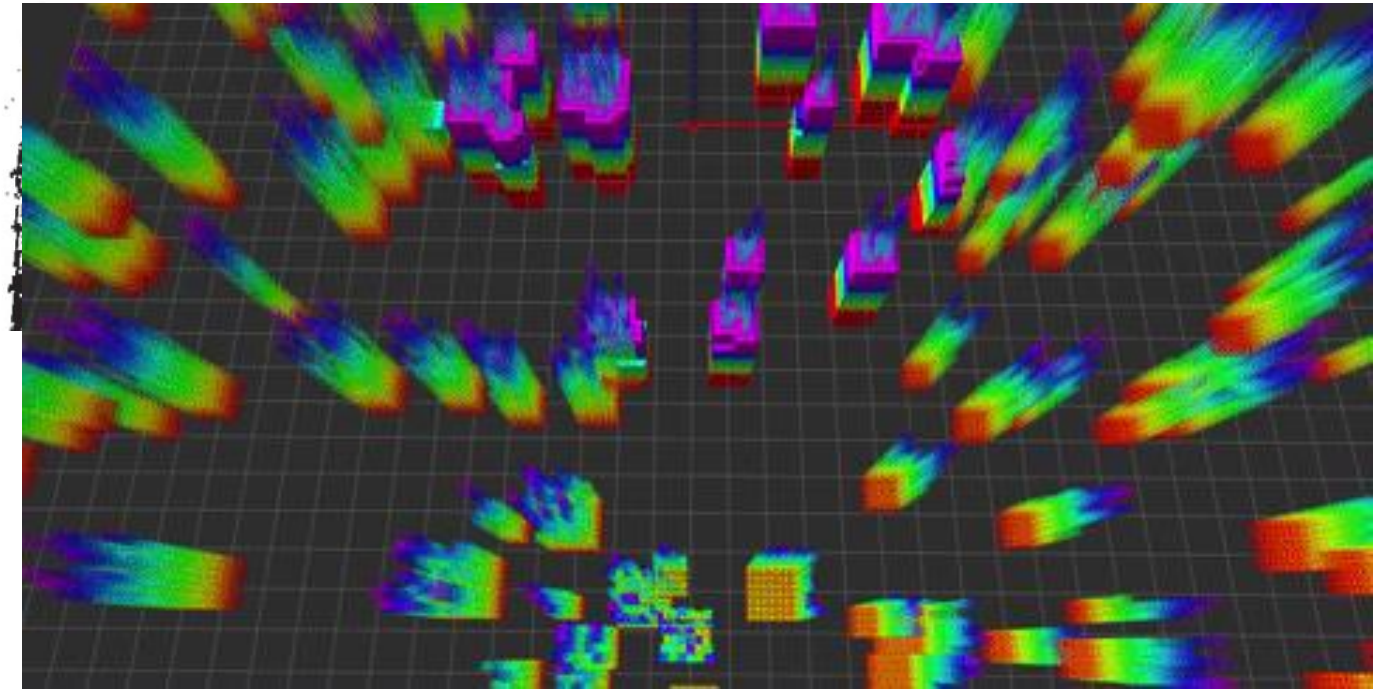


Why trajectory generation/optimization

Ask: We have the front-end (path finding?), why a back-end necessary?

Ask: The front-end is kinodynamic feasible, why a back-end necessary?

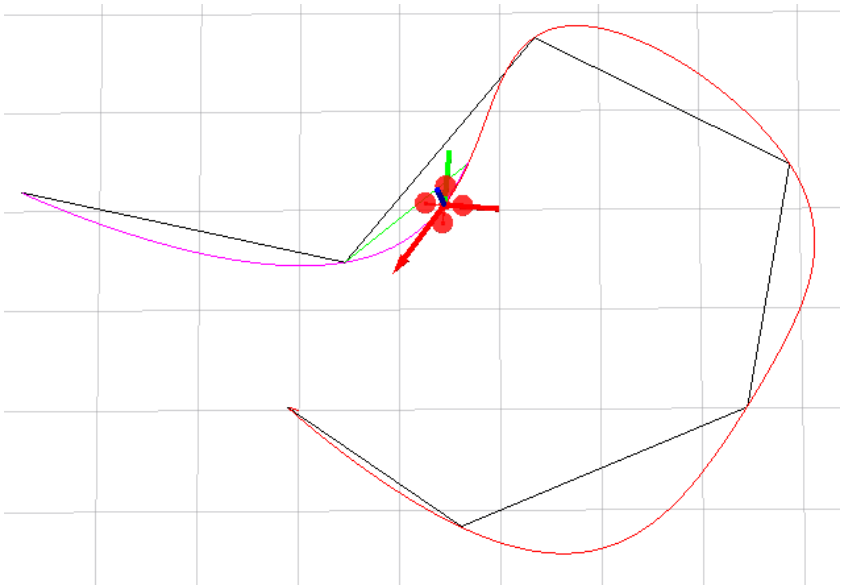




- Path quality.
- Time efficiency.

Smooth trajectory generation

- Boundary condition: start, goal positions (orientations)
- Intermediate condition: waypoint positions (orientations)
  - Waypoints can be found by path planning (A*, RRT*, etc.)
  - Introduced in previous 3 lectures
- Smoothness criteria
  - Generally translates into minimizing rate of change of “input”



Minimum Snap Optimization

Differential Flatness

Differential Flatness

12 dims → 4 dims

The states and the inputs of a quadrotor can be written as algebraic functions of four carefully selected flat outputs and their derivatives

- Enables automated generation of trajectories
- Any smooth trajectory in the space of flat outputs (with reasonably bounded derivatives) can be followed by the under-actuated quadrotor
- A possible choice:
  - $\boldsymbol{\sigma} = [x, y, z, \psi]^T$
- Trajectory in the space of flat outputs:
  - $\boldsymbol{\sigma}(t) = [T_0, T_M] \rightarrow \mathbb{R}^3 \times SO(2)$

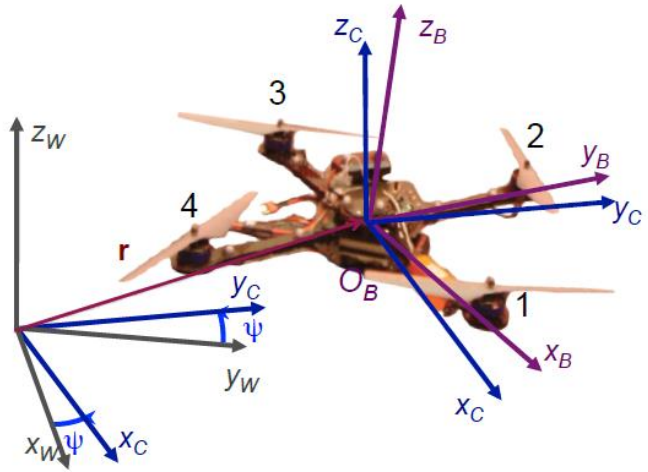
Minimum Snap Trajectory Generation and Control for Quadrotors, Daniel Mellinger and Vijay Kumar

Quadrotor dynamics

- Quadrotor states

  Position, orientation, linear velocity, angular velocity

$$\mathbf{X} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \omega_x, \omega_y, \omega_z]^T$$



- Nonlinear dynamics

Newton Equation: $$m\ddot{\boldsymbol{p}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \boldsymbol{R} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix} u_1$$

Euler Equation: $$\boldsymbol{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \boldsymbol{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} \begin{matrix} \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{matrix}$$

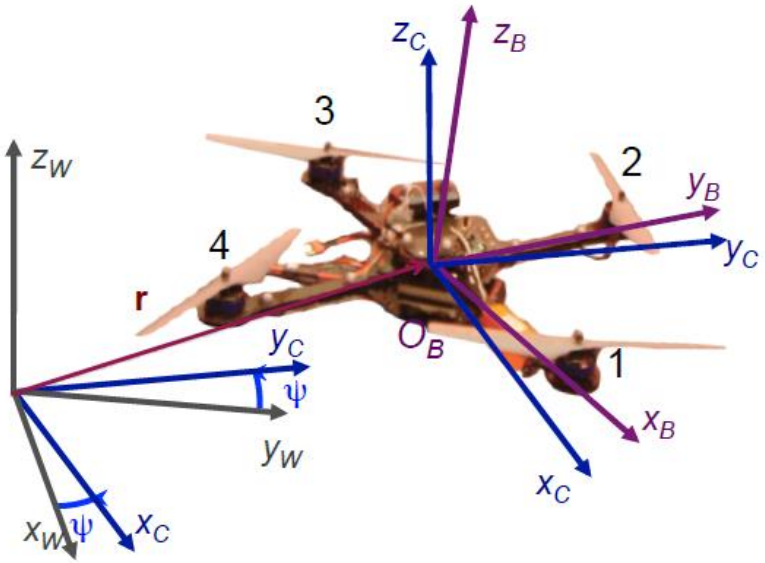
Differential Flatness

- **Quadrotor states**
  - Position, orientation, linear velocity, angular velocity

$$\mathbf{X} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \omega_x, \omega_y, \omega_z]^T$$

角加速度可以通过roll、pitch解算

Body angular velocity viewed in the body frame



  - Equation of motions:

$$m\ddot{\mathbf{p}} = -mg\mathbf{z}_W + u_1\mathbf{z}_B.$$

$$\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \dot{\boldsymbol{\omega}}_B = \boldsymbol{I}^{-1}\left[-\boldsymbol{\omega}_B \times \boldsymbol{I} \cdot \boldsymbol{\omega}_B + \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix}\right]$$

- Position, velocity, and acceleration are simply derivatives of the flat outputs

Minimum Snap Trajectory Generation and Control for Quadrotors, Daniel Mellinger and Vijay Kumar

Differential Flatness

- **Orientation**
  - Quadrotor state:

$$\mathbf{X} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \omega_x, \omega_y, \omega_z]^T$$

  - From the equation of motion:

$$\mathbf{z}_B = \frac{\mathbf{t}}{\|\mathbf{t}\|}, \mathbf{t} = [\ddot{\boldsymbol{\sigma}}_1, \ddot{\boldsymbol{\sigma}}_2, \ddot{\boldsymbol{\sigma}}_3 + g]^T$$

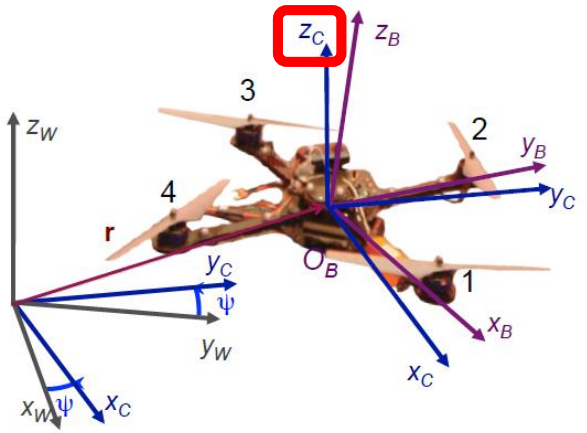
  - Define the yaw vector (Z-X-Y Euler):

$$\mathbf{x}_C = [cos\boldsymbol{\sigma}_4, sin\boldsymbol{\sigma}_4, 0]^T$$

  - Orientation can be expressed in terms of flat outputs

$$\mathbf{y}_B = \frac{\mathbf{z}_B \times \mathbf{x}_C}{\|\mathbf{z}_B \times \mathbf{x}_C\|}, \quad \mathbf{x}_B = \mathbf{y}_B \times \mathbf{z}_B \quad \boldsymbol{R}_B = [\mathbf{x}_B \ \mathbf{y}_B \ \mathbf{z}_B]$$

A “no tilt” frame



$$\boldsymbol{\sigma} = [x, y, z, \psi]^T$$

Differential Flatness

- **Angular velocity**
  - Quadrotor state:

    $\mathbf{X} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \omega_z, \omega_y, \omega_z]^T$

  - Take the derivative of the equation of motion

    $m\ddot{\boldsymbol{p}} = -mg\mathbf{z}_W + u_1\mathbf{z}_B.$ ⟶ $m\dot{\boldsymbol{a}} = \dot{u}_1\mathbf{z}_B + \boldsymbol{\omega}_{BW} \times u_1\,\mathbf{z}_B$

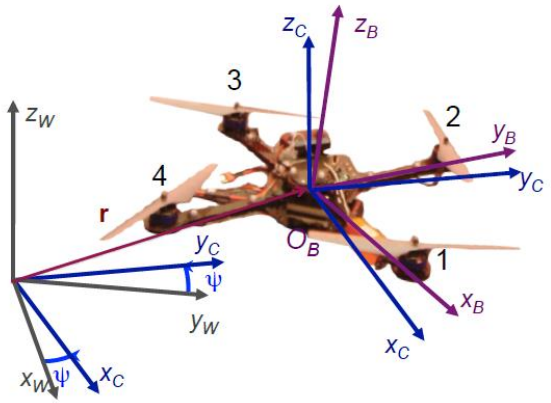
    Body angular velocity viewed in the world frame

  - Quadrotors only have vertical thrust:

    $\dot{u}_1 = \mathbf{z}_B \cdot m\dot{\boldsymbol{a}}$

  - We have:

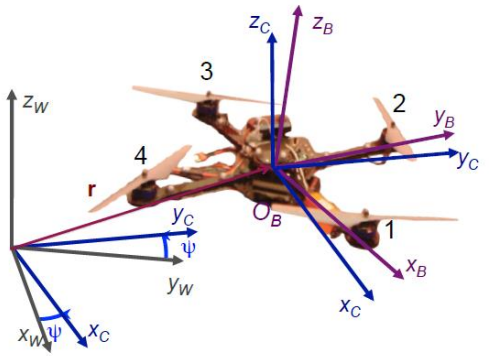
    $\mathbf{h}_\omega = \boldsymbol{\omega}_{BW} \times \mathbf{z}_B = \frac{m}{u_1}(\dot{\boldsymbol{a}} - (\mathbf{z}_B \cdot \dot{\boldsymbol{a}})\mathbf{z}_B).$



Differential Flatness

- **Angular velocity**
  - Quadrotor state:

$$\mathbf{X} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \omega_x, \omega_y, \omega_z]^T$$



  - We have:

$$\mathbf{h}_\omega = \boldsymbol{\omega}_{BW} \times \mathbf{z}_B = \frac{m}{u_1}(\dot{\boldsymbol{a}} - (\mathbf{z}_B \cdot \dot{\boldsymbol{a}})\mathbf{z}_B).$$

  - We know that:

$$\boldsymbol{\omega}_{BW} = \omega_x \mathbf{x}_B + \omega_y \mathbf{y}_B + \omega_z \mathbf{z}_B$$

  - Angular velocities along x_B and y_B directions can be found as:

$$\omega_x = -\mathbf{h}_\omega \cdot \mathbf{y}_B, \quad \omega_y = \mathbf{h}_\omega \cdot \mathbf{x}_B$$

Differential Flatness

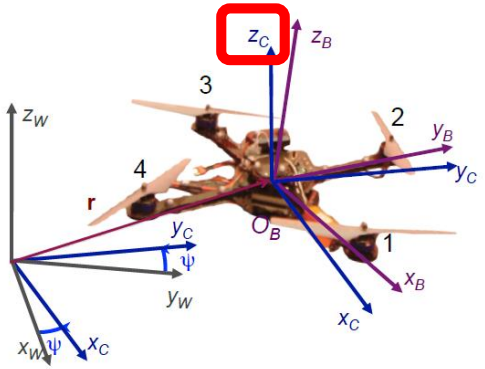
- Angular velocity
  - Quadrotor state:

$$\mathbf{X} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \omega_x, \omega_y, \omega_z]^T$$

  - We have:

$$\mathbf{h}_\omega = \boldsymbol{\omega}_{BW} \times \mathbf{z}_B = \frac{m}{u_1}(\dot{\boldsymbol{a}} - (\mathbf{z}_B \cdot \dot{\boldsymbol{a}})\mathbf{z}_B).$$

A “no tilt” frame



  - Since $\boldsymbol{\omega}_{BW} = \boldsymbol{\omega}_{BC} + \boldsymbol{\omega}_{CW}$, where $\boldsymbol{\omega}_{BC}$ has no $\mathbf{z}_B$ component:

$$\omega_z = \boldsymbol{\omega}_{BW} \cdot \mathbf{z}_B = \boldsymbol{\omega}_{CW} \cdot \mathbf{z}_B = \dot{\psi}\mathbf{z}_W \cdot \mathbf{z}_B.$$

Differential Flatness

Drone states can be represent by Flat output

12 dims

4 dims

- **Summary**
  - Quadrotor state:

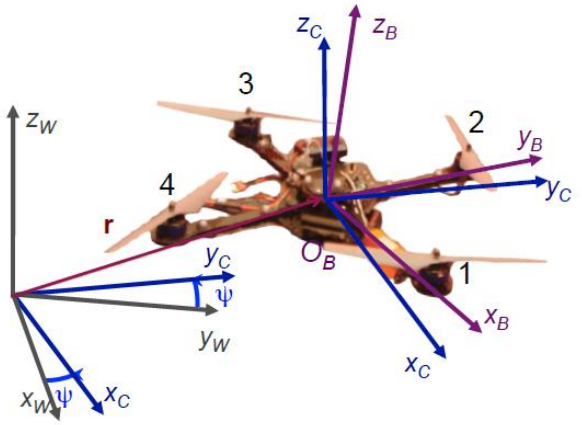
$$\mathbf{X} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \omega_x, \omega_y, \omega_z]^T$$

  - Flat outputs:
    - $\boldsymbol{\sigma} = [x, y, z, \psi]^T$
  - Position, velocity, acceleration:
    - Derivatives of flat outputs
  - Orientation:

$$\mathbf{x}_C = [cos\sigma_4, sin\sigma_4, 0]^T \longrightarrow \boldsymbol{R}_B = [\mathbf{x}_B \ \mathbf{y}_B \ \mathbf{z}_B]$$

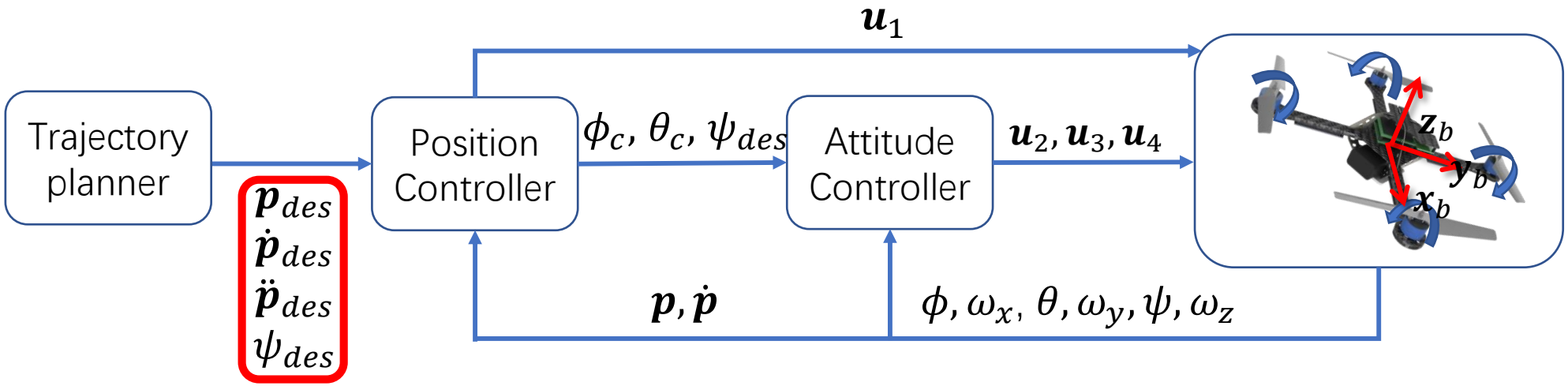
  - Angular velocity:

$$\omega_x = -\mathbf{h}_\omega \cdot \mathbf{y}_B, \quad \omega_y = \mathbf{h}_\omega \cdot \mathbf{x}_B, \quad \omega_z = \dot{\psi}\mathbf{z}_W \cdot \mathbf{z}_B$$



Done. You should only remember the planning of a UAV can be done in x, y, z, ψ.

Close the planning-control loop



Nonlinear dynamics

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Euler Equation: $$\boldsymbol{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \boldsymbol{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$$

Polynomial Trajectories

- Flat outputs:
  - $\boldsymbol{\sigma} = [x, y, z, \psi]^T$

- Trajectory in the space of flat outputs:
  - $\boldsymbol{\sigma}(t) = [T_0, T_M] \rightarrow \mathbb{R}^3 \times SO(2)$

- Polynomial functions can be used to specify trajectories in the space of flat outputs
  - Easy determination of smoothness criterion with polynomial orders
  - Easy and closed form calculation of derivatives
  - Decoupled trajectory generation in three dimensions

Minimum-snap

Smooth 1D Trajectory

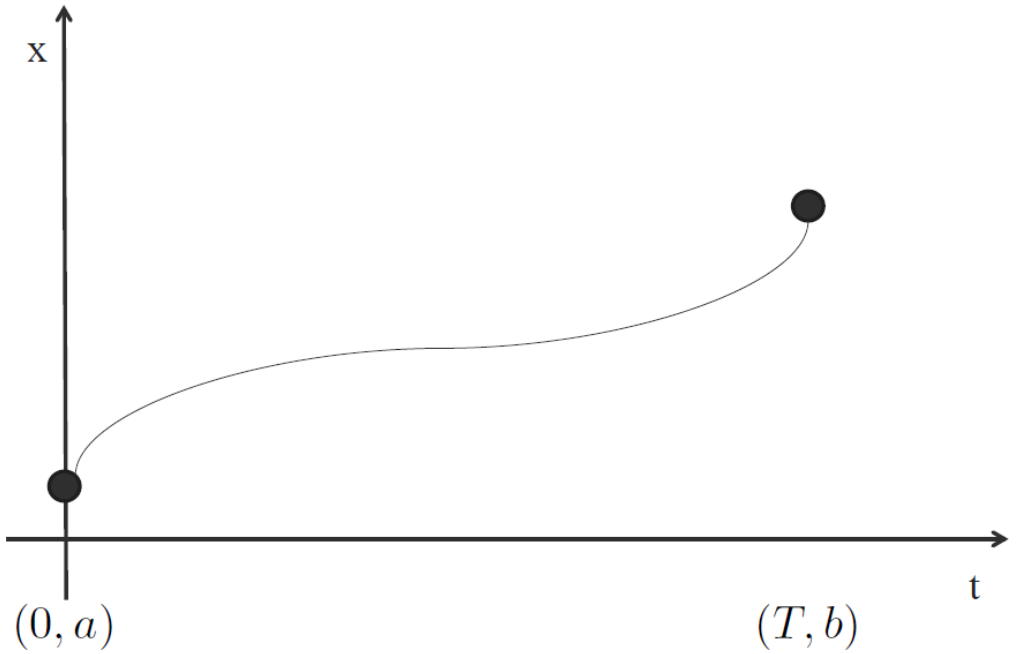
Boundary Value Problem

Just a simple BVP

- Design a trajectory $x(t)$ such that:
  - $x(0) = a$
  - $x(T) = b$

  } Boundary condition

- Smoothness ensured by parametrization — Smoothness criteria



Smooth 1D Trajectory

- 5^{th} order polynomial trajectory:
  - $x(t) = p_5 t^5 + p_4 t^4 + p_3 t^3 + p_2 t^2 + p_1 t + p_0$ → Trajectory parametrization
- Boundary conditions — No intermediate condition

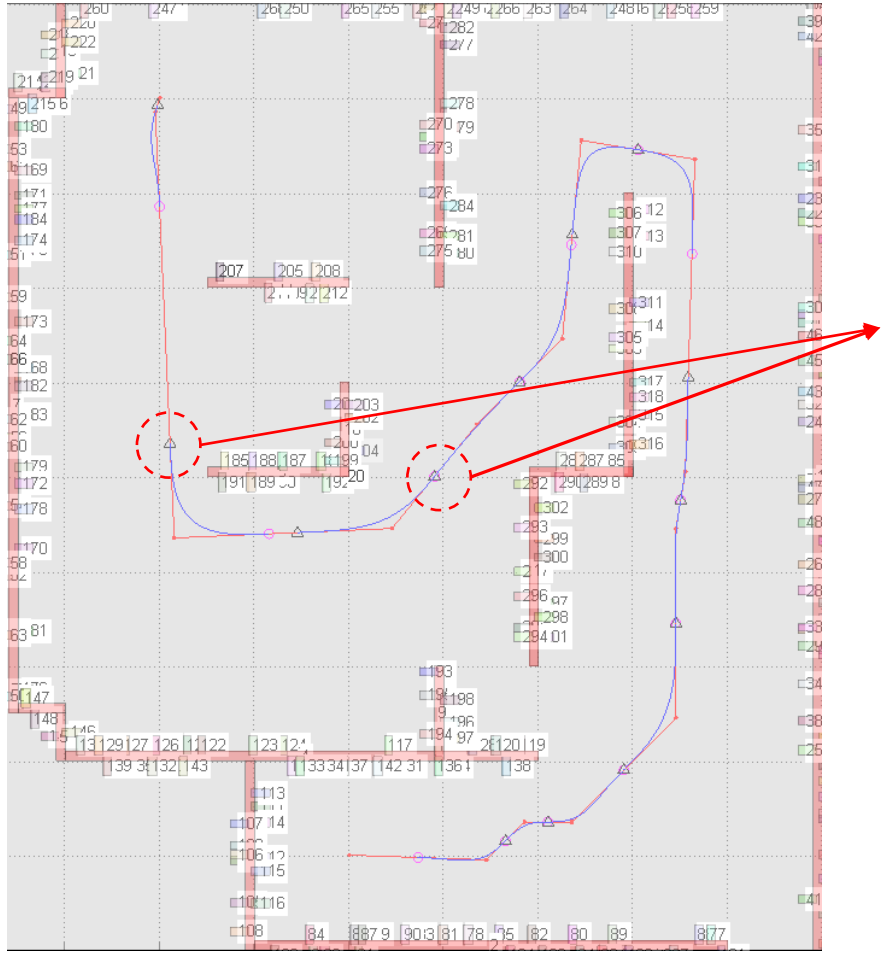
	Position	Velocity	Acceleration
t = 0	a	0	0
t = T	b	0	0

- Solve:

$$\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_5 \\ p_4 \\ p_3 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix}$$

Smooth Multi-Segment Trajectory

- Smooth the corners of straight line segments.
- Preferred constant velocity motion at v.
- Preferred zero acceleration.
- Requires special handling of short segments.



Smooth Multi-Segment 1D Trajectory

- Generate each 5^{th} order polynomial independently:
  - $x(t) = p_5 t^5 + p_4 t^4 + p_3 t^3 + p_2 t^2 + p_1 t + p_0$
- Boundary conditions

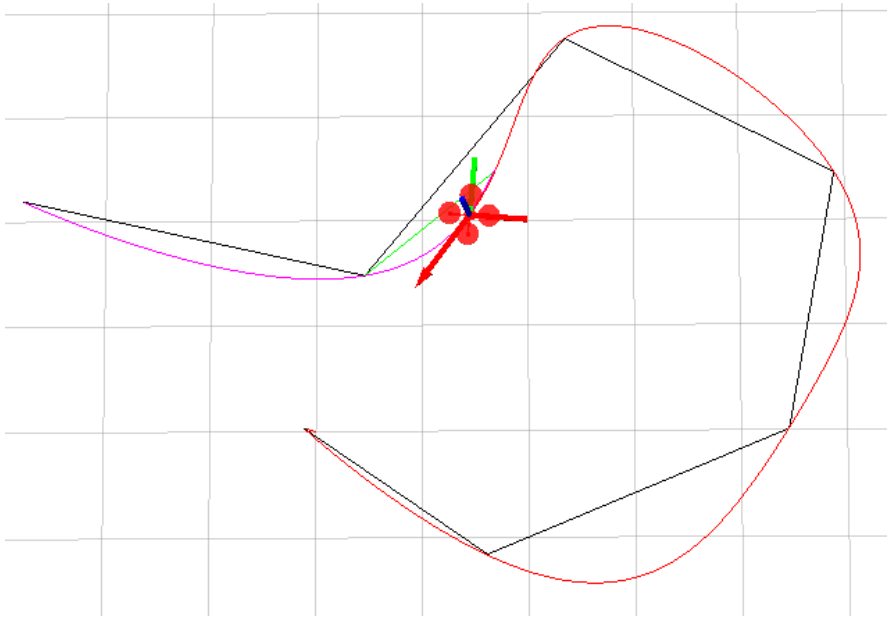
	Position	Velocity	Acceleration
t = 0	a	v_0	0
t = T	b	v_T	0

- Solve:

$$\begin{bmatrix} a \\ b \\ v_0 \\ v_T \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_5 \\ p_4 \\ p_3 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix}$$

Smooth 3D Trajectory

- Boundary condition: start, goal positions (orientations)
- Intermediate condition: waypoint positions (orientations)
  - Waypoints can be found by path planning (A*, RRT*, etc)
  - Introduced in previous 3 lectures
- Smoothness criterion
  - Generally translates into minimizing rate of change of “input”



Optimization-based Trajectory Generation

- Explicitly minimize certain derivatives in the space of flat outputs
- Quadrotor dynamics

Derivative	Translation	Rotation	Thrust
0	Position		
1	Velocity		
2	Acceleration	Rotation	
3	Jerk	Angular Velocity	Thrust
4	Snap	Angular Acceleration	Differential Thrust

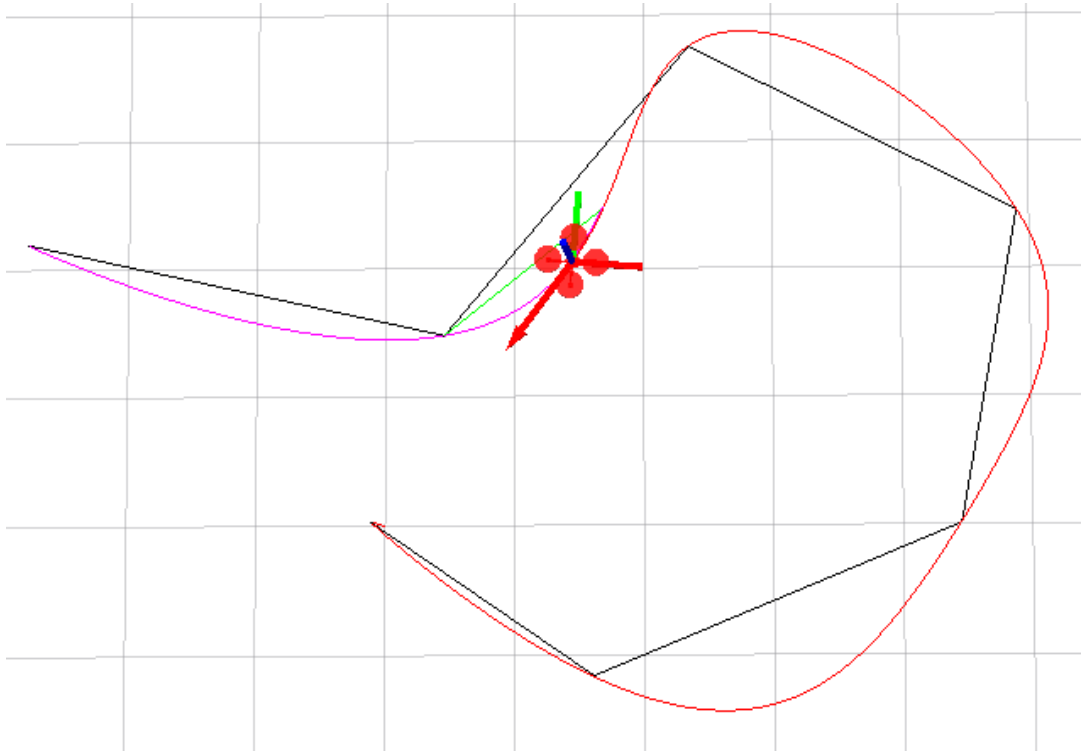
Optimization-based Trajectory Generation

- Explicitly minimize certain derivatives in the space of flat outputs
  - Minimum jerk: minimize angular velocity, good for visual tracking
  - Minimum snap: minimize differential thrust, saves energy

Derivative	Translation	Rotation	Thrust
0	Position		
1	Velocity		
2	Acceleration	Rotation	
3	Jerk	Angular Velocity	Thrust
4	Snap	Angular Acceleration	Differential Thrust

Optimization-based Trajectory Generation

- Multi-segment minimum snap trajectory



Minimum Snap Trajectory Generation

Formulation:

$$f(t) = \begin{cases} f_1(t) \doteq \sum_{i=0}^{N} p_{1,i} t^i & T_0 \le t \le T_1 \\ f_2(t) \doteq \sum_{i=0}^{N} p_{2,i} t^i & T_1 \le t \le T_2 \\ \vdots & \vdots \\ f_M(t) \doteq \sum_{i=0}^{N} p_{M,i} t^i & T_{M-1} \le t \le T_M \end{cases}$$

- Each segment is a polynomial. 默认每段轨迹多项式阶数相同，且持续时间已知
- No need to fix the order, but keep the same order make this problem simpler.
- **Time durations** for each segment must be known!

对闭环记

Minimum Snap Trajectory Generation

Constraints:

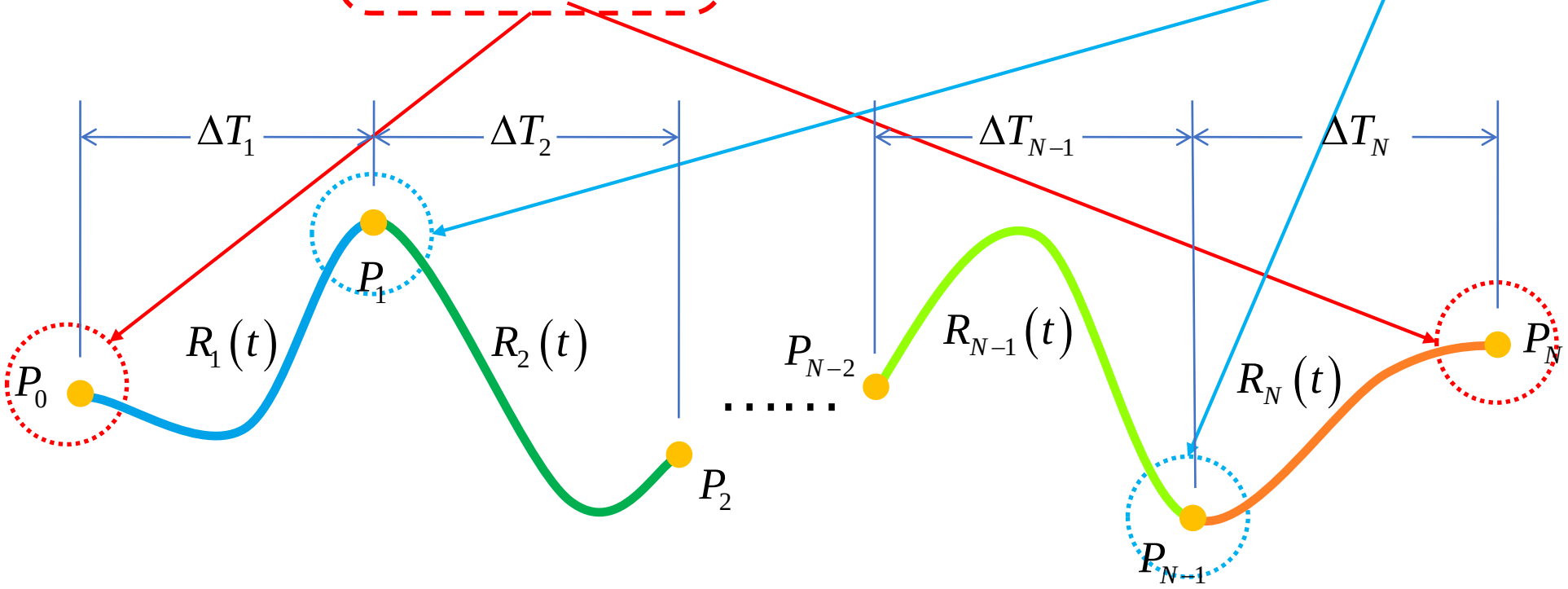
导数有界

- Derivative constraints:

$$\begin{cases} f_j^{(k)}(T_{j-1}) = x_{0,j}^{(k)} \\ f_j^{(k)}(T_j) = x_{T,j}^{(k)} \end{cases}$$

轨迹连续

- Continuity constraints: $f_j^{(k)}(T_j) = f_{j+1}^{(k)}(T_j)$



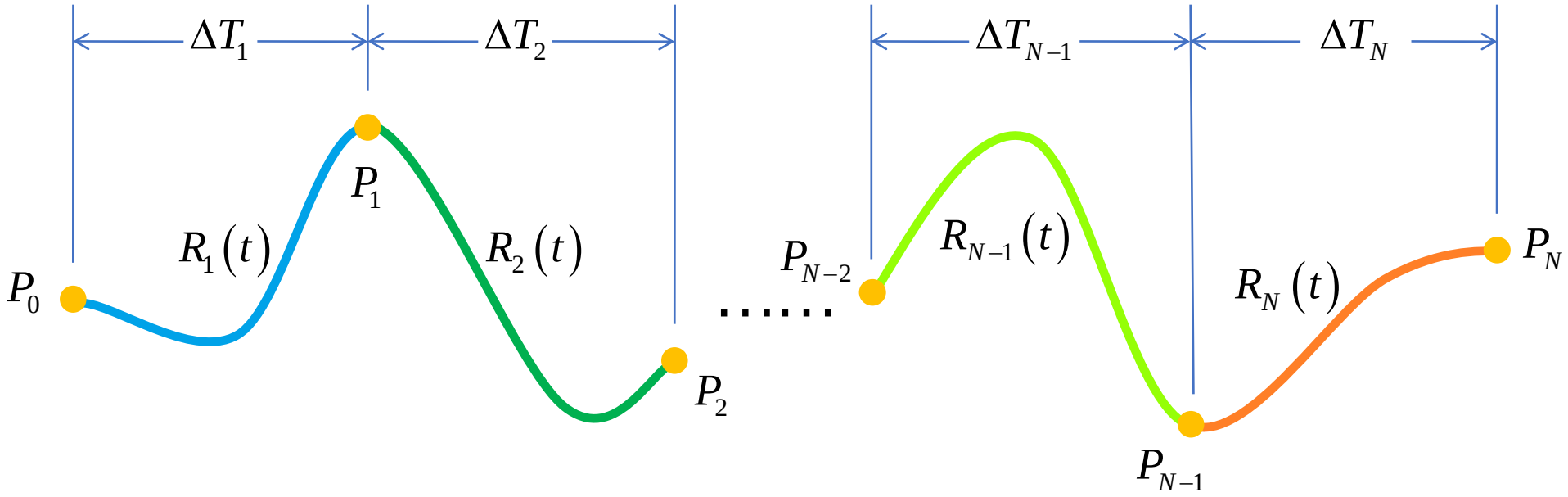
Minimum Snap Trajectory Generation

How to determine the trajectory order?

- Ensure smoothness at an order.
- Ensure continuity at an order.
- Minimize control input at an order.

This three items are not coupled!

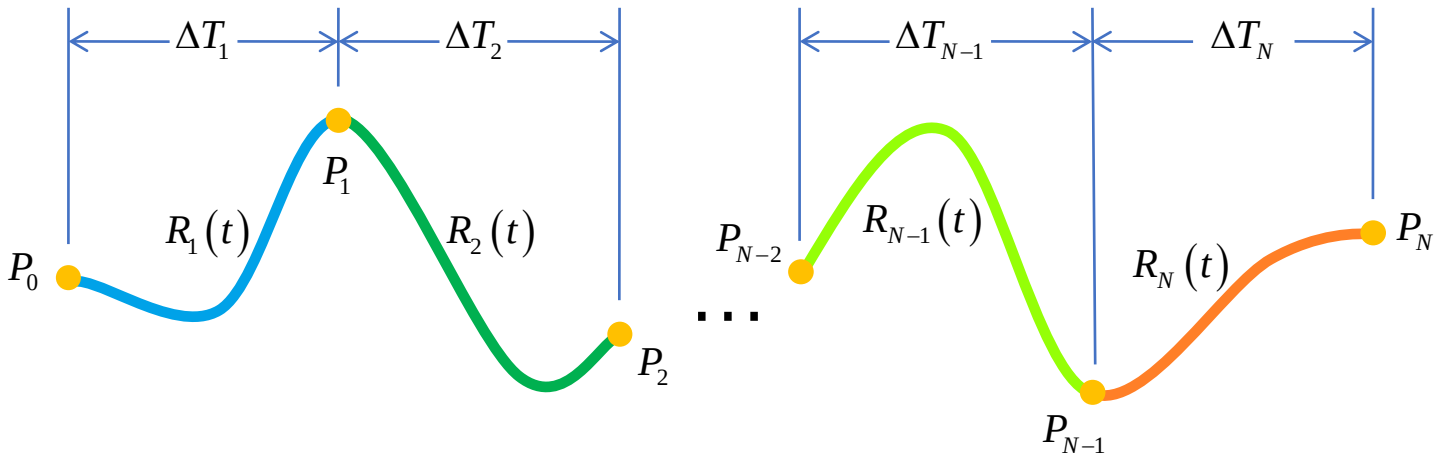
Smoothness means its derivative is continuous!



Minimum Snap Trajectory Generation

- Minimum degree polynomial to ensure smoothness for one-segment trajectory:
  - Minimum jerk: N = 2 * 3(jerk) - 1 = 5
  - Minimum snap: N = 2 * 4(snap) - 1 = 7

- Minimum degree polynomial to ensure smoothness for k-segment trajectory:



Minimum jerk:

Constraints num: 3 + 3 + (k-1) = k + 5

Unknowns num: (N+1) * k

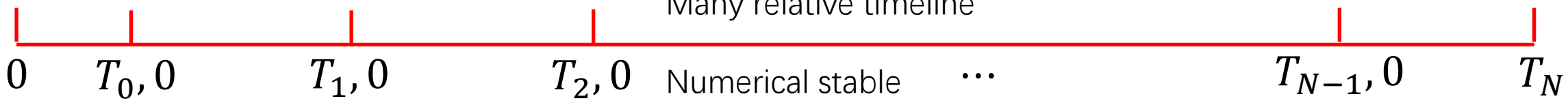
$$(N+1) * k = k+5 \qquad N = \frac{5}{k}$$

Minimum Snap Trajectory Generation

Note different timeline 时间分配

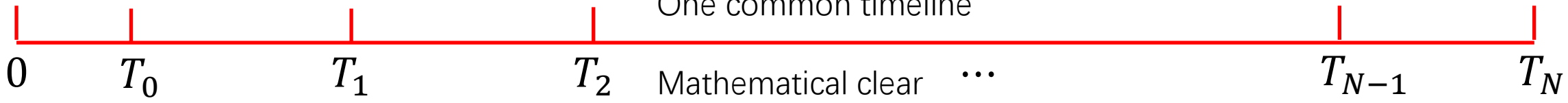
✓ Implementation, homework

Many relative timeline



✓ Problem formulation, lecture slides

One common timeline



Minimum Snap Trajectory Generation

- Cost function for one polynomial segment:

$$f(t) = \sum_i p_i t^i$$

$$\Rightarrow f^{(4)}(t) = \sum_{i\geq 4} i(i-1)(i-2)(i-3)t^{i-4}p_i$$

$$\Rightarrow \left(f^{(4)}(t)\right)^2 = \sum_{i\geq 4, l\geq 4} i(i-1)(i-2)(i-3)l(l-1)(l-2)(l-3)t^{i+l-8}p_i\, p_l$$

$$\Rightarrow J(T) = \int_{T_{j-1}}^{T_j} \left(f^4(t)\right)^2 dt = \sum_{i\geq 4, l\geq 4} \frac{i(i-1)(i-2)(i-3)j(l-1)(l-2)(l-3)}{i+l-7}\left(T_j^{i+l-7} - T_{j-1}^{i+l-7}\right)p_i\, p_l$$

$$\Rightarrow J(T) = \int_{T_{j-1}}^{T_j} \left(f^4(t)\right)^2 dt$$

$$= \begin{bmatrix} \vdots \\ p_i \\ \vdots \end{bmatrix}^T \begin{bmatrix} \cdots & \vdots & \cdots \\ \cdots & \frac{i(i-1)(i-2)(i-3)l(l-1)(l-2)(l-3)}{i+l-7}T^{i+l-7} & \cdots \\ \cdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ p_l \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}$$

$$\Rightarrow \boxed{J_j(T) = \mathbf{p}_j^T \mathbf{Q}_j \mathbf{p}_j}$$ Minimize this!

目标函数

Minimum Snap Trajectory Generation

- Derivative constraint for one polynomial segment
  - Also models waypoint constraint (0^{th} order derivative)

$$f_j^{(k)}(T_j) = x_j^{(k)}$$

$$\Rightarrow \sum_{i \geq k} \frac{i!}{(i-k)!} T_j^{i-k} p_{j,i} = x_{T,j}^{(k)}$$

$$\Rightarrow \begin{bmatrix} \cdots & \frac{i!}{(i-k)!} T_j^{i-k} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,i} \\ \vdots \end{bmatrix} = x_{T,j}^{(k)}$$

$$\Rightarrow \begin{bmatrix} \cdots & \frac{i!}{(i-k)!} T_{j-1}^{i-k} & \cdots \\ & \vdots & \\ \cdots & \frac{i!}{(i-k)!} T_j^{i-k} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,i} \\ \vdots \end{bmatrix} = \begin{bmatrix} x_{0,j}^{(k)} \\ \\ x_{T,j}^{(k)} \end{bmatrix}$$

$$\Rightarrow \mathbf{A}_j \mathbf{p}_j = \mathbf{d}_j$$

等式约束

$$x(t) = p_5 t^5 + p_4 t^4 + p_3 t^3 + p_2 t^2 + p_1 t + p_0$$

$$\begin{aligned} x(0) &= \cdots, x(T) = \cdots \\ \dot{x}(0) &= \cdots, \dot{x}(T) = \cdots \\ \ddot{x}(0) &= \cdots, \ddot{x}(T) = \cdots \\ &\vdots \end{aligned}$$

$$p_0 = \cdots,$$
$$p_5 T^5 + p_4 T^4 + p_3 T^3 + p_2 T^2 + p_1 T + p_0 = \cdots$$

$$[T^5, T^4, T^3, T^2, T, 1] \begin{bmatrix} p_5 \\ p_4 \\ p_3 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \cdots$$

Minimum Snap Trajectory Generation

- Continuity constraint between two segments:
  - Ensures continuity between trajectory segments when no specific derivatives are given

$$f_j^{(k)}(T_j) = f_{j+1}^{(k)}(T_j)$$

$$\Rightarrow \sum_{i \geq k} \frac{i!}{(i-k)!} T_j^{i-k} p_{j,i} - \sum_{l \geq k} \frac{l!}{(l-k)!} T_j^{l-k} p_{j+1,l} = 0$$

$$\Rightarrow \begin{bmatrix} \cdots & \frac{i!}{(i-k)!} T_j^{i-k} & \cdots & -\frac{l!}{(l-k)!} T_j^{l-k} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,i} \\ \vdots \\ p_{j+1,l} \\ \vdots \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \mathbf{A}_j & -\mathbf{A}_{j+1} \end{bmatrix} \begin{bmatrix} \mathbf{p}_j \\ \mathbf{p}_{j+1} \end{bmatrix} = 0$$

等式约束

Minimum Snap Trajectory Generation

- Constrained quadratic programming (QP) formulation:

$$\min \quad \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}$$

$$\text{s.t.} \quad \mathbf{A}_{\text{eq}} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix} = \mathbf{d}_{eq}$$

⊂ 凸优化
↓
一定存在全局极值

It's a typical **convex optimization** program.

Experimental validation

Convex Optimization

Convex function and convex set

Convex function

- A function $f: R^n \longrightarrow R$ is said to be **convex** if the domain, dom f, is convex and for any $x, y \in$ dom f and $0 \leq \theta \leq 1$,

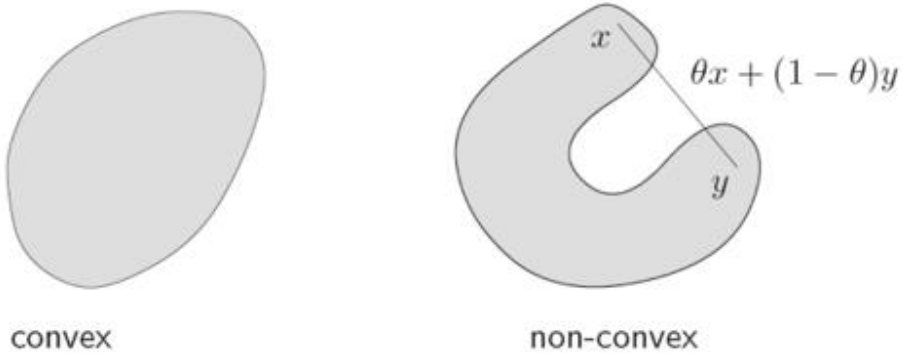
$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

- f is strictly convex if the inequality is strict for $0 < \theta < 1$.
- f is concave if $-f$ is convex.

Convex set

- A set $y \in R^n$ is said to be **convex** if the line segment between any two points is in the set: for any $x, y \in C$ and $0 \leq \theta \leq 1$,

$$\theta x + (1-\theta)y \in C.$$



Courtesy: *Convex Optimization*, Daniel Palomar, HKUST

Convex optimization

- Optimization problem in standard form

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \ldots, m \\ & h_i(x) = 0 \quad i = 1, \ldots, p \end{array}$$

$x \in R^n$ is the optimization variable

$f_0 \colon R^n \to R$ is the objective function

$f_i \colon R^n \to R, i = 1, \ldots, m$ are inequality constraint functions

$h_i \colon R^n \to R, i = 1, \ldots, p$ are equality constraint functions

- Convex optimization problem in standard form:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \ldots, m \\ & \mathrm{A}x = b \end{array}$$

where $f_0, f_1, \ldots, f_m$ are convex and equality constraints are affine.

- **Local and goal optima**: any locally optimal point of a convex problem is globally optimal.
- Most problems are not convex when formulated.
- Reformulating a problem in convex form is an art, there is no systematic way.

Convex optimization

- Optimization problem in standard form

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \dots, m \\ & h_i(x) = 0 & i = 1, \dots, p \end{array}$$

$x \in R^n$ is the optimization variable

$f_0 \colon R^n \to R$ is the objective function

$f_i \colon R^n \to R, i = 1, \dots, m$ are inequality constraint functions

$h_i \colon R^n \to R, i = 1, \dots, p$ are equality constraint functions

- Convex optimization problem in standard form:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \dots, m \\ & Ax = b & \end{array}$$

where $f_0, f_1, \dots, f_m$ are convex and equality constraints are affine.

- **Local and goal optima**: any locally optimal point of a convex problem is globally optimal.
- Most problems are not convex when formulated.
- Reformulating a problem in convex form is an art, there is no systematic way.

Disciplined convex optimization programs

Linear Programming (LP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + d \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

- Convex problem: affine objective and constraint functions.

Quadratic Programming (QP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

- Convex problem (assuming $P \in S^n \geq 0$): convex quadratic objective and affine constraint functions.

可转化

Quadratically Constrained QP (QCQP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0 \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- Convex problem (assuming $P_i \in S^n \geq 0$): convex quadratic objective and constraint functions.

Second-Order Cone Programming (SOCP)

二阶锥优化

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^T x \\ \text{subject to} & \|A_i x + b_i\| \leq c_i^T x + d_i \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

- Convex problem: linear objective and second-order cone constraints
- For A_i row vector, it reduces to an LP.
- For $c_i = 0$, it reduces to a QCQP.
- More general than QCQP and LP.

Closed-form Solution to Minimum Snap

Decision variable mapping

- Direct optimization of polynomial trajectories is numerically unstable
- A change of variable that instead optimizes segment endpoint derivatives is preferred

v, a, ---, not p

- We have $\boldsymbol{M}_j \mathbf{p}_j = \mathbf{d}_j$, where $\boldsymbol{M}_j$ is a mapping matrix that maps polynomial coefficients to derivatives

$$J = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix} \qquad J = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_M \end{bmatrix}^T \begin{bmatrix} \boldsymbol{M}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M}_M \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \boldsymbol{M}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M}_M \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_M \end{bmatrix}$$

Polynomial Trajectory Planning for Aggressive Quadrotor Flight in Dense Indoor Environments, Charles Richter, Adam Bry, and Nicholas Roy

Decision variable mapping

$$J = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix} \quad + \quad \boldsymbol{M}_j \mathbf{p}_j = \mathbf{d}_j$$

$$J = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_M \end{bmatrix}^T \begin{bmatrix} \boldsymbol{M}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M}_M \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \boldsymbol{M}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M}_M \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_M \end{bmatrix}$$

$$x(t) = p_5 t^5 + p_4 t^4 + p_3 t^3 + p_2 t^2 + p_1 t + p_0$$
$$x'(t) = 5p_5 t^4 + 4p_4 t^3 + 3p_3 t^2 + 2p_2 t + p_1$$
$$x''(t) = 20p_5 t^3 + 12p_4 t^2 + 6p_3 t + 2p_2$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix}$$

Polynomial Trajectory Planning for Aggressive Quadrotor Flight in Dense Indoor Environments, Charles Richter, Adam Bry, and Nicholas Roy

Fixed and free variable separation

- Use a selection matrix $\mathbf{C}$ to separate free ($\mathbf{d}_P$) and constrained ($\mathbf{d}_F$) variables
  - Free variables : derivatives unspecified, only enforced by continuity constraints

$$\mathbf{C}^T \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_M \end{bmatrix} \quad \Longrightarrow \quad J = \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \underbrace{\mathbf{C}\boldsymbol{M}^{-T}\mathbf{Q}\boldsymbol{M}^{-1}\mathbf{C}^T}_{\mathbf{R}} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix} = \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \begin{bmatrix} \mathbf{R}_{FF} & \mathbf{R}_{FP} \\ \mathbf{R}_{PF} & \mathbf{R}_{PP} \end{bmatrix} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}$$

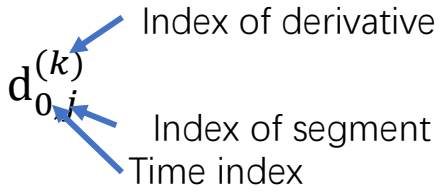
分块

- Turned into an unconstrained quadratic programming that can be solved in closed form:

$$J = \mathbf{d}_F^T \mathbf{R}_{FF} \mathbf{d}_F + \mathbf{d}_F^T \mathbf{R}_{FP} \mathbf{d}_P + \mathbf{d}_P^T \mathbf{R}_{PF} \mathbf{d}_F + \mathbf{d}_P^T \mathbf{R}_{PP} \mathbf{d}_P$$

$$\frac{\partial J}{\partial \mathbf{d}_P} = 0 \Rightarrow \mathbf{d}_P^* = -\mathbf{R}_{PP}^{-1} \mathbf{R}_{FP}^T \mathbf{d}_F$$

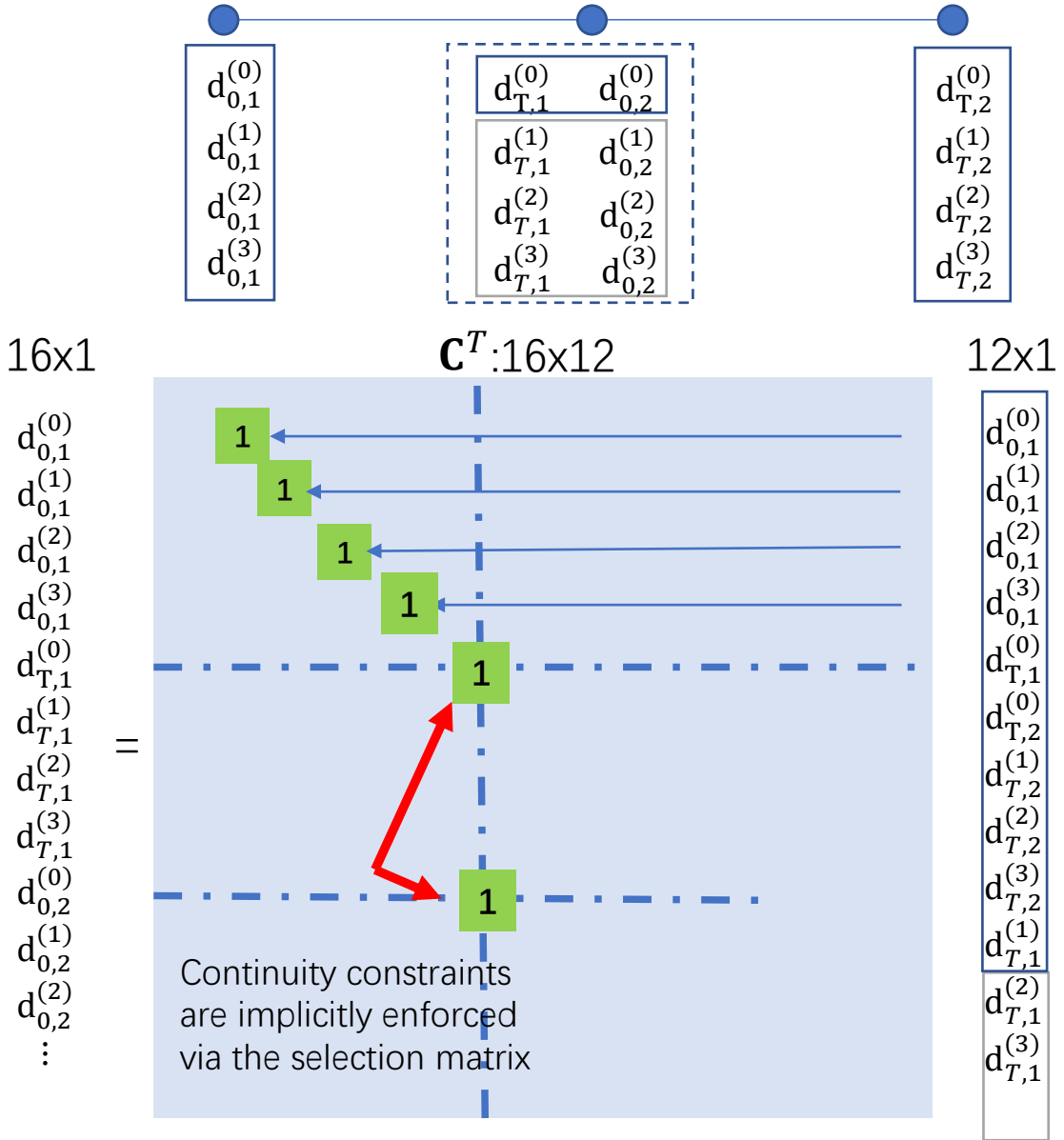
Build the selection matrix



Fixed derivatives: fixed start, goal state, and intermediate positions

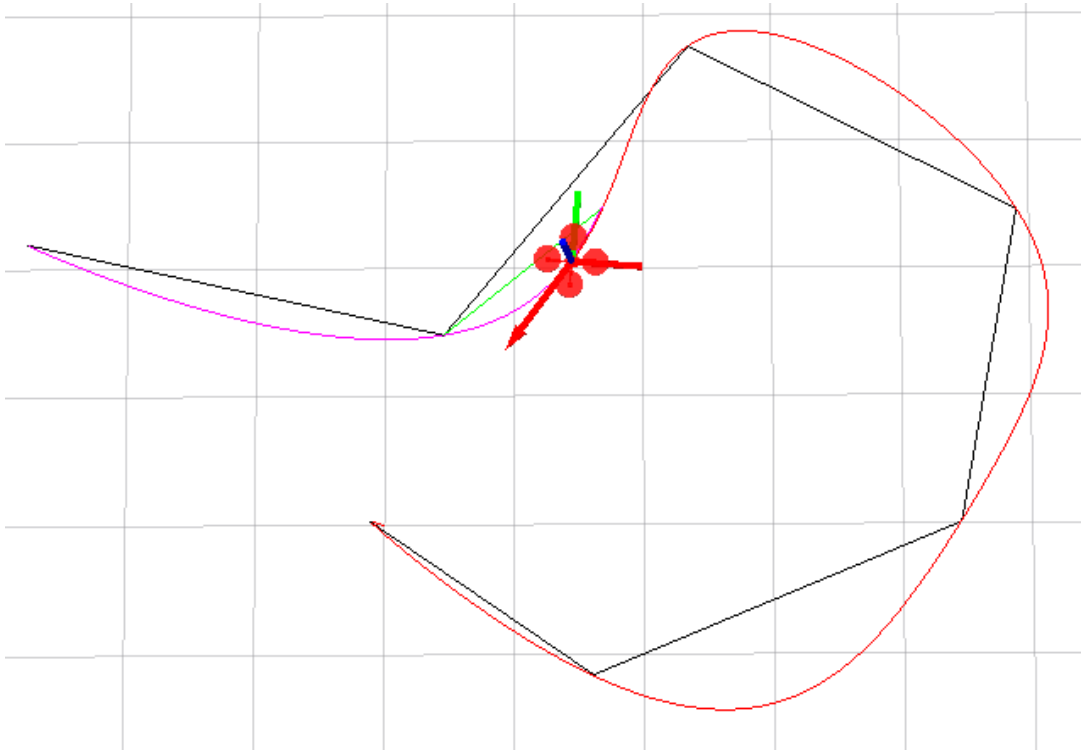
Free derivatives: all derivatives at intermediate connections.

$$\begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_M \end{bmatrix} = \mathbf{C}^T \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}$$



Same results as convex optimization

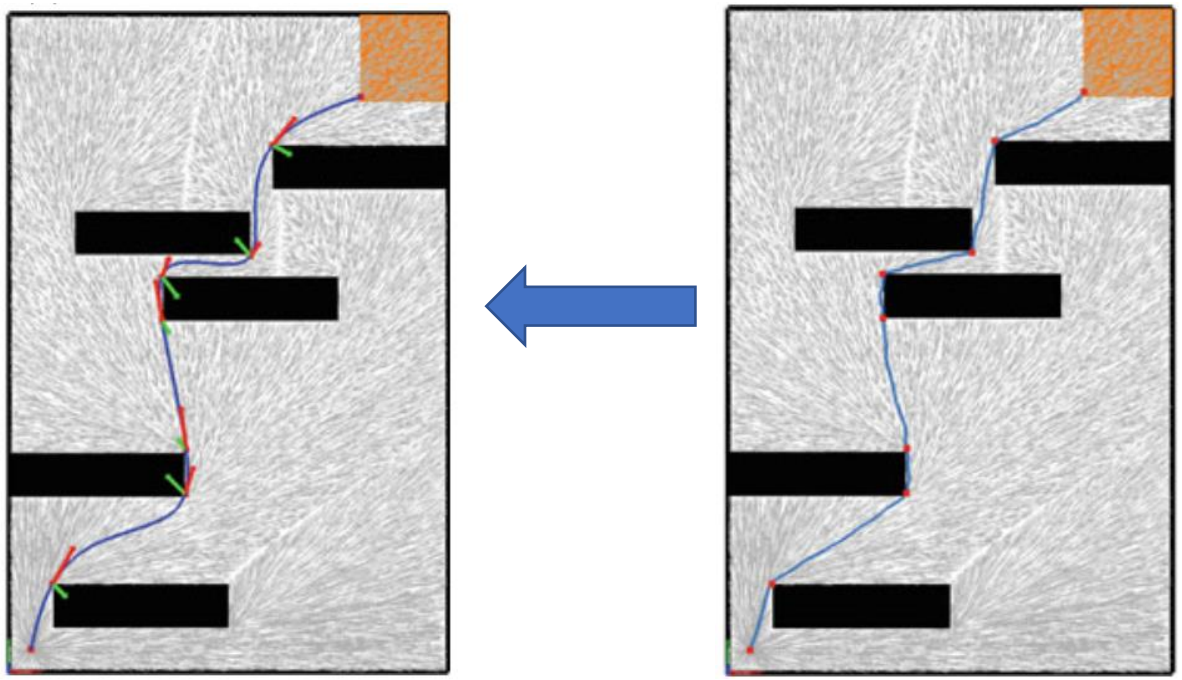
- Final trajectory



Ask: how to get these waypoints?

Hierarchical approach

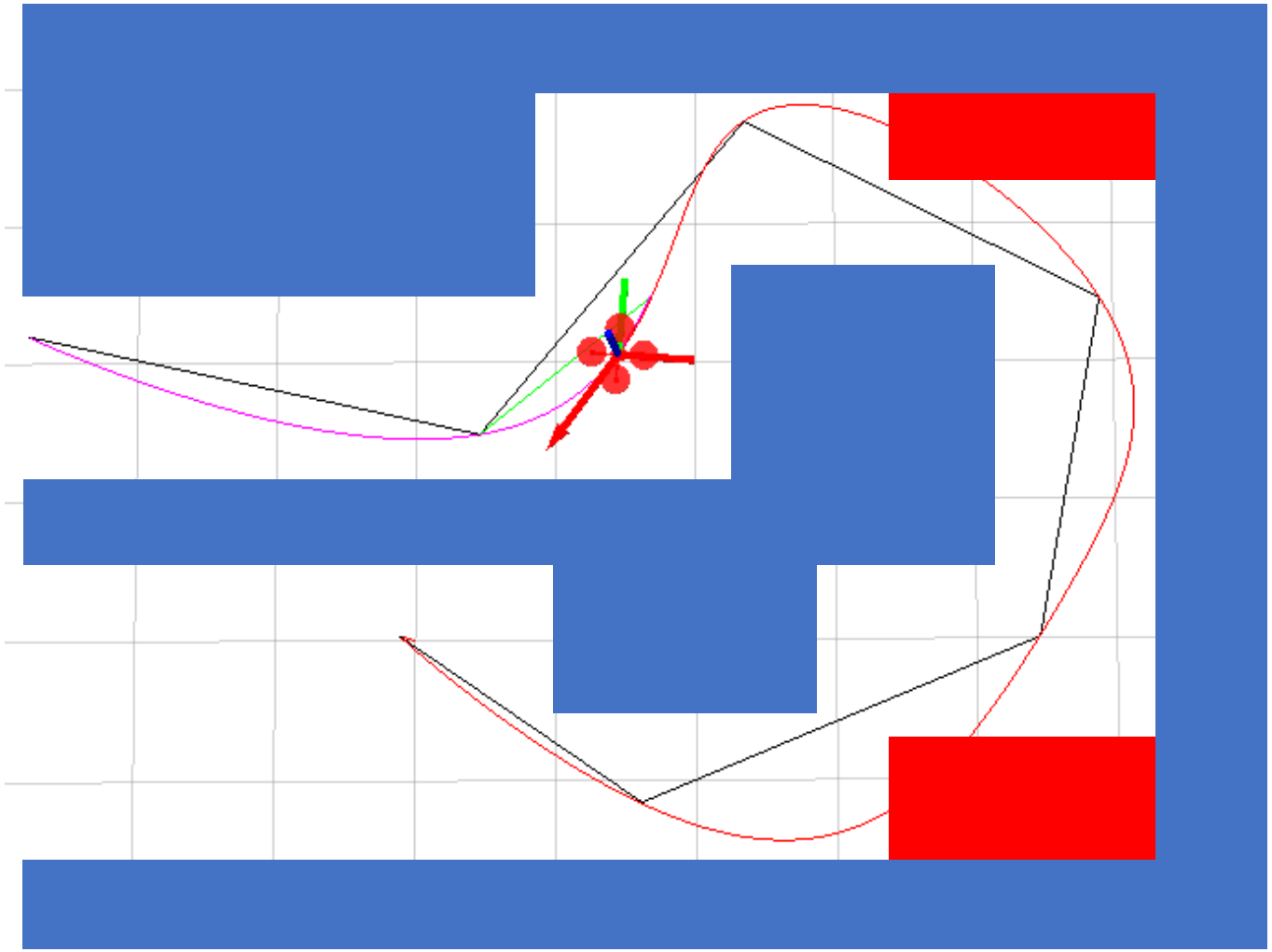
- path planning + trajectory generation 路径规划+轨迹生成
  - Low complexity solution
    - Path planning can be more efficient since it's in a much lower dimension state space.



We already know how to fit the polynomial for given waypoints

Then how to get these collision-free waypoints? → the role of path planning

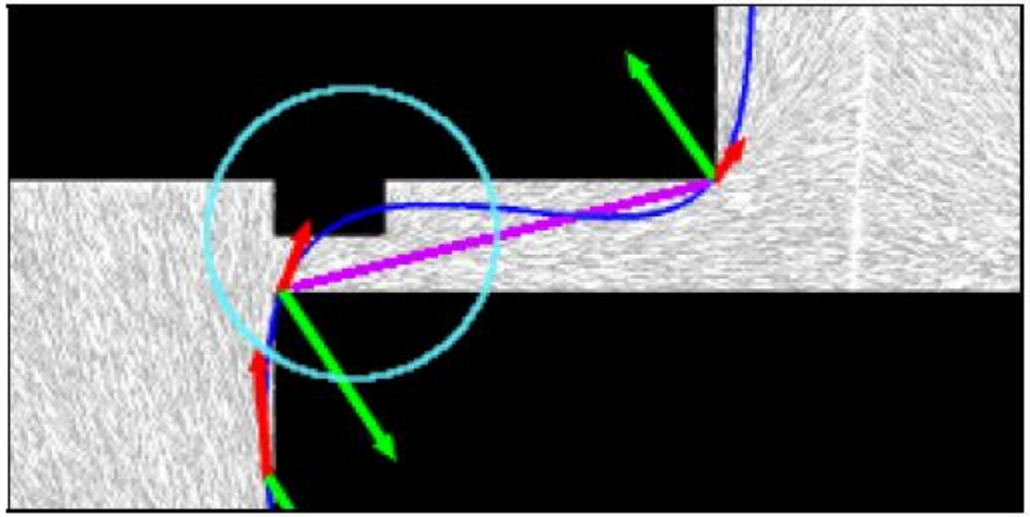
Problem: safety issue

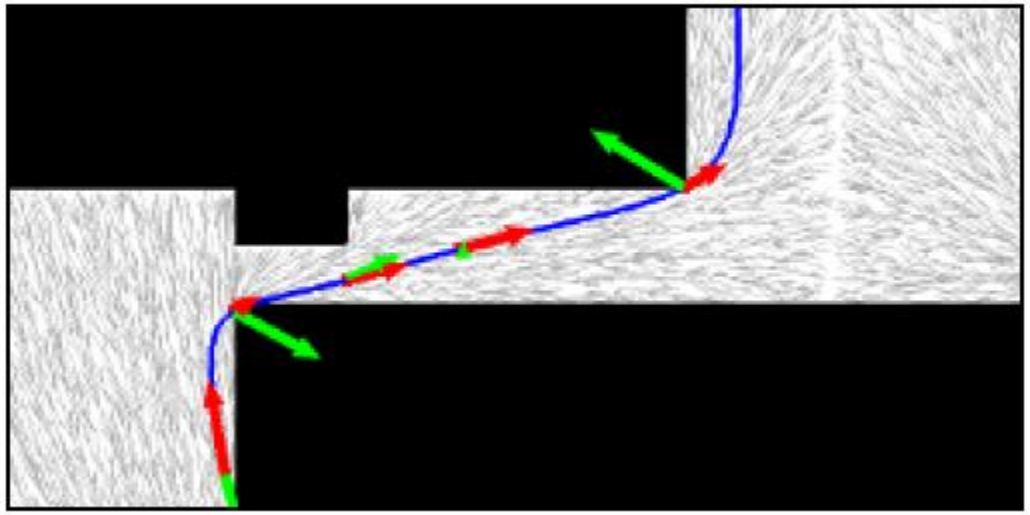


Ask: How to Ensure Collision-Free Trajectories?

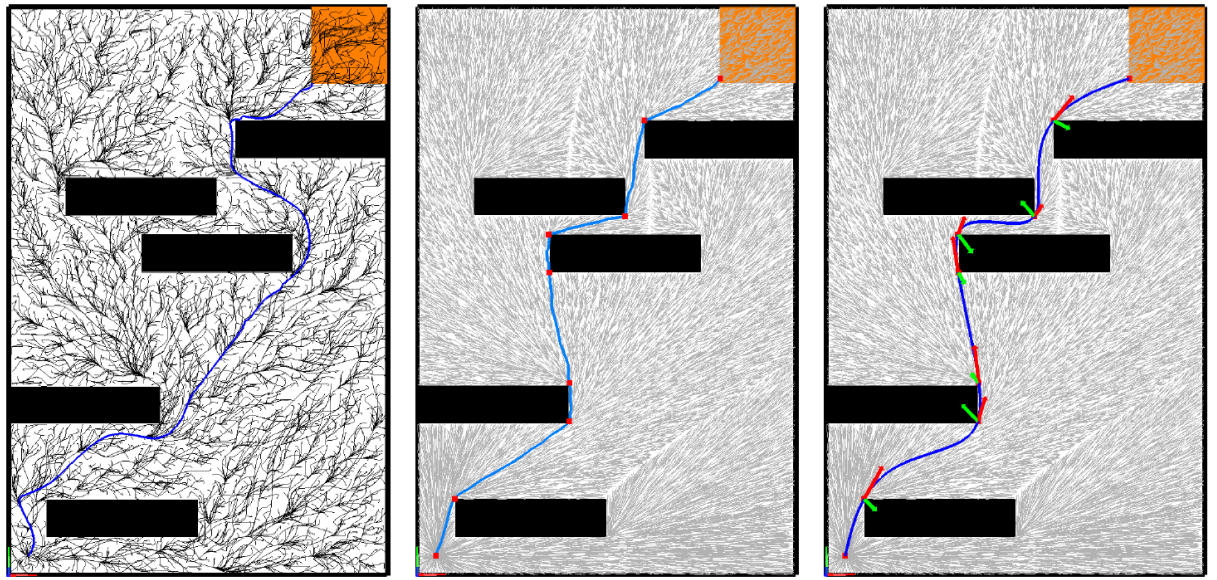
Iterative approach

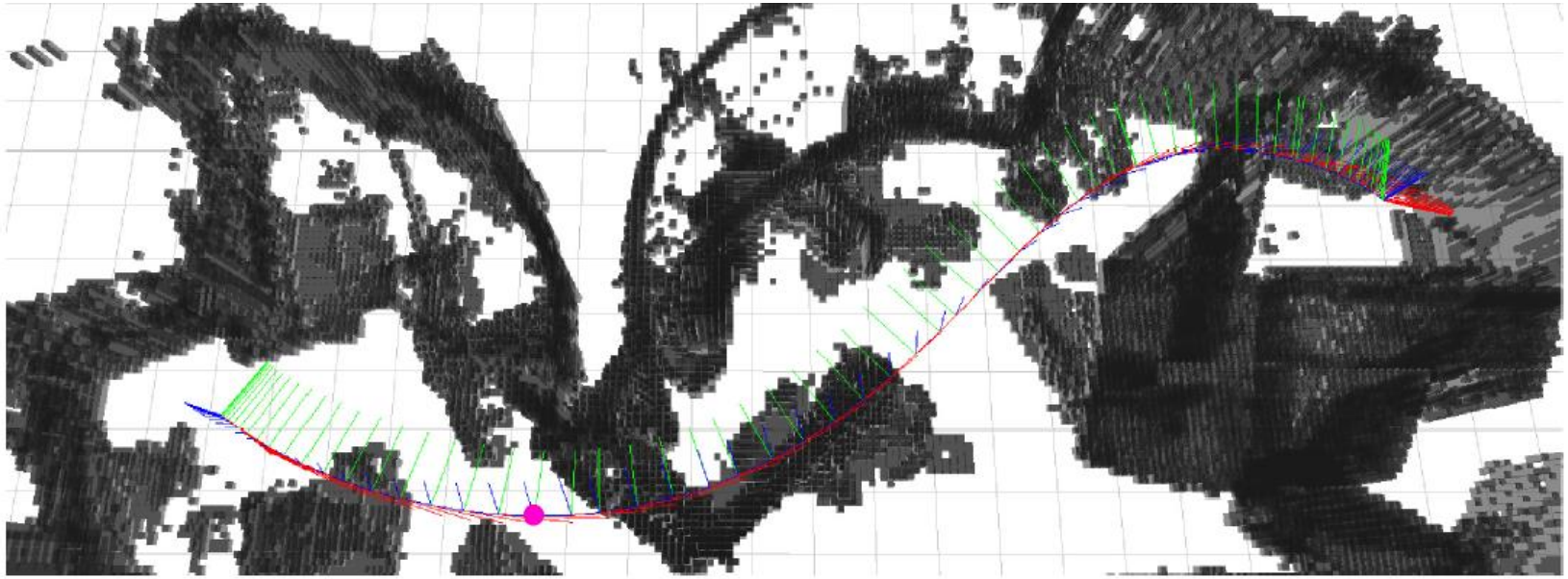
- The initial path is collision-free.
- We can approach the trajectory to the path iteratively.
- This is done by adding intermediate waypoints



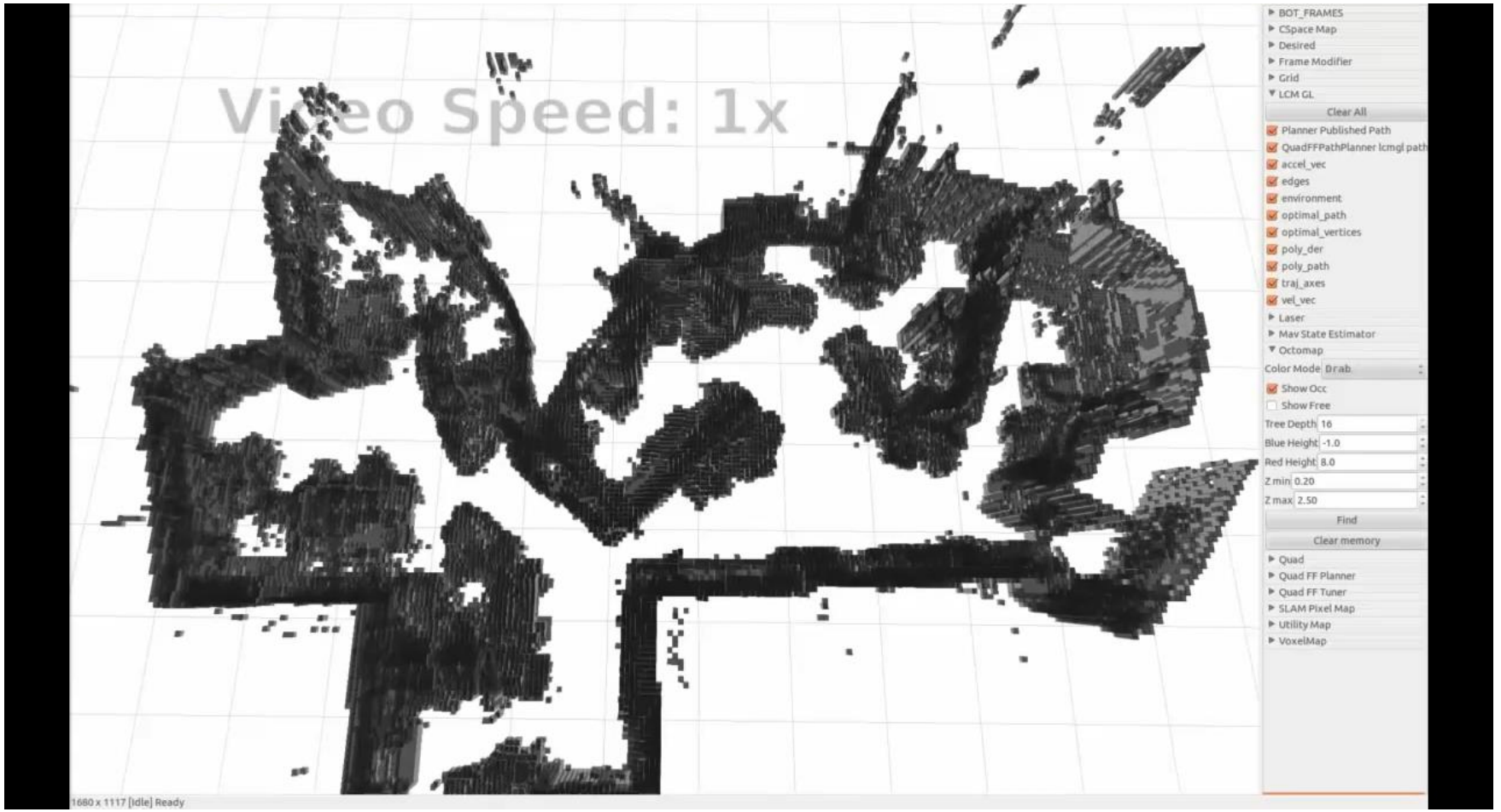


RRT* + minimum snap

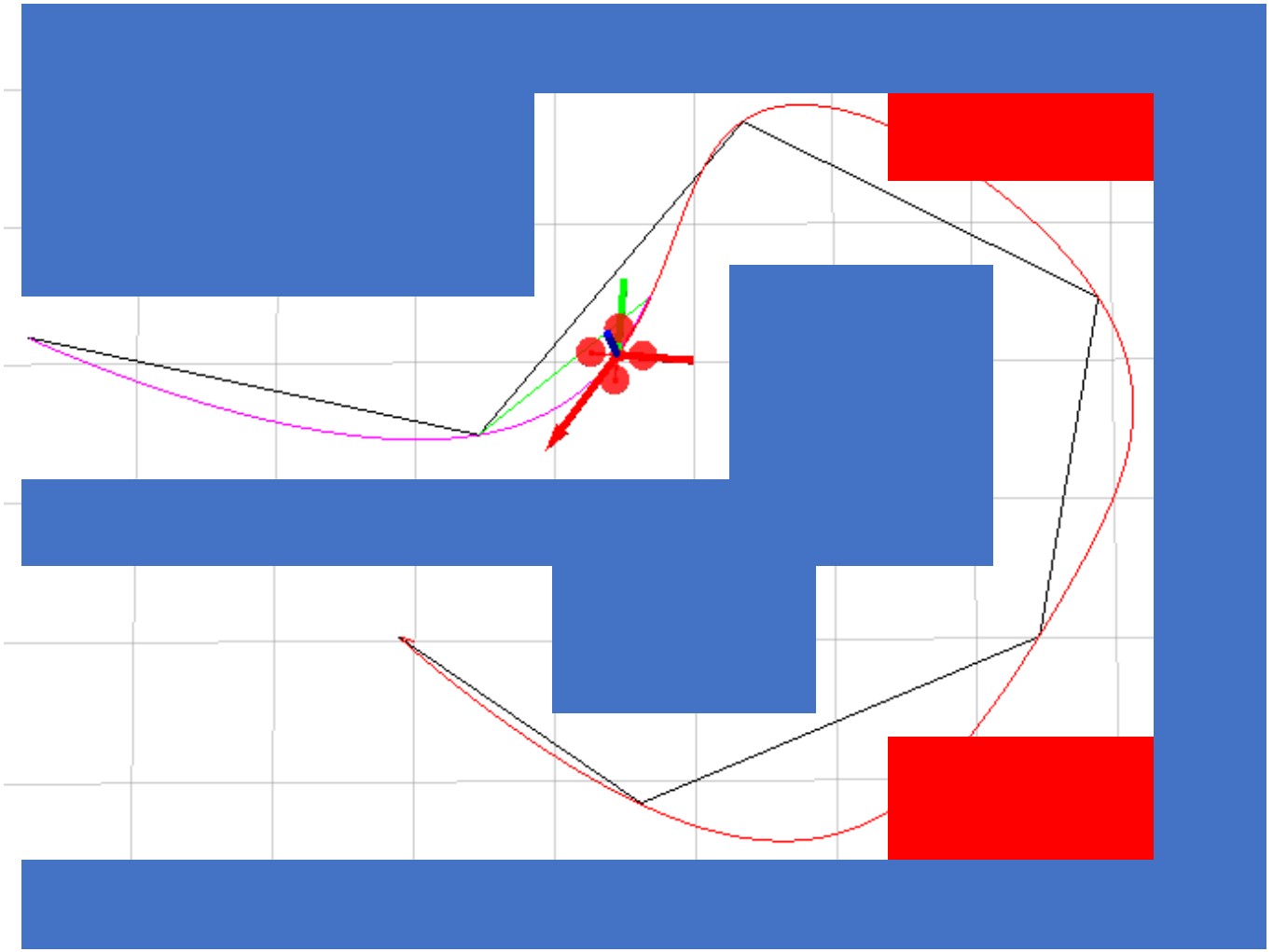




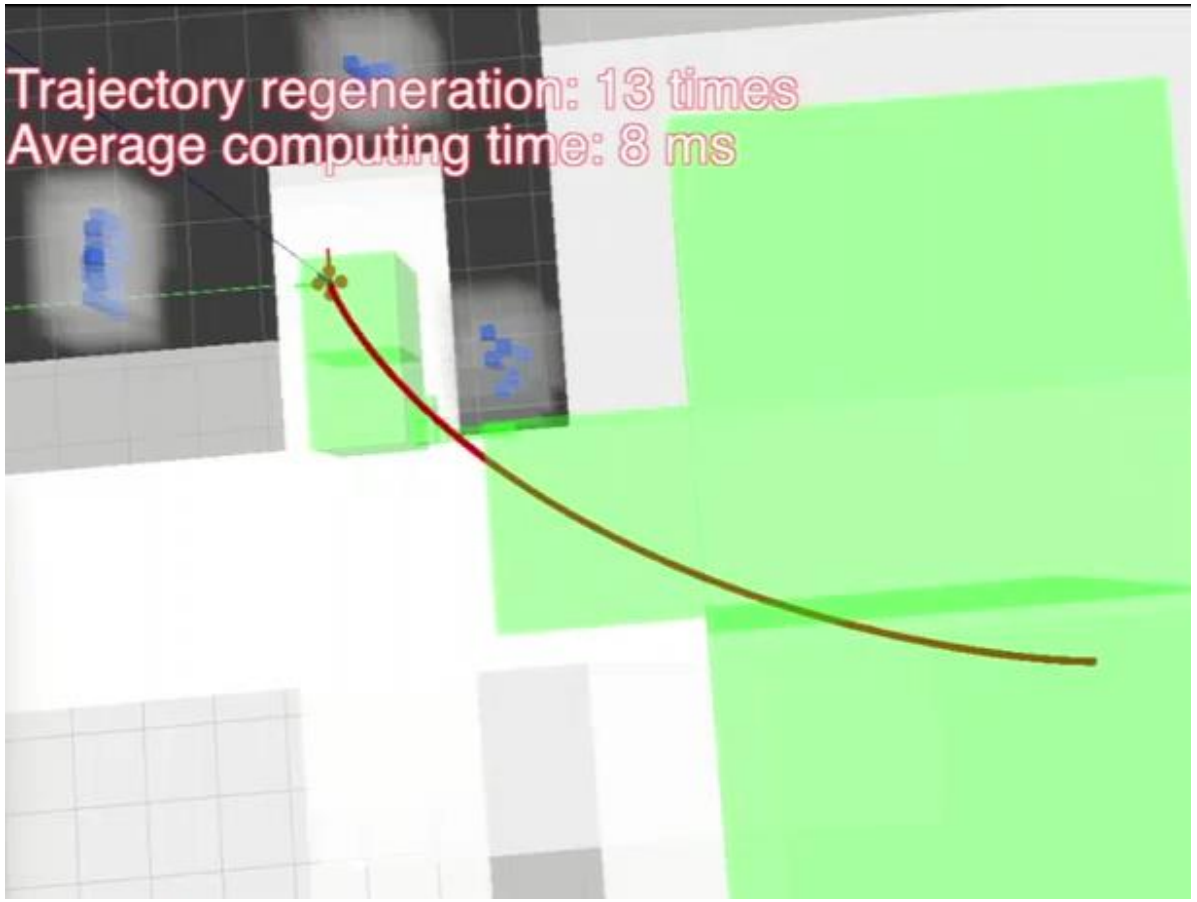
Experimental validation

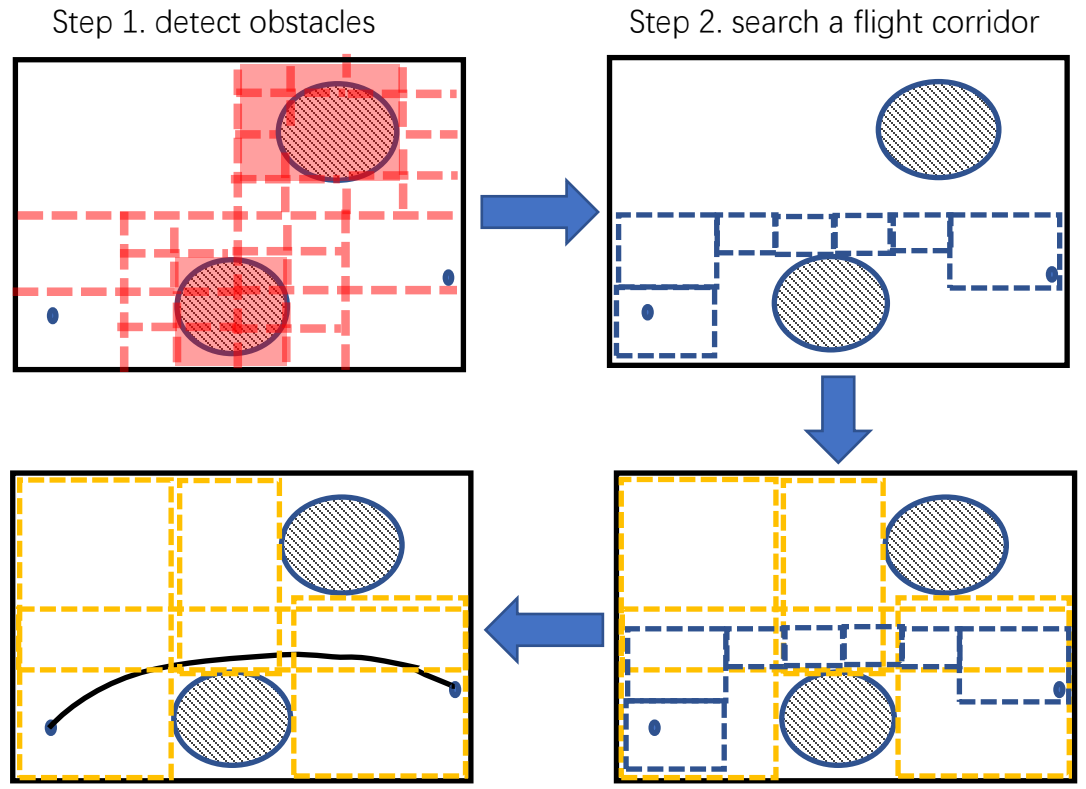


Better solutions?



Smooth Trajectory Generation with Guaranteed Obstacle Avoidance

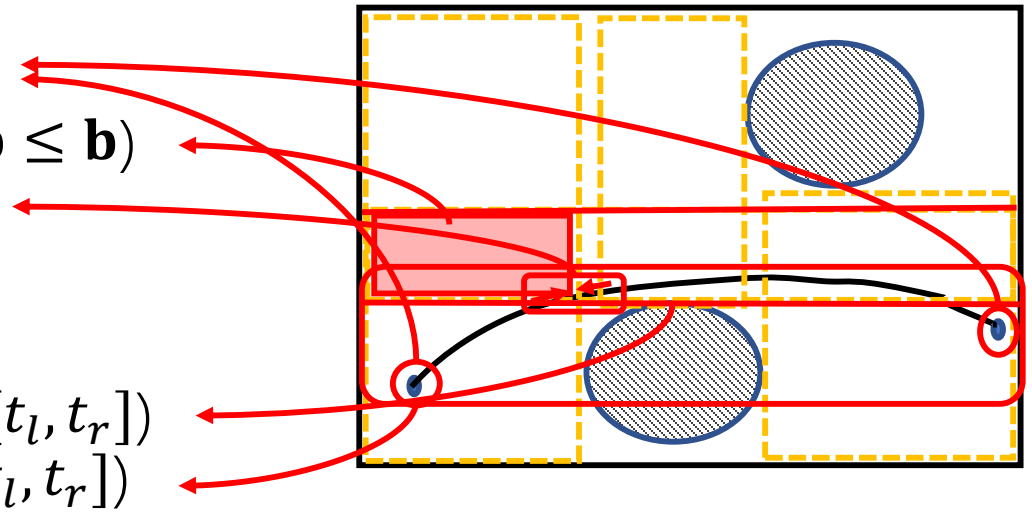




Smooth Trajectory Generation with Guaranteed Obstacle Avoidance

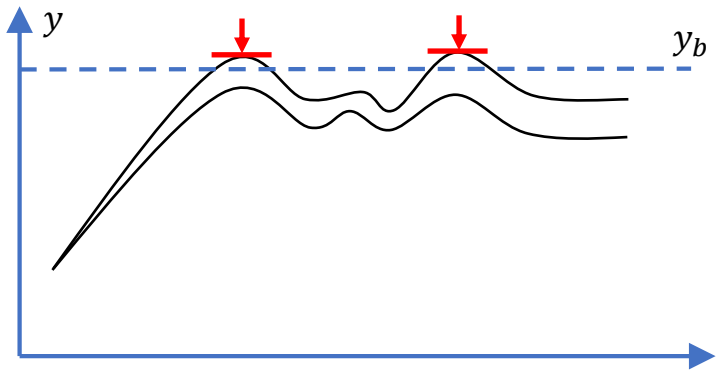
- Instant linear constraints:
  - Start, goal state constraint ($\mathbf{Ap} = \mathbf{b}$)
  - Transition point constraint ($\mathbf{Ap} = \mathbf{b}, \mathbf{Ap} \leq \mathbf{b}$)
  - Continuity constraint($\mathbf{Ap_i} = \mathbf{Ap_{i+1}}$)

- Interval linear constraints:
  - Boundary constraint ($\mathbf{A}(t)\mathbf{p} \leq \mathbf{b}, \forall t \in [t_l, t_r]$)
  - Dynamic constraint ($\mathbf{A}(t)\mathbf{p} \leq \mathbf{b}, \forall t \in [t_l, t_r]$)
    - Velocity constraints
    - Acceleration constraints



How to make constraints globally activated

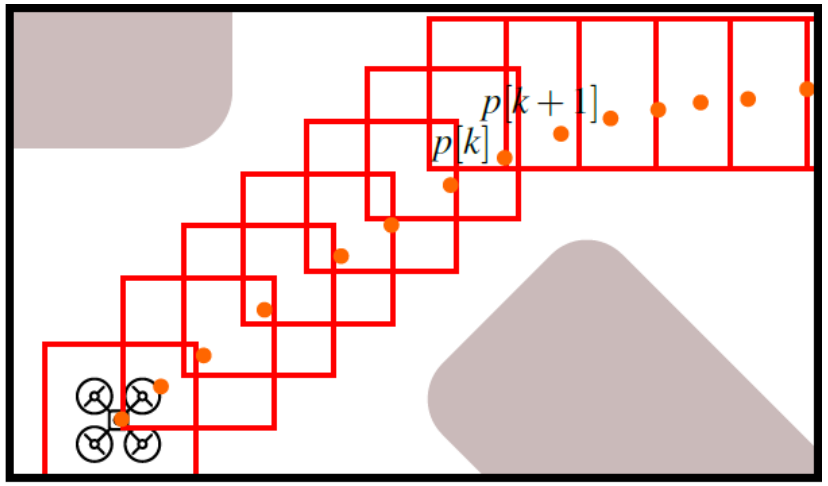
- Iteratively check extremum and add extra constraints.



- Iterative solving is time consuming.
- If strictly no feasible solution meets all constraints. We have to run 10 iterations to determine the status of the solution ?

Online generation of collision-free trajectories for quadrotor flight in unknown cluttered environments, J. Chen, **ICRA** 2016

- Adding numerous constraints at discrete time ticks.



- Always generates over-conservative trajectories.
- Too many constraints, the computational burden is high.

A hybrid method for online trajectory planning of mobile robots in cluttered environments, L Campos-Macías, **RAL** 2017

Solution 1

Solution 2

Implementation Details

Convex Solvers

Solve a convex trajectory generation program

- You target is to formulate a trajectory generation problem into **Disciplined convex optimization programs** in P.41.
- Many off-the-shelf can help you solve them.
- Choose a proper solver according to your requirement.

CVX

http://cvxr.com/cvx/

Matlab wrapper. Let you write down the convex program like mathematical equations, then call other solvers to solve the problem.

Mosek

https://www.mosek.com/

Very robust convex solvers, can solve almost all kinds of convex programs. Can apply free academic license. Only library available (x86).

OOQP

http://pages.cs.wisc.edu/~swright/ooqp/

Very fast, robust QP solver. Open sourced.

GLPK

https://www.gnu.org/software/glpk/

Very fast, robust LP solver. Open sourced.

Numerical Stability

Normalization

对时间变量或空间变量做归一化，防止数值解不稳定（过大、过小）

- Time normalization
  - Some very small time durations may break the generation entirely.
  - Scale short time durations to a normal number (1.0).
  - Or adding scale factor to all piece of the curve.

$$f(t) = \begin{cases} \sum_{i=0}^{N} p_{1,i} \left(\frac{t - T_0}{T_1 - T_0} \right)^i & T_0 \le t \le T_1 \\ \sum_{i=0}^{N} p_{2,i} \left(\frac{t - T_1}{T_2 - T_1} \right)^i & T_1 \le t \le T_2 \\ \vdots & \vdots \\ \sum_{i=0}^{N} p_{M,i} \left(\frac{t - T_{M-1}}{T_M - T_{M-1}} \right)^i & T_{M-1} \le t \le T_M \end{cases}$$

Use relative timeline!

- Problem scale (spatial) normalization
  - If the problem is underlying for large-scale scene.
  - Such as waypoints with $x = 100.0\,m$
  - Consider solve a tiny problem (a sandbox), and re-scale the solution back.

These two operations highly increase the numerical stability in practice.

Other engineering stuff

1. Solve 3 axis independently, or together?

$$\min \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}$$

$$\text{s.t. } \mathbf{A}_{\text{eq}} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix} = \mathbf{d}_{eq}$$

- Typically, solving 3 small-scaled problems is better (stable, faster) than 1 large-scaled problem.
- Coupled generation may add different weighting into 3 axis.

2. Is closed-form solution always better?

- When matrix operation is expansive, the numerical convex solver is much more robust.
- Modern solver (Mosek) has pretty good stability.

Efficiency comparison

Benchmark Problem: 3-Segment Joint Optimization	
Method	Solution Time (ms)
MATLAB `quadprog.m`	9.5
MATLAB Constrained	1.7
MATLAB Unconstrained (Dense)	2.7
C++/Eigen Constrained	0.18
C++/Eigen Unconstrained (Dense)	0.34

3. Is polynomial can do anything?

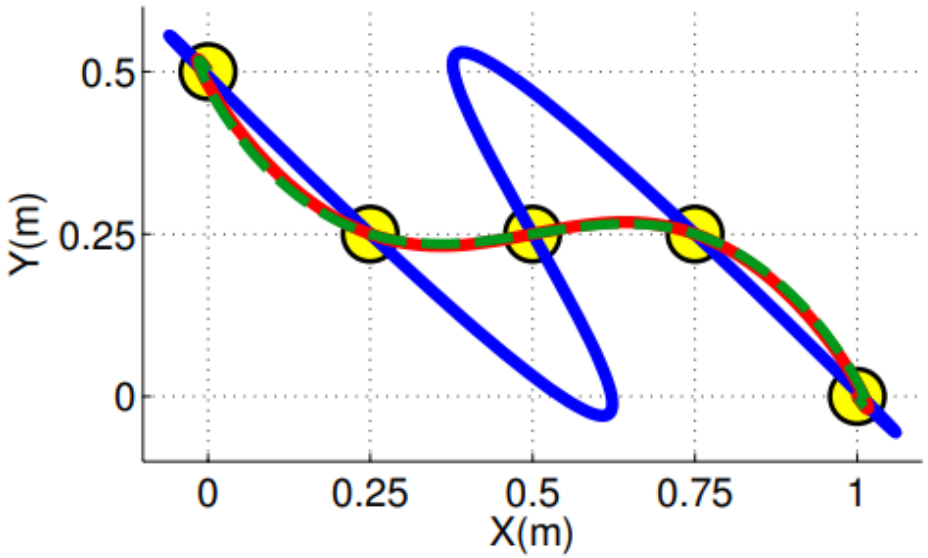
- Almost, but not all.
- One can prove that the polynomial function (with several order) is the best solution for minimize single squared control input. But when things come to it's not (out of this course's scope).

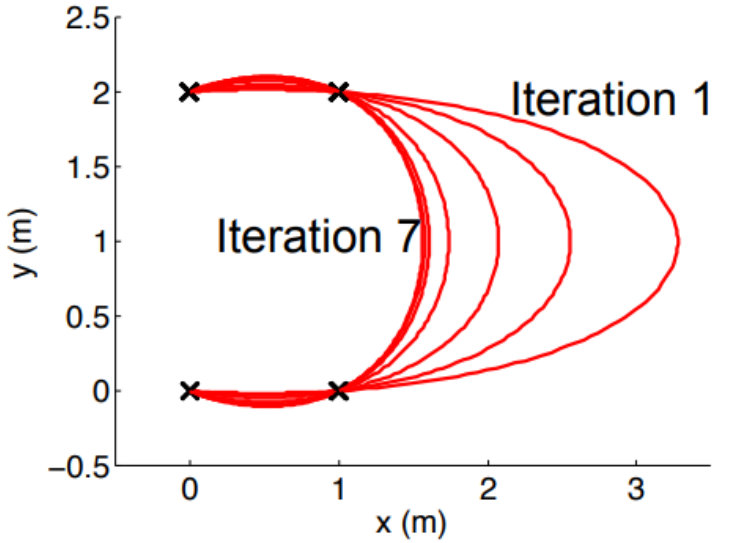
$$J = \int_0^T \rho_1 \cdot jerk^2(t) + \rho_1 \cdot snap^2(t) dt.$$

Time Allocation

Problem Definition

- Piecewise trajectory depends on a piecewise time allocation.
- Time allocation significantly affect the final trajectory.
- How to get a proper time allocation?





Naïve solution

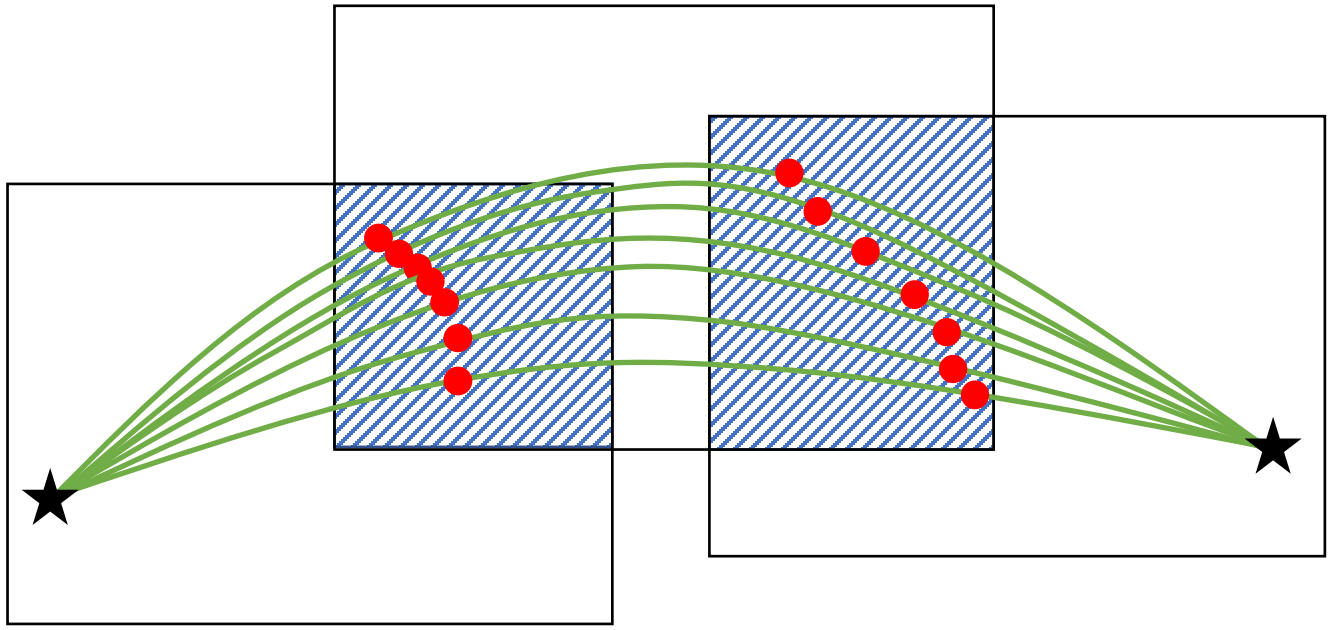
- Use “trapezoidal velocity” time profile to get time durations for each pieces.
  - Assume in every piece, accelerate to max. velocity -> de-accelerate to 0.
  - Accelerate + max. velocity + de-accelerate.
- Use expected average velocity to get time durations for each pieces.

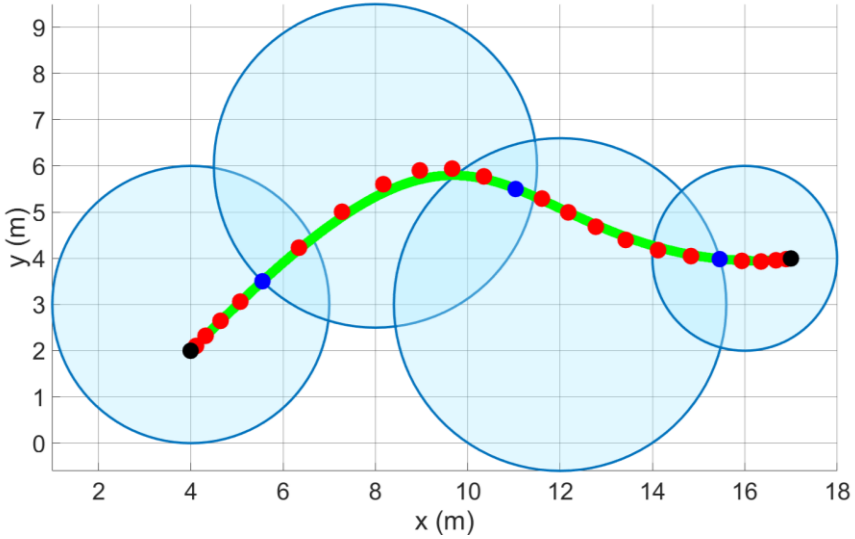
- Far away from optimal.
- Only get conservative time profile.
- Has no reaction to environmental styles

Online generation of collision-free trajectories for quadrotor flight in unknown cluttered environments, J. Chen, **ICRA** 2016
Fast, dynamic trajectory planning for a dynamically stable mobile robot, M. Shomin, **IROS** 2014

Naïve solution

- Looks stupid for one piece.
- Works fine in corridor based trajectory generation.
- Overlapping regions in corridor provides large auto-adjustment space.



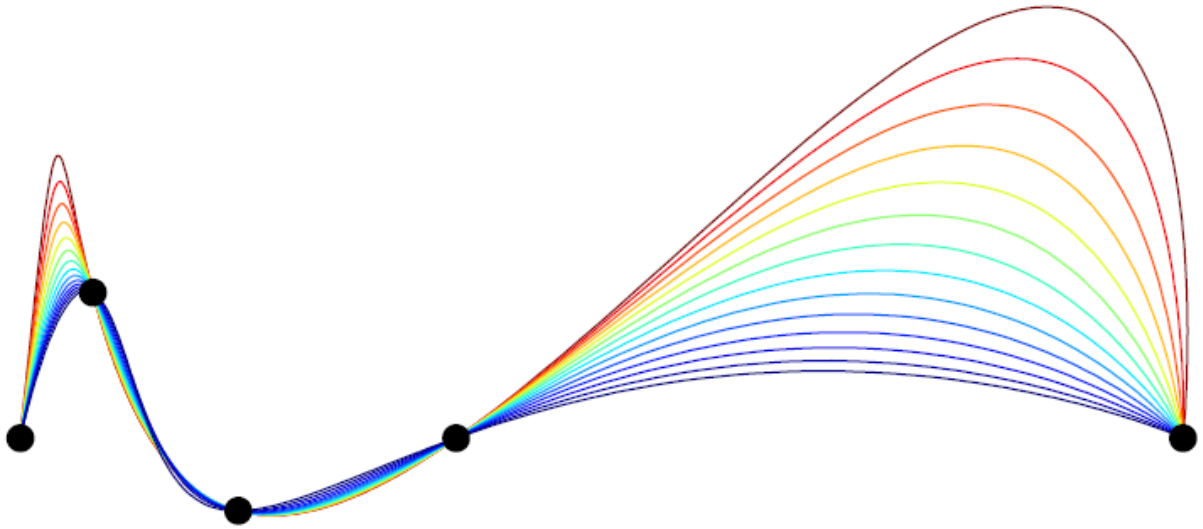


Iterative numerical solution

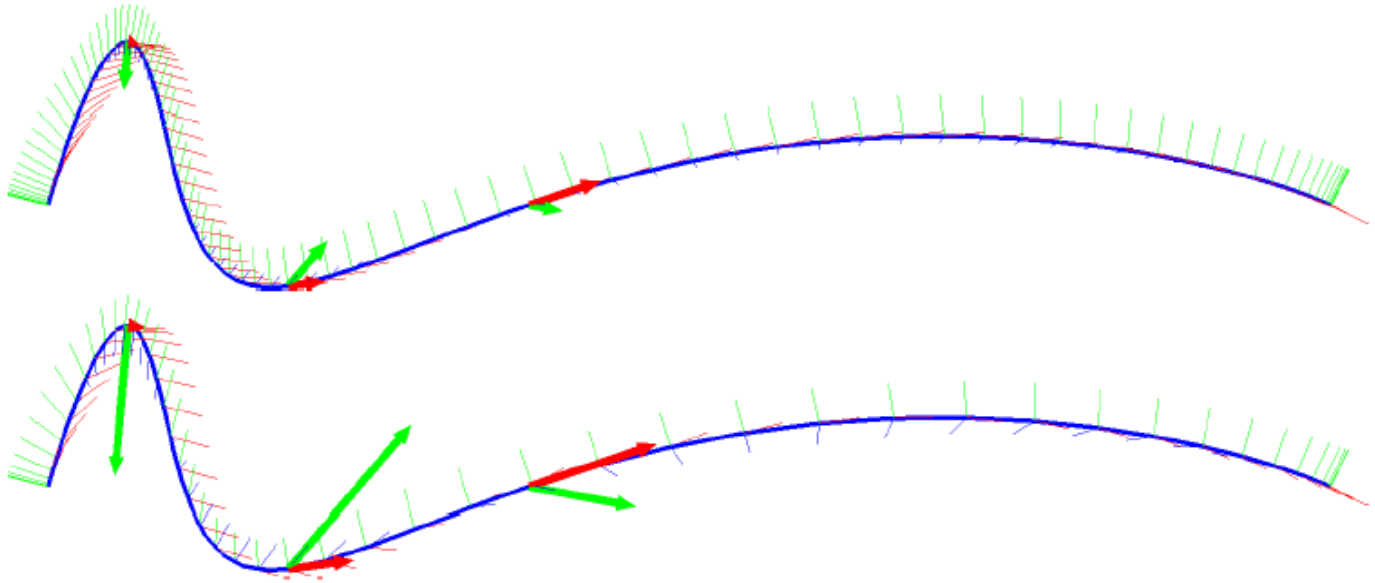
$$J_T = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} Q_1(T_1) & & \\ & \ddots & \\ & & Q_M(T_M) \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T + k_T \sum_{i=1}^{M} T_i$$

Penalize time duration in the overall cost

- Minimize this objective function **J**
- Get the gradient to T numerically



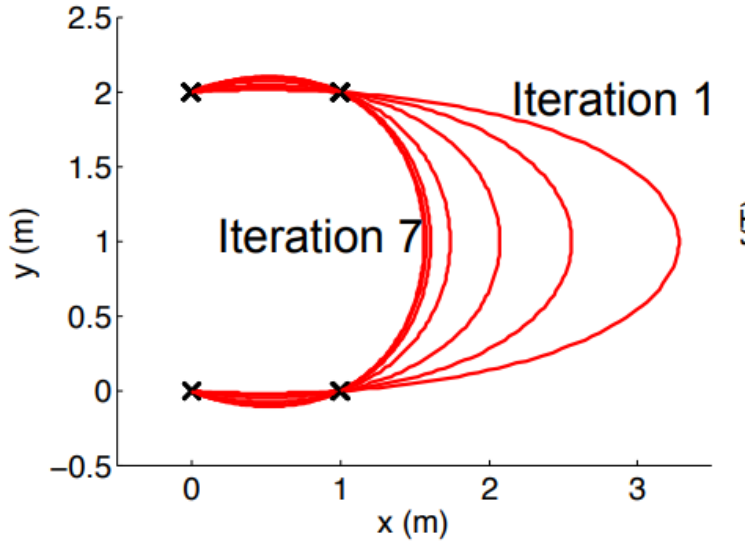
Iterative numerical solution



- Varying segment time ratio
- Varying total time

Dynamic limit?

- Unconstrained, find the best time allocation, fix the ratio.
- scale the total trajectory time preserving the optimal ratio.
- until a constraint becomes active.



- Varying segment time ration
- Fixed total time

Homework

Implement minimum snap in Matlab or C++/ROS

Homework 1.1: In matlab, use the ‘quadprog’ QP solver, write down a minimum snap trajectory generator

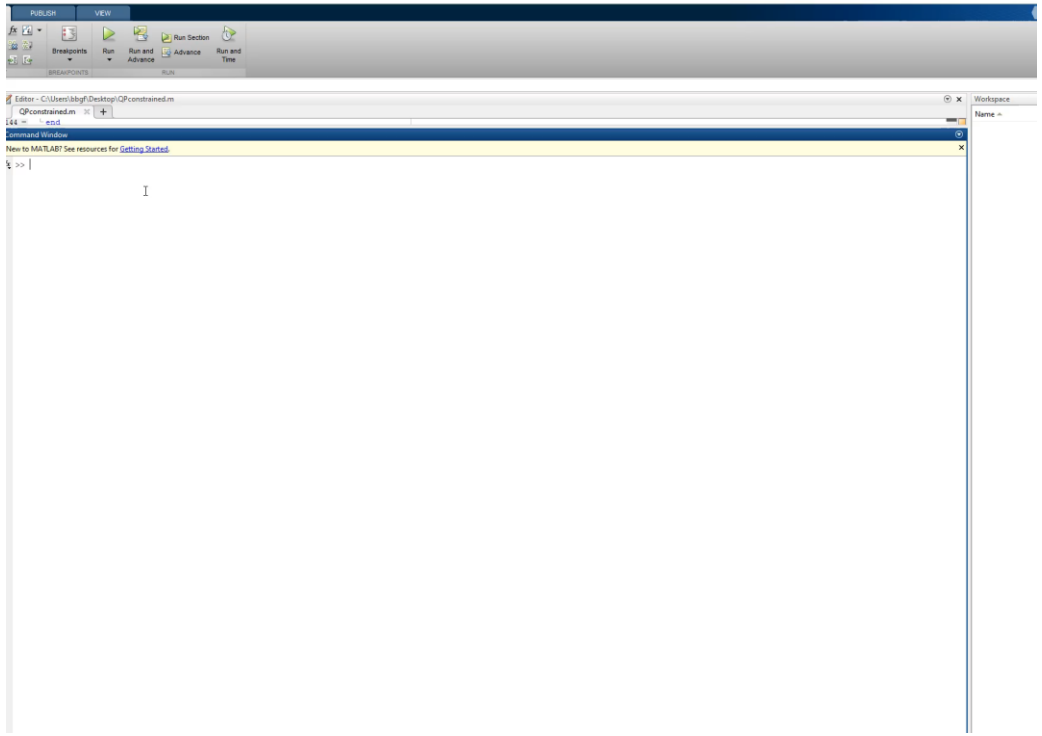
Homework 1.2: In matlab, generate minimum snap trajectory based on the closed form solution

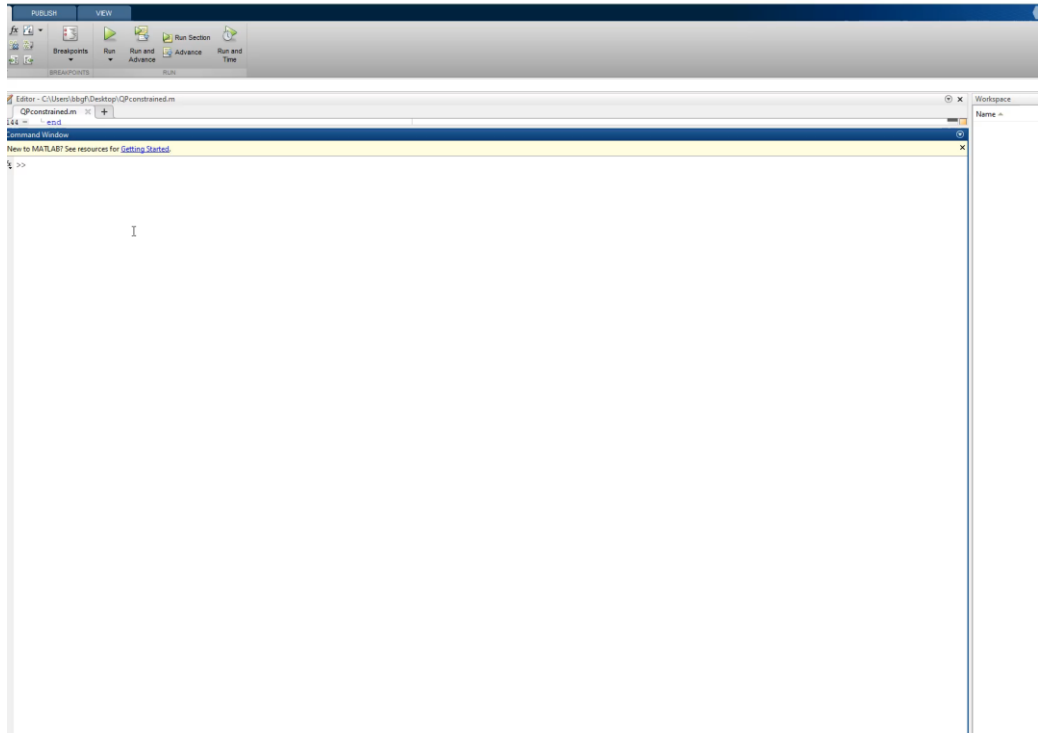
Homework 2.1: In C++/ROS, use the OOQP solver, write down a minimum snap trajectory generator

Homework 2.2: In C++/ROS, use Eigen, generate minimum snap trajectory based on the closed form solution

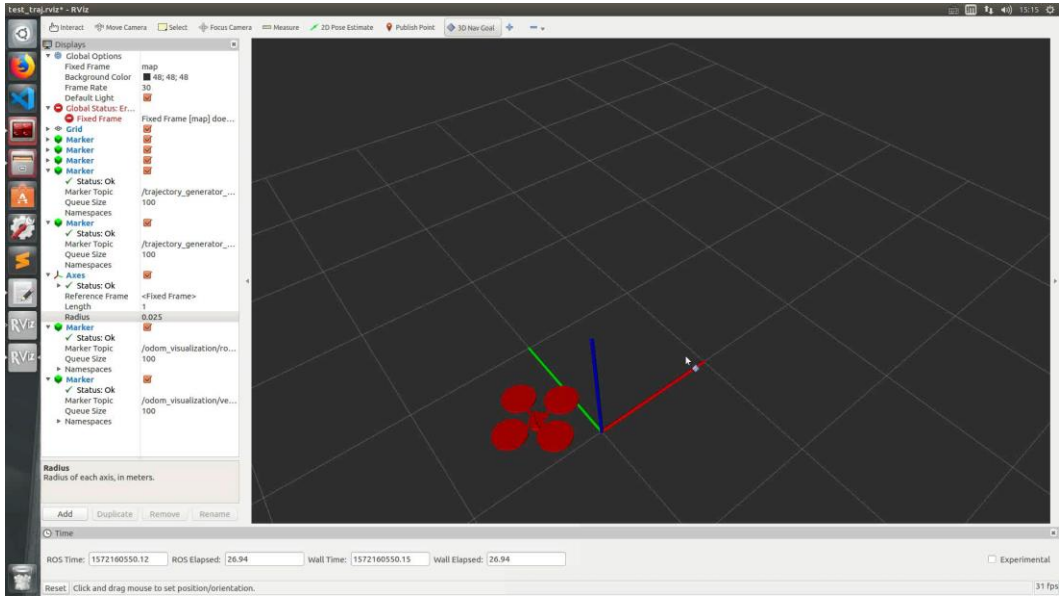
- Choose one homework. I suggest the first one.
- I highly suggest you finish all these options, since they are more than fundamental in planning.

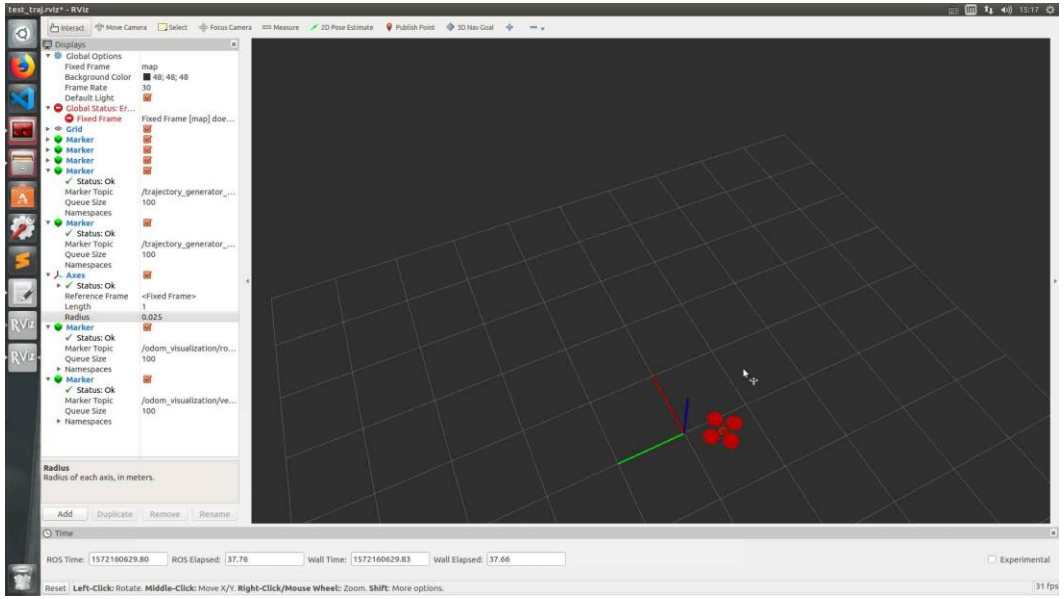
Matlab homework expectation

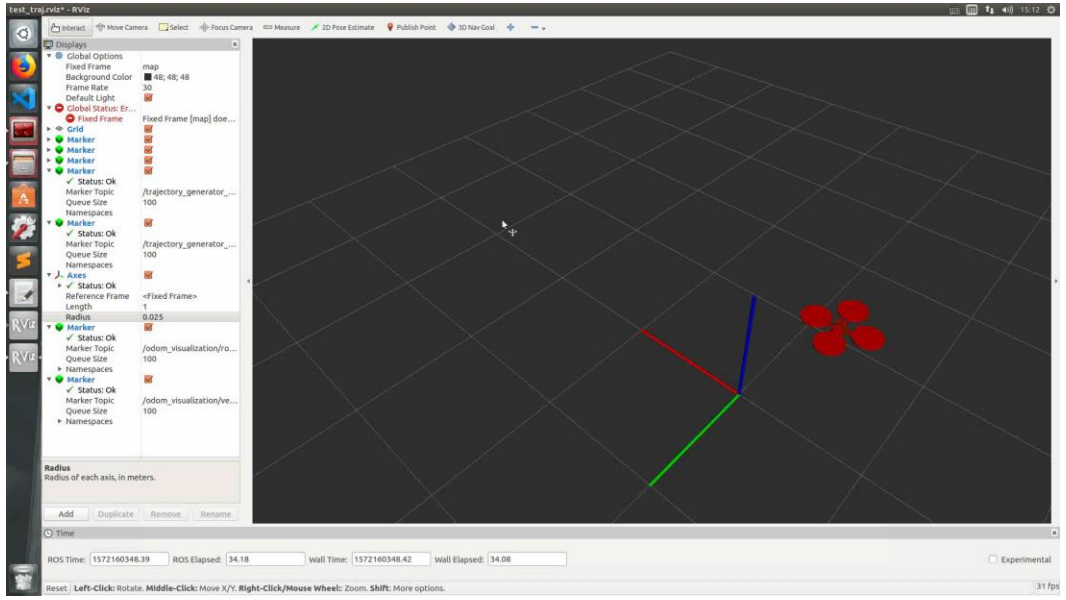




C++/ROS homework expectation







Thanks for Listening!