MODEL THEORY 25/01/2021

$\mathfrak{U}$ is the monster model of a complete theory T

(5.5) **Proposition** Let $A \subseteq \mathfrak{U}$. For all $\phi(x) \in L(\mathfrak{U})$, tfae:

(i) there is $\psi(x) \in L(A)$ such that
$$\models \forall x \, [\psi(x) \leftrightarrow \phi(x)]$$
(ii) $\phi(\mathfrak{U})$ is invariant over A.

(5.6) **Definition** An elem $a \in \mathfrak{U}$ is *definable over* $A \subseteq \mathfrak{U}$ if $\{a\}$ is definable over A (i.e. there is $\phi(x) \in L(A)$ such $\phi(\mathfrak{U}) = \{a\}$).

An element $a \in \mathfrak{U}$ is *algebraic over* $A \subseteq \mathfrak{U}$ if there is $\phi(x) \in L(A)$ such $a \in \phi(\mathfrak{U})$ and $|\phi(\mathfrak{U})| < \omega$.
A formula $\phi(x)$ is *algebraic* if $|\phi(\mathfrak{U})| < \omega$.

The *definable closure of* $A \subseteq \mathfrak{U}$ is the set
$$dcl(A) = \{a \in \mathfrak{U} : a \text{ is definable over } A\}$$
The *algebraic closure* of $A \subseteq \mathfrak{U}$ is
$$acl(A) = \{a \in \mathfrak{U} : a \text{ is algebraic over } A\}$$

Clearly $a \in dcl(A) \Rightarrow a \in acl(A)$

If $A = acl(A)$, A is stb *algebraically closed*

(5.7) **Proposition** For $a \in \mathfrak{U}$ and $A \subseteq \mathfrak{U}$ tfae:

(i) $a \in dcl(A)$
(ii) $O(a/A) = \{a\}$

Pf: (i) $\Rightarrow$ (ii) If $\phi(x) \in L(A)$ s.t. $\phi(\mathfrak{U}) = \{a\}$, then – since $\phi(\mathfrak{U})$ is invariant over A – we have $O(a/A) \subseteq \phi(\mathfrak{U})$.
(ii) $\Rightarrow$ (i) Since $O(a/A)$ is finite, it is definable (by $x = a$) and invariant over A. By Prop. 5.5, $O(a/A)$ is definable by $\phi(x) \in L(A)$. Then $a \in dcl(A)$. $\square$

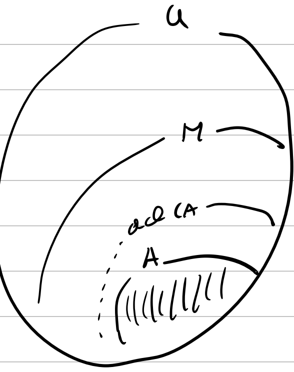
Reminder: in $\mathfrak{U}$, any type $p(x) \subseteq L(A)$ such that $|p(\mathfrak{U})| \geq \omega$ is such that $|p(\mathfrak{U})| = |\mathfrak{U}|$.

Also if $p(x)$ is closed under conjunction (i.e. $\phi(x), \psi(x) \in p(x) \Rightarrow \phi(x) \wedge \psi(x) \in p(x)$) then $|p(\mathfrak{U})| \geq \omega \Leftrightarrow$ for all $\phi(x) \in p(x)$, we have $|\phi(\mathfrak{U})| \geq \omega$.
($\Rightarrow$) is clear, since $p(\mathfrak{U}) \subseteq \phi(\mathfrak{U})$ for all $\phi(x) \in p(x)$
($\Leftarrow$) if $|p(\mathfrak{U})| < \omega$, say $p(\mathfrak{U}) = \{a_1, \dots, a_n\}$, then
$$p(x) \rightarrow \bigvee_{i=1}^{n} x = a_i$$
was $p(x)$ closed under conjunction

Then (by compactness and saturation) there is $\phi(x) \in p(x)$ such that
$$\models \phi(x) \rightarrow \bigvee_{i=1}^{n} x = a_i$$

$p(x) \cup \{x \neq a_i : i = 1, \dots, n\}$

and so $|\varphi(\mathfrak{U})| < \omega$.



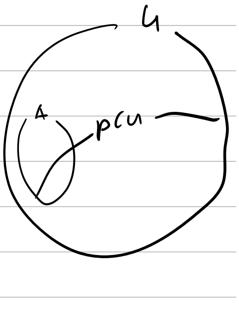
(5.8) Theorem Let $A \subseteq \mathfrak{U}$. Then tfae:
(i) $a \in acl(A)$
(ii) $O(a/A)$ is finite
(iii) $a \in M$ for any model M s.t. $A \subseteq M$

Pf (i) $\Rightarrow$ (ii) Let $\varphi(x) \in L(A)$ be such that $\models \varphi(a)$ and $|\varphi(\mathfrak{U})| < \omega$. Since $\varphi(\mathfrak{U})$ is invariant / A, so $O(a/A) \subseteq \varphi(\mathfrak{U})$.

(ii) $\Rightarrow$ (i) Since $O(a/A)$ is finite, it is definable. Also, it is invariant over A, hence there is $\varphi(x) \in L(A)$ such that $\varphi(\mathfrak{U}) = O(a/A)$. Since $a \in \varphi(\mathfrak{U})$, and $|\varphi(\mathfrak{U})| < \omega$, so $a \in acl(A)$

(i) $\Rightarrow$ (iii) Let $\varphi(x) \in L(A)$ be such that $\models \varphi(a) \wedge \exists^{=n} x\, \varphi(x)$
Then for any M s.t. $A \subseteq M$, $M \models \exists^{=n} x\, \varphi(x)$ (because $M \preceq \mathfrak{U}$). But any realization of $\varphi(x)$ in M is also a realization in $\mathfrak{U}$, and so $\varphi(M) = \varphi(\mathfrak{U})$. In particular, $a \in M$.

(iii) $\Rightarrow$ (i) Suppose $a \notin acl(A)$, and let $p(x) = tp(a/A)$. Since $a \notin acl(A)$, we have $|\varphi(\mathfrak{U})| \geq \omega$ for all $\varphi(x) \in p(x)$. Then $|p(\mathfrak{U})| \geq \omega$, so $|p(\mathfrak{U})| = |\mathfrak{U}|$ (by Fact 4.13).



If M is a model cont. A, by cardinality considerations we must have $p(\mathfrak{U}) \setminus M \neq \emptyset$. Let $b \in p(\mathfrak{U}) \setminus M$.
Then by homogeneity of $\mathfrak{U}$ there is $\alpha \in Aut(\mathfrak{U}/A)$ such that $\alpha(b) = a$. Then $\alpha[M]$ is a model s.t. $A \subseteq \alpha[M]$ but $a \notin \alpha[M]$. $\square$

(5.9) Proposition Let $a \in \mathfrak{U}$, $A \subseteq \mathfrak{U}$. Then:

(i) if $a \in acl(A)$, then $a \in acl(A_0)$ where $A_0 \subseteq A$, $|A_0| < \omega$
(ii) $A \subseteq acl(A)$
(iii) if $A \subseteq B$ then $acl(A) \subseteq acl(B)$
(iv) $acl(acl(A)) = acl(A)$
(v) $acl(A) = \bigcap_{A \subseteq M} M$ (M ranges over models containing A)

Pf (i) A_0 contains exactly the params appearing in the fmla in $L(A)$ that makes a algebraic
(ii) if $a \in A$, then $a \in dcl(A) \subseteq acl(A)$ $\quad A \subseteq acl(A)$
(iii) clear
(iv) $acl(A) \subseteq acl(acl(A))$: follows from (ii) and (iii)
For the converse inclusion, by Theorem 5.8 we have
$A \subseteq M \Leftrightarrow acl(A) \subseteq M$
Also, $a \in acl(acl(A)) \Rightarrow a \in M$ for all $M \supseteq acl(A)$
Therefore $a \in acl(acl(A)) \Rightarrow a \in M$ for all $M \supseteq A$, so $a \in acl(A)$.
(v) follows from Theorem 5.8.

5.10 <u>Fact</u> Let $\beta \in Aut(\mathfrak{U})$, $A \subseteq \mathfrak{U}$. Then

$$\beta[acl(A)] = acl(\beta[A])$$

<u>Proof</u>: ($\subseteq$) For $a \in acl(A)$, let $\varphi(x, z) \in L$, $b \in A^{|z|}$ be such that $a \in \varphi(\mathfrak{U}, b)$ and $|\varphi(\mathfrak{U}, b)| < \omega$.
Then $\models \varphi(\beta(a), \beta(b))$ and $|\varphi(\mathfrak{U}, \beta(b))| < \omega$, so $\beta(a)$ is algebr. over $\beta(b)$, hence $\beta(a) \in acl(\beta[A])$.
($\supseteq$) similar.

6 STRONGLY MINIMAL THEORIES

Recall: a theory T has quantifier elimination (q.e.) if for every $\phi(x) \in L$ there is a quantifier-free fmla $\psi(x) \in L$ such that
$$T \vdash \forall x\, [\phi(x) \leftrightarrow \psi(x)].$$

6.1 Definition Let M is an L-structure; $A \subseteq M$ is cofinite if $M \setminus A$ is finite

Relevant: finite and cofinite sets are definable in every structure.

6.2 Definition An L-structure is said to be minimal if all its definable sets are finite or cofinite
M is strongly minimal if it is minimal and all its elementary extensions are minimal.
If T is a complete theory w/o finite models, T is strongly minimal if for every fmla $\phi(x, z)$ where $|x| = 1$ there is $n \in \omega$ s.t.

$$T \vdash \forall z\, [\exists^{\leq n} x\, \phi(x, z) \vee \exists^{\leq n} x\, \neg \phi(x, z)].$$

Example Let $L = \{E\}$ binary relation. Let M be an infinite L-structure where E is interpr. as an eq. rel. with exactly one class of size n for each $n \in \omega \setminus \{0\}$, and no infinite classes.
It can be shown that M is minimal (needs some work!)
However, there is $M' \succcurlyeq M$ where E has an infinite class.
If $a \in$ infinite class, $E(a, x)$ defines an infinite / coinfinite subset of M.

FROM NOW ON, WORK IN MONSTER MODEL $\mathcal{U}$ OF A S.M. THEORY T.

6.3 Definition Let $a \in \mathcal{U}$, $B \subseteq \mathcal{U}$. Then a is independent from B if $a \notin \mathrm{acl}(B)$.
The set B is independent if every $b \in B$ is indep. from $B \setminus \{b\}$.

MAIN EXAMPLES OF S.M. THEORIES

(1) Let K be an infinite field, $L_K = \{+, -, 0, \{\lambda\}_{\lambda \in K}\}$. Then vector spaces / K are L_K-structures, where for $\lambda \in K$, λ is unary fn symbol interpreted as multiplication by the scalar λ (we write λv for $\lambda(v)$).
The theory T_{VSK} includes the following axioms.
- axioms for abelian gp (using $+, -, 0$)
- axioms for scalar product - for example:

  for each $\lambda \in K$: $\forall x y\, [\lambda (x+y) = \lambda x + \lambda y]$

  for $\lambda_1, \lambda_2, \mu \in K$ such that $\lambda_1 \lambda_2 = \mu$ $\quad \forall x\, [\lambda_1(\lambda_2 x) = \mu x]$

  ....
- $\forall x\, (1 \cdot x = x]$
- $\exists x\, (x \neq 0]$

Can show that T_{VSK} is complete and has q.e. Then:

- terms are linear combinations $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$
- atomic fmlas are equalities between linear combinations
- atomic fmlas with one free variable and parameters are equiv. to fmlas of the form $\lambda x = a$. Therefore atomic fmlas of this kind define singletons

Therefore: q.f. fmlas with one var and parameters define finite or cofinite sets.

But by q.e. of T_{VSK}, T_{VSK} is strongly minimal. Can show
- $acl(A) = \langle A \rangle$ (the linear span of A)
- $a \notin acl(A) \iff a$ is linearly indep from A
- A independent (as in Def. 6.3) $\iff$ A is linearly independent

There is also a theory $T_{VSK}^{\sim}$ of inf. dim. vector spaces over a finite field — similar considerations apply

(2) Let $L_{ring} = \{+, \cdot, -, 0, 1\}$. Then ACF is the L_{ring}-theory that includes:
- axioms for abelian gp $(+, -, 0)$
- axioms for commutative monoid (using $\cdot$, 1)
- field axioms ($\cdot$ distributes over $+$, ...)
- axioms for algebraic closure: for all $n < \omega$
  $\forall x_0 \dots x_n \exists y [x_0 + x_1 y + \cdots + x_n y^n = 0]$

Let $\chi_p \equiv 1 + \cdots + 1 = 0$ (expresses the char of the field).

Then $ACF_p = ACF \cup \{\chi_p\}$ where p is prime
$ACF_0 = ACF \cup \{\neg \chi_n : n < \omega\}$