Assume that T is a complete theory with a monster model $\mathcal{U}$ and work in $\mathcal{U}$ unless otherwise specified.

(The result in Question 1 is used in the proof of Fact 9.6. The result in Question 2 is used in the proof of Theorem 9.12.)

Question 1. Let $A \subseteq \mathcal{U}$ be a small set, and let $p(\bar{x}, \bar{z})$ be a type in $L(A)$. Show that there is $q(\bar{x})$ in $L(A)$ such that for $\bar{a} \in \mathcal{U}^{|\bar{x}|}$ the following are equivalent:

(i) there is $b \in \mathcal{U}^{|\bar{z}|}$ such that $\bar{a} \models p(\bar{x}, \bar{b})$

(ii) $\bar{a} \models q(\bar{x})$.

Question 2. Let $\lambda < |\mathcal{U}|$ and let $\{p_i(x) : i < \lambda\}$ be a collection of complete types such that

1. $\mathcal{U} = \bigcup_{i<\lambda} p_i(\mathcal{U})$
2. $p_i(x)$ is isolated by $\varphi_i(x)$ for all $i < \lambda$.

Show that there is $n \in \omega$ such that

$$\mathcal{U} = \bigcup_{j=1}^{n} \varphi_{i_j}(\mathcal{U}).$$

Question 3. Suppose that $p(x)$ and $q(x)$ are types such that $p(\mathcal{U}) = \mathcal{U} \setminus q(\mathcal{U})$ – that is, $p(x)$ defines a set whose complement is type-definable. Show that $p(\mathcal{U})$ is a definable set.

Question 4. Prove directly the implication (ii) $\Rightarrow$ (i) in Theorem 6.5 in the special case $\lambda = |\mathcal{N}|$, that is, suppose that $\mathcal{N}$ is an L-structure such that $|\mathcal{N}| \geq |L|$. Suppose further that whenever $f : \mathcal{M} \mapsto \mathcal{N}$ is a partial elementary map such that $|f| < |\mathcal{N}|$ and $a \in M$, there is a partial elementary map $\hat{f}$ that extends f and such that $a \in \mathrm{dom}(\hat{f})$. Show that $\mathcal{N}$ is saturated. [*Do not assume the existence of a monster model.*]