The second examples class will take place on Thursday 27 February. An update to this sheet will be posted by Monday 17 February.

If you spot any typos or problems, please email `silvia.barbina@open.ac.uk`.

Question 1. Let N be a countable random graph, that is, $N \models T_{\mathrm{rg}}$. If x is a vertex in N, a vertex y is a *neighbour* of x if $R(x, y)$. Suppose that N' is obtained from N by erasing finitely many vertices from N and all their neighbours. Is N' a random graph?

Question 2. Let N_1 and N_2 be two countable random graphs. Let N be the *free amalgamation* of N_1 and N_2, that is, the graph with domain $N = N_1 \sqcup N_2$ and let $R^N = R^{N_1} \sqcup R^{N_2}$, where $\sqcup$ denotes the disjoint union.

(i) Is N a random graph?

(ii) Find an L_{gph}-formula $\psi(x, y) \in L$ which is true in N if either both x and y belong to N_1, or both belong to N_2.

(iii) Let $M \equiv N$. Is M the free amalgamation of two random graphs?

Question 3. Let $L = \{<, E\}$, where $<$ and E are binary relations. Let T_0 be the theory that says that $<$ is a strict linear order and that E is an equivalence relation. Find a theory $T_1 \supseteq T_0$ such that every $N \models T_1$ has the following property: for every $M \models T_0$, every $b \in M$, every finite partial embedding $p : M \to N$ has an extension to a partial embedding defined in b.

Question 4. Prove that the theory T_1 in Exercise 3 is ω-categorical.

Question 5. Let N be a saturated L-structure, and let $p(x)$ be a type (with one free variable) in $L(A)$, where $A \subset N$ and $|A| < |N|$. Let

$$p(N) = \{a \in N : N \models p(a)\}.$$

Are the following conditions equivalent?

(i) $p(N)$ is infinite;

(ii) $|p(N)| = |N|$.

Are (i) and (ii) equivalent for a type $q(\bar{x})$, where $|\bar{x}|$ is possibly infinite?

Question 6. Let N be saturated and let $p(x)$ be a type with parameters in $A \subseteq N$ such that $|A| < |N|$. Suppose that $p(x)$ is closed under conjunction (that is, if $\varphi(x), \psi(x) \in p(x)$, then $\varphi(x) \wedge \psi(x) \in p(x)$). Prove that the following are equivalent:

(i) $p(x)$ has finitely many realizations;

(ii) $p(x)$ contains a formula with finitely many realizations.

Question 7. Let $N \models T_{\mathrm{lo}}$ be saturated. Prove that the following are equivalent:

(i) there is a sequence $\langle a_i : i < \omega \rangle$ such that $a_i < a_{i+1}$ for every $i < \omega$;

(ii) there is a sequence $\langle a_i : i < \omega \rangle$ such that $a_i > a_{i+1}$ for every $i < \omega$.