ROB 501 HW3

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1

(a) Not a subspace. Reason: Not closed under multiplication by a constant, such as -1.

(b) A subspace. Reason: Closed under vector addition and scalar multiplication.

(c) Not a subspace. Reason: Not closed under addition, such as $[0, 1]^T + [1, 0]^T = [1, 1]^T$.

(d) A subspace. Reason: Closed under vector addition and scalar multiplication.

(e) Not a subspace. Reason: Zero vector is not included.

(f) Not a subspace. Reason: Zero vector is not included.

2

Definition 2.1. Let $\mathcal{S}$ be a subset of a vector space$(X,\ \mathcal{F})$. The span of $\mathcal{S}$ is the set of all linear combinations of elements of $\mathcal{S}$. That is

$$span\{\mathcal{S}\} := \{x \in X \mid \exists n \geq 1, \alpha_1, \ldots, \alpha_n \in \mathcal{F}, v_1, \ldots, v_n \in \mathcal{S},\ \text{s.t. } x = \alpha_1 v_1 + \cdots + \alpha_n v_n\}$$

Theorem 2.1. Given a finite set $\mathcal{S}$ in a vector space X, $span\{\mathcal{S}\}$ is a subspace of X.

Proof. Let $u_1, \ldots, u_n \in \mathcal{S}_1$, $v_1, \ldots, v_n \in \mathcal{S}_2$, then $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathcal{S}_1 \cup \mathcal{S}_2$.

$$span\{\mathcal{S}_1\} := \{u \in X \mid u = \alpha_1 u_1 + \cdots + \alpha_n u_n\}$$

$$span\{\mathcal{S}_2\} := \{v \in X \mid v = \beta_1 v_1 + \cdots + \beta_n v_n\}$$

Then $span\{\mathcal{S}_1\}$ and $span\{\mathcal{S}_2\}$ are subspaces in X. The addition of the subspaces is given by

$$span\{\mathcal{S}_1\} + span\{\mathcal{S}_2\} = \{w \in X \mid w = \alpha_1 u_1 + \cdots + \alpha_n u_n + \beta_1 v_1 + \cdots + \beta_n v_n\}$$

Since $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathcal{S}_1 \cup \mathcal{S}_2$, we have $span\,\{S_1\} + span\,\{S_2\} \subset span\,\{S_1 \cup S_2\}$.

$$span\{\mathcal{S}_1 \cup \mathcal{S}_2\} := \{z \in X \mid z = \gamma_1 z_1 + \cdots + \gamma_n z_n\}$$

Since z can be represented by a linear combination of u and v, we have $span\,\{S_1 \cup S_2\} \subset span\,\{S_1\} + span\,\{S_2\}$.

Hence $span\,\{S_1 \cup S_2\} = span\,\{S_1\} + span\,\{S_2\}$. $\square$

3

3.1

Linear dependent

$$\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

3.2

Linear dependent

$$\begin{bmatrix}0\\5\\4\end{bmatrix} = 4\begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\0\\6\end{bmatrix} - 4\begin{bmatrix}1\\1\\1\end{bmatrix}$$

3.3

Linear independent, since the third component of the first vector can not be expressed as a linear combination of the rest.

4

$$\begin{aligned}
\alpha_1\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} + \alpha_2\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} + \alpha_3\begin{bmatrix}4 & -1\\-1 & 1\end{bmatrix} &= \mathbf{0}\\
\begin{bmatrix}\alpha_1+2\alpha_2+4\alpha_3 & 2\alpha_1+\alpha_2-\alpha_3\\ 2\alpha_1+\alpha_2-\alpha_3 & \alpha_1+\alpha_2+\alpha_3\end{bmatrix} &= \mathbf{0}\\
\underbrace{\begin{bmatrix}1 & 2 & 4\\2 & 1 & -1\\1 & 1 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}}_{x} &= \mathbf{0}\\
\Rightarrow\ det(A) &= 0
\end{aligned}$$

Choose $\alpha_1 = 2$, $\alpha_2 = -3$, $\alpha_1 = 1$ solved $Ax = \mathbf{0}$. Hence the set is linearly dependent.

5

Let $(X,\, \mathcal{F})$ be a vector space and $\mathcal{S} \subset X$. Prove that if Y is a subspace of X and $\mathcal{S} \subset Y$, then $span\{\mathcal{S}\} \subset Y$.

Proof.

$$span\{\mathcal{S}\} := \{x \in X \mid \exists n \geq 1, \alpha_1, \ldots, \alpha_n \in \mathcal{F}, v_1, \ldots, v_n \in \mathcal{S},\ \text{s.t. } x = \alpha_1 v_1 + \cdots + \alpha_n v_n\}$$

Since $S \subset Y$, then $v_1, \ldots, v_n \in Y$. Since Y is closed under vector multiplication and scalar addition, $x \in Y$. Hence, $span\{\mathcal{S}\} \subset Y$. □

6

Let $(X,\, \mathcal{F})$ be a vector space and V and W are subspaces of X. Prove the following two statements are equivalent.

(a) $V \cap W = \{0\}$

(b) $\forall x \in V + W, \exists$ unique $v \in V$ and $w \in W$ s.t. $x = v + w$

Proof. ($a \Rightarrow b$) Since for every $x \in V + W$, there exist $v_1 \in V$ and $w_1 \in W$ s.t. $x = v_1 + w_1$. Suppose there exist other vectors $v_2 \in V$ and $w_2 \in W$ such that $x = v_2 + w_2$. Then,

$$\mathbf{0} = (v_1 - v_2) + (w_1 - w_2) \Leftrightarrow (v_1 - v_2) = -(w_1 - w_2)$$

Therefore $v_1 - v_2 \in W$ and so $v_1 - v_2 \in V \cap W$. Since $V \cap W = \{0\}$, we then conclude that $v_1 = v_2$, which also says $w_1 = w_2$. Then $\forall x \in V + W, \exists$ unique $v \in V$ and $w \in W$ s.t. $x = v + w$.
($b \Rightarrow a$) Suppose that $x_0 \in V \cap W$, then on the one hand, there exists $v_0 \in V$ such that $x_0 = v_0 + \mathbf{0}$; on the other hand, there is $w_0 \in W$ such that $x_0 = \mathbf{0} + w_0$. Therefore, $v_0 = \mathbf{0}$ and $w_0 = \mathbf{0}$, so $V \cap W = \{0\}$. □