IOE 611: Final Exam

Yulun Zhuang

December 18, 2024

Problem 1

(a) Let $B(0,1) = \{x \in \mathbb{R}^n \mid \|x\|_2^2 \leq 1\}$, we have

$$\begin{aligned} E &= \{x \in \mathbb{R}^n \mid \|Q^{-\frac{1}{2}}x\|_2^2 = x^\top Q^{-1} x \leq 1\} \\ &= Q^{\frac{1}{2}} B(0,1) \\ &= \{Q^{\frac{1}{2}} x \mid \|x\|_2^2 \leq 1\} \end{aligned}$$

Hence, $E \subseteq S$ is equivalent to $B(0,1) \subseteq Q^{-\frac{1}{2}} S = \{Q^{-\frac{1}{2}} x \mid |a_i^\top x| \leq 1, i = 1, \ldots, k\}$.

Note that $Q^{-\frac{1}{2}} S = \{y \in \mathbb{R}^n : |a_i^\top Q^{\frac{1}{2}} y|^2 \leq 1, i = 1, \ldots, k\}$

If $B(0,1) \subseteq Q^{\frac{1}{2}} S$, let $y_i = \frac{Q^{\frac{1}{2}} a_i}{\|Q^{\frac{1}{2}} a_i\|_2}$, then we have

$$a_i^\top Q^{\frac{1}{2}} y_i = a_i^\top Q a_i / \sqrt{a_i^\top Q a_i} = \left(a_i^\top Q a_i\right)^{\frac{1}{2}} \leq 1, \quad \forall 1 \leq i \leq k.$$

If $a_i^\top Q a_i \leq 1, \forall 1 \leq i \leq k$, by Cauchy-Schwarz inequality, for any $y \in B(0,1), 1 \leq i \leq k$

$$\begin{aligned} a_i^\top Q^{\frac{1}{2}} y &= \left\langle Q^{\frac{1}{2}} a_i, y \right\rangle \\ &\leq \left\| Q^{\frac{1}{2}} a_i \right\|_2 \cdot \|y\|_2 \\ &= \sqrt{a_i^\top Q a_i} \cdot \|y\|_2 \\ &\leq 1. \end{aligned}$$

Therefore, $B(0,1) \leq Q^{-\frac{1}{2}} S$.

(b) Recall that the volume of $E = C_n \sqrt{\det Q}$, where $C_n > 0$ is a constant w.r.t. Q. Since $y = \log x$ is increasing in $x > 0$, maximizing the volume of E given $E \subseteq S$ is equivalent to

$$\begin{aligned} \max \quad & \log \det Q \\ \text{s.t.} \quad & a_i^\top Q a_i \leq 1, \forall 1 \leq i \leq k. \end{aligned}$$

The Lagrange is

$$
\begin{aligned}
L(Q,\lambda) &= \log\det Q + \sum_{i=1}^{k} \lambda_i (1 - a_i^\top Q a_i) \\
&= \log\det Q + \sum_{i=1}^{k} \lambda_i - \operatorname{tr}((\sum_{i=1}^{k} \lambda_i a_i a_i^\top) Q),
\end{aligned}
$$

Therefore, since $\frac{\partial}{\partial Q} \log\det Q = Q^{-\top} = Q^{-1}$, $\frac{\partial}{\partial Q} \operatorname{tr}\left(A^\top Q\right) = A$,

$$
\frac{\partial}{\partial Q} L(Q,\lambda) = Q^{-1} - \sum_{i=1}^{k} \lambda_i a_i a_i^\top,
$$

The dual problem is:

$$
\begin{aligned}
\min_{\lambda \in \mathbb{R}^n} \quad & -\log\det(\sum_{i=1}^{k} \lambda_i a_i a_i^\top) + \sum_{i=1}^{k} \lambda_i - n \\
\text{s.t.} \quad & \lambda \geq 0.
\end{aligned}
$$

Problem 2

Let $y = 0$, we have $Ay = 0, y^\top \nabla^2 f(\bar{x}) y = 1 < 1$, so the Slater's condition is satisfied.

The Lagrangian is

$$L(y, \lambda, \nu) = \nabla f(\bar{x})^\top y - \lambda(1 - y^\top \nabla^2 f(\bar{x}) y) + \nu^\top A y,$$

The KKT conditions at the optimal solution $\bar{y}$ are

$$\nabla f(\bar{x}) + 2\lambda \nabla^2 f(\bar{x}) \bar{y} + A^\top \nu = 0 \tag{1}$$

$$\bar{y}^\top \nabla^2 f(\bar{x}) \bar{y} \leq 1 \tag{2}$$

$$A\bar{y} = 0 \tag{3}$$

$$\lambda \geq 0 \tag{4}$$

$$\lambda(\bar{y}^\top \nabla^2 f(\bar{x}) \bar{y} - 1) = 0 \tag{5}$$

From (5) we know that

$$\begin{aligned} &\nabla f(\bar{x}) + \nabla^2 f(\bar{x}) \Delta x + A^\top w = 0 \\ &A^\top \Delta x = 0 \end{aligned}$$

Let $\bar{y} = \frac{\Delta x}{2\lambda}, w = \nu$, we have $A\bar{y} = 0$ and (1) is satisfied.

Let $\lambda = \frac{1}{2}\sqrt{\Delta x^\top \nabla^2 f(\bar{x}) \Delta x} = \frac{1}{2}\lambda(\bar{x})$, we have (4), (2) and (5) are satisfied, given that

$$\bar{y}^\top \nabla^2 f(x) y = \frac{\Delta x^\top \nabla^2 f(\bar{x}) \Delta x}{\Delta x^\top \nabla^2 f(\bar{x}) \Delta x} = 1$$

Therefore, $\bar{y} = \frac{\Delta x}{\lambda(\bar{x})}$.

Problem 3

(a) Since (P1) is convex, $f_i(x)$ is convex for all $0 \leq i \leq m$.

Since the maximum of convex functions is convex, we have

$$\begin{aligned} & f_i^+(x) = \max\{0, f_i(x)\} \\ \Rightarrow \quad & \max_{1\leq i\leq m} f_i^+(x) \text{ is convex.} \end{aligned}$$

Since $y = |x|$ is convex and $a_j^\top x - b_j$ is affine, we have

$$\begin{aligned} & \left|a_j^\top x - b_j\right| \text{ is convex} \\ \Rightarrow \quad & \max_{1\leq j\leq p} \left|a_j^\top x - b_j\right| \text{ is convex.} \end{aligned}$$

Given $t_1, t_2 > 0$, the sum of convex functions is also convex.

Thus, $f(x) := f_0(x) + t_1 \max_{1\leq i\leq m} f_1^+(x) + t_2 \max_{1\leq j\leq p} |a_j^\top x - t_j|$ is convex.

(b)

" $\Rightarrow$ "

Let x^* be the optimal solution of (P1-relaxed), we can define

$$y^* = \max_{1\leq i\leq m} f^+(x^*), \ z^* = \max_{1\leq j\leq p} \left|a_j^\top x^* - b_j\right|,$$

such that (x^*, y^*, z^*) is a feasible solution of (P2).

First we show the objectives are equivalent at (x^*, y^*, z^*)

$$v_2(x^*, y^*, z^*) = f_0(x^*) + t_1 y^* + t_2 z^* = f(x^*) = v_1(x^*).$$

Note that for any feasible (x, y, z) of (P2),

$$\begin{aligned} & y \geq \max\{0, f_i(x), \ldots, f_m(x)\} = \max_{1\leq i\leq m} f_i^+(x), \\ & z \geq \max_{1\leq j\leq p} \left|a_j^\top x - b_j\right| \\ \Rightarrow \quad & v_2(x, y, z) \geq v_1(x). \end{aligned}$$

Since x^* is the minima of (P1-relaxed) $\forall x$, we have $v_1(x) \geq v_1(x^*)$.

For any feasible (x, y, z) of (P2), $v_2(x, y, z) \geq v_1(x) \geq v_1(x^*) = v_2(x^*, y^*, z^*)$.

Thus, (x^*, y^*, z^*) is the optimal of (P2).

" $\Leftarrow$ "

If (x^*, y^*, z^*) is the optimal of (P2), we must have

$$y^* = \max_{1\leq i\leq m} f_i^+(x^*), \quad z^* = \max_{1\leq j\leq p} \left|a_j^\top x^* - b_j\right|.$$

Otherwise, we can find

$$\bar{y} = \max_{1\le i\le m} f_i^+(x^*), \quad \bar{z} = \max_{1\le j\le p} \left| a_j^\top x^* - b_j \right|,$$

such that $(x^*, \bar{y}, \bar{z})$ is feasible in (P2), and $y^* - \bar{y} \ge 0, z^* - \bar{z} \ge 0$.

However, the objective value is $v_2(x^*, \bar{y}, \bar{z}) < v_2(x^*, y^*, z^*)$, which leads to the contradiction.

For any x,

$$\begin{aligned} v_1(x) &= v_2(x, \max_{1\le i\le m} f_i^+(x), \max_{1\le j\le p} \left| a_j^\top x - b_j \right|) \\ &\ge v_2(x^*, y^*, z^*) \\ &= v_1(x^*) \end{aligned}$$

Thus, x^* is the optimal solution of (P1-relaxed) and the objective value $v_1^* = v_2^*$.

Finally we can conclude that (P1-relaxed) is equivalent to (P2).

(c) The Lagrangian of (P2) is

$$\begin{aligned} L\left(x, y, z, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\right) =& f_0(x) + t_1 y + t_2 z + \sum_{i=1}^m \lambda_i^{(1)} (f_i(x) - y) \\ &+ \sum_{j=1}^m \lambda_j^{(2)} \left(a_j^\top x - b_j - z\right) + \sum_{j=1}^m \lambda_{j+p}^{(2)} \left(b_j - a_j^\top x - z\right) \\ &- \lambda_1^{(3)} y - \lambda_2^{(3)} z, \end{aligned}$$

The partial derivatives are

$$\begin{aligned} \frac{\partial L}{\partial x} &= \nabla f_0(x) + \sum_{i=1}^m \lambda_i^{(m)} \nabla f_i(x) + \sum_{j=1}^m \left(\lambda_j^{(2)} - \lambda_{j+m}^{(2)}\right) a_j \\ \frac{\partial L}{\partial y} &= t_1 - \sum_{i=1}^m \lambda_i^{(1)} - \lambda_1^{(3)} \\ \frac{\partial L}{\partial z} &= t_2 - \sum_{j=1}^{2p} \lambda_j^{(2)} - \lambda_2^{(3)} \end{aligned}$$

The dual problem of (P2) can be formulated as (D2)

$$\begin{aligned} \max \quad & \bar{g}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = -(f_0 + \sum_{i=1}^m \lambda_i^{(1)} f_i)^*(-\sum_{j=1}^m (\lambda_j^{(2)} - \lambda_{j+p}^{(2)}) a_j) + \sum_{j=1}^m (\lambda_{j+p}^{(2)} - \lambda_j^{(2)}) b_j \\ \text{s.t.} \quad & t_1 = \sum_{i=1}^m \lambda_i^{(1)} + \lambda_1^{(3)}, \\ & t_2 = \sum_{j=1}^{2p} \lambda_j^{(2)} + \lambda_2^{(3)}, \\ & \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)} \ge 0 \end{aligned}$$

For any x, we can let y, z to be sufficiently large, then the inequality constraints are strict, and the Slater's condition will be satisfied. Therefore, (P2) and (D2) have strong duality.

Note that for the optimal solution x^* in (P1), $(x^*, 0, 0)$ is feasible in (P2) and $f_0(x^*) = v_2(x^*, 0, 0)$. By the strong duality,

$$d_1^* = f_0(x^*) = v_2(x^*, 0, 0) \geq v_2^* = d_2^*$$

(d) The KKT conditions of (P1) are

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{j=1}^{p} \nu_j^* a_j = 0 \tag{6}$$

$$f_i(x^*) \leq 0, \ 1 \leq i \leq m \tag{7}$$

$$a_j^\top x - b_j = 0, \ 1 \leq j \leq p \tag{8}$$

$$\lambda_i^* f_i(x^*) = 0, \ 1 \leq i \leq m \tag{9}$$

$$\lambda^* \geq 0 \tag{10}$$

If (x^*, λ^*, ν^*) is the primal-dual pair of (P1) with $\sum_{i=1}^{m} < t_1, \sum_{j=1}^{n} |\nu_j^*| < t_2$, we can define the followings to satisfy the KKT conditions of (P2)

$$\begin{aligned}
&\bar{x} = x^*, \bar{y} = 0, \bar{z} = 0 \\
&\bar{\lambda}_i^{(1)} = \lambda_i^*, \ 1 \leq i \leq m \\
&\bar{\lambda}_j^{(2)} = (\nu_j^*)^+, \ \bar{\lambda}_{j+p}^{(2)} = (\nu_j^*)^-, \ 1 \leq j \leq p \\
&\bar{\lambda}_1^{(3)} = t_1 - \sum_{i=1}^{m} \lambda_i^* > 0 \\
&\bar{\lambda}_2^{(3)} = t_2 - \sum_{j=1}^{p} |\nu_j^*| > 0,
\end{aligned}$$

such that $(x^*, 0, 0, \bar{\lambda}^{(1)}, \bar{\lambda}^{(2)}, \bar{\lambda}^{(3)})$ forms a primal-dual pair of (P2) and (D2), and $(\bar{\lambda}^{(1)}, \bar{\lambda}^{(2)}, \bar{\lambda}^{(3)})$ is the optimal solution of (D2).

Note that $\bar{\lambda}^{(3)} > 0$, if $(\bar{x}, \bar{y}, \bar{z})$ is the optimal solution of (P2), we must have $\bar{\lambda}_1^{(3)} \bar{y} = 0, \bar{\lambda}_2^{(3)} \bar{z} = 0 \Rightarrow \bar{y} = \bar{z} = 0$, indicating that all optimal solution of (P2) must satisfy

$$y = z = 0. \tag{$*$}$$

From part (b), if $\bar{x}$ is the optimal solution of (P1-relax), then $(\bar{x}, \bar{y}, \bar{z})$ is the optimal solution to (P2).

However, by conclusion $(*)$, we know that every solution to (P2) must have the form $(x, 0, 0)$, so we have

$$\begin{cases} 0 = \bar{y} = \max_i f_i^+(\bar{x}) \ \Leftrightarrow \ \forall 1 \leq i \leq m, \ f_i(\bar{x}) = 0, \\ 0 = \bar{z} = \max_j \left| a_j^\top \bar{x} - b_j \right| \ \Leftrightarrow \ \forall 1 \leq j \leq p, \ a_j^\top \bar{x} - b_j = 0, \end{cases}$$

which shows that $\bar{x}$ is feasible in (P1), and the objective value $f(\bar{x}) = f_0(\bar{x})$.

However, for any feasible solution x of (P1), x is also feasible in (P1-relax), and $f(x) = f_0(x)$, so the optimal value of (P1-relax) v_1^* is smaller them that of (P1) v_0^*.

Consequently, $f_0(\bar{x}) = v_1^* \leq v_0^*$, but $\bar{x}$ is feasible in (P1) $\Rightarrow$ $f_0(\bar{x}) \geq v_0^*$, so $f_0(\bar{x}) = v_0^*$, showing that $\bar{x}$ is the optimal solution of (P1).

Problem 4

(a) Given the positive definite and tridiagonal matrix $M \in \mathbb{S}^n$,

$$M = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix} = LL^\top,\ L = \begin{bmatrix} u_1 & & & & \\ v_1 & u_2 & & & \\ & v_2 & u_3 & & \\ & & \ddots & \ddots & \\ & & & v_{n-1} & u_n \end{bmatrix}$$

and by the Cholesky factorization, we have

$$\begin{aligned} &u_1^2 = a_1 \\ &u_i v_i = b_i,\ 1 \le i \le n-1 \\ &v_i^2 + u_{i+1}^2 = a_{i+1},\ 1 \le i \le n-1 \end{aligned}$$

Then we can calculate u, v in the following order

- $u_1 = \sqrt{a_1}$
- $v_1 = \frac{b_1}{u_1}, u_2 = \sqrt{a_2 - v_1^2}$
- $v_2 = \frac{b_2}{u_2}, u_3 = \sqrt{a_3 - v_2^2}$
- $\dots$
- $v_{n-1} = \frac{b_{n-1}}{u_{n-1}}, u_n = \sqrt{a_n - v_{n-1}^2}$

Thus, the time complexity becomes $O(n)$.

(b) Given that $\alpha, \beta > 0, y = x^2$ is convex, and $a_i^\top x - b_i$ for $i = 1, \dots, k, x_{j+1} - x_j$ for $j = 1, \dots, n-1$ are affine, we know the objective function $f(x)$ is convex.

Since $f(x)$ is differentiable, x^* is optimal if and only if $\nabla f(x^*) = 0$. Note that

$$\begin{aligned} &\frac{\partial}{\partial x_1} \sum_{j=1}^{n-1} (x_j - x_{j+1})^2 = 2\left(x_l - x_2\right) \\ &\frac{\partial}{\partial x_l} \sum_{j=1}^{n-1} (x_j - x_{j+1})^2 = 2\left(2x_l - x_{l-1} - x_{l+1}\right),\ 2 \le l \le n-1 \\ &\frac{\partial}{\partial x_n} \sum_{j=1}^{n-1} (x_j - x_{j+1})^2 = 2\left(x_n - x_{n-1}\right), \end{aligned}$$

We can pick C to be

$$C = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

such that $\nabla \sum_{j=1}^{n-1}(x_j - x_{j+1})^2 = 2Cx$. Therefore,

$$\begin{aligned}
\nabla f(x) &= \sum_{i=1}^{k} a_i(a_i^\top x - b_i) + 2\alpha x + 2\beta C x \\
&= (\sum_{i=1}^{k} a_i a_i^\top + 2\alpha I + 2\beta C)x - \sum_{i=1}^{k} a_i b_i \\
\Rightarrow \quad x^* &= (\sum_{i=1}^{k} a_i a_i^\top + 2\alpha I + 2\beta C)^{-1}(\sum_{i=1}^{k} a_i b_i)
\end{aligned}$$

(c) We can now calculate x^* in the following order

- Apply Cholesky factorization to $2\alpha I + 2\beta C = LL^\top$, since the LHS is tridiagonal $\Leftarrow O(n)$;
- Calculate $B^{-1} = (LL^\top)^{-1}$. Since L is in lower tridiagonal form, let L^{-1} be

$$L^{-1} = \begin{bmatrix} s_1 & & & & \\ t_1 & s_2 & & & \\ & t_2 & s_3 & & \\ & & \ddots & \ddots & \\ & & & t_{n-1} & s_n \end{bmatrix} \Rightarrow \begin{cases} s_i = u_i^{-1} & 1 \le i \le n \\ t_i = -\frac{v_i s_i}{u_{i+1}} & 1 \le i \le n-1 \end{cases}$$

  Thus, the cost of inverting L is $O(n)$.

  Since B^{-1} should also be tridiagonal, the cost of inverting it is also $O(n)$.

- Calculate $\bar{b} = \sum_{i=1}^{k} a_i b_i \Leftarrow O(kn)$
- Let $A = [a_1 \ \cdots \ a_k] \in \mathbb{R}^{n\times k}$, we can write x^* into the following form

$$(B + AA^\top)x^* = \bar{b} \quad \Leftrightarrow \quad \begin{bmatrix} B & A \\ A^\top & -I \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} \bar{b} \\ 0 \end{bmatrix},$$

  We can solve the linear system via block elimination

$$\begin{cases} (I + A^\top B^{-1} A)y^* = A^\top B^{-1}\bar{b} \\ x^* = B^{-1}\bar{b} - B^{-1}Ay^* = L^{-1}L^{-\top}\bar{b} - L^{-1}L^{-\top}Ay^* \end{cases}$$

Therefore, the total time complexity is $O(k^2 n + k^3)$.

Problem 5

(a) The gradient and Hessian of $f(x)$ are

$$\nabla f(x) = c - \frac{1}{x}$$
$$\nabla^2 f(x) = \operatorname{diag}\left(\frac{1}{x^2}\right),$$

where $\frac{1}{x}$ and $\frac{1}{x^2}$ are element-wise, i.e., $\left[\frac{1}{x}\right]_j = \frac{1}{x_j}$, $\left[\frac{1}{x^2}\right]_j = \frac{1}{x_j^2}$. The equality constraint Newton system is derived from the Karush-Kuhn-Tucker (KKT) conditions

$$\nabla f(x) + A^\top \lambda = 0$$
$$Ax - b = 0,$$

where $\lambda \in \mathbb{R}^m$ are the Lagrange multipliers. Rewriting in matrix form

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ b - Ax \end{bmatrix},$$

where Δx is the Newton step, and $\Delta\lambda$ is the Newton decrement.

To solve the KKT system efficiently, we exploit the structure of the equations. The Schur complement can be used to eliminate $\Delta\lambda$ and solve for Δx directly

$$\Delta x = -H^{-1}(\nabla f(x) + A^\top \Delta\lambda),$$

where $H = \nabla^2 f(x)$. Substituting back, we solve

$$(AH^{-1}A^\top)\Delta\lambda = AH^{-1}\nabla f(x) - (b - Ax).$$

This approach avoids directly inverting the full KKT matrix, improving the computational efficiency. At each iteration:

- Solve the Newton system to obtain Δx and $\Delta\lambda$.
- Perform a backtracking line search to ensure feasibility, reducing the step size α until $x + \alpha\Delta x > 0$.
- Update the primal and dual variables: $x \leftarrow x + \alpha\Delta x,\ \lambda \leftarrow \lambda + \alpha\Delta\lambda$.

The stopping criterion is: $\|\Delta x\|^2/2 \leq \epsilon$, where ϵ is a small tolerance.

As shown in Fig. 1, the implemented method converges quadratically on the generated instance.

(b) As shown in Fig. 2, smaller μ leads to more Newton steps and thus slower convergence, while larger μ leads to fewer Newton steps but higher duality gap. Based on this observation, we pick $\mu = 200$ in the implementation.

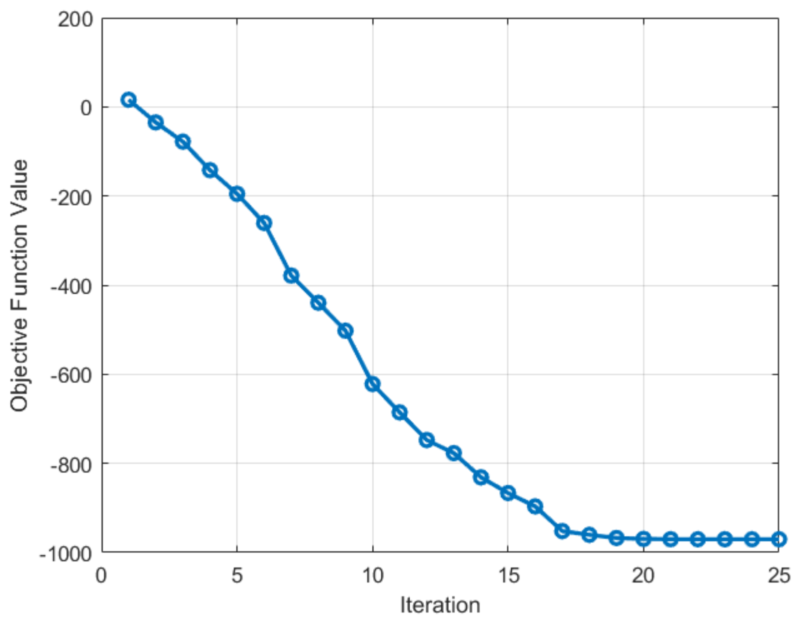

Figure 1: Convergence of Newton Method

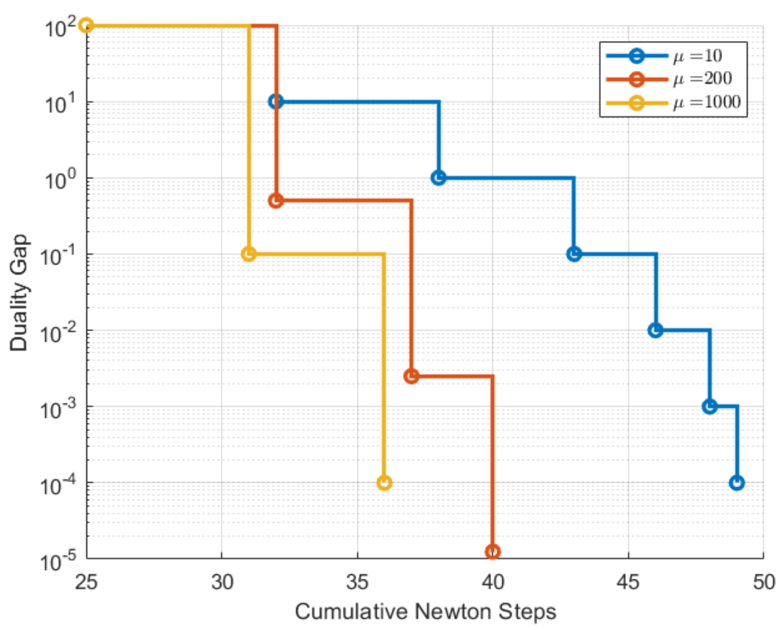

Figure 2: Convergence of Barrier Method