If you spot any typos or problems, please email `silvia.barbina@open.ac.uk`.

Question 5. Let N be a saturated L-structure, and let $p(x)$ be a type (with one free variable) in $L(A)$, where $A \subset N$ and $|A| < |N|$. Let

$$p(N) = \{a \in N : N \models p(a)\}.$$

Are the following conditions equivalent?

(i) $p(N)$ is infinite;

(ii) $|p(N)| = |N|$.

Are (i) and (ii) equivalent for a type $q(\bar{x})$, where $|\bar{x}| = \omega$ and $|N| > 2^{\omega}$?

Solution Clearly (ii) $\Rightarrow$ (i).

We claim that (i) $\Rightarrow$ (ii). Let $|p(N)| \geq \omega$, and let

$$q(x) = p(x) \cup \{x \neq a : a \in p(N)\}.$$

Then $q(x)$ is finitely satisfiable and it has parameters in $p(N) \cup A$.

If $|p(N)| < |N|$, then $|p(N) \cup A| < |N|$, and so $q(x)$ is satisfiable by saturation of N. Let $b \models q(x)$. Then in particular $b \models p(x)$, and $b \neq a$ for all $a \in p(N)$, which is a contradiction.

The equivalence does not hold for a type $q(\bar{x})$ where $|\bar{x}| = \omega$ and $|N| > 2^{\omega}$. Let $L = \{P\}$, where P is a unary predicate, and let N be a saturated model such that

$$N \models \exists xy\,[P(x) \land P(y) \land x \neq y \land \forall z\,[P(z) \to (z = x \lor z = y)]],$$

that is, $|P(N)| = 2$. Let $|\bar{x}| = \omega$. Then

$$p(\bar{x}) = \{P(x_i) : i \in \omega\}$$

has 2^{ω} realizations.

Question 8.

(a) Let I be an infinite set. An ultrafilter F on I is said to be *principal* if there is $x \in I$ such that

$$F = \{A \subseteq I : x \in A\}.$$

Show that if the ultrafilter F is nonprincipal, then F contains the filter $G = \{A \subseteq I : I \setminus A \text{ is finite}\}$.

(b) Let F be a nonprincipal ultrafilter on ω. Let $p(x) = \{\varphi_i(x) : i \in \omega\}$ be a type in L, and let $\langle M_i : i \in \omega \rangle$ be a collection of L-structures such that for all $n \in \omega$

$$M_n \models \exists x \bigwedge_{i=0}^{n} \varphi_i(x).$$

Prove that the ultraproduct $\prod_{i \in \omega} M_i \big/ \sim_F$ realizes the type $p(x)$.

Solution

(a) Suppose that F is an ultrafilter that contains a finite set $\{a_1, \ldots, a_n\}$. By repeated applications of Fact 5.2(b), F must contain a singleton $\{a\}$. Then if $B \in F$ is such that $a \notin B$, we have $B \cap \{a\} = \emptyset \in F$, which is a contradiction.

Therefore if F contains a finite set, then it is principal, that is, a non principal ultrafilter only contains infinite sets. By the defining property of an ultrafilter (Definition 5.1), the ultrafilter F must contain all cofinite sets.

(b) We have that $M_n \models \exists x \bigwedge_{i=0}^{k} \varphi_i(x)$ for all $n \geq k$. For $n \in \omega$, let a_n be such that

$$M_n \models \bigwedge_{i=0}^{n} \varphi_i(a_n).$$

Let $a_F = \langle a_n : n \in \omega \rangle_F$.

Claim: $a_F \models p(x)$ in the ultraproduct. Let $\varphi_n \in p(x)$. If $k \geq n$, then $M_k \models \varphi_n(a_k)$, so the set

$$\{i \in \omega : M_i \models \varphi_n(a_i)\}$$

is cofinite. Since F is nonprincipal, by part (a) this set is in F. By Łoś's Theorem

$$\prod_{i \in \omega} M_i \big/ \sim_F \models \varphi_n(a_F).$$