ROB 501 HW5

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1

1.1

$$A_3 = \begin{bmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A_3 - \lambda I = \begin{bmatrix} 3-\lambda & 10 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Characteristic polynomial of A_3 is $det(A_3 - \lambda I) = (3-\lambda)(2-\lambda)(1-\lambda)$.

$$det(A_3 - \lambda I) = 0 \Rightarrow \lambda = 1, 2, 3$$

When $\lambda = 3$, $(A_3 - \lambda I)x_1 = \mathbf{0} \Rightarrow x_1 = [1, 0, 0]^T$

When $\lambda = 2$, $(A_3 - \lambda I)x_2 = \mathbf{0} \Rightarrow x_2 = [-10, 1, 0]^T$

When $\lambda = 1$, $(A_3 - \lambda I)x_3 = \mathbf{0} \Rightarrow x_3 = [20, -4, 1]^T$

Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$,

$$\underbrace{\begin{bmatrix} 1 & -10 & 20 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$
$$\Rightarrow det(A) = 1 > 0$$
$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

The e-vectors of A are linear independent.

1.2

$$A_4 = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

The e-values: $\lambda = 5, 5, 2$.

When $\lambda = 5$, $(A_4 - \lambda I)x_1 = \mathbf{0} \Rightarrow x_1 = [1, 0, 0]^T$

When $\lambda = 2$, $(A_4 - \lambda I)x_2 = \mathbf{0} \Rightarrow x_2 = [0, -1, 1]^T$

No, since $span\{x_1, x_2\} \neq \mathbb{R}^3$. E.g. $v = [0, 0, 1]$ can not be represented by any combination of x_1 and x_2.

2

Given $B = P^{-1}AP$, show that $det(A - \lambda I) = det(B - \lambda I)$.

Proof.

$$\begin{aligned} & det(B - \lambda I) \\ &= det(P^{-1}AP - \lambda I) \\ &= det(P^{-1}AP - P^{-1}\lambda IP) \\ &= det(\left[P^{-1}(A - \lambda I)P\right]) \\ &= det(P^{-1})det(A - \lambda I)det(P) \\ &= det(A - \lambda I) \end{aligned}$$

□

3

Given A_3, whose e-values are $\lambda = 3, 2, 1$ and cooresponding e-vectors are $[1, 0, 0]^T,\ [-10, 1, 0]^T,\ [20, -4, 1]^T$, show that A is similar to a diagonal matrix.

Proof.

$$\begin{aligned} A_3[x_1 \mid x_2 \mid x_3] &= [\lambda_1 x_1 \mid \lambda_2 x_2 \mid \lambda_3 x_3] \\ &= \underbrace{[x_1 \mid x_2 \mid x_3]}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}}_{\Lambda} \end{aligned}$$

$$\Rightarrow A_3 = P\Lambda P^{-1}$$

□

4

4.1

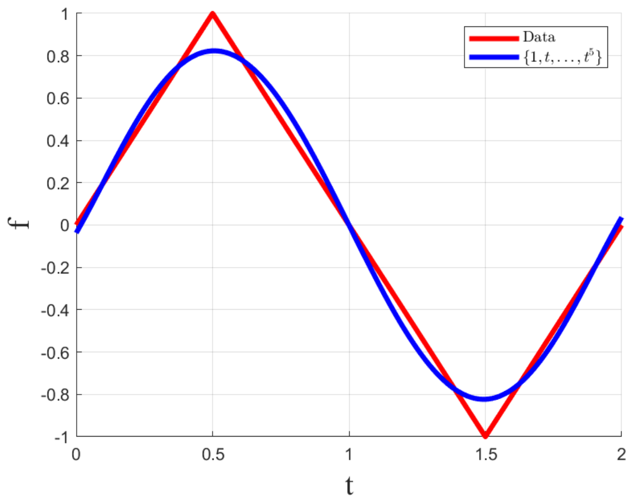

Figure 1: Least square fit of the data using $\{1, t, \ldots, t^5\}$

Coefficients: $\boldsymbol{\alpha} = [-0.0370, 2.2158, 2.8901, -11.2271, 7.6978, -1.5396]^T$.

4.2

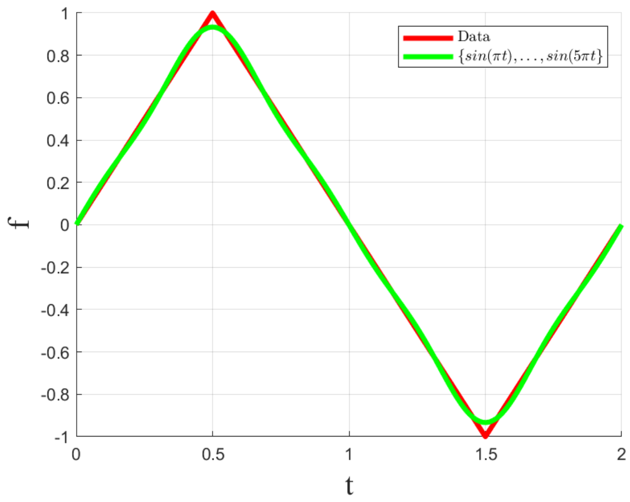

Figure 2: Least square fit of the data using $\{sin(\pi t), \ldots, sin(5\pi t)\}$

Coefficients: $\boldsymbol{\alpha} = [0.8106, 0.0000, -0.0901, 0.0000, 0.0325]^T$.

5

Function: $f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$

$$f'(t) = \frac{df(t)}{dt} = 3\alpha_3 t^2 + 2\alpha_2 t + \alpha_1$$
$$f'(0.3) = -0.4016$$

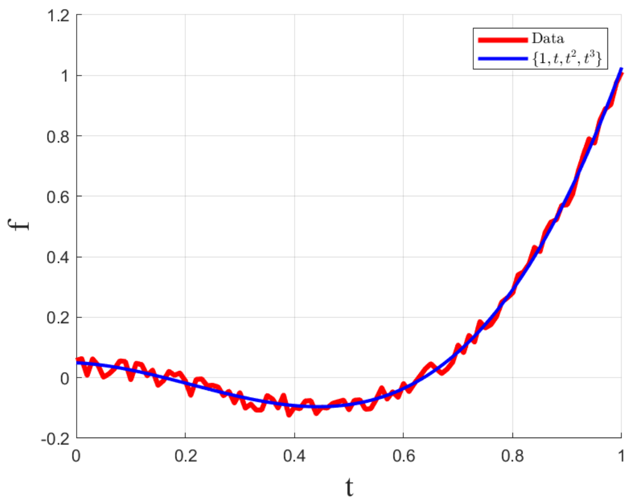

Figure 3: Least square fit of the data using $\{1, t, t^2, t^3\}$

Coefficients: $\boldsymbol{\alpha} = [0.0495, -0.0847, -1.8287, 2.8900]^T$.

6

Show that on $(\mathbb{C}^n, \mathbb{C})$, $< x, y >= x^T \bar{y}$ satisfies the difinition of inner product used in lecture, while $< x, y >= \bar{x}^T y$ satisfies the difinition of inner product in Nagy's book.

For part I,

(a) **Hermitian Symmetry**: $\forall x, y \in \mathbb{C}^n, \overline{< y, x >} = \overline{y^T \bar{x}} = \overline{\Sigma_{i=1}^n y_i \bar{x}_i} = \Sigma_{i=1}^n x_i \bar{y}_i = x^T \bar{y} =< x, y >$

(b) **Non-negativity**: $\forall x \in \mathbb{C}^n, < x, x >= x^T \bar{x} = \Sigma_{i=1}^n x_i \bar{x}_i \geq 0$; when $< x, x >= 0, |x_i|^2 = 0 \Rightarrow x = \mathbf{0}$

(c) **Linearity in the first argument**:

$\forall \alpha_1, \alpha_2 \in \mathbb{C}, x_1, x_2, y \in \mathbb{C}^n$,

$$< \alpha_1 x_1 + \alpha_2 x_2,\ y >= (\alpha_1 x_1 + \alpha_2 x_2)^T \bar{y} = \alpha_1 {x_1}^T \bar{y} + \alpha_2 {x_2}^T \bar{y} = \alpha_1 < x_1, y > + \alpha_2 < x_2, y >$$

For part II,

(a) **Hermitian Symmetry**: $\forall x, y \in \mathbb{C}^n, \overline{< y, x >} = \overline{\bar{y}^T x} = \overline{\Sigma_{i=1}^n \bar{y}_i x_i} = \Sigma_{i=1}^n \bar{x}_i y_i = \bar{x}^T y =< x, y >$

(b) **Non-negativity**: $\forall x \in \mathbb{C}^n, < x, x >= \bar{x}^T x = \Sigma_{i=1}^n \bar{x}_i x_i \geq 0$; when $< x, x >= 0, |x_i|^2 = 0 \Rightarrow x = \mathbf{0}$

(c) **Linearity in the second argument**:

$\forall \alpha_1, \alpha_2 \in \mathbb{C}, x, y_1, y_2 \in \mathbb{C}^n$,

$$< x,\ \alpha_1 y_1 + \alpha_2 y_2 >= \bar{x}^T (\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 \bar{x}^T y_1 + \alpha_2 \bar{x}^T y_2 = \alpha_1 < x, y_1 > + \alpha_2 < x, y_2 >$$

7

Define the inner product $< p, q >= \int_{-1}^1 p(x) q(x) dx$ in $\mathbb{P}_3([-1, 1])$.

Show that the set $P = \{p_0, p_1, p_2, p_3\}$ is an orthogonal basis in $\mathbb{P}_3$.

Proof. (a) P is linear independent Let $\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$,

$$\frac{5}{2}\alpha_3 x^3 + \frac{3}{2}\alpha_2 x^2 + (\alpha_1 - \frac{3}{2}\alpha_3)x + (\alpha_0 - \frac{1}{2}\alpha_2) = 0$$

$$\begin{cases} \alpha_3 = 0 \\ \alpha_2 = 0 \\ \alpha_1 - \frac{3}{2}\alpha_3 = 0 \\ \alpha_0 - \frac{1}{2}\alpha_2 = 0 \end{cases} \Rightarrow \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$$

(b) $span\{p_0, \ldots, p_3\} = \mathbb{P}_3$ $\forall x \in \mathbb{P}_3, x = \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0$, we show that x can be represented by P.

$$\begin{cases} \frac{5}{2}\alpha_3 = \beta_3 \\ \frac{3}{2}\alpha_2 = \beta_2 \\ \alpha_1 - \frac{3}{2}\alpha_3 = \beta_1 \\ \alpha_0 - \frac{1}{2}\alpha_2 = \beta_0 \end{cases} \Rightarrow \begin{cases} \alpha_0 = \beta_0 + \frac{1}{3}\beta_2 \\ \alpha_1 = \beta_1 + \frac{3}{5}\beta_3 \\ \alpha_2 = \frac{2}{3}\beta_2 \\ \alpha_3 = \frac{2}{5}\beta_0 \end{cases}$$

(c) The inner product of between elements in P equal to 0.

$$\langle p_0, p_3 \rangle = \int_{-1}^1 \frac{1}{2}\left(5x^3 - 3x\right) dx = \int_{-1}^1 \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) dx = \left(\frac{5}{8}x^4 - \frac{3}{4}x^2\right)\Big|_{-1}^1 = 0$$

$$\langle p_1, p_2 \rangle = \int_{-1}^1 \frac{1}{2}x\left(3x^2 - 1\right) dx = \int_{-1}^1 \left(\frac{3}{2}x^3 - \frac{1}{2}x\right) dx = \left(\frac{3}{8}x^4 - \frac{1}{4}x^2\right)\Big|_{-1}^1 = 0$$

□

8

8.1

$$
\begin{aligned}
&(A+BCD)\left[A^{-1}-A^{-1}B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}\right]\\
=&\left[I-B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}\right]+\left[BCDA^{-1}-BCDA^{-1}B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}\right]\\
=&\left[I+BCDA^{-1}\right]-\left[B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}+BCDA^{-1}B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}\right]\\
=&I+BCDA^{-1}-\left(B+BCDA^{-1}B\right)\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}\\
=&I+BCDA^{-1}-BC\left(C^{-1}+DA^{-1}B\right)\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}\\
=&I+BCDA^{-1}-BCDA^{-1}\\
=&I
\end{aligned}
$$

8.2

$$
\begin{aligned}
A&=diag([0.5,1,1,0.5,1])\\
A^{-1}&=diag([2,1,1,2,1])
\end{aligned}
$$

$$
\begin{aligned}
&A^{-1}-A^{-1}B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1}\\
=&A^{-1}-A^{-1}\begin{bmatrix}3\\0\\2\\0\\1\end{bmatrix}\left(4+\begin{bmatrix}3&0&2&0&1\end{bmatrix}A^{-1}\begin{bmatrix}3\\0\\2\\0\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}3&0&2&0&1\end{bmatrix}A^{-1}\\
=&A^{-1}-\begin{bmatrix}6\\0\\2\\0\\1\end{bmatrix}(27)^{-1}\begin{bmatrix}6&0&2&0&1\end{bmatrix}\\
=&\begin{bmatrix}\frac{2}{3}&0&-\frac{4}{9}&0&-\frac{2}{9}\\0&1&0&0&0\\-\frac{4}{9}&0&\frac{23}{27}&0&-\frac{2}{27}\\0&0&0&2&0\\-\frac{2}{9}&0&-\frac{2}{27}&0&\frac{26}{27}\end{bmatrix}
\end{aligned}
$$

9

9.1

Define $f(x)=\left(x^TA_x\right)^{\frac{1}{2}}$ in $(\mathbb{R}^n,\mathbb{R})$

A is positive definite $\Rightarrow$ $x^TAx>0,\forall x\neq 0$ and there exists an invertible matrix B s.t. $A=BB^T$.

Show that $f(x)$ is a norm.

(a) **Non-negative**: $\forall x\in\mathbb{R}^n, f(x)=\left(x^TAx\right)^{\frac{1}{2}}\geqslant 0$ and $f(x)=0\Leftrightarrow x=0$

(b) **Scalability**: $\forall\alpha\in\mathbb{R},\forall x\in\mathbb{R}^n, f(\alpha x)=\left[(\alpha x)^TA(\alpha x)\right]^{\frac{1}{2}}=|\alpha|\left(x^TAx\right)^{\frac{1}{2}}$

(c) **Triangular inequality**: $\forall x, y \in \mathbb{R}^n$

$$\begin{aligned}
f(x+y) &= \left[(x+y)^T A(x+y)\right]^{\frac{1}{2}} \\
&= \left[\left(x^T+y^T\right) A(x+y)\right]^{\frac{1}{2}} \\
&= \left[x^T A(x+y)+y^T A(x+y)\right]^{\frac{1}{2}} \\
&= \left[x^T A x+x^T A y+y^T A x+y^T A y\right]^{\frac{1}{2}} \\
&= \left[x^T A x+y^T A y+2 x^T A y\right]^{\frac{1}{2}} \Leftarrow x^T A y=y^T A x, x^T A y=(B^T x)^T(B^T y) \\
&\leq \left[x^T A x+y^T A y+2 \sqrt{x^T A x \cdot y^T A y}\right] \Leftarrow |<x, y>| \leq<x, x>^{1/2}<y, y>^{1/2} \\
&= \left[x^T A x\right]^{\frac{1}{2}}+\left[y^T A y\right]^{\frac{1}{2}} \\
&\text{Equality holds when } x=y \\
&= f(x)+f(y)
\end{aligned}$$

If A is replaced by $2A$, it is still a positive definite matrix, so proof holds.

9.2

Given $(\mathbb{R}^n, \mathbb{R}, \|\cdot\|_V)$ is a norm space.

Show that in $(\mathbb{R}^{n\times n}, \mathbb{R})$, $f_V(A) = \sup_{x\neq 0} \frac{\|Ax\|_V}{\|x\|_V}$ $(x \in \mathbb{R}^n)$ is a norm.

(a) **Non-negative**: $\forall A$ in $\mathbb{R}^{n\times n}$, $f_V(A) \geqslant 0$, since $\|\cdot\|_V \geqslant 0$. $f_V(A) = 0$ holds when $A = 0$

(b) **Scalability**: $\forall \alpha \in \mathbb{R}$, $\forall A \in \mathbb{R}^{n\times n}$, $f_v(\alpha A) = \sup_{x\neq 0} \frac{\|\alpha Ax\|_V}{\|x\|_V} = \sup_{x\neq 0} \frac{|\alpha|\|A_x\|_V}{\|x\|_V} = |\alpha| f_V(A)$

(c) **Triangular inequality**: $\forall A, B \in \mathbb{R}^{n\times n}$

$$\begin{aligned}
f_v(A+B) &= \sup_{x\neq 0} \frac{\|(A+B)x\|_V}{\|x\|_V} \\
&= \sup_{x\neq 0} \frac{\|Ax+B_x\|_V}{\|x\|_V} \\
&\leq \sup_{x\neq 0} \frac{\|Ax\|_V+\|Bx\|_V}{\|x\|_V} \text{[Equality form } (\mathbb{R}^n, \mathbb{R}, \|\cdot\|_V)] \\
&= \sup_{x\neq 0} \frac{|\, Ax\|_V}{\|x\|_V} + \underbrace{\sup_{x\neq 0} \frac{\|B_x\|_V}{\|x\|_V}} \\
&= f_V(A)+f_V(B)
\end{aligned}$$

Hence,

$$f_1(A) = \sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \sup_{x\neq 0} \frac{\sum_{i=1}^n \left|\sum_{j=1}^n a_{ij} x_j\right|}{\sum_{i=1}^n |x_i|} = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|$$

$$f_\infty(A) = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sup_{x\neq 0} \frac{\max_{1\leq i\leq n} \left|\sum_{j=1}^n a_{ij} x_j\right|}{\max_{1\leq i\leq n} |x_i|} = \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}|$$