Università di Udine - PhD training 2020/21

**Model theory**

# Comments and errata

**Definition 2.2** (lecture 1)

Condition (ii) should read as follows (thanks to Joshua for pointing this out):

(ii) for all $n$-ary function symbols $f$, for all $a \in \mathrm{dom}(p)^n$ and $b \in \mathrm{dom}(p)$,

$$f^M(a) = b \text{ if and only if } f^N(p(a)) = p(b)\,.$$

**Reminder after Proposition 5.7** (lecture 5)

Here is a correct statement (thanks for spotting the mistake!) and a full argument.

Suppose the type $p(x)$ is closed under conjunction (that is, $\phi(x) \wedge \psi(x) \in p(x)$ whenever $\phi(x), \psi(x) \in p(x)$). Then the following are equivalent:

(i) $|p(\mathcal{U})| \geq \omega$

(ii) $|\phi(\mathcal{U})| \geq \omega$ for all $\phi(x) \in p(x)$.

(i)$\Rightarrow$ (ii) is clear, since $p(\mathcal{U}) \subseteq \phi(\mathcal{U})$ for all $\phi(x) \in p(x)$.

For the converse implication, suppose $|p(\mathcal{U})| < \omega$, say $p(\mathcal{U}) = \{a_1, \ldots, a_n\}$. Then

$$p(x) \to \bigvee_{i=1}^{n} x = a_i\,.$$

Then the type $p(x) \cup \{x \neq a_i : i = 1, \ldots, n\}$ is not satisfiable in $\mathcal{U}$, hence it is not finitely satisfiable in $\mathcal{U}$ (by saturation of $\mathcal{U}$). Therefore, since $p(x)$ is closed under conjunction, there is $\phi(x) \in p(x)$ such that $\{\phi(x)\} \cup \{x \neq a_i : i = 1, \ldots, n\}$ is not satisfiable in $\mathcal{U}$, that is

$$\models \phi(x) \to \bigvee_{i=1}^{n} x = a_i\,.$$

Hence $|\phi(\mathcal{U})| < \omega$.

*Remark*: in the model theoretic literature, it is quite common to say that a claim, or a step in a proof, holds 'by compactness' when it is, in fact, a consequence of the saturation of the monster model. So, for example, in the proof above it would be ok to write

> $p(x) \to \bigvee_{i=1}^{n} x = a_i$, and since $p(x)$ is closed under conjunction, by compactness there is $\phi(x) \in p(x)$ such that $\{\phi(x)\} \cup \{x \neq a_i : i = 1, \ldots, n\}$ is inconsistent.

**Chapter 6: choosing a language for $T_{\mathrm{VSK}}$**

We could describe vector spaces over a given field $K$ as a two-sorted structure, with one sort for the scalars and another for the vectors. However:

- the theory of vector spaces over $K$ would then contain the theory of the field $K$ as a subtheory. Unless $K$ is finite, or algebraically closed, $\mathrm{Th}(K)$ is highly intractable – for example, it does not have quantifier elimination, so the theory of vector spaces over $K$ would not have q.e., either;
- we would lose strong minimality, even if we only considered the vector sort: the linear span of a vector $v$ would be definable (e.g. by the formula $\exists x\,(y = xv)$, where $x$ is a variable in the field sort).

**Chapter 7: an example of a weakly $\omega$-homogeneous, non $\omega$-homogeneous structure.**

Let the language $L$ contain one binary relation symbol $E$ and nothing else. Let $T$ be the theory that says that $E$ is an equivalence relation with exactly two classes, each of which is infinite. Then $T$ has quantifier elimination (this requires a little work). Let $M \models T$ be such that the equivalence class $E_1$ is countably infinite and the equivalence class $E_2$ is uncountable.

Since both equivalence classes are infinite, any finite partial embedding (i.e., by q.e., any finite elementary map) from $M$ to itself can be extended to include any element of $M$ in the domain.

However, the map $\{\langle m, n\rangle\}$, where $m \in E_1$ and $n \in E_2$, is elementary (by q.e.) but it cannot be extended to an automorphism of $M$.