HW 9

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1. $(\mathcal{X}, \mathbb{R}, \langle\cdot,\cdot\rangle)$ be a finite-dimension inner-product space. $\{y_1, \cdots, y_p\}$ be a linear independent set in $\mathcal{X}$, and $c_1, \cdots, c_p$ be real constants.

Define $V = \{x \in \mathcal{X} \mid \langle x, y_i\rangle = c_i,\ 1 \le i \le p\}$.

(a) There exist a unique $x_0 \in \text{span}\{y_1, \cdots, y_p\}$ s.t. $\langle x_0, y_i\rangle = c_i$, $1 \le i \le p$.

proof:

Let $x_0 = \sum_{i=1}^{p} \alpha_i y_i$, $\alpha_i \in \mathbb{R}$

Since $x_0 \in V$, $\langle x_0, y_i\rangle = c_i$, $1 \le i \le p$.

$$\begin{aligned}\langle x_0, y_i\rangle &= \langle \textstyle\sum_{i=1}^{p} \alpha_i y_i, y_i\rangle \\ &= \alpha_1\langle y_1, y_i\rangle + \alpha_2\langle y_2, y_i\rangle + \cdots + \alpha_p\langle y_p, y_i\rangle \\ &= [\langle y_1, y_i\rangle \ \cdots \ \langle y_p, y_i\rangle]\begin{bmatrix}\alpha_1\\ \vdots\end{bmatrix}\end{aligned}$$

$$\Rightarrow \underbrace{\begin{bmatrix}\langle x_0, y_1\rangle \\ \vdots \\ \langle x_0, y_p\rangle\end{bmatrix}}_{\beta} = \underbrace{\begin{bmatrix}\langle y_1, y_1\rangle & \cdots & \langle y_p, y_1\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1, y_p\rangle & \cdots & \langle y_p, y_p\rangle\end{bmatrix}}_{G^T} \underbrace{\begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_p\end{bmatrix}}_{\alpha}$$

G is invertibe because the set $\{y_1, \cdots y_p\}$ is linear independent,

Hence $G^T\alpha = \beta$ has one unique solution

$$\alpha = G^{-T}\beta$$

□

(b) Define $M = (\text{span}\{y_1, \cdots, y_p\})^\perp$, $V = x_0 + M$. Or $x \in V$ iff $(x - x_0) \perp \text{span}\{y_1, \cdots, y_p\}$

proof: Suppose $x \in V$, $\langle x, y_i\rangle = c_i$ for $1 \le i \le p$.

Since $x_0 \in V$, we have $\langle x_0, y_i\rangle = c_i$ for $1 \le i \le p$.

$\Leftrightarrow \langle x - x_0, y_i\rangle = c_i - c_i = 0$, $1 \le i \le p$

$\Leftrightarrow (x - x_0) \perp y_i$, $1 \le i \le p$

$\Leftrightarrow (x - x_0) \perp \text{span}\{y_1, \cdots, y_p\}$ □

(c) $\exists$ unique $v^* \in V$, s.t. $v^* = \arg\min_{v \in V} \|v\|$, and $v^* \perp \text{span}\{y_1, \cdots, y_p\}$

$\forall\ v \in V$, iff $v = x_0 + m$, $m \in M$.

$\Rightarrow v = x_0 - m \quad (m \in M \Leftrightarrow -m \in M)$

$\Rightarrow \inf_{v \in V} \|v\| = \inf_{m \in M} \|x_0 - m\| = d(x_0, M)$

From the Projection Theorem, there exist unique $m^* \in M$,

s.t. $\|x_0 - m^*\| = d(x_0, M) = \min_{m \in M} \|x_0 - m\|$

And $m^* \in M$ is characterized by $x_0 - m^* \perp M$.

$\Rightarrow v^* = x_0 - m^*$, and $v^* \perp M$ $\quad \square$

2, Prove that $v^* = x_0$.

$v^* \perp M \Leftrightarrow v^* \in \mathrm{span}\{y_1, \cdots, y_p\}$ ①

$x_0 \in \mathrm{span}\{y_1, \cdots, y_p\}$ ②

$M = (\mathrm{span}\{y_1, \cdots, y_p\})^\perp$ is a subspace $\Leftrightarrow \{0\} \in M$ ③

$\Rightarrow v^* = x_0 + \{0\} = x_0 = \sum_{i=1}^{p} \alpha_i y_i$, $1 \le i \le p$

and from 1.(a) $G^T \alpha = \beta$

$$\underbrace{\begin{bmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_p, y_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle y_1, y_p \rangle & \cdots & \langle y_p, y_p \rangle \end{bmatrix}}_{G^T} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}}_{\beta} \quad \square$$

3, Suppose X and Y are jointly distributed normal R.V.

$$\mu = \begin{bmatrix} E(X) \\ E(Y) \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}, \quad \Sigma = \mathrm{cov}\left(\begin{bmatrix} X \\ Y \end{bmatrix}, \begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} P & PC^T \\ CP & CPC^T + Q \end{bmatrix}$$

(a)
$$\mu_{X|Y=y} = \mu_X + \Sigma_{XY} \Sigma_Y^{-1} (y - \bar{y})$$
$$= \bar{x} + PC^T (CPC^T + Q)^{-1} (y - \bar{y})$$

$$\Sigma_{X|Y=y} = \Sigma_X - \Sigma_{XY} \Sigma_Y^{-1} \Sigma_{YX}$$
$$= P - PC^T (CPC^T + Q)^{-1} CP$$

$\mu_{X|Y=y}$ is equivilant to $\hat{x}$ in MVE when $\bar{x}=\bar{y}=0$

(b) Schur Complements of CPC^T+Q

$$P - PC^T(CPC^T+Q)^{-1}CP$$

this is equivilant to $\Sigma_{X|Y=y}$.

4. $\mathcal{X} = \text{span}\{1, t, t^2, \sin(\pi t)\}$, $\langle f, g\rangle = \int_0^2 f(\tau)g(\tau)d\tau$.

(a) Find the minimum norm $f \in \mathcal{X}$, s.t. $\langle f, t\rangle = 2$

$$\langle f, t\rangle = \alpha\langle t, t\rangle = 2$$

$$\Rightarrow \alpha = \frac{3}{4}, \quad f = \alpha t$$

(b) Find the minimum norm $f \in \mathcal{X}$, s.t. $\langle f, t\rangle = 2$, $\langle f, \sin(\pi t)\rangle = \pi$

$$\begin{bmatrix} \langle t, t\rangle & \langle \sin(\pi t), t\rangle \\ \langle t, \sin(\pi t)\rangle & \langle t, t\rangle \end{bmatrix} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}}_{\alpha} = \begin{pmatrix} 2 \\ \pi \end{pmatrix}$$

$$\Rightarrow \alpha = \begin{pmatrix} -\frac{6\pi^3(\cos(2\pi)+1)(2\pi-\sin(2\pi))}{\sigma_1} \\ -\frac{4\pi^2(6\pi\cos(2\pi)-3\sin(2\pi)+4\pi^3)}{\sigma_1} \end{pmatrix} \quad \alpha = \begin{bmatrix} \frac{3\pi^2}{2\pi^2-3} \\ \frac{2\pi^2+3\pi}{2\pi^2-3} \end{bmatrix}$$

where

$$\sigma_1 = 12\pi^2\cos(4\pi) - 12\pi\sin(4\pi) - 3\cos(4\pi) + 4\pi^3\sin(4\pi) + 12\pi^2 - 16\pi^4 + 3$$

$$f = \alpha_1 t + \alpha_2 \sin(\pi t)$$

5. Underdetermined Equations $Ax=b$, $b \in \mathbb{R}^p$, $x \in \mathbb{R}^n$, $n > p$

(a) let $A = \begin{bmatrix} y_1^T \\ \vdots \\ y_p^T \end{bmatrix}$, $y_i \in \mathbb{R}^n$, $b = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$

$$\Rightarrow Ax = b$$

$$\begin{bmatrix} y_1^T \\ \vdots \\ y_p^T \end{bmatrix} x = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

$$\begin{bmatrix} y_1^T x \\ \vdots \\ y_p^T x \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

$\Rightarrow$ the rows of LHS is $\langle y_i, x\rangle = \langle x, y_i\rangle = c_i$, $1 \le i \le p$

Hence, $\{y_1, \cdots y_p\}$ is a linear independent set, b is a constant vector, The minimum norm solution $\hat{x}$ is

$$\hat{x} = \arg\min_{Ax=b} \|x\|$$

$$= \sum_{i=1}^{p} \alpha_i y_i$$

where α_i satisfied

$$\underbrace{\begin{bmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_p, y_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle y_1, y_p \rangle & \cdots & \langle y_p, y_p \rangle \end{bmatrix}}_{G^T} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}}_{b}$$

$\Rightarrow \quad G^T = AA^T$

$\Rightarrow \quad \alpha = (AA^T)^{-1} b$

$\Rightarrow \quad \hat{x} = \sum_{i=1}^{p} \alpha_i y_i$

$$= [\, y_1 \quad \cdots \quad y_p \,] \alpha$$

$$= A^T (AA^T)^{-1} b$$

(b) Assum $\langle x, z \rangle = x^T Q z$, $Q > 0$, thus $\|x\| = (x^T Q x)^{\frac{1}{2}}$.

Let $A = \begin{bmatrix} y_1^T \\ \vdots \\ y_p^T \end{bmatrix}$, $y_i \in \mathbb{R}^n$, $b = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$

Define $v_i = (y_i^T Q^{-1})^T = Q^{-1} y_i$, $v_i \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} y_1^T x \\ \vdots \\ y_p^T x \end{bmatrix}$$

$$= \begin{bmatrix} y_1^T Q^{-1} Q x \\ \vdots \\ y_p^T Q^{-1} Q x \end{bmatrix}$$

$$= \begin{bmatrix} (Q^{-1} y_1)^T Q x \\ \vdots \\ (Q^{-1} y_p)^T Q x \end{bmatrix}$$

$$\left| (Q^{-1} y_p)^T Q x \right|$$

$$= \begin{bmatrix} \langle v_1, x \rangle \\ \vdots \\ \langle v_p, x \rangle \end{bmatrix}$$

$\Rightarrow$ The rows of $Ax = b$ is $\langle v_i, x \rangle = c_i$, $1 \le i \le p$

Hence, $\{v_1, \cdots, v_p\}$ is a linear independent set, b is a constant vector, the minimum norm solution $\hat{x}$ is

$$\hat{x} = \underset{Ax=b}{\arg\min} \|x\|$$
$$= \sum_{i=1}^{p} \alpha_i v_i$$

where α_i satisfied

$$\underbrace{\begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_p, v_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle v_1, v_p \rangle & \cdots & \langle v_p, v_p \rangle \end{bmatrix}}_{G^T} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}}_{b}$$

$$\Rightarrow G^T = A Q^{-1} A^T$$
$$\Rightarrow \alpha = (A Q^{-1} A^T)^{-1} b$$
$$\Rightarrow \hat{x} = \sum_{i=1}^{p} \alpha_i v_i$$
$$= [v_1 \cdots v_p] \alpha$$
$$= (A Q^{-1})^T \alpha$$
$$= Q^{-1} A^T (A Q^{-1} A^T)^{-1} b$$

6, QR factorization

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Let $A_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $\|A_1\| = \sqrt{35}$, $\|A_2\| = 2\sqrt{14}$

$$v_1 = \frac{A_1}{\|A_1\|} = \frac{1}{\sqrt{35}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$v_2' = A_2 - \langle A_2, v_1 \rangle v_1$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \frac{44}{\sqrt{35}} \times \frac{1}{\sqrt{35}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7429 \\ 0.2286 \\ -0.2857 \end{bmatrix}$$

$$v_2 = \frac{v_2'}{\|v_2'\|} = \begin{bmatrix} 0.8971 \\ 0.2760 \\ -0.3450 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 0.1690 & 0.8971 \\ 0.5071 & 0.2760 \\ 0.8452 & -0.3450 \end{bmatrix}$$

$$\Rightarrow R = \begin{bmatrix} \langle A_1, v_1 \rangle & \langle A_2, v_1 \rangle \\ 0 & \langle A_2, v_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 5.9161 & 7.4374 \\ 0 & 0.8281 \end{bmatrix}$$

These [Q, R] is the same as qr(A, 0),

qr(A) is full QR decompostion.