ROB 501 HW6

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1

$$
\begin{aligned}
y_1 &= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, y_2 = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, y_3 = \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} \\
v_1 &= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
\|v_1\| &= \sqrt{6} \\
v_2 &= y_2 - \frac{\langle v_1, y_2 \rangle}{\|v_1\|^2} \cdot v_1 \\
&= \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} - \frac{3}{6} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \frac{7}{2} \\ -\frac{3}{2} \end{bmatrix} \\
\|v_2\| &= \sqrt{\frac{31}{2}} \\
v_3 &= y_3 - \frac{\langle v_1, y_3 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_2, y_3 \rangle}{\|v_2\|^2} v_2 \\
&= \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \frac{19}{31} \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} 100/31 \\ 40/31 \\ 160/31 \end{bmatrix}
\end{aligned}
$$

2

2.1

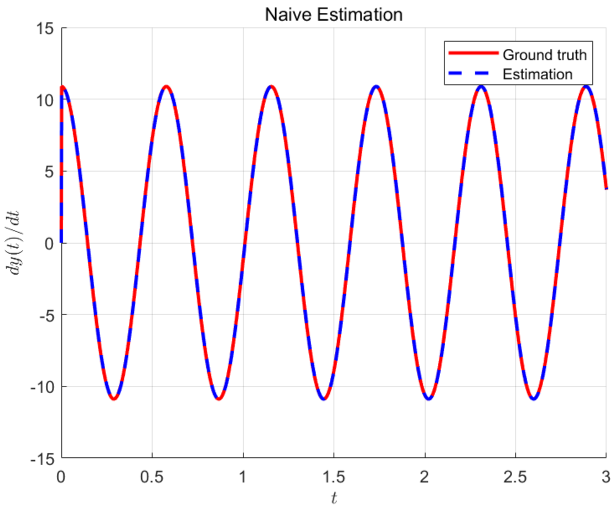

Figure 1: Naive estimation of the derivative of $y(t)$

2.2

The function I used is $\hat{y}(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$. And the window size is 7.

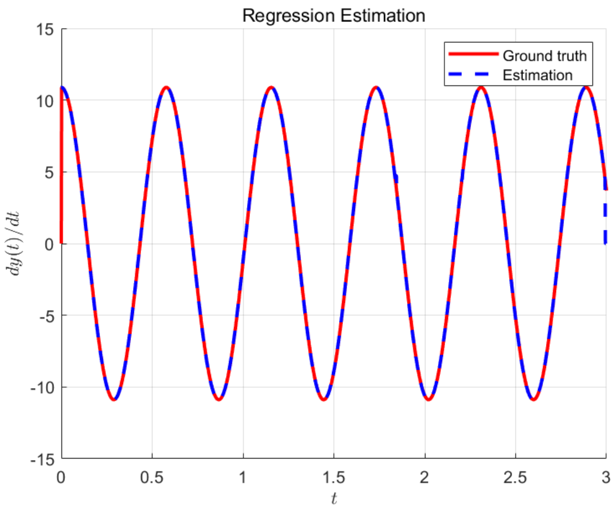

Figure 2: Regression estimation of the derivative of $y(t)$

3

3.1

The function I used is $\hat{y}(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$. And the window size is 14.

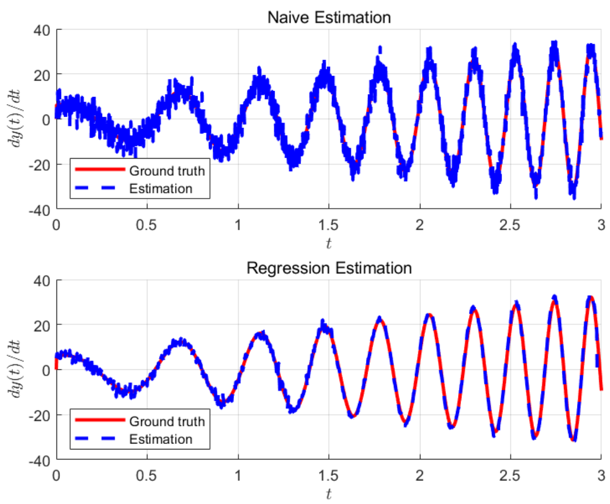

Figure 3: Estimations of the derivative of $y(t)$

3.2

The Root mean Square Error(RMSE) for the naive estimation is 3.5854, and the RMSE for the regression estimation is 1.3508.

4

Given $\mathcal{X} = \mathbb{R}^{2,2}$, $< A, B >= tr(A^T B)$ and $M = span\,\{y_1, y_2\}$

$$y_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix},\ y_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solve $\hat{x} = arg\min_{y\in M} \|x - y\|$ when $x = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

Apply the Normal Equations,

$$\begin{aligned}
&G^\top \alpha = \beta \\
&\quad G = \left[\begin{array}{cc} \langle y^1, y^1 \rangle, & \langle y^1, y^2 \rangle \\ \langle y^2, y^1 \rangle, & \langle y^2, y^2 \rangle \end{array}\right] = \left[\begin{array}{cc} 5 & 3 \\ 3 & 4 \end{array}\right] \\
&\quad \beta = \left[\begin{array}{c} \langle x, y^1 \rangle \\ \langle x, y^2 \rangle \end{array}\right] = \left[\begin{array}{c} 4 \\ 1 \end{array}\right] \\
&\Rightarrow\ \alpha = \left[\begin{array}{cc} \langle y^1, y^1 \rangle & \langle y^2, y^1 \rangle \\ \langle y^1, y^2 \rangle & \langle y^2, y^2 \rangle \end{array}\right]^{-1} \left[\begin{array}{c} \langle x, y^1 \rangle \\ \langle x, y^2 \rangle \end{array}\right] \\
&\qquad = \frac{1}{11}\left[\begin{array}{cc} 4 & -3 \\ -3 & 5 \end{array}\right]\left[\begin{array}{c} 4 \\ 2 \end{array}\right] \\
&\qquad = \begin{bmatrix} 13/11 \\ -7/11 \end{bmatrix} \\
&\Rightarrow\ \hat{x} = \alpha_1 y^1 + \alpha_2 y^2 = \begin{bmatrix} 6/11 & -7/11 \\ 19/11 & -7/11 \end{bmatrix}
\end{aligned}$$

5

Given $(x, \mathbb{R}, \|\cdot\|)$ is strictly normed. M is a subspace of x, Show that if exists $m^* \in M$,

$$\text{s.t. } \|x - m^*\| = d(x, M) := \inf_{y\in M} \|x - y\|$$

then m is unique.

Proof. Suppose $m_1, m_2 \in M$ satisfy $\|x - m_i\| = d(x, M), i = 1, 2$.
Let $\gamma = d(x, M)$, and note that $(m_1 + m_2)/2 \in M$.

$$\begin{aligned}
\gamma &= \inf_{y \in M} \|x - y\| \\
&\leq \left\| x - \frac{m_1 + m_2}{2} \right\| \\
&= \left\| \frac{x - m_1}{2} + \frac{x - m_2}{2} \right\| \\
&\leq \frac{1}{2} \|x - m_1\| + \frac{1}{2} \|x - m_2\| \\
&= \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma
\end{aligned}$$

implies $\left\| \frac{x-m_1}{2} + \frac{x-m_2}{2} \right\| = \frac{1}{2} \|x - m_1\| + \frac{1}{2} \|x - m_2\|$
Since the norm space is strictly normed,

$$\frac{x - m_1}{2} = \alpha \frac{x - m_2}{2} \Rightarrow \left\| \frac{x - m_1}{2} \right\| = \alpha \left\| \frac{x - m_2}{2} \right\|.$$

Hence $\alpha = 1$ and $\frac{x-m_1}{2} = \frac{x-m_2}{2} \Rightarrow m_1 = m_2$ □

6

Given $x \in \mathbb{R}^2,\ x = [x_1, x_2]^T$.

(a) $\|x\|_1 = |x_1| + |x_2|$

$$\begin{aligned}
\text{Let } x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}&,\ y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\|x + y\|_1 &= 2 \\
\|x\|_1 + \|y\|_1 &= 1 + 1 = 2
\end{aligned}$$

Since $\|x + y\| = \|x\| + \|y\|$, but x and y is not related by a non-negative factor.

(b) $\|x\|_\infty = \max\{|x_1|, |x_2|\}$

$$\begin{aligned}
\text{Let } x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}&,\ y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\|x + y\|_\infty &= 1 \\
\|x\|_\infty + \|y\|_\infty &= 0 + 1 = 1
\end{aligned}$$

Since $\|x + y\| = \|x\| + \|y\|$, but x and y is not related by a non-negative factor.

7

```
function result = MatrixInversionLemma(A_inv, B, C, D)
    result = A_inv - A_inv * B * inv(inv(C) + D * A_inv * B) * D * A_inv;
end
```

8

Given

$$x = \{f \mid f : \mathbb{R} \to \mathbb{R}\}, F = \mathbb{R}$$

$$\text{Define } \langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$$

$$M = \text{span}\left\{1, t, \frac{1}{2}\left(3t^2 - 1\right)\right\}, \quad x = e^t$$

Find $\hat{x} = \arg\min_{y \in M} \|x - y\|$

$$\begin{aligned}
G^T\alpha = \beta \text{ where } G_{ij} &= \left\langle y^i, y^j \right\rangle, \beta_i = \left\langle x, y^i \right\rangle \\
G &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \\
\beta &= \begin{bmatrix} e - e^{-1} \\ 2e^{-1} \\ e - 7e^{-1} \end{bmatrix} \\
\alpha &= G^{-T}\beta \\
&= \begin{bmatrix} \frac{1}{2} & 0 & 2 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} e - e^{-1} \\ 2e^{-1} \\ e - 7e^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}\left(e - e^{-1}\right) \\ 3e^{-1} \\ \frac{5}{2}\left(e - 7e^{-1}\right) \end{bmatrix} \\
\hat{x} &= \alpha_1 y^1 + \alpha_2 y^2 + \alpha_3 y^3 \\
&= \frac{1}{2}\left(e - e^{-1}\right) + 3e^{-1}t + \frac{5}{4}\left(e - 7e^{-1}\right)\left(3t^2 - 1\right)
\end{aligned}$$

Discussion:

In this problem's setup, we use orthogonal basis to compute the normal equations, while in recitation, we use naive polynomial basis $\{t, t^2, t^3\}$. Note that when using orthogonal basis, the G matrix is diagonal, which is easier to compute its inverse and gives reliable results comparing with naive polynomial basis.