# ME424 HW4

Zhuang Yulun 11811126

December 23, 2020

## 1

### 1.1

The system is asymptotically stable if and only if all eigenvalues of A satisfiles $|\lambda_i| < 1$

Since $\lambda = \begin{bmatrix} 0.9 \\ 1 \\ 1 \end{bmatrix}$ ,eigenvalues of A are not all less than 1, matrix A is not asymptotically stable.

### 1.2

$H(z) = C(zI - A)^{-1}B + D$

The system is BIBO stable if and only if the poles of every entry in $H(z)$ lies inside the unit circle.

$$(zI - A)^{-1} = \begin{bmatrix} \frac{1}{z-0.9} & 0 & \frac{1}{(z-0.9)(z-1)} \\ 0 & \frac{1}{z-1} & 0 \\ 0 & 0 & \frac{1}{z-1} \end{bmatrix}$$

Choose $B = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$ , such that $H(z) = \frac{1}{z-0.9}$ and the pole of $H(z)$ is $0.9 < 1$.

## 2

### 2.1

$$\begin{aligned} H(z) &= C(zI - A)^{-1}B \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{z-2} & \frac{2}{(z-2)(z-0.5)} \\ 0 & \frac{1}{z-0.5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{z-2} & \frac{3z-1}{(z-2)(z-0.5)} \\ \frac{1}{z-2} & \frac{3z-1}{(z-2)(z-0.5)} \\ \frac{2}{z-2} & \frac{5z}{(z-2)(z-0.5)} \end{bmatrix} \end{aligned}$$

There are many poles of $H(z)$ larger than 1, so the system is not BIBO stable.

### 2.2

$$\begin{aligned} H(z) &= C(zI - A)^{-1}B \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} z-0.5 & 0 & 0 \\ 0 & z-1 & -1 \\ 0 & 0 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{(z-0.5)} \\ \frac{1}{z-0.5} \end{bmatrix} \end{aligned}$$

The poles of $H(z)$ are both 0.5, so the system is BIBO stable.

# 3

## 3.1

$$\begin{aligned} M_c &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 3 & 0 & 9 & 0 & 27 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 & 0 & 21 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The rank of $M_c$ is 3, which is less than 4, so the system is not controllable.

## 3.2

Since $x_f - A^k x_0 = [1,1,1,0]^T \in range(M_c)$, the system is able to reach $x_f$ within finite steps.
If we look at $M_c$, we can choose the control input as $u(3) = [1,0]^T, u(2) = [0,1]^T, u(1) = [0,0]^T, u(0) = [0,0]^T$ to drive the system from $x_0$ to $x_f$ within 4 steps. Thus,the minimum steps is less or equal to 4.
Then I check

$$\begin{bmatrix} A & AB & A^2B \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = x_f$$

solutions given by MATLAB are $u(2) = [0, 4/5]^T, u(1) = [0,0]^T, u(0) = [1/9, 1/5]^T$. Chech $\begin{bmatrix} A & AB \end{bmatrix}$ in the same way and found no solutions. Thus, the minimum steps is 3 with control inputs showed above.

# 4

## 4.1

$$\begin{aligned} M_o &= \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 9 & 0 & 0 & 0 \\ 9 & 0 & 1 & 3 \\ 27 & 0 & 0 & 0 \\ 27 & 0 & 1 & 7 \end{bmatrix} \end{aligned}$$

The rank of $M_o$ is 3, which is less than 4, so the system is not observable.

## 4.2

Given

$$\begin{aligned} u(0) &= u(1) = [0,0]^T \\ y(0) &= [1,2]^T \\ y(1) &= [3,4]^T \end{aligned}$$

Find two different $x_0$

$$
\begin{aligned}
M_o x_0 &= Y_2 - T_2 U_2 \\
\begin{bmatrix} C \\ CA \end{bmatrix} x_0 &= \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{bmatrix} x_0 &= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\end{aligned}
$$

Choose $x_0^{(1)} = [1, 0, 1, 0]^T$ and $x_0^{(2)} = [1, 1, 1, 0]^T$

# 5

SYS1: $\begin{cases} x_1(k+1) = A_1 x_1(k) + B_1 u_1(k) \\ y_1(k) = C_1 x_1(k) + D_1 u_1(k) \end{cases}$ SYS2: $\begin{cases} x_2(k+1) = A_2 x_2(k) + B_2 u_2(k) \\ y_2(k) = C_2 x_2(k) + D_2 u_2(k) \end{cases}$

Define overall state $x(k) = [x_1(k), x_2(k)]^T$. Then the overall closed-loop dynamics can be written as:

$$
u_1(k) = u(k) - y(k) \tag{1}
$$

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} \\
&= \begin{bmatrix} A_1 x_1(k) + B_1 u_1(k) \\ B_2 C_1 x_1(k) + A_2 x_2(k) + B_2 D_1 u_1(k) \end{bmatrix} \\
y(k) &= (I + D_2 D_1)^{-1} \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} x(k) + (I + D_2 D_1)^{-1} D_2 D_1 u(k)
\end{aligned}
$$

Subsitude (**??**) and $y(k)$ into $x(k+1)$,

$$
\begin{aligned}
x(k+1) = &\begin{bmatrix} A_1 - B_1 (I + D_2 D_1)^{-1} D_2 C_1 & -B_1 (I + D_2 D_1)^{-1} C_2 \\ B_2 C_1 - B_2 D_1 (I + D_2 D_1)^{-1} D_2 C_1 & A_2 - B_2 D_1 (I + D_2 D_1)^{-1} C_2 \end{bmatrix} x(k) \\
&+ \begin{bmatrix} B_1 (I + D_2 D_1)^{-1}) \\ B_2 D_1 (I + D_2 D_1)^{-1}) \end{bmatrix} u(k)
\end{aligned}
$$

Thus, the overall dynamics system is $\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$

where

$$
\begin{aligned}
A &= \begin{bmatrix} A_1 - B_1 (I + D_2 D_1)^{-1} D_2 C_1 & -B_1 (I + D_2 D_1)^{-1} C_2 \\ B_2 C_1 - B_2 D_1 (I + D_2 D_1)^{-1} D_2 C_1 & A_2 - B_2 D_1 (I + D_2 D_1)^{-1} C_2 \end{bmatrix} \\
B &= \begin{bmatrix} B_1 (I + D_2 D_1)^{-1}) \\ B_2 D_1 (I + D_2 D_1)^{-1}) \end{bmatrix} \\
C &= (I + D_2 D_1)^{-1} \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \\
D &= (I + D_2 D_1)^{-1} D_2 D_1
\end{aligned}
$$

# 6

## 6.1

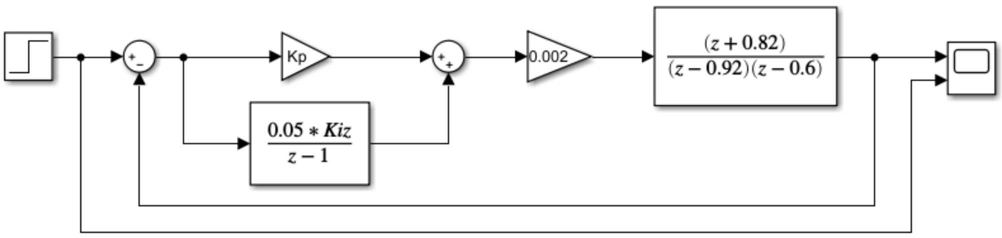
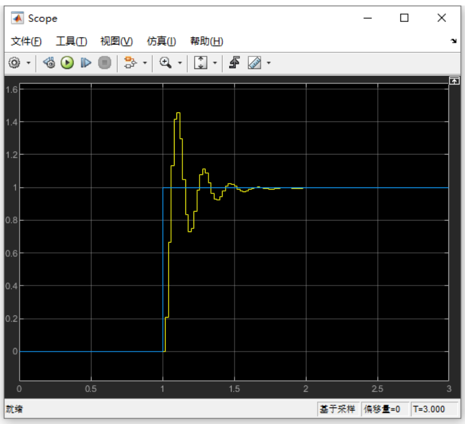
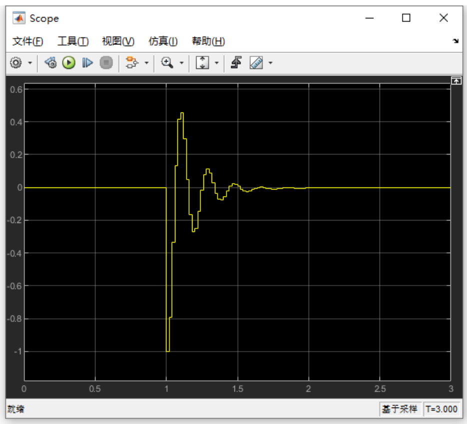



Figure 1: Simulink Model for DC Motor

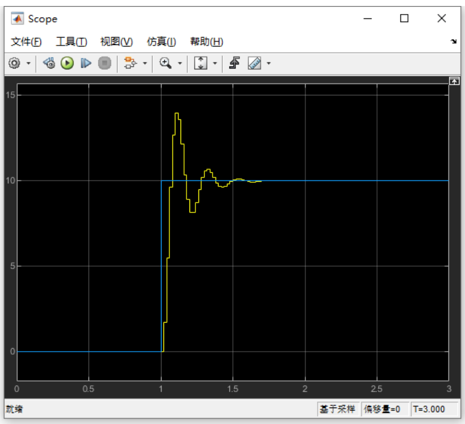
## 6.2



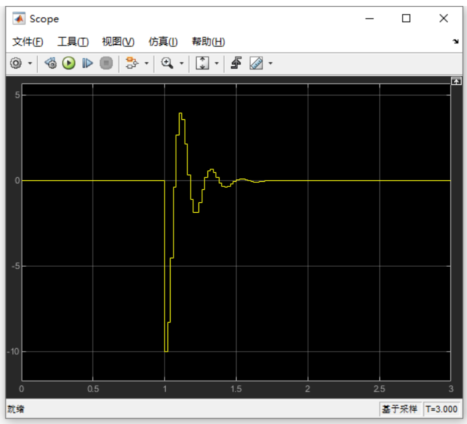
(a) $K_I = 100, K_P = 100 \quad and \quad \delta(t) = 1$



(b) Tracting Error for (a)



(c) $K_I = 100, K_P = 80 \quad and \quad \delta(t) = 10$



(d) Tracting Error for (b)

Figure 2: Simulation Results (T = 0.02)

## 6.3

For the first set:
Open loop transfer function is $G(z) = \frac{0.002(105z-100)(z+0.82)}{(z-1)(z-0.92)(z-0.6)}$
By MATLAB, the closed loop poles are $0.99, 0.66 \pm 0.53i$, which are all less than 1, so the new system

is BIBO stable.
On the other hand,

$$\begin{aligned}
e(\infty) &= \lim_{z\to 1}(z-1)\frac{\frac{z}{z-1}}{1+G(z)} \\
&= \lim_{z\to 1}\frac{z(z-1)(z-0.92)(z+0.82)}{0.002(105z-100)(z+0.82)+(z-1)(z-0.92)(z+0.82)} \\
&= 0
\end{aligned}$$

For the second set, we have the same conditions for closed loop poles, and the steady state error $e(\infty)$ converge to 0 in a similar process as I showed above.