ROB 501 HW1

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1

Let A be an $n \times m$ matrix and B an $m \times p$ matrix. Denote the i-th row of A by a_i and the j-th column of B by b_j.

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_p \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_p \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \dots & a_n b_p \end{bmatrix} \tag{2}$$

1.1

According to eq (**??**), we can collect b_j for each column of AB, which leads to

$$\begin{aligned} AB &= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} b_1 + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} b_2 + \dots + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} b_p \\ &= \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_p \end{bmatrix} \end{aligned}$$

1.2

According to eq (**??**), we can collect a_i for each row of AB, which leads to

$$\begin{aligned} AB =& a_1 \begin{bmatrix} b_1 & b_2 & \dots & b_p \end{bmatrix} + \\ & a_2 \begin{bmatrix} b_1 & b_2 & \dots & b_p \end{bmatrix} + \dots + \\ & a_n \begin{bmatrix} b_1 & b_2 & \dots & b_p \end{bmatrix} \\ =& \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_n B \end{bmatrix} \end{aligned}$$

1.3

According to eq (**??**), the entry of i-the row and j-th column of AB is $a_i b_j$, i.e.

$$[AB]_{ij} = a_i b_j$$

2

Given $A \in \mathbb{R}^{n \times n}$, $tr(A) = \sum_{i=1}^{n} a_i i$.

2.1

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\
tr(A) &= 1 + 5 + 9 = 15
\end{aligned}$$

2.2

$$\begin{aligned}
x &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \\
xx^T &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \\
&= \begin{bmatrix} x_1x_1 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2x_2 & \dots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \dots & x_nx_n \end{bmatrix} \\
tr(xx^T) &= \sum_{i=1}^{n} x_i x_i
\end{aligned}$$

2.3

Given $K \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{n \times n}$. Let k_i be the i-th column of K.

$$\begin{aligned}
K^TQK &= \begin{bmatrix} k_1^T \\ k_2^T \\ \vdots \\ k_m^T \end{bmatrix} Q \begin{bmatrix} k_1 & k_2 & \dots & k_m \end{bmatrix} \\
&= \begin{bmatrix} k_1^TQ \\ k_2^TQ \\ \vdots \\ k_m^TQ \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \dots & k_m \end{bmatrix} \\
&= \begin{bmatrix} k_1^TQk_1 & k_1^TQk_2 & \dots & k_1^TQk_m \\ k_2^TQk_1 & k_2^TQk_2 & \dots & k_2^TQk_m \\ \vdots & \vdots & \ddots & \vdots \\ k_m^TQk_1 & k_m^TQk_2 & \dots & k_m^TQk_m \end{bmatrix} \\
tr(K^TQK) &= \sum_{i=1}^{m} k_i^T Q k_i
\end{aligned}$$

3

A real matrix M is symmetric if it is equal to its transpose: $M^T = M$.

3.1

$$
\begin{aligned}
M =& \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \\
det(M - \lambda I) =& 0 \\
\begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} =& 0 \\
\Rightarrow \lambda_1 = \frac{5-\sqrt{5}}{2},\ \lambda_2 =& \frac{5+\sqrt{5}}{2}
\end{aligned}
$$

3.2

When $\lambda = \lambda_1$,

$$
\begin{aligned}
(M - \lambda_1 I)v_1 =& 0 \\
\Rightarrow v_1 =& \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}
\end{aligned}
$$

When $\lambda = \lambda_2$,

$$
\begin{aligned}
(M - \lambda_2 I)v_2 =& 0 \\
\Rightarrow v_2 =& \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}
\end{aligned}
$$

$$
v_1^T v_2 = 0
$$

3.3

Show that $M = A^T A$ is symmetric for any real $n \times m$ matrix A.

$$
M^T = (A^T A)^T = A^T A = M
$$

Thus, $\forall A \in \mathbb{R}^{n \times m},\ M = A^T A$ is symmetric.

3.4

- The inner product of eigenvectors $v_i^T v_i$ with $i \neq j$ is **zero**.
- The sum of all eigenvalues is the **same** as the trace of the matrix.
- The product of all eigenvalues is the **same** as the determinant of the matrix.

4

Given $X \sim N(\mu, \sigma^2)$,

$$
f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2}) \tag{3}
$$

4.1

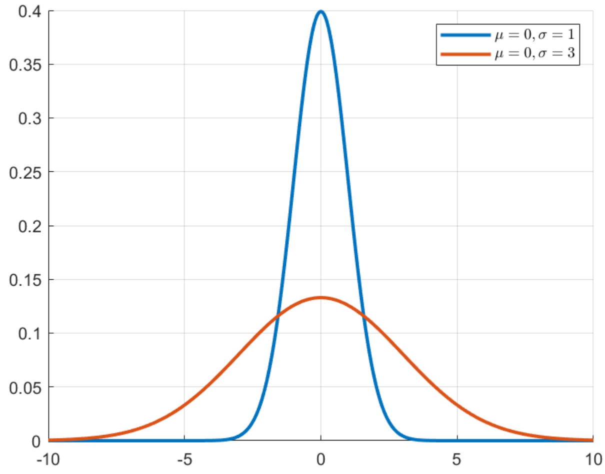

Figure 1: Density plots for different σ

4.2

For $\mu = 4$ and $\sigma = 5$,

$$P\{X \geq 4\} = \int_4^\infty f_X(x) = 0.3446$$
$$P\{-2 \leq X \leq 4\} = \int_{-2}^4 f_X(x) = 0.4436$$
$$P\{X \in A | A = [-2, 4] \cup [8, 100]\} = \int_{-2}^4 f_X + \int_8^{100} f_X = 0.5586$$

4.3

Given $Y = 2X + 4$, we have $\mu' = 2\mu + 4 = 8$ and $\sigma' = 2\sigma = 10$. Thus

$$f_X(x) = \frac{1}{10\sqrt{2\pi}} exp(-\frac{(x-8)^2}{200})$$

5

Given $f_{XY}(x, y) = K(x + y)^2,\ 0 \leq x \leq 1,\ 0 \leq y \leq 2.$

5.1

$$\begin{aligned}
\int_0^1 \int_0^2 K(x+y)^2 dydx =& 1 \\
\int_0^1 K\left(2x^2 + 4x + \frac{8}{3}\right) dx =& 1 \\
K\left(\frac{2}{3} + 2 + \frac{8}{3}\right) =& 1 \\
K =& \frac{3}{16}
\end{aligned}$$

5.2

Marginal densities and distributions

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{3}{16}(x+y)^2 dx \\ &= \frac{3}{16}y^2 + \frac{3}{16}y + \frac{1}{16} \\ F_Y(y) &= \int_0^y f_Y(v)dv \\ &= \frac{1}{16}y^3 + \frac{3}{32}y^2 + \frac{1}{16}y \\ f_X(x) &= \int_0^2 \frac{3}{16}(x+y)^2 dy \\ &= \frac{3}{8}x^2 + \frac{3}{4}x + \frac{1}{2} \\ F_X(x) &= \int_0^x f_X(u)du \\ &= \frac{1}{8}x^3 + \frac{3}{8}x^2 + \frac{1}{4}x \end{aligned}$$

5.3

Conditional densities and distributions

$$\begin{aligned} f_{X|Y}(x|Y=y) =& \frac{f_{XY}(x,y)}{f_Y(y)} &= \frac{(x+y)^2}{y^2+y+1/3} \\ F_{X|Y}(x|Y=y) =& \int_0^x f_{X|Y}(u|Y=y)du &= \frac{x^3+3yx^2+3y^2x}{3y^2+3y+1} \end{aligned}$$

6

$$\begin{aligned} min\ & x_1^2 + x_2^2 \\ s.t.\ & x_1 + 3x_2 = 4 \end{aligned}$$

where $x_1, x_2 \in \mathbb{R}$.

Let

$$f(x) = x_1^2 + x_2^2,\ g(x) = x_1 + 3x_2 - 4$$

and

$$L(x, \lambda) = f(x) - \lambda g(x)$$

where $\lambda > 0$.

$$\begin{aligned} \nabla L &= \nabla f(x) - \lambda \nabla g(x) \\ &= \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 - \lambda \\ 2x_2 - 3\lambda \end{bmatrix} \end{aligned}$$

Solve $\nabla L = \mathbf{0}$,

$$\begin{cases} 2x_1 - \lambda = 0 \\ 2x_2 - 3\lambda = 0 \\ x_1 + 3x_2 = 4 \end{cases} \Rightarrow \begin{cases} x_1 = 2/5 \\ x_2 = 6/5 \\ \lambda = 4/5 \end{cases}$$

Check the Hessian matrix

$$\nabla^2 L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

which is positive definite. Thus $x_0 = [2/5,\ 6/5]^T$ is a local minimum point of $f(x)$.

7

7.1

Marginal densities and distributions

$$\begin{aligned}
X \sim & N(1,3) \\
f_X(x) = & \frac{1}{\sqrt{6\pi}} exp(-\frac{(x-1)^2}{6}) \\
F_X(x) = & \int_{-\infty}^{x} \frac{1}{\sqrt{6\pi}} exp(-\frac{(u-1)^2}{6}) du \\
Y \sim & N(2,2) \\
f_Y(y) = & \frac{1}{2\sqrt{\pi}} exp(-\frac{(y-2)^2}{4}) \\
F_Y(y) = & \int_{-\infty}^{y} \frac{1}{2\sqrt{\pi}} exp(-\frac{(v-2)^2}{4}) dv
\end{aligned}$$