Please hand in solutions to **Questions 4** and **7** by 3 pm on Wed 11 March, for discussion in class on Thursday 12 March. You are welcome to email a pdf (typeset or scanned) if you prefer.

If you spot any typos or problems, please email silvia.barbina@open.ac.uk.

Assume that T is a complete theory with a monster model $\mathcal{U}$ and work in $\mathcal{U}$ unless otherwise specified.

Question 1. Show that the following are equivalent (and hence equivalent to quantifier elimination for T):

(i) every partial embedding $p : \mathcal{U} \to \mathcal{U}$ is elementary.

(ii) for every finite partial embedding $p : \mathcal{U} \to \mathcal{U}$ and $b \in \mathcal{U}$, there is a partial embedding $\hat{p}$ such that $p \subseteq \hat{p}$ and $b \in \mathrm{dom}(\hat{p})$.

Hint: Start by assuming that the language is countable. For (ii) ⇒ (i) use a back-and-forth argument and adapt the construction in the proof of the Downward Löwenheim-Skolem Theorem to build models M and N and an isomorphism $h : M \to M$ that extends p.

Then show that the general case (L has any cardinality) follows from the countable case – for example, consider for a contradiction a formula whose truth is not preserved by p.

Question 2. Let $p(x)$ be a type with parameters in A, and let $\varphi(x, y)$ be an $L(A)$-formula that defines, when restricted to $p(\mathcal{U})$, a *finite* equivalence relation - that is, an equivalence relation with finitely many classes. Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x, y)$ on $p(\mathcal{U})$.

Question 3. Let T be a consistent theory. Suppose that all complete theories containing T are of the form $T \cup S$ for some set S of quantifier-free sentences (for example, $T = \mathrm{ACF}$). Prove that if all complete theories containing T have elimination of quantifiers, so does T.

Question 4. Let $\varphi(z)$ be a consistent $L(A)$-formula. Prove that if $a \in \mathrm{acl}(A, b)$ for every $b \models \varphi(z)$, then $a \in \mathrm{acl}(A)$. Prove that the same holds when $a \in \mathrm{acl}(A, b)$ for every $b \models p(z)$, where $p(z)$ is a satisfiable type with parameters in A.

Question 5. Let $\bar{c}$ be en enumeration of $\mathcal{U}$. Let $\mathcal{V}$ be the set enumerated by some $\bar{a} \models \mathrm{tp}(\bar{c}/\varnothing)$. Prove that $\mathcal{V} \preceq \mathcal{U}$.

Question 6. Let $a \in \mathcal{U} \smallsetminus \mathrm{acl} A$. Prove that $\mathcal{U}$ is isomorphic over A to some $\mathcal{V} \preceq \mathcal{U}$ such that $a \notin \mathcal{V}$.

Question 7. Let C be a finite set. Prove that if $C \cap M \neq \varnothing$ for every model M containing A, then $C \cap \mathrm{acl}(A) \neq \varnothing$.

Question 8. Let $\varphi(x) \in L(\mathcal{U})$ and fix an arbitrary set A. Prove that the following are equivalent

(i) there is some model M containing A and such that $M \cap \varphi(\mathfrak{U}) = \varnothing$;

(ii) there is no satisfiable $L(A)$-formula $\psi(z_1, \ldots, z_n)$ such that

$$\psi(z_1, \ldots, z_n) \to \bigvee_{i=1}^{n} \varphi(z_i).$$

Question 9. Prove that in a strongly minimal theory every infinite algebraically closed set is a model.

Question 10. Prove that a countable strongly minimal theory is either ω-categorical or has infinitely many countable models.