ME424 HW5

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1 Probabilistic modeling of uncertainties

Define event A = $\{S_a$ is accepted$\}$ and event B = $\{S_b$ is accepted$\}$
$\Omega = \{(A, B), (A, \bar{B}), (\bar{A}, B), (\bar{A}, \bar{B})\}$

1.1

Given

$$P(A) = 0.8,\ P(B) = 0.6,\ P(AB) = P(A)P(B)$$

We have

$$\begin{aligned} P(AB) &= P(A)P(B) = 0.48 \\ P(A\bar{B}) &= P(A)P(\bar{B}) = 0.32 \\ P(\bar{A}B) &= P(\bar{A})P(B) = 0.12 \\ P(\bar{A}\bar{B}) &= P(\bar{A})P(\bar{B}) = 0.16 \end{aligned}$$

1.2

Given

$$P(A|B) = 1,\ P(A|\bar{B}) = 0.3\ , P(\bar{A}\bar{B}) = 0.35$$

We have$P(B) = \frac{P(\bar{A}\bar{B})}{P(\bar{A}|\bar{B})} = 0.5$
Thus,

$$\begin{aligned} &P(AB) = P(A|B)P(B) = 0.5 \\ &P(A\bar{B}) = P(A|\bar{B})P(\bar{B}) = 0.15 \\ &P(\bar{A}B) = P(\bar{A}|B)P(B) = 0 \\ &P(\bar{A}\bar{B}) = 0.35 \end{aligned}$$

2 Conditional Probability and Expectation

2.1

Given

$$\begin{aligned} p_X(x) &= \frac{1}{n+1}\ for\ x = 0, 1, \cdots, n \\ p_{Y|X=x}(y|x) &= \frac{1}{x+1}\ for\ y = 0, 1, \cdots, x \end{aligned}$$

(a) $E(Y|X = x) = x/2$

(b) $E(Y) = \Sigma_{i=0}^{n} E(Y|X = x)p_X(x) = n/4$

(c) $p_{XY}(x, y) = p_X(x)p_{Y|X=x}(y|x) = \frac{1}{(n+1)(x+1)}\ for\ x \geq y,\ otherwise\ 0$

(d) $p_Y(y) = \Sigma_{x=y}^{n} p_{XY}(x,y) = \Sigma_{x=y}^{n} \frac{1}{(n+1)(x+1)}\ for\ x \geq y,\ otherwise\ 0$

(e)

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n = 100;
E = 0;
for y = 0:n
    p = 0;
    for x = y:n
        p = p+1/(n+1)/(x+1);
    end
    E = E + y*p;
end
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The result $E(Y) = 25$ for n = 100, which is the same to (b)

(f) $E(g(X)Y) = 2E(Y|X=1)p_X(1) + 2E(Y|X=n)p_X(n) = 1$

3 Conditional Density and Expectation

(a) $E(Y|X=x) = \frac{x-6}{2}$

(b) $f_X(x) = \int_0^2 f_{XY}(x,y)dy = x + \frac{1}{2},\ x \in [0,1]$
$f_Y(y) = \int_0^1 f_{XY}(x,y)dx = \frac{1}{4}(y+1),\ y \in [0,2]$
$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2x+y}{y+1},\ x \in [0,1]$
$\Rightarrow$
$E(X) = \int_0^1 x(f_X(x))dx = \frac{7}{12}$
$E(X|Y=\frac{1}{2}) = \int_0^1 x f_{X|Y}(x|\frac{1}{2})dx = \frac{11}{18}$

(c) Given

$$\begin{aligned} f(x_1|x_2) &= f(x_1) \\ f(x_1|x_3) &= f(x_1) \\ f(x_2,x_3) &= f(x_2,x_3|x_1) = \frac{f(x_1,x_2,x_3)}{f(x_1)} \end{aligned}$$

We have

$$\begin{aligned} f(x_1,x_2|x_3) &= \frac{f(x_1,x_2,x_3)}{f(x_3)} \\ &= \frac{f(x_1)f(x_2,x_3)}{f(x_3)} \\ &= f(x_1|x_3)f(x_2|x_3) \end{aligned}$$

4 Random Vectors

(a)

$$\begin{aligned} E(W) &= \begin{bmatrix} E(X_1) \\ E(X_3) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ Cov(W) &= \begin{bmatrix} Cov(X_1,X_1) & Cov(X_1,X_3) \\ Cov(X_3,X_1) & Cov(X_3,X_3) \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \end{aligned}$$

(b)

$$Cov(W,X_2) = \begin{bmatrix} Cov(X_1,X_2) \\ Cov(X_3,X_2) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(c)

$$
\begin{aligned}
Cov(V,V) &= \begin{bmatrix} Cov(X_2-1, X_2-1) & Cov(X_2-1, X_1+X_3) \\ Cov(X_2-1, X_1+X_3) & Cov(X_1+X_3, X_1+X_3) \end{bmatrix} \\
&= \begin{bmatrix} Cov(X_2, X_2) & Cov(X_2, X_1) + Cov(X_2, X_3) \\ Cov(X_2, X_1) + Cov(X_2, X_3) & Cov(X_1) + Cov(X_3) + 2Cov(X_1, X_3) \end{bmatrix} \\
&= \begin{bmatrix} 9 & 5 \\ 5 & 22 \end{bmatrix}
\end{aligned}
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