ROB 422 HW 5

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Questions

Derive $P(\neg a) = 1 - P(a)$

Proof.

$$\begin{aligned} 1 &= P(a \vee \neg a) \\ &= P(a) + P(\neg a) - P(a \wedge \neg a) \\ &= P(a) + P(\neg a) \Leftarrow P(a \wedge \neg a) = 0 \\ \Rightarrow P(\neg a) &= 1 - P(a) \end{aligned}$$

□

AI Book, Chapter 13, Ex. 13.8

a. $P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$

b. $P(Cavity) =< 0.2, 0.8 >$

c. $P(Toothache|cavity) =< 0.6, 0.4 >$

d. $P(Cavity|toothache \vee catch) =< 0.4615, 0.5384 >$

AI Book, Chapter 13, Ex. 13.16

a. *Proof.*

$$\begin{aligned} P(X|Y, e)P(Y|e) &= \frac{P(X, Y, e)}{P(Y, e)} \frac{P(Y, e)}{P(e)} \\ &= \frac{P(X, Y, e)}{P(e)} \\ &= P(X, Y|e) \end{aligned}$$

□

b. *Proof.*

$$\begin{aligned} \frac{P(X|Y, e)P(Y|e)}{P(X|e)} &= \frac{P(X, Y|e)}{P(X|e)} \\ &= P(Y|X, e) \end{aligned}$$

□

AI Book, Chapter 14, Ex. 14.5 part a

Let X be the set of all variables in the Bayesian Network except for Y and $MB(Y)$. Given $MB(Y)$, we have $P(X|Y, mb(Y)) = P(X|mb(Y)) = \alpha P(X, mb(Y))$. Also parents of Y 's children are a subset of $Y \cup MB(Y)$, without including any variables in X. Thus, all the CPT entries for Y 's children including the expand of $P(X, mb(Y))$, are constants and can be subsumed in α.

AI Book, Chapter 15, Ex. 15.2

a. $\forall t$, we have $\mathbf{P}(R_t|u_{1:t}) = \alpha\mathbf{P}(u_t|R_t)\sum_{R_{t-1}}\mathbf{P}(R_t|R_{t-1})P(R_{t-1}|u_{1:t-1})$ which increases monotonically.
For the fixed point, let its probabilities be $<\rho, 1-\rho>$, plug in $\mathbf{P}(R_t|u_{1:t}) = \mathbf{P}(R_{t-1}|u_{1:t-1})$ and we have $\langle\rho, 1-\rho\rangle = \alpha\langle 0.9, 0.2\rangle\langle 0.7, 0.3\rangle\rho + \langle 0.3, 0.7\rangle(1-\rho)$. Solving it we find $\rho \approx 0.8933$.

b.

$$\begin{aligned}
\mathbf{P}(R_{2+k}|U_1, U_2) &= \langle 0.7, 0.3\rangle P(r_{2+k-1}|U_1, U_2) + \langle 0.3, 0.7\rangle P(\neg r_{2+k-1}|U_1, U_2) \\
\mathbf{P}(r_{2+k}|U_1, U_2) &= 0.7P(r_{2+k-1}|U_1, U_2) + 0.3(1 - P(r_{2+k-1}|U_1, U_2)) \\
&= 0.4P(r_{2+k-1}|U_1, U_2) + 0.3 \\
\Rightarrow P(r2+k|U_1, U_2) &= 0.5
\end{aligned}$$

AI Book, Chapter 15, Ex. 15.13

Let S_t denotes whether a student gets enough sleep, R_t denotes whether a student has red eyes in class, and C_t denotes whether a student sleeps in class. S_t is a parent of S_{t+1}, R_t and C_t.

Given

$$\begin{aligned}
P(s_0) &= 0.7 \\
P(s_{t+1}|s_t) &= 0.8 \\
P(s_{t+1}|\neg s_t) &= 0.3 \\
P(r_t|s_t) &= 0.2 \\
P(r_t|\neg s_t) &= 0.7 \\
P(c_t|s_t) &= 0.1 \\
P(c_t|\neg s_t) &= 0.3
\end{aligned}$$

To reformulate it as a hidden Markov model, define a new variable O_t that represent the daily observation from the professor, combining the information about red eyes and sleeping in class.

$$O_t = \begin{cases} 1 & \text{if } \neg R_t \text{ and } \neg C_t \\ 2 & \text{if } R_t \text{ and } \neg C_t \\ 3 & \text{if } \neg R_t \text{ and } C_t \\ 4 & \text{if } R_t \text{ and } C_t \end{cases}$$

The probability tables for this HMM are:

1. Initial state probabilities:

$$P(s_0) = 0.7$$

2. Transition probabilities:

$$\begin{aligned}
P(s_{t+1}|s_t) &= 0.8 \\
P(s_{t+1}|\neg s_t) &= 0.3
\end{aligned}$$

3. Emission probabilities:

$$\begin{aligned}
P(o_t = 1|s_t) &= P(\neg r_t|s_t)P(\neg c_t|s_t) = 0.56 \\
P(o_t = 2|s_t) &= P(r_t|s_t)P(\neg c_t|s_t) = 0.14 \\
P(o_t = 3|s_t) &= P(\neg r_t|s_t)P(c_t|s_t) = 0.24 \\
P(o_t = 4|s_t) &= P(r_t|s_t)P(c_t|s_t) = 0.06
\end{aligned}$$

PF vs. EKF vs. UKF

a. **Particle Filter**: High computational cost when simulating large amount of particles, can handle highly non-linear dynamics with multi-model state distributions.

b. **Extended Kalman Filter**: Polynomial computational cost in state and measurement dimensionalities, can handle non-linear dynamics using first order local linear approximation, but can diverge if too large non-linearity, can handle only single-model state distributions.

c. **Unscented Kalman Filter**: Same computational complexity as EKF, can handle single-model distributed states with non-linear dynamics using second order Taylor approximation.

Implementation

Kalman Filter

a. System derivations

Define the state $\mathbf{x} = [x, y]^T$, the input $\mathbf{u} = [u_1, u_2]^T$. The system models can be derived as following.

$$\mathbf{x}_{t+1} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \mathbf{x}_t + \underbrace{\begin{bmatrix} 1.5 & 0.1 \\ 0.2 & -0.5 \end{bmatrix}}_{B} \mathbf{u} + \begin{bmatrix} \zeta_x & \\ & \zeta_y \end{bmatrix}$$

$$\mathbf{z} = \underbrace{\begin{bmatrix} 1.05 & 0.01 \\ 0.01 & 0.9 \end{bmatrix}}_{C} \mathbf{x} + \begin{bmatrix} \delta_1 & \\ & \delta_2 \end{bmatrix}$$

b. Noise covariance estimation

The motion noise covariance R and the sensor noise covariance Q are estimated as following.

$$R = \begin{bmatrix} 2.5069e^{-3} & 1.7995e^{-5} \\ 1.7995e^{-5} & 2.5106e^{-3} \end{bmatrix}$$

$$Q = \begin{bmatrix} 4.8695e^{-2} & 5.8636e^{-3} \\ 5.8636e^{-3} & 1.0121e^{-0} \end{bmatrix}$$

c. KF execution

The total errors (sum of the norm of state difference for each frame) are 21.8194.

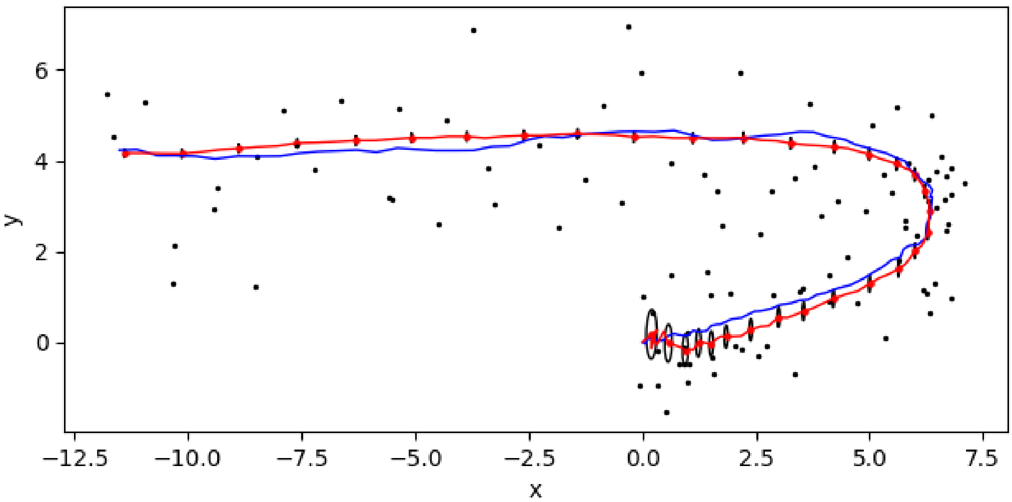

Figure 1: The comparison of estimated (in red) and ground truth (in blue) trajectories. The black dots shows the original measurements while black ellipses are the estimated pose covariance.