IOE 611: Homework 2

Yulun Zhuang

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Problem 1

Inverse of an increasing convex function. Suppose $f : \mathbb{R} \to \mathbb{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse, i.e., the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?

Proof. Since g and f are both differentiable, by the chain rule we have

$$
\begin{aligned}
& g(f(x)) = x \\
& g'(f(x))f'(x) = 1 \\
\Rightarrow\ & g'(f(x)) = \frac{1}{f'(x)} > 0 \\
& g''(f(x))f'(x)^2 + g'(f(x))f''(x) = 0 \\
& g''(f(x))f'(x)^2 + \frac{f''(x)}{f'(x)} = 0 \\
\Rightarrow\ & g''(f(x)) = -\frac{f''(x)}{f'(x)^3} < 0
\end{aligned}
$$

Since $f(x)$ is increasing, $f'(x) > 0$, such that $g' > 0$. Also because $f(x)$ is convex, $f''(x) >= 0, \forall x \in \textbf{dom}\, f$, such that $g'' <= 0$.

Thus, g is concave. □

Problem 2

A family of concave utility functions. For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$$

with $\mathbf{dom}\, u_\alpha = \mathbb{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\, u_0 = \mathbb{R}_{++}$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$.

Proof. Since at $\lim_{\alpha \to 0} u_\alpha(x)$, both numerator and denominator go to zero, by the L'Hopital's rule, we have

$$\begin{aligned}
\lim_{\alpha \to 0} u_\alpha(x) &= \lim_{\alpha \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}\alpha}(x^\alpha - 1)}{\frac{\mathrm{d}}{\mathrm{d}\alpha}\alpha} \\
&= \lim_{\alpha \to 0} \frac{x^\alpha \log(x)}{1} \\
&= \log(x)
\end{aligned}$$

□

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

Proof. For $\alpha \in (0, 1]$

$$\begin{aligned}
&u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = 0 \\
&u'_\alpha(x) = x^{\alpha - 1} \geq 0 \Rightarrow \text{monotone increasing} \\
&u''_\alpha(x) = (\alpha - 1)x^{\alpha - 2} \leq 0 \Rightarrow \text{concave}
\end{aligned}$$

□

Problem 3

For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$

Solution.

$$\begin{aligned} f'(x) &= e^x > 0 \Rightarrow \text{monotonic} \Rightarrow \text{quasilinear} \\ f''(x) &= e^x > 0 \Rightarrow \text{convex, not concave} \end{aligned}$$

□

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

Solution.

$$\begin{aligned} \nabla f &= \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \\ \nabla^2 f &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &\Rightarrow \text{indefinite} \Rightarrow \text{not convex or concave} \end{aligned}$$

The superlevel sets $\{x_1, x_2 \mid x_1 x_2 \geq \alpha\}$ on $\mathbb{R}^2_{++}$ are convex, so f is quasiconcave but not quasiconvex.

□

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

Solution.

$$\begin{aligned} \nabla f &= \begin{bmatrix} -\frac{1}{x_1^2 x_2} \\ -\frac{1}{x_1 x_2^2} \end{bmatrix} \\ \nabla^2 f &= \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix} \succ 0 \\ &\Rightarrow \text{convex, not concave; quasiconvex, not quasiconcave} \end{aligned}$$

□

(d) $f(x_1, x_2) = x_1 / x_2$ on $\mathbb{R}^2_{++}$.

Solution.

$$\begin{aligned} \nabla f &= \begin{bmatrix} \frac{1}{x_2} \\ -\frac{x_1}{x_2^2} \end{bmatrix} \\ \nabla^2 f &= \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix} \\ &\Rightarrow \text{indefinite} \Rightarrow \text{not convex or concave} \end{aligned}$$

The sublevel sets of both f and $-f$ are convex, so it is quasilinear. □

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

Solution.

$$\nabla f = \begin{bmatrix} \frac{2x_1}{x_2} \\ -\frac{x_1^2}{x_2^2} \end{bmatrix}$$
$$\nabla^2 f = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$$

The leading principal minors of $\nabla^2 f$ are one positive and one zero, so $\nabla^2 f$ is positive semidefinite and f is convex and quasiconvex, but not concave and quasiconcave. □

(f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}_{++}^2$.

Solution.

$$\begin{aligned}
\nabla f &= \begin{bmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{bmatrix} \\
\nabla^2 f &= \begin{bmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{bmatrix} \\
&= \alpha(1-\alpha) x^{\alpha} x^{1-\alpha} \begin{bmatrix} -\frac{1}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & -\frac{1}{x_2^2} \end{bmatrix} \\
&= -\alpha(1-\alpha) x^{\alpha} x^{1-\alpha} \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_2} \end{bmatrix} \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_2} \end{bmatrix}^{\top} \preceq 0 \\
&\Rightarrow \text{concave, not convex; quasiconcave, not quasiconvex}
\end{aligned}$$

□

Problem 4

Adapt the proof of concavity of the log-determinant function to show the following.

(a) $f(X) = \mathrm{tr}(X^{-1})$ is convex on $\mathbf{dom}\, f = \mathbb{S}^n_{++}$.

Proof. Define $g(t) = f(Z + tV)$ where $Z \succ 0$ and $V \in \mathbb{S}^n$.

$$\begin{aligned} g(t) &= f(Z + tV) \\ &= \mathrm{tr}\left((Z + tV)^{-1}\right) \\ &= \mathrm{tr}\left(Z^{-1}\left(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}\right)^{-1}\right) \\ &= \sum_{i=1}^{n} Z_{ii}^{-1}(1 + t\lambda_i)^{-1} \end{aligned}$$

where $\lambda_i, i = 1, \ldots, n$ are eigenvalues of $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$.

We have $Z_{ii}^{-1} > 0$ because the inverse of a positive definite matrix is also positive definite. $g(t)$ is convex since it is a positive weighted sum of convex functions $(1+t\lambda_i)^{-1}$. Thus, $f(X)$ is convex. □

(b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbb{S}^n_{++}$.

Proof. Define $g(t) = f(Z + tV)$ where $Z \succ 0$ and $V \in \mathbb{S}^n$.

$$\begin{aligned} g(t) &= (\det(Z + tV))^{1/n} \\ &= \left(\det Z^{1/2} \det\left(I + tZ^{-1/2}VZ^{-1/2}\right) \det Z^{1/2}\right)^{1/n} \\ &= (\det Z)^{1/n}\left(\prod_{i=1}^{n}(1 + t\lambda_i)\right)^{1/n} \end{aligned}$$

where $\lambda_i, i = 1, \ldots, n$ are eigenvalues of $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$.

$g(t)$ is concave since $\det Z > 0$ and the geometric mean $(\prod_{i=1}^{n} x_i)^{1/n}$ is concave on $\mathbb{R}^n_{++}$. Thus $f(X)$ is concave. □

Problem 5

Perspective of a function.

(a) Show that for $p > 1$,

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{\|x\|_p^p}{t^{p-1}}$$

is convex on $\{(x,t) \mid t > 0\}$

Proof. Define $g(x) = \|x\|_p^p, \mathbf{dom}\, g = \{x \mid x \in \mathbb{R}^n\}$ which is convex. The perspective function of $g(x)$ on $\{(x,t) \mid x/t \in \mathbf{dom}\, g, t > 0\}$ is

$$tg(x/t) = t\frac{\|x\|_p^p}{t^p} = \frac{\|x\|_p^p}{t^{p-1}} = f(x,t)$$

Thus, $f(x,t), \mathbf{dom}\, f = \{(x,t) \mid t > 0\}$ is the perspective function of a convex function $\|x\|_p^p$, so it is convex. $\square$

(b) Show that

$$f(x) = \frac{\|Ax + b\|_2^2}{c^\top x + d}$$

is convex on $\{x \mid c^\top x + d > 0\}$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

Proof. Let $g(x) = \|Ax + b\|_2^2, \mathbf{dom}\, g = \{x \mid x \in \mathbb{R}^n\}$, then $\nabla^2 g(x) = 2A^\top A \succeq 0$, which implies g is convex.

Therefore, the perspective function

$$h(x,t) = \frac{\|Ax + \|_2^2}{t}, \quad \mathbf{dom}\, h = \{(x,t) \mid x \in \mathbb{R}^n, t > 0\}$$

is also convex. Since $f(x) = h\left(x, c^\top x + d\right), \mathbf{dom}\, f = \left\{x \mid c^\top x + d > 0\right\}$ is a composition of an affine function and a convex function, then it is also a convex function. $\square$

Problem 6

Subgradient. A vector $g \in \mathbb{R}^n$ is a subgradient of function $f : \mathbb{R}^n \to \mathbb{R}$ at point $x \in \mathbf{dom}\, f$ if

$$f(y) \geq f(x) + g^\top (y - x)$$

for any $y \in \mathbf{dom}\, f$.

(a) Suppose f is a convex function, and $x \in \mathbf{int}\,\mathbf{dom}\, f$. Prove that there exists a subgradient of f at x.

Proof. Since f is convex, $\mathbf{epi}\, f = \{(x, t) \mid f(x) \leq t, x \in \mathbf{dom}\, f\}$ is a convex set in $\mathbb{R}^{n+1}$. Therefore, for any $x_0 \in \mathbf{dom}\, f$, $(x_0, f(x_0))$ is on the boundary of $\mathbf{epi}\, f$, thus there exists a supporting hyperplane on x_0, i.e. $\exists (a, b) \in \mathbb{R}^{n+1}$ nonzero, such that for any $(x, t) \in \mathbf{epi}\, f$,

$$a^\top x + bt \geq a^\top x_0 + bf\left(x_0\right)$$

If $b = 0$, then $a \neq 0$. We have $a^\top x \geq a^\top x_0$ for all $x \in \mathbf{dom}\, f$. However, since $x_0 \in \mathbf{int}\,\mathbf{dom}\, f$, there exists $\epsilon > 0$, such that $B(x_0, \epsilon) \subseteq \mathbf{dom}\, f$. Consider $x = x_0 - \epsilon a$, we have

$$a^\top x = a^\top x_0 - \epsilon \|a\|_2^2 < a^\top x_0$$

which violates the inequality.

If $b < 0$, then t can be arbitrarily large, such that $a^\top x + bt$ is arbitrarily small which will violate the inequality.

Therefore, b has to be positive. Let $x = y, t = f(y)$, we have

$$\begin{aligned} a^\top y + bf(y) &\geq a^\top x_0 + bf(x_0) \\ bf(y) &\geq bf(x_0) + a^\top (x_0 - y) \\ f(y) &\geq f(x_0) + \frac{-a^\top}{b}(y - x_0) \end{aligned}$$

where $g = -a^\top / b$ is the subgradient of f at x_0. □

(b) Suppose $x \in \mathbf{dom}\, f$, but is not an interior point of the domain. Does there (always) exist a subgradient of f at x?

Proof. Not always.

Consider a convex function $f(x), \mathbf{dom}\, f = (-\infty, 1]$

$$f(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

However, for any $g \in \mathbb{R}$, let $y = 1 - \frac{1}{|g|} < 1, f(y) = 0$, then

$$\begin{aligned} f(1) + g(y - 1) &= 1 - \frac{g}{|g|} \\ &\geq 0 = f(y) \end{aligned}$$

which indicates subgradient function may not exist at boundaries. □

Problem 7

Show that the so-called logarithmic barrier for the second order cone, $f(x,t) = -log(t^2 - x^\top x)$ with $\mathbf{dom}\, f = \{(x,t) \in \mathbb{R}^{n+1} | t > \|x\|_2\}$, is convex. While this can be done by demonstrating that the Hessian of f is positive semidefinite everywhere on its domain, the following is an outline of a much simpler and more elegant proof:

(a) Show that $t - \frac{u^\top u}{t}$ is a concave function on $\mathbf{dom}\, f$.

Proof. Let $g(u) = u^\top u, \mathbf{dom}\, g = \{u \in \mathbb{R}^n\}$, which is convex, hence $h(u,t) = u^\top u/t, \mathbf{dom}\, h = \{(u,t) \in \mathbb{R}^{n+1} \mid u/t \in \mathbf{dom}\, g, t > \|u\|_2\}$ is the perspective of $g(u)$ so it is convex. Note that $-\frac{u^\top u}{t}$ is concave and $t - \frac{u^\top u}{t} > 0$ is a combination of an affine function and a concave function so it is also concave. □

(b) Show that $-\log(t - \frac{u^\top u}{t})$ is a convex function on $\mathbf{dom}\, f$.

Proof. Since the negative logarithm is convex, its extended-value extension is non-increasing. By the composition rules, given $h(u,t)$ is concave, $-\log(h(u,t))$ is convex. □

(c) Show that f is convex.

Proof.

$$
\begin{aligned}
f(x,t) &= -\log(t^2 - x^\top x) \\
&= -\log(t \cdot h(x,t)) \\
&= -\log(t) - \log(h(x,t))
\end{aligned}
$$

which is a sum of two convex function and thus also convex on $\mathbf{dom}\, f$. □

Problem 8

Suppose that $f(x) : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function (but not necessarily convex). Show that if $\bar{x}$ is a local minimum of $f(x)$, then we must have $\nabla f(\bar{x}) = 0$.

Proof. Given $\bar{x}$ is a local minimum of $f(x)$, there exists $\epsilon > 0, y \in \mathbb{R}^n, y \neq x$, such that $f(y) \geq f(\bar{x})$ for $\|y - \bar{x}\| \leq \epsilon$.

Since $f(x)$ is twice differentiable, expand $f(y) = f(\bar{x}) + \nabla f(\bar{x})(y - \bar{x}) \geq f(\bar{x})$, i.e.

$$\nabla f(\bar{x})(y - \bar{x}) \geq 0$$

Choose $y_1 > \bar{x}$ and $y_2 < \bar{x}$, the inequality must holds for both y, thus

$$\nabla f(\bar{x})(y - \bar{x}) = 0 \Rightarrow \nabla f(\bar{x}) = 0$$

$\square$

Problem 9

Given a set of points $\{v_1, v_2, \ldots, v_k\}$, define

$$\mathbf{conv}\{v_1, v_2, \ldots, v_k\} = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_1 + \ldots \alpha_k = 1, \alpha_1, \ldots, \alpha_k \geq 0\}$$

Show that the maximum of a convex function f over $\mathbf{conv}\{v_1, v_2, \ldots, v_k\}$ is achieved at one of its vertices, i.e.,

$$\sup_{x \in \mathbf{conv}\{v1, \ldots, v_k\}} f(x) = \max_{1 \leq i \leq k} f(v_i)$$

Proof. Let $x = \alpha_1 v_1 + \cdots + \alpha_k v_k \in \mathbf{conv}\{v_1, v_2, \ldots, v_k\}$,

$$\begin{aligned} f(x) &= f(\alpha_1 v_1 + \cdots + \alpha_k v_k) \\ &\leq \alpha_1 f(v_1) + \cdots + \alpha_k f(v_k) \\ &\leq \max_{1 \leq i \leq k} f(v_i) \\ &= \sup_x f(x) \end{aligned}$$

That is, the maximum of $f(x)$ is achieved at one of its vertices $f(v_i)$. $\square$