ARCAD Lab Typings

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Lagrangian and KKT

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{v} \end{bmatrix}$$

$$L(\boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\alpha}) = \frac{1}{2}\boldsymbol{z}^\top \boldsymbol{W}\boldsymbol{z} + \boldsymbol{g}^\top \boldsymbol{z} + \boldsymbol{\alpha}(\boldsymbol{A}\boldsymbol{z} - \boldsymbol{b}) + \boldsymbol{\gamma}(\boldsymbol{M}\boldsymbol{z} - \boldsymbol{f})$$

$$\begin{aligned}
&\nabla_{\boldsymbol{z}} L = \boldsymbol{W}\boldsymbol{z} + \boldsymbol{g} + \boldsymbol{A}^\top \boldsymbol{\alpha} + \Sigma_{i \in \mathcal{A}} \gamma_i \boldsymbol{m}_i = 0 \\
&\nabla_{\boldsymbol{\gamma}} L = \boldsymbol{A}\boldsymbol{z} - \boldsymbol{b} = 0 \\
&\nabla_{\boldsymbol{\alpha}} L = \boldsymbol{M}\boldsymbol{z} - \boldsymbol{f} = 0
\end{aligned}$$

MPC and WBC

$$\boldsymbol{q} = [\boldsymbol{q}_f^\top,\ \boldsymbol{q}_j^\top]^\top$$

$$\begin{aligned}
\min_{\ddot{\boldsymbol{q}},\mathbf{f},\boldsymbol{\tau}} \quad & \|\ddot{\mathbf{x}} - \ddot{\mathbf{x}}^{\text{cmd}}\|^2 + \|\mathbf{f} - \mathbf{f}^{\text{cmd}}\| \\
\text{s.t.} \quad & A\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} = S\boldsymbol{\tau} + J^\top \mathbf{f} \\
& \ddot{\mathbf{x}}^{\text{cmd}} = K_p(\mathbf{x} - \mathbf{x}^{\text{cmd}}) + K_d(\dot{\mathbf{x}} - \dot{\mathbf{x}}^{\text{cmd}}) \\
& \ddot{\mathbf{x}} = J\ddot{\boldsymbol{q}} + \dot{J}\dot{\boldsymbol{q}} \\
& |\mathbf{f}_x| \leq \mu \mathbf{f}_z,\ |\mathbf{f}_y| \leq \mu \mathbf{f}_z \\
& 0 \leq \mathbf{f}_z \leq \mathbf{c}\ \mathbf{f}_z^{\max}
\end{aligned}$$

$$\mathbf{x} = [\boldsymbol{q}_f^\top,\ \dot{\boldsymbol{q}}_f^\top]^\top$$

$$\begin{aligned}
\min_{\mathbf{x}_k,\mathbf{f}_k} \quad & \sum_{k=0}^{N} \|\mathbf{x}_{k+1} - \mathbf{x}_{k+1}^{\text{cmd}}\|_Q^2 + \|\mathbf{f}_k\|_R^2 \\
\text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{x}_k + \dot{\mathbf{x}}_k\ \Delta t \\
& |\mathbf{f}_{k,x}| \leq \mu \mathbf{f}_{k,z},\ |\mathbf{f}_{k,y}| \leq \mu \mathbf{f}_{k,z} \\
& 0 \leq \mathbf{f}_{k,z} \leq \mathbf{c}\ \mathbf{f}_z^{\max} \\
& \mathbf{x}_0 = \mathbf{x}_{\text{init}}
\end{aligned}$$

$$\Rightarrow \dot{\mathbf{x}}_k = f(\mathbf{x}_k, \mathbf{u}_k)$$

$$\in \mathbb{R}^6$$

$$\begin{aligned}
\mathbf{x} &= \text{FK}(\boldsymbol{q}) \in \mathbb{R}^{n_{\mathbf{x}}} \\
\dot{\mathbf{x}} &= \underbrace{\frac{\partial}{\partial \boldsymbol{q}} FK(\boldsymbol{q})}_{J \in \mathbb{R}^{n_{\mathbf{x}} \times n_{\boldsymbol{q}}}}\ \dot{\boldsymbol{q}} \\
\ddot{\mathbf{x}} &= J\ddot{\boldsymbol{q}} + \dot{J}\dot{\boldsymbol{q}}
\end{aligned}$$

3 DOF Inverse Kinematics

$$p_{\text{foot}} = \begin{bmatrix} -l_1 - l_3\,\sin(\theta_2+\theta_3) - l_2\,\sin\theta_2 \\ \sin\theta_1\,(l_3\,\cos(\theta_2+\theta_3) + l_2\,\cos\theta_2) \\ -\cos\theta_1\,(l_3\,\cos(\theta_2+\theta_3) + l_2\,\cos\theta_2) \end{bmatrix}$$

$$\begin{aligned} a &= \|p_{\text{foot}} - p_{\text{hip}}\| \\ b &= \|p_{\text{foot}} - p_{\text{abad}}\| \\ a^2 &= l_2^2 + l_3^2 - 2l_2l_3\cos\theta_3 \\ l_3^2 &= a^2 + l_2^2 - 2al_2\cos\beta_1 \\ b^2 &= a^2 + l_1^2 - 2al_1\cos\beta_2 \\ \theta_2 &= \beta_1 + \beta_2 \\ \frac{p_{\text{foot},y}}{p_{\text{foot},z}} &= -\tan\theta_1 \end{aligned}$$

Momentum Observer

$$\begin{aligned} &p = M\dot{q} \\ &\hat{\tau} = B^{-\top}K_O(p-\hat{p}) \\ &\dot{\hat{p}} = C^\top\dot{q} - G - A^\top\lambda + B^\top\tau_{\text{motor}} + B^\top\hat{\tau} \\ &\hat{p} = \hat{p} + \dot{\hat{p}}\;\Delta t \end{aligned}$$

$$\mathrm{f}_{\text{est}} = (J^\top)^\dagger\hat{\tau}$$

Floating Body Kalman Filter

Every variable is in the world frame

$$x = \begin{bmatrix} p \\ v \end{bmatrix}, \quad u = a, \quad y = \begin{bmatrix} p_z \\ v \end{bmatrix}$$

Linear system dynamics

$$\dot{x} = \begin{bmatrix} v \\ a \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{I} \end{bmatrix}}_{B} u = Ax + Bu$$
$$y = \begin{bmatrix} p_z \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \end{bmatrix}}_{C} x = Cx$$

Measurements

$$p^{\mathcal{W}}_{foot} = p^{\mathcal{W}}_{com} + R^{\mathcal{W}}_{\mathcal{B}} \mathrm{FK}(q)^{\mathcal{B}}$$
$$v^{\mathcal{W}}_{foot} = v^{\mathcal{W}}_{com} + R^{\mathcal{W}}_{\mathcal{B}} [\omega^{\mathcal{B}}_{com}]_{\times} \mathrm{FK}(q)^{\mathcal{B}} + R^{\mathcal{W}}_{\mathcal{B}} J(q)^{\mathcal{B}} \dot{q}$$

When any foot is in contact with ground, $p^{\mathcal{W}}_{foot} = v^{\mathcal{W}}_{foot} = 0$, so that the height and velocity of CoM can be measured.

Prediction from IMU

$$x_{k+1} = x_k + \Delta t\ \dot{x}_k = (I + \Delta t\ A)x_k + \Delta t\ Bu_k$$

Correction from joint encoders

$$y_k = Cx_k$$