If M is a structure and $a = \langle a_i : i < |M| \rangle$, $q(x) = tp(a) = \{\phi(x) \in L : M \models \phi(a)\}$
Then $q(x)$ is satisfiable in N iff then there is $\beta : M \to N$ elem embedding.

(3.6) <u>Theorem</u> If M is an L-structure and $p(x) \subseteq L(M)$ is fin. sat in M, then $p(x)$ is realized in some $N \succeq M$.

Example: $\overset{M}{\simeq}\langle (0,1), < \rangle \models T_{dlo}$. Let $a_n = 1 - \frac{1}{n}$ for $n \in \omega \setminus \{0\}$.
Let $q(x) = \{x > a_n : n \in \omega \setminus \{0\}\}$ is a type that finitely sat in M, but not satisfiable. However, $q(x)$ is realized in $\langle \mathbb{Q}, < \rangle$ and we know $(0,1) \preceq \mathbb{Q}$.

<u>Proof</u>: Enumerate M, say via $a = \langle a_i : i < |M| \rangle$, and let
$$q(z) = \{\phi(z) : M \models \phi(a)\} \subseteq L$$
where $|z| = |M| = |a|$ and the z_i are new variables, so in part $z \cap x = \emptyset$.
Write $p(x)$ as $p'(x, a)$ where $p'(x, z) \subseteq L$.
Then $p'(x, z) \cup q(z)$ is fin sat in M (because $p'(x, a)$ is, and $q(z)$ is realized by a). By compactness (Thm 3.2), there are N, and $c \in N^{|x|}$, $d \in N^{|z|}$ such that $N \models p'(c, d) \cup q(d)$.
In part. $N \models q(d)$ so by Lemma 3.4 $\beta : a_i \mapsto d_i$ is an elem embedding of M into N. Wma $M \preceq N$.

(3.7) <u>Theorem</u> (Upward Löwenheim Skolem)
Let M be an infinite L-structure. Then for any $\lambda \geq |M| + |L|$, there is $N \succeq M$ such that $|N| = \lambda$.
Proof: Let $x = \langle x_i : i < \lambda \rangle$ be distinct variables.
Consider $p(x) = \{x_i \neq x_j \text{ for } i < j < \lambda\}$
Clearly $p(x)$ is fin. sat. in M, so by Thm 3.6 $p(x)$ is realized in some $N \succeq M$, and $|N| \geq \lambda$. By DLS we may assume $|N| = \lambda$.

(4) SATURATION

(4.1) Definition Let λ be an infinite cardinal, M an L-structure.
Then M is <u>λ-saturated</u> if it realizes every type $p(x) \subseteq L(A)$, where $|x| = 1$ and $A \subseteq M$ such that
(i) $p(x)$ is finitely satisfiable in M
(ii) $|A| < \lambda$.
M is <u>saturated</u> if it is $|M|$-saturated.

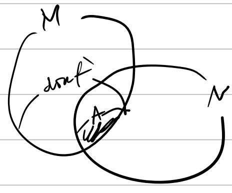
Remark: If $\lambda > |M|$, M cannot be λ-saturated.
Consider $p(x) = \{ x \neq a : a \in M \}$. Then $p(x)$ is fin sat in M, but not satisfiable - here the parameter set is M itself - too large!

(4.2) Definition Let M be an L-str, $A \subseteq M$, $b \in M^{|b|}$. The <u>type of b over A</u> is
$$tp_M(b/A) = \{ \varphi(x) \in L(A) : M \models \varphi(b) \}$$

M is omitted when clear from the context

(4.3) Remarks (i) $tp(b/A)$ is complete, that is, for all $\varphi(x) \in L(A)$ either $\varphi(x) \in tp(b/A)$ or $\neg\varphi(x) \in tp(b/A)$
(ii) If $A \subseteq M \preceq N$ and $b \in M^{|b|}$ then
$$tp_M(b/A) = tp_N(b/A)$$

(4.4) Facts (types and elem maps)
Let $f : dom(f) \subseteq M \to N$ be elem map. Then:
(i) $M \equiv N$ (and if $M \equiv N$ the empty map $\emptyset$ is elementary as it preserves sentences)
(ii) If a enumerates $dom(f)$, then
$$tp(a/\emptyset) = tp(f(a)/\emptyset)$$



More generally, if $A \subseteq dom(f) \cap N$ and $f{\restriction_A} = id$ then for tuples $b \in dom(f)$
$$tp(b/A) = tp(f(b)/A)$$

(iii) If a enumerates $dom(f)$ and $p(x,a) \subseteq L(A)$ is finitely sat in M then $p(x, f(a))$ is fin. sat in N.
Suppose $\{ \varphi_1(x,a), \dots, \varphi_n(x,a) \} \subseteq p(x,a)$. Then
$$M \models \exists x \bigwedge_{i=1}^{n} \varphi_i(x,a)$$
Therefore $$N \models \exists x \bigwedge_{i=1}^{n} \varphi_i(x, f(a))$$

<u>Warning</u>: $p(x,a)$ satisfiable in M does not imply $p(x, f(a))$ sat. in N.
$$\Downarrow$$
$p(x, f(a))$ fin. sat in N.

(4.5) Theorem Let N be such that $|L| \leq \lambda \leq |N|$. Tfae

(i) N is λ-saturated

(ii) If $f : M \to N$ is an elem map s.t. $|f| < \lambda$ and $b \in M$, then there is $\hat{f} \supseteq f$ which is elem and such that $b \in \mathrm{dom}(\hat{f})$

(iii) If $p(\bar{z})$ is a type in $L(A)$ where $A \subseteq N$ and $|A| < \lambda$, $|\bar{z}| \leq \lambda$ and $p(\bar{z})$ is fin. sat. in N, then $p(\bar{z})$ is realized in N.

Proof : (i) $\Rightarrow$ (ii) Let f be as in (ii), let $b \in M$.

Let $a = \langle a_i : i < |\mathrm{dom}(f)| \rangle$ enumerate $\mathrm{dom}(f)$, let

$$p(x, a) = tp_M(b/a). \quad (|x| = 1,\ |a| = |\mathrm{dom}(f)| < \lambda)$$

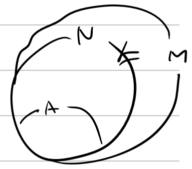
Since $p(x,a)$ is satisfiable in M, $p(x, f(a))$ is fin. sat in N.

But N is λ-saturated and $|f(a)| < \lambda$. Therefore $p(x, f(a))$ is realized in N, say $N \models p(c, f(a))$. Then $\hat{f} = f \cup \{\langle b, c\rangle\}$ is the required extension of f. Let $\varphi(x, a) \in L(a)$

$M \models \varphi(b,a) \Leftrightarrow \varphi(x,a) \in p(x,a) \Leftrightarrow \varphi(x, f(a)) \in p(x, f(a)) \Leftrightarrow N \models \varphi(c, a)$.

(ii) $\Rightarrow$ (iii) Let $p(\bar{z})$ be as in (iii). By Theorem 3.6, $p(\bar{z})$ is realized in some $M \succeq N$ by a tuple a, where $|a| = |\bar{z}| \leq \lambda$.

Since $A \subseteq N \preceq M$, the map $id_{\restriction A} : M \to N$ is elementary



We extend $id_{\restriction A}$ to an elem map $f : M \to N$ such that $a \subseteq \mathrm{dom}(f)$. Let $f_0 = id_{\restriction A}$

At stage $i+1$, we use (ii) to include a_i in $\mathrm{dom}(f_{i+1})$

At limit stages $\mu < \lambda$, let $f_\mu = \bigcup_{i<\mu} f_i$

Then $\bigcup_{i<|a|} f_i$ is the required elementary map.

(iii) $\Rightarrow$ (i) trivial

(4.6) Corollary If M, N are saturated, $M \equiv N$ and $|M| = |N|$ then any elementary $f : M \to N$, $|f| < |M|$, extends to an isomorphism from M to N. In particular, two saturated models of the same complete theory that have the same cardinality are isomorphic.

Proof Use Theorem 6.5 (ii) to extend f to an isom using a back and forth construction. This can be done with $f = \emptyset$

(4.7) Corollary Models of T_{dlo} and T_{rg} are ω-saturated. Proof By the extension lemmas (Lemmas 2.4 and 2.15) and Prop 2.11 and Corollary 2.17.

So $(\mathbb{Q}, <)$ is saturated and $(\mathbb{R}, <)$ is ω-saturated but not saturated: the type $\{x > q : q \in \mathbb{Q}\}$ is fin. satisfiable in $\mathbb{R}$, but it is not realized.

Recall that an isomorphism $\alpha : N \to N$ is called an automorphism of N

(4.8) Notation The autom of N form a gp denoted by $\mathrm{Aut}(N)$

If $A \subseteq N$ then

$$\mathrm{Aut}(N/A) = \{\alpha \in \mathrm{Aut}(N) : \alpha_{\restriction A} = id_A\}$$

$\hookrightarrow$ α fixes A setwise

4.9 <u>Definition</u> An L-structure N is

(i) <u>λ-universal</u> if for every $M \equiv N$ such that $|M| \leq \lambda$ there is an $\beta : M \to N$ elem embedding

N is <u>universal</u> if it is $|N|$-universal

(ii) λ-homogeneous if every $f : N \to N$ elementary and such that $|f| < \lambda$ extends to an automorphism of N

N is <u>homogeneous</u> if it is $|N|$ homogeneous.

<u>Warning</u>: In some texts (books / papers)

- λ-universality is called λ^+-universality
- λ-homogeneity as defined in (ii) is called <u>strong homogeneity</u> (as opposed to what we'll call "weak homogeneity", which is similar to cond (ii) in Thm 4.5