MODEL THEORY

① REVIEW: LANGUAGES AND STRUCTURES

1.1 Definition: a language L consists of
(i) a set $\mathcal{F}$ of function symbols, and for each $f \in \mathcal{F}$ a positive integer n_f
(ii) a set $\mathcal{R}$ of relation symbols, and for each $R \in \mathcal{R}$ a positive integer n_R
(iii) a set $\mathcal{C}$ of constant symbols

The numbers n_f and n_R are the arities of the function symbol f and relation symbol R, respectively

Any of the sets $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{C}$ may be empty.

Examples:

constant

(a) $L_{gp} = \{ \{ \cdot , {}^{-1} \} , \{1\} \}$ here $\mathcal{R} = \emptyset$

unary function

binary function

$L_{lo} = \{ < \}$ here $\mathcal{F} = \mathcal{C} = \emptyset$

binary relation

1.2 Definition Let L be a language. Then an L-structure consists of
(i) a set M, the domain
(ii) for each $f \in \mathcal{F}$, a function
$$f^M : M^{n_f} \longrightarrow M$$
(iii) for each $R \in \mathcal{R}$, a relation (subset) $R^M \subset M^{n_R}$
(iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M are called the interpretations of f, M, c respectively

1.3 Remarks on notation: we often omit the distinction between a symbol and its interpretation in a structure M when M is understood

We may write $\mathcal{M} = \langle M, \{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K} \rangle$ for a structure in the language
$\{ \{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K} \}$

In practice, we often denote a structure by its domain

(1.4) Examples:

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, ^{-1}\}, 1 \rangle$ is an L_{gp}-structure

$\mathbb{R}^+$: +ve real numbers

(b) $\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is an L_{gp}-structure

Formally: $\cdot^{\mathcal{Z}} = +$, $(^{-1})^{\mathcal{Z}} = -$, $1^{\mathcal{Z}} = 0$

(note that $\mathcal{R}$ and $\mathcal{Z}$ are groups, but an L_{gp}-structure needn't be a group)

(c) $\mathcal{Q} = \langle \mathbb{Q}, \{<\} \rangle$ is an L_{lo}-structure

(1.5) Definition Let $L = \{\mathcal{F}, \mathcal{R}, \mathcal{C}\}$ be a language, and let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.

An embedding of $\mathcal{M}$ into $\mathcal{N}$ is a one-to-one mapping

$$\alpha : M \longrightarrow N$$

such that

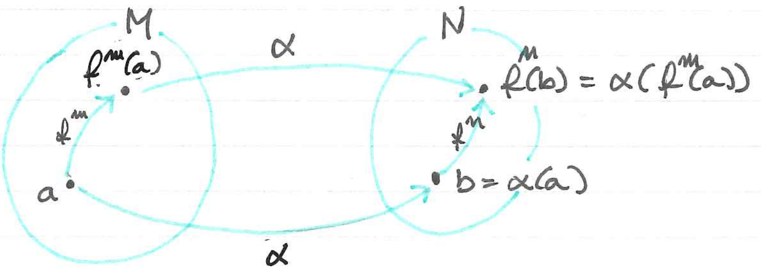
(i) for all $f \in \mathcal{F}$ and $a_1 \dots a_{m_f} \in M$

$$\alpha(f^{\mathcal{M}}(a_1 \dots a_{m_f})) = f^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_{m_f}))$$

(ii) for all $R \in \mathcal{R}$ and $a_1 \dots a_{m_R} \in M$,

$$(a_1, \dots, a_{m_R}) \in R^{\mathcal{M}} \iff (\alpha(a_1), \dots, \alpha(a_{m_R})) \in R^{\mathcal{N}}$$

(iii) for all $c \in \mathcal{C}$, $\qquad \alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$



A surjective (onto) embedding is said to be an isomorphism.

(1.6) Exercise Let G_1, G_2 be groups and describe them as L_{gp}-structures. Check that $G_1 \cong G_2$ (in the sense of group theory) if and only if G_1 and G_2 are isomorphic in the sense of Def 1.5

(2) REVIEW: TERMS, FORMULAE AND THEIR INTERPRETATIONS

Idea: a language L allows us to construct formulae using the symbols in L as well as:
- logical connectives $\wedge, \neg$ (also $\vee, \rightarrow, \leftrightarrow$)
- quantifier(s) $\exists$ (also $\forall$)
- variables $x_1, x_2, \dots, x_n, \dots$ (infinitely many).
- punctuation (,) (ensure unique readability of fmlae)
- equality symbol $\dot{=}$

Variables allow us to define L-terms

(2.1) Definition L-terms are defined recursively as follows:

(i) any variable x_i is an L-term
(ii) any constant symbol c is an L-term
(iii) whenever f is a fn symbol and $t_1, \dots, t_{n_f}$ are L-terms $f(t_1, \dots, t_{n_f})$ is an L-term
(iv) nothing else is an L-term.

Notation: we write $t(x_1, \dots, x_n)$ to indicate that the variables that appear in t are among $x_1, \dots, x_n$

The complexity of a term t is the number of symbols of L that appear in t.

Example: in $\langle \mathbb{R}^+, \cdot, ^{-1}, 1 \rangle$ $\quad \cdot(\cdot(x_1, x_2), x_3)$ is a term
$$\cdot(\cdot(x_1, x_2), x_3) = (x_1 \cdot x_2) \cdot x_3$$
and so is $(\cdot(1, x_1))^{-1} = (1 \cdot x_1)^{-1}$

(2.2) Definition If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1 \dots x_k)$ we assign a function
$$t^{\mathcal{M}} : M^k \to M$$
defined as follows:
(i) if t is a variable x_i, $t^{\mathcal{M}}(a_1 \dots a_k) = a_i$
(ii) if t is a constant symbol c, then $t^{\mathcal{M}}(a_1 \dots a_k) = c^{\mathcal{M}}$
(iii) if $t(x_1 \dots x_k) = f(t_1(x_1 \dots x_k), \dots, t_{n_f}(x_1 \dots x_k))$, then
$$t^{\mathcal{M}}(a_1 \dots a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1 \dots a_k), \dots, t_{n_f}^{\mathcal{M}}(a_1 \dots a_k))$$

Notation: we write $\bar{x}$ for $x_1 \dots x_n$, $\bar{a}$ for $a_1 \dots a_n$. The length of a tuple of variables, or of elements of a structure $\mathcal{M}$, is usually understood (and made explicit if not)

Idea: given a structure $\mathcal{M}$ and a sequence $a_1 \cdots a_k$ of elements of $\mathcal{M}$, we consider $a_1 \cdots a_k$ as assignments to the variables $x_1, \dots, x_k$. If we replace each variable x_i occurring in a term t by a_i, then t determines an element $t^{\mathcal{M}}$ of $\mathcal{M}$ in an obvious way.

Remarks:

(1) The recursive definition of $t^{\mathcal{M}}$ works because a term has a unique decomposition into its building blocks, that is: if $f(t_1, \dots, t_k) = f(t_1', \dots, t_k')$, we can prove (by induction on the complexity of terms) that $t_i = t_i'$ for $i = 1, \dots, k$.

(2) As stated, Definition 2.2 depends on the variables $x_1, \dots, x_k$ in the notation $t(x_1, \dots, x_k)$. The notation is not unique, in that we allow $\{x_1, \dots, x_k\}$ to contain variables that do not appear in the term t. Therefore, different functions may be assigned to a term t. However, it can be proved that $t^{\mathcal{M}}(\bar{a})$ only depends on a_i if x_i occurs in t, and so the ambiguity is harmless.

For example, the term $x_2 \cdot x_3$ in L_{gp} can be described as $t_1(x_1, x_2, x_3)$, or as $t_2(x_1, x_2, x_3, x_4)$.

Then $t_1^{\mathcal{M}} : M^3 \to M$ and $t_2^{\mathcal{M}} : M^4 \to M$, and both functions give the product of the second and third entry of the input tuple as output.

2.3 Fact Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let
$$\alpha : \mathcal{M} \to \mathcal{N}$$
be an embedding. recall notation explained at the end of Lecture 1

Then for any term $t(\bar{x})$ and $\bar{a} \in M^k$ (where k is the length of
$$\alpha(t^{\mathcal{M}}(\bar{a})) = t^{\mathcal{N}}(\alpha(\bar{a})),$$
where $\alpha(\bar{a}) = (\alpha(a_1), \dots, \alpha(a_k))$.

Proof by induction on the complexity of t. Let $\bar{a} \in M^k$. Then:

(i) if $t = x_i$, then $t^{\mathcal{M}}(\bar{a}) = a_i$, $t^{\mathcal{N}}(\alpha(\bar{a})) = \alpha(a_i)$ and so the desired conclusion holds.

(ii) if $t = c$, then $t^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$, $t^{\mathcal{N}}(\alpha(\bar{a})) = c^{\mathcal{N}}$, and $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$ because α is an embedding.

(iii) if $t = f(t_1(\bar{x}), \dots, t_{n_f}(\bar{x}))$, then
$$\alpha(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{n_f}^{\mathcal{M}}(\bar{a}))) = f^{\mathcal{N}}(\alpha(t_1^{\mathcal{M}}(\bar{a})), \dots, \alpha(t_{n_f}^{\mathcal{M}}(\bar{a})))$$
because α is an embedding.

Now apply the induction hypothesis to $t_1(\bar{x}), \dots, t_{n_f}(\bar{x})$.

2.4 Exercise: conclude the proof of Fact 2.3

(2.5) <u>Definition</u> The set of <u>atomic formulas</u> of L is defined as follows:

(i) if t_1 and t_2 are terms, then $t_1 = t_2$ is an atomic fmla
(ii) if R is a relation symbol and $t_1, \dots, t_{n_R}$ are terms, then $R(t_1, \dots, t_{n_R})$ is an atomic formula
(iii) nothing else is an atomic formula.

(2.6) <u>Definition</u> The set of <u>formulas of L</u> is defined as follows:

(i) any atomic formula is an L-formula
(ii) if ϕ and ψ are L-formulas, so is $\phi \wedge \psi$
(iii) if ϕ is an L-formula, so is $\neg \phi$
(iv) if ϕ is an L-formula, for any $i \geq 1$ $\exists x_i \, \phi$ is an L-formula
(v) nothing else is an L-formula.

Examples: in L_{gp}, $x_1 = x_2 \cdot x_3$
$1 \cdot x_1 = x_1$ are atomic fmlas
$\exists x_1 \, (x_1 \cdot x_2 = x_2)$ is an L_{gp}-fmla.

We say that a variable occurs <u>freely</u> in a formula if it is not within the scope of a quantifier. Otherwise the variable is <u>bound</u>.
So in $\exists x_1 \, (x_1 \cdot x_2 = x_2)$ x_1 is a bound variable, x_2 is free.

A <u>sentence</u> is a formula with no free variables.

We write $\phi(x_1, \dots, x_k)$ to signify that the free variables in ϕ are among $x_1, \dots, x_k$.

We'd like to define what it means for a formula to be true in a structure. Formulas with free variables express properties that may or may not be true of specific elements. For example,
$$\exists x_1 \, (x_1 + x_1 = x_2)$$
is true of 2 in $(\mathbb{Z}, +, -, 0)$ but not of 1.

(2.7) <u>Definition</u> Let $\phi(x_1, \dots, x_k)$ be an L-formula and let $\mathcal{M}$ be an L-structure. Let $\bar{a} \in M^k$.
We define $\mathcal{M} \models \phi(\bar{a})$ as follows:

(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$
(ii) if ϕ is $R(t_1, \dots, t_{n_R})$, then $\mathcal{M} \models \phi(\bar{a})$ iff
$$(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$$
(iii) if ϕ is $\psi \wedge \chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \chi(\bar{a})$
(iv) if ϕ is $\neg \chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \chi(\bar{a})$

(v) if ϕ is $\exists x_j\ \chi(x_1, \dots, x_k, x_j)$ (where $j \notin \{1, \dots, k\}$) then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ such that

$$\mathcal{M} \models \chi(\bar{a}, b)$$

We say "$\mathcal{M}$ models $\phi(\bar{a})$" or "$\mathcal{M}$ satisfies $\phi(\bar{a})$" or "$\phi(\bar{a})$ is true in $\mathcal{M}$" for $\mathcal{M} \models \phi(\bar{a})$.

Examples: $\mathcal{R} = \langle \mathbb{R}^*, \cdot, ^{-1}, 1 \rangle$. Then $\mathcal{R} \models \phi(2)$ but $\mathcal{R} \not\models \phi(-2)$, where

$$\phi(x_1) \equiv \exists x_2\ (x_2 \cdot x_2 = x_1)$$

2.8 Notation (useful abbreviations)

Let ϕ and ψ be L-formulas. We use the following abbreviations:

- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$
- $\phi \to \psi$ for $\neg\phi \vee \psi$
- $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$
- $\forall x_i\ \phi$ for $\neg\exists x_i\ (\neg\phi)$

When doing induction on the complexity of a formula, it is enough to treat the cases involving $\wedge$, $\neg$ and $\exists$.

2.9 Proposition Suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures, and $\alpha : \mathcal{M} \to \mathcal{N}$ is an embedding.
Then for any atomic formula $\phi(\bar{x})$ and $\bar{a} \in M^k$

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a})).$$

Proof (sketch): by induction on ϕ and Fact 2.3

(i) if ϕ is $t_1 = t_2$ for terms $t_1(\bar{x})$ and $t_2(\bar{x})$, use Fact 2.3.

(ii) if ϕ is $R(t_1(\bar{x}), \dots, t_{n_R}(\bar{x}))$, then $\mathcal{M} \models R(t_1^{\mathcal{M}}(\bar{a}), \dots, t^{\mathcal{M}}_{n_R}(\bar{a}))$ iff $(t_1^{\mathcal{M}}(\bar{a}), \dots, t^{\mathcal{M}}_{n_R}(\bar{a})) \in R^{\mathcal{M}}$

Since α is an embedding, ...

2.10 Exercise: complete the proof of Proposition 2.9, and show that the Proposition also holds when ϕ is a formula with no quantifiers.

2.11 Remark: an embedding does not necessarily preserve L-formulas.

Let $\mathcal{Z} = (\mathbb{Z}, <)$ and $\mathcal{Q} = (\mathbb{Q}, <)$ ($L_{<}$ - structures)

Then $\quad \alpha : \mathbb{Z} \longrightarrow \mathbb{Q}$
$\quad \alpha(n) = n \quad$ (the inclusion map)
is an embedding.

Let $\phi(x_1, x_2) \equiv \exists x_3 (x_1 < x_3 \wedge x_3 < x_2)$
Then
$$\mathcal{Z} \models \neg \phi(1,2) \text{ but } \mathcal{Q} \models \phi(1,2).$$

On the other hand, isomorphisms preserve all formulas

2.12 <u>Fact</u> Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism.
Then: if $\phi(\bar{x})$ is an L-fmla and $\bar{a} \in M^k$ then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a})).$$
(recall that $\alpha(\bar{a}) := \alpha(a_1, \dots, a_k) = (\alpha(a_1), \dots, \alpha(a_k))$).

Proof: exercise (Ex 2 on Worksheet 1)

(3) THEORIES AND ELEMENTARITY

Let L be a language.

(3.1) Definition An <u>L-theory</u> T is a set of L-sentences.
An L-structure $\mathcal{M}$ is a <u>model</u> of T, written $\mathcal{M} \models T$, if $\mathcal{M} \models \phi$ for all $\phi \in T$.

The class of all models of T is denoted by $\mathrm{Mod}(T)$.

If $\mathcal{M}$ is an L-structure, the <u>theory of $\mathcal{M}$</u> is
$$\mathrm{Th}(\mathcal{M}) = \{ \phi : \phi \text{ is an } L\text{-sentence and } \mathcal{M} \models \phi \}$$

In general, describing $\mathrm{Th}(\mathcal{M})$ is a non-trivial problem!

(3.2) Example Let T_{gp} be the theory containing the three L_{gp}-sentences

(i) $\forall x_1 x_2 x_3 \; (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$

(ii) $\forall x_1 \; (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$

(iii) $\forall x_1 \; (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$

Clearly, if G is a group then $G \models T_{gp}$. However, $\mathrm{Th}(G)$ is much larger than T_{gp}!
In general, describing the theory $\mathrm{Th}(\mathcal{M})$ of a structure $\mathcal{M}$ is a highly non-trivial task.

(3.3) Definition Let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures. Then:
$\mathcal{M}$ and $\mathcal{N}$ are <u>elementarily equivalent</u>, written
$$\mathcal{M} \equiv \mathcal{N}$$
if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$, that is, if $\mathcal{M}$ and $\mathcal{N}$ model exactly the same L-sentences.

Clearly, if $\mathcal{M} \simeq \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$. In general, establishing whether two structures are elementarily equivalent is, again, non-trivial.
One example we'll see is $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$

(3.4) Definition If $\mathcal{M}$ and $\mathcal{N}$ are two L-structures, then

(i) an embedding $\beta : \mathcal{M} \to \mathcal{N}$ is <u>elementary</u> if for all formulas $\phi(\bar{x})$ and $a \in M^{|\bar{x}|}$,
$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\beta(\bar{a}))$$

(ii) if $M \subseteq N$ and
$$id : \mathcal{M} \longrightarrow \mathcal{N} \quad \text{(the inclusion map)}$$
is an embedding, then $\mathcal{M}$ is said to be a <u>substructure</u> of $\mathcal{N}$, $\mathcal{M} \subseteq \mathcal{N}$.

(iii) if $M \subseteq N$ and $id : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then $\mathcal{M}$ is said to be an elementary substructure of $\mathcal{N}$, written $\mathcal{M} \preccurlyeq \mathcal{N}$.

Intuition: a substructure of $\mathcal{N}$ is some subset of the domain N that is closed under the interpretations of all the function symbols of L, and that contains the interpretations of all the constants. This is the content of Exercise 4 on Worksheet 1. There isn't a similar intuition of what an elementary substructure is - nor of what an elementary embedding which isn't an isomorphism is.

(3.5) Example Recall that $L_{lo} = \{<\}$ (binary rel. symbol). Consider the closed intervals $[0,1], [0,2] \subseteq \mathbb{R}$. Then $[0,1]$ and $[0,2]$ are isomorphic as L_{lo}-structures where $<$ is interpreted as the usual ordering on the real numbers. It follows that $[0,1] \equiv [0,2]$ as L_{lo}-structures. It is clear that $[0,1]$ is a substructure of $[0,2]$ (the ordering on $[0,1]$ is precisely the ordering inherited from $[0,2]$). However, $[0,1] \not\preccurlyeq [0,2]$. Let $\phi(x) \equiv \forall y \, (y < x \vee y = x)$. Then $\phi(1)$ holds in $[0,1]$ but not in $[0,2]$.
So we have two structures $\mathcal{M}$ and $\mathcal{N}$ such that
- $\mathcal{M} \subseteq \mathcal{N}$
- $\mathcal{M} \simeq \mathcal{N}$, and so $\mathcal{M} \equiv \mathcal{N}$
- $\mathcal{M} \not\preccurlyeq \mathcal{N}$.

Moral: the existence of an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ does not imply that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$!

(3.6) Definition If $\mathcal{M}$ is an L-structure and $A \subseteq M$ is a subset of M, we write $L(A)$ for the language obtained by adding to L a constant symbol c_a for each $a \in A$:

$$L(A) = L \cup \{ c_a : a \in A \}.$$

It is understood that an interpretation of $\mathcal{M}$ as an L-structure is extended to an interpretation of $\mathcal{M}$ as an $L(A)$-structure, where the constant c_a is interpreted as the element $a \in A \subseteq \mathcal{M}$. In this context, the elements of A are called parameters.

If $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $A \subseteq M \cap N$, we write

$$\mathcal{M} \equiv_A \mathcal{N}$$

if for every sentence $\phi \in L(A)$, $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$

(3.7) Remark $\mathcal{M} \preccurlyeq \mathcal{N} \Leftrightarrow \mathcal{M} \equiv_M \mathcal{N}$ (exercise)

3.8 Lemma (Tarski-Vaught test)

Let M be an L-structure and let $A \subseteq M$. Then the following are equivalent:

(i) A is the domain of an elementary substructure $\mathcal{M} \preccurlyeq M$

(ii) for every $L(A)$-formula $\phi(x)$ (one free variable)

$$M \models \exists x\, \phi(x) \Rightarrow M \models \phi(b) \text{ for some } b \in A$$

<u>Proof</u> (i) $\Rightarrow$ (ii) Suppose $M \models \exists x\, \phi(x)$.

Then by elementarity $\mathcal{M} \models \exists x\, \phi(x)$, so

$\mathcal{M} \models \phi(b)$ for some $b \in \mathcal{M}$.

Again by elementarity, $M \models \phi(b)$, as required.

(ii) $\Rightarrow$ (i) First we show that A is the domain of a substructure of M. By Exercise 4 on Worksheet 1, it suffices to check that:

(a) $c^M \in A$ for all constants c

(b) $f^M \bar{a} \in A$ for all function symbols f and $\bar{a} \in A^{n_f}$

For (a), use (ii) with the formula $\exists x\,(x = c)$

For (b), use (ii) with the formula $\exists x\,(f\bar{a} = x)$

So let $\mathcal{M} \subseteq M$ have domain A, and let $\chi(\bar{z})$ be an L-formula. We show that for all $\bar{a} \in A^{|\bar{z}|}$

$$\mathcal{M} \models \chi(\bar{a}) \Leftrightarrow M \models \chi(\bar{a}) \quad (*)$$

We argue by induction of the complexity of $\chi(\bar{z})$:

- if $\chi(\bar{z})$ is atomic, then $(*)$ follows from the fact that $\mathcal{M} \subseteq M$.
- the cases $\chi(\bar{z}) \equiv \neg\psi(\bar{z})$ and $\chi(\bar{z}) \equiv \psi(\bar{z}) \wedge \xi(\bar{z})$ are straightforward
- if $\chi(\bar{z}) \equiv \exists y\, \psi(\bar{z}, y)$, suppose that $\mathcal{M} \models \chi(\bar{a})$ for $\bar{a} \in A^{|\bar{z}|}$.

  Then $\mathcal{M} \models \exists y\, \psi(\bar{a}, y)$, hence

  $\mathcal{M} \models \psi(\bar{a}, b)$ for some $b \in \mathcal{M}$.

  By inductive hypothesis, $M \models \psi(\bar{a}, b)$ and so $M \models \exists y\, \psi(\bar{a}, y)$, that is, $M \models \chi(\bar{a})$, as required.

Now suppose $M \models \chi(\bar{a})$, that is, $M \models \exists y\, \psi(\bar{a}, y)$.

Then by (ii) $M \models \psi(\bar{a}, b)$ for some $b \in \mathcal{M}$.

By inductive hypothesis, $\mathcal{M} \models \psi(\bar{a}, b)$, and so $\mathcal{M} \models \exists y\, \psi(\bar{a}, y)$, that is, $\mathcal{M} \models \chi(\bar{a})$, as required. $\square$

3.9 <u>Remark</u> Assume that the set of variables is countably infinite. Then:

- the set of L-formulas has cardinality $|L| + \omega$
- if A is a set of parameters, the set of $L(A)$-formulas has cardinality $|L| + |A| + \omega$

(3.10) <u>Definition</u> Let λ be an ordinal. Then a <u>chain of sets of length λ</u> is a sequence
$$\langle M_i : i < \lambda \rangle$$
such that, for all $i < j < \lambda$, $M_i \subseteq M_j$.

A <u>chain of L-structures of length λ</u> is a sequence
$$\langle \mathcal{M}_i : i < \lambda \rangle$$
such that for all $i < j < \lambda$ $\quad \mathcal{M}_i \subseteq \mathcal{M}_j$ (substructure)

The <u>union of the chain</u> $\langle \mathcal{M}_i : i < \lambda \rangle$ is the L-structure $\mathcal{M}$ defined as follows:
- $M = \bigcup_{i<\lambda} M_i$ (domain)
- if c is a constant, then $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for any $i < \lambda$
- if f is a function symbol and $\bar{a} \in M^{|\bar{a}|}$, then
$$f^{\mathcal{M}} \bar{a} = f^{\mathcal{M}_i} \bar{a}$$
for any $i < \lambda$ such that $\bar{a} \in M_i^{n_f}$
- if R is a relation symbol, then $R^{\mathcal{M}} = \bigcup_{i<\lambda} R^{\mathcal{M}_i}$

(check that for all $i < \lambda$ $\quad \mathcal{M}_i \subseteq \mathcal{M}$).

(3.11) <u>Theorem</u> (Downward Löwenheim - Skolem)

Let $\mathcal{M}$ be an L-structure such that $|M| \geq |L| + \omega$, and let $A \subseteq M$. Then for every cardinal λ such that
$$|L| + |A| + \omega \leq \lambda \leq |M|$$
there is an elementary substructure $\mathcal{N} \preccurlyeq \mathcal{M}$ such that
(i) $A \subseteq N$ (ii) $|N| = \lambda$

<u>Proof</u> We build inductively a chain $\langle A_i : i < \omega \rangle$ of subsets of M containing A such that $|A_i| = \lambda$.
(Our goal: define $N = \bigcup A_i$).
Let $A_0 \subseteq M$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$.
At stage $i+1$, assume that A_{i+1} has been built with $|A_i| = \lambda$.
Let $\langle \varphi_k(x) : k < \lambda \rangle$ be an enumeration of all $L(A_i)$-formulas such that $\mathcal{M} \models \exists x\, \varphi_k(x)$.
For $k < \lambda$, let a_k be such that $\mathcal{M} \models \varphi_k(a_k)$, and let $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$
← this set has cardinality at most λ
Then $|A_{i+1}| = \lambda$.
Now let $N = \bigcup_{i\in\omega} A_i$. We use the Tarski-Vaught test (Lemma 3.8) to show that N is the domain of a structure $\mathcal{N} \preccurlyeq \mathcal{M}$.

This is notation for "$\exists x\, \phi(x, \bar{a})$ is an $L(M)$ - formula"

Claim: if $\exists x\, \phi(x, \bar{a}) \in L(M)$, then $M \models \exists x\, \phi(x, \bar{a})$ implies $M \models \phi(b, \bar{a})$ for some $b \in M$.

So let $M \models \exists x\, \phi(x, \bar{a})$. Since $\bar{a}$ is a finite tuple, $\bar{a} \in A_i$ for some $i < \lambda$, and so $\phi(x, \bar{a})$ is among the formulas that have a witness in A_{i+1}. Hence

$$M \models \phi(b, \bar{a}) \text{ for some } b \in A_{i+1} \subseteq M. \qquad \square$$

Remark: it may help to run through the proof with the case $|A| = |L| = \omega$ and $|M|$ uncountable in mind.

(4) TWO RELATIONAL STRUCTURES

(4.1) Dense linear orders : definitions

Recall $L_{lo} = \{<\}$ where $<$ is a binary relation symbol.
An L_{lo} - structure is a linear order if it satisfies
(i) $\forall x \neg (x < x)$ (irreflexivity)
(ii) $\forall xyz \ ((x < y \wedge y < z) \rightarrow x < z)$ (transitivity)
(iii) $\forall xy \ (x < y \vee y < x \vee x = y)$ (linearity)

A linear order is dense if it satisfies
(iv) $\exists xy \ (x < y)$ (non triviality)
(v) $\forall xy \ (x < y \rightarrow \exists z \ (x < z < y))$ (density)

A linear order has no endpoints if
(vi) $\forall x \ (\exists y (x < y) \wedge \exists z (z < x))$ (no endpoints)

Then T_{lo} is the theory which includes axioms (i), (ii) and (iii)
T_{dlo} is the theory which includes axioms (i) to (vi).

$(\mathbb{Q}, <) \models T_{lo}$ and $(\mathbb{Q}, <) \models T_{dlo}$.

Remark: axioms (iv) and (v) imply that any model of T_{dlo} is infinite

(4.2) Definition If $\mathcal{M}, \mathcal{N} \models T_{lo}$, then an injective map
$$p : A \subseteq M \rightarrow N$$
is a partial embedding if for all $a, b \in \mathrm{dom}(p)$
$$\mathcal{M} \models a < b \iff \mathcal{N} \models p(a) < p(b).$$

(4.3) Lemma Let $\mathcal{M} \models T_{lo}$ and $\mathcal{N} \models T_{dlo}$. Let
$$p : M \rightarrow N$$
be a partial embedding such that $|\mathrm{dom}(p)| < \omega$. Let $c \in M$.
Then there is a partial embedding $\hat{p}$ such that $c \in \mathrm{dom}(\hat{p})$ and $p \subseteq \hat{p}$.

Proof Since $|\mathrm{dom}(p)| < \omega$, we can write
$\mathrm{dom}(p) = \{a_0, \dots, a_n\}$ so that $a_i < a_j$ if $i < j$.
If $c \in \mathrm{dom}(p)$, nothing to do.
If $c \notin \mathrm{dom}(p)$, then:
Case 1: $c < a_0$. By axiom (vi), there is $d \in N$ such that $d < p(a_0)$
Let $\hat{p} = p \cup \{\langle c, d \rangle\}$

Case 2: $a_i < c < a_{i+1}$ for some $i \in \{0, \dots, n\}$. Then by axiom (v) (density), there is $d \in N$ such that
$$p(a_i) < d < p(a_{i+1})$$
Let $\hat{p} = p \cup \{\langle c, d \rangle\}$

Case 3: $a_n < c$. Similar to case 1.

4.4 <u>Theorem</u> Let $\mathcal{M}, \mathcal{N} \models T_{dlo}$ be countable, and let $p : M \to N$ be a finite partial embedding (i.e. $|\mathrm{dom}(p)| < \omega$). Then there is an isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ that extends p.

<u>Proof</u> Enumerate M and N, say $M = \langle a_i : i < \omega \rangle$ and $N = \langle b_i : i < \omega \rangle$.
We define inductively a chain of finite partial embeddings $\langle p_i : i < \omega \rangle$ so that, for all i, $a_i \in \mathrm{dom}(p_{i+1})$ and $b_i \in \mathrm{rng}(p_{i+1})$.
Let $p_0 = p$. At stage $i+1$, p_i is given.
Forth step: by Lemma 4.3, extend p_i to $p_{i+\frac{1}{2}}$ so that $a_i \in \mathrm{dom}(p_{i+\frac{1}{2}})$
Back step: by Lemma 4.3 applied to $(p_{i+\frac{1}{2}})^{-1}$, extend $(p_{i+\frac{1}{2}})^{-1}$ to $(p_{i+1})^{-1}$ so that $b_i \in \mathrm{dom}(p_{i+1})^{-1}$. Then p_{i+1} extends p_i in the required way.

Let $\pi = \bigcup_{i<\omega} p_i$. It is easy to check that π is an order-preserving bijection, hence an isomorphism.

4.5 <u>Definition</u> An L-theory is said to be <u>consistent</u> if it has a model.
If T is an L-theory and φ is an L-sentence, we write $T \vdash \varphi$ if for every L-structure $\mathcal{M}$
$$\mathcal{M} \models T \Rightarrow \mathcal{M} \models \varphi$$
An L-theory T is <u>complete</u> if for all L-sentences φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.
If $\mathcal{M}$ is an L-structure, $Th(\mathcal{M})$ is complete.
It is usually hard to determine what sentences are in the theory $Th(\mathcal{M})$ of a given structure $\mathcal{M}$. In some cases, it is possible to find a simpler theory S such that $\mathcal{M} \models S$ and S is complete. This is the case with $Th(\langle \mathbb{Q}, < \rangle)$

4.6 <u>Definition</u> Let T be a theory in a countable language and assume that T has an infinite model. Then T is <u>ω-categorical</u> if any two countable models of T are isomorphic.

(4.7) Corollary (of Theorem 4.4) : T_{dlo} is ω-categorical

Proof : let $\mathcal{M}, \mathcal{N} \models T_{dlo}$ and $|M| = |N| = \omega$.
Then $\emptyset$ (the empty map) is a finite partial embedding.
By Theorem 4.4, $\emptyset$ extends to an isomorphism.
(if $\emptyset$ worries you, consider $a \in M$ and $b \in N$, and then use Thm 4.4 on the partial embedding $\{\langle a, b\rangle\}$).

(4.8) Theorem If T is an ω-categorical theory in a countable language, and suppose T has no finite models. Then T is complete.

Proof : Suppose $\mathcal{M} \models T$ and φ is an L-formula.
Let $\mathcal{M} \models \varphi$ and suppose $\mathcal{N} \models T$. By Theorem 3.11, there are countable $\mathcal{M}' \preceq \mathcal{M}$ and $\mathcal{N}' \preceq \mathcal{N}$. By Corollary 4.7, $\mathcal{M}' \cong \mathcal{N}'$ and so in particular $\mathcal{M}' \simeq \mathcal{N}'$. By elementarity, $\mathcal{M} \equiv \mathcal{N}$ and so $\mathcal{N} \models \varphi$.
Similarly, if $\mathcal{M} \models \neg\varphi$ then $\mathcal{N} \models \neg\varphi$. $\square$

(4.9) Corollary T_{dlo} is complete.

(4.10) Definition If $\mathcal{M}$ and $\mathcal{N}$ are L-structures, a map f such that $\mathrm{dom}(f) \subseteq \mathcal{M}$ and $\mathrm{img}(f) \subseteq \mathcal{N}$ is a (partial) elementary map if for all L-formulas $\varphi(\bar{x})$ and finite tuples $\bar{a} \in \mathrm{dom}(f)^{|\bar{x}|}$,

$$\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(f(\bar{a}))$$

(4.11) Remark A map f is elementary if and only if every finite restriction of f is elementary
Proof : ($\Rightarrow$) clear
($\Leftarrow$) Suppose f is not elementary. Then there are $\varphi(\bar{x})$ and $\bar{a} \in \mathrm{dom}(f)^{|\bar{x}|}$ such that $\mathcal{M} \models \varphi(\bar{a}) \not\Leftrightarrow \mathcal{N} \models \varphi(f(\bar{a}))$.
Then $f \restriction_{\bar{a}}$ is a finite restriction of f which is not elementary.

(4.12) Proposition Let $\mathcal{M}, \mathcal{N} \models T_{dlo}$ and let $p : A \subseteq M \to N$ be a partial embedding. Then p is an elementary map.

Proof By Remark 4.11, it suffices to consider the case when p is finite.
By The Downward Löwenheim - Skolem Theorem (3.11), there are countable $\mathcal{M}'$ and $\mathcal{N}'$ such that $\mathrm{dom}(p) \subseteq \mathcal{M}'$, $\mathrm{img}(p) \subseteq \mathcal{N}'$, $\mathcal{M}' \preceq \mathcal{M}$ and $\mathcal{N}' \preceq \mathcal{N}$.
By Theorem 4.4, there is an isomorphism $\pi : \mathcal{M}' \to \mathcal{N}'$ which extends p. In particular, π is an elementary map from $\mathcal{M}$ to $\mathcal{N}$. $\square$

(4.13) Corollary $(\mathbb{Q}, <) \not\preccurlyeq (\mathbb{R}, <)$

Proof: Apply Proposition 4.12 to the inclusion map $id : \mathbb{Q} \to \mathbb{R}$. □

(4.14) Definition (graphs and random graphs)

Let $L_{gph} = \{R\}$, where R is a binary relation symbol.
A graph is an L-gph structure which satisfies
(i) $\forall x \ \neg R(x,x)$ (irreflexivity)
(ii) $\forall xy \ (R(x,y) \to R(y,x))$ (symmetry)

If $\mathcal{G} = \langle G, R \rangle$ is a graph, the elements of G are called vertices, and 2-subsets $\{a, b\}$ such that $\langle a, b \rangle \in R^{\mathcal{G}}$ are called edges

A graph is a random graph if, in addition, it satisfies the following axioms for all $n \in \omega$
(iii) $\exists xy \ (x \neq y)$
(r_n) $\forall x_0 \dots x_n y_0 \dots y_n \left(\bigwedge_{i,j=0}^{n} x_i \neq y_j \to \exists z \left(\bigwedge_{i=0}^{n} (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$

Let T_{gph} be the theory which consists of axioms (i) and (ii), and T_{rg} the theory which consists of axioms (i), (ii), (iii) and (r_n) for all n
Remark: axiom (iii) and axiom schema (r_n) imply that any model of T_{rg} is infinite. Once we prove that random graphs are infinite, we can also prove that given a finite subset of a random graph, there is a vertex that is joined to all the vertices in the subset (and similarly for "not joined")

(4.15) Fact T_{rg} is consistent.
Pf Define a graph on ω as follows: for $i < j$, $\{i, j\}$ is an edge if and only if when j is written as a sum of distinct powers of 2, 2^i occurs in this sum.
(exercise: check that (r_n) holds for each n!)

(4.16) Definition Let $\mathcal{M}, \mathcal{N} \models T_{gph}$. Then an injective map $p : A \subseteq \mathcal{M} \to \mathcal{N}$ is a partial embedding if for all $a, b \in \text{dom}(p)$,
$$\mathcal{M} \models R(a,b) \iff \mathcal{N} \models R(a,b)$$

(4.17) <u>Lemma (extension)</u> Let $\mathcal{M} \models T_{gph}$, $\mathcal{N} \models T_{rg}$.
Let $p : A \subset M \to N$ be a finite partial embedding and let $c \in M$. Then there is a finite partial embedding $\hat{p}$ such that $p \leq \hat{p}$ and $c \in \mathrm{dom}(p)$.

Proof: Assume $c \notin \mathrm{dom}(p)$, and let

$$U = \{a \in \mathrm{dom}(p) : R(a,c)\}$$
$$V = \{b \in \mathrm{dom}(p) : \neg R(b,c)\}.$$

p is finite, hence U and V are finite.
Since $\mathcal{N}$ is a random graph, we can find $d \in N \setminus (p(U) \cup p(V))$ such that
(i) $R(d, p(a))$ for all $a \in U$
(ii) $\neg R(d, p(b))$ for all $b \in V$
Then $p \cup \{\langle c, d\rangle\}$ is the required extension of p.

(4.18) <u>Theorem</u> Let $\mathcal{M}, \mathcal{N} \models T_{rg}$ be countable, and let $p : A \subseteq M \to N$ be a finite partial embedding. Then there is an isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ which extends p.
Proof: The same as proof of Theorem 4.4, but with Lemma 4.17 instead of Lemma 4.3.

Proofs of this kind are called "back and forth" arguments.

(4.19) <u>Corollary</u> T_{rg} is ω-categorical and complete.
Every partial embedding between models of T_{rg} is an elementary map.

(4.20) <u>Remarks</u> The (unique up to isomorphism) countable model of T_{rg} is called the <u>countable random graph</u>, or the <u>Rado graph</u>.
The random graph embeds every finite or countable graph, and it is ultrahomogeneous, i.e. every finite isomorphism between subgraphs extends to an automorphism (i.e. an isomorphism of the random graph to itself).

(5) COMPACTNESS

(5.1) Definition Let T be an L-theory

(i) T is finitely satisfiable if every finite subset of T has a model

(ii) T is maximal if for all L-formulas ϕ, either $\phi \in T$ or $\neg\phi \in T$

(iii) T has the witness property (WP) if for all L-formulas $\phi(x)$ (with one free variable) there is a constant symbol c such that $(\exists x\, \phi(x)) \rightarrow \phi(c) \in T$

(5.2) Lemma Let T be maximal and finitely satisfiable. Let ϕ be an L-sentence. If $\Delta \subseteq T$ is finite and $\Delta \vdash \phi$, then $\phi \in T$.

Proof If $\phi \notin T$ then $\neg\phi \in T$ (by maximality). But then $\Delta \cup \{\neg\phi\}$ is a finite subset of T which does not have a model. $\square$

(5.3) Lemma Let T be a maximal finitely satisfiable theory with the witness property. Then T has a model. Moreover, if λ is a cardinal and L has at most λ constant symbols, there is $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \lambda$.

Proof: For $c, d \in \mathcal{C}$ (constant symbols of L), we define $c \sim d$ iff $c = d \in T$.

Claim: $\sim$ is an equivalence relation. For transitivity, suppose $c = d \in T$ and $d = e \in T$. Then by Lemma 5.2 $c = e \in T$ (similarly for reflexivity). Reflexivity follows from maximality.

We now define an L-structure with domain $M = \mathcal{C}/\sim$
Clearly, $|M| \leq \lambda$ if $|\mathcal{C}| \leq \lambda$.
We define interpretations for the symbols of L.
If $c \in \mathcal{C}$, define $c^{\mathcal{M}} = c/\sim$, the eq class of c.
For brevity, we'll write c^* for $c/\sim$.

If $R \in \mathcal{R}$, define $R^{\mathcal{M}} := \{(c_1^*, \dots, c_{n_R}^*) : R(c_1, \dots, c_{n_R}) \in T\}$
Claim: $R^{\mathcal{M}}$ is well defined.
Suppose $\bar{c}, \bar{d} \in \mathcal{C}^{|n_R|}$ and suppose $c_i \sim d_i$ for each $i = 1, \dots, n_R$
Then $c_i = d_i \in T$ for each i, and so by Lemma 5.2
$R(\bar{c}) \in T \Leftrightarrow R(\bar{d}) \in T$

If $f \in \mathcal{F}$ (function symbols), then $\exists x\, (f(\bar{c}) = x) \in T$
and so, by WP and Lemma 5.2, $f(\bar{c}) = c \in T$ for some $d \in \mathcal{C}$.
Define $f^{\mathcal{M}}(\bar{c}^*) = c^*$

Claim: $f^{\mathcal{M}}$ is well defined. Suppose $\bar{c}, \bar{d} \in \mathcal{C}^{n_f}$ are such that $c_i \sim d_i$ for all i. Then $c_i = d_i \in T$; since $f(\bar{d}) = d \in T$ for some $d \in \mathcal{C}$, by Lemma 5.2 $c = d \in T$, and so $c \sim d$, hence $c^* = d^*$.

Claim: if $t(x_1, \dots, x_n)$ is a term and $c_1, \dots, c_n, d \in \mathcal{C}$, then $t(c_1, \dots, c_n) = d \in T$ iff $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$.
($\Rightarrow$) by induction on the complexity of t.
($\Leftarrow$) Suppose $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$.
By WP and Lemma 5.2, there is $e \in \mathcal{C}$ such that $t(c_1, \dots, c_n) = e \in T$.
Using ($\Rightarrow$), $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = e^*$. Hence $d^* = e^*$ and so $d = e \in T$.
By Lemma 5.2, $t(c_1, \dots, c_n) \in T$.

Claim: For all L-formulas $\varphi(x_1, \dots, x_n)$ and $c_1, \dots, c_n \in \mathcal{C}$,
$$\mathcal{M} \models \varphi(\bar{c}^*) \iff \varphi(\bar{c}) \in T$$

Proof: induction on φ.
This shows that $\mathcal{M} \models T$. $\square$

(5.4) Lemma Let T be a finitely satisfiable L-theory. Then there are a language L^* and a finitely satisfiable L^*-theory T^* such that:

(i) $L \subseteq L^*$ and $|L^*| = |L| + \omega$

(ii) $T \subseteq T^*$ and any L^*-theory extending T^* has the witness property

Proof: We define chains $\langle L_i : i < \omega \rangle$ of languages and $\langle T_i : i < \omega \rangle$ of theories and eventually let $L^* = \bigcup L_i$ and $T^* = \bigcup T_i$.

Let $L_0 = L$ and $T_0 = T$.

At stage $i+1$, L_i and T_i are given with $|L_i| = |L| + \omega$ and T_i a f.s. theory containing T.

For each L_i-formula $\phi(x)$ with one free variable, let c_ϕ be a new constant symbol (not in L_i). Let

$L_{i+1} := L_i \cup \{ c_\phi : \phi(x) \text{ is an } L_i\text{-fmla with one free var} \}$.

Clearly $|L_{i+1}| = |L_i| + \omega = |L| + \omega$.

For each L_i-fmla $\phi(x)$ with one free variable, let Φ_ϕ be the L_{i+1}-sentence

$$\exists x\, \phi(x) \rightarrow \phi(c_\phi)$$

Let $T_{i+1} := T_i \cup \{ \Phi_\phi : \phi(x) \text{ is an } L_i\text{-fmla with one free var} \}$.

Claim: T_{i+1} is f.s.

Suppose $\Delta \subseteq T_{i+1}$ is finite. Then $\Delta = \Delta_0 \cup \{ \Phi_{\phi_1}, \dots, \Phi_{\phi_n} \}$ where $\Delta_0 \subseteq T_i$

Let $\mathcal{M} \models \Delta_0$. We define an L_{i+1}-structure $\mathcal{M}'$ on $\mathcal{M}$ by interpreting constant as follows: if $\mathcal{M} \models \exists x\, \phi(x)$, then let a be s.t. $\mathcal{M} \models \phi(a)$, and let $(c_\phi)^{\mathcal{M}'} := a$.

Otherwise let $(c_\phi)^{\mathcal{M}}$ be arbitrary.

Then $\mathcal{M}' \models \Delta$ and so T_{i+1} is f.s.

Let $L^* = \bigcup L_i$ and $T^* = \bigcup_{i<\omega} T_i$. By construction, T^* has WP.

If $\Delta \subseteq T^*$ is finite, then $\Delta \subseteq T_i$ for some i, and so Δ has a model. It follows that T^* is f.s.

(5.5) Lemma If T is a finitely satisfiable L-theory, then there is a maximal finitely satisfiable L-theory $T' \supseteq T$.

Proof: Let $I = \{ S : S \text{ is a finitely sat } L\text{-theory and } T \subseteq S \}$.

I is partially ordered by inclusion.

If $\langle C_i : i < \lambda \rangle$ is a chain, then $\bigcup_{i<\lambda} C_i$ is a theory which is finitely satisfiable (a finite subset of $\bigcup C_i$ is contained in one of the C_i). Clearly $\bigcup C_i$ contains T. Hence $\bigcup C_i$ is an upper bound for I. By Zorn's lemma, there is a maximal element T' of I. For any L-sentence σ, either $T' \cup \{\sigma\}$

or $T \cup \{\neg\sigma\}$ is finitely satisfiable (check this). By the maximality of T' in the partial order I, either $\sigma \in T'$ or $\neg\sigma \in T'$. Hence T' is a maximal theory. □

5.6 <u>Theorem (Compactness)</u> If T is a finitely satisfiable L-theory and $\lambda \geq |L| + \omega$, then there is a model of T of cardinality at most λ.

Proof: By Lemma 5.4, T can be extended to a finitely satisfiable L^*-theory T^*, where $L^* \supseteq L$, $|L^*| = |L| + \omega$, and any extension of T^* has WP.
By Lemma 5.5, there is $T' \supseteq T^*$ which is maximal and finitely satisfiable. Also, T' has WP. By Lemma 5.3, T' has a model $\mathcal{M}$ such that $|\mathcal{M}| \leq \lambda$. Then $\mathcal{M}$ is the required model of T. □

5.7 <u>Definition</u> An <u>L-type</u> $p(\bar{x})$ is a set of L-formulas whose free variables are among $\bar{x} = \langle x_i : i < \lambda \rangle$.
An L-type $p(x)$ is:
- <u>satisfiable in an L-structure $\mathcal{M}$</u> if there is $\bar{a} \in \mathcal{M}$ such that $\mathcal{M} \models \phi(\bar{a})$ for all fmlas $\phi(\bar{x}) \in p(\bar{x})$. We write $\mathcal{M} \models p(\bar{a})$ or $\mathcal{M}, \bar{a} \models p(\bar{x})$. We say $\bar{a}$ <u>realizes</u> $p(x)$ in $\mathcal{M}$
- <u>satisfiable</u> if it is satisfiable in some $\mathcal{M}$
- <u>finitely satisfiable</u> if all its finite subsets are satisfiable
- <u>finitely satisfiable in $\mathcal{M}$</u> if all its finite subsets are satisfiable in $\mathcal{M}$

Warning: we might say "<u>consistent</u>" instead of "satisfiable" throughout the definition (some authors prefer the terms "consistent", "finitely consistent", etc).

Remark: a type may be finitely satisfiable in $\mathcal{M}$ but not satisfiable in $\mathcal{M}$. For example, let $\mathcal{M} = \langle \mathbb{N}, < \rangle$ and let $\phi_n(x)$ say "there are at least n elements less than x".
Let $p(x) = \{\phi_n(x) : n < \omega\}$. Then $p(x)$ is finitely satisfiable in $\mathcal{M}$, but it is not satisfiable in $\mathcal{M}$.

5.8 <u>Theorem (Compactness for types)</u> Every finitely satisfiable L-type is satisfiable
Proof: Let $p(\bar{x})$ be an L-type with $\bar{x} = \langle x_i : i < \lambda \rangle$. Expand L to $L' = L \cup \{c_i : i < \lambda\}$ where the c_i are distinct constants not in L.
Then $p(\bar{c})$ is a finitely satisfiable theory in L'. By Theorem 5.6, there is an L'-structure $\mathcal{M}'$ such that $\mathcal{M}' \models p(\bar{c})$. But $\mathcal{M}'$ can be reduced to an L-structure $\mathcal{M}$ (simply forget the interpretations of the c_i). Then $\mathcal{M}, \bar{c}^{\mathcal{M}'} \models p(\bar{x})$. □

(5.9) <u>Lemma</u> Let $\mathcal{M}$ be an L-structure, and let $\bar{a}$ be an enumeration of $\mathcal{M}$, say $\bar{a} = \langle a_i : i < \lambda \rangle$.
Let $q(\bar{x}) = \{ \varphi(\bar{x}) : \mathcal{M} \models \varphi(\bar{a}) \}$ where $|\bar{x}| = \lambda$.
Then $q(\bar{x})$ is satisfiable in $\mathcal{N}$ if and only if there is an elementary embedding $\beta : \mathcal{M} \to \mathcal{N}$.

Proof : ($\Rightarrow$) If $q(\bar{x})$ is satisfiable in $\mathcal{N}$, then there is a tuple $\bar{b} \in N^{|\bar{x}|}$ such that $\mathcal{N} \models \varphi(\bar{b})$ for all $\varphi(\bar{x}) \in q(\bar{x})$.
Then $\beta : \mathcal{M} \to \mathcal{N}$ defined by $a_i \mapsto b_i$ $(i < \lambda)$ is an elementary embedding.
The fact that β is an embedding follows from the fact that certain specific atomic formulas are preserved (e.g. $f(a_{i_1}, \ldots a_{i_n}) = a_j$)
Elementarity is clear because

$$\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(\bar{b}) \text{ and } \beta(\bar{a}) = \bar{b}.$$

($\Leftarrow$) If $\beta : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\beta(\bar{a})$ is a realisation of $q(\bar{x})$ in $\mathcal{N}$. $\square$

(5.10) <u>Remark</u> If $\mathcal{M}$ is an L-structure, we can consider types in $L(A)$ for any $A \subseteq \mathcal{M}$. In particular, we can consider types in $L(M)$. Types of this kind are said to have <u>parameters</u> in A (resp. $\mathcal{M}$). A type with parameters in A is sometimes said to be <u>over A</u>.
If $p(\bar{x})$ is a type in $L(M)$, then there are an enumeration $\bar{a}$ of $\mathcal{M}$ and a type $q(\bar{x}, \bar{z})$ in L such that $p(\bar{x}) = q(\bar{x}, \bar{a})$.

(5.11) <u>Theorem</u> If $\mathcal{M}$ is an L-structure and $p(\bar{x})$ is a type in $L(M)$ that is finitely satisfiable in $\mathcal{M}$, then $p(\bar{x})$ is realized in some elementary extension of $\mathcal{M}$.

Example : Let $\mathcal{M} = (\mathbb{Q}, <)$. Let $\langle a_i : i < \omega \rangle$ be a sequence of rational numbers that converges to $\sqrt{2}$ from below.
Let $\langle b_i : i < \omega \rangle$ be a sequence of rational numbers that converges to $\sqrt{2}$ from above.
Let $\varphi_n(x)$ be the formula that says $a_n < x < b_n$.
Let $p(x) = \{ \varphi_n(x) : n < \omega \}$.
Then $p(x)$ is a type in $L(\mathbb{Q})$ and $p(x)$ is finitely satisfiable in $\mathcal{M}$, but not satisfiable.
However, $p(x)$ is realized in $(\mathbb{R}, <)$, and we now know that $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$.

Proof of Theorem 5.11 Let $\bar{a} = \langle a_i : i < \lambda \rangle$ be an enumeration of $\mathcal{M}$, and let

$$q(\bar{z}) = \{ \varphi(\bar{z}) : \mathcal{M} \models \varphi(\bar{a}) \}$$

where $|\bar{z}| = \lambda$ and the z_i are new variables.

Write $p(\bar{x})$ as $p'(\bar{x},\bar{a})$, where $p'(\bar{x},\bar{z})$ is an L-type.
Then $p'(\bar{x},\bar{z}) \cup q(\bar{z})$ is finitely consistent in $\mathcal{M}$ (because $p'(\bar{x},\bar{a})$ is finitely consistent and $q(\bar{z})$ is realized by $\bar{a}$).
By the compactness Theorem for types (Theorem 5.8) there are an L-structure $\mathcal{N}'$ and tuples $\bar{c} \in N^{|\bar{x}|}$, $\bar{a}' \in N^{|\bar{z}|}$ such that

$$\mathcal{N}' \models p'(\bar{c},\bar{a}') \cup q(\bar{a}')$$

Since $\mathcal{N}' \models q(\bar{a}')$, $\mathcal{M}, \bar{a} \models q(\bar{z})$, and so by Lemma 5.9 $\beta : \mathcal{M} \to \mathcal{N}'$ defined by

$$a_i \mapsto a_i'$$

is an elementary embedding. Then $\beta(\mathcal{M}) \preccurlyeq \mathcal{N}'$, and there is $\mathcal{N} \succcurlyeq \mathcal{M}$ such that $\mathcal{N} \cong \mathcal{N}'$. Then $\mathcal{N}$ is the required elementary extension of $\mathcal{M}$.

5.12 Theorem (Upward Löwenheim-Skolem)
If $\mathcal{M}$ is an L-structure and $|\mathcal{M}| \geqslant \omega$, then for any $\lambda \geqslant |\mathcal{M}| + |L|$ there is $\mathcal{N}$ such that
(i) $\mathcal{M} \preccurlyeq \mathcal{N}$ (ii) $|\mathcal{N}| = \lambda$.

Proof: Let $\bar{x} = \langle x_i : i < \lambda \rangle$ be a tuple of distinct variables, and let

$$p(\bar{x}) := \{ x_i \neq x_j : i < j < \lambda \}.$$

Then $p(\bar{x})$ is finitely satisfiable in $\mathcal{M}$ (and indeed in any infinite structure).
By Theorem 5.11, $p(\bar{x})$ is realised in some $\mathcal{N} \succcurlyeq \mathcal{M}$.
By using the Downward Löwenheim-Skolem theorem if necessary (Theorem 3.11), we may assume $|\mathcal{N}| = \lambda$.

6 SATURATION

6.1 Definition Let λ be an infinite cardinal and $\mathcal{M}$ an L-structure such that $|\mathcal{M}| \geq \omega$. Then $\mathcal{M}$ is <u>λ-saturated</u> if it realizes every type $p(x)$ (one free variable) such that
(i) $p(x)$ has parameters in some $A \subseteq M$ with $|A| < \lambda$
(ii) $p(x)$ is finitely satisfiable in $\mathcal{M}$.

$\mathcal{M}$ is <u>saturated</u> if it is $|\mathcal{M}|$-saturated

Remark: If $|\mathcal{M}| < \lambda$, $\mathcal{M}$ cannot be λ-saturated. For example, $\mathcal{M}$ does not realize the following type with parameters in M:

$$p(x) = \{ x \neq a \ : a \in M \}.$$

6.2 Definition Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, $\bar{b}$ a tuple in $\mathcal{M}$ (possibly infinite). Then the <u>type of $\bar{b}$ over A</u> is

$$tp_{\mathcal{M}}(\bar{b}/A) = \{ \varphi(\bar{x}) \in L(A) : \mathcal{M} \models \varphi(\bar{b}) \}$$

$\mathcal{M}$ is often omitted if clear from the context.

6.3 Remarks (i) $tp_{\mathcal{M}}(\bar{b}/A)$ is complete, that is, for all $L(A)$-formulas $\varphi(\bar{x})$, either $\varphi(\bar{x}) \in tp_{\mathcal{M}}(\bar{b}/A)$ or $\neg\varphi(\bar{x}) \in tp_{\mathcal{M}}(\bar{b}/A)$.
(ii) If $\mathcal{M} \preceq \mathcal{N}$, $A \subseteq M$ and $\bar{b}$ is a tuple in $\mathcal{M}$, then

$$tp_{\mathcal{M}}(\bar{b}/A) = tp_{\mathcal{N}}(\bar{b}/A)$$

6.4 Facts (types and elementary maps)
Recall Definition 4.10.
(i) If $f : A \subseteq \mathcal{M} \to \mathcal{N}$ is an elementary map, then in particular $\mathcal{M} \equiv \mathcal{N}$
If $\mathcal{M} \equiv \mathcal{N}$, then the empty map is elementary (it preserves sentences).
(ii) If $f : A \subseteq \mathcal{M} \to \mathcal{N}$ is elementary and $\bar{a}$ is an enumeration of $\mathrm{dom}(f)$, then $tp(\bar{a}/\emptyset) = tp(f(\bar{a})/\emptyset)$.

More generally, if $f : \mathcal{M} \to \mathcal{N}$ is (partial) elementary and there is $A \subseteq M \cap N$ such that f is the identity on A, then for every tuple $\bar{b}$ in $\mathrm{dom}(f)$,

$$tp(\bar{b}/A) = tp(f(\bar{b})/A)$$

(iii) Let $\bar{a}$ enumerate $A = \mathrm{dom}(f)$ for some $f : \mathcal{M} \to \mathcal{N}$ elementary. Then if $p(\bar{x}, \bar{a})$ is an $L(A)$-type that is finitely satisfiable in $\mathcal{M}$, then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in $\mathcal{N}$.

Let $\{\varphi_1(\bar{x}, \bar{a}), \ldots, \varphi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a})$. Then $\mathcal{M} \models \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x}, \bar{a})$

By the elementarity of f, $M \models \exists \bar{x} \bigwedge_{i=1}^{n} \phi_i(\bar{x}, f(\bar{a}))$

(6.5) Theorem Let M be such that $|M| \geq \lambda \geq |L| + \omega$. The following are equivalent:

(i) M is λ-saturated

(ii) if $\mathcal{M} \equiv M$, $b \in M$ and $f : \mathcal{M} \to M$ is a partial elementary map such that $|f| < \lambda$, then there is a partial elementary $\hat{f} \supseteq f$ such that $b \in \mathrm{dom}(\hat{f})$

(iii) if $p(\bar{z})$ is an $L(A)$-type that is finitely satisfiable in M and $|\bar{z}| \leq \lambda$ and $|A| < \lambda$, then $p(\bar{z})$ is realized (i.e. satisfiable) in M.

Proof: (i) $\Rightarrow$ (ii) Let f be as in the hypothesis of (ii) and let $\bar{a}$ be an enumeration of $\mathrm{dom}(f)$, so $|\bar{a}| < \lambda$. For $b \in M$, let $p(x, \bar{a}) = tp_{\mathcal{M}}(b/\bar{a})$. Since $p(x, \bar{a})$ is (finitely) satisfiable in $\mathcal{M}$, $p(x, f(\bar{a}))$ is finitely satisfiable in M (by Fact 6.4 (iii)).
Since M is λ-saturated and $|f(\bar{a})| < \lambda$, there is a realization c of $p(x, f(\bar{a}))$ in M.
Then $f \cup \{\langle b, c\rangle\}$ is the required extension of f.
(Note that $f \cup \{\langle b,c\rangle\}$ is elementary because $\mathcal{M} \models \phi(b, \bar{a})$
$\Leftrightarrow \phi(x, \bar{a}) \in p(x, \bar{a}) \Leftrightarrow \phi(x, f(\bar{a})) \in p(x, f(\bar{a})) \Leftrightarrow M \models \phi(c, f(\bar{a}))$)

(ii) $\Rightarrow$ (iii) Let $p(\bar{z})$ be as in the hypothesis of (iii). By Theorem 5.11, $p(\bar{z})$ is realized in some elementary extension $\mathcal{M} \succeq M$ by some tuple $\bar{a}$.
Since $M \preceq \mathcal{M}$, the map $id_A : A \subseteq \mathcal{M} \to M$ is elementary.
We extend id_A to an elementary map $f : \mathcal{M} \to M$ such that $\bar{a} \in \mathrm{dom}(f)$. Then $f(\bar{a})$ realizes $p(\bar{z})$ in M.
We extend id_A by building a chain $\langle f_i : i < |\bar{a}|\rangle$.
Let $f_0 = id_A$.
At stage $i+1$, extend f_i to f_{i+1} so that $a_i \in \mathrm{dom}(f_{i+1})$. This can be done by (ii).
At limit stages $\mu < |\bar{a}|$, let $f_\mu = \bigcup_{i<\mu} f_i$.
Then $f = \bigcup_{i<|\bar{a}|} f_i$ is an elementary map (by Remark 4.11).

(iii) $\Rightarrow$ (i) trivial □

Remark: in (ii) $\Rightarrow$ (iii), one could also:
- find $\mathcal{M}' \succeq M$ where $p(\bar{z})$ is realized by $\bar{a}$.
- use Downward Löwenheim Skolem to get $\mathcal{M} \preceq \mathcal{M}'$ such that $A \cup \bar{a} \subseteq \mathcal{M}$ and $|\mathcal{M}| \leq |M|$
- notice that $id_A : \mathcal{M} \to M$ is elementary (as $\mathcal{M}, M \preceq \mathcal{M}'$)
- extend id_A to an elementary embedding $\beta : \mathcal{M} \to M$ by using (ii).
Then $\beta(\bar{a})$ realizes $p(\bar{z})$ in M.

(6.6) Corollary If $\mathcal{M}$ and $\mathcal{N}$ are saturated, $\mathcal{M} \equiv \mathcal{N}$ and $|\mathcal{M}| = |\mathcal{N}|$, then every elementary map $f : \mathcal{M} \to \mathcal{N}$ extends to an isomorphism.

Proof: use Theorem (6.5) (ii) to extend f to $\beta : \mathcal{M} \to \mathcal{N}$ by back-and-forth (take unions at limit stages).

(6.7) Corollary Models of T_{dlo} and T_{rg} are ω-saturated.
Proof: by Theorem 6.5 and Lemma 4.3 for T_{dlo}, Lemma 4.17 for T_{rg}.

(6.8) Definition An isomorphism $\alpha : \mathcal{N} \to \mathcal{N}$ is called an automorphism of $\mathcal{N}$.

The automorphisms of $\mathcal{N}$ form a group denoted by $\mathrm{Aut}(\mathcal{N})$.
If $A \subseteq \mathcal{N}$, then $\mathrm{Aut}(\mathcal{N}/A) = \{\alpha \in \mathrm{Aut}(\mathcal{N}) : \alpha \restriction_A = \mathrm{id}\}$

α restricted to A is the identity, i.e. α fixes A pointwise

(6.9) Definition (i) An L-structure $\mathcal{N}$ is λ-universal if for every $\mathcal{M} \equiv \mathcal{N}$ such that $|\mathcal{M}| \leq \lambda$ there is an elementary embedding $\beta : \mathcal{M} \to \mathcal{N}$
$\mathcal{N}$ is universal if it is $|\mathcal{N}|$-universal
(ii) $\mathcal{N}$ is λ-homogeneous if every elementary map $f : A \subseteq \mathcal{N} \to \mathcal{N}$ such that $|f| < \lambda$ extends to an automorphism of $\mathcal{N}$
$\mathcal{N}$ is homogeneous if it is $|\mathcal{N}|$-homogeneous.

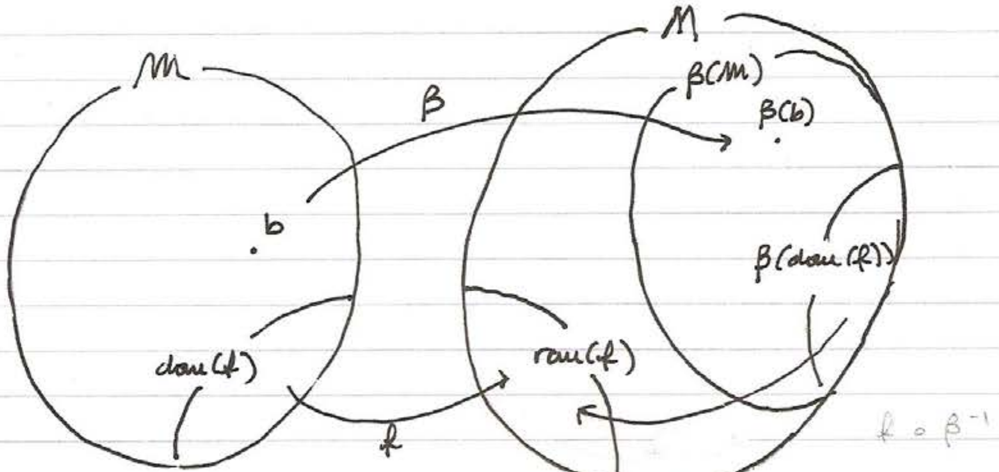
Warning: in some textbooks / papers, λ-universality according to (i) is called λ^+-universality. Homogeneity is sometimes called "strong homogeneity", and "weak homogeneity" is a property akin to the one in Theorem 6.5 (ii) (with $\mathcal{M} = \mathcal{N}$). There is also a notion of "ultrahomogeneity," which involves partial embeddings rather than elementary maps. Theorems 4.4 and 4.18 prove that $(\mathbb{Q}, <)$ and the random graph are ultrahomogeneous.

Countable models of T_{dlo} and T_{rg} are universal and homogeneous

(6.10) Theorem Let $\mathcal{N}$ be such that $|\mathcal{N}| \geq |L| + \omega$. Then the following are equivalent:
(i) $\mathcal{N}$ is saturated
(ii) $\mathcal{N}$ is universal and homogeneous.

Proof: (i) $\Rightarrow$ (ii) Assume that $\mathcal{M}$ is saturated and $\mathcal{N} \equiv \mathcal{M}$ is such that $|\mathcal{N}| \leq |\mathcal{M}|$.
Let $tp(\bar{x}) = tp(\bar{a}/\emptyset)$, where $\bar{a}$ enumerates $\mathcal{N}$. Then $p(\bar{x})$ is finitely satisfiable in $\mathcal{N}$, hence in $\mathcal{M}$ (because $\mathcal{N} \equiv \mathcal{M}$).
Since $|\bar{x}| \leq |\mathcal{M}|$, $\mathcal{M}$ realizes $p(\bar{x})$ by saturation (use Theorem 6.5). If $\bar{b}$ realizes $p(\bar{x})$, then $a_i \mapsto b_i$ $(i < |\mathcal{N}|)$ is elementary.
Homogeneity follows from Corollary 6.6.

(ii) $\Rightarrow$ (i) We show that if $\mathcal{N} \equiv \mathcal{M}$, $b \in \mathcal{N}$ and $f : \mathcal{N} \to \mathcal{M}$ is elementary with $|f| < |\mathcal{M}|$, then f has an extension defined in b.
By working in $\mathcal{N}' \preccurlyeq \mathcal{N}$ s.t. $dom(f) \cup \{b\} \subseteq \mathcal{N}'$ if necessary, (use Downward Löwenheim-Skolem), we may assume $|\mathcal{N}| \leq |\mathcal{M}|$.
Since $\mathcal{N} \equiv \mathcal{M}$, by universality there is an elementary embedding $\beta : \mathcal{N} \to \mathcal{M}$. Then $\beta(\mathcal{N}) \preccurlyeq \mathcal{M}$.



Then the map $f \circ \beta^{-1} : \beta(dom(f)) \to ran(f)$ is elementary.
By homogeneity, there is $\alpha \in Aut(\mathcal{M})$ such that $f \circ \beta^{-1} \subseteq \alpha$.
Then $f \cup \{\langle b, \alpha(\beta(b))\rangle\}$ is elementary (it is a restriction of $\alpha \circ \beta$). $\square$

(6.11) <u>Definition</u> Let $\bar{a}$ be a tuple in a structure $\mathcal{M}$ and let $A \subseteq N$. Then the <u>orbit</u> of $\bar{a}$ over A is the set
$$O_{\mathcal{M}}(\bar{a}/A) = \{\alpha(\bar{a}) : \alpha \in Aut(\mathcal{M}/A)\}$$

If $\varphi(\bar{x})$ is an $L(A)$-formula, then
$$\varphi(\mathcal{M}) = \{\bar{b} \in N^{|\bar{x}|} \text{ such that } \mathcal{M} \models \varphi(\bar{b})\}$$
is the set <u>defined</u> by $\varphi(\bar{x})$.
A set is <u>definable over A</u> if it is defined by some $L(A)$-formula.

A type $p(\bar{x})$ in $L(A)$ defines the set
$$p(\mathcal{M}) = \{\bar{b} \in N^{|\bar{x}|} \text{ such that } \mathcal{M} \models p(\bar{b})\}$$

A subset of $N^{|\bar{x}|}$ is <u>type-definable</u> over A if it is defined by some type $p(\bar{x})$ in $L(A)$.

6.12 <u>Remark</u> If $\bar{a}$ and $\bar{b}$ are tuples in $\mathcal{M}$ and $|\bar{a}| = |\bar{b}|$ and $A \subseteq N$, then the following are equivalent:
(i) $tp(\bar{a}/A) = tp(\bar{b}/A)$
(ii) $\{\langle a_i, b_i \rangle : i < |\bar{a}|\} \cup id_A$ is an elementary map from $\mathcal{M}$ to $\mathcal{M}$

6.13 <u>Proposition</u> Let $\mathcal{M}$ be λ-homogeneous, $A \subseteq \mathcal{M}$ such that $|A| < \lambda$ and let $\bar{a}$ be a tuple in $\mathcal{M}$ such that $|\bar{a}| < \lambda$. Then $O_{\mathcal{M}}(\bar{a}/A) = p(\mathcal{M})$, where $p(\bar{x}) = tp_{\mathcal{M}}(\bar{a}/A)$.

Proof: If $\alpha(\bar{a}) = \bar{b}$ for some $\alpha \in Aut(\mathcal{M}/A)$, then clearly $tp_{\mathcal{M}}(\bar{a}/A) = tp_{\mathcal{M}}(\bar{b}/A)$.

If $tp_{\mathcal{M}}(\bar{a}/A) = tp_{\mathcal{M}}(\bar{b}/A)$, then $\{\langle a_i, b_i \rangle : i < |\bar{a}|\} \cup id_A$ is elementary. By λ-homogeneity, there is $\alpha \in Aut(\mathcal{M})$ that extends this elementary map. Clearly, $\alpha \in Aut(\mathcal{M}/A)$. Hence $\bar{b} \in O_{\mathcal{M}}(\bar{a}/A)$. $\square$

7 THE MONSTER MODEL

Given a complete theory T with an infinite model, it is often convenient to work in a saturated structure $\mathcal{U}$ that is large enough that any model of T we might be interested in is an elementary substructure of $\mathcal{U}$. We call $\mathcal{U}$ the <u>monster model</u>, and we denote its cardinality by κ.

The existence of arbitrarily large saturated models for general theories may depend on extra set-theoretic assumptions (such as the existence of inaccessible cardinals). However, in specific cases such assumptions can be avoided by suitable ad-hoc arguments. In practice, the monster model is an expository device that allows to prove results about all the models of a given theory T by proving results about elementary substructures of saturated models of T.

7.1 <u>Terminology and conventions</u>

When working in a monster model $\mathcal{U}$ of a theory T, we say that

- $\phi(\bar{x})$ holds if $\mathcal{U} \models \forall\bar{x}\, \phi(\bar{x})$
- $\phi(\bar{x})$ is consistent if $\mathcal{U} \models \exists\bar{x}\, \phi(\bar{x})$
- a type $p(\bar{x})$ is <u>consistent / satisfiable</u> if $\mathcal{U}, \bar{a} \models p(\bar{x})$ for some $\bar{a} \in \mathcal{U}^{|\bar{x}|}$
- a cardinality is <u>small</u> if it is strictly less than κ
- a <u>model</u> is $\mathcal{M} \preceq \mathcal{U}$ such that $|\mathcal{M}|$ is small

Conventions:

- tuples have length $< \kappa$, unless otherwise specified
- formulas have parameters in $\mathcal{U}$, unless o/w specified
- definable sets have the form $\phi(\mathcal{U})$ for some $L(\mathcal{U})$-formula $\phi(\bar{x})$
- type-definable sets have the form $p(\mathcal{U})$ for some $L(A)$-type $p(\bar{x})$, where $|A| < \kappa$

Notation:

- the symbols $A, B, C, \ldots$ denote sets of parameters of small cardinality
- $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}_{\mathcal{U}}(\bar{a}/A)$
- $O(\bar{a}/A)$ is the orbit of $\bar{a}$ in $\mathcal{U}$ under $\mathrm{Aut}(\mathcal{U}/A)$

If $p(\bar{x})$ and $q(\bar{x})$ are types, we write $p(\bar{x}) \rightarrow q(\bar{x})$ to mean that every realization of $p(\bar{x})$ is a realization of $q(\bar{x})$.
In general, we extend the semantics of formulas to types, where types are thought of as infinite conjunctions of formulas. However, these infinite conjunctions are NOT formulas and so truth is not preserved by elementary maps.

(7.2) Fact Let $p(\bar{x})$ be a consistent $L(A)$-type and let $q(\bar{x})$ be a consistent $L(B)$-type. Suppose that $p(\bar{x}) \to \neg q(\bar{x})$ (that is, $p(\bar{x})$ and $q(\bar{x})$ have no realizations in common). Then there are a conjunction $\phi(\bar{x})$ of fmlas in $p(\bar{x})$ and a conjunction $\psi(\bar{x})$ of formulas in $q(\bar{x})$ such that

$$\phi(\bar{x}) \to \neg \psi(\bar{x}).$$

Proof If $p(\bar{x}) \to \neg q(\bar{x})$ then $p(\bar{x}) \cup q(\bar{x})$ is not realized in $\mathcal{U}$. Hence it is not finitely satisfiable (by saturation of $\mathcal{U}$). Let $\{\phi_1(\bar{x}), \dots, \phi_m(\bar{x})\} \subseteq p(\bar{x})$ and $\{\psi_1(\bar{x}), \dots, \psi_m(\bar{x}) \subseteq q(\bar{x})$ be such that $\bigwedge \phi_i(\bar{x}) \wedge \bigwedge \psi_i(\bar{x})$ is not satisfiable.
Then $\bigwedge \phi_i(\bar{x}) \to \neg(\bigwedge \psi_i(\bar{x}))$, as required. □

(7.3) Remark Let $\phi(\mathcal{U}, \bar{b})$ be a definable set, where $\phi(\bar{x}, \bar{z})$ is an L-formula and $\bar{b} \in \mathcal{U}^{|\bar{z}|}$. If $\alpha \in Aut(\mathcal{U})$, then

$$\begin{aligned}\alpha[\phi(\mathcal{U}, \bar{b})] &= \{\alpha(\bar{a}) : \phi(\bar{a}, \bar{b}),\ \bar{a} \in \mathcal{U}^{|\bar{x}|}\} \\ &= \{\alpha(\bar{a}) : \phi(\alpha(\bar{a}), \alpha(\bar{b})),\ \bar{a} \in \mathcal{U}^{|\bar{x}|}\} \\ &= \phi(\mathcal{U}, \alpha(\bar{b})).\end{aligned}$$

So $Aut(\mathcal{U})$ acts on the definable sets of $\mathcal{U}$ in a natural way.
Similarly, if $p(\bar{x}, \bar{z})$ is an L-type and $\bar{b} \in \mathcal{U}^{|\bar{z}|}$, then

$$\alpha[p(\mathcal{U}, \bar{b})] = p(\mathcal{U}, \alpha(\bar{b})).$$

(7.4) Definition A set $D \subseteq \mathcal{U}^d$ is invariant over A if $\alpha(D) = D$ for all $\alpha \in Aut(\mathcal{U}/A)$, or, equivalently, if $O(\bar{a}/A) \subseteq D$ for all $\bar{a} \in D$.

If $\bar{a} \in D$ and $q(\bar{x}) = tp(\bar{a}/A)$, then for $\bar{b} \models q(\bar{x})$ there is $\alpha \in Aut(\mathcal{U}/A)$ such that $\alpha(\bar{a}) = \bar{b}$ (by homogeneity), and so $b \in O(\bar{a}/A)$.
So the invariance of D is equivalent to the following condition: for all $\bar{a} \in D$, if $\bar{b} \equiv_A \bar{a}$ then $\bar{b} \in D$.

Definability and invariance are closely related in $\mathcal{U}$.

(7.5) Proposition For every $L(\mathcal{U})$-formula $\phi(\bar{x})$ the following are equivalent:
(i) $\phi(\bar{x})$ is equivalent to some formula $\psi(\bar{x})$ with parameters in A for some $A \subseteq \mathcal{U}$
(ii) $\phi(\mathcal{U})$ is invariant over A.

Proof: (i) ⇒ (ii) obvious
(ii) ⇒ (i) Suppose $\phi(\bar{x}, \bar{z})$ is an L-fmla such that $\phi(\mathcal{U}, \bar{b})$ is invariant over A for $\bar{b} \in \mathcal{U}^{|\bar{z}|}$.

Let $q(\bar{z}) = tp(\bar{b}/A)$. Then if $\bar{c} \models q(\bar{z})$, by homogeneity there is $\alpha \in Aut(\mathcal{U}/A)$ such that $\alpha(\bar{b}) = \bar{c}$.

Then $\alpha[\phi(\mathcal{U}, \bar{b})] = \phi(\mathcal{U}, \bar{c})$ (by Remark 7.3)

$$\begin{array}{c} \| \\ \phi(\mathcal{U}, \bar{b}) \text{ (by invariance)} \end{array}$$

Hence $q(\bar{z}) \rightarrow \forall \bar{x}\, [\phi(\bar{x}, \bar{z}) \leftrightarrow \phi(\bar{x}, \bar{b})]$. By (an argument similar to the one in) Fact 7.2, there is $\theta(\bar{z}) \in q(\bar{z})$ such that $\theta(\bar{z}) \rightarrow \forall \bar{x}\, [\phi(\bar{x}, \bar{z}) \leftrightarrow \phi(\bar{x}, \bar{b})]$.

Then $\exists \bar{z}\, [\theta(\bar{z}) \wedge \phi(\bar{x}, \bar{z})]$ is an $L(A)$-formula that defines $\phi(\mathcal{U}, \bar{b})$, as required. □

7.6 <u>Definition</u> (partial embedding, general case) An injective map $p : A \subseteq M \rightarrow M$ is a <u>partial embedding</u> if for all tuples $\bar{a}$ from A, p satisfies conditions (i), (ii) and (iii) of Definition 1.5.

In particular, a partial embedding preserves the truth of quantifier-free formulas on elements of its domain

7.7 <u>Proposition</u> Let $\phi(\bar{x})$ be an L-formula. Then the following are equivalent:

(i) $\phi(\bar{x})$ is equivalent to a quantifier-free formula $\psi(\bar{x})$ (that is, $\mathcal{U} \models \forall \bar{x}\, [\phi(\bar{x}) \leftrightarrow \psi(\bar{x})]$)

(ii) for all partial embeddings $p : \mathcal{U} \rightarrow \mathcal{U}$, for all tuples $\bar{a}$ from $dom(p)$,
$$\phi(\bar{a}) \leftrightarrow \phi(p(\bar{a}))$$

Proof: (i) $\Rightarrow$ (ii): clear

(ii) $\Rightarrow$ (i) For a tuple $\bar{a}$ from $\mathcal{U}$, let
$$qftp(\bar{a}) = \{\psi(\bar{x}) \;\; \psi(\bar{x}) \text{ is a quantifier-free } L\text{-fmla and } \psi(\bar{a}) \text{ holds}\}$$
Let $D = \{q(\bar{x}) : q(\bar{x}) = qftp(\bar{a}) \text{ for } \bar{a} \in \phi(\mathcal{U})\}$

Claim: $\phi(\mathcal{U}) = \bigcup_{q(\bar{x}) \in D} q(\mathcal{U})$

$\subseteq$ is clear (by our choice of D)

For $\supseteq$, let $q(\bar{x}) \in D$, say $q(\bar{x}) = qftp(\bar{a})$, and let $\bar{b} \models q(\bar{x})$. Then $\{\langle a_i, b_i \rangle : i \leq |\bar{a}|\}$ is a partial embedding, and so, by (ii), $\phi(\bar{b})$ holds.

Hence $q(\mathcal{U}) \subseteq \phi(\mathcal{U})$, and therefore $\bigcup_{q(\bar{x}) \in D} q(\mathcal{U}) \subseteq \phi(\mathcal{U})$.

Then in particular $q(\bar{x}) \rightarrow \phi(\bar{x})$. By (an argument similar to that in) Fact 7.2, there is $\psi_q(\bar{x}) \in q(\bar{x})$ such that $\psi_q(\bar{x}) \rightarrow \phi(\bar{x})$.

Then $\phi(\bar{x}) \leftrightarrow \bigvee_{q(\bar{x}) \in D} \{ \psi_q(\bar{x}) : \psi_q(\bar{x}) \to \phi(\bar{x}) \text{ and } \psi_q(\bar{x}) \in q(\bar{x}) \}$

Again by (a version of) Fact 7.2, there are $\psi_{q_1}(\bar{x}), \dots, \psi_{q_n}(\bar{x})$ such that

$$\phi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{n} \psi_{q_i}(\bar{x})$$

and $\bigvee \psi_{q_i}(\bar{x})$ is the required quantifier-free formula. $\square$

(7.8) <u>Definition</u> An L-theory T has <u>quantifier elimination</u> (q.e.) if for every L-formula $\phi(\bar{x})$ there is a quantifier-free L-formula $\psi(\bar{x})$ such that

$$T \vdash \forall \bar{x} \, [\phi(\bar{x}) \leftrightarrow \psi(\bar{x})]$$

If a theory has q.e., then the type of a tuple is determined by the quantifier-free type, and so types of tuples are much easier to describe and understand.

(7.9) <u>Theorem</u> Let T be a complete theory with an infinite model. The following are equivalent:

(i) T has quantifier elimination

(ii) every partial embedding $p : \mathfrak{U} \to \mathfrak{U}$ is elementary

(iii) for every partial embedding $p : \mathfrak{U} \to \mathfrak{U}$ such that $|p| < |\mathfrak{U}|$ and for every $b \in \mathfrak{U}$ there is a partial embedding $\hat{p} \supseteq p$ s.t. $b \in \operatorname{dom}(\hat{p})$.

Proof: (i) $\Rightarrow$ (ii) Since every formula is equivalent to a quantifier-free one, any partial embedding preserves all formulas, and so it is elementary.
(ii) $\Rightarrow$ (i) Since all partial embeddings are elementary, any formula is preserved by all partial embeddings. By Proposition 7.7, any formula is equivalent to a quantifier-free formula.

(ii) $\Rightarrow$ (iii) If $p : \mathfrak{U} \to \mathfrak{U}$ is as in (iii), then, by (ii), it is an elementary map. Since $|p| < |\mathfrak{U}|$, by homogeneity of $\mathfrak{U}$ p extends to $\alpha \in \operatorname{Aut}(\mathfrak{U})$. Then $p \cup \{\langle b, \alpha(b) \rangle\}$ is the required extension of p
(iii) $\Rightarrow$ (ii) Given $p : \mathfrak{U} \to \mathfrak{U}$ as in (ii), consider a finite restriction $p_0 \subseteq p$ and show that p_0 is elementary by extending p_0 to $\alpha \in \operatorname{Aut}(\mathfrak{U})$ by back and forth (use saturation). Then p_0 is elementary. This holds for any finite restriction p_0 of p. Hence p is elementary.

Remark: another condition equivalent to (i), (ii) and (iii) is
(iv) for every finite partial embedding $p : \mathfrak{U} \to \mathfrak{U}$ and $b \in \mathfrak{U}$, there is $\hat{p} \supseteq p$ such that $\hat{p}$ is a partial embedding and $b \in \operatorname{dom}(\hat{p})$.

So in Chapter 4 we proved q.e. for T_{rg} and T_{dlo}.

7.10 Definition An element $a \in \mathcal{U}$ is <u>definable over $A \subseteq \mathcal{U}$</u> if there is an $L(A)$-formula $\phi(x)$ such that $\phi(\mathcal{U}) = \{a\}$.
An element $a \in \mathcal{U}$ is <u>algebraic over $A \subseteq \mathcal{U}$</u> if there is an $L(A)$-formula $\phi(x)$ such that $a \in \phi(\mathcal{U})$ and $|\phi(\mathcal{U})| < \omega$.

The <u>definable closure</u> of A is the set
$$dcl(A) = \{a \in \mathcal{U} : a \text{ is definable over } A\}$$
The <u>algebraic closure</u> of A is the set
$$acl(A) = \{a \in \mathcal{U} : a \text{ is algebraic over } A\}.$$

If $A = acl(A)$, A is said to be <u>algebraically closed</u>.

7.11 Proposition For $a \in \mathcal{U}$ and $A \subseteq \mathcal{U}$ the following are equivalent:
(i) $a \in dcl(A)$
(ii) $O(a/A) = \{a\}$ (i.e. a is fixed by $Aut(\mathcal{U}/A)$).
Proof: Let $\phi(x)$ define a over A. Then $\phi(\mathcal{U})$ is invariant over A, and so $O(a/A) = \{a\} = \phi(\mathcal{U})$.
For the other direction, use that $O(a/A)$ is invariant over A and Proposition 7.5.

7.12 Theorem Let $A \subseteq \mathcal{U}$ and $a \in \mathcal{U}$. The following are equivalent.
(i) $a \in acl(A)$
(ii) $O(a/A)$ is finite
(iii) $a \in \mathcal{M}$ for any model $\mathcal{M}$ s.t. $A \subseteq \mathcal{M}$.

Proof: (i) $\Rightarrow$ (ii) If $a \in acl(A)$, then there is an $L(A)$-formula $\phi(x)$ such that $\phi(a)$ holds and $|\phi(\mathcal{U})| < \omega$.
Since $\phi(\mathcal{U})$ is invariant over A, $O(a/A) \subseteq \phi(\mathcal{U})$, and so $|O(a/A)| < \omega$.
(ii) $\Rightarrow$ (i) Since $O(a/A)$ is finite, it is definable (by the formula $\bigvee_{i=1}^{n} x = a_i$, where $O(a/A) = \{a_1, \dots a_n\}$).
Since $O(a/A)$ is invariant over A, by Proposition 7.5 there is an $L(A)$-formula $\phi(x)$ that defines $O(a/A)$.
Then $|\phi(\mathcal{U})| < \omega$ and $\phi(a)$ holds, hence $a \in acl(A)$.

(i) $\Rightarrow$ (iii) If $a \in acl(A)$ then there is an $L(A)$-formula such that $\phi(a)$ holds and $\mathcal{U} \models \exists^{=n} x\ \phi(x)$ for some $n \in \omega$.
By elementarity, $\exists^{=n} x\ \phi(x)$ holds in any model containing A, and the n solutions of $\phi(x)$ in $\mathcal{M}$ are solutions in $\mathcal{U}$. Hence $a \in A$.
(iii) $\Rightarrow$ (i) Suppose $a \notin acl(A)$ and let $p(x) = tp(a/A)$.
Then for all $\phi(x) \in p(x)$, $|\phi(\mathcal{U})| \geq \omega$. Then $|p(\mathcal{U})| \geq \omega$ (see Exercise on Worksheet 2). By an argument similar to that for Exercise , $|p(\mathcal{U})| = \mathcal{U}$.
Let $\mathcal{M}$ be any model such that $A \subseteq \mathcal{M}$.

Then $p(\mathfrak{U}) \setminus \mathcal{M} \neq \emptyset$ (by cardinality considerations). Let $b \in p(\mathfrak{U}) \setminus \mathcal{M}$.
By homogeneity, there is $\alpha \in \mathrm{Aut}(\mathfrak{U}/A)$ such that $\alpha(b) = a$.
Then $\alpha[\mathcal{M}]$ is a model that contains A but does not contain $a = \alpha(b)$. □

7.13 Proposition Let $a \in \mathfrak{U}$ and $A \subseteq \mathfrak{U}$. Then

(i) if $a \in \mathrm{acl}(A)$ then $a \in \mathrm{acl}(A_0)$ for some finite $A_0 \subseteq A$

(ii) $A \subseteq \mathrm{acl}(A)$

(iii) if $A \subseteq B$, then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$

(iv) $\mathrm{acl}(A) = \mathrm{acl}(\mathrm{acl}(A))$

(v) $\mathrm{acl}(A) = \bigcap_{A \subseteq \mathcal{M}} \mathcal{M}$

Proof: (iv) By (i), $\mathrm{acl}(A) \subseteq \mathrm{acl}(\mathrm{acl}(A))$.
By Theorem 7.12, $A \subseteq \mathcal{M} \Leftrightarrow \mathrm{acl}(A) \subseteq \mathcal{M}$. Moreover, $a \in \mathrm{acl}(\mathrm{acl}(A)) \Rightarrow a \in \mathcal{M}$ for every $\mathcal{M} \supseteq \mathrm{acl}(A)$. Hence $a \in \mathrm{acl}(\mathrm{acl}(A)) \Rightarrow a \in \mathcal{M}$ for every $\mathcal{M} \supseteq A \Rightarrow a \in \mathrm{acl}(A)$

(v) follows from Theorem 7.12.

Properties (i) to (iv) say that acl is a closure operator with finite character.

7.14 Proposition Let $\beta \in \mathrm{Aut}(\mathfrak{U})$ and $A \subseteq \mathfrak{U}$. Then
$\beta[\mathrm{acl}(A)] = \mathrm{acl}(\beta[A])$.

Proof: ($\subseteq$) Let $a \in \mathrm{acl}(A)$, and $\phi(x, \bar{z})$ be an L-formula and $\bar{b}$ a tuple in A such that $\phi(a, \bar{b})$ holds and $|\phi(\mathfrak{U}, \bar{b})| < \omega$. Then $\phi(\beta(a), \beta(\bar{b}))$ holds and $|\phi(\mathfrak{U}, \beta(\bar{b}))| < \omega$, and so $\beta(a)$ is algebraic over $\beta(\bar{b})$.
This proves $\beta[\mathrm{acl}(A)] \subseteq \mathrm{acl}(\beta[A])$.
The same proof with f^{-1} in place of f and $f(A)$ in place of A gives the converse inclusion.

(8) STRONGLY MINIMAL THEORIES

(8.1) <u>Definition</u> Let M be an infinite L-structure. A subset $A \subseteq M$ is <u>cofinite</u> if $M \setminus A$ is finite

(8.2) <u>Remark</u> Finite and cofinite sets are definable in every structure. In this chapter, we will look at structures where these are all the definable sets.

(8.3) <u>Definition</u> A structure M is said to be <u>minimal</u> if all its definable subsets are finite or cofinite.
M is <u>strongly minimal</u> if it is minimal and all its elementary extensions are minimal.
If T is a consistent theory without finite models, then T is <u>strongly minimal</u> if for every formula $\varphi(x, \bar{z})$ there is $n \in \omega$ such that

$$T \vdash \forall \bar{z} \, [\, \exists^{\leq n} x \; \varphi(x, \bar{z}) \vee \exists^{\leq n} x \; \neg\varphi(x, \bar{z})] .$$

Example: let $L = \{E\}$ (binary relation) and let M be the L-structure where E is interpreted as an equivalence relation with one class of size n for $n = 1, 2, \ldots$ and no infinite classes.
It can be shown that M is minimal. However, there is an elementary extension N of M where there is an element a whose equivalence class is infinite. Then $E(x, a)$ defines an infinite, coinfinite subset of the domain. Hence M is not strongly minimal

For the remainder of this chapter, we assume that T is a complete strongly minimal theory with an infinite model.

(8.4) <u>Definition</u> Let $a \in \mathcal{U}$, $B \subseteq \mathcal{U}$. Then a is <u>independent from B</u> if $a \notin \operatorname{acl}(B)$.
The set B is <u>independent</u> if every $a \in B$ is independent from $B \setminus \{a\}$

<u>Notation</u>: we write a, B for $\{a\} \cup B$ and $B \setminus a$ for $B \setminus \{a\}$

(8.5) <u>Theorem</u> Let $B \subseteq \mathcal{U}$ and $a, b \in \mathcal{U} \setminus \operatorname{acl}(B)$ Then

$$b \in \operatorname{acl}(a, B) \Leftrightarrow a \in \operatorname{acl}(b, B).$$

Proof. Let $a, b \in \mathcal{U} \setminus \operatorname{acl}(B)$. Assume for a contradiction that $b \notin \operatorname{acl}(a, B)$ and $a \in \operatorname{acl}(b, B)$. Let $\varphi(x, y)$ be an $L(B)$-formula such that for some $n \in \omega$

$$\varphi(a, b) \wedge \exists^{\leq n} x \; \varphi(x, b)$$

Since $b \notin \operatorname{acl}(a, B)$, the formula

$$\psi(a, y) \equiv \varphi(a, y) \wedge \exists^{\leq n} x \; \varphi(x, y)$$

has infinitely many realizations. Hence $|\varphi(a, U)| = |U|$. By strong minimality, the set of realizations of $\neg\varphi(a, y)$ is finite. Therefore every model $\mathcal{M}$ containing B also contains a realization c of $\varphi(a, y)$.
Then $a \in acl(c, B)$ and so $a \in \mathcal{M}$. Hence $a \in \mathcal{M}$ for any $\mathcal{M} \supseteq B$. By Theorem 7.12, $a \in acl(B)$ ※ $\square$

Examples of strongly minimal theories

① Let K be an infinite field. The language $L_K = \{+, -, \underline{0}, \{\lambda\}_{\lambda \in K}\}$ can be used to axiomatize the theory of vector spaces over K, where for $\lambda \in K$, the symbol λ in the language represents a unary function (to be interpreted as multiplication by the scalar λ). We write $\lambda(v)$ as λv.
Then the theory T_{VSK} includes the following axioms / axiom schemata:
- axioms for abelian groups (using $+, -, \underline{0}$)
- the following axioms for scalar product:

$\forall xy\, [\lambda(x+y) = \lambda x + \lambda y]$ for each $\lambda \in K$

$\forall x\, [\lambda_1(\lambda_2 x) = \mu x]$ for all $\lambda_1, \lambda_2, \mu \in K$ such that $\lambda_1\lambda_2 = \mu$

$\forall x\, [\lambda_1 x + \lambda_2 x = \nu x]$ for all $\lambda_1, \lambda_2, \nu \in K$ such that $\lambda_1 + \lambda_2 = \nu$

$\forall x\, [1x = x]$

$\exists x\, [x \neq \underline{0}]$ (avoids the model whose domain is $\{\underline{0}\}$)

It can be shown (with some work!) that T_{VSK} is complete and has q.e.
Then:
- atomic fmlas are equalities between linear combinations;
- an atomic formula with one variable and with parameters is equivalent to one of the form $\lambda x = a$. Therefore atomic fmlas with one variable (and parameters) define singletons;
- therefore, quantifier free fmlas with one variable and parameters define finite or cofinite sets.
- by q.e., T_{VSK} is strongly minimal.

One can also show that, for $A \subseteq \mathcal{M} \models T_{VSK}$ and $a \in \mathcal{M}$.
- $acl(A) = \langle A \rangle$, the linear span
- $a \notin acl(A) \iff a$ is linearly independent from A
- A is independent if and only if it is linearly independent

② Let $L_{ring} = \{+, \cdot, -, 0, 1\}$. Then ACF is the theory that includes:
- axioms for abelian groups (using $+, -, 0$)
- axioms for commutative monoids (using $\cdot, 1$)
- the following field axioms:

$\forall xyz\, [x \cdot (y+z) = x \cdot y + x \cdot z]$

$0 \neq 1$

$\forall x \; [x = 0 \vee \exists y \, (x \cdot y = 1)]$

- axioms for algebraic closure: for all $n < \omega$

$\forall x_0 \cdots x_n \, \exists y \;\; [x_0 + x_1 y + \cdots + x_n y^n = 0]$

Let $\chi_p = \underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} = 0$. Then $ACF_p = ACF \cup \{\chi_p\}$, where p is prime.

$ACF_0 = ACF \cup \{\neg \chi_n : n < \omega\}$.

ACF_p and ACF_0 are both complete and have q.e. (showing this requires some work). Then:

- atomic formulas are polynomial equations;
- if $A \subseteq \mathcal{M} \models ACF_p$ (or ACF_0), an atomic fmla with one variable and parameters in A is equivalent to a polynomial equation $p(x) = 0$, where $p(x) \in F[x]$ and F is the subfield of K generated by A
- therefore, atomic formulas with parameters and one free variable define finite sets;
- quantifier-free formulas with parameters and one free variable define finite or cofinite sets;
- by quantifier elimination, ACF_p and ACF_0 are strongly minimal.

(8.6) Definition Let $B \subseteq C \subseteq \mathfrak{U}$. Then B is a basis of C if B is an independent set and $C \subseteq acl(B)$ (or, equivalently, if $acl(B) = acl(C)$).

(8.7) Lemma If B is an independent set and $a \notin acl(B)$, then $\{a\} \cup B$ is independent.

Proof: Suppose that $a \notin acl(B)$ and that $\{a\} \cup B$ is not independent. Then there is $b \in B$ such that $b \in acl(a, B \setminus b)$. Since B is independent, $b \notin acl(B \setminus b)$. Also, $a \notin acl(B \setminus b)$. So by Theorem 8.5 we have $a \in acl(b, B \setminus b) = acl(B)$ ※.

(8.8) Corollary If $B \subseteq C \subseteq \mathfrak{U}$, the following are equivalent:

(i) B is a basis of C

(ii) B is a maximal independent subset of C (i.e. if $B \subseteq B' \subset C$ and B' is independent, then $B = B'$)

(8.9) Theorem Let $C \subseteq \mathfrak{U}$. Then:

(i) every independent subset $B \subseteq C$ can be extended to a basis of C;

(ii) if A, B are bases of C, then $|A| = |B|$

Proof (i) If $\langle B_i : i < \lambda \rangle$ is a chain of independent subsets of C such that, for all i, $B \subseteq B_i$, then $\bigcup_{i<\lambda} B_i$ is

independent (if not, let $b \in \bigcup B_i$ be such that $b \in \operatorname{acl}(\bigcup B_i \setminus b)$. Then there is a finite $A \subseteq B_i \setminus b$ such that $b \in \operatorname{acl}(A \setminus b)$. But this implies $b \in \operatorname{acl}(B_i \setminus b)$ for some i ✗).
By Zorn's Lemma, there is a maximal independent subset of C that contains B. By Corollary 8.8, this maximal subset is the required basis of C.

(ii) Suppose A and B are bases of C and B is infinite.
Suppose for a contradiction that $|A| < |B|$. For $a \in A$, let $D_a \subseteq B$ be finite and such that $a \in \operatorname{acl}(D_a)$. Let $D = \bigcup_{a \in A} D_a$.
Then $A \subseteq \operatorname{acl}(D)$, and $|D| < |B|$.
Since $A \subseteq \operatorname{acl}(D)$, $C \subseteq \operatorname{acl}(D)$, which contradicts the independence of B.

If A and B are finite, show that $|A| \leq |B|$ by using the following: if there is $a \in A \setminus B$, then there is $b \in B \setminus A$ such that the set $\{b\} \cup (A \setminus \{a\})$ is independent. This is because if $a \in A \setminus B$, then $a \in \operatorname{acl}(B)$, and so $B \not\subseteq \operatorname{acl}(A \setminus a)$ (otherwise A is not independent). So let $b \in B \setminus \operatorname{acl}(A \setminus a)$. Then $\{b\} \cup (A \setminus \{a\})$ is independent by Lemma 8.7. □

8.10 Definition Let $C \subseteq \mathcal{U}$ be algebraically closed. Then the dimension of C, $\dim(C)$, is $|A|$ for any basis A of C.

8.11 Proposition Let $f : \mathcal{U} \to \mathcal{U}$ be a (partial) elementary map. Let $b \notin \operatorname{acl}(\operatorname{dom}(f))$ and $c \notin \operatorname{acl}(\operatorname{ran}(f))$. Then $f \cup \{\langle b, c \rangle\}$ is elementary.

Proof: Let $\bar{a}$ enumerate $\operatorname{dom}(f)$, and let $\phi(x, \bar{a})$ be a formula with parameters in $\bar{a}$.
Claim: $\phi(b, \bar{a}) \leftrightarrow \phi(c, \bar{a})$.
Case 1: $|\phi(\mathcal{U}, \bar{a})| < \omega$. Then $|\phi(\mathcal{U}, f(\bar{a}))| < \omega$ (by elementarity of f). Since $b \notin \operatorname{acl}(\bar{a})$ and $c \notin \operatorname{acl}(f(\bar{a}))$, we have that
$\neg\phi(b, \bar{a}) \wedge \neg\phi(c, \bar{a})$.
Case 2: $|\phi(\mathcal{U}, \bar{a})| \geq \omega$. Then, by strong minimality, $|\neg\phi(\mathcal{U}, \bar{a})| < \omega$ and so $|\neg\phi(\mathcal{U}, f(\bar{a}))| < \omega$. As above,
$\phi(b, \bar{a}) \wedge \phi(c, \bar{a})$. □

8.12 Corollary Every bijection between independent sets in $\mathcal{U}$ is elementary.

Proof: Let $A, B \subseteq \mathcal{U}$ be independent. Let $f : A \to B$ be a bijection and let $\bar{a}$ enumerate A. Let $b_i = f(a_i)$.
Then $a_0 \notin \operatorname{acl}(\emptyset)$ and $b_0 \notin \operatorname{acl}(\emptyset)$ (otherwise A, B are not independent). By Proposition 8.11, $\{\langle a_0, b_0 \rangle\}$ is an elementary map.

At stage $i+1$, $\{\langle a_j, b_j\rangle : j \leq i\}$ is elementary, $a_{i+1} \notin acl(a_0 \dots a_i)$ and $b_{i+1} \notin acl(b_0 \dots b_i)$. By Proposition 8.11, $\{\langle a_j, b_j\rangle : j \leq i+1\}$ is an elementary map. $\square$

8.13 <u>Remark</u> If $M \subseteq \mathcal{U}$ is a model, then by Proposition 7.13 (v) M is algebraically closed.

8.14 <u>Theorem</u> Suppose $M, N \subseteq \mathcal{U}$ are such that $\dim(M) = \dim(N)$. Then $M \simeq N$.

Proof. Let A, B be bases of M, N respectively. Let $f: A \to B$ be a bijection. By Corollary 8.12, f is elementary. By homogeneity of $\mathcal{U}$, f extends to $\alpha \in Aut(\mathcal{U})$. Then

$$\alpha(M) = \alpha(acl(A)) = acl(\alpha(A)) = acl(B) = N. \qquad \square$$

by Proposition 7.14

8.15 <u>Corollary</u> Let T be strongly minimal and let $\lambda > |L|$. Then T is λ-categorical

Proof: If $A \subseteq \mathcal{U}$, then $|acl(A)| \leq L(A)$. This is because there are finitely many elements of $acl(A)$ that are solutions of a given formula in $L(A)$, and there are at most $|L(A)|$ formulas.
If $|M| = \lambda > |L|$, then a basis of M must have cardinality λ. Hence by Theorem 8.14 any two models of cardinality λ are isomorphic. $\square$

So models of strongly minimal theories are determined up to isomorphism by their dimension, hence by their cardinality if the cardinality is greater than that of the language. Dimension in s.m. theory is an example of the kind of invariant that is useful in model-theoretic classification theory.