We assume that T is a complete theory with a monster model $\mathcal{U}$ and we work in $\mathcal{U}$.

Question 1. Show that the following are equivalent (and hence equivalent to quantifier elimination for T):

(i) every partial embedding $p : \mathcal{U} \to \mathcal{U}$ is elementary.

(ii) for every finite partial embedding $p : \mathcal{U} \to \mathcal{U}$ and $b \in \mathcal{U}$, there is a partial embedding $\hat{p}$ such that $p \subseteq \hat{p}$ and $b \in \mathrm{dom}(\hat{p})$.

Solution (Sketch)

(ii) $\Rightarrow$ (i) Assume that the language is countable and let $p : \mathcal{U} \to \mathcal{U}$ be a partial embedding. It suffices to show that every finite restriction p_0 of p is elementary.

We construct two countable models $M \preceq \mathcal{U}$ and $N \preceq \mathcal{U}$ together with an isomorphism $h : M \to N$ that extends p_0. Then h is an elementary map.

The models $\mathcal{M}$ and $\mathcal{N}$ are constructed at the same time as h by defining a chain of finite partial embeddings $\langle h_i : i{<}\omega \rangle$ in such a way that eventually

$$h = \bigcup_{i<\omega} h_i ,$$

and $\mathrm{dom}(h)$ and $\mathrm{ran}(h)$ satisfy the Tarski-Vaught test, so that we set $M = \mathrm{dom}(h)$ and $N = \mathrm{ran}(h)$.

Let $h_0 = p_0$. Assume h_i is given. Let $\varphi(x)$ be an $L(\mathrm{dom}(h_i))$-formula that is consistent and pick a solution $b \in \mathcal{U}$. By (ii) there is a $c \in \mathcal{U}$ such that $h_i \cup \{\langle b, c\rangle\}$ is a partial embedding. Let $h_{i+1/2} = h_i \cup \{\langle b, c\rangle\}$.

Similarly, we extend $h_{i+1/2}$ to h_{i+1} by using a similar argument with $h^{-1}_{i+1/2}$.

Let $M = \mathrm{dom}(h)$. By the Tarski-Vaught test we get $\mathcal{M} \preceq \mathcal{U}$ if all $L(M)$-formulas are eventually considered. A similar consideration holds for $\mathcal{N}$.

This is achieved if at stage i we choose $\varphi(x)$ appropriately. Let $\pi : \omega^2 \to \omega$ be a bijection such that $\pi(m, n) \geq m, n$ for all m, n. The formula we pick at stage $i = \pi(n, m)$ is the n-th $L(\mathrm{dom}(h_m))$-formula in a suitable enumeration.

The result can be extended to the case when L is uncountable as follows. Suppose p is a partial embedding and assume for a contradiction that there is a formula $\varphi(x)$ that is not preserved by p. Let $L' \subseteq L$ be countable and such that φ is an L'-formula. Let $\mathcal{U}'$ be the structure obtained by restricting the interpretation of $\mathcal{U}$ to L'. Then $\mathcal{U}'$ is saturated, and we can use the previous result to obtain a contradiction.

Question 2. Let $p(x)$ be a type with parameters in A, and let $\varphi(x,y)$ be an $L(A)$-formula that defines, when restricted to $p(\mathfrak{U})$, a *finite* equivalence relation – that is, an equivalence relation with finitely many classes. Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x,y)$ on $p(\mathfrak{U})$.

Solution Let $\chi(x,y,z) \equiv \varphi(x,x) \wedge [\varphi(x,y) \to \varphi(y,x)] \wedge [(\varphi(x,y) \wedge \varphi(y,z)) \to \varphi(x,z)]$.

We assume

$$[p(x) \wedge p(y) \wedge p(z)] \to \left[\chi(x,y,z) \wedge \left(\bigvee_{i=1}^{n} \varphi(x,b_i)\right)\right],$$

where the b_i are representatives of the equivalence classes defined by φ in $p(\mathfrak{U})$.

By compactness/saturation there are $\psi_1(x), \psi_2(x), \psi_3(x) \in p(x)$ such that

$$[\psi_1(x) \wedge \psi_2(y) \wedge \psi_3(z)] \to \left[\chi(x,y,z) \wedge \left(\bigvee_{i=1}^{n} \varphi(x,b_i)\right)\right].$$

Let $\psi(x) \equiv \bigwedge_{i=1}^{3} \psi_i(x)$. Then

$$[\neg\psi(x) \wedge \neg\psi(y)] \vee [\psi(x) \wedge \psi(y) \wedge \varphi(x,y)]$$

defines the required equivalence relation.

Question 3. Let T be a consistent theory. Suppose that all complete theories containing T are of the form $T \cup S$ for some set S of quantifier-free sentences (for example, $T = \mathrm{ACF}$). Prove that if all complete theories containing T have elimination of quantifiers, so does T.

Solution Let $\varphi(\bar{x})$ be a formula and $T \cup S$ be a completion of T. Then

$$T \cup S \vdash \forall \bar{x}\, [\varphi(\bar{x}) \leftrightarrow \psi_S(\bar{x})]$$

for some $\psi_S(\bar{x})$ that is quantifier-free. By compactness, there is a finite $\Sigma_S \subseteq S$ such that

$$T \cup \Sigma_S \vdash \forall \bar{x}\, [\varphi(\bar{x}) \leftrightarrow \psi_S(\bar{x})].$$

Let σ_S be the conjunction of the sentences in Σ_S. Since

$$T \vdash \bigvee \{\sigma_S : S \cup T \text{ is a completion of } T\},$$

there are $\sigma_{S_1}, \ldots, \sigma_{S_n}$ such that $T \vdash \bigvee_{i=1}^{n} \sigma_{S_i}$. Wlog we may choose the σ_{S_i} to be mutually exclusive (that is, $\sigma_{S_i} \to \neg\sigma_{S_j}$ for $i \neq j$). Let $\psi(\bar{x}) \equiv \bigwedge_{i=1}^{n} (\sigma_{S_i} \to \psi_{S_i}(\bar{x}))$. Then

$$T \vdash \forall \bar{x}\, [\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})].$$

Question 4. Let $\varphi(z)$ be a consistent $L(A)$-formula. Prove that if $a \in \operatorname{acl}(A,b)$ for every $b \models \varphi(z)$, then $a \in \operatorname{acl}(A)$. Prove that the same holds when $a \in \operatorname{acl}(A,b)$ for every $b \models p(z)$, where $p(z)$ is a satisfiable type with parameters in A.

Solution Let $\varphi(z)$ be consistent, so $\mathcal{U} \models \exists z\,\varphi(z)$. Let $\mathcal{M}$ be a model such that $A \subseteq \mathcal{M}$. By elementarity, $\mathcal{M}$ contains a witness b of $\varphi(z)$. Therefore $\operatorname{acl}(A,b) \subseteq \mathcal{M}$, and so $a \in \mathcal{M}$. It follows that $a \in \bigcap_{A\subseteq\mathcal{M}} \mathcal{M}$, which is equivalent to $a \in \operatorname{acl}(A)$.

Now suppose $a \in \operatorname{acl}(A,b)$ for every $b \models p(z)$, where $p(z)$ is a type with parameters in A.

Claim: $a \in \operatorname{acl}(A)$.

Suppose for a contradiction that $a \notin \operatorname{acl}(A)$. Then $|\operatorname{Orb}(a/A)| = |\mathcal{U}|$ (this is because $\operatorname{Orb}(a/A) = q(\mathcal{U})$, where $q(x) = \operatorname{tp}(a/A)$). Let $\mathcal{M}$ be a model that contains $A \cup \{b\}$ for some $b \models p(z)$. By cardinality considerations, there is $a' \in \operatorname{Orb}(a/A) \smallsetminus \mathcal{M}$. Let $\beta \in \operatorname{Aut}(\mathcal{U}/A)$ be such that $a = \beta(a')$. Then $a \notin \beta(\mathcal{M})$, and $\beta(\mathcal{M})$ is a model that contains A and contains $\beta(b) \models p(z)$. However, $a \notin \beta(\mathcal{M})$ contradicts $a \in \operatorname{acl}(A\beta(b))$.

Question 5. Let $\bar{c}$ be en enumeration of $\mathcal{U}$. Let $\mathcal{V}$ be the set enumerated by some $\bar{a} \models \operatorname{tp}(\bar{c}/\varnothing)$. Prove that $\mathcal{V} \preceq \mathcal{U}$.

Solution

Let $\overline{c}$ enumerate $\mathcal{U}$, let $p(\overline{x}) = \operatorname{tp}(\overline{c}/\varnothing)$ (so $|\overline{x}| = |\mathcal{U}|$). Suppose $\overline{a} \models p(\overline{x})$. Claim: $\overline{a}$ enumerates $\mathcal{V} \preceq \mathcal{U}$.

We use the Tarski-Vaught test. Suppose $\mathcal{U} \models \exists y\,\varphi(\overline{a}, y)$. Since $\operatorname{tp}(\overline{a}/\varnothing) = \operatorname{tp}(\overline{c}/\varnothing)$, this implies
$\mathcal{U} \models \exists y\,\varphi(\overline{c}, y)$ (because $\exists y\,\varphi(\overline{x}, y) \in p(\overline{x})$). Then $\mathcal{U} \models \varphi(\overline{c}, c_i)$ for some i, so $\varphi(\overline{x}, x_i) \in p(\overline{x})$.
Then $\mathcal{U} \models \varphi(\overline{a}, a_i)$.

Question 6. Let $a \in \mathcal{U} \smallsetminus \operatorname{acl} A$. Prove that $\mathcal{U}$ is isomorphic over A to some $\mathcal{V} \preceq \mathcal{U}$ such that $a \notin \mathcal{V}$.

Solution We want to find $\mathcal{V} \cong \mathcal{U}$, $\mathcal{V} \preceq \mathcal{U}$ such that $a \notin \mathcal{V}$.

Let $\overline{c}$ enumerate $\mathcal{U}$ and let $p(\overline{x}) = \operatorname{tp}(\overline{c}/A)$. Let $q(\overline{x}) = p(\overline{x}) \cup \{x_i \neq a : i < |\overline{x}|\}$.

Claim: $q(\overline{x})$ is finitely satisfiable. If not, there are $\varphi(\overline{x}) \in p(\overline{x})$ and $i_1, \dots, i_n$ such that

$$\varphi(\overline{x}) \to \bigvee_{j=1}^{n} x_{i_j} = a.$$

Let $\mathcal{M} \supseteq A$ be a model. Since $\varphi(\overline{x})$ is satisfiable in $\mathcal{U}$, we have that $\mathcal{M} \models \exists \overline{x}\,\varphi(\overline{x})$, and so $\mathcal{M} \models \varphi(\overline{b})$ for some $\overline{b}$. Then $b_i = a$ for some i, and so $a \in \mathcal{M}$. It follows that $a \in \mathcal{M}$ whenever $\mathcal{M} \supseteq A$, that is, $a \in \operatorname{acl}(A)$, a contradiction.

So $q(\overline{x})$ is finitely satisfiable, hence satisfied by some $\overline{d}$, say. Since $\overline{d} \models q(\overline{x})$ we have that $\overline{d} \models p(\overline{x})$, so $\overline{d}$ enumerates $\mathcal{V} \cong \mathcal{U}$. By Question 5, we also have $\mathcal{V} \preceq \mathcal{U}$. By construction, $a \notin \mathcal{V}$.

Question 7. Let C be a finite set. Prove that if $C \cap M \neq \varnothing$ for every model $\mathcal{M}$ containing A, then $C \cap \operatorname{acl}(A) \neq \varnothing$.

Solution *Method 1* Let $C = \{c_1, \ldots, c_n\}$ be such that $C \cap \mathcal{M} \neq \emptyset$ for every $\mathcal{M} \supseteq A$, and suppose for a contradiction that $C \cap \operatorname{acl}(A) = \emptyset$. By Question 6, there are $\mathcal{V}_1, \ldots, \mathcal{V}_n$ such that, for each i,

(i) $\mathcal{V}_i \cong_A \mathcal{V}_{i-1}$

(ii) $\mathcal{V}_i \preceq \mathcal{V}_{i-1}$ (where $\mathcal{V}_0 = \mathcal{U}$)

(iii) $c_i \notin \mathcal{V}_i$.

By the Downward Löwenheim-Skolem Theorem, there is $\mathcal{M}$ such that $A \subseteq \mathcal{M} \preceq \mathcal{V}_n$, hence $C \cap \mathcal{M} = \emptyset$, which is a contradiction.

Method 2 By induction on $|C|$.
Let $|C| = 1$, that is, $C = \{c\}$. Then $C \cap \mathcal{M} \neq \emptyset$ for all $\mathcal{M} \supseteq A$ implies that $c \in \mathcal{M}$ for all $\mathcal{M} \supseteq A$, and so $c \in \operatorname{acl}(A)$.

Now let $C = \{c_1, \ldots, c_{n+1}\}$. If $C \subseteq \operatorname{acl}(A)$, nothing to do. So let $c_j \in C \smallsetminus \operatorname{acl}(A)$. By Question 6, there is $\mathcal{V} \cong \mathcal{U}$ such that $\mathcal{V} \preceq \mathcal{U}$, $A \subseteq \mathcal{V}$ and $c_j \notin \mathcal{V}$. Let $C' = C \cap \mathcal{V}$. Then:

(i) $C' \neq \emptyset$ because $C \cap \mathcal{V} \neq \emptyset$ and $c_j \notin \mathcal{V}$

(ii) $|C'| \leq n$

(iii) if $\mathcal{M} \preceq \mathcal{V}$ and $A \subseteq \mathcal{M}$ then also $A \subseteq \mathcal{M} \preceq \mathcal{U}$, and so $C' \cap \mathcal{M} \neq \emptyset$.

By inductive hypothesis, $C' \cap \operatorname{acl}(A) \neq \emptyset$. But $C' \cap \operatorname{acl}(A) \subseteq C \cap \operatorname{acl}(A)$, so we have the required result.

Question 8. Let $\varphi(x) \in L(\mathcal{U})$ and fix an arbitrary set A. Prove that the following are equivalent

(i) there is some model $\mathcal{M}$ containing A and such that $\mathcal{M} \cap \varphi(\mathcal{U}) = \varnothing$;

(ii) there is no satisfiable $L(A)$-formula $\psi(z_1, \ldots, z_n)$ such that

$$\psi(z_1, \ldots, z_n) \to \bigvee_{i=1}^{n} \varphi(z_i).$$

Solution (i) $\Rightarrow$ (ii) Assume there is a satisfiable $\psi(z_1, \ldots, z_n)$ such that

$$\psi(z_1, \ldots, z_n) \to \bigvee_{i=1}^{n} \varphi(z_i),$$

Let $\bar{z} = z_1, \ldots, z_n$. Then $\mathcal{U} \models \exists \bar{z}\, \psi(\bar{z})$, and

$$\mathcal{U} \models \forall \bar{z}\, [\psi(\bar{z}) \to \bigvee_{i=1}^{n} \varphi(z_i)].$$

Let $\mathcal{M}$ be a model containing A. By elementarity, $\mathcal{M} \models \psi(\bar{b})$ for some $\bar{b}$. Again by elementarity, $\mathcal{M} \models \bigvee_{i=1}^{n} \varphi(b_i)$, hence $\mathcal{M} \cap \varphi(\mathcal{U}) \neq \emptyset$.

(ii) $\Rightarrow$ (i) Let $\mathcal{N} \supseteq A$ be a model, let $\bar{a}$ enumerate $\mathcal{N}$, and let $p(\bar{z}) = \mathrm{tp}(\bar{a}/A)$ (so $|\bar{z}| = |\mathcal{N}|$).

Claim: $p(\bar{z}) \cup \{\neg\varphi(z_i) : i < |\bar{z}|\}$ is satisfiable.

If not, there are $\psi(z_1, \ldots, z_n) \in p(\bar{z})$ and $\neg\varphi(z_{i_1}), \ldots, \neg\varphi(z_{i_k})$ such that

$$\{\psi(z_1, \ldots, z_n)\} \cup \{\neg\varphi(z_{i_1}), \ldots, \neg\varphi(z_{i_k})\}$$

is inconsistent. By padding the second set if necessary, we may assume that the variables $z_1, \ldots, z_n$ and $z_{i_1}, \ldots, z_{i_k}$ coincide. Then

$$\mathcal{U} \models \forall \bar{z}\, [\psi(z_1, \ldots, z_n) \to \bigvee_{i=1}^{n} \varphi(z_i)].$$

But by assumption no such $\psi(\bar{z})$ exists. Hence $p(\bar{z}) \cup \{\neg\varphi(z_i)\}$ is satisfied by a tuple $\bar{c}$, say. By $\bar{c}$ enumerates a model $\mathcal{M}$ that contains A and is such that $\mathcal{M} \cap \varphi(\mathcal{U}) = \emptyset$.

Question 9. Prove that in a strongly minimal theory every infinite algebraically closed set is a model.

Solution Let T be strongly minimal, $A \subseteq \mathcal{U}$ infinite (and small) such that $A = \mathrm{acl}(A)$. We use the Tarski-Vaught Test.

Let $\varphi(x)$ be an $L(A)$-formula such that $\mathcal{U} \models \exists x\, \varphi(x)$.
Case 1: $|\varphi(\mathcal{U})| < \omega$. Then all realizations of $\varphi(x)$ are algebraic over A, and so $\varphi(\mathcal{U}) \subseteq A$.

Case 2: $|\varphi(\mathcal{U})| \geq \omega$. Then by strong minimality $|\neg\varphi(\mathcal{U})| < \omega$, and so every infinite set intersects $\varphi(\mathcal{U})$, in particular A.

In either case, if $\mathcal{U} \models \exists x\, \varphi(x)$, then there is $b \in A$ such that $\mathcal{U} \models \varphi(b)$. By the Tarski-Vaught Test, A is the universe of an elementary substructure of $\mathcal{U}$, as required.

Question 10. Prove that a countable strongly minimal theory is either ω-categorical or has infinitely many countable models.

Solution Let T be strongly minimal and let $\mathcal{M}$ and $\mathcal{N}$ be two countable models of T such that $\mathcal{M} \ncong \mathcal{N}$. Then one of $\dim(\mathcal{M})$, $\dim(\mathcal{N})$ must be finite, say $\dim(\mathcal{M}) = n < \omega$.

Claim: there is a countable model that has dimension $n+1$. Let A be a basis of $\mathcal{M}$, let $b \in \mathcal{U} \setminus \mathcal{M}$. Then $b \notin \mathrm{acl}(A)$, and so $A \cup \{b\}$ is an independent set. Since A is a basis for $\mathcal{M}$, $\mathrm{acl}(A, b)$ is infinite. By Question 9, $\mathrm{acl}(A, b)$ is a model and it has dimension $n+1$. Since the language is countable, $|\mathrm{acl}(A, b)| = \omega$.