9 Imaginaries

T complete theory, $\mathfrak{U}$ monster model

9.1 def

$D \subseteq \mathfrak{U}^n$ definable. $c \in \mathfrak{U}^{|c|}$ fin is a canonical parameter for D if for all $\alpha \in \mathrm{Aut}(\mathfrak{U})$

$$\alpha(c) = c \text{ iff } \alpha[D] = D$$

9.2 Rmks

(i) d can par for D, then D is $\{d\}$-definable

(ii) D def, d can par for D, then:

$$c \text{ is can par for } D \text{ iff } c \in \mathrm{dcl}(d) \text{ and } d \in \mathrm{dcl}(c)$$

9.3 def

M model. An imaginary in M is an eq. class a/E where $a \in M^n$, E 0-def eq rel on M^n

9.4 def

T eliminates imaginaries (or has E.o.I) if for all n, all 0-def eq rel on $\mathfrak{U}^n$, every imag. a/E has a can par

9.5 Thm

If T has E.o.I, then any def set has a can. par.

9.6 Thm

T has E.o.I, $A \subseteq \mathcal{U}$, D def. TFAE:

(i) D is definable over $acl(A)$
(ii) $O(D/A) = \{\alpha[D] : \alpha \in Aut(\mathcal{U}/A)\}$ is finite
(iii) D is the union of eq classes of an eq rel that is definable over A and has finitely many classes.

proof

(i) $\Rightarrow$ (ii) $\quad D = \varphi(\mathcal{U}, b) \quad \varphi(x,y) \in L$, $b \in acl(A)^{|b|}$

take $\alpha \in Aut(\mathcal{U}/A)$, $\quad \alpha[D] = \varphi(\mathcal{U}, \alpha(b))$

and by Thm 5.8 $O(b/A)$ is finite

(ii) $\Rightarrow$ (i) D has a can par (Thm 9.5) d, and D is $\{d\}$-defin.
(Rmk 9.2(i))

CLAIM: $d \in acl(A)$, i.e. $O(d/A)$ is finite

$$O(D/A) = \{\alpha_i[D]\}_{i=1,\dots,n} \qquad \alpha_i \in Aut(A/\mathcal{U})$$

Take
$\beta \in Aut(\mathcal{U}/A)$: $\quad \beta(d) = \alpha_i(d)$ for some i iff $d = \beta^{-1}\alpha_i(d)$

iff $D = \beta^{-1}\alpha_i[D]$ iff $\beta[D] = \alpha_i[D]$

but this is true for some i

(iii) $\Rightarrow$ (ii) $\quad E \subseteq \mathcal{U}^n \times \mathcal{U}^n$ eq rel, $E = \psi(\mathcal{U}, a)$, $\psi(x,y) \in L$
$a \in A^{|a|}$

$|\mathcal{U}^n/E| < \omega$. Consider $c/E \in \mathcal{U}^n/E$

$$\mathcal{O}(c/E/A) = \{\alpha[c/E] : \alpha \in Aut(\mathcal{U}/A)\}$$ is finite

$\alpha[c/E] = \alpha(c)/E$ ← E is A-def: $\alpha(d) E \alpha(c)$ iff $\models \psi(\alpha(d), \alpha(c), \underset{\alpha(a)}{a})$ iff $\models \psi(d,c,a)$ iff dEc

$$D = \bigcup_{i=1}^{k} c_i/E \qquad \alpha[D] = \bigcup_{1}^{k} \alpha[c_i/E] = \bigcup_{1}^{k} \alpha(c_i)/E$$

(ii) ⟹ (iii) $\mathcal{O}(D/A) = \{D = D_1, \ldots D_n\}$ $\qquad D_i = \alpha_i[D]$

$\alpha_i \in Aut(\mathcal{U}/A)$

- define $E \subseteq \mathcal{U}^n \times \mathcal{U}^n$, $(c,c') \in E$ iff [for all i, $c \in D_i$ iff $c' \in D_i$]
  E is an eq rel, and it is finite (if $b \in D_{i_1} \cap \ldots \cap D_{i_k}$, $b \notin D_{j_1} \cup \ldots \cup D_{j_\ell}$ then bEc for all $c \in \bigcap D_i \setminus \bigcup D_j$)

- E is def over A iff E is def and A-invariant (prop 5.5)

  - $D = \varphi(\mathcal{U}, d)$, $d \in acl(A)^{|d|}$, $D_i = \varphi(\mathcal{U}, \alpha_i(d))$
    $$\theta(x,y) := \bigwedge_{i=1}^{n} \left(\varphi(x, \alpha_i(d)) \longleftrightarrow \varphi(y, \alpha_i(d)) \right)$$
    $E = \theta(\mathcal{U})$, so it is definable.
  - E is A-inv: $\alpha \in Aut(\mathcal{U}/A)$, $(a,b) \in E$, then $(\alpha(a), \alpha(b)) \in E$.
    Indeed, α induces a permutation of $\{\alpha_i(d)\}_{i=1}^{n}$

- $c \in D$, $c'Ec$, then $c' \in D$ $\quad (i=1)$, hence D is the union of eq classes. □

Now, take M an infinite L-struct., $T = Th(M)$; define

$$S = \{E : E \text{ is a } 0\text{-def eq rel on } M^n \text{ for some } n\}.$$

Define M^{eq} a many sorted structure, with sorts S, where M^{eq}_E is M^n/E.

The elements of M^{eq} are all imaginaries in M.

Since $=$ is a 0-def eq rel we can identify M with $M/_=$ in M^{eq}; we call $M/_=$ the home-sort; elements of M are called real elements.

We take an expansion L^{eq} of the language L where we add for each $E \in S$ a function

$$\pi_E : M^n \longrightarrow M^n/_E$$
$$a \longmapsto a/_E$$

$T^{eq} := Th(\mathcal{U}^{eq})$

9.7 Rmk

Elements of M^{eq} are definable with parameters in M.

Indeed, an elem. of M^{eq} is an elem. of M^{eq}_E for some $E \in S$, i.e. $a/_E$, $a \in M^n$. Take

$$\varphi(\underset{E\text{-sort}}{x}, \underset{\text{home-sort}}{y}) := [x = \pi_E(y)]$$

Clearly, $\{a/_E\} = \varphi(M^{eq}, a)$.

9.8 Lemma

For every formula $\varphi(\underset{\text{home sort}}{y}, \underset{E_i\text{-sort, } E_i \text{ equal on } m_i\text{-tuples}}{x_1^{E_1}, \ldots, x_n^{E_n}}) \in L^{eq}$, there is a formula $\psi(y, \underset{\text{new } m_i\text{-tuple}}{y_1}, \ldots, y_n) \in L$ s.t. for all tuples $a, a_1, \ldots, a_n$ in $\mathcal{U}$ of the corresponding lenghts,

$$\mathcal{U}^{eq} \models \varphi(a, \pi_{E_1}(a_1), \ldots \pi_{E_n}(a_n)) \text{ iff } \mathcal{U} \models \psi(a, a_1, \ldots a_n)$$

proof

We replace $\exists x_i^{E_i} \rightsquigarrow \exists y_i$

$$x_i^{E_i} \longrightarrow \pi_{E_i}(y_i)$$

$$\pi_{E_i}(z) = \pi_{E_i}(z') \rightsquigarrow E_i(z, z')$$

(No new relation symbols were added in L^{eq} so, the only relation symbol that can appear with terms with sort E_i is $=$) □

9.9 Proposition

(i) The 0-defin. set on the home-sort of $\mathcal{U}^{eq}$ are exactly the same as those in $\mathcal{U}$.

(ii) If $\alpha \in Aut(\mathcal{U})$, then α extends uniquely to $\hat{\alpha} \in Aut(\mathcal{U}^{eq})$, so we identify $Aut(\mathcal{U})$ with $Aut(\mathcal{U}^{eq})$.

(iii) T^{eq} eliminates imaginaries.

proof

(i) Take $\varphi(\mathcal{U}^{eq})$, where $\varphi(\overset{\text{h-s.}}{x}) \in L^{eq}$; then there exists $\psi(x) \in L$ (Lemma 9.8) s.t.

$$\psi(\mathcal{U}) = \varphi(\mathcal{U}^{eq})$$

(ii) Take $\alpha \in Aut(\mathcal{U})$. Define

$$\hat{\alpha}: \mathcal{U}^{eq} \longrightarrow \mathcal{U}^{eq}$$
$$a/_E \longmapsto \alpha(a)/_E$$
$$E \in S$$

$$\hat{\alpha}(\pi_E(a)) = \pi_E(\alpha(a))$$

- $\hat{\alpha}$ is well defined
- $\hat{\alpha}$ is an homom.
- $\hat{\alpha}$ is an isom (elem. emb. by 9.8 + surj)
- $\hat{\alpha}$ is unique

Fact $\mathcal{U}^{eq}$ is the monster model of T^{eq}

We prove that $\mathcal{U}^{eq}$ is saturated: take $p(y) \in L^{eq}(A)$, WLOG $A \subseteq \mathcal{U}$, $|y|=1$, y variable of sort E, $p(y)$ fin sat in $\mathcal{U}^{eq}$

Define $\Sigma(x) = p(\pi_E(x)) \ni \varphi(\pi_E(x), a)$, $\varphi(y,a) \in p(y)$
home-sorted

$\Sigma^*(x) = \{\psi\}_{\varphi \in \Sigma}$ where each ψ is defined from φ by Lemma 9.8
$\cap$
L

$p(y)$ is fin sat, so $\Sigma(x)$ is fin sat in $\mathcal{U}^{eq}$, and so $\Sigma^*(x)$ is fin sat in $\mathcal{U}$
But $\mathcal{U}$ is saturated, so take a realization $c \models \Sigma^*(x)$, hence $\pi_E(c) \models p(y)$.

(iii)
Take E a 0-def eq rel on $\mathcal{U}^{eq^m}$, $E = \varphi(\mathcal{U}^{eq})$
$\varphi(\bar{x},\bar{y}) \in L^{eq}$, $\bar{x} = (x, x_1^{E_1}, \dots x_n^{E_n})$, sim. $\bar{y}$, $|(\bar{x},\bar{y})| = 2m$

$\bar{a} = (a, \pi_{E_1}(a_1), \dots \pi_{E_n}(a_n))$, $\bar{b}$, $(\bar{a},\bar{b}) \in E$ iff $\mathcal{U}^{eq} \models \varphi(\bar{a},\bar{b})$

Use Lemma 9.8 to define $\psi(x, z_1, \dots z_n, y, w_1, \dots w_n) \in L$
from $\varphi(x, x_1^{E_1}, \dots y, y_1^{E_1}, \dots) \in L^{eq}$

Given $\bar{a}$ as above, write $\underline{a} = (a, a_1, \dots a_n)$
$\mathcal{U}^{eq} \models \varphi(\bar{a},\bar{b})$ iff $\mathcal{U} \models \psi(\underline{a},\underline{b})$
$\psi(\mathcal{U})$ is a 0-def eq rel on $\mathcal{U}^m$, say $E_\psi := \psi(\mathcal{U}) \in S$

<u>CLAIM</u>: $\pi_{E_\psi}(\underline{a})$ is a canpar for $\bar{a}/E$

Take $\hat{\alpha} \in \mathrm{Aut}(\mathfrak{U}^{eq}) \longleftrightarrow \alpha \in \mathrm{Aut}(\mathfrak{U})$

$$\underbrace{\hat{\alpha}(\pi_{E_\varphi}(\underline{a}))}_{= \pi_{E_\varphi}(\alpha(\underline{a}))} = \pi_{E_\varphi}(\underline{a}) \text{ iff } \alpha(\underline{a})\ E_\varphi\ \underline{a} \text{ iff}$$

$$\hat{\alpha}(\bar{a})\ E\ \bar{a} \text{ iff}$$

$$\hat{\alpha}(\bar{a})/E = \bar{a}/E$$

E is ∅-def. $\longrightarrow$ $\hat{\alpha}(\bar{a})/E = \hat{\alpha}[\bar{a}/E]$

□

9.10 Corollary

The theory T has E.o.I iff in T^{eq} every imag is interdef. with a real tuple

__Proof__

T has E.o.I. iff each e/E has can par d in T iff d in T^{eq} iff (*)

But in T^{eq}, $\pi_E(e)$ is a can par for e/E

$\left(\hat{\alpha}(\pi_E(e)) = \pi_E(e) \text{ iff } \alpha(e)\ E\ e \text{ iff } \hat{\alpha}[e/E] = e/E \right)$

So by Rmk 9.2.(ii), (*) iff $\pi_E(e)$ and d are interdef.

□