

Model Theory

Part III 2019/20

Silvia Barbina

Preliminaries and review	1
Maps that preserve formulas	3
Theories and elementarity	5
Two relational structures	10
Compactness	14
Saturation	20
The monster model	25
Strongly minimal theories	31
Countable models	36

MODEL THEORY

① PRELIMINARIES AND REVIEW

1.1 Definition A language L consists of

(i) a set \mathcal{F} of function symbols, and for each $f \in \mathcal{F}$ a positive integer n_f

(ii) a set \mathcal{R} of relation symbols, and for each $R \in \mathcal{R}$ a positive integer n_R

(iii) a set \mathcal{C} of constant symbols.

The numbers n_f , n_R are the arities of the fn symbol f and relation symbol R , respectively.

Any of the sets \mathcal{F} , \mathcal{R} and \mathcal{C} may be empty.

Examples : (a) $L_{\text{gp}} = \{\cdot, -^1\}, \{1\} \xrightarrow{\substack{\text{constant} \\ \text{unary function}}}$

(b) $L_{\text{eo}} = \{<\} \xrightarrow{\text{binary relation}}$

1.2 Definition Let L be a language. Then an L -structure consists of:

(i) a set M , The domain ($M \neq \emptyset$)

(ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \rightarrow M$

(iii) for each $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$

(iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M , R^M and c^M are called the interpretations of f , R , c respectively.

1.3 Motivation and conventions We omit the distinction between:

- a structure and its domain

- a symbol f and its interpretation in a structure M where M is understood.

We write $\langle M, \{f_i\}_{i \in \mathbb{Z}}, \{R_j\}_{j \in \mathbb{Z}}, \{c_k\}_{k \in \mathbb{Z}} \rangle$ for a structure in the language $\{\{f_i\}, \{R_j\}, \{c_k\}\}$.

1.4 Examples (a) $\langle \mathbb{R}^+, \{\cdot, -^1\}, 1 \rangle$ is an L_{gp} -structure
 $\xrightarrow{\text{+ve real numbers}}$

(b) $\langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is an L_{gp} -structure

(c) $\langle \mathbb{Q}, \{<\} \rangle$ is an L_{eo} -structure

Note that the structures in (a) and (b) are groups, but an L_{gp} -structure needn't be a group.

Given a language L , we assume familiarity with the first-order logic of L . We use the symbols of L and the following logical symbols:

- connectives \wedge, \neg (also \vee, \rightarrow and \leftrightarrow)
- quantifiers \forall, \exists
- variables $x_0, x_1, x_2, x_3 \dots$ (infinitely many)
- punctuation $(,)$ (ensures unique readability)
- equality $=$

to define

- L -terms
- L -formulas (in particular, atomic formulas and sentences)

Motivation: the letters x, y, z denote variables, the letters a, b, c, \dots denote constants (or elements of the domain of a structure).

The notation $t(x_0 \dots x_n)$ (term), $\phi(x_0 \dots x_n)$ (formula) means that the free variables of t , ϕ are in $\{x_0, x_1, \dots, x_n\}$

If $\phi(x_1, \dots, x_n)$ is an L -formula, M is an L -structure and $a_1, \dots, a_n \in M$, then

$$M \models \phi(a_1, \dots, a_n)$$

means that ϕ holds in M under the assignment $x_i \mapsto a_i$.
This notion is also defined recursively.

(You can revise all these notions on the following textbooks)

- J. Kirby, An Invitation to Model Theory (CUP, 2019):
Chapter 1 Section 1.1, Chapter 2 Section 2.1,
Chapter 3 Sections 3.1 - 3.3
- D. Marker, Model Theory: an Introduction (Springer 2002),
Chapter 1, Section 1.1
- Last year's Part III notes (available from the course webpage),
Chapter 2, 2.1, 2.2, 2.5, 2.6, 2.7

We often write \bar{x} for a tuple of variables x_1, \dots, x_n , and \bar{a} for a tuple of constants/elements a_1, \dots, a_n , so for example

$$t(\bar{x}), \phi(\bar{x}), \phi(\bar{a})$$

(2) MAPS THAT PRESERVE FORMULAS

2.1 Definition Let $L = \{ \mathcal{F}, \mathcal{R}, \mathcal{C} \}$ be a language, and let M and N be L -structures. An embedding of M into N is an injective mapping

$$\alpha: M \rightarrow N$$

such that

(i) for all $f \in \mathcal{F}$ and $a_1, \dots, a_m \in M$

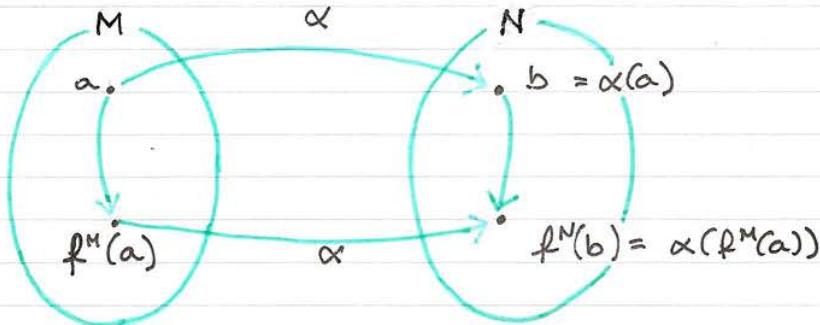
$$\alpha(f^M(a_1, \dots, a_m)) = f^N(\alpha(a_1), \dots, \alpha(a_m))$$

(ii) for all $R \in \mathcal{R}$ and $a_1, \dots, a_m \in M$

$$(a_1, \dots, a_m) \in R^M \text{ if and only if } (\alpha(a_1), \dots, \alpha(a_m)) \in R^N$$

(iii) for all $c \in \mathcal{C}$, $\alpha(c^M) = c^N$

For a unary function f , condition (i) is illustrated by a commutative diagram.



A surjective (onto) embedding is said to be an isomorphism.

2.2 Examples If G_1, G_2 are groups treated as L_{gp} -structures, then a mapping $\alpha: G_1 \rightarrow G_2$ is an embedding if and only if it is an injective group homomorphism.

If A and B are linear orders treated as L_{eo} -structures, then $\alpha: A \rightarrow B$ is an embedding if and only if it is an order-preserving injective map.

2.3 Fact Let M and N be L -structures, and let $\alpha: M \rightarrow N$ be an embedding. Let $\bar{a} \in M^k$, and let $t(x)$ be an L -term, where $|x| = |\bar{a}| = k$. Then

$$\alpha(t^M(\bar{a})) = t^N(\alpha(\bar{a})), \text{ where } \alpha(\bar{a}) = (\alpha(a_1), \dots, \alpha(a_k)).$$

Proof: by induction on the complexity of t . Let $\bar{a} \in M^k$. Then

(i) if $t = x_i$, then $t^M(\bar{a}) = a_i$ and $t^N(\alpha(\bar{a})) = \alpha(a_i)$.

(ii) if $t = c$, then $t^M(\bar{a}) = c^M$, $t^N(\alpha(\bar{a})) = c^N$, and

$\alpha(c^M) = c^N$ because α is an embedding.

(iii) if $t = f(t_1(\bar{x}), \dots, t_{m_f}(\bar{x}))$, then

$\alpha(f^M(t_1^M(\bar{\alpha}), \dots, t_{m_f}^M(\bar{\alpha}))) = f^N(\alpha(t_1^M(\bar{\alpha})), \dots, \alpha(t_{m_f}^M(\bar{\alpha})))$
because α is an embedding. By the inductive hypothesis applied to $t_1(\bar{x}), \dots, t_{m_f}(\bar{x})$ we are done \square

2.4 Proposition Suppose that M and N are L -structures, and let $\alpha: M \rightarrow N$ be an embedding. Then for any atomic L -formula $\varphi(\bar{x})$ and $\bar{\alpha} \in M^{|\bar{x}|}$

$$M \models \varphi(\bar{\alpha}) \Leftrightarrow N \models \varphi(\alpha(\bar{\alpha})).$$

Proof: Case 1 $\varphi(\bar{x})$ is $t_1(\bar{x}) = t_2(\bar{x})$, where t_1 and t_2 are L -terms. Then the statement follows by

Case 2: $\varphi(\bar{x})$ is $R(t_1(\bar{x}), \dots, t_{m_R}(\bar{x}))$ for a relation R and L -terms $t_1(\bar{x}), \dots, t_{m_R}(\bar{x})$.

Let $\bar{\alpha} \in M^{|\bar{x}|}$. Then

$$\begin{aligned} M \models R(t_1(\bar{\alpha}), \dots, t_{m_R}(\bar{\alpha})) \\ \Leftrightarrow (t_1^M(\bar{\alpha}), \dots, t_{m_R}^M(\bar{\alpha})) \in R^M \\ \Leftrightarrow (\alpha(t_1^M(\bar{\alpha})), \dots, \alpha(t_{m_R}^M(\bar{\alpha}))) \in R^N \text{ because } \alpha \text{ is an embedding} \\ \Leftrightarrow (t_1^N(\alpha(\bar{\alpha})), \dots, t_{m_R}^N(\alpha(\bar{\alpha}))) \in R^N \text{ by Fact 2.3} \\ \Leftrightarrow N \models R(t_1(\alpha(\bar{\alpha})), \dots, t_{m_R}(\alpha(\bar{\alpha}))) \end{aligned} \quad \square$$

2.5 Exercise Show that Proposition 2.4 holds if we replace "atomic L -formula" with "quantifier-free L -formula"

2.6 Remark An embedding does not necessarily preserve L -formulas. Consider two L_{\leq} -structures (\mathbb{Z}, \leq) and (\mathbb{Q}, \leq) .

Then the inclusion maps $\alpha: \mathbb{Z} \rightarrow \mathbb{Q}$

$$\alpha: n \mapsto n$$

is an embedding.

Let $\varphi(x, y) \equiv \exists z (x < z \wedge z < y)$.

Then $\mathbb{Z} \models \neg \varphi(1, 2)$ but $\mathbb{Q} \models \varphi(1, 2)$.

On the other hand, isomorphisms preserve all formulas.

2.7 Fact Let M, N be L -structures, and let $\alpha: M \rightarrow N$ be an isomorphism. Then, for any L -formula $\varphi(\bar{x})$ and $\bar{\alpha} \in M^{|\bar{x}|}$

$$M \models \varphi(\bar{\alpha}) \Leftrightarrow N \models \varphi(\alpha(\bar{\alpha})).$$

Proof: exercise (use induction on $\varphi(\bar{x})$).

2.8 Remark $\alpha: M \rightarrow N$ is an embedding $\Leftrightarrow \alpha$ preserves atomic formulas

(3) THEORIES AND ELEMENTARITY.

Let L be a language.

3.1 Definition An L -theory T is a set of L -sentences.

An L -structure M is a model of T , written

$$M \models T$$

if $M \models \phi$ for all $\phi \in T$.

The class of all models of T is denoted by $\text{Mod}(T)$.

If M is an L -structure, the Theory of M is

$$\text{Th}(M) = \{\phi : \phi \text{ is an } L\text{-sentence} \text{ and } M \models \phi\}$$

In general, describing the theory of a given structure is a non-trivial problem!

3.2 Example Let T_{gp} be the theory containing exactly the following L_{gp} sentences

$$(i) \forall x_1 x_2 x_3 ((x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3))$$

$$(ii) \forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$$

$$(iii) \forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$$

If G is a group, then clearly $G \models T_{\text{gp}}$. However, $\text{Th}(G)$ is much larger than T_{gp} .

3.3 Definition Let M and N be two L -structures. Then

M and N are said to be elementarily equivalent, written

$$M \equiv N$$

if $\text{Th}(M) = \text{Th}(N)$, that is, if M and N model exactly the same L -sentences.

Clearly, if $M \cong N$ then $M \equiv N$. However, in general establishing whether two structures are elementarily equivalent is highly non-trivial. We'll see that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$

3.4 Definition If M and N are two L -structures, then

(i) an embedding $\beta : M \rightarrow N$ is elementary if for all L -formulas $\phi(x)$ and $\bar{a} \in M^{|\bar{x}|}$

$$M \models \phi(\bar{a}) \iff N \models \phi(\beta(\bar{a}))$$

(ii) if $M \subseteq N$ and

$$\text{id} : M \rightarrow N \quad (\text{the inclusion map})$$

is an embedding, then M is said to be a substructure of N , written $M \subseteq N$

(iii) if $M \subseteq N$ and $\text{id} : M \rightarrow N$ is an elementary embedding, then M is said to be an elementary substructure of N , written $M \preccurlyeq N$.

Intuition: a substructure of N is a subset of the domain of N that:

- is closed under the interpretations of all the function symbols of L
- inherits the relations that hold in N
- contains the interpretations of all the constants of L in N . (Exercise)

There isn't an intuition of what an elementary substructure is - nor of what an elementary embedding other than an isomorphism is.

3.5 Example Let $L_{\leq} = \{\leq\}$, and consider the closed intervals $[0, 1]$, $[0, 2] \subseteq \mathbb{R}$ as L_{\leq} -structures (with the usual ordering). Then $[0, 1] \cong [0, 2]$, and so $[0, 1] \equiv [0, 2]$.

It is clear that $[0, 1]$ is a substructure of $[0, 2]$.

Let $\phi(x) \equiv \forall y (y < x \vee y = x)$. Then

$$[0, 1] \models \phi(1) \text{ but } [0, 2] \not\models \phi(1).$$

Therefore $[0, 1]$ is NOT an elementary substructure of $[0, 2]$.

So we have two L_{\leq} -structures M and N such that

- $M \subseteq N$
- $M \cong N$ and so $M \equiv N$
- $M \not\preccurlyeq N$.

Remember: the existence of an isomorphism between M and N when $M \subseteq N$ does not guarantee that $M \preccurlyeq N$, although it implies that $M \equiv N$.

3.6 Definition Let M be an L -structure and let A be a subset of the domain of M . Then $L(A)$ is the language obtained by adding to L a constant symbol c_a for each $a \in A$:

$$L(A) = L \cup \{c_a : a \in A\}$$

It is understood that an interpretation of M as an L -structure is extended to an interpretation of M as an $L(A)$ structure by interpreting c_a as the element $a \in A$. In this context, the elements of A are called parameters.

Motivation: if M, N are L -structures and $A \subseteq M \cap N$, we write

$$M \equiv_A L$$

if for every $L(A)$ -sentence φ

$$M \models \varphi \iff N \models \varphi$$

3.7 Remark $M \leq N \Leftrightarrow M \equiv_N N$ (exercise)

3.8 Lemma (Tarski-Vaught test)

Let N be an L -structure and let $A \subseteq N$. Then the following are equivalent:

- (i) A is the domain of an elementary substructure of N
- (ii) for every $L(A)$ -formula $\varphi(x)$ with one free variable
 $N \models \exists x \varphi(x) \Rightarrow N \models \varphi(b)$ for some $b \in A$. (*)

Proof (i) \Rightarrow (ii) Suppose $N \models \exists x \varphi(x)$. Then, by elementarity
 $A \models \exists x \varphi(x)$,
so $A \models \varphi(b)$ for some $b \in A$. Again by elementarity, $N \models \varphi(b)$.

(ii) \Rightarrow (i) First we check that A is the domain of a substructure.

It suffices to show:

- (a) for all constants c , $c^N \in A$
- (b) for all function symbols f and $\bar{a} \in A^{|\bar{x}|}$, $f^N(\bar{a}) \in A$

For (a) : $N \models \exists x c = x$. Then, by property (*),
 $N \models c = b$ for some $b \in A$.

Hence $c^N = b \in A$.

For (b) : similar, with $N \models \exists x f(\bar{a}) = x$.

Now we show that for all L -formulas $\chi(\bar{x})$ and $\bar{a} \in A^{|\bar{x}|}$
 $A \models \chi(\bar{a}) \Leftrightarrow N \models \chi(\bar{a})$.

We argue by induction on the complexity of $\chi(\bar{x})$.

- if $\chi(\bar{x})$ is atomic, the claim follows from $A \subseteq N$ (just proved)
- if $\chi(\bar{x}) \equiv \neg \psi(\bar{x})$, then

$$A \models \neg \psi(\bar{a})$$

$$\Leftrightarrow A \not\models \psi(\bar{a})$$

$$\Leftrightarrow N \not\models \psi(\bar{a}) \text{ by inductive hyp}$$

$$\Leftrightarrow N \models \neg \psi(\bar{a})$$

- if $\chi(\bar{x}) \equiv \psi(\bar{x}) \wedge \xi(\bar{x})$, again easy induction

- if $\chi(\bar{x}) \equiv \exists y \psi(\bar{x}, y)$, then for $\bar{a} \in A^{|\bar{x}|}$ $\exists y \psi(\bar{a}, y)$ is an $L(A)$ -formula.

Suppose $A \models \exists y \psi(\bar{a}, y)$. Then

$A \models \psi(\bar{a}, b)$ for $b \in A$. By inductive hypothesis

$N \models \psi(\bar{a}, b)$, so

$N \models \exists y \psi(\bar{a}, y)$.

Now suppose $N \models \exists y \psi(\bar{a}, y)$. Then by property (*)
 $N \models \psi(\bar{a}, b)$ for some $b \in A$.

By ind. hyp., $A \models \psi(\bar{a}, b)$ and so $A \models \exists y \psi(\bar{a}, y)$, as required.

3.9 Definition and remarks We define the cardinality of the language L to be the cardinality of the set of L -formulas,
 $|L| = |\{\varphi(x) : \varphi(x) \text{ is an } L\text{-formula}\}|$

Note that we have

$$|L| \geq \omega \quad (\text{we often use } \omega \text{ both as a cardinal and as an ordinal})$$

If A is a set of parameters, then

$$|L(A)| = |L| + |A|$$

3.10 Definition Let λ be an ordinal. Then a chain of sets of length λ is a sequence

$$\langle A_i : i < \lambda \rangle$$

such that, for all $i < j < \lambda$, $A_i \subseteq A_j$

A chain of L -structures of length λ is a sequence

$$\langle M_i : i < \lambda \rangle$$

such that, for all $i < j < \lambda$, $M_i \subseteq M_j$ (substructure)

The union of the chain $\langle M_i : i < \lambda \rangle$ is the L -structure M defined as follows:

- $M = \bigcup_{i<\lambda} M_i$ (domain)

- if c is a constant, then $c^M = c^{M_i}$ for any $i < \lambda$

- if f is a function symbol and $\bar{a} \in M^{\text{out}}$, then
 $f^M(\bar{a}) = f^{M_i}(\bar{a})$

for any $i < \lambda$ such that $\bar{a} \in M_i$

- if R is a relation symbol, then $R^M = \bigcup_{i<\lambda} R^{M_i}$.

3.11 Theorem (Downward Löwenheim Skolem)

Let N be an L -structure such that $|N| \geq |L| + \omega$, and let $A \subseteq N$. Then for every cardinal λ such that

$$|L| + |A| \leq \lambda \leq |N|$$

there is an elementary substructure $M \trianglelefteq N$ such that

(i) $A \subseteq M$ (ii) $|M| = \lambda$.

Proof We build inductively a chain $\langle A_i : i < \omega \rangle$ of subsets of N containing A and such that $|A_i| = \lambda$. Eventually we take $M = \bigcup A_i$.

Let A_0 be such that $A \subseteq A_0$ and $|A_0| = \lambda$.

At stage $i+1$, A_i has been built with $|A_i| = \lambda$. Then $|L(A_i)| = \lambda$

Let $\langle \varphi_k(x) : k < \lambda \rangle$ be an enumeration of $L(A_i)$ -formulas with one free variable such that $N \models \exists x \varphi_k(x)$.

For $k < \lambda$, let a_k be such that $N \models \varphi_k(a_k)$ and let

$$A_{i+1} := A_i \cup \{a_k : k < \lambda\}$$

This set has cardinality at most λ .

Then $|A_{i+1}| = \lambda$.

Now let $M = \bigcup_{i < \lambda} A_i$. We use the Tarski-Vaught test to prove that $M \leq N$.
(Lemma 3.8) \Leftrightarrow

Let $\varphi(x, y)$ be an L-formula.

Claim: if $N \models \exists y \varphi(\bar{a}, y)$ for $\bar{a} \in M^{(1)}_{\leq i}$, then $N \models \varphi(\bar{a}, b)$ for some $b \in M$.

If $N \models \exists y \varphi(\bar{a}, y)$, then the finite tuple $\bar{a} \in M^{(1)}_{\leq i}$ is contained in some A_i for $i < \lambda$, and so $\varphi(\bar{a}, y)$ is among the formulas that have a witness in A_{i+1} . Hence

$N \models \varphi(\bar{a}, b)$ where $b \in A_i \subseteq M$.

□

Remark: The following version of the Downward Löwenheim-Skolem Theorem, which you might have encountered in undergraduate logic courses, follows from the statement of Theorem 3.11:

If L is a countable language and T is a theory with an infinite model, then T has a countable model.

④ TWO RELATIONAL STRUCTURES

4.1 Definition (Dense linear orders)

An L_≤-structure is a linear order if it satisfies

1. $\forall x \sim(x < x)$ irreflexivity
 2. $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$ transitivity
 3. $\forall x \forall y (x = y \vee x < y \vee y < x)$ linearity
- A linear order is dense if it satisfies
4. $\exists x \forall y (x < y)$ non triviality
 5. $\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$ density
- A linear order has no endpoints if
6. $\forall x (\exists y (x < y) \wedge \exists z (z < x))$

T_{\leq} is the theory that includes axioms 1, 2 and 3

T_{dlo} is the theory that includes axioms 1 to 6

$(\mathbb{Q}, <) \models T_{\leq}$ and $(\mathbb{Q}, <) \models T_{\text{dlo}}$

Clearly any model of T_{dlo} is infinite

4.2 Definition If M, N are L-structures, then a partial embedding from M to N is an injective map
 $p: \text{dom}(p) \subset M \rightarrow N$
 that satisfies conditions (i), (ii) and (iii) in Definition 2.1.

M, N are said to be partially isomorphic if there is a nonempty collection I of partial embeddings such that

1. $p \in I$ and $a \in M \Rightarrow$ there is $\hat{p} \in I$ s.t. $p \subseteq \hat{p}$ and $a \in \text{dom}(\hat{p})$
2. $p \in I$ and $b \in N \Rightarrow$ there is $\hat{p} \in I$ s.t. $p \subseteq \hat{p}$ and $b \in \text{ran}(\hat{p})$.

4.3 Lemma (Back & forth) If $|M| = |N| = \omega$ and M, N are partially isomorphic via I , then $M \simeq N$

Proof Enumerate M and N , say $M = \langle a_i : i < \omega \rangle$ and

$N = \langle b_i : i < \omega \rangle$. We define inductively a chain
 $\langle p_i : i < \omega \rangle$ such that $p_i \in I$ and $a_{i+1} \in \text{dom}(p_i)$, $b_{i+1} \in \text{ran}(p_i)$.

Let p_0 be any map in I . At stage $i+1$, p_i is given.

Use property 1 in Def 4.2 to find $\hat{p} \supseteq p_i$ such that $a_i \in \text{dom}(\hat{p})$, and property 2 to find $p_{i+1} \supseteq \hat{p}$ such that $b_i \in \text{ran}(p_{i+1})$.

Then $\bigcup p_i$ is the required isomorphism between M and N .

4.4 Lemma (Extension) Let $M \models T_{\leq}$ and $N \models T_{\text{dlo}}$. Let

$p: M \rightarrow N$ be a partial embedding such that $|\text{dom}(p)| < \omega$.

Let $c \in M$. Then there is $\hat{p} \supseteq p$ s.t. $|\text{dom}(\hat{p})| < \omega$ and $c \in \text{dom}(\hat{p})$.

Proof Let $\text{dom}(p) = \{a_0, \dots, a_n\}$ with $a_i < a_j$ if $i < j$. If $c \in \text{dom}(\hat{p})$

Case 1: $c < a_0$. By axiom 6, there is $d \in N$ s.t. $d < p(a_0)$

Case 2: $a_i < c < a_{i+1}$. By axiom 5, there is $d \in N$ s.t. $p(a_i) < d < p(a_{i+1})$.

Case 3: $c > a_n$: similar to Case 1.

Then $\hat{p} = p \vee \{c, d\}$ is the required extension of p .

(4.5) Theorem Let $M, N \models T_{\text{alo}}$, $|M| = |N| = \omega$. Then $M \cong N$

Proof: the empty map \emptyset is a finite partial embedding between M and N (equivalently, $\{\langle m, n \rangle\}$ for any $m \in M, n \in N$ is a finite partial embedding). Therefore the collection I of finite partial embeddings between M and N is non-empty. By Lemma 4.4, I satisfies properties 1 and 2 in Definition 4.2, so Lemma 4.3 applies.

(4.6) Definition An L -theory T is consistent if it has a model.

If φ is an L -sentence, then $T \vdash \varphi$ if for all L -structures M

$$M \models T \rightarrow M \models \varphi$$

T is complete if for all sentences φ either $T \vdash \varphi$ or $T \vdash \neg \varphi$.

Remarks:

- $\text{Th}(M)$ is complete for any structure M .

- In specific cases, determining whether a given formula φ is in $\text{Th}(M)$ is non-trivial!

- We often seek $S \subseteq \text{Th}(M)$ s.t. $M \models S$ and S is complete (an axiomatization of $\text{Th}(M)$).

(4.7) Definition If $|L| = \omega$, an L -theory T is ω -categorical if T has an infinite model, and any two countable models are isomorphic.

So Theorem 4.5 says that T_{alo} is ω -categorical

(4.8) Theorem If T is an ω -categorical theory with no finite models, then T is complete.

Proof: Suppose $M, N \models T$ and φ is an L -sentence such that $M \models \varphi$. By Theorem 3.11 (Downward Löwenheim-Skolem) there are $M', N' \models T$ such that $|M'| = |N'| = \omega$ and $M' \leq M, N' \leq N$. By Theorem 4.5, $M' \cong N'$, so $N' \models \varphi$. By elementarity, $N \models \varphi$.

(4.9) Corollary T_{alo} is complete

(4.10) Definition If M, N are L -structures, a map $f: \text{dom}(f) \subseteq M \rightarrow N$ is said to be elementary if for all L -formulas $\varphi(x)$ and $\bar{a} \in \text{dom}(f)^{\mathbb{N}}$

$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a}))$$

(4.11) Remark A map f is elementary if and only if every finite restriction of f is elementary.

Proof (\Rightarrow) clear

(\Leftarrow) Let $\varphi(x)$ and $\bar{a} \in \text{dom}(f)^{\mathbb{N}}$ be such that
 $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a}))$

Then $f \upharpoonright_{\bar{a}}$ is a finite restriction of f which is not elementary.

4.12 Proposition Let $M, N \models T_{\text{defo}}$, and suppose $p : M \rightarrow N$ is a partial embedding. Then p is an elementary map.

Proof: By Remark 4.11, it suffices to consider the case $|p| < \omega$.

By Theorem 3.11 (Downward Löwenheim-Skolem), there are $M' \subseteq M$, $N' \subseteq N$ such that $|M'| = |N'| = \omega$, $\text{dom}(p) \subseteq M'$, $\text{ran}(p) \subseteq N'$.

By a back-and-forth argument (proof of Theorem 4.3 with $p_0 = p$ and I the collection of finite partial embeddings from M' to N'), p can be extended to an isomorphism $\pi : M' \rightarrow N'$, so in particular p is an elementary map.

4.13 Corollary $(\mathbb{Q}, \leq) \leq (\mathbb{R}, \leq)$

Proof: The inclusion map is a partial embedding, so by Theorem 4.12 it is an elementary map.

4.14 Definition (graphs and random graph)

Let $L_{\text{gph}} = \{R\}$, where R is a binary relation symbol.

A graph is an L_{gph} -structure which satisfies

1. $\forall x \neg R(x, x)$ (irreflexivity)
2. $\forall x y (R(x, y) \rightarrow R(y, x))$ (symmetry)

If G is a graph, the elements of G are called vertices, and 2-subsets $\{a, b\}$ such that $R(a, b)$ are called edges.

A graph is a random graph if, in addition to axioms 1 and 2, it satisfies

3. $\exists x y (x \neq y)$

together with axioms r_n for $n \in \omega$:

$$r_n : \forall x_0 \dots x_n y_0 \dots y_n (\bigwedge_{i,j=0}^n x_i \neq y_i \rightarrow \exists z (\bigwedge_{i=0}^n (z \neq x_i) \wedge (z \neq y_i))$$

$$\wedge R(z, x_i) \wedge \neg R(z, y_i)))$$

T_{gph} is the theory that includes axioms 1 and 2

T_{rg} is the theory that includes axioms 1, 2, 3 and $r_n \vee n \in \omega$.

Axiom 3 and the axiom scheme r_n imply that a model of T_{rg} is infinite.

4.15 Fact T_{rg} is consistent

Proof Define a graph on ω as follows: for $i, j \in \omega$ with $i < j$, $\{i, j\}$ is an edge iff the i th digit in the binary expansion of j is a 1. (we count digits from right to left).

Exercise: check this L_{gph} -structure is a model of T_{rg}

4.16 Lemma (Extension) Let $M \models T_{\text{graph}}$, $N \models T_{\text{rg}}$. Let $p: \text{dom}(p) \subseteq M \rightarrow N$ be a finite partial embedding and let $c \in M$. Then there is a finite partial embedding $\hat{p} \supseteq p$ such that $c \in \text{dom}(\hat{p})$.

Proof Assume $c \notin \text{dom}(p)$ and let

$$U = \{a \in \text{dom}(p) : R(a, c)\}$$

$$V = \{b \in \text{dom}(p) : \neg R(b, c)\}$$

p is finite, hence U and V are finite.

Since N is a random graph, we can find $d \in N \setminus (p(U) \cup p(V))$ such that

- (i) $R(d, p(a))$ for all $a \in U$
- (ii) $\neg R(d, p(b))$ for all $b \in V$.

Then $\hat{p} = p \cup \{(c, d)\}$. \square

4.17 Theorem Let $M, N \models T_{\text{rg}}$ be such that $|M| = |N| = \omega$. Then $M \simeq N$.

Proof: The same as the proof of Theorem 4.5, with Lemma 4.16 instead of Lemma 4.4.

4.18 Corollary T_{rg} is ω -categorical and complete.

Every partial embedding between models of T_{rg} is elementary.

4.19 Remarks The (unique up to isomorphism) countable model Γ of T is called the countable random graph, or Rado's graph. Γ embeds every finite and countable graph, and every finite partial isomorphism extends to an automorphism of Γ (i.e. an isomorphism from Γ to itself).

(5) COMPACTNESS

5.1 Definition Let I be a set. Then a filter F on I is a collection of subsets of I such that

1. $I \in F$
2. if X and Y are in F , then $X \cap Y \in F$
3. if $X \in F$ and $X \subseteq Y \subseteq I$, then $Y \in F$

power set

A filter on I is proper if and only if $F \neq P(I)$ - equivalently, if $\emptyset \notin F$

An ultrafilter on I is a proper filter U such that for all $X \subseteq I$, either $X \in U$ or $I \setminus X \in U$.

5.2 Fact The following conditions are equivalent to the proper filter U being an ultrafilter:

- (a) U is maximal among proper filters
- (b) if $X \cup Y \in U$, then either $X \in U$ or $Y \in U$

5.3 Definition Let $\langle M_i : i \in I \rangle$ be a family of L -structures.

The direct product of the M_i in the set

$$X = \prod_{i \in I} M_i = \{ f : I \rightarrow \bigcup M_i : f(i) \in M_i \text{ for all } i \in I \}$$

not necessarily an ordered set

We will write $a = \langle a(i) : i \in I \rangle$ for a typical element of X .

Let U be an ultrafilter on I . Define a relation \sim_U on X as follows:

$$a \sim_U b \Leftrightarrow \{i \in I : a(i) = b(i)\} \in U$$

5.4 Fact \sim_U is an equivalence relation on X

Proof Reflexivity and symmetry are immediate.

Suppose $a, b, c \in X$ and $a \sim_U b$, $b \sim_U c$. Then:

$$A = \{i \in I : a(i) = b(i)\} \in U$$

$$B = \{i \in I : b(i) = c(i)\} \in U$$

Then $A \cap B \in U$, and

$$A \cap B \subseteq C = \{i \in I : a(i) = c(i)\}, \text{ hence } C \in U.$$

So $a \sim_U c$.

Motivation: we write a_U for the equivalence class of $a \in X$ under U .

5.5 Fact Let $a^k, b^k \in X$ for $k = 1, \dots, n$ be such that

$$a^k \sim_U b^k$$

Then:

- (a) If f is an n -ary function symbol, then

$$\langle f^{M_i}(a^1(i), \dots, a^n(i)) : i \in I \rangle \sim_u \langle f^{M_i}(b^1(i), \dots, b^n(i)) : i \in I \rangle$$

(b) if R is an n -ary relation symbol, then
 $\{i \in I : (a^1(i), \dots, a^n(i)) \in R^{M_i}\} \in U \Leftrightarrow$
 $\{i \in I : (b^1(i), \dots, b^n(i)) \in R^{M_i}\} \in U$

Proof (a) For ease of notation, we only consider the case $n=1$.
So let $a, b \in X$ be such that $a \sim_u b$.

$$A = \{i \in I : a(i) = b(i)\}$$

Then $C = \{i \in I : f^{M_i}(a(i)) = f^{M_i}(b(i))\} \supseteq A$, and so $C \in U$.

(b) similar.

5.6 Definition Given a collection $\langle M_i : i \in I \rangle$ of L -structures and an ultrafilter U on I , the ultraproduct of the M_i is the L -structure defined as follows.

(i) the domain is X / \sim_u (where $X = \prod_{i \in I} M_i$)

(for ease of notation, write X_u)

(ii) if c is a constant, then

$$c^{X_u} := \langle c^{M_i} : i \in I \rangle_u$$

(iii) if f is a function symbol and $a^1_u, \dots, a^{n_e}_u \in X_u$, then

$$f^{X_u}(a^1_u, \dots, a^{n_e}_u) = \langle f^{M_i}(a^1(i), \dots, a^{n_e}(i)) : i \in I \rangle_u$$

(iv) if R is a relation symbol and $a^1_u, \dots, a^{n_r}_u \in X_u$, then
 $(a^1_u, \dots, a^{n_r}_u) \in R^{X_u} \Leftrightarrow \{i \in I : (a^1(i), \dots, a^{n_r}(i)) \in R^{M_i}\} \in U$.

Fact 5.5 ensures that the interpretations of f and R in X_u is well defined.

5.7 Theorem (tos) Let U be an ultrafilter on a set I , and let $\langle M_i : i \in I \rangle$ be a collection of L -structures. Then

1. for all L -terms $t(x_1, \dots, x_n)$ and $a^1_u, \dots, a^n_u \in X_u$,
 $t^{X_u}(a^1_u, \dots, a^n_u) = \langle t^{M_i}(a^1(i), \dots, a^n(i)) : i \in I \rangle_u$
2. for all L -formulas $\varphi(x_1, \dots, x_n)$ and $a^1_u, \dots, a^n_u \in X_u$
 $X_u \models \varphi(a^1_u, \dots, a^n_u) \Leftrightarrow \{i \in I : M_i \models \varphi(a^1(i), \dots, a^n(i))\} \in U$
3. for all L -sentences σ

$$X_u \models \sigma \Leftrightarrow \{i \in I : M_i \models \sigma\} \in U$$

Proof : 1 is an easy induction on the complexity of $t(x_1, \dots, x_n)$
3 is immediate from 2.

For 2 : The cases where φ is atomic are an easy induction.

If $\varphi \equiv \neg X$ for an L -formula $X(x_1, \dots, x_n)$, let

$$A_X = \{i \in I : M_i \models X(a^1(i), \dots, a^n(i))\}. \text{ By inductive hypothesis}$$

$$X_u \models X(a^1_u, \dots, a^n_u) \Leftrightarrow A_X \in U, \text{ so}$$

$$X_u \not\models X(a^1_u, \dots, a^n_u) \Leftrightarrow A_X \notin U.$$

But \mathcal{U} is an ultrafilter, hence $A_x \notin \mathcal{U} \Leftrightarrow I \setminus A_x \in \mathcal{U}$. Therefore $X_u \models \neg K(a^u_1, \dots, a^u_n) \Leftrightarrow \{i \in I : M_i \models \neg K(a^*(i), \dots, a^*(i))\} \in \mathcal{U}$.

If $\phi \equiv \psi \wedge K$, let $A_\phi = \{i \in I : M_i \models \phi(a^*(i), \dots, a^*(i))\}$, and let A_ψ and A_K be defined similarly.

Then $A_\phi = A_\psi \wedge A_K$ and, since \mathcal{U} is an ultrafilter,
 $A_\phi \in \mathcal{U} \Leftrightarrow A_\psi \in \mathcal{U}$ and $A_K \in \mathcal{U}$.

The required result follows by induction hypothesis.

If $\phi \equiv \exists y \psi(\bar{x} y)$, define A_ϕ as usual.

Suppose there is $b_u \in X_u$ such that $X_u \models \psi(a^u_1, \dots, a^u_n, b_u)$.

Then $\{i \in I : M_i \models \psi(a^*(i), \dots, a^*(i), b(i))\} \subseteq A_\phi$

By induction hypothesis, the first set is in \mathcal{U} , therefore so is A_ϕ .

Now suppose $A_\phi \in \mathcal{U}$. For $i \in A_\phi$, let $b_i \in M_i$ be a witness of $\exists y \psi(a^*(i), \dots, a^*(i), y)$.

For $i \in I \setminus A_\phi$, let b_i be arbitrary.

Define $b \in X$ by $b(i) = b_i$.

Let $A_\psi = \{i \in I : M_i \models \psi(a^*(i), \dots, a^*(i), b(i))\}$.

Then $A_\phi \subseteq A_\psi$, so $A_\psi \in \mathcal{U}$.

By ind. hyp, $X_u \models \psi(a^u_1, \dots, a^u_n, b_u)$, hence $X_u \models \exists y \psi(a^u_1, \dots, a^u_n, y)$, as required.

5.8 Definition A subset $S \subseteq P(I)$ has the finite intersection property (FIP) if whenever $A_1, \dots, A_n \in S$

$$\bigcap_{i=0}^n A_i \neq \emptyset$$

Remark: proper filters have FIP.

5.9 Lemma

1. If $S \subseteq P(I)$ has FIP, then S can be extended to a proper filter.

2. A proper filter can be extended to an ultrafilter.

Proof:

1. Let F be the extension of S defined as follows:

$F = \{X \subseteq I : X \text{ contains the intersection of a finite collection of elements of } S\}$.

Then F is a proper filter

2. Let F be a proper filter, and let

$\mathcal{G} = \{G \subseteq P(I) : G \text{ is a proper filter containing } F\}$

Then \mathcal{G} is partially ordered by inclusion. It is easy to check that the union of a chain in \mathcal{G} is a proper filter containing F , hence an upper bound in \mathcal{G} for the chain.

By Zorn's Lemma, \mathcal{G} has a maximal element F' , say.

By Fact 5.2, F' is an ultrafilter and F' extends F , as required

(5.10) Definition A theory T is said to be

(i) consistent, or satisfiable, if it has a model

(ii) finitely consistent, or finitely satisfiable, if every finite subset of T has a model

(5.11) Theorem (Compactness)

An L -theory T is consistent if and only if it is finitely consistent.

Proof (\Rightarrow) is clear.

For the converse, if $S \subseteq T$ is finite, let $M_S \models S$.

Let I be the collection of finite subsets of T . Idea : define an ultrafilter U on T so that $\bigcap_{S \in I} M_S / \sim_U$ is a model of T .

By Theorem 5.9 (to \check{s}), it is enough to find U such that for all $\varphi \in T$ $\{S : M_S \models \varphi\} \in U$.

For $\varphi \in T$, let $A_\varphi = \{S \in I : \varphi \in S\}$. Claim : the set $\{A_\varphi : \varphi \in T\}$ has FIP.

Let $\varphi_1, \dots, \varphi_n \in T$. Then $\{\varphi_1, \dots, \varphi_n\} \in I$, and $\{\varphi_1, \dots, \varphi_n\} \subseteq \bigcap_{i=1}^n A_{\varphi_i}$, so $\bigcap A_{\varphi_i} \neq \emptyset$.

By Lemma 5.9, $\{A_\varphi : \varphi \in T\}$ extends to an ultrafilter U .

By to \check{s} 's Theorem, $X_U \models \varphi \Leftrightarrow \{S : M_S \models \varphi\} \in U$.

But $A_\varphi \in U$, and $A_\varphi \subseteq \{S : M_S \models \varphi\}$. So we are done.

(5.12) Definition A type $p(\bar{x})$ in L is a set of L -formulas whose free variables are among $\bar{x} - \langle x_i : i < \lambda \rangle$.

A type $p(\bar{x})$ is

- satisfiable in an L -structure M if there is $\bar{a} \in M^{|\bar{x}|}$ such that $M \models \varphi(\bar{a})$ for all $\varphi(\bar{x}) \in p(\bar{x})$. We write

$M \models p(\bar{x})$, or $M, \bar{a} \models p(\bar{x})$

and we say \bar{a} realizes $p(\bar{x})$ in M

- satisfiable if it is satisfiable in some M

- finitely satisfiable in M if all its finite subsets are satisfiable in M

- finitely satisfiable if all its finite subsets are satisfiable

Warning : we may say "consistent", "finitely consistent" etc throughout the definition.

Remark : a type may be finitely satisfiable in M but not satisfiable in M . For example, let $M = \langle \mathbb{N}, < \rangle$ and let $\varphi_n(x)$ say "There are at least n elements less than x ".

Let $p(x) = \{d_n(x) : n < \omega\}$. Then $p(x)$ is finitely satisfiable in M , but not satisfiable in M .

5.13 Theorem (compactness for types)

Every finitely satisfiable type in L is satisfiable.

Proof: Let $p(\bar{x})$ be a type in L with $\bar{x} = \langle \bar{x}_i : i < \lambda \rangle$.

Expand L to $L' = L \cup \{c_i : i < \lambda\}$, where the c_i are distinct constants not in L .

Then $p(\bar{c})$ is a finitely satisfiable theory in L' .

By the Compactness Theorem, there is an L' -structure M' such that $M' \models p(\bar{c})$. But M' can be reduced to an L -structure (simply forget the interpretations of the c_i).

Then $M', \bar{c}^{M'} \models p(\bar{x})$.

5.14 Lemma

Let M be an L -structure, and let \bar{a} be an enumeration of M , say $\bar{a} = \langle a_i : i < \lambda \rangle$.

Let $q(\bar{x}) = \{\varphi(\bar{x}) : M \models \varphi(\bar{a})\}$, where $|\bar{x}| = \lambda$.

Then $q(\bar{x})$ is satisfiable in an L -structure N if and only if there is an elementary embedding $\beta : M \rightarrow N$

Proof: (\Rightarrow) If $q(\bar{x})$ is satisfiable in N , then there is a tuple $\bar{b} \in N^{|\bar{x}|}$ such that $N \models \varphi(\bar{b})$ for all $\varphi(\bar{x}) \in q(\bar{x})$.

Then $\beta : M \rightarrow N$ defined by $a_i \mapsto b_i$ ($i < \lambda$) is an elementary embedding: let $\varphi(\bar{x})$ be an L -tuple. Then

$$M \models \varphi(\bar{a}) \Leftrightarrow \varphi(\bar{x}) \in q(\bar{x}) \Leftrightarrow N \models \varphi(\bar{b}) \text{ and } \bar{b} = \beta(\bar{a}).$$

(\Leftarrow) If $\beta : M \rightarrow N$ is an elementary embedding, then $N, \beta(\bar{a}) \models q(\bar{x})$.

5.15 Remark If M is an L -structure and $A \subseteq M$, we can work with types in $L(A)$. In particular, we sometimes work with types in $L(M)$. Types of this kind are said to have parameters in A (resp M), or to be over A (resp M).

If $p(\bar{x})$ is a type in $L(M)$, then there are an enumeration \bar{a} of M and a type $q(\bar{x}, \bar{a})$ in L such that $p(\bar{x}) = q(\bar{x}, \bar{a})$.

Lemma 5.14 is sometimes called Diagram Lemma and stated as follows.

Let $\text{Th}(M_N)$ be the $L(M)$ -theory of M . Suppose $N \models \text{Th}(M_N)$. Then M embeds elementarily in N .

5.16 Theorem If M is an L -structure and $p(x)$ is a type in $L(M)$ that is finitely satisfiable in M , then $p(x)$ is realized in some N such that $M \leq N$.

Example Let $(0, 1) \subseteq \mathbb{Q}$, let $M = \langle (0, 1), < \rangle$.

Let $a_n = 1 - \frac{1}{n}$ for $n \in \omega \setminus \{0\}$.

Let $\varphi_n(x) = a_n < x$. Then $p(x) = \{\varphi_n(x) : n \in \omega \setminus \{0\}\}$ is a type in $L(M)$ that is finitely satisfiable in M , but not satisfiable. However, $p(x)$ is realized in $\langle \mathbb{Q}, < \rangle$, and $M \leq \langle \mathbb{Q}, < \rangle$

Proof of Theorem 5.16 Let $\bar{a} = \langle a_i : i < \lambda \rangle$ be an enumeration of M , and let

$$q(\bar{z}) = \{d(\bar{z}) : M \models d(\bar{a})\}$$

where $|\bar{z}| = \lambda$ and the z_i are new variables (so $\bar{z} \cap \bar{x} = \emptyset$). Write $p(z)$ as $p'(\bar{z}, \bar{a})$, where $p'(\bar{z}, \bar{z})$ is a type in L .

Then $p'(\bar{z}, \bar{z}) \cup q(\bar{z})$ is finitely consistent in M , because $p'(\bar{z}, \bar{z})$ is finitely consistent and $q(\bar{z})$ is realized by \bar{a} .

By compactness for types, there are an L -structure N and tuples $\bar{c} \in N^{|\bar{z}|}$ and $\bar{b} \in N^{|\bar{z}|}$ such that

$$N \models p'(\bar{c}, \bar{b}) \cup q(\bar{b}).$$

In particular, $N \models q(\bar{b})$. By Lemma 5.14, $\beta : a_i \mapsto b_i$ is an elementary embedding. We may assume $M \leq N$.

(5.17) Theorem (Upward Löwenheim-Skolem)

Let M be an infinite L -structure. Then for any $\lambda \geq |M| + |L|$, there is N such that $M \leq N$ and $|N| = \lambda$.

Proof: Let $\langle \bar{x} : i < \lambda \rangle$ be a tuple of (distinct) variables.

$$\text{Let } p(\bar{x}) = \{x_i \neq x_j : i < j < \lambda\}.$$

Then $p(\bar{x})$ is finitely consistent in M .

By Theorem 5.16, $p(\bar{x})$ is realized in some $M \leq N$, and so $|N| \geq \lambda$.

By Downward Löwenheim-Skolem, we may assume $|N| = \lambda$.

⑥ SATURATION

6.1 Definition Let λ be an infinite cardinal and M an infinite L -structure. Then M is said to be λ -saturated if it realizes every type $p(x)$ with one free variable which

- (i) is finitely satisfiable in M
- (ii) has parameters in $A \subseteq M$ with $|A| < \lambda$

M is said to be saturated if it is $|M|$ -saturated

Remark: if $\lambda > M$, M cannot be λ -saturated: consider the type $p(x) = \{x+a : a \in M\}$, which is finitely satisfiable in M , but not satisfiable in M . This type has $|M|$ parameters.

6.2 Definition Let M be an L -structure, $A \subseteq M$, \bar{b} a tuple in M (possibly infinite). Then the type of \bar{b} over A is

$$\text{tp}_M(\bar{b}/A) = \{\varphi(\bar{x}) \in L(A) : M \models \varphi(\bar{b})\}$$

M is usually omitted when it is clear from the context

6.3 Remarks (i) $\text{tp}(\bar{b}/A)$ is complete: for all $L(A)$ -formulas $\varphi(\bar{x})$, either $\varphi(\bar{x}) \in \text{tp}(\bar{b}/A)$ or $\neg \varphi(\bar{x}) \in \text{tp}(\bar{b}/A)$
(ii) If $M \leq N$, $A \subseteq M$ and $\bar{b} \in M^{|\bar{b}|}$, then

$$\text{tp}_M(\bar{b}/A) = \text{tp}_N(\bar{b}/A)$$

6.4 Facts (types and elementary maps)

Let $f : A \subseteq M \rightarrow N$ be an elementary map. Then

- (i) $M \equiv N$ (and if $M \equiv N$ then the empty map is elementary, as it preserves sentences)
 - (ii) If \bar{a} is an enumeration of $\text{dom}(f)$, then
- $$\text{tp}(\bar{a}/\emptyset) = \text{tp}(f(\bar{a})/\emptyset)$$

More generally, if there is $A \subseteq \text{dom}(f) \cap N$ and f is the identity on A , then for every tuple \bar{b} in $\text{dom}(f)$

$$\text{tp}(\bar{b}/A) = \text{tp}(f(\bar{b})/A)$$

- (iii) Let \bar{a} enumerate $\text{dom}(f)$. If $p(\bar{x}, \bar{a})$ is a finitely satisfiable type in $L(A)$, then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in N .
Let $\{\varphi_i(\bar{x}, \bar{a}), \dots, \varphi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a})$. Then $M \models \exists \bar{x} \bigwedge_{i=1}^n \varphi_i(\bar{x}, \bar{a})$ so by elementarity of f

$$N \models \exists \bar{x} \varphi_i(\bar{x}, f(\bar{a})).$$

However, if $p(\bar{x}, \bar{a})$ is satisfiable in M , it does not follow that $p(\bar{x}, f(\bar{a}))$ is satisfiable in N (though $p(\bar{x}, f(\bar{a}))$ will still be finitely satisfiable in N).

6.5 Theorem Let N be such that $|L| \leq \lambda \leq |N|$. The following are equivalent:

(i) N is λ -saturated

(ii) If $f : M \rightarrow N$ is an elementary map ($\Rightarrow M = N$), $|f| < \lambda$ and $b \in M$, then there is $\tilde{f} \supseteq f$ that is elementary and $b \in \text{dom}(\tilde{f})$

(iii) If $p(\bar{x})$ is a type in $L(A)$, where $A \subseteq N$, $|A| < \lambda$, $|\bar{x}| \leq \lambda$, and $p(\bar{x})$ is finitely satisfiable in N , then $p(\bar{x})$ is satisfiable in N .

Proof: (i) \Rightarrow (ii) Let f be as in the hypothesis of (ii) and let $b \in M$. Let \bar{a} be an enumeration of $\text{dom}(f)$, so $|\bar{a}| < \lambda$. Let $p(x, \bar{a}) = \text{tp}_M(b / \bar{a})$.

Since $p(x, \bar{a})$ is finitely satisfiable in M , $p(x, f(\bar{a}))$ is finitely satisfiable in N . Since N is λ -saturated and $|f| < \lambda$, $p(x, f(\bar{a}))$ is satisfiable in N , say $N \models p(c, f(\bar{a}))$. Then $f \cup \{b, c\}$ is the required extension of f :

$$M \models d(b, \bar{a}) \Leftrightarrow d(x, \bar{a}) \in p(x, \bar{a}) \Leftrightarrow d(x, f(\bar{a})) \in p(x, f(\bar{a})) \Leftrightarrow N \models d(c, f(\bar{a})).$$

(ii) \Rightarrow (iii) Let $p(\bar{x})$ be as in (iii). By Theorem 5.16, $p(\bar{x})$ is realized in some elementary $M \succcurlyeq N$ by some tuple \bar{a} , where $|\bar{a}| = |\bar{x}| \leq \lambda$.

Since $N \leq M$, the map $\text{id}_A : M \rightarrow N$ is elementary.

Idea: we extend id_A to an elementary map $f : M \rightarrow N$ such that $\bar{a} \in \text{dom}(f)$. Then $N \models p(f(\bar{a}))$.

We build f in stages. Let $f_0 = \text{id}_A$.

At stage $i+1$, use (ii) to put a_i in $\text{dom}(f_{i+1})$.

At limit stages $\mu < \lambda$, let $f_\mu = \bigcup_{i < \mu} f_i$.

Then $\bigcup_{i < \lambda} f_i$ is the required elementary map.

(iii) \Rightarrow (i) trivial

6.6 Corollary If M and N are saturated, $M = N$ and $|M| = |N|$, then any elementary $f : M \rightarrow N$, $|f| < |M|$, extends to an isomorphism. In particular, if $M = N$ and $|M| = |N|$ and M, N are saturated, then $M \cong N$.

Proof: Use Theorem 6.5 (ii) to extend $f : M \rightarrow N$ to an isomorphism back and forth. This can also be done with $f = \emptyset$ (since $M = N$, \emptyset is elementary).

6.7 Corollary Models of Tdlo and Trg are w -saturated.

Proof By Theorem 6.5, Lemma 4.4 and Proposition 4.12 (linear orders) and Lemma 4.16 and 4.18 (graphs).

So (\mathbb{Q}, \subset) is saturated. On the other hand, (\mathbb{R}, \subset) is not ω_1 -saturated: the type $\{\bar{x} > q : q \in \mathbb{Q}\}$ in $L(\mathbb{Q})$ is finitely satisfiable in \mathbb{R} but not satisfiable.

6.8 Definition An isomorphism $\alpha : N \rightarrow N$ is called an automorphism. The automorphisms of N form a group denoted by $\text{Aut}(N)$.

If $A \subseteq N$, then

$$\text{Aut}(N/A) = \{\alpha \in \text{Aut}(N) : \alpha = \text{id}_A\}$$

(automorphisms that fix A pointwise)

6.9 Definition Let N be an L -structure

(i) N is λ -universal if for every $M \equiv N$ such that $|M| \leq \lambda$ there is an elementary embedding $\beta : M \rightarrow N$.

N is universal if it is $|N|$ -universal

(ii) N is λ -homogeneous if every elementary map $f : N \rightarrow N$ such that $|f| < \lambda$ extends to an automorphism of N .

N is homogeneous if it is $|N|$ -homogeneous

Warning: in some books/papers, λ -universality according to (i) is called λ^+ -universality

Homogeneity according to (ii) is called strong homogeneity (as opposed to weak homogeneity, a property similar to the one in Theorem 6.5(ii) with $N = M$).

There is also a notion of "ultrahomogeneity" defined for partial embeddings, rather than elementary maps, as in (ii). In Chapter 4 we actually saw that (\mathbb{Q}, \subset) and the countable random graph are ultrahomogeneous.

Countable models of T_{SO} and T_{RG} are universal and homogeneous.

6.10 Theorem Let N be such that $|N| \geq |L|$. Then the following are equivalent:

(i) N is saturated

(ii) N is universal and homogeneous.

Proof: (i) \rightarrow (ii) Assume N is saturated, and let $M \equiv N$ be such that $|M| \leq |N|$. Let $\bar{a} = \langle a_i : i < |M| \rangle$ be an enumeration of M , and let $p(\bar{z}) = \text{tp}(\bar{a} / \emptyset)$ (so $|\bar{z}| = |\bar{a}| = |M|$). Since $p(\bar{z})$ is finitely satisfiable in M , $p(\bar{z})$ is finitely satisfiable in N (if $\varphi_1(\bar{z}), \dots, \varphi_n(\bar{z}) \in p(\bar{z})$ then $M \models \exists \bar{z} \bigwedge \varphi_i(\bar{z})$). Since $M \equiv N$, $N \models \exists \bar{z} \bigwedge \varphi_i(\bar{z})$.

Since $|\bar{z}| \leq |N|$, N realizes $p(\bar{z})$ by saturation (Theorem 6.5(iii)).

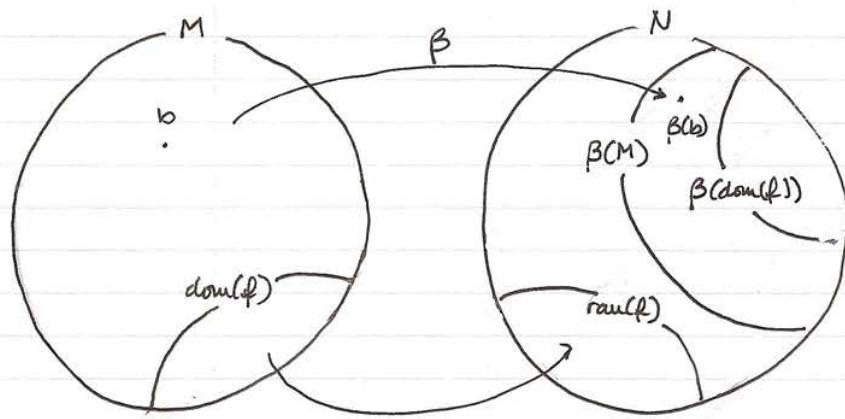
Then if $N \models p(\bar{b})$, $a_i \mapsto b_i$ is an elementary embedding from M to N .

Homogeneity follows from Corollary 6.6 with $M = N$.

(ii) \Rightarrow (i) We show that if $M \equiv N$, $b \in M$ and $f: M \rightarrow N$ is an elementary map such that $|f| < |N|$, then there is $\hat{f} \supseteq f$ which is elementary and defined on b . By Theorem 6.5, this shows that N is saturated.

By working in $M' \leq M$ such that $\text{dom}(f) \cup \{b\} \subseteq M'$ if necessary (use Downward Löwenheim-Skolem), we may assume $|M| \leq |N|$.

Since $M \equiv N$, by universality there is an elementary embedding $\beta: M \rightarrow N$, so $\beta(M) \leq N$.



The map $f \circ \beta^{-1}: \beta(\text{dom}(f)) \rightarrow \text{ran}(f)$ is elementary, and $|f \circ \beta^{-1}| < |N|$. By homogeneity, there is $\alpha \in \text{Aut}(N)$ such that $f \circ \beta^{-1} \subseteq \alpha$. Then $f \cup \{\langle b, \alpha(\beta(b)) \rangle\}$ is elementary, as it is a restriction of $\alpha \circ \beta$.

6.11 Definition Let \bar{a} be a tuple in N and $A \subseteq N$.

The orbit of \bar{a} over the set A is

$$O_N(\bar{a}/A) := \{\alpha(\bar{a}) : \alpha \in \text{Aut}(N/A)\}$$

If $\varphi(x)$ is an $L(A)$ -formula, then

$$\varphi(N) := \{\bar{a} \in N^{|x|} : N \models \varphi(\bar{a})\}$$

is the set defined by $\varphi(x)$.

A set is definable over A if it is defined by some $L(A)$ -formula.

There are analogous notions of sets defined by types / type-definable sets.

Recall that if \bar{a}, \bar{b} are tuples in N , $A \subseteq N$ and $|\bar{a}| = |\bar{b}|$, the following are equivalent

$$(i) \text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A)$$

(ii) $\{\bar{a}_i \mapsto \bar{b}_i : i < |\bar{a}|\} \cup \text{id}_A$ is an elementary map from N to N .

6.12 Proposition Let N be λ -homogeneous, $A \subseteq N$, $|A| < \lambda$, and let $\bar{\alpha}$ be a tuple in N such that $|\bar{\alpha}| < \lambda$. Then

$$O_N(\bar{\alpha}/A) = p(N)$$

where $p(\bar{x}) = tp_N(\bar{x}/A)$, and $p(N) = \{ \bar{b} \in N^{|\bar{\alpha}|} : N \models p(\bar{b}) \}$

Proof If $\bar{b} = \alpha(\bar{\alpha})$ with $\alpha \in \text{Aut}(N/A)$, then $tp_N(\bar{\alpha}/A) = tp_N(\bar{b}/A)$. Conversely, if $N \models p(\bar{b})$, then $tp_N(\bar{\alpha}/A) = tp_N(\bar{b}/A)$ and so the map $\{ \langle a_i, b_i \rangle : i < |\bar{\alpha}| \} \cup \text{id}_A$ is elementary. By λ -homogeneity, this elementary map extends to $\alpha \in \text{Aut}(N)$, and in particular $\alpha \in \text{Aut}(N/A)$, so $\bar{b} \in O_N(\bar{\alpha}/A)$.

7 THE MONSTER MODEL

Given a complete theory T with an infinite model, it is often convenient to work in a saturated model of T large enough to embed elementarily any other model of T we might be interested in. We call such a large saturated model of T the monster model. The monster model is usually denoted by \mathbb{U} , or M , and its cardinality is denoted by κ .

The existence of arbitrarily large saturated models for general theories may depend on extra set-theoretic assumptions on the desired cardinality (e.g. the cardinality is inaccessible). In specific cases, such assumptions can be avoided by ad-hoc arguments. The monster model is an expository device that allows us to prove results about all the models of a theory by proving results about elementary substructures of saturated models of T .

7.1 Terminology and conventions

When working in the monster model \mathbb{U} of a theory T , we say that

- $\varphi(\bar{x})$ holds if $\mathbb{U} \models \forall \bar{x} \varphi(\bar{x})$
- $\varphi(\bar{x})$ is consistent if $\mathbb{U} \models \exists \bar{x} \varphi(\bar{x})$
- a type $p(\bar{x})$ is consistent / satisfiable if there is $\bar{a} \in \mathbb{U}^{\bar{x}}$ such that $\mathbb{U} \models p(\bar{a})$
- a cardinality is small if it is strictly less than $\kappa = |\mathbb{U}|$
- a model is $M \leq \mathbb{U}$ such that $|M|$ is small

Conventions

- tuples have length $< \kappa$ unless otherwise specified
- formulas have parameters in \mathbb{U} , unless otherwise specified
- definable sets have the form $\varphi(\mathbb{U})$ for some $L(\mathbb{U})$ -formula $\varphi(\bar{x})$
- type-definable sets have the form $p(\mathbb{U})$ for some type $p(\bar{x})$ in $L(A)$, where $|A| < \kappa$

Notation

- the symbols A, B, C, \dots denote sets of parameters of small cardinality
- $\text{tp}(\bar{a}/A) = \text{tp}_{\mathbb{U}}(\bar{a}/A)$
- $O(\bar{a}/A)$ is the orbit of \bar{a} in \mathbb{U} under $\text{Aut}(\mathbb{U}/A)$

If $p(\bar{x})$ and $q(\bar{x})$ are types, we write $p(\bar{x}) \rightarrow q(\bar{x})$ to mean that every realization of $p(\bar{x})$ is also a realization of $q(\bar{x})$

We extend the semantics of formulas to types, where a type is thought of as an infinite conjunction of formulas. However, such infinite conjunctions are not formulas, and their "truth" is not preserved by elementary maps

7.2 Fact Let $p(\bar{x})$ be a satisfiable type in $L(A)$ and $q(\bar{x})$ a satisfiable type in $L(B)$. Suppose $p(\bar{x}) \rightarrow \neg q(\bar{x})$ (that is, $p(\bar{x})$ and $q(\bar{x})$ have no realizations in common). Then there are a conjunction $\phi(\bar{x})$ of formulas in $p(\bar{x})$ and a conjunction $\psi(\bar{x})$ of formulas in $q(\bar{x})$ such that
 $\models \phi(\bar{x}) \rightarrow \neg \psi(\bar{x})$

Proof If $p(\bar{x}) \rightarrow \neg q(\bar{x})$ then $p(\bar{x}) \cup q(\bar{x})$ is not satisfiable in U , hence it is not finitely satisfiable (by saturation of U).

Let $\{\phi_1(\bar{x}), \dots, \phi_n(\bar{x})\} \subseteq p(\bar{x})$

$\{\psi_1(\bar{x}), \dots, \psi_m(\bar{x})\} \subseteq q(\bar{x})$

be such that $\bigwedge_{i=1}^n \phi_i(\bar{x}) \wedge \bigwedge_{i=1}^m \psi_i(\bar{x})$ is not satisfiable.

Then $\models \bigwedge_{i=1}^n \phi_i(\bar{x}) \rightarrow \neg \bigwedge_{i=1}^m \psi_i(\bar{x})$, as required

7.3 Remark Let $\phi(U, \bar{b})$ be a definable set, where $\phi(\bar{x}, \bar{z})$ is an L -formula and $\bar{b} \in U^{|\bar{z}|}$. Let $\alpha \in \text{Aut}(U)$. Then
 $\alpha[\phi(U, \bar{b})] = \phi(U, \alpha(\bar{b}))$

If $\bar{c} \in \alpha[\phi(U, \bar{b})]$, then $\bar{c} = \alpha(\bar{a})$ for some \bar{a} s.t. $\models \phi(\bar{a}, \bar{b})$.

Then $\models \phi(\alpha(\bar{a}), \alpha(\bar{b}))$, that is, $\models \phi(\bar{c}, \alpha(\bar{b}))$ and so

$\bar{c} \in \phi(U, \alpha(\bar{b}))$.

The other containment is similar.

So $\text{Aut}(U)$ acts on the definable sets in a natural way

Similarly, if $p(\bar{x}, \bar{z})$ is a type in L and $\bar{b} \in U^{|\bar{z}|}$, then

$$\alpha[p(U, \bar{b})] = p(U, \alpha(\bar{b}))$$

7.4 Definition A set $D \subseteq U^\lambda$ is invariant over A if for all $\alpha \in \text{Aut}(U/A)$

$$\alpha(D) = D$$

or, equivalently, if for all $\bar{a} \in D$ $O(\bar{a}/A) \subseteq D$.

Remark if $\bar{a} \in D$, $q(\bar{z}) = tp(\bar{a}/A)$ and $b \models q(\bar{z})$, then by homogeneity there is $\alpha \in \text{Aut}(U/A)$ such that $\alpha(\bar{a}) = \bar{b}$, and so $\bar{b} \in O(\bar{a}/A)$. So the invariance of D is equivalent to the following condition:

for all $\bar{a} \in D$, if $tp(\bar{b}/A) = tp(\bar{a}/A)$ then $\bar{b} \in D$.

Definability and invariance are closely related in U .

7.5 Proposition Let $A \subseteq U$. For every $L(U)$ -formula $\phi(\bar{x})$ the following are equivalent.

(i) there is a formula $\psi(\bar{x})$ in $L(A)$ such that

$$\models \forall \bar{z} [\psi(\bar{z}) \leftrightarrow \phi(\bar{z})]$$

(ii) $\phi(U)$ is invariant over A

Proof (i) \Rightarrow (ii) Clear, since $\psi(u)$ is fixed setwise by $\alpha \in \text{Aut}(U/A)$ (by Remark 7.3)

(ii) \Rightarrow (i) Suppose $\phi(\bar{x}, \bar{z})$ is an L-formula such that $\phi(U, b)$ is invariant over A for some $b \in U^{|\bar{z}|}$.

Let $g(\bar{z}) = \text{tp}(b/A)$. Then if $\bar{c} \models g(\bar{z})$, by homogeneity there is $\alpha \in \text{Aut}(U/A)$ such that $\alpha(b) = \bar{c}$. Then

$$\alpha[\phi(U, b)] = \phi(U, \bar{c}) \text{ by Remark 7.3}$$

"

$\phi(U, b)$ by invariance

and so $\phi(U, b) = \phi(U, \bar{c})$. Therefore

$$g(\bar{z}) \rightarrow \forall \bar{x} [\phi(\bar{x}, \bar{z}) \leftrightarrow \phi(\bar{x}, b)]$$

By (an argument similar to the one in) Fact 7.2, there is $\Theta(\bar{z}) \in q(\bar{z})$ such that $\Theta(\bar{z}) \rightarrow \forall \bar{x} [\phi(\bar{x}, \bar{z}) \leftrightarrow \phi(\bar{x}, b)]$

Then the formula

$$\exists \bar{z} [\Theta(\bar{z}) \wedge \phi(\bar{z}, \bar{z})]$$

defines $\phi(U, b)$ and it is in $L(A)$, as required. \square

Recall that a partial embedding preserves the truth of quantifier-free formulas (see Definition 4.2)

7.6 Proposition Let $\phi(\bar{z})$ be an L-formula. The following are equivalent:

(i) There is a quantifier-free L-formula $\psi(\bar{z})$ such that

$$\models \forall \bar{z} [\phi(\bar{z}) \leftrightarrow \psi(\bar{z})]$$

(ii) for all partial embeddings $g: U \rightarrow U$, for all tuples \bar{a} in $\text{dom}(g)$

$$\models \phi(\bar{a}) \leftrightarrow \phi(g(\bar{a}))$$

Proof (i) \Rightarrow (ii) clear, since partial embeddings preserve q.f. formulas.

(ii) \Rightarrow (i) For a tuple \bar{a} in U, let

$$q \text{f tp}(\bar{a}) = \{\psi(\bar{z}) \in \text{tp}(\bar{a}) : \psi(\bar{z}) \text{ is quantifier-free}\}$$

Let $D = \{q(\bar{z}) : q(\bar{z}) = q \text{f tp}(\bar{a}) \text{ for } \bar{a} \in \phi(U)\}$.

$$\text{Claim: } \phi(U) = \bigcup_{q(\bar{z}) \in D} q(U)$$

\subseteq is clear (by the way D is defined)

For \bar{z} , let $q(\bar{z}) \in D$, say $q(\bar{z}) = q \text{f tp}(\bar{a})$, and let $b \models q(\bar{z})$.

Then $a_i \mapsto b_i$ is a partial embedding, and so, by (ii), $\phi(b)$ holds. Hence $q(U) \subseteq \phi(U)$, and therefore

$$\bigcup_{q \in D} q(U) \subseteq \phi(U)$$

Then in particular $q(x) \rightarrow \phi(x)$. By an argument similar to that in Fact 7.2, there is $\psi_q(x) \in q(x)$ such that
 $\vdash \psi_q(x) \rightarrow \phi(x)$.

Then $\phi(x) \leftrightarrow \bigvee_{q(x) \in D} \{\psi_q(x) : \psi_q(x) \rightarrow \phi(x) \text{ and } \psi_q(x) \in q(x)\}$

Again by a version of Fact 7.2, there are $\psi_{q_1}(x), \dots, \psi_{q_n}(x)$ such that

$$\phi(x) \leftrightarrow \bigvee_{i=1}^n \psi_{q_i}(x)$$

and $\bigvee_{i=1}^n \psi_{q_i}(x)$ is the required quantifier-free formula. \square

7.7 Definition An L-theory T has quantifier elimination (q.e.) if for every L-formula $\phi(x)$ there is a quantifier-free L-formula $\psi(x)$ such that

$$T \vdash \forall x [\phi(x) \leftrightarrow \psi(x)]$$

Remark: if a theory has q.e., then the type of a tuple in U is determined by the quantifier-free type, and so types of tuples are easier to describe and understand - and so are definable sets.

7.8 Theorem Let T be a complete theory with an infinite model. The following are equivalent:

- (i) T has quantifier elimination
- (ii) every partial embedding $p: U \rightarrow U$ is elementary
- (iii) for every partial embedding $p: U \rightarrow U$ such that $|p| < |U|$ and for every $b \in U$ there is a partial embedding $\hat{p} \supseteq p$ such that $b \in \text{dom}(\hat{p})$.

Proof: (i) \Rightarrow (ii) Clear, since every formula is equivalent to a quantifier-free one and partial embeddings preserve q.f. formulas
(ii) \Rightarrow (i) By (ii) and Proposition 7.6 (since all partial embeddings are elementary, any formula is preserved by all partial embeddings, and so it is equivalent to a q.f. rule by Proposition 7.6)

(ii) \Rightarrow (iii) Let $p: U \rightarrow U$ be such that $|p| < |U|$. Then by (ii) p is elementary, so it extends to $\alpha \in \text{Aut}(U)$, and $p \cup \{b, \alpha(b)\}$ is the required extension of p.

(iii) \Rightarrow (ii) Let $p: U \rightarrow U$ be a partial embedding. Consider $p_0 \subseteq p$ finite. Extend p_0 to $\alpha \in \text{Aut}(U)$ by back-and-forth and saturation of U. Then p_0 is elementary. This can be done for all finite restrictions of p, so p is elementary.

Remark: a fourth condition equivalent to (i), (ii) and (iii) is
(iv) for every finite partial embedding $p: U \rightarrow U$ and $b \in U$,
there is $\hat{p} \supseteq p$ such that \hat{p} is a partial embedding and $b \in \text{dom}(\hat{p})$.
Proof: exercise, later.

This condition and the results in Chapter 4 show that Trg and Tde
have quantifier elimination.

Remark: if a theory T has q.e. Then all embeddings between models of T are elementary (that is, T is model-complete)

7.9 Definition An element $a \in U$ is definable over $A \subseteq U$
if the singleton $\{a\}$ is a definable set (i.e. there is an $L(A)$ -formula $\varphi(x)$ such that $\varphi(U) = \{a\}$).

An element $a \in U$ is algebraic over $A \subseteq U$ if there is an $L(A)$ -formula $\varphi(x)$ such that $a \in \varphi(U)$ and $|\varphi(U)| < \omega$
(such a formula is said to be algebraic)

The definable closure of A is the set

$$\text{dcl}(A) = \{a \in U : a \text{ is definable over } A\}$$

The algebraic closure of A is the set

$$\text{acl}(A) = \{a \in U : a \text{ is algebraic over } A\}$$

If $A = \text{acl}(A)$, A is said to be algebraically closed.

7.10 Proposition For $a \in U$ and $A \subseteq U$ the following are equivalent:

$$(i) a \in \text{dcl}(A)$$

$$(ii) O(a/A) = \{a\} \text{ (i.e. } a \text{ is fixed by all } \alpha \in \text{Aut}(U/A))$$

Proof Let $\varphi(x)$ define a over A . Then $\varphi(U)$ is invariant under $\text{Aut}(U/A)$, and so $O(a/A) \subseteq \{a\}$.

For the other direction, use that $O(a/A)$ is invariant over A and so by Proposition 7.5 there is an $L(A)$ -formula that defines $O(a/A)$.

7.11 Theorem Let $A \subseteq U$ and $a \in U$. The following are equivalent:

$$(i) a \in \text{acl}(A)$$

$$(ii) O(a/A) \text{ is finite}$$

$$(iii) a \in M \text{ for any model } M \text{ such that } A \subseteq M$$

Proof (i) \Rightarrow (ii) If $a \in \text{acl}(A)$ then there is $\varphi(x)$ in $L(A)$ such that $\varphi(a)$ holds and $|\varphi(U)| < \omega$.

Since $\varphi(U)$ is invariant over A , $O(a/A) \subseteq \varphi(U)$, and so $|O(a/A)| < \omega$

(ii) \Rightarrow (i) Since $O(a/A)$ is finite, it is definable (by the formula $\bigvee_{i=1}^n x = a_i$, where $\{a_1, \dots, a_n\} = O(a/A)$). But $O(a/A)$ is invariant over A , so by Proposition 7.5 there is an $L(A)$ -formula $\phi(x)$ that defines $O(a/A)$. Then $|\phi(U)| < \omega$ and $a \in \text{acl}(A)$, so $a \in \text{acl}(A)$.

(i) \Rightarrow (iii) If $a \in \text{acl}(A)$ then there is an $L(A)$ -formula $\phi(x)$ such that $\models \phi(a) \wedge \exists^{=n} x \phi(x)$. By elementarity, $\exists^{=n} x \phi(x)$ holds in any model containing A , and the n solutions in M are also the solutions in U , therefore $a \in M$.

(iii) \Rightarrow (i) Suppose $a \notin \text{acl}(A)$ and let $p(x) = \text{tp}(a/A)$. Then for all $\phi(x) \in p(x)$ we have $|\phi(U)| \geq \omega$. Then $|p(U)| \geq \omega$ (see Q6 on Example Sheet 2), and we can show $|p(U)| = |U|$ (see Example Sheet 2). If M is a model such that $A \subseteq M$, by cardinality considerations $p(U) \setminus M \neq \emptyset$. Let $b \in p(U) \setminus M$. Then by homogeneity there is $\alpha \in \text{Aut}(U/A)$ such that $\alpha(b) = a$. Then $\alpha[M]$ is a model that contains A but does not contain $a = \alpha(b)$.

7.12 Proposition

Let $a \in U$ and $A \subseteq U$. Then

- (i) if $a \in \text{acl}(A)$ then $a \in \text{acl}(A_0)$ for $A_0 \subseteq A$, A_0 finite
- (ii) $A \subseteq \text{acl}(A)$
- (iii) if $A \subseteq B$ then $\text{acl}(A) \subseteq \text{acl}(B)$
- (iv) $\text{acl}(A) = \text{acl}(\text{acl}(A))$
- (v) $\text{acl}(A) = \bigcap_{M \supseteq A} M$ (where M ranges over models containing A).

Proof (ii) if $a \in A$ then $a \in \text{acl}(A) \subseteq \text{acl}(A)$

(iv) By (ii) and (iii), $\text{acl}(A) \subseteq \text{acl}(\text{acl}(A))$. For the converse inclusion: by Theorem 7.11, $A \subseteq M \Leftrightarrow \text{acl}(A) \subseteq M$.

Moreover, $a \in \text{acl}(\text{acl}(A)) \Rightarrow a \in M$ for every $M \supseteq \text{acl}(A)$.

Therefore $a \in \text{acl}(\text{acl}(A)) \Rightarrow a \in M$ for every $M \supseteq A$, so $a \in \text{acl}(A)$.

(v) follows from Theorem 7.11.

Properties (i) to (iv) say that acl is a closure operator with finite character.

7.13 Proposition

Let $\beta \in \text{Aut}(U)$ and $A \subseteq U$. Then

$$\beta[\text{acl}(A)] = \text{acl}(\beta[A]).$$

Proof: (\subseteq) For $a \in \text{acl}(A)$ let $\phi(x, \bar{z})$ be an L -formula and $\bar{b} \in A^{|\bar{z}|}$ such that $a \in \phi(U, \bar{b})$ and $|\phi(U, \bar{b})| < \omega$. Then $\models \phi(\beta(a), \beta(\bar{b}))$ and $|\phi(U, \beta(\bar{b}))| < \omega$, so $\beta(a)$ is algebraic over $\beta(\bar{b})$.

This proves $\beta[\text{acl}(A)] \subseteq \text{acl}(\beta[A])$.

The same proof with β^{-1} in place of β and $\beta[A]$ in place of A gives the converse inclusion.

(8) STRONGLY MINIMAL THEORIES

(8.1) Definition Let M be an infinite L -structure. A subset $A \subseteq M$ is cofinite if $M \setminus A$ is finite.

(8.2) Remark Finite and cofinite sets are definable in every structure. In this chapter, we'll look at structures where these are the only definable sets.

(8.3) Definition An L -structure M is said to be minimal if all its definable subsets are finite or cofinite. M is strongly minimal if it is minimal and all its elementary extensions are minimal.

If T is a consistent theory without finite models, then T is strongly minimal if for every L -formula $\phi(x, \bar{z})$ there is $n \in \omega$ such that

$$T \vdash \forall \bar{z} [\exists^{\leq n} x \phi(x, \bar{z}) \vee \exists^{\leq n} x \neg \phi(x, \bar{z})].$$

Example: let $L = \{E\}$ where E is a binary relation, and let M be an L -structure where E is interpreted as an equivalence relation with exactly one equivalence class of size $n \in \omega \setminus \{0\}$ and no infinite classes. It can be shown that M is minimal.

However, there is an elementary extension $N \models M$ where E has an infinite class. Let a be in this infinite class. Then $E(a, x)$ defines an infinite/cofinite subset of the domain. Hence M is not strongly minimal.

From now on, T is a complete strongly minimal theory with monster model U .

(8.4) Definition Let $a \in U$, $B \subseteq U$. Then a is independent from B if $a \notin \text{acl}(B)$.

The set B is independent if every $a \in B$ is independent from $B \setminus \{a\}$.

Notation: we write aB for $\{a\} \cup B$ and $B \setminus a$ for $B \setminus \{a\}$.

(8.5) Theorem Let $B \subseteq U$, $a, b \in U \setminus \text{acl}(B)$. Then $b \in \text{acl}(aB) \Leftrightarrow a \in \text{acl}(bB)$.

Proof Assume for a contradiction that $a \in \text{acl}(bB)$ and $b \notin \text{acl}(aB)$.

Let $\phi(x, y)$ be an $L(B)$ -formula such that

$$\models \phi(a, b) \wedge \exists^{\leq n} x \phi(x, b)$$

for some $n \in \omega \setminus \{0\}$

$\in L(aB)$

Consider the formulae $\psi(a, y) \equiv \phi(a, y) \wedge \exists^{\leq n} x \phi(x, y)$.

Since $b \notin \text{acl}(aB)$, $|\psi(a, b)| \geq \omega$, so $|\psi(a, b)| = |\omega|$.

By strong minimality, $|\neg\psi(a, b)| < \omega$, so any model M containing B also contains $c \in \psi(a, b)$.

Then $a \in \text{acl}(cB)$, and so $a \in M$. So $a \in M$ whenever $M \models B$. By Theorem 7.12(v), $a \in \text{acl}(B)$. \square

Main examples of strongly minimal theories.

(1) Let K be a field. The language $L_K = \{+, -, 0, \{\lambda\}_{\lambda \in K}\}$ can be used to axiomatise vector spaces over K , where for $\lambda \in K$ the symbol λ is a unary function symbol that is interpreted as multiplication by the scalar λ . We write λv for $\lambda(v)$.

The theory T_{VK} includes the following axioms / axiom schemata:

- axioms for abelian groups (using $+, -, 0$)

- axioms for scalar product:

for each $\lambda \in K$, $\forall x y [\lambda(x+y) = \lambda x + \lambda y]$

for $\lambda_1, \lambda_2, \mu \in K$ such that $\lambda_1 \lambda_2 = \mu$ $\forall x [\lambda_1(\lambda_2 x) = \mu x]$

for $\lambda_1, \lambda_2, \nu \in K$ s.t. $\lambda_1 + \lambda_2 = \nu$ $\forall x [\lambda_1 x + \lambda_2 x = \nu x]$

- $\forall x [1x = x]$

- $\exists x [x \neq 0] \rightarrow$ it is categorical in every $\lambda > |K|$

If K is an infinite field, it can be shown (with some work!) that T_{VK} is complete and has q.e.. Then

- linear combinations $\lambda_1 x_1 + \dots + \lambda_n x_n$ are terms

- atomic formulas are equalities between linear combinations

- atomic formulas with one variable and parameters are equivalent to formulas of the form $\lambda x = a$. Therefore such formulas define singletons.

- therefore, quantifier-free formulas with one variable and parameters define finite or cofinite sets.

By q.e., T_{VK} is strongly minimal. Moreover one can show

- $\text{acl}(A) = \langle A \rangle$, the linear span

- $a \notin \text{acl}(A) \Leftrightarrow a$ is linearly independent from A

- A is independent \Leftrightarrow it is linearly independent

(If K is finite, there is a theory T_{VK}^∞ of infinite dimensional v. spaces)

(2) Let $L_{\text{ring}} = \{+, \cdot, -, 0, 1\}$. Then ACF is the L_{ring} -Theory that includes:

- axioms for abelian groups (using $+, -, 0$)

- axioms for commutative monoids (using $\cdot, 1$)

- field axioms:

$\forall x y z [x \cdot (y + z) = x \cdot y + x \cdot z]$

$\forall x [x = 0 \vee \exists y (x \cdot y) = 1]$

- axioms for algebraic closure: for all $n < \omega$

$\forall x_0 \dots x_n \exists y [x_0 + x_1 y + \dots + x_n y^n = 0]$.

Let $X_p = \underbrace{1+1+\dots+1}_{p \text{ times}} = 0$. Then $\text{ACF}_p = \text{ACF} \cup \{X_p\}$ (p prime)

and $\text{ACF}_0 = \text{ACF} \cup \{X_n : n < \omega\}$

ACF_0 and ACF_p for a given p are both complete and have q.e. (showing this requires some work). Then:

- atomic formulas are polynomial equations;
- if $A \subseteq M \models \text{ACF}_p$ or ACF_0 , an atomic formula in $\text{Lang}(A)$ with one free variable is equivalent to a polynomial equation $p(x) = 0$, where $p(x) \in F[x]$ and F is the subfield of K generated by A ;
- therefore, atomic formulas with parameters and one free variable define finite sets.
- quantifier-free formulas with parameters and one free variable define finite or infinite sets
- by q.e., ACF_0 and ACF_p are strongly minimal.

(8.6) Definition Let $B \subseteq C \subseteq U$. Then B is a basis of C if B is an independent set and $C \subseteq \text{acl}(B)$, or, equivalently, if $\text{acl}(B) = \text{acl}(C)$.

(8.7) Lemma If B is an independent set and $a \notin \text{acl}(B)$, then $\{a\} \cup B$ is an independent set.

Proof: Suppose $a \notin \text{acl}(B)$ and that $\{a\} \cup B$ is not independent. Then there is $b \in B$ such that $b \in \text{acl}(a, B \setminus b)$. Since B is independent, $b \notin \text{acl}(B \setminus b)$. Also, $a \notin \text{acl}(B \setminus b)$. So by Theorem 8.5 we have $a \in \text{acl}(b, B \setminus b) = \text{acl}(B)$. \therefore

(8.8) Corollary If $B \subseteq C \subseteq U$, the following are equivalent:

- B is a basis of C
- B is a maximal independent subset of C (i.e. if $B \subseteq B' \subseteq C$ and B' is independent, then $B = B'$).

(8.9) Theorem Let $C \subseteq U$. Then

- every independent subset $B \subseteq C$ can be extended to a basis of C
- if A, B are bases of C , then $|A| = |B|$

Proof (i) If $\langle B_i : i < \lambda \rangle$ is a chain of independent subsets of C such that for all i $B \subseteq B_i$, then $\bigcup_{i < \lambda} B_i$ is independent.

If not, let $b \in \bigcup_{i < \lambda} B_i$ be such that $b \in \text{acl}(\bigcup_{i < \lambda} B_i \setminus b)$. Then there is finite $A \subseteq B_i \setminus b$ for some i such that $b \in \text{acl}(A \setminus b)$. But this implies $b \in \text{acl}(B_i \setminus b)$ for some i . \times

Therefore, by Zorn's Lemma, there is a maximal independent subset of C . By Corollary 8.8, this maximal subset is the required basis.

(ii) Suppose A and B are bases of C and B is infinite. Assume for a contradiction that $|A| < |B|$. For $a \in A$, let $D_a \subseteq B$ be finite and such that $a \notin \text{acl}(D_a)$. Let $D = \bigcup_{a \in A} D_a$. Then $A \subseteq \text{acl}(D_a)$, and $|D| < |B|$.

Since $A \subseteq \text{acl}(D)$ and A is a basis, $C \subseteq \text{acl}(D)$. But then $B \subseteq \text{acl}(D)$, which contradicts the independence of B .

If A and B are finite, show that $|A| \leq |B|$ by using the following result: if there is $a \in A \setminus B$, then there is $b \in B \setminus A$ such that the set $\{b\} \cup (A \setminus \{a\})$ is independent. This is because if $a \in A \setminus B$, then $a \in \text{acl}(B)$, as B is a basis. Then $B \not\subseteq \text{acl}(A \setminus \{a\})$, otherwise A is not independent. Let $b \in B \setminus \text{acl}(A \setminus \{a\})$. Then $\{b\} \cup A \setminus \{a\}$ is independent by Lemma 8.7.

8.10 Definition Let $C \subseteq U$ be algebraically closed. Then the dimension of C , $\dim(C)$, is $|A|$ for any basis A of C .

8.11 Proposition Let $f: U \rightarrow U$ be a partial elementary map.

Let $b \notin \text{acl}(\text{dom}(f))$, $c \notin \text{acl}(\text{ran}(f))$. Then $f \cup \{b, c\}$ is elementary.

Proof Let $\bar{\alpha}$ be an enumeration of $\text{dom}(f)$, and let $\varphi(x, \bar{\alpha})$ be an $L(\bar{\alpha})$ -formula.

Claim: $\models \varphi(b, \bar{\alpha}) \leftrightarrow \varphi(c, f(\bar{\alpha}))$.

Case 1: $|\varphi(U, \bar{\alpha})| < \omega$. Then $|\varphi(U, f(\bar{\alpha}))| < \omega$ by elementarity of f . Since $b \notin \text{acl}(\bar{\alpha})$ and $c \notin \text{acl}(f(\bar{\alpha}))$, we have

$$\vdash \neg \varphi(b, \bar{\alpha}) \wedge \neg \varphi(c, f(\bar{\alpha}))$$

Case 2: $|\varphi(U, \bar{\alpha})| \geq \omega$. By strong minimality, $|\varphi(U, \bar{\alpha})| < \omega$ and the argument in Case 1 applies to give

$$\models \varphi(b, \bar{\alpha}) \wedge \varphi(c, f(\bar{\alpha})).$$

8.12 Corollary Every bijection between independent subsets of U is elementary.

Proof: Let $A, B \subseteq U$ be independent and let $f: A \rightarrow B$ be a bijection. Let $\bar{\alpha}$ enumerate A and let $\bar{b} = f(\bar{\alpha})$. Then $a_0 \notin \text{acl}(\emptyset)$ and $b_0 \notin \text{acl}(\emptyset)$ (otherwise neither A nor B is independent). By Proposition 8.11, $\{a_0, b_0\}$ is an elementary map. The step $i+1$ is similar, since $a_{i+1} \notin \text{acl}(a_0, \dots, a_i)$, and similarly for b_{i+1} .

8.13 Remark If $M \subseteq U$ is a model, then by Proposition 7.12(v) M is algebraically closed.

8.14 Theorem Suppose $M, N \subseteq U$ are such that $\dim(M) = \dim(N)$. Then $M \sim N$.

Proof: Let A, B be bases of M and N respectively.

Let $f: A \rightarrow B$ be a bijection. By Corollary 8.12, f is elementary.

By homogeneity, there is $\alpha \in \text{Aut}(U)$ that extends f . Then

$$\alpha(M) = \alpha(\text{acl}(A)) = \text{acl}(\alpha(A)) = \text{acl}(B) = N$$

↪ by Proposition 7.13.

(8.15) Corollary Let T be strongly minimal and let $\lambda > |L|$.

Then T is λ -categorical.

Proof: if $A \subseteq U$, then $|\text{acl}(A)| \leq |L(A)|$ because:

- There are at most $|L(A)|$ formulas
- There are finitely many elements in $\text{acl}(A)$ that are solutions of a given formula in $L(A)$.

If $|M| = \lambda > |L|$, then a basis of M must have cardinality λ .

By Theorem 8.14, any two models of cardinality λ are isomorphic.

Models of a strongly minimal theory are determined up to isomorphism by their dimension, and hence by their cardinality if the cardinality is greater than the cardinality of the language.

(8.16) Theorem Let N be a model, $|N| \geq |L|$. The following are equivalent:

- (i) N is saturated
- (ii) $\dim(N) = |N|$.

Proof (i) \Rightarrow (ii) Let B be a basis of N , so $N = \text{acl}(B)$, and let $p(x)$ be the type in $L(B)$ defined by

$$p(x) = \{\neg\phi(x) \text{ in } L(B) : \phi(x) \text{ is algebraic}\}$$

Then $p(x)$ is closed under conjunction. Let $\neg\phi(x) \in p(x)$. By strong minimality, $|\phi(U)| < \omega$, so N must contain a realization of $\neg\phi(x)$. Hence $p(x)$ is finitely satisfiable.

But $p(x)$ is not realized in N , since no realization of $p(x)$ is algebraic over B .

By saturation of N , we must then have $|B| = |N|$.

(ii) \Rightarrow (i) Let $f: M \rightarrow N$ be elementary with $|f| < |N|$. Let $b \in M$. We want to find $\hat{f} \supseteq f$ that is elementary and such that $b \in \text{dom}(\hat{f})$. By Theorem 6.5, this proves that N is saturated.

If $b \in \text{acl}(\text{dom}(f))$, let $\alpha \in \text{Aut}(U)$ extend f . Then

$$\alpha(\text{acl}(\text{dom}(f))) = \text{acl}(\alpha(\text{dom}(f))) = \text{acl}(\text{ran}(f))$$

by Proposition 7.13. Then $f \cup \{\langle b, \alpha(b) \rangle\}$ is the required extension.

If $b \notin \text{acl}(\text{dom}(f))$, then let $c \in N \setminus \text{acl}(\text{ran}(f))$, which is nonempty as $|f| < |N| = \dim(N)$. By Proposition 8.11, $f \cup \{\langle b, c \rangle\}$ is elementary.

9 COUNTABLE MODELS

Let T be a complete L -theory with a monster model \mathcal{U} , and $|\bar{x}| < \omega$.

9.1 Definition An L -formula $\varphi(\bar{x})$ is said to isolate a type $p(\bar{x})$ in L if : (i) $\varphi(\bar{x})$ is consistent
(ii) $\varphi(\bar{x}) \rightarrow p(\bar{x})$ (that is, $\bar{a} \models p(\bar{x})$ whenever $\models \varphi(\bar{a})$).

A set Δ of L -formulas isolates $p(\bar{x})$ if there is $\varphi(\bar{x}) \in \Delta$ which isolates $p(\bar{x})$. When Δ is the set of $L(A)$ formulas with free variables in \bar{x} , we say that the set A isolates $p(\bar{x})$ or, if A is clear from the context, that $p(\bar{x})$ is isolated.

Isolated types are sometimes called principal

A model M is said to omit the type $p(\bar{x})$ if $p(\bar{x})$ is not realized in M .

9.2 Remark If M is a model and $p(\bar{x})$ is a type in $L(M)$, then M realizes $p(\bar{x}) \Leftrightarrow M$ isolates $p(\bar{x})$.

(\Leftarrow) Let $\varphi(\bar{x})$ be a consistent $L(M)$ -formula that isolates $p(\bar{x})$.

Then $M \models \varphi(\bar{a})$ for some \bar{a} , and \bar{a} is a realization of $p(\bar{x})$.

(\Rightarrow) Let $\bar{a} \models p(\bar{x})$. Then $p(\bar{x})$ is isolated by $\bar{x} = \bar{a}$.

9.3 Lemma Let $|L(A)| = \omega$. Let $p(\bar{x})$ be a type in $L(A)$ and suppose that A does not isolate $p(\bar{x})$. Let $\psi(z)$ be a consistent $L(A)$ -formula with one free variable.

Then there is $a \in \mathcal{U}$ such that

- (i) $\psi(a)$
- (ii) $A \cup \{a\}$ does not isolate $p(\bar{x})$.

Proof We build a sequence $\langle \psi_i(\bar{z}) : i < \omega \rangle$ of $L(A)$ -formulas such that :

- (i) $\psi_i(\bar{z})$ is consistent
- (ii) $\psi_{i+1}(\bar{z}) \rightarrow \psi_i(\bar{z})$
- (iii) a realization of the type $\{\psi_i(\bar{z}) : i < \omega\}$ is the required solution of $\psi(z)$.

Let $\langle \chi_i(\bar{x}, z) : i < \omega \rangle$ be an enumeration of formulas with free variables in $\bar{x} \cup \{z\}$ and parameters in A , and let

$$\psi_0(z) = \psi(z).$$

At stage $i+1$:

- if $\chi_i(\bar{x}, z)$ is inconsistent, let $\psi_{i+1}(x) = \psi_i(x)$

- otherwise let $\varphi(\bar{x}) \in p(\bar{x})$ be such that

$$\psi_i(z) \wedge \exists \bar{x} (\chi_i(\bar{x}, z) \wedge \neg \varphi(\bar{x}))$$

is consistent. Let $\psi_{i+1}(z)$ be this conjunction. Then at stage $i+1$

we have guaranteed that a realization of ψ_{ini} does not violate the type. We must show that $\varphi(\bar{x}) \in p(\bar{x})$ can always be found.

Suppose no $\varphi(\bar{x}) \in p(\bar{x})$ makes $\psi_{\text{ini}}(\bar{z})$ consistent. Then

$$\chi_i(\bar{x}, \bar{z}) \wedge \psi_i(\bar{z}) \rightarrow \varphi(\bar{x})$$

for all $\varphi(\bar{x}) \in p(\bar{x})$. But this implies that

$$\exists \bar{z} [\chi_i(\bar{x}, \bar{z}) \wedge \psi_i(\bar{z})] \rightarrow p(\bar{x})$$

that is, $p(\bar{x})$ is isolated by an L(A)-formula with free variables in \bar{x} .

9.4 Theorem (Omitting types) Let L(A) be countable and let $p(\bar{x})$ be a consistent type in L(A). Then the following are equivalent:

- (i) all models containing A realize $p(\bar{x})$
- (ii) A isolates $p(\bar{x})$.

Proof (ii) \Rightarrow (i) Let $\varphi(\bar{x})$ be an L(A)-formula that violates $p(\bar{x})$.

Then $\models \exists \bar{z} \varphi(\bar{z})$, so if M is a model containing A then there is $\bar{a} \in M^{|\bar{x}|}$ such that $M \models \varphi(\bar{a})$, and \bar{a} is a realization of $p(\bar{x})$.

(i) \Rightarrow (ii) We argue by contraposition. Suppose that A does not isolate $p(\bar{x})$. We build a chain $\langle A_i : i < \omega \rangle$ of sets such that

- (i) $A_0 = A$
- (ii) $|A_i| \leq \omega$ for all i
- (iii) A_i does not isolate $p(\bar{x})$.

We eventually take $M = \bigcup_{i \in \omega} A_i$ and show that M is a model that omits $p(\bar{x})$.

At stage $i+1$, enumerate consistent $L(A_i)$ -formulas with one free variable, say $\langle \psi_k(z) : k < \omega \rangle$. For $k \in \omega$, find $a_k^i \in U$ such that $A_i \cup \{a_0^i, \dots, a_k^i\}$ does not isolate $p(\bar{x})$.

This is possible by Lemma 9.3. We let

$$A_{i+1} := A_i \cup \{a_k^i : k < \omega\}.$$

Let $M = \bigcup_{i \in \omega} A_i$. Claim: (i) $M \leq U$ (ii) M omits $p(\bar{x})$.

For (i), we use the Tarski-Vaught Test: let $\psi(\bar{z})$ be a consistent $L(M)$ -formula. Then $\psi(\bar{z})$ is an $L(A_i)$ -formula for some i , and so it is realized at stage $i+1$.

By construction, M does not isolate $p(\bar{x})$. By Remark 9.2, M does not realize $p(\bar{x})$.

Remark: the hypothesis that $|L(A)| = \omega$ is necessary. There are counterexamples to the OTT where a model M is the disjoint union of two uncountable sets and the set of all bijections between these two sets.

9.5 Corollary Let T be an ω -categorical theory. Then every type $p(\bar{x})$ (over \emptyset) is isolated.

(9.5) Definition Let M be a model and $A \subseteq M$. Then

(i) M is prime over A if for every $N \supseteq A$ there is an elementary embedding $f: M \rightarrow N$ that fixes A pointwise. When $A = \emptyset$, M is said to be prime.

(ii) M is atomic over A if for all n and all $\bar{a} \in M^n$, the type $\text{tp}(\bar{a}/A)$ is isolated. When $A = \emptyset$, M is said to be atomic.

(9.6) Fact Let $\bar{a}, \bar{b} \in U^n$, $A \subseteq U$, and suppose that $\text{tp}(\bar{b}\bar{a}/A)$ is isolated. Then $\text{tp}(\bar{b}/A\bar{a})$ and $\text{tp}(\bar{a}/A)$ are isolated.

Proof Let $p(x, \bar{z}) = \text{tp}(\bar{b}\bar{a}/A)$. Then $\text{tp}(\bar{b}/A\bar{a}) = p(\bar{x}, \bar{a})$, and $\text{tp}(\bar{a}/A) = \{ \exists \bar{x} \varphi(x, \bar{z}) : \varphi(\bar{x}, \bar{z}) \text{ is a conjunction of formulas in } p(\bar{x}, \bar{z}) \} = \{ \exists \bar{x} \varphi(x, \bar{z}) : \varphi(\bar{x}, \bar{z}) \in p(\bar{x}, \bar{z}) \}$ (check)

Suppose $\varphi(x, \bar{z})$ isolates $p(\bar{x}, \bar{z})$. Then

(i) $\varphi(\bar{x}, \bar{a})$ isolates $\text{tp}(\bar{b}/A\bar{a})$

(ii) $\exists \bar{x} \varphi(x, \bar{z})$ isolates $\text{tp}(\bar{a}/A)$.

(9.7) Proposition If M is atomic over A , then for every finite tuple $\bar{a} \in M^n$, M is atomic over $A\bar{a}$.

Proof Let \bar{b} be a finite tuple. Then $\text{tp}(\bar{b}\bar{a}/A)$ is isolated, and so $\text{tp}(\bar{b}/A\bar{a})$ is isolated by Fact 9.6.

(9.8) Proposition Let $f: M \rightarrow N$ be an elementary map, and let M be atomic over $\text{dom}(f)$. Then for every $b \in M$ there is $c \in N$ such that $f \cup \{ \langle b, c \rangle \}$ is elementary.

Proof Let \bar{a} enumerate $\text{dom}(f)$ and let $p(x, \bar{z}) = \text{tp}(b\bar{a}/\emptyset)$.

Let $\varphi(x, \bar{a})$ isolate $p(x, \bar{a})$. By elementarity, $\varphi(x, f(\bar{a}))$ isolates $p(x, f(\bar{a}))$. Then any solution of $\varphi(x, f(\bar{a}))$ is the required c .

(9.9) Proposition Any two countable models that are atomic over A are isomorphic.

Proof By a back-and-forth argument using Propositions 9.7 and 9.8.

(9.10) Theorem Assume that $L(A)$ is countable. Then for any model $M \supseteq A$ the following are equivalent:

(i) M is countable and atomic over A .

(ii) M is prime over A .

Proof (i) \Rightarrow (ii) Let N be a model containing A . Then $\text{id}_A: M \rightarrow N$ is elementary, and it can be extended to an embedding $g: M \rightarrow N$ by Propositions 9.8 and 9.9.

(ii) \Rightarrow (i) By Downward Löwenheim-Skolem, we know there is a countable model containing A . Since M embeds in this model, M is itself countable. Suppose that there is $\bar{b} \in M^n$ such that $\text{tp}(\bar{b}/A)$ is not isolated. By Theorem 9.4 (Omitting Types), there is a countable model N that contains A and omits $\text{tp}(\bar{b}/A)$. But then

M does not embed in $N \cdot X$.

(9.11) Definition For new and $A \subseteq U$, $S_n(A)$ denotes the collection of complete, consistent types with n variables and parameters in A . When $A = \emptyset$, we write $S_n(T)$ for complete consistent types with no parameters and n free variables.

(9.12) Theorem (Ryll-Mordzinski, Engeler, Svenonius) For a countable theory T , the following are equivalent:

- (i) T is ω -categorical
- (ii) for all n , every type $p(x) \in S_n(T)$ is isolated
- (iii) for all n , $|S_n(T)| < \omega$
- (iv) $\text{Aut}(U)$ has finitely many orbits on U^n for all $n \in \omega$.

Proof (Sketch) (i) \Rightarrow (ii) Suppose $p(x) \in S_n(T)$ is not isolated. Then there is a countable model M that realizes $p(x)$ (by Downward Löwenheim-Skolem) and there is a countable model N that omits $p(x)$ (by Omitting Types Theorem). Then $M \not\cong N$.

(ii) \Rightarrow (i) If all types are isolated, then all countable models are atomic. By Proposition 9.9, all countable models are isomorphic.

(ii) \Rightarrow (iii) We have $U^n = \bigcup p(U)$. If all types in $S_n(T)$ are isolated, then $U^n = \bigcup_{p \in S_n(T)} q_p(U)$, where $q_p(x)$ isolates $p(x)$.

By compactness

$$U^n = \bigcup_{i=1}^k q_{p_i}(x) \quad (\text{check this!})$$

for some $k \in \omega$, hence $|S_n(T)| \leq \omega$.

(iii) \Rightarrow (ii) Let $p(x) \in S_n(T)$. If $|S_n(T)| < \omega$, then $U \setminus p(U)$ is the union of finitely many type-definable sets, hence it is type-definable. By an argument similar to that in Fact 7.2, there are formulas $\phi(x)$ and $\chi(x)$ such that $\phi(U) = p(U)$ and $\chi(x) \supset U \setminus p(U)$. It follows that $\phi(x)$ isolates $p(x)$.

(iii) \Leftrightarrow (iv) This follows from the fact that for $\bar{a}, \bar{b} \in U^n$
 $t_p(\bar{a}) = t_p(\bar{b}) \Leftrightarrow O(\bar{a}/\phi) = O(\bar{b}/\phi)$.

Remark We can prove that if M is a countable model of an ω -categorical theory, then $\text{Aut}(M)$ is oligomorphic (i.e. it has finitely many orbits on M^n for all $n \in \omega$). The converse also holds.

(9.13) Definition A complete theory T is small if for all $n \in \omega$
 $|S_n(T)| \leq \omega$.

9.14 Proposition If T is countable, complete and has no finite models, the following are equivalent:

- (i) T is small
- (ii) For all $n < \omega$ and finite $A \subseteq U$ we have $|S_n(A)| \leq \omega$
- (iii) For all finite $A \subseteq U$, $|S_1(A)| \leq \omega$
- (iv) T has a countable saturated model.

Proof (sketch) (i) \Rightarrow (ii) Let A be finite and let \bar{a} enumerate A .

Let $q(\bar{q}) = \text{tp}(\bar{x})$, and

$$X = \{p(\bar{x}, \bar{a}) \in S_{n+m}(T) : q(\bar{q}) \subseteq p(\bar{x}, \bar{a})\},$$

where $|\bar{q}| = |\bar{a}| = m$ and $|\bar{x}| = n$. Then X is countable, and the map $p(\bar{x}, \bar{q}) \mapsto p(\bar{x}, \bar{a})$ from $S_{n+m}(T)$ to $S_n(A)$ is surjective. Hence $|S_n(A)| \leq |X| \leq \omega$

(ii) \Rightarrow (iii) clear

(iii) \Rightarrow (iv) Build a chain $\langle A_n : n < \omega \rangle$ where each A_i is such that all types over $A \subseteq A_{i+1}$, $|A| \leq \omega$, are realized.

Since $|S_1(A)| \leq \omega$ for all finite A , each A_i can be chosen countable. Then $M = \bigcup_{i \in \omega} A_i$ is the required countable saturated model.

(iv) \Rightarrow (i) Since all $p(\bar{x}) \in S_n(T)$ are realized in the countable saturated model, there are at most countably many realizations.

9.15 Corollary Countable models of ω -categorical theories are saturated.