ROB 501 HW10

Yulun Zhuang
yulunz@umich.edu

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1

1.1

Given $f(x_1, x_2, x_3) = 3x_1\left(2x_2 - x_3^3\right) + \frac{1}{3}x_2^4$ and $x^* = [1, 3, -1]^T$

$$\begin{aligned}
J_f &= \nabla^\top f \\
&= \left[\frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2}\ \frac{\partial f}{\partial x_3}\right] \\
&= \left[6x_2 - 3x_3^3 \middle| 6x_1 + \frac{4}{3}x_2^3 \middle| -9x_1x_3^2\right] \\
J_f(x^*) &= [21,\ 42,\ -9]
\end{aligned}$$

1.2

```
clc; clear;
syms x y z;
grad = jacobian(3 * x * (2 * y - z^3) + y^4 / 3, [x y z]);

f = @(x) 3 * x(1) * (2 * x(2) - x(3)^3) + x(2)^4 / 3;
x_star = [1 3 -1]';

grad_star = double(subs(grad, [x y z], x_star'))
```

```
grad_star = 1×3
    21    42    -9
```

```
delta = [1e-3 1e-3 1e-3]';

grad_hat = zeros(1, 3);

e = eye(3);

for i = 1:3
    grad_hat(i) = (f(x_star + delta(i)*e(:, i)) ...
        - f(x_star - delta(i)*e(:, i))) / (2*delta(i));
end

grad_hat
```

```
grad_hat = 1×3
   21.0000   42.0000   -9.0000
```

1.3

```
clc; clear;

x_star = [1 1 1 1 1]';

delta = [1e-5 1e-3 1e-6 1e-3 1e-3]';

grad_hat = zeros(1, 5);

e = eye(5);

for i = 1:5
    grad_hat(i) = (funcPartC(x_star + delta(i)*e(:, i)) ...
        - funcPartC(x_star - delta(i)*e(:, i))) / (2*delta(i));
end

grad_hat
```

```
grad_hat = 1×5
  117.3525  -41.3695 -636.6017   -3.8520  -11.2049
```

2

2.1

```
clc; clear; close all;

load SegwayData4KF.mat

whos
```

```
  Name        Size            Bytes  Class     Attributes

  A           4x4               128  double              
  B           4x1                32  double              
  C           1x4                32  double              
  D           1x1                 8  double              
  G           4x4               128  double              
  N           1x1                 8  double              
  P0          4x4               128  double              
  Q           1x1                 8  double              
  R           4x4               128  double              
  Ts          1x1                 8  double              
  t         500x1              4000  double              
  u         500x1              4000  double              
  x0          4x1                32  double              
  y         500x1              4000  double              
```

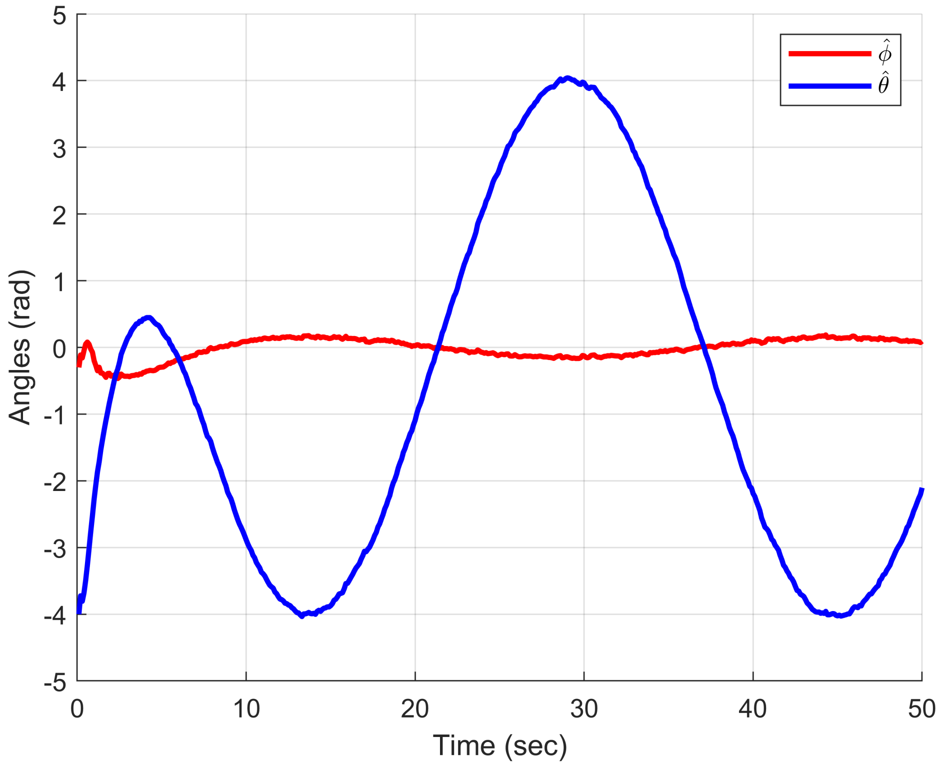
```
x_hat = zeros(length(x0), length(y));
xk = x0;
Pk = P0;
Kt = zeros(length(x0), length(y));

for i=1:length(y)
    x_hat(:, i) = xk;
    Kk = (Pk * C') * inv(C * Pk * C' + Q);
    Kt(:, i) = Kk;
    xk = A * xk + B * u(i) + A * Kk * (y(i) - C * xk);
    Pk = A * (Pk - Kk * C * Pk) * A' + G * R * G';
end
```

2.2

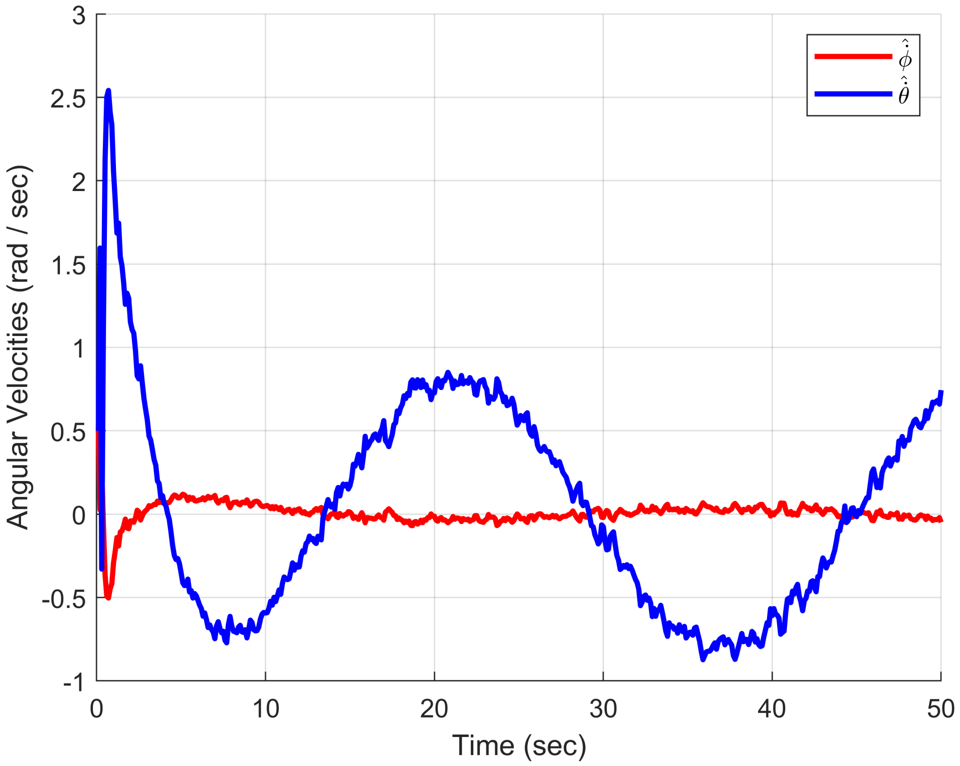
```
figure()
plot(t, x_hat(1, :), 'r', "LineWidth",2)
hold on; grid on; box off;
plot(t, x_hat(2, :), 'b', "LineWidth",2)
hold off;
legend(["$$\hat{\phi}$$", "$$\hat{\theta}$$"], 'interpreter', 'latex')

xlabel("Time (sec)")
ylabel("Angles (rad)")
```



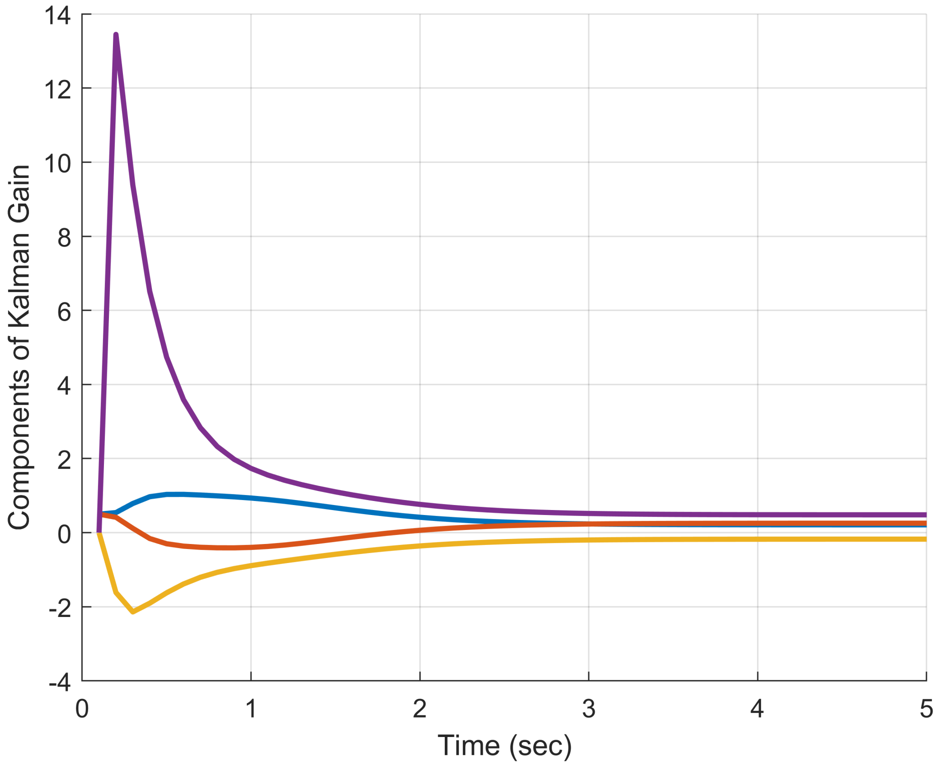

```
figure()
plot(t, x_hat(3, :), 'r', "LineWidth",2)
hold on; grid on; box off;
plot(t, x_hat(4, :), 'b', "LineWidth",2)
hold off;
legend(["$$\hat{\dot\phi}$$", "$$\hat{\dot\theta}$$"], 'interpreter', 'latex')

xlabel("Time (sec)")
ylabel("Angular Velocities (rad / sec)")
```




```
figure()
plot(t, Kt(1, :), "LineWidth",2)
hold on; grid on; box off;
plot(t, Kt(2, :), "LineWidth",2)
plot(t, Kt(3, :), "LineWidth",2)
plot(t, Kt(4, :), "LineWidth",2)
hold off;
xlim([0, 5])

xlabel("Time (sec)")
ylabel("Components of Kalman Gain")
```



2.3

```
Kk
```

```
Kk = 4×1
    0.2113
    0.2559
   -0.1744
    0.4816
```

```
[Kss,Pss] = dlqe(A,G,C,R,Q);
Kss
```

```
Kss = 4×1
    0.2113
    0.2559
   -0.1744
    0.4816
```

3

$$\begin{cases} x_0 \sim N(1,\ 0.25) & \Rightarrow \hat{x_0} = 1,\ P_0 = 0.25 \\ u_1 \sim N(10,\ 16) & \Rightarrow \hat{u}_1 = 10,\ R = 16 \end{cases}$$

$$\begin{aligned} \Rightarrow & A = 1,\ B = 0.1 \\ & x_1 = x_0 + u_1 \cdot \delta_t \\ & z_t = -\frac{2}{c} x_t + \frac{10}{c} \Rightarrow C = -\frac{2}{c} \end{aligned}$$

Given $z_1 = 2.2 \times 10^{-8}$, $Q = 10^{-18}$ and $t = 0.1$.

Prediction Step:

$$\begin{aligned} \hat{x}_{1|0} &= A\hat{x}_0 + B\hat{u}_1 \\ &= 1 + 0.1 \times 10 \\ &= 2 \\ P_{1|0} &= AP_0A^\top + BRB^\top \\ &= 0.25 + 0.01 \times 16 \\ &= 0.41 \end{aligned}$$

Measurement Update Step:

$$\begin{aligned} K_1 &= P_{1|0}C^\top \left(CP_{1|0}C^\top + Q\right)^{-1} \\ &= 0.41 \times -\frac{2}{c}\left(\frac{4}{c^2} \times 0.41 + Q\right)^{-1} \\ &= -1.422 \times 10^8 \\ \hat{z}_{1|0} &= \frac{2}{c}\left(5 - \hat{x}_{1|0}\right) \\ &= \frac{6}{c} \\ \hat{x}_{1|1} &= \hat{x}_{1|0} + K_1\left(z_1 - \hat{z}_{1|0}\right) \\ &= 2 + K_1\left(2.2 \times 10^{-8} - \frac{6}{c}\right) \\ &= 1.7156 \\ P_{1|1} &= P_{1|0} - K_1CP_{1|0} \\ &= 0.41 + K_1 \times \frac{2}{c} \times 0.41 \\ &= 0.0213 \end{aligned}$$

Hence, $x_1 \sim N(\hat{x}_{1|1},\ P_{1|1})$, where $\hat{x}_{1|1} = 1.7156$ and $P_{1|1} = 0.0213$.

4

$$\hat{x} = arg \min_{x^Tx=1} x^T A^T A x$$

Show that if A is real, then $\hat{x}$ is given by the last column of V where $A = U\Sigma V^T$ is the SVD of A.

Proof. Let $M = A^T A$, M is an $n \times n$ real symmetric matrix. Apply SVD to A, we have $A = U\Sigma V^T$, where $V \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and its columns are eigenvectors of M in descending order according to its eigenvalues.

Construct $L(x, \lambda) = x^T Mx - \lambda(x^T x - 1)$, and $\lambda > 0$.

$$\nabla L = \frac{dL}{dx} = 2Mx - 2\lambda x$$

Solve $\nabla L = 0$, we have $Mx = \lambda x$.

Therefore, λ is the eigenvalue of M and x is the corresponding eigenvector.

$$x^T Mx = x^T(\lambda x) = \lambda x^T x = \lambda$$

Solve $det(M - \lambda I) = 0$. Since the eigenvalues are real and finite in number, there exists a largest eigenvalue, denoted λ_{max}, and a smallest eigenvalue, denoted λ_{min}, i.e. $\boldsymbol{\lambda} = [\lambda_{max}, \ldots, \lambda_{min}]^T$

Hence, $\hat{x}$ is its corresponding eigenvector given by $(M - \lambda_{min} I)x = 0$, which is the last column of V.

□

5

```
clc; clear;
A = [4.041 7.046 3.014;
     10.045 17.032 7.027;
     16.006 27.005 11.048];

[U V D] = svd(A);
V
```

```
V = 3×3
   40.2854         0         0
         0    0.1859         0
         0         0    0.0051
```

```
A_hat = U(:, 1:2) * V(1:2, 1:2) * D(:, 1:2)'
```

```
A_hat = 3×3
    4.0420    7.0450    3.0150
   10.0426   17.0345    7.0245
   16.0073   27.0037   11.0493
```

```
A_delta = A - A_hat
```

```
A_delta = 3×3
   -0.0010    0.0010   -0.0010
    0.0024   -0.0025    0.0025
   -0.0013    0.0013   -0.0013
```