08/02/21

8.10 Theorem Assume $|L(A)| \leq \omega$.
Let M be a model and $A \subseteq M$. Tfae:
(i) M is countable and atomic over A
(ii) M is prime over A

Proof (i) $\Rightarrow$ (ii) Let $N \supseteq A$. Then $id_A : M \to N$ is elementary map. By Prop 8.8, id_A extends to $\beta : M \to N$, an elem. emb. of M into N.
(ii) $\Rightarrow$ (i) There is a model that contains A and is countable. Since M embeds in this model, M is countable.
Now let $b \in M^n$ be such that $tp(b/A)$ is not isolated. By OTT, there is a ctble model N that omits $tp(b/A)$. But then $M \not\preceq N$ ✗. $\square$

Recall $S_n(A) = \{ p(x) \subseteq L(A) : p(x) \text{ is consistent, complete, } |x| = n \}$
When $A = \emptyset$, we write $S_n(T)$ for $S_n(\emptyset)$.

8.11 Theorem (Ryll-Nardzewski / Engeler / Svenonius).

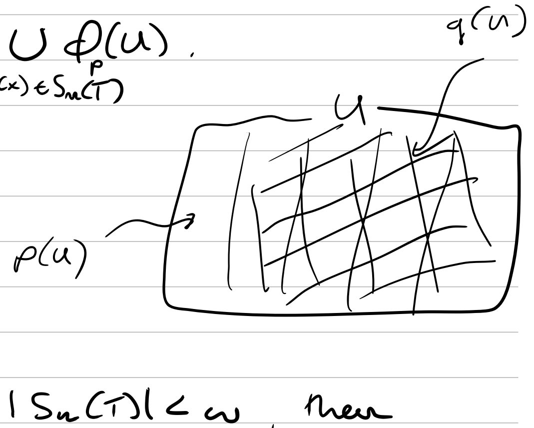
Let T be ctble. Tfae
(i) T is ω-categorical
(ii) for all $n \in \omega$, every type in $S_n(T)$ is isolated
(iii) for all $n \in \omega$ $|S_n(T)| < \omega$
(iv) for all $n \in \omega$, $Aut(U)$ has finitely many orbits on U^n
($O(a_1, \dots a_n) = \{ (\beta(a_1), \dots, \beta(a_n)) : \beta \in Aut(U) \}$

Proof (Sketch) (i) $\Rightarrow$ (ii) Let $p(x) \in S_n(T)$ be non-isolated.
Then there is M s.t. $|M| = \omega$ and M realizes $p(x)$
By OTT, there is N s.t. $|N| = \omega$ and N omits $p(x)$.
Then $M \not\cong N$.
(ii) $\Rightarrow$ (i) By Prop. 8.9

(ii) $\Rightarrow$ (iii) We have $U^n = \bigcup_{p(x) \in S_n(T)} p(U)$. Let $p(x) \in S_n(T)$, and let $\varphi_p(x)$ isolate $p(x)$. Then $U^n = \bigcup_{p(x) \in S_n(T)} \varphi_p(U)$.

By compactness, there is $k \in \omega$ s.t.

$$U^n = \bigcup_{i=1}^{k} \varphi_{p_i}(U)$$



It follows that $|S_n(T)| < \omega$.

(iii) $\Rightarrow$ (ii) Let $p(x) \in S_n(T)$. If $|S_n(T)| < \omega$, then there is $q(x) \subseteq L$ such that $q(U) = U \setminus p(U)$
(to see this, show if $q_1(x), \dots, q_k(x)$ are such $a \models q_1(x)$ or $a \models q_2(x)$ $\cdots$ or $a \models q_k(x)$ then $a \models q(x)$ where $q(x) = \{ \varphi_1(x) \vee \cdots \vee \varphi_k(x) : \varphi_i(x) \in q_i(x) \}$)

$\rightsquigarrow \quad p(\mathfrak{U}) \cap q(\mathfrak{U}) = \emptyset$

Then $p(x) \to \neg q(x)$ and so by Fact 5.2 there are $\varphi(x) \in p(x)$ and $\chi(x) \in q(x)$ such that
$\varphi(\mathfrak{U}) = p(\mathfrak{U})$ and $\chi(\mathfrak{U}) = q(\mathfrak{U}) = \mathfrak{U} \setminus p(\mathfrak{U})$.
Then $\varphi(x)$ isolates $p(x)$.

(iii) $\Leftrightarrow$ (iv) follows from the fact that if $a, b \in \mathfrak{U}^n$ are s.t. $tp(a) = tp(b)$ $\Leftrightarrow$ $O(a) = O(b)$. ☐

8.12 <u>Remark</u> Condition (iv) is usually stated as follows:

If M is a ctble model then $Aut(M)$ has finitely many orbits on M^n for all n. (We say that $Aut(M)$ is <u>oligomorphic</u>)

This version of cond. (iv) can be deduced (exercise). It is a group-theoretic characterization of ω-categoricity.

8.13 <u>Definition</u> A complete theory T is <u>small</u> if for all $n \in \omega$

$|S_n(T)| \leq \omega$.

(in particular, ω-categorical theories are small.

8.14 <u>Proposition</u> If T is ctble, complete and has no finite models, then tfae:

(i) T is small
(ii) For all $n \in \omega$ and $A \overset{fin}{\subseteq} \mathfrak{U}$, we have $|S_n(A)| \leq \omega$
(iii) For all finite $A \subseteq \mathfrak{U}$, $|S_1(A)| \leq \omega$
(iv) T has a countable saturated model.

<u>Proof</u> (sketch) (i) $\Rightarrow$ (ii) Let $A \subseteq \mathfrak{U}$ be finite, let a enumerate A. Let $q(y) = tp(a)$ (so $|y| = |a| = |A| = m$). Let

$$X = \{ p(x, y) \in S_{n+m}(T) : q(y) \subseteq p(x, y) \}.$$

Then $|X| \leq \omega$. But the map $: S_{n+m}(T) \to S_n(A)$
$p(x, y) \mapsto p(x, a)$
is surjective, and so $|S_{n+m}(T)| \leq \omega \Rightarrow |S_n(A)| \leq \omega$.

(ii) $\Rightarrow$ (iii) clear

(iii) $\Rightarrow$ (iv) Build a chain $\langle B_i : i < \omega \rangle$ such that for all i $|B_i| \leq \omega$ and for all i, for all $A \overset{fin}{\subseteq} B_i$ every type in $S_1(A)$ is realized in B_{i+1}. Since $|S_1(A)| \leq \omega$ B_{i+1} can be chosen to be countable.

Then $M = \bigcup_{i \in \omega} B_i$, by TVT $M \preccurlyeq \mathfrak{U}$. By construction, M is saturated. Also $|M| = \omega$. ☐

(vi) $\Rightarrow$ (i) Since for all n, all $p(x) \in S_n(T)$ are realized in the countable sat. model, so there are at most ctbly many realizations, therefore there are at most ctbly many types. $\square$

8.15 Corollary Countable models of ω-categorical theories are saturated.

9 IMAGINARIES

We work in the monster model $\mathcal{U}$ of a complete theory T.
Recall
<u>Proposition 5.5</u> Let $A \subseteq \mathcal{U}$. Then for every fmla $\phi(x) \in L(\mathcal{U})$, tfae
(i) there is $\psi(x) \in L(A)$ s.t.
$\models \forall x\, [\phi(x) \leftrightarrow \psi(x)]$ (i.e. $\phi(\mathcal{U})$ is A-definable)
(ii) $\phi(\mathcal{U})$ is invariant over A.

9.1 <u>Definition</u> Let $D \subseteq \mathcal{U}^n$ be a definable. Then $c \in \mathcal{U}^{|c|}$ is a <u>canonical parameter</u> for D if for all $\alpha \in \mathrm{Aut}(\mathcal{U})$
$\alpha(c) = c \iff \alpha[D] = D$.

9.2 <u>Remarks</u> (i) If d is a can. par for D, then D is $\{d\}$-defble.

(ii) Let D be defble, d a can. par for D. Then
c is a can par for $D \iff c \in dcl(d)$ and $d \in dcl(c)$

This is because
c can. par for $D \iff [\forall \alpha \in \mathrm{Aut}(\mathcal{U}),\ \alpha(c) = c \iff \alpha(d) = d] \iff$

$\iff c \in dcl(d)$ and $d \in dcl(c)$.

9.3 <u>Definition</u> Let M be a model. Then an <u>imaginary</u> in M is an equiv. class a/E where $a \in M^n$ and E is a 0-definable equivalence relation E on M^n.

Examples: In $(\mathbb{Z}^*, \cdot)$ define $(x, y) E (w, z) \leftrightarrow xz = yw$
Then an imaginary is a rational number (the eq class of (a,b) is a/b).

In a gp $\langle G, \cdot, ^{-1}, 1\rangle$ define $x E y \leftrightarrow \exists z\, [y = z^{-1} x z]$.
Here an imaginary is a conjugacy class of G.

9.4 <u>Definition</u> T <u>eliminates imaginaries</u> if for all n, all 0-definable eq. relations E on $\mathcal{U}^n$, every imaginary a/E has a canonical parameter.
(We also say "T has elimination of imaginaries, or EoI).

9.5 <u>Theorem</u> If T eliminates imaginaries, then any definable set has a canonical parameter.

<u>Proof</u>: Let $D = \phi(\mathcal{U}, b)$ for $\phi(x, y) \in L$ and $b \in \mathcal{U}^{|y|}$.

Define E_ϕ on $U^{|y|}$ by
$$w E_\phi z \iff \forall x \ [\phi(x, w) \leftrightarrow \phi(x, z)]$$

(this is saying that $b E_\phi c \iff \phi(U, b) = \phi(U, c)$).

Then E_ϕ is a 0-def.ble equiv. relation.
Claim: the canonical param for b/E_ϕ (exists by EoI) is also a canonical param for D.

Let d be the can. par for b/E_ϕ. Let $\alpha \in \mathrm{Aut}(U)$.

$$\alpha(d) = d \iff \alpha(b/E_\phi) = \alpha(b)/E_\phi \iff b E_\phi \alpha(b) \iff$$

$$\iff \phi(U, b) = \phi(U, \alpha(b)) \iff \alpha[D] = D. \quad \square.$$

9.6 Theorem Let T eliminate imaginaries, let $A \subseteq U$, let D be definable. Tfae:
(i) D is def.ble over $\mathrm{acl}(A)$
(ii) $O(D/A) = \{\alpha[D] : \alpha \in \mathrm{Aut}(U/A)\}$ is finite. (α fixes A pointwise.)
(iii) D is the union of equiv. classes of an equiv. relation that is definable over A and has finitely many classes.

Proof: later?

The eq-expansion

Let S be an index set, then a many-sorted structure with sorts S is a structure whose domain N is partitioned in $\{N_s : s \in S\}$ (a sort).
N is a structure in a language where:
- if R is n-ary rel symbol, $R^N \subseteq N_{s_1} \times \dots \times N_{s_n}$
- if f is n-ary fn symbol, then there are $s_0, \dots, s_n \in S$ s.t. $f : N_{s_1} \times \dots \times N_{s_n} \to N_{s_0}$

Let M be an infinite L-structure, let $T = \mathrm{Th}(M)$, let

$$S = \{E : E \text{ is a 0-def.ble eq. rel. on } M^n \text{ for some } n\}$$

Then M^{eq} is a many sorted structure with sorts S, where the sort M^{eq}_E is M^n/E (set of equiv. classes of E).

So the elements of M^{eq} are all the imaginaries in M.

Since $=$ is an eq. rel, so we can identify M with $M\backslash=$ in M^{eq}; we call $M\backslash=$ the home sort

We describe M^{eq} in L^{eq}, an expansion of L where we add for each $E \in S$ a function $\pi_E : M^n \to M^n/E$

$$\pi_E(a) = a/E$$

In this way, the elements of M^n/E are definable with parameters in M.

Let $T^{eq} = Th(M^{eq})$ in the language L^{eq}.
We have that:

- if $\mathcal{U}$ is the monster model of T, $\mathcal{U}^{eq}$ is the monster model T^{eq}
- the 0-def relations on the home sort of $\mathcal{U}^{eq}$ are the same as those $\mathcal{U}$
- if $\alpha \in Aut(\mathcal{U})$ then α extends uniquely to $\hat{\alpha} \in Aut(\mathcal{U}^{eq})$, so we identify $Aut(\mathcal{U}^{eq})$ with $Aut(\mathcal{U})$.
- T^{eq} eliminates imaginaries.