ROB 501 HW2

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1

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

1.1

$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
F	F	F
F	T	T
T	F	T
T	T	T

1.2

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
F	F	F
F	T	F
T	F	F
T	T	T

2

(a) There exist an interger n, $2n + 1$ is even.

(b) For every interger n, $2^n + 1$ is composite.

(c) $\forall v \in \mathbb{R}^n, v \neq 0$ s.t. $Av \neq \lambda v$

(d) $\exists \eta > 0, \forall \delta > 0$ s.t. $|x| \leq \delta \cancel{\Rightarrow} |f(x)| > \eta |x|$

3

Prove that $\sqrt{7}$ is irrational.

Proof. Assum $\sqrt{7}$ is rational. Then there exist natural numbers m and n such that m and n have no common factor, $n \neq 0$ and

$$\sqrt{7} = \frac{m}{n} \tag{1}$$

Square both sides of (1)

$$(7n^2 = m^2) \tag{2}$$

This implies 7 divides m^2, so 7 also divides m.

Let $m = 7k, k \in \mathbb{N}$,

$$\begin{aligned} 7n^2 =& (7k)^2 \\ n^2 =& 7k^2 \end{aligned}$$

This implies 7 divides n^2, so 7 also divides n.

Therefore, m and n both have a common factor 7, which leads to a contradiction.

Hence, $\sqrt{7}$ is irrational.

□

4

Let A be a square matrix. Prove that if $det(A) = 0$, then A is not invertible.

Definition 4.1. A square matrix A has an inverse A^{-1} iff $AA^{-1} = A^{-1}A = I$.

Proof. p: $det(A) = 0$, q: A is not invertible

Now we prove its contrapositive, $\neg q \implies \neg p$.

We have A is invertible,

$$\begin{aligned} det(AA^{-1}) =& det(A)det(A^{-1}) = det(I) = 1 \\ \Rightarrow \quad & det(A) \neq 0 \end{aligned}$$

□

5

Prove that for all integers $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

Proof. $P(n) = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

When $n = 1$, $P(1) = \frac{1}{1+1} = \frac{1}{2}$

When $n = m$, assum $P(m) = \sum_{k=1}^{m} \frac{1}{k(k+1)} = \frac{m}{m+1}$ is true.

When $n = m + 1$,

$$\begin{aligned}P(m+1) =& P(m) + \frac{1}{(m+1)(m+2)} \\ =& \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} \\ =& \frac{(m+1)^2}{(m+1)(m+2)} \\ =& \frac{m+1}{(m+1)+1}\end{aligned}$$

□

6

$P(n) : \forall n \in \mathbb{Z}, n \geq 12, \exists k_1, k_2 \in \mathbb{Z}, k_1, k_2 \geq 0, s.t.\ n = 4k_1 + 5k_2$

Proof. If $n = 12$, $k_1 = 3$ and $k_2 = 0$. If $n = 13$, $k_1 = 2$ and $k_2 = 1$. If $n = 14$, $k_1 = 1$ and $k_2 = 2$. If $n = 15$, $k_1 = 0$ and $k_2 = 3$.

Induction: Assume $P(n)$ is true for $12 \leq n \leq k$, show that $P(k+1)$ is true.

Start with $k + 1 \geq 16$,

$$(k+1) - 4 \geq 12$$

By inductive hypothesis,

$$(k+1) - 4 = 4u + 5v$$

where u and v are non-negtive integers.

Therefore,

$$k + 1 = 4(u+1) + 5v$$

□

$P(n)$ is not true for $n \geq 8$, since no combinations of '4' and '5' can add up to 11.

6.1

$P(n) : \forall n = 2m, m \in \mathbb{Z}, n \geq 6, \exists k_1, k_2 \in \mathbb{Z}, k_1, k_2 \geq 0, s.t.\ n = 3k_1 + 5k_2$

Proof. If $n = 6$, $k_1 = 2$ and $k_2 = 0$. If $n = 8$, $k_1 = 1$ and $k_2 = 1$. If $n = 10$, $k_1 = 0$ and $k_2 = 2$. If $n = 12$, $k_1 = 4$ and $k_2 = 0$.

Induction: Assume $P(n)$ is true for $6 \leq n \leq k$, show that $P(k+1)$ is true.

Start with $k + 1 \geq 12$,

$$(k+1) - 6 \geq 6$$

By inductive hypothesis,

$$(k+1) - 6 = 3u + 5v$$

where u and v are non-negtive integers.

Therefore,

$$k + 1 = 3(u + 2) + 5v$$

$\square$

7

Given M is an $n \times n$ real symmetric matrix. $f(x) = x^T M x$, subject to the constraint $x^T x = 1$.

Let $L(x, \lambda) = f(x) - \lambda(x^T x - 1) = x^T M x - \lambda x^T x - \lambda$, and $\lambda > 0$.

$$\nabla L = \frac{dL}{dx} = 2Mx - 2\lambda x$$

Solve $\nabla L = 0$, we have $Mx = \lambda x$.

Therefore, λ is the eigenvalue of M and x is the corresponding eigenvector.

$$f(x) = x^T M x = x^T(\lambda x) = \lambda x^T x = \lambda$$

Solve $det(M - \lambda I) = 0$. Since the eigenvalues are real and finite in number, there exists a largest eigenvalue, denoted λ_{max}, and a smallest eigenvalue, denoted λ_{min}, i.e. $\boldsymbol{\lambda} = [\lambda_{max}, \ldots, \lambda_{min}]^T$

7.1

$max(f(x)) = \lambda_{max}$ and x_{max} is its corresponding eigenvector given by $(M - \lambda_{max} I)x = 0$.

7.2

$min(f(x)) = \lambda_{min}$ and x_{min} is its corresponding eigenvector given by $(M - \lambda_{min} I)x = 0$.