ROB 501 HW4

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1

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ 2 & 0 & 8 & 1 & 3 \\ -1 & 0 & -4 & 1 & 0 \\ 3 & 2 & 8 & 1 & 6 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = 0$$

Since $rank(A)$ is 3, and through observations

$$\begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} = -6 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} 4 \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}$$

The dimension of the space spanned by columns of A is 3.

2

$$x = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix} = 8e_1 + 7e_2 + 4e_3$$

$$[x]_S = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 8 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 = 7 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1 = 9 \\ \alpha_2 = 2 \\ \alpha_3 = -3 \end{cases}$$

$$[x]_U = \begin{bmatrix} 9 \\ 2 \\ -3 \end{bmatrix}$$

3

Theorem 3.1. There exists an invertible matrix P, with colefficients in $\mathcal{F}$, such that $\forall x \in (\mathcal{X}, \mathcal{F}), [x]_{\bar{u}} = P[x]_u$, where, $P = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}$ and its i^{th} column is given by $P_i := \left[u^i\right]_{\bar{u}} \in \mathcal{F}^n$, and $\left[u^i\right]_{\bar{u}}$ is the representation of u^i with respect to $\bar{u}$. Similarly, there exists an invertible matrix $\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 & \cdots & \bar{P}_n \end{bmatrix}$ with $\bar{P}_i = \left[\bar{u}^i\right]_u$, the representation of $\bar{u}^i$ with respect to u, and $P \cdot \bar{P} = \bar{P} \cdot P = I$

From standard basis to the new basis, we have

$$\begin{aligned} s_1 &= 2u_1 - u_2 \\ s_2 &= -u_1 + 2u_2 - u_3 \\ s_3 &= -u_2 + u_3 \end{aligned}$$

Hence,

$$P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

4

Find the change of basis matrix P from the world frame (X_W, Y_W) to the robot's frame (X_R, Y_R), such that $[x]_R = P[x]_W$.

$$\begin{aligned} &P_i = [w_i]_R \\ &w_1 = [1, 0]^T,\ w_2 = [0, 1]^T \\ &[w_1]_R = [cos\theta, -sin\theta]^T,\ [w_2]_R = [sin\theta, cos\theta]^T \\ \Rightarrow &P = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \end{aligned}$$

5

5.1

To show that M is a basis of $\mathbb{R}^{2,2}$, we have to show

(a) M is linear independent

(b) $span\{M\} = \mathbb{R}^{2,2}$

Proof. For part (a)

$$\begin{aligned}
&\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = 0 \\
\Rightarrow\ &\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0
\end{aligned}$$

For part (b), given an arbitrary element U in $\mathbb{R}^{2,2}$

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

It can be written as a linear combination by elements in M.

$$\begin{aligned}
U =& \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 \\
\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} =& \alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
=& \begin{bmatrix} \alpha_3 + \alpha_4 & \alpha_1 - \alpha_2 \\ \alpha_1 + \alpha_2 & \alpha_3 - \alpha_4 \end{bmatrix} \\
\Rightarrow& \begin{cases} \alpha_1 = (u_{12} + u_{21})/2 \\ \alpha_2 = (u_{21} - u_{12})/2 \\ \alpha_3 = (u_{11} + u_{22})/2 \\ \alpha_4 = (u_{11} - u_{22})/2 \end{cases}
\end{aligned}$$

Hence, $span\{M\} = \mathbb{R}^{2,2}$. □

5.2

Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We have

$$[A]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

To find the change of basis matrix P ($[A]_M = P[A]_S$) from standard basis S to new basis M, we compute $\bar{P}$ where $P\bar{P}^{-1} = I$ and $\bar{P}_i = [m_i]_S$.

$$\begin{aligned}
[M_1]_S = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, [M_2]_S = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, [M_3]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [M_4]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \\
\Rightarrow \bar{P} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \\
\Rightarrow P = \bar{P}^{-1} = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & -0.5 \end{bmatrix} \\
\Rightarrow [A]_M = P[A]_S = \begin{bmatrix} 2.5 \\ 0.5 \\ 2.5 \\ -1.5 \end{bmatrix}
\end{aligned}$$

6

6.1

Find the representation of $r(x)$ w.r.t. basis S.

$$r(x) = 2 + 3x - x^2 = 2p_0 + 3p_1 - p_2$$
$$[r(x)]_S = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

6.2

Find the representation of $r(x)$ w.r.t. basis Q.

$$[r(x)]_Q = P[r(x)]_S$$
$$P_i = [S_i]_Q$$
$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, P_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$
$$\Rightarrow P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$[r(x)]_Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$$

7

Let $\mathcal{F} = \mathbb{R}$ and let $\mathcal{X}$ be the set of 2×2 matrices with real coefficients. Define $L : \mathcal{X} \to \mathcal{X}$ by

$$L(M) = 2\left(M + M^\top\right)$$

7.1

Proof.

$$\begin{aligned} L(\alpha A + \beta B) &= 2(\alpha A + \beta B + \alpha A^T + \beta B^T) \\ &= \alpha 2(A + A^T) + \beta 2(B + B^T) \\ &= \alpha L(A) + \beta L(B) \end{aligned}$$

□

7.2

Definition 7.1. Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be finite dimensional vector spaces, and $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ be a linear operator. A matrix representation of $\mathcal{L}$ with respect to a basis $u := \left\{u^1, \ldots, u^m\right\}$ for $\mathcal{X}$ and $v := \left\{v^1, \ldots, v^n\right\}$ for $\mathcal{Y}$ is an $n \times m$ matrix A, with coefficients in $\mathcal{F}$, such that $\forall x \in \mathcal{X}$, $[\mathcal{L}(x)]_v = A[x]_u$.

Theorem 7.1. Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be finite dimensional vector spaces, $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ a linear operator, $u := \left\{u^1, \ldots, u^m\right\}$ a basis for $\mathcal{X}$ and $v := \left\{v^1, \ldots, v^n\right\}$ a basis for $\mathcal{Y}$, then $\mathcal{L}$ has a matrix representation $A = \left[\begin{array}{ccc} A_1 & \cdots & A_m \end{array}\right]$, where the i^{th} column of A is given by

$$A_i := \left[\mathcal{L}\left(u^i\right)\right]_v, 1 \le i \le m$$

$$
\begin{aligned}
A_i &= [L(E_i)]_E \\
L(E_1) &= \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = 4E_1 \\
L(E_2) &= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = 2E_2 + 2E_3 \\
L(E_3) &= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = 2E_2 + 2E_3 \\
L(E_4) &= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = 4E_4
\end{aligned}
$$

$$
\Rightarrow A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}
$$

8

Let A be an $n \times n$ matrix with possibly complex coefficients. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ by $L(x) = Ax$. Note that the field is $\mathcal{F} = \mathbb{C}$.

8.1

Compute the matrix representation of L when the "natural" (also called canonical) basis is used in $\mathbb{C}^n$. Call your representation $\hat{A}$ and find its relation to the original matrix A.

The matrix representation $\hat{A}$ of L satisfies $[L(x)]_S = \hat{A}[x]_S$.

Define the natural basis in $\mathbb{C}^n$ to be

$$
S = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}
$$

$$
\begin{aligned}
\hat{A} &= [\hat{A}_1 \mid \hat{A}_2 \mid \cdots \mid \hat{A}_n] \\
\hat{A}_i &= [L(e_i)]_S \\
&= [Ae_i]_S \Leftarrow L(x) = Ax \\
&= [A_i]_S \Leftarrow e_i = [0, \dots, 1, \dots, 0]^T \\
&= A_i \\
\Rightarrow \hat{A} &= A
\end{aligned}
$$

8.2

Suppose that the e-values of A are distinct. Compute the matrix representation L with respect to a basis constructed from the e-vectors of A. Call your representation $\hat{A}$.

Recall $L(x) = Ax$, $Ax = \lambda x$ and the matrix representation $\hat{A}$ of L satisfies $[L(x)]_S = \hat{A}[x]_S$.

Define a set of basis constructed from the e-vectors of A to be

$$
V = \{v_1, v_2, \dots, v_n\}
$$

$$
\begin{aligned}
\hat{A} &= [\hat{A}_1 \mid \hat{A}_2 \mid \cdots \mid \hat{A}_n] \\
\hat{A}_i &= [L(v_i)]_V \\
&= [Av_i]_V \Leftarrow L(x) = Ax \\
&= [\lambda_i v_i]_V \Leftarrow Ax = \lambda x \\
&= \lambda_i [v_i]_V \\
&= \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \\
\hat{A} &= \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}
\end{aligned}
$$