ME424 HW6

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1

Given

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^T$$
$$\Sigma_X = \begin{bmatrix} 2 & 0 & \sigma_{13} \\ 0 & 2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 2 \end{bmatrix}$$

1.1

$$W \sim N(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix})$$

Given Z = 10.

$$\begin{aligned}
\mu_{W|Z=10} &= \mu_W + \Sigma_{WZ}\Sigma_Z^{-1}(10 - \mu_Z) \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 2^{-1} * 10 \\
&= \begin{bmatrix} 5 \\ 5 \end{bmatrix} \\
\Sigma_{W|Z=10} &= \Sigma_X - \Sigma_{XZ}\Sigma_Z^{-1}\Sigma_{ZX} \\
&= \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}
\end{aligned}$$

If I observed Z = -10 instead, the conditional mean of W will be negative according to the formula above. On the other hand, Σ_{WZ} shows that W is positive correlated with Z, and Σ_Z^{-1} shows that the measurement of Z has little noisy. Thus, the conditional mean of W is likely to be negative if Z was observed negative.

1.2

X_1 is independent of X_2 because X_1 and X_2 are jointly Gaussian and $E(X_1X_2^T) = E(X_1)E(X_2)^T = 0$, $Cov(X_1, X_2) = 0$.
Define

$$Q = (W|Z = 10)$$

We have

$$Q \sim N(\begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix})$$

Since $Cov(Q_1, Q_2) = -1/2$, they are not independent.
If $\sigma_{13} = \sigma_{31} = 0$ then $Cov(Q_1, Q_2) = 0$, so they are independent.

1.3

The probability density function for $W|Z = 10$ is

$$g(x) = \frac{1}{2\pi|\Sigma|^{-\frac{1}{2}}} \exp^{-\frac{1}{2}(X-\mu)^T\Sigma(X-\mu)}$$

The MATLAB code is shown below.

```
X = [linspace(0,10,50);linspace(0,10,50)];
Y = zeros(50,50);
mu = [5;5];
sigma = [3/2,-1/2;-1/2,3/2];
g = @(x,mu,sigma) 1./(2*pi*det(sigma)^(1/2))*exp(-1/2*(x-mu)'*sigma^(-1)*(x-mu));
for i = 1:size(X,2)
    for j = 1:size(X,2)
        Y(i,j) = g([X(1,i);X(1,j)],mu,sigma);
    end
end
surf(X(1,:),X(2,:),Y);
colormap('jet');
xlabel('X1');ylabel('X2');
zlabel('Probability Density');
```

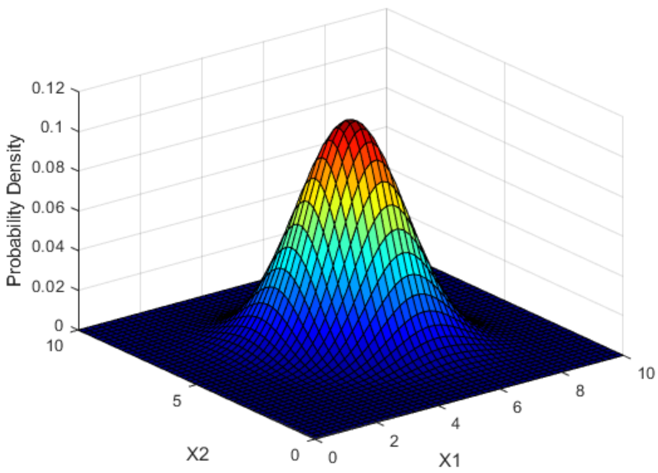

Figure 1: PDF for W—Z = 10

2

2.1

i

x_1	1	2
$p(x_1\|X_2 = -1, X_3 = 3)$	$\frac{2}{7}$	$\frac{5}{7}$

ii

$$\begin{array}{lccc} x_1, x_3 & 1,3 & 2,2 & 2,3 \\ p(x_1, x_3|X_2=-1) & \frac{1}{4} & \frac{1}{8} & \frac{5}{8} \end{array}$$

$$P(X_3=3|X_2=-1) = \frac{7}{8}$$

$$p(x_1|X_2=-1, X_3=3) = \frac{p(x_1, x_3|X_2=-1)}{P(X_3=3|X_2=-1)}$$

$$\begin{array}{lcc} x_1 & 1 & 2 \\ p(x_1|X_2=-1, X_3=3) & \frac{2}{7} & \frac{5}{7} \end{array}$$

2.2

$$\begin{aligned} \hat{X}_{1MMSE} &= E(x_1|X_2=-1, X_3=3) \\ &= \frac{12}{7} \end{aligned}$$

3

When $\mu = [1,2]^T$,

$$\begin{aligned} \mu^{(1)}_{X|Y=1} &= \mu_x + \Sigma_{XY}\Sigma_Y^{-1}(1-\mu_Y) \\ &= 1 + (-1) \times \frac{1}{2} \times (1-2) \\ &= \frac{3}{2} \end{aligned}$$

When $\mu = [2,1]^T$,

$$\begin{aligned} \mu^{(2)}_{X|Y=1} &= \mu_x + \Sigma_{XY}\Sigma_Y^{-1}(1-\mu_Y) \\ &= 2 \end{aligned}$$

Thus,

$$\begin{aligned} X_{MMSE} &= \mu^{(1)}_{X|Y=1} Prob(\mu = \mu^{(1)}) + \mu^{(2)}_{X|Y=1} Prob(\mu = \mu^{(2)}) \\ &= 1.9 \end{aligned}$$

4

Given

$$\begin{aligned} &X \sim N(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}) \\ &V \sim N(0,3) \\ &Z = HX + V,\ where\ H = [1\ 2] \end{aligned}$$

4.1

We have

$$
\begin{aligned}
E(Z) &= HE(X) + E(V) \\
&= 5 \\
Cov(Z) &= Cov(HX + V, HX + V) \\
&= HCov(X)H^T + Cov(V) \\
&= 8 \\
Coc(X, Z) &= Cov(X, HX + V) \\
&= Cov(X)H^T \\
&= \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
Cov(Z, X) &= Cov(HX + V, X) \\
&= HCov(X) \\
&= [-1\ 3]
\end{aligned}
$$

Thus

$$
\begin{aligned}
E(X|Z = 4) &= \mu_X + \Sigma_{XZ}\Sigma_Z^{-1}(4 - \mu_Z) \\
&= \begin{bmatrix} 9/8 \\ 13/8 \end{bmatrix} \\
Cov(X|Z = 4) &= \Sigma_X - \Sigma_{XZ}\Sigma_Z^{-1}\Sigma_{ZX} \\
&= \begin{bmatrix} 7/8 & -5/8 \\ -5/8 & 7/8 \end{bmatrix}
\end{aligned}
$$

4.2

I need update the mean and covariance of X given Z = 4.

$$
\begin{Bmatrix} E(X) \\ Cov(X) \end{Bmatrix} \xrightarrow{measurement\ update} \begin{Bmatrix} E(X|Z = 4) \\ Cov(X|Z = 4) \end{Bmatrix}
$$

According to Kalman formula,

$$
\begin{aligned}
K &= \Sigma_X H^T (H\Sigma_X H^T + \Sigma_V)^{-1} \\
&= \begin{bmatrix} -1/8 \\ 3/8 \end{bmatrix} \\
E(X|Z = 4) &= \mu_X + K(4 - H\mu_X) \\
&= \begin{bmatrix} 9/8 \\ 13/8 \end{bmatrix} \\
Cov(X|Z = 4) &= (I - KH)\Sigma_X \\
&= \begin{bmatrix} 7/8 & -5/8 \\ -5/8 & 7/8 \end{bmatrix}
\end{aligned}
$$