ROB 501 HW7

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1

Prove, for a matrix $A \in \mathbb{R}^{m \times n}$, $rank(A) + nullity(A) = n$.

Proof. Assume $rank(A) = r$, $\forall x \in N(A),\ Ax = 0$.
$\Rightarrow$ Rows of A is orthogonal to x
$\Rightarrow$ Columns of A^T is orthogonal to x
$\Rightarrow$ All $x \in N(A)$ are orthogonal to $y \in R(A^T)$
$\Rightarrow N(A) = R(A^T)^\perp$
$\Rightarrow N(A) + R(A^T) = \mathbb{R}^n$ and $N(A) \cap R(A^T)^\perp = \{0\}$
$\Rightarrow dim(N(A)) + dim(R(A^T)) = n$
$\Rightarrow nullity(A) + rank(A) = n$ $\square$

2

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -0.5 & 0.707 & 0.5 \\ -0.707 & 0 & -0.707 \\ 0.5 & 0.707 & -0.5 \end{bmatrix} \begin{bmatrix} 0.586 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3.414 \end{bmatrix} \begin{bmatrix} -0.5 & 0.707 & 0.5 \\ -0.707 & 0 & -0.707 \\ 0.5 & 0.707 & -0.5 \end{bmatrix} \\
&= O \Lambda O^T
\end{aligned}
$$

3

$$O^T A O = diag([2, -1, 2])$$

4

4.1

Find $n = 34$.

4.2

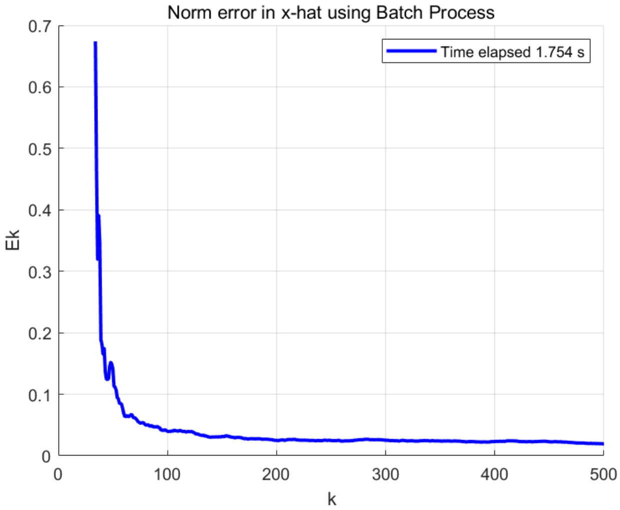

Figure 1: Norm error in x-hat using Batch Process

4.3

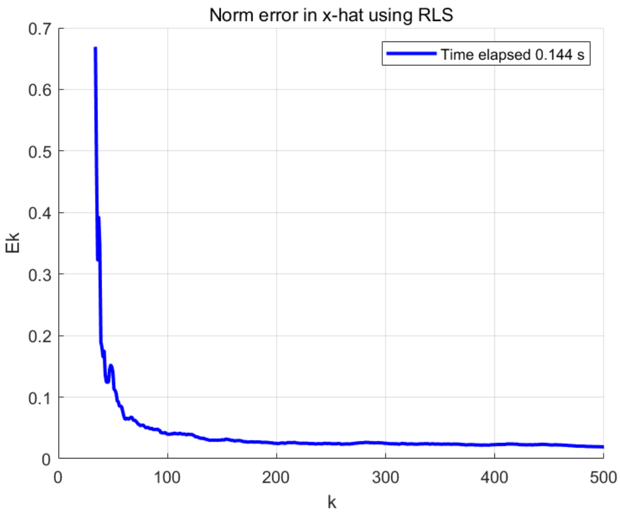

Figure 2: Norm error in x-hat using Recursive Least Squares (RLS)

4.4

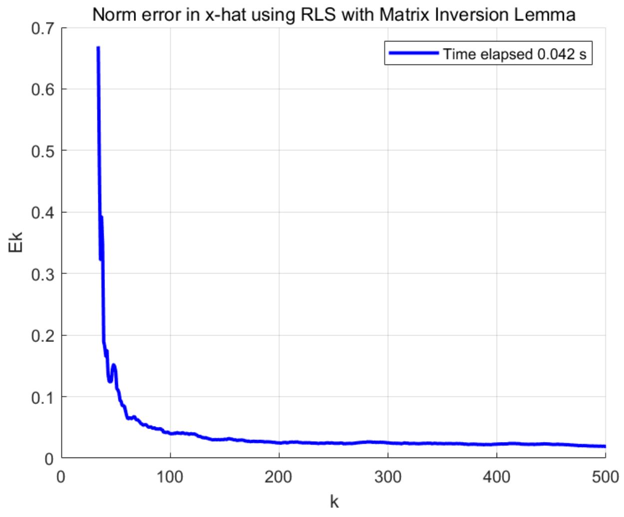

Figure 3: Norm error in x-hat using RLS with Matrix Inversion Lemma

5

5.1

Find $n = 7$.

5.2

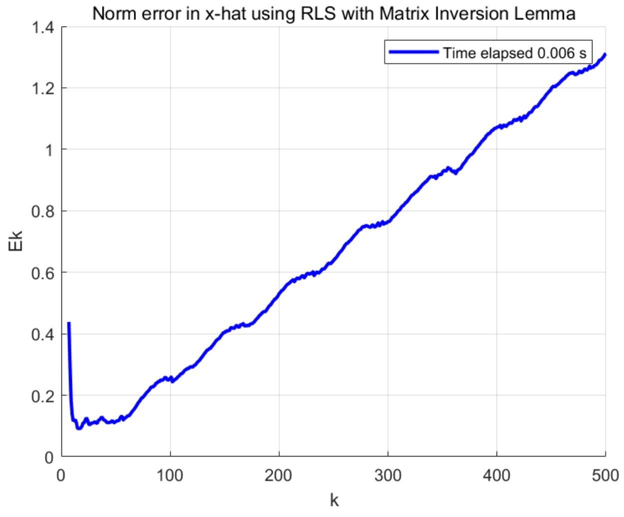

Figure 4: Norm error in x-hat using RLS with Matrix Inversion Lemma

5.3

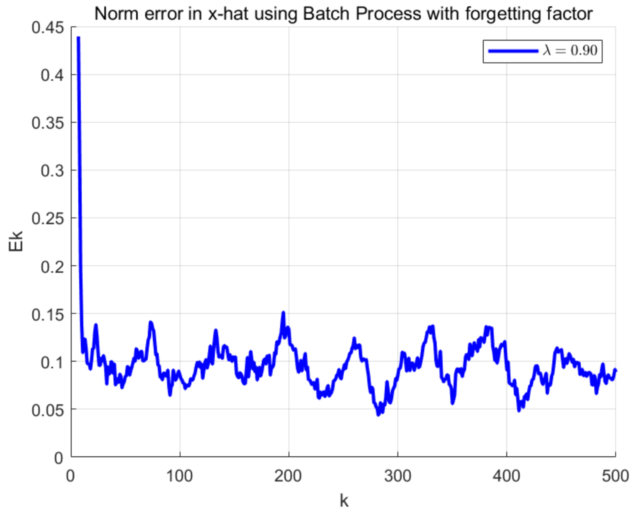

Figure 5: Norm error in x-hat using Batch Process with forgetting factor

5.4

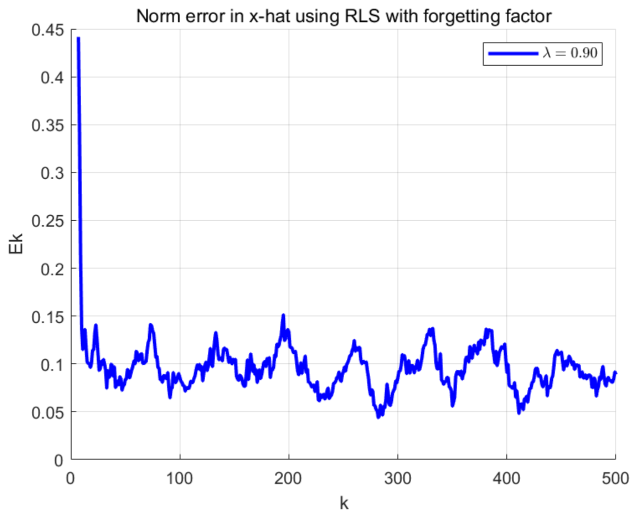

Figure 6: Norm error in x-hat using RLS with forgetting factor

6

6.1

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}, \text{e-values: } 10,\ 0 \\
\Rightarrow A & \text{ is positive semi-definite.} \\
A &= O\Lambda O^T \\
&= O\Lambda^{\frac{1}{2}T}\Lambda^{\frac{1}{2}}O^T \\
&= (\Lambda^{\frac{1}{2}}O^T)^T(\Lambda^{\frac{1}{2}}O^T) \\
\Rightarrow \Lambda^{\frac{1}{2}}O^T &= \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}
\end{aligned}
$$

6.2

$$B = \begin{bmatrix} 6 & 10 & 11 \\ 10 & 19 & 19 \\ 11 & 19 & 21 \end{bmatrix} \text{, e-values: 0.1856, 1.0842, 44.7302}$$

$$\begin{aligned} &\Rightarrow B \text{ is positive definite.} \\ &B = O\Lambda O^T \\ &\quad = (\Lambda^{\frac{1}{2}}O^T)^T(\Lambda^{\frac{1}{2}}O^T) \\ &\Rightarrow \Lambda^{\frac{1}{2}}O^T = \begin{bmatrix} -0.3767 & -0.0105 & 0.2087 \\ 0.3405 & -0.7991 & 0.5743 \\ 2.3963 & 4.2850 & 4.5417 \end{bmatrix} \end{aligned}$$

6.3

$$C = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} \text{, e-values: -2.9165, 0, 32.9165}$$

$$\Rightarrow C \text{ is neigher positive definite or positive semi-definite.}$$

7

Recall the Schur Complements: suppose that A, B and C are real matrices, $A \in \mathbb{R}^{n\times n}$ is symmetric, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times m}$ is symmetric. We have

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

the following statements are equivalent:

(a) $M > 0$.
(b) $A > 0$, and $C - B^\top A^{-1} B > 0$.
(c) $C > 0$, and $A - BC^{-1}B^\top > 0$.

7.1

$$\begin{aligned} &A_1 = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} \\ &A = 1 > 0 \\ &C - B^\top A^{-1} B = 8 - 3 \times 1 \times 3 = -1 < 0 \\ &\Rightarrow A_1 \text{ is not positue definite} \end{aligned}$$

7.2

$$\begin{aligned} &A_2 = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 4 & 7 \\ 6 & 7 & 10 \end{bmatrix} \\ &\text{Let } A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \quad C = 10 \\ &C = 10 > 0 \\ &A - BC^{-1}B^\top = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 6 \\ 7 \end{bmatrix}\begin{bmatrix} 6 & 7 \end{bmatrix} \\ &= \begin{bmatrix} -2.6 & -4.2 \\ -4.2 & -0.9 \end{bmatrix} < 0 \\ &\Rightarrow A_2 \text{ is not positive definite} \end{aligned}$$

7.3

$$\begin{aligned}
&A_3 = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 6 & 7 & a \end{bmatrix} \\
&\text{Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 7 \end{bmatrix}, C = a \\
&\begin{cases} C > 0 \\ A - BC^{-1}B^\top > 0 \end{cases} \\
\Rightarrow\ & a > 0 \text{ and} \\
&\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} - \frac{1}{a}\begin{bmatrix} 36 & 42 \\ 42 & 49 \end{bmatrix} > 0 \\
&\begin{bmatrix} 1 - 36/a & 2 - 42/a \\ 2 - 42/a & 5 - 49/a \end{bmatrix} > 0 \\
&\text{Let } A = 1 - \frac{36}{a}, B = 2 - \frac{42}{a}, C = 5 - \frac{49}{a} \\
&\begin{cases} A > 0 \\ C - B^\top A^{-1} B > 0 \end{cases} \\
\Rightarrow\ & a > 36 \text{ and} \\
&5 - \frac{49}{a} - \left(2 - \frac{42}{a}\right)\frac{a}{a-36}\left(2 - \frac{42}{a}\right) > 0 \\
&5a - 49 - \frac{4a^2 - 168a + 42^2}{a - 36} > 0 \\
&(5a - 49)(a - 36) > 4a^2 - 168a + 42^2 \\
&a^2 - 61a > 0 \\
\Rightarrow\ & a < 0 \text{ or } a > 61
\end{aligned}$$

Hence, $a > 61$.

8

Given

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find the minimum norm solution x, s.t. $Ax = b$

8.1

$$\tilde{x} = A^\top \left(AA^\top\right)^{-1} b = \begin{bmatrix} -0.0952 \\ 0,0476 \\ 0.4762 \end{bmatrix}$$

8.2

Let $M = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix}$, note that M is symmetric with $M = M^\top$ under the inner product convension $\langle x, y \rangle = x^\top M y$.

Denote the least norm solution as x_{ln}. We find $x$ln as the solution of a constrained optimization problem.

$$\begin{aligned} \min \quad & \|x\|^2 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Then remove the constraints using Lagrange multiplier for matrix (i.e. $\lambda \in \mathbb{R}^2$ and $\lambda > 0$).

$$
\begin{aligned}
& L(x) = x^\top M x - \lambda^\top (Ax - b) \\
& \nabla L(x) = 2x^\top M - \lambda^\top A \\
& \begin{cases} 2x^\top M - \lambda^\top A = 0 \\ Ax - b = 0 \end{cases} \\
& x = \frac{1}{2} M^{-1} A^\top \lambda \\
& \frac{1}{2} \underbrace{AM^{-1}A^\top}_{\text{invertible}} \lambda = b \\
& \lambda = 2 \left(AM^{-1}A^\top \right)^{-1} b \\
\Rightarrow\ & x_{\text{ln}} = M^{-1}A^\top \left(AM^{-1}A^\top \right)^{-1} b \\
& x_{\text{ln}} = \begin{bmatrix} -0.6497 \\ 0.3248 \\ 0.3376 \end{bmatrix}
\end{aligned}
$$