IOE 611: Homework 3

Yulun Zhuang

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Problem 1

Prove that $x^* = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & -1 \leq x_i \leq 1, \quad i = 1, 2, 3 \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}, \quad r = 1$$

Proof. The gradient of the objective function at x^* is

$$\begin{aligned} \nabla f_0(x) &= Px + q \\ \nabla f_0(x^*) &= [-1, 0, 2]^\top \end{aligned}$$

The optimality condition is

$$\begin{aligned} \nabla f_0(x^*)^\top (y - x^*) &= (-1)(y_1 - 1) + 2(y_3 + 1) \\ &= -y_1 + 2y_3 + 3 \end{aligned}$$

which is always non-negative for the feasible region $\{y_i \mid -1 \leq y_i \leq 1\}$. $\square$

Problem 2

Equivalent convex problems. Show that the following three convex problems are equivalent. Carefully explain how the solution of each problem is obtained from the solution of the other problems. The problem data are the matrix $A \in \mathbb{R}^{m \times n}$ (with rows $a_i^\top$), the vector $b \in \mathbb{R}^m$, and the constant $M > 0$.

(a) The *robust least-squares problem*

$$\text{minimize} \quad \sum_{i=1} m\Phi(a_i^\top x - b_i)$$

with variable $x \in \mathbb{R}^n$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is define as

$$\Phi(u) = \begin{cases} u^2, & |u| \leq M \\ M(2|u| - M), & |u| > M \end{cases}$$

(b) The *least-squares problem with variable weights*

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} (a_i^\top x - b_i)^2/(w_i + 1) + M^2 \mathbf{1}^\top w \\ \text{subject to} \quad & w \succeq 0 \end{aligned}$$

with variables $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, and domain $D = \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^m \mid w \succ -1\}$.

(c) The *quadratic program*

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} (u_i^2 + 2Mv_i) \\ \text{subject to} \quad & -u - v \preceq Ax - b \preceq u + v \\ & 0 \preceq u \preceq M\mathbf{1} \\ & v \succeq 0 \end{aligned}$$

Proof. TODO $\square$

Problem 3

4.13

Robust LP with interval coefficients. Consider the problem, with variable $x \in \mathbb{R}^n$,

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax \preceq b, \ \forall A \in \mathcal{A} \end{array}$$

where $\mathcal{A} \subseteq \mathbb{R}^{m \times n}$ is the set

$$\mathcal{A} = \{A \in \mathbb{R}^{m \times n} \mid \bar{A}_{ij} - V_{ij} \leq A_{ij} \leq \bar{A}_{ij} + V_{ij}, i = 1, \ldots, m, j = 1, \ldots, n\}$$

(The matrices $\bar{A}$ and V are given.) This problem can be interpreted as an LP where each coefficient of A is only known to lie in an interval, and we require that x must satisfy the constraints for all possible values of the coefficients.

Express this problem as an LP.

Solution. Since x must satisfy $Ax \preceq b$ for all possible values of A, the problem is equivalent to

$$\begin{array}{ll} \min\limits_{x} & c^\top x \\ \text{s.t.} & \bar{A}x + V|x| \succeq b \end{array}$$

where $|x| = [|x_1|, |x_2|, \ldots, |x_n|]^\top$.

By introducing an additional variable $z \in \mathbb{R}^n$, it can be transformed as LP at the expense of an additional inequality constraint

$$\begin{array}{ll} \min\limits_{x} & c^\top x \\ \text{s.t.} & \bar{A}x + Vz \preceq b \\ & -z \preceq x \preceq z \end{array}$$

□

8.16

Maximum volume rectangle inside a polyhedron. Formulate the following problem as a convex optimization problem. Find the rectangle

$$\mathcal{R} = \{x \in \mathbb{R}^n \mid l \preceq x \preceq u\}$$

of maximum volume, enclosed in a polyhedron $\mathcal{P} = \{x \mid Ax \preceq b\}$. The variables are $l, u \in \mathbb{R}^n$.

Solution. Define $a_{ij}^+ = \max\{a_{ij}, 0\}, a_{ij}^- = \max\{-a_{ij}, 0\}$, then the original problem can be formulated as follows

$$\begin{array}{ll} \max\limits_{l,u} & (\prod\limits_{i=1}^{n}(u_i - l_i))^{1/n} \\ \text{s.t.} & \sum\limits_{i=1}^{n}(a_{ij}^+ u_j - a_{ij}^- l_j) \leq b_i \\ & i = 1, \ldots, m \end{array}$$

□

Problem 4

Minimum fuel optimal control. We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbb{R}, t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t),\ t = 0, \ldots, N-1$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}$ are given. We assume that the initial state is zero, i.e., $x(0) = 0$.

The *minimum fuel optimal control problem* is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t))$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and x_{des} is the (given) desired final or target state. The function $f : \mathbb{R} \to \mathbb{R}$ is the fuel use map for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

Formulate the minimum fuel optimal control problem as an LP.

Solution.

$$\begin{aligned} \min_{u} \quad & \sum_{t=0}^{N-1} f(u(t)) \\ \text{s.t.} \quad & x_{\text{des}} = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{bmatrix} u \end{aligned}$$

Define $z(t)$, such that $z(t) \geq f(u(t))$. Then we push the objective into inequality constraints

$$\begin{aligned} \min_{z,u} \quad & \sum_{t=0}^{N-1} z(t) \\ \text{s.t.} \quad & x_{\text{des}} = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{bmatrix} u \\ & z \succeq |u| \\ & z \succeq 2|u| - 1 \end{aligned}$$

which is also equivalent to a LP

$$\begin{aligned} \min_{z,u} \quad & \sum_{t=0}^{N-1} z(t) \\ \text{s.t.} \quad & x_{\text{des}} = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{bmatrix} u \\ & -z \preceq u \preceq z \\ & -(z+1)/2 \preceq 2u \preceq (z+1)/2 \end{aligned}$$

$\square$

Problem 5

Linear separation of two sets of ellipsoids. Suppose we are given $K + L$ ellipsoids

$$\mathcal{E}_i = \{P_i u + q_i \mid \|u\|_2 \leq 1\}, \quad i = 1, \ldots, K + L,$$

where $P_i \in \mathbb{S}^n$. We are interested in finding a hyperplane that strictly separates $\mathcal{E}_1, \ldots, \mathcal{E}_K$ from $\mathcal{E}_{K+1}, \ldots, \mathcal{E}_{K+L}$, i.e., we want to compute $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ such that

$$\begin{aligned} a^\top x + b > 0 \quad &\text{for} \quad x \in \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_K, \\ a^\top x + b < 0 \quad &\text{for} \quad x \in \mathcal{E}_{K+1} \cup \ldots \cup \mathcal{E}_{K+L}, \end{aligned}$$

or prove that no such hyperplane exists. Express this problem as an SOCP feasibility problem.

Solution. To form the separating hyperplane, a and b must satisfy

$$\begin{aligned} 0 < \min_u \ a^\top (P_i u + q_i) + b, i = 1, \ldots, K \\ 0 > \max_u \ a^\top (P_i u + q_i) + b, i = K+1, \ldots, K+L \end{aligned}$$

Given that $|a^\top P_i u| \leq \|P_i^\top a\|$,

$$\begin{aligned} \min_u \ a^\top (P_i u + q_i) + b \geq -\|P_i^\top a\|_2 + a^\top q_i + b, i = 1, \ldots, K \\ \max_u \ a^\top (P_i u + q_i) + b \leq \|P_i^\top a\|_2 + a^\top q_i + b, i = K+1, \ldots, K+L \end{aligned}$$

Thus, the SOCP problem can be formulated as follows

$$\begin{aligned} \min_{a,b} \quad & 0 \\ \text{s.t.} \quad & \|P_i^\top a\|_2 < a^\top q_i + b, i = 1, \ldots, K \\ & \|P_i^\top a\|_2 < -a^\top q_i - b, i = K+1, \ldots, K+L \end{aligned}$$

$\square$

Problem 6

Hyperbolic constraints as SOC constraints. Verify that $x \in \mathbb{R}^n, y, z \in \mathbb{R}$ satisfy

$$x^\top x \leq yz, \quad y \geq 0, \quad z \geq 0$$

if and only if

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z, \quad y \geq 0, \quad z \geq 0$$

Use this observation to cast the following problems as SOCPs

(a) *Maximizing harmonic mean.*

$$\max_x \quad \left(\sum_{i=1}^m 1/(a_i^\top x - b_i) \right)^{-1}$$

with domain $\{x \mid Ax \succ b\}$, where $a_i^\top$ is the ith row of A.

(b) *Maximizing geometric mean.*

$$\max_x \quad \left(\prod_{i=1}^m (a_i^\top x - b_i) \right)^{1/m}$$

with domain $\{x \mid Ax \succ b\}$, where $a_i^\top$ is the ith row of A.

Solution. For $y \geq 0, z \geq 0$,

$$\begin{aligned} & \left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z \\ \Leftrightarrow \quad & 4x^\top x + (y - z)^2 \leq (y + z)^2 \\ \Leftrightarrow \quad & x^\top x \leq yz \end{aligned}$$

For part (a), the original optimization is equivalent to

$$\begin{aligned} \min_x \quad & \sum_{i=1}^m 1/(a_i^\top x - b_i) \\ \text{s.t.} \quad & Ax \succ b \end{aligned}$$

Define $t \in \mathbb{R}^m$, such that $1/(a_i^\top x - b_i) \leq t_i$, and push the objective into constraints

$$\begin{aligned} \min_x \quad & \sum_{i=1}^m t_i \\ \text{s.t.} \quad & Ax \succ b \\ & t_i(a_i^\top x - b_i) \geq 1 \\ & t_i \geq 0 \\ & i = 1, \ldots, m \end{aligned}$$

Use the given observation, and transform it as a SOCP

$$
\begin{aligned}
\min_{x} \quad & \sum_{i=1}^{m} t_i \\
\text{s.t.} \quad & Ax \succ b \\
& \left\| \begin{bmatrix} 2 \\ a_i^\top x - b_i - t_i \end{bmatrix} \right\|_2 \leq a_i^\top x - b_i + t_i \\
& t_i \geq 0 \\
& i = 1, \ldots, m
\end{aligned}
$$

$\square$

Problem 7

Robust quadratic programming. Consider the robust variation of the (convex) quadratic program

$$\begin{aligned} \min_x \quad & (1/2)x^\top Px + q^\top x + r \\ \text{s.t.} \quad & Ax \preceq b \end{aligned}$$

For simplicity we assume that only the matrix P is subject to errors, and the other parameters (q, r, A, b) are exactly known. The robust quadratic program is defined as

$$\begin{aligned} \min_x \quad & \sup_{P \in \mathcal{E}} ((1/2)x^\top Px + q^\top x + r) \\ \text{s.t.} \quad & Ax \preceq b \end{aligned}$$

where $\mathcal{E}$ is the set of possible matrices P.

For each of the following sets $\mathcal{E}$, express the robust QP as a convex problem. Be as specific as you can. If the problem can be expressed in a standard form (e.g., QP, QCQP, SOCP, SDP), say so.

(a) A finite set of matrices: $\mathcal{E} = \{P_1, \ldots, P_K\}$, where $P_i \in \mathbb{S}^n_+, i = 1, \ldots, K$.

(b) A set specified by a nominal value $P_0 \in \mathbb{S}^n_+$ plus a bound on the eigenvalues of the deviation $P - P_0$:

$$\mathcal{E} = \{P \in \mathbb{S}^n \mid -\gamma I \preceq P - P_0 \preceq \gamma I\}$$

where $\gamma \in \mathbb{R}$ and $P_0 \in \mathbb{S}^n_+$.

Solution. For part (a), it can be expressed as a QCQP as follows

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & Ax \preceq b \\ & (1/2)x^\top P_i x + q^\top x + r \leq t, \ i = 1, \ldots, K \end{aligned}$$

For part (b), given that $P \preceq P_0 + \gamma I$, it can be expressed as a QP as follows

$$\begin{aligned} \min_x \quad & (1/2)x^\top (P_0 + \gamma I)x + q^\top x + r \\ \text{s.t.} \quad & Ax \preceq b \end{aligned}$$

$\square$

Problem 8

Express the following problems as convex optimization problems

(a) Minimize $\max\{p(x), q(x)\}$, where p and q are posynomials.

(b) Minimize $exp(p(x)) + exp(q(x))$, where p and q are posynomials.

(c) Minimize $p(x)/(r(x) - q(x))$, subject to $r(x) > q(x)$, where p, q are posynomials, and r is a monomial.

Solution. (a)

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & p(x)/t \leq 1 \\ & q(x)/t \leq 1 \\ & x_i = e^{y_i} \end{aligned}$$

(b)

$$\begin{aligned} \min \quad & e^{t_1} + e^{t_2} \\ \text{s.t.} \quad & p(x)/t \leq 1 \\ & q(x)/t \leq 1 \\ & x_i = e^{y_i} \end{aligned}$$

(c)

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \frac{p(x)/t + q(x)}{r(x)} \leq 1 \\ & x_i = e^{y_i} \end{aligned}$$

$\square$

Problem 9

LPs, QPs, QCQPs, and SOCPs as SDPs. Express the following problems as SDPs.

Recall the semidefinite programming (SDP),

$$\begin{aligned} \min_x \quad & c^\top x + d \\ \text{s.t.} \quad & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{aligned}$$

where $G, F_1, \ldots, F_n \in \mathbb{S}^k$, and $A \in \mathbb{R}^{p \times n}$.

Recall the Schur Complement, for $A \in \mathbb{S}^r_{++}, C \in \mathbb{S}^s, B \in \mathbb{R}^{r \times s}$, we have

$$C - B^\top A^{-1} B \succeq 0 \quad \Leftrightarrow \quad \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \succeq 0$$

(a) The LP

$$\begin{aligned} \text{minimize} \quad & c^\top x + d \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

Rewrite the inequality constraint as a diagonal matrix form

$$\begin{aligned} Gx \preceq h \quad & \Leftrightarrow \quad g_i^\top x - h_i \leq 0, \;\; i = 1, \ldots, n \\ & \Leftrightarrow \quad \begin{bmatrix} g_1^\top x - h_1 & & \\ & \ddots & \\ & & g_n^\top x - h_n \end{bmatrix} \preceq 0 \\ & \Leftrightarrow \quad x_1 \begin{bmatrix} g_{11} & & \\ & \ddots & \\ & & g_{n1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} g_{1n} & & \\ & \ddots & \\ & & g_{nn} \end{bmatrix} - \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix} \preceq 0 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \min_x \quad & c^\top x + d \\ \text{s.t.} \quad & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b \end{aligned}$$

(b) The QP

$$\begin{aligned} \text{minimize} \quad & (1/2) x^\top P x + q^\top x + r \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

Decompose $P = WW^\top$ where $W \in \mathbb{R}^{n\times r}$. Define t, such that $x^\top WW^\top x \le t$.

$$t - (W^\top x)(I)^{-1}(W^\top x) \ge 0 \quad \Leftrightarrow \quad \begin{bmatrix} I & W^\top x \\ x^\top W & t \end{bmatrix} \succeq 0$$

$\Leftrightarrow$

$$\begin{aligned} \min_{x,t} \quad & (1/2)t + q^\top x + r \\ \text{s.t.} \quad & \begin{bmatrix} I & W^\top x \\ x^\top W & t \end{bmatrix} \succeq 0 \\ & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b \end{aligned}$$

(c) The QCQP

$$\begin{aligned} \text{minimize} \quad & (1/2)x^\top P_0 x + q_0^\top x + r_0 \\ \text{subject to} \quad & (1/2)x^\top P_i x + q_i^\top x + r_i \le 0,\ i = 1, \dots, m \\ & Ax = b \end{aligned}$$

Decompose $P_i = W_i W_i^\top$ where $W_i \in \mathbb{R}^{n\times r_i}$. Define t_i, such that $x^\top W_i W_i^\top x \le t_i,\ i = 0, \dots, m$.

$\Leftrightarrow$

$$\begin{aligned} \min_{x,t_i} \quad & (1/2)t_0 + q_0^\top x + r_0 \\ \text{s.t.} \quad & \begin{bmatrix} I & W_i^\top x \\ x^\top W_i & t_i \end{bmatrix} \succeq 0,\ i = 0, \dots, m \\ & (1/2)t_i + q_i^\top x + r_i,\ i = 1, \dots, m \\ & Ax = b \end{aligned}$$

(d) The SOCP

$$\begin{aligned} \text{minimize} \quad & f^\top x \\ \text{subject to} \quad & \|A_i x + b_i\|_2 \le c_i^\top x + d_i,\ i = 1, \dots, m \\ & Fx = g \end{aligned}$$

Assume $c_i^\top x + d_i > 0$, square both sides and divide it by $(c_i^\top x + d_i)$

$$\|A_i x + b_i\|_2^2 \le (c_i^\top x + d_i)^2 \quad \Leftrightarrow \quad (c_i^\top x + d_i) - (A_i x + b_i)^\top((c_i^\top x + d_i)I)^{-1}(A_i x + b_i) \ge 0$$

$\Leftrightarrow$

$$\begin{aligned} \min_{x,t_i} \quad & f^\top x \\ \text{s.t.} \quad & \begin{bmatrix} (c_i^\top x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^\top & c_i^\top x + d_i \end{bmatrix} \succeq 0,\ i = 1, \dots, m \\ & Fx = g \end{aligned}$$