MODEL THEORY

1 PRELIMINARIES AND REVIEW

1.1 Definition A language L consists of
(i) a set $\mathcal{F}$ of function symbols, and for each $f \in \mathcal{F}$ a positive integer n_f
(ii) a set $\mathcal{R}$ of relation symbols, and for each $R \in \mathcal{R}$ a positive integer n_R
(iii) a set $\mathcal{C}$ of constant symbols.
The numbers n_f, n_R are the arities of the fn symbol f and relation symbol n_R, respectively.

Any of the sets $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{C}$ may be empty.

Examples: (a) $L_{gp} = \{\{\cdot, {}^{-1}\}, \{1\}\}$ constant
unary function
binary function

(b) $L_{lo} = \{<\}$
binary relation

1.2 Definition Let L be a language. Then an L-structure consist of:
(i) a set M, the domain $(M \neq \emptyset)$
(ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \to M$
(iii) for each $R \in \mathcal{R}$, a relation $R^M \in M^{n_R}$
(iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.
f^M, R^M and c^M are called the interpretations of f, M, c respectively.

1.3 Notation and conventions We omit the distinction between:
- a structure and its domain
- a symbol f and its interpretation in a structure M when M is understood.

We write $\langle M, \{f_i\}_{i \in I}, \{R_j\}_{j \in J}, \{c_k\}_{k \in K} \rangle$ for a structure in the language $\{\{f_i\}, \{R_j\}, \{c_k\}\}$.

1.4 Examples (a) $\langle \mathbb{R}^+, \{\cdot, {}^{-1}\}, 1 \rangle$ is an L_{gp}-structure
+ve real numbers

(b) $\langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is an L_{gp}-structure
(c) $\langle \mathbb{Q}, \{<\} \rangle$ is an L_{lo}-structure

Note that the structures in (a) and (b) are groups, but an L_{gp}-structure needn't be a group.

Given a language L, we assume familiarity with the first-order logic of L. We use the symbols of L and the following logical symbols:
- connectives $\wedge$, $\neg$ (also $\vee$, $\rightarrow$ and $\leftrightarrow$)
- quantifiers $\forall$, $\exists$
- variables $x_0, x_1, x_2, x_3 \ldots$ (infinitely many)
- punctuation (,) (ensures unique readability)
- equality $=$

to define recursively
- L - terms
- L - formulas (in particular, atomic formulas and sentences)

Notation: the letters x, y, z denote variables, the letters $a, b, c, \ldots$ denote constants (or elements of the domain of a structure).
The notation $t(x_0 \cdots x_n)$ (term), $\phi(x_0 \cdots x_n)$ (formula) means that the free variables of t, ϕ are in $\{x_0, x_1, \ldots, x_n\}$

If $\phi(x_1, \ldots, x_n)$ is an L-fmla, M is an L-structure and $a_1, \ldots, a_n \in M$, then
$$M \models \phi(a_1 \ldots a_n)$$
means that ϕ holds in M under the assignment $x_i \mapsto a_i$.
This notion is also defined recursively.
(You can revise all these notions on the following textbooks:
- J. Kirby, An Invitation to Model Theory (CUP, 2019): Chapter 1 Section 1.1, Chapter 2 Section 2.1, Chapter 3 Sections 3.1 - 3.3
- D. Marker, Model Theory: an Introduction (Springer 2002), Chapter 1, Section 1.1
- last year's Part III notes (available from the course webpage), Chapter 2, 2.1, 2.2, 2.5, 2.6, 2.7

We often write $\bar{x}$ for a tuple of variables $x_1, \ldots, x_n$, and $\bar{a}$ for a tuple of constants/elements $a_1, \ldots, a_n$, so for example
$$t(\bar{x}), \quad \phi(\bar{x}), \quad \phi(\bar{a})$$

(2) MAPS THAT PRESERVE FORMULAS

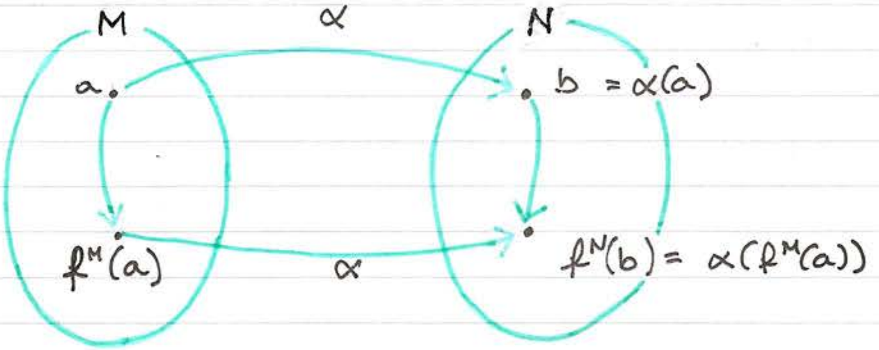
(2.1) Definition Let $L = \{\mathcal{F}, \mathcal{R}, \mathcal{C}\}$ be a language, and let M and N be L-structures. An embedding of M into N is an injective mapping

$$\alpha : M \to N$$

such that

(i) for all $f \in \mathcal{F}$ and $a_1, \dots, a_{n_f} \in M$
$$\alpha(f^M(a_1 \dots a_{n_f})) = f^N(\alpha(a_1), \dots, \alpha(a_{n_f}))$$
(ii) for all $R \in \mathcal{R}$ and $a_1, \dots, a_{n_R} \in M$
$$(a_1, \dots, a_{n_R}) \in R^M \text{ if and only if } (\alpha(a_1), \dots, \alpha(a_{n_R})) \in R^N$$
(iii) for all $c \in \mathcal{C}$, $\alpha(c^M) = c^N$

For a unary function f, condition (i) is illustrated by a commutative diagram.



A surjective (onto) embedding is said to be an isomorphism

(2.2) Examples If G_1, G_2 are groups treated as L_{gp}-structures, then a mapping $\alpha : G_1 \to G_2$ is an embedding if and only if it is an injective group homomorphism.

If A and B are linear orders treated as L_{lo}-structures, then $\alpha : A \to B$ is an embedding if and only if it is an order-preserving injective map.

(2.3) Fact Let M and N be L-structures, and let $\alpha : M \to N$ be an embedding. Let $\bar{a} \in M^k$, and let $t(\bar{x})$ be an L-term, where $|\bar{x}| = |\bar{a}| = k$. Then
$$\alpha(t^M(\bar{a})) = t^N(\alpha(\bar{a})), \text{ where}$$
$$\alpha(\bar{a}) = (\alpha(a_1), \dots, \alpha(a_k)).$$

Proof: by induction on the complexity of t. Let $\bar{a} \in M^k$. Then

(i) if $t = x_i$, then $t^M(\bar{a}) = a_i$ and $t^N(\alpha(\bar{a})) = \alpha(a_i)$.

(ii) if $t = c$, then $t^M(\bar{a}) = c^M$, $t^N(\alpha(\bar{a})) = c^N$, and
$\alpha(c^M) = c^N$ because α is an embedding

(iii) if $t = f(t_1(\bar{x}), \dots, t_{n_f}(\bar{x}))$, then

$$\alpha(f^M(t_1^M(\bar{a}), \dots, t_{n_f}^M(\bar{a})) = f^N(\alpha(t_1^M(\bar{a})), \dots, \alpha(t_{n_f}^M(\bar{a})))$$

because α is an embedding. By the inductive hypothesis applied to $t_1(\bar{x}), \dots, t_{n_f}(\bar{x})$ we are done $\square$

(2.4) __Proposition__ Suppose that M and N are L-structures, and let $\alpha : M \to N$ be an embedding. Then for any atomic L-formula $\varphi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\alpha(\bar{a})).$$

Proof: Case 1 $\varphi(\bar{x})$ is $t_1(\bar{x}) = t_2(\bar{x})$, where t_1 and t_2 are L-terms. Then the statement follows by

Case 2: $\varphi(\bar{x})$ is $R(t_1(\bar{x}), \dots, t_{n_R}(\bar{x}))$ for a relation R and L-terms $t_1(\bar{x}), \dots, t_{n_R}(\bar{x})$.

Let $\bar{a} \in M^{|\bar{x}|}$. Then

$$\begin{aligned} & M \models R(t_1(\bar{a}), \dots, t_{n_R}(\bar{a})) \\ \iff & (t_1^M(\bar{a}), \dots, t_{n_R}^M(\bar{a})) \in R^M \\ \iff & (\alpha(t_1^M(\bar{a})), \dots, \alpha(t_{n_R}^M(\bar{a})) \in R^N \quad \text{because } \alpha \text{ is an embedding} \\ \iff & (t_1^N(\alpha(\bar{a})), \dots, t_{n_R}^N(\alpha(\bar{a}))) \in R^N \quad \text{by Fact 2.3} \\ \iff & N \models R(t_1(\alpha(\bar{a})), \dots, t_{n_R}(\alpha(\bar{a}))) \end{aligned}$$

$\square$

(2.5) __Exercise__ Show that Proposition 2.4 holds if we replace "atomic L-formula" with "quantifier-free L-formula"

(2.6) __Remark__ An embedding does not necessarily preserve L-formulas. Consider two L_{lo}-structures $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$. Then the inclusion map $\alpha : \mathbb{Z} \to \mathbb{Q}$

$$\alpha : n \to n$$

is an embedding.

Let $\varphi(x, y) \equiv \exists z \; (x < z \;\wedge\; z < y)$.

Then $\mathbb{Z} \models \neg\varphi(1,2)$ but $\mathbb{Q} \models \varphi(1,2)$.

On the other hand, isomorphisms preserve all formulas.

(2.7) __Fact__ Let M, N be L-structures, and let $\alpha : M \to N$ be an isomorphism. Then, for any L-formula $\varphi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\alpha(\bar{a})).$$

Proof: exercise (use induction on $\varphi(\bar{x})$).

(2.8) Remark $\alpha : M \to N$ is an embedding $\iff$ α preserves atomic formulas

(3) THEORIES AND ELEMENTARITY.

Let L be a language.

(3.1) <u>Definition</u> An L-theory T is a set of L-sentences. An L-structure M is a <u>model</u> of T, written

$$M \models T,$$

if $M \models \phi$ for all $\phi \in T$.

The class of all models of T is denoted by $\mathrm{Mod}(T)$.
If M is an L-structure, the <u>theory of M</u> is

$$Th(M) = \{\phi : \phi \text{ is an } L\text{-sentence and } M \models \phi\}$$

In general, describing the theory of a given structure is a non-trivial problem!

(3.2) <u>Example</u> Let T_{gp} be the theory containing exactly the following L-gp sentences

(i) $\forall x_1 x_2 x_3 \ ((x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3))$

(ii) $\forall x_1 \ (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$

(iii) $\forall x_1 \ (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$

If G is a group, then clearly $G \models T_{gp}$. However, $Th(G)$ is much larger than T_{gp}.

(3.3) <u>Definition</u> Let M and N be two L-structures. Then M and N are said to be <u>elementarily equivalent</u>, written

$$M \equiv N$$

if $Th(M) = Th(N)$, that is, if M and N model exactly the same L-sentences.

Clearly, if $M \simeq N$ then $M \equiv N$. However, in general establishing whether two structures are elementarily equivalent is highly non-trivial. We'll see that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$

(3.4) Definition If M and N are two L-structures, then

(i) an embedding $\beta : M \to N$ is <u>elementary</u> if for all L-formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$

$$M \models \phi(\bar{a}) \iff N \models \phi(\beta(\bar{a}))$$

(ii) if $M \subseteq N$ and

$$id : M \to N \quad \text{(the inclusion map)}$$

is an embedding, then M is said to be a <u>substructure</u> of N, written $M \subseteq N$

(iii) if $M \subseteq N$ and $id : M \to N$ is an elementary embedding, then M is said to be an <u>elementary substructure</u> of N, written $M \preccurlyeq N$.

Intuition: a substructure of N is a subset of the domain of N that:
- is closed under the interpretations of all the function symbols of L
- inherits the relations that hold in N
- contains the interpretations of all the constants of L in N. (exercise)

There isn't an intuition of what an elementary substructure is - nor of what an elementary embedding other than an isomorphism is.

3.5 <u>Example</u> Let $L_{\leq} = \{<\}$, and consider the closed intervals $[0,1]$, $[0,2] \subseteq \mathbb{R}$ as $L_{\leq}$-structures (with the usual ordering).
Then $[0,1] \simeq [0,2]$, and so $[0,1] \equiv [0,2]$.

It is clear that $[0,1]$ is a substructure of $[0,2]$.
Let $\phi(x) \equiv \forall y \, (y < x \vee y = x)$. Then

$$[0,1] \models \phi(1) \quad \text{but} \quad [0,2] \not\models \phi(1).$$

Therefore $[0,1]$ is NOT an elementary substructure of $[0,2]$.
So we have two $L_{\leq}$-structures M and N such that
- $M \subseteq N$
- $M \simeq N$ and so $M \equiv N$
- $M \not\preccurlyeq N$.

Remember: the existence of an isomorphism between M and N when $M \subseteq N$ does not guarantee that $M \preccurlyeq N$, although it implies that $M \equiv N$.

3.6 <u>Definition</u> Let M be an L-structure and let A be a subset of the domain of M. Then $L(A)$ is the language obtained by adding to L a constant symbol c_a for each $a \in A$:

$$L(A) = L \cup \{c_a : a \in A\}$$

It is understood that an interpretation of M as an L-structure is extended to an interpretation of M as an $L(A)$ structure by interpreting c_a as the element $a \in A$. In this context, the elements of A are called <u>parameters</u>.

Notation: if M, N are L-structures and $A \subseteq M \cap N$, we write

$$M \equiv_A L$$

if for every $L(A)$-sentence ϕ

$$M \models \phi \iff N \models \phi$$

(3.7) Remark $M \preccurlyeq N \iff M \equiv_M N$ (exercise)

(3.8) Lemma (Tarski - Vaught test)

Let N be an L-structure and let $A \subseteq N$. Then the following are equivalent:
(i) A is the domain of an elementary substructure of N
(ii) for every $L(A)$-formula $\phi(x)$ with one free variable
$$N \models \exists x\, \phi(x) \Rightarrow N \models \phi(b) \text{ for some } b \in A. \quad (*)$$

Proof (i) $\Rightarrow$ (ii) Suppose $N \models \exists x\, \phi(x)$. Then, by elementarity
$$A \models \exists x\, \phi(x),$$
so $A \models \phi(b)$ for some $b \in A$. Again by elementarity, $N \models \phi(b)$.

(ii) $\Rightarrow$ (i) First we check that A is the domain of a substructure.
It suffices to show:
(a) for all constants c, $c^N \in A$
(b) for all function symbols f and $\bar{a} \in A^{n_f}$, $f^N(\bar{a}) \in A$

For (a): $N \models \exists x\; c = x$. Then, by property $(*)$,
$N \models c = b$ for some $b \in A$.
Hence $c^N = b \in A$.
For (b): similar, with $N \models \exists x\; f(\bar{a}) = x$.

Now we show that for all L-formulas $\chi(\bar{z})$ and $\bar{a} \in A^{|\bar{z}|}$
$$A \models \chi(\bar{a}) \iff N \models \chi(\bar{a}).$$
We argue by induction on the complexity of $\chi(\bar{z})$.
- if $\chi(\bar{z})$ is atomic, the claim follows from $A \subseteq N$ (just proved)
- if $\chi(\bar{z}) \equiv \neg\psi(\bar{z})$, then
$$\begin{aligned} & A \models \neg\psi(\bar{a}) \\ \iff & A \not\models \psi(\bar{a}) \\ \iff & N \not\models \psi(\bar{a}) \quad \text{by inductive hyp} \\ \iff & N \models \neg\psi(\bar{a}) \end{aligned}$$
- if $\chi(\bar{z}) \equiv \psi(\bar{z}) \wedge \xi(\bar{z})$, again easy induction
- if $\chi(\bar{z}) \equiv \exists y\, \psi(\bar{z}, y)$, then for $\bar{a} \in A^{|\bar{z}|}$ $\exists y\, \psi(\bar{a}, y)$ is an $L(A)$-formula.

Suppose $A \models \exists y\, \psi(\bar{a}, y)$. Then
$A \models \psi(\bar{a}, b)$ for $b \in A$. By inductive hypothesis
$N \models \psi(\bar{a}, b)$, so
$N \models \exists y\, \psi(\bar{a}, y)$.

Now suppose $N \models \exists y\, \psi(\bar{a}, y)$. Then by property $(*)$
$N \models \psi(\bar{a}, b)$ for some $b \in A$.
By ind. hyp, $A \models \psi(\bar{a}, b)$ and so $A \models \exists y\, \psi(\bar{a}, y)$, as required.

(3.9) Definition and remarks We define the cardinality of the language L to be the cardinality of the set of L-formulas,

$$|L| = |\{\phi(\bar{x}) : \phi(\bar{x}) \text{ is an } L\text{-formula}\}|$$

Note that we have

$|L| \geqslant \omega$ (we often use ω both as a cardinal and as an ordinal)

If A is a set of parameters, then

$$|L(A)| = |L| + |A|$$

(3.10) Definition Let λ be an ordinal. Then a chain of sets of length λ is a sequence

$$\langle A_i : i < \lambda \rangle$$

such that, for all $i < j < \lambda$, $A_i \subseteq A_j$

A chain of L-structures of length λ is a sequence

$$\langle M_i : i < \lambda \rangle$$

such that, for all $i < j < \lambda$, $M_i \subseteq M_j$ (substructure)

The union of the chain $\langle M_i : i < \lambda \rangle$ is the L-structure M defined as follows:

- $M = \bigcup_{i<\lambda} M_i$ (domain)
- if c is a constant, then $c^M = c^{M_i}$ for any $i < \lambda$
- if f is a function symbol and $\bar{a} \in M^{n_f}$, then
  $$f^M(\bar{a}) = f^{M_i}(\bar{a})$$
  for any $i < \lambda$ such that $\bar{a} \in M_i$
- if R is a relation symbol, then $R^M = \bigcup_{i<\lambda} R^{M_i}$.

(3.11) Theorem (Downward Löwenheim Skolem)

Let N be an L-structure such that $|N| \geqslant |L| + \omega$, and let $A \subseteq N$. Then for every cardinal λ such that

$$|L| + |A| \leq \lambda \leq |N|$$

there is an elementary substructure $M \preccurlyeq N$ such that
(i) $A \subseteq M$ (ii) $|M| = \lambda$.

Proof We build inductively a chain $\langle A_i : i < \omega \rangle$ of subsets of N containing A and such that $|A_i| = \lambda$. Eventually we take $M = \bigcup A_i$.

Let A_0 be such that $A \subseteq A_0$ and $|A_0| = \lambda$.
At stage $i+1$, A_i has been built with $|A_i| = \lambda$. Then $|L(A_i)| = \lambda$.
Let $\langle \phi_k(x) : k < \lambda \rangle$ be an enumeration of $L(A_i)$-formulas with one free variable such that $N \models \exists x\, \phi_k(x)$.
For $k < \lambda$, let a_k be such that $N \models \phi_k(a_k)$ and let

$$A_{i+1} := A_i \cup \{a_k : k < \lambda\}$$

This set has cardinality at most λ.

Then $|A_{i+1}| = \lambda$.
Now let $M = \bigcup_{i<\lambda} A_i$. We use the Tarski-Vaught test to prove that $M \preceq N$. (Lemma 3.8)
Let $\varphi(\bar{x}, y)$ be an L-formula.
Claim: if $N \models \exists y\, \varphi(\bar{a}, y)$ for $\bar{a} \in M^{|\bar{x}|}$, then $N \models \varphi(\bar{a}, b)$ for some $b \in M$.
If $N \models \exists y\, \varphi(\bar{a}, y)$, then the finite tuple $\bar{a} \in M^{|\bar{x}|}$ is contained in some A_i for $i < \lambda$, and so $\varphi(\bar{a}, y)$ is among the formulas that have a witness in A_{i+1}. Hence

$$N \models \varphi(\bar{a}, b) \text{ where } b \in A_i \subseteq M. \qquad \square$$

Remark: the following version of the Downward Löwenheim-Skolem Theorem, which you might have encountered in undergraduate logic courses, follows from the statement of Theorem 3.11:
If L is a countable language and T is a theory with an infinite model, then T has a countable model.

(4) TWO RELATIONAL STRUCTURES

(4.1) Definition (Dense linear orders)

An L_{lo}-structure is a linear order if it satisfies

1. $\forall x \ \neg(x < x)$ irreflexivity
2. $\forall xyz \ ((x<y \wedge y<z) \to x<z)$ transitivity
3. $\forall xy \ (x=y \vee x<y \vee y<x)$ linearity

A linear order is dense if it satisfies

4. $\exists xy \ (x<y)$ non triviality
5. $\forall xy \ (x<y \to \exists z \ (x<z<y))$ density

A linear order has no endpoints if

6. $\forall x \ (\exists y \ (x<y) \wedge \exists z \ (z<x))$

T_{lo} is the theory that includes axioms 1, 2 and 3

T_{dlo} is the theory that includes axioms 1 to 6

$(\mathbb{Q}, <) \models T_{lo}$ and $(\mathbb{Q}, <) \models T_{dlo}$

Clearly any model of T_{dlo} is infinite

(4.2) Definition If M, N are L-structures, then a partial embedding from M to N is an injective map

$$p : \mathrm{dom}(p) \subseteq M \to N$$

that satisfies conditions (i), (ii) and (iii) in Definition 2.1.

M, N are said to be partially isomorphic if there is a nonempty collection I of partial embeddings such that

1. $p \in I$ and $a \in M \Rightarrow$ there is $\hat{p} \in I$ s.t. $p \subseteq \hat{p}$ and $a \in \mathrm{dom}(\hat{p})$
2. $p \in I$ and $b \in N \Rightarrow$ there is $\hat{p} \in I$ s.t. $p \subseteq \hat{p}$ and $b \in \mathrm{ran}(\hat{p})$.

(4.3) Lemma (Back & forth) If $|M| = |N| = \omega$ and M, N are partially isomorphic via I, then $M \simeq N$

Proof Enumerate M and N, say $M = \langle a_i : i < \omega \rangle$ and $N = \langle b_i : i < \omega \rangle$. We define inductively a chain $\langle p_i : i < \omega \rangle$ such that $p_i \in I$ and $a_{i-1} \in \mathrm{dom}(p_i)$, $b_{i-1} \in \mathrm{ran}(p_i)$.
Let p_0 be any map in I. At stage $i+1$, p_i is given.
Use property 1 in Def 4.2 to find $\hat{p} \supseteq p_i$ such that $a_i \in \mathrm{dom}(\hat{p})$, and property 2 to find $p_{i+1} \supseteq \hat{p}$ such that $b_i \in \mathrm{ran}(p_{i+1})$.
Then $\bigcup p_i$ is the required isomorphism between M and N.

(4.4) Lemma (Extension) Let $M \models T_{lo}$ and $N \models T_{dlo}$. Let $p : M \to N$ be a partial embedding such that $|\mathrm{dom}(p)| < \omega$.
Let $c \in M$. Then there is $\hat{p} \supseteq p$ s.t. $|\mathrm{dom}(\hat{p})| < \omega$ and $c \in \mathrm{dom}(\hat{p})$.
Proof Let $\mathrm{dom}(p) = \{a_0, \dots, a_n\}$ with $a_i < a_j$ if $i < j$. If $c \in \mathrm{dom}(\hat{p})$
Case 1: $c < a_0$. By axiom 6, there is $d \in N$ s.t. $d < p(a_0)$

Case 2: $a_i < c < a_{i+1}$. By axiom 5, there is $d \in N$ s.t. $p(a_i) < d < p(a_{i+1})$.
Case 3: $c > a_n$: similar to Case 1.
Then $\hat{p} = p \cup \{\langle a, d\rangle\}$ is the required extension of p.

(4.5) <u>Theorem</u> Let $M, N \models T_{dlo}$, $|M| = |N| = \omega$. Then $M \simeq N$
Proof: the empty map $\emptyset$ is a finite partial embedding between M and N (equivalently, $\{\langle m, n\rangle\}$ for any $m \in M$, $n \in N$ is a finite partial embedding)
Therefore the collection I of finite partial embeddings between M and N is non-empty. By Lemma 4.4, I satisfies properties 1 and 2 in Definition 4.2, so Lemma 4.3 applies.

(4.6) <u>Definition</u> An L-theory T is consistent if it has a model.
If φ is an L-sentence, then $T \vdash \varphi$ if for all L-structures M
$M \models T \Rightarrow M \models \varphi$.
T is complete if for all sentences φ either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Remarks:
- $Th(M)$ is complete for any structure M.
- In specific cases, determining whether a given formula φ is in $Th(M)$ is non-trivial!
- We often seek $S \subseteq Th(M)$ s.t. $M \models S$ and S is complete (an axiomatization of $Th(M)$).

(4.7) <u>Definition</u> If $|L| = \omega$, an L-theory T is <u>ω-categorical</u> if T has an infinite model, and any two countable models are isomorphic.

So Theorem 4.5 says that T_{dlo} is ω-categorical

(4.8) <u>Theorem</u> If T is an ω-categorical theory with no finite models, then T is complete.
Proof: Suppose $M, N \models T$ and φ is an L-sentence such that $M \models \varphi$. By Theorem 3.11 (Downward Löwenheim-Skolem) there are $M', N' \models T$ such that $|M'| = |N'| = \omega$ and $M' \preceq M$, $N' \preceq N$.
By Theorem 4.5, $M' \simeq N'$, so $N' \models \varphi$. By elementarity, $N \models \varphi$.

(4.9) <u>Corollary</u> T_{dlo} is complete

(4.10) <u>Definition</u> If M, N are L-structures, a map $f: \mathrm{dom}(f) \subseteq M \to N$ is said to be elementary if for all L-formulas $\varphi(\bar{x})$ and $\bar{a} \in \mathrm{dom}(f)^{|\bar{x}|}$
$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a}))$$

(4.11) Remark A map f is elementary if and only if every finite restriction of f is elementary
Proof ($\Rightarrow$) clear

($\Leftarrow$) Let $\varphi(\bar{z})$ and $\bar{a} \in \mathrm{dom}(f)^{|\bar{z}|}$ be such that
$$M \models \varphi(\bar{a}) \not\Leftrightarrow N \models \varphi(f(\bar{a}))$$
Then $f\restriction_{\bar{a}}$ is a finite restriction of f which is not elementary.

4.12 Proposition Let $M, N \models T_{dlo}$, and suppose $p : M \to N$ is a partial embedding. Then p is an elementary map.
Proof: By Remark 4.11, it suffices to consider the case $|p| < \omega$. By Theorem 3.11 (Downward Löwenheim-Skolem), there are $M' \preceq M$, $N' \preceq N$ such that $|M'| = |N'| = \omega$, $\mathrm{dom}(p) \subseteq M'$, $\mathrm{ran}(p) \subseteq N'$. By a back-and-forth argument (proof of Theorem 4.3 with $p_0 = p$ and I the collection of finite partial embeddings from M' to N'), p can be extended to an isomorphism $\pi : M' \to N'$, so in particular p is an elementary map.

4.13 Corollary $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$
Proof: The inclusion map is a partial embedding, so by Theorem 4.12 it is an elementary map.

4.14 Definition (graphs and random graph)
Let $L_{gph} = \{R\}$, where R is a binary relation symbol.
A graph is an L_{gph}-structure which satisfies
1. $\forall x \ \neg R(x,x)$ (irreflexivity)
2. $\forall xy \ (R(x,y) \to R(y,x))$ (symmetry)

If G is a graph, the elements of G are called vertices, and 2-subsets $\{a,b\}$ such that $R(a,b)$ are called edges.

A graph is a random graph if, in addition to axioms 1 and 2, it satisfies
3. $\exists xy \ (x \neq y)$
together with axioms r_n for $n \in \omega$:

$$r_n. \ \forall x_0 \cdots x_n y_0 \cdots y_n \left(\bigwedge_{i,j=0}^{n} x_i \neq y_j \to \exists z \left(\bigwedge_{i=0}^{n} (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right)\right)$$

T_{gph} is the theory that includes axioms 1 and 2
T_{rg} is the theory that includes axioms 1, 2, 3 and r_n $\forall n \in \omega$.
Axiom 3 and the axiom scheme r_n imply that a model of T_{rg} is infinite.

4.15 Fact T_{rg} is consistent
Proof Define a graph on ω as follows: for $i, j \in \omega$ with $i < j$, $\{i, j\}$ is an edge iff the ith digit in the binary expansion of j is a 1. (we count digits from right to left).
Exercise: check this L_{gph}-structure is a model of T_{rg}

(4.16) Lemma (extension) Let $M \models T_{gph}$, $N \models T_{rg}$. Let $p : \mathrm{dom}(p) \subseteq M \to N$ be a finite partial embedding and let $c \in M$. Then there is a finite partial embedding $\hat{p} \supseteq p$ such that $c \in \mathrm{dom}(\hat{p})$.

Proof Assume $c \notin \mathrm{dom}(p)$ and let

$U = \{a \in \mathrm{dom}(p) : R(a,c)\}$

$V = \{b \in \mathrm{dom}(p) : \neg R(b,c)\}$

p is finite, hence U and V are finite.

Since N is a random graph, we can find $d \in N \setminus (p(U) \cup p(V))$ such that

(i) $R(d, p(a))$ for all $a \in U$

(ii) $\neg R(d, p(b))$ for all $b \in V$.

Then $\hat{p} = p \cup \{\langle c,d \rangle\}$. □

(4.17) Theorem Let $M, N \models T_{rg}$ be such that $|M| = |N| = \omega$. Then $M \simeq N$.

Proof: The same as the proof of Theorem 4.5, with Lemma 4.16 instead of Lemma 4.4.

(4.18) Corollary T_{rg} is ω-categorical and complete. Every partial embedding between models of T_{rg} is elementary.

(4.19) Remarks The (unique up to isomorphism) countable model Γ of T is called the countable random graph, of Rado's graph. Γ embeds every finite and countable graph, and every finite partial isomorphism extends to an automorphism of Γ (i.e. an isomorphism from Γ to itself).

(5) COMPACTNESS

(5.1) Definition Let I be a set. Then a filter F on I is a collection of subsets of I such that

1. $I \in F$
2. if X and Y are in F, then $X \cap Y \in F$
3. if $X \in F$ and $X \subseteq Y \subseteq I$, then $Y \in F$

power set

A filter on I is proper if and only if $F \neq \mathcal{P}(I)$ - equivalently, if $\emptyset \notin F$

An ultrafilter on I is a proper filter U such that for all $X \subseteq I$, either $X \in U$ or $I \setminus X \in U$.

(5.2) Fact The following conditions are equivalent to the proper filter U being an ultrafilter:

(a) U is maximal among proper filters

(b) if $X \cup Y \in U$, then either $X \in U$ or $Y \in U$

not necessarily an ordered set

(5.3) Definition Let $\langle M_i, i \in I \rangle$ be a family of L-structures. The direct product of the M_i in the set

$$X = \prod_{i \in I} M_i = \{ f : I \to \bigcup M_i : f(i) \in M_i \text{ for all } i \in I \}$$

We will write $a = \langle a(i) : i \in I \rangle$ for a typical element of X.

Let U be an ultrafilter on I. Define a relation $\sim_u$ on X as follows:

$$a \sim_u b \iff \{ i \in I : a(i) = b(i) \} \in U$$

(5.4) Fact $\sim_u$ is an equivalence relation on X

Proof Reflexivity and symmetry are immediate.

Suppose $a, b, c \in X$ and $a \sim_u b$, $b \sim_u c$. Then:

$A = \{ i \in I : a(i) = b(i) \} \in U$

$B = \{ i \in I : b(i) = c(i) \} \in U$

Then $A \cap B \in U$, and

$A \cap B \subseteq C = \{ i \in I : a(i) = c(i) \}$, hence $C \in U$.

So $a \sim_u c$.

Notation: we write a_u for the eq class of $a \in X$ under U.

(5.5) Fact Let $a^k, b^k \in X$ for $k = 1, \dots, n$ be such that $a^k \sim_u b^k$

Then:

(a) If f is an n-ary function symbol, then

$\langle f^{M_i}(a^1(i), \dots, a^n(i)) : i \in I \rangle \sim_U \langle f^{M_i}(b^1(i), \dots, b^n(i)) : i \in I \rangle$

(b) if R is an n-ary relation symbol, then
$\{i \in I : (a^1(i), \dots, a^n(i)) \in R^{M_i}\} \in U \iff$
$\{i \in I : (b^1(i), \dots, b^n(i)) \in R^{M_i}\} \in U$

Proof (a) For ease of notation, we only consider the case $n = 1$.
So let $a, b \in X$ be such that $a \sim_U b$.
Let $A = \{i \in I : a(i) = b(i)\}$.
Then $C = \{i \in I : f^{M_i}(a(i)) = f^{M_i}(b(i))\} \supseteq A$, and so $C \in U$.
(b) similar.

5.6 Definition Given a collection $\langle M_i : i \in I \rangle$ of L-structures and an ultrafilter U on I, the ultraproduct of the M_i is the L-structure defined as follows.
(i) the domain is $X/\sim_U$ (where $X = \prod_{i \in I} M_i$)
(for ease of notation, write X_U)
(ii) if c is a constant, then
$c^{X_U} := \langle c^{M_i} : i \in I \rangle_U$
(iii) if f is a function symbol and $a^1_U, \dots, a^{n_f}_U \in X_U$, then

$$f^{X_U}(a^1_U, \dots, a^{n_f}_U) = \langle f^{M_i}(a^1(i), \dots, a^{n_f}(i)) : i \in I \rangle_U$$
(iv) if R is a relation symbol and $a^1_U, \dots, a^{n_R}_U \in X_U$, then
$(a^1_U, \dots, a^{n_R}_U) \in R^{X_U} \iff \{i \in I : (a^1(i), \dots, a^{n_R}(i)) \in R^{M_i}\} \in U$.

Fact 5.5 ensures that the interpretations of f and R in X_U is well defined.

5.7 Theorem (Łoś) Let U be an ultrafilter on a set I, and let $\langle M_i : i \in I \rangle$ be a collection of L-structures. Then
1. for all L-terms $t(x_1, \dots, x_n)$ and $a^1_U, \dots, a^n_U \in X_U$,
$t^{X_U}(a^1_U, \dots, a^n_U) = \langle t^{M_i}(a^1(i), \dots, a^n(i)) : i \in I \rangle_U$
2. for all L-formulas $\phi(x_1, \dots, x_n)$ and $a^1_U, \dots, a^n_U \in X_U$
$X_U \models \phi(a^1_U, \dots, a^n_U) \iff \{i \in I : M_i \models \phi(a^1(i), \dots, a^n(\,))\} \in U$
3. for all L-sentences σ
$X_U \models \sigma \iff \{i \in I : M_i \models \sigma\} \in U$

Proof: 1 is an easy induction on the complexity of $t(x_1 \dots x_n)$
3 is immediate from 2.
For 2: the cases where ϕ is atomic are an easy induction.
If $\phi \equiv \neg \chi$ for an L-fmla $\chi(x_1 \dots x_n)$, let
$A_\chi = \{i \in I : M_i \models \chi(a^1(i), \dots, a^n(i))\}$. By inductive hypothesis
$X_U \models \chi(a^1_U, \dots, a^n_U) \iff A_\chi \in U$, so
$X_U \not\models \chi(a^1_U, \dots, a^n_U) \iff A_\chi \notin U$.

But $\mathcal{U}$ is an ultrafilter, hence $A_\chi \notin \mathcal{U} \Leftrightarrow I \setminus A_\chi \in \mathcal{U}$. Therefore
$$X_u \models \neg\chi(a^1_u, \dots, a^n_u) \Leftrightarrow \{i \in I : M_i \models \neg\chi(a^1(i), \dots, a^n(i))\} \in \mathcal{U}.$$

If $\phi \equiv \psi \wedge \chi$, let $A_\phi = \{i \in I : M_i \models \phi(a^1(i), \dots, a^n(i))\}$, and let A_ψ and A_χ be defined similarly.
Then $A_\phi = A_\psi \cap A_\chi$ and, since $\mathcal{U}$ is an ultrafilter,
$$A_\phi \in \mathcal{U} \Leftrightarrow A_\psi \in \mathcal{U} \text{ and } A_\chi \in \mathcal{U}.$$
The required result follows by inductive hypothesis

If $\phi \equiv \exists y\ \psi(\bar{x} y)$, define A_ϕ as usual.
Suppose there is $b_u \in X_u$ such that $X_u \models \psi(a^1_u, \dots, a^n_u, b_u)$.
Then $\{i \in I : M_i \models \psi(a^1(i), \dots, a^n(i), b(i)\} \subseteq A_\phi$
By inductive hypothesis, the first set is in $\mathcal{U}$, therefore so is A_ϕ.

Now suppose $A_\phi \in \mathcal{U}$. For $i \in A_\phi$, let $b_i \in M_i$ be a witness of $\exists y\ \psi(a^1(i), \dots, a^n(i), y)$.
For $i \in I \setminus A_\phi$, let b_i be arbitrary.
Define $b \in X$ by $b(i) = b_i$.
Let $A_\psi = \{i \in I : M_i \models \psi(a^1(i), \dots, a^n(i), b(i))\}$.
Then $A_\phi \subseteq A_\psi$, so $A_\psi \in \mathcal{U}$.
By ind. hyp, $X_u \models \psi(a^1_u, \dots, a^n_u, b_u)$, hence $X_u \models \exists y\ \psi(a^1_u, \dots, a^n_u, y)$, as required.

5.8 Definition A subset $S \subseteq P(I)$ has the <u>finite intersection property</u> (FIP) if whenever $A_1, \dots, A_n \in S$
$$\bigcap_{i=0}^{n} A_i \neq \emptyset$$

Remark: proper filters have FIP.

5.9 Lemma
1. If $\mathcal{S} \subseteq P(I)$ has FIP, then $\mathcal{S}$ can be extended to a proper filter.
2. A proper filter can be extended to an ultrafilter.

Proof:
1. Let F be the extension of $\mathcal{S}$ defined as follows:
$$F = \{X \subseteq I : X \text{ contains the intersection of a finite collection of elements of } \mathcal{S}\}.$$
Then F is a proper filter
2. Let F be a proper filter, and let
$$\mathcal{F} = \{G \subseteq P(I) : G \text{ is a proper filter containing } F\}$$
Then $\mathcal{F}$ is partially ordered by inclusion. It is easy to check that the union of a chain in $\mathcal{F}$ is a proper filter containing F, hence an upper bound in $\mathcal{F}$ for the chain.
By Zorn's Lemma, $\mathcal{F}$ has a maximal element F', say.

By Fact 5.2, F' is an ultrafilter and F' extends F, as required

(5.10) <u>Definition</u> A theory T is said to be
(i) <u>consistent</u>, or <u>satisfiable</u>, if it has a model
(ii) <u>finitely consistent</u>, or <u>finitely satisfiable</u>, if every finite subset of T has a model

(5.11) <u>Theorem</u> (Compactness)
An L-theory T is consistent if and only if it is finitely consistent

Proof ($\Rightarrow$) is clear.
For the converse, if $S \subseteq T$ is finite, let $M_S \models S$.
Let I be the collection of finite subsets of T. Idea: define an ultrafilter $\mathcal{U}$ on T so that $\prod_{S \in I} M_S / \sim_{\mathcal{U}}$ is a model of T.

By Theorem 5.9 (Łoš), it is enough to find $\mathcal{U}$ such that for all $\phi \in T$ $\{S : M_S \models \phi\} \in \mathcal{U}$.

For $\phi \in T$, let $A_\phi = \{S \in I : \phi \in S\}$. Claim: the set $\{A_\phi : \phi \in T\}$ has FIP.
Let $\phi_1, \dots, \phi_n \in T$. Then $\{\phi_1, \dots, \phi_n\} \in I$, and
$\{\phi_1, \dots, \phi_n\} \subseteq \bigcap_{i=1}^{n} A_{\phi_i}$, so $\bigcap A_{\phi_i} \neq \emptyset$.

By Lemma 5.9, $\{A_\phi : \phi \in T\}$ extends to an ultrafilter $\mathcal{U}$.
By Łoš's Theorem, $X_{\mathcal{U}} \models \phi \iff \{S : M_S \models \phi\} \in \mathcal{U}$.
But $A_\phi \in \mathcal{U}$, and $A_\phi \subseteq \{S : M_S \models \phi\}$. So we are done.

(5.12) <u>Definition</u> A <u>type</u> $p(\bar{x})$ in L is a set of L-formulas whose free variables are among $\bar{x} = \langle x_i : i < \lambda \rangle$.
A type $p(\bar{x})$ is
- <u>satisfiable in an L-structure M</u> if there is $\bar{a} \in M^{|\bar{x}|}$ such that $M \models \phi(\bar{a})$ for all $\phi(\bar{x}) \in p(\bar{x})$. We write
$M \models p(\bar{x})$, or $M, \bar{a} \models p(\bar{x})$
and we say $\bar{a}$ <u>realizes</u> $p(\bar{x})$ in M
- <u>satisfiable</u> if it is satisfiable in some M
- <u>finitely satisfiable in M</u> if all its finite subsets are satisfiable in M
- <u>finitely satisfiable</u> if all its finite subsets are satisfiable

Warning: we may say "consistent", "finitely consistent" etc throughout the definition

Remark: a type may be finitely satisfiable in M but not satisfiable in M. For example, let $M = \langle \mathbb{N}, < \rangle$ and let $\phi_n(x)$ say "There are at least n elements less than x".

Let $p(x) = \{\varphi_n(x) : n < \omega\}$. Then $p(x)$ is finitely satisfiable in M, but not satisfiable in M.

5.13 Theorem (compactness for types)
Every finitely satisfiable type in L is satisfiable

Proof: Let $p(\bar{x})$ be a type in L with $\bar{x} = \langle x_i : i < \lambda \rangle$.
Expand L to $L' = L \cup \{c_i : i < \lambda\}$, where the c_i are distinct constants not in L.
Then $p(\bar{c})$ is a finitely satisfiable theory in L'.
By the Compactness Theorem, there is an L'-structure M' such that $M' \models p(\bar{c})$. But M' can be reduced to an L-structure (simply forget the interpretations of the c_i).
Then $M', \bar{c}^{M'} \models p(\bar{x})$.

5.14 Lemma Let M be an L-structure, and let $\bar{a}$ be an enumeration of M, say $\bar{a} = \langle a_i : i < \lambda \rangle$.
Let $q(\bar{x}) = \{\varphi(\bar{x}) : M \models \varphi(\bar{a})\}$, where $|\bar{x}| = \lambda$.
Then $q(\bar{x})$ is satisfiable in an L-structure N if and only if there is an elementary embedding $\beta : M \to N$

Proof: ($\Rightarrow$) If $q(\bar{x})$ is satisfiable in N, then there is a tuple $\bar{b} \in N^{|\bar{x}|}$ such that $N \models \varphi(\bar{b})$ for all $\varphi(\bar{x}) \in q(\bar{x})$.
Then $\beta : M \to N$ defined by $a_i \mapsto b_i$ $(i < \lambda)$ is an elementary embedding: let $\varphi(\bar{x})$ be an L-fmla. Then
$$M \models \varphi(\bar{a}) \Leftrightarrow \varphi(\bar{x}) \in q(\bar{x}) \Leftrightarrow N \models \varphi(\bar{b}) \text{ and } \bar{b} = \beta(\bar{a}).$$
($\Leftarrow$) If $\beta : M \to N$ is an elementary embedding, then $N, \beta(\bar{a}) \models q(\bar{x})$.

5.15 Remark If M is an L-structure and $A \subseteq M$, we can work with types in $L(A)$. In particular, we sometimes work with types in $L(M)$. Types of this kind are said to have parameters in A (resp M), or to be over A (resp M).
If $p(\bar{x})$ is a type in $L(M)$, then there are an enumeration $\bar{a}$ of M and a type $q(\bar{x}, \bar{z})$ in L such that $p(\bar{x}) = q(\bar{x}, \bar{a})$

Lemma 5.14 is sometimes called Diagram Lemma and stated as follows.
Let $Th(M_M)$ be the $L(M)$-theory of M. Suppose $N \models Th(M_M)$.
Then M embeds elementarily in N.

5.16 Theorem If M is an L-structure and $p(x)$ is a type in $L(M)$ that is finitely satisfiable in M, then $p(x)$ is realized in some N such that $M \preceq N$.

Example Let $(0,1) \subseteq \mathbb{Q}$, let $M = \langle (0,1), < \rangle$.
Let $a_n = 1 - \frac{1}{n}$ for $n \in \omega \setminus \{0\}$.

Let $\phi_n(x) \equiv a_n < x$. Then $p(x) = \{\phi_n(x) : n \in \omega \setminus \{0\}\}$ is a type in $L(M)$ that is finitely satisfiable in M, but not satisfiable. However, $p(x)$ is realized in $\langle \mathbb{Q}, < \rangle$, and $M \preceq \langle \mathbb{Q}, < \rangle$

Proof of Theorem 5.16 Let $\bar{a} = \langle a_i : i < \lambda \rangle$ be an enumeration of M, and let
$$q(\bar{z}) = \{\phi(\bar{z}) : M \models \phi(\bar{a})\}$$
where $|\bar{z}| = \lambda$ and the z_i are new variables (so $\bar{z} \cap \bar{x} = \emptyset$)
Write $p(\bar{x})$ as $p'(\bar{x}, \bar{a})$, where $p'(\bar{x}, \bar{z})$ is a type in L.
Then $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is finitely consistent in M, because $p'(\bar{x}, \bar{z})$ is finitely consistent and $q(\bar{z})$ is realized by $\bar{a}$.
By compactness for types, there are an L-structure N and tuples $\bar{c} \in N^{|\bar{x}|}$ and $\bar{b} \in N^{|\bar{z}|}$ such that
$$N \models p'(\bar{c}, \bar{b}) \cup q(\bar{b}).$$
In particular, $N \models q(\bar{b})$. By Lemma 5.14, $\beta : a_i \mapsto b_i$ is an elementary embedding. We may assume $M \preceq N$.

(5.17) Theorem (Upward Löwenheim-Skolem)
Let M be an infinite L-structure. Then for any $\lambda \geq |M| + |L|$, there is N such that $M \preceq N$ and $|N| = \lambda$.
Proof: Let $\langle \bar{x} : i < \lambda \rangle$ be a tuple of (distinct) variables.
Let $p(\bar{x}) = \{x_i \neq x_j : i < j < \lambda\}$.
Then $p(\bar{x})$ is finitely consistent in M.
By Theorem 5.16, $p(\bar{x})$ is realized in some $M \preceq N$, and so $|N| \geq \lambda$.
By Downward Löwenheim-Skolem, we may assume $|N| = \lambda$.

6 SATURATION

6.1 Definition Let λ be an infinite cardinal and M an infinite L-structure. Then M is said to be λ-saturated if it realizes every type $p(x)$ with one free variable which
(i) is finitely satisfiable in M
(ii) has parameters in $A \subseteq M$ with $|A| < \lambda$

M is said to be saturated if it is $|M|$-saturated

Remark: if $\lambda > M$, M cannot be λ-saturated: consider the type $p(x) = \{x \neq a : a \in M\}$, which is finitely satisfiable in M, but not satisfiable in M. This type has $|M|$ parameters.

6.2 Definition Let M be an L-structure, $A \subseteq M$, $\bar{b}$ a tuple in M (possibly infinite). Then the type of $\bar{b}$ over A is
$$tp_M(\bar{b}/A) = \{\varphi(\bar{x}) \in L(A) : M \models \varphi(\bar{b})\}$$
M is usually omitted when it is clear from the context

6.3 Remarks (i) $tp(\bar{b}/A)$ is complete: for all $L(A)$-formulas $\varphi(\bar{x})$, either $\varphi(\bar{x}) \in tp(\bar{b}/A)$ or $\neg\varphi(\bar{x}) \in tp(\bar{b}/A)$
(ii) If $M \preceq N$, $A \subseteq M$ and $\bar{b} \in M^{|\bar{b}|}$, then
$$tp_M(\bar{b}/A) = tp_N(\bar{b}/A)$$

6.4 Facts (types and elementary maps)
Let $f : A \subseteq M \to N$ be an elementary map. Then
(i) $M \equiv N$ (and if $M \equiv N$ then the empty map is elementary, as it preserves sentences)
(ii) If $\bar{a}$ is an enumeration of $dom(f)$, then
$$tp(\bar{a}/\emptyset) = tp(f(\bar{a})/\emptyset)$$
More generally, if there is $A \subseteq dom(f) \cap N$ and f is the identity on A, then for every tuple $\bar{b}$ in $dom(f)$
$$tp(\bar{b}/A) = tp(f(\bar{b})/A)$$
(iii) Let $\bar{a}$ enumerate $dom(f)$. If $p(\bar{x}, \bar{a})$ is a finitely satisfiable type in $L(A)$, then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in N.
Let $\{\varphi_1(\bar{x}, \bar{a}), \dots, \varphi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a})$. Then $M \models \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x}, \bar{a})$ so by elementarity of f
$$N \models \exists \bar{x}\ \varphi_i(\bar{x}, f(\bar{a})).$$

However, if $p(\bar{x}, \bar{a})$ is satisfiable in M, it does not follow that $p(\bar{x}, f(\bar{a}))$ is satisfiable in N (though $p(\bar{x}, f(\bar{a}))$ will still be finitely satisfiable in N).

6.5 Theorem Let N be such that $|L| \leq \lambda \leq |N|$. The following are equivalent:
(i) N is λ-saturated
(ii) If $f: M \to N$ is an elementary map (so in particular $M \equiv N$), $|f| < \lambda$ and $b \in M$, then there is $\hat{f} \supseteq f$ that is elementary and $b \in \mathrm{dom}(\hat{f})$
(iii) If $p(\bar{z})$ is a type in $L(A)$, where $A \subseteq N$, $|A| < \lambda$, $|\bar{z}| \leq \lambda$, and $p(\bar{z})$ is finitely satisfiable in N, then $p(\bar{z})$ is satisfiable in N.

Proof: (i) $\Rightarrow$ (ii) Let f be as in the hypothesis of (ii) and let $b \in M$. Let $\bar{a}$ be an enumeration of $\mathrm{dom}(f)$, so $|\bar{a}| < \lambda$.
Let $p(x, \bar{a}) = tp_M(b/\bar{a})$.
Since $p(x,\bar{a})$ is finitely satisfiable in M, $p(x, f(\bar{a}))$ is finitely satisfiable in N. Since N is λ-saturated and $|f| < \lambda$, $p(x, f(\bar{a}))$ is satisfiable in N, say $N \models p(c, f(\bar{a}))$. Then $f \cup \{\langle b, c\rangle\}$ is the required extension of f:
$M \models \varphi(b, \bar{a}) \Leftrightarrow \varphi(x,\bar{a}) \in p(x,\bar{a}) \Leftrightarrow \varphi(x, f(\bar{a})) \in p(x, f(\bar{a}))$
$\Leftrightarrow N \models \varphi(c, f(\bar{a}))$.
(ii) $\Rightarrow$ (iii) Let $p(\bar{z})$ be as in (iii). By Theorem 5.16, $p(\bar{z})$ is realized in some elementary $M \succcurlyeq N$ by some tuple $\bar{a}$, where $|\bar{a}| = |\bar{z}| \leq \lambda$.
Since $N \preccurlyeq M$, the map $id_A : M \to N$ is elementary.
Idea: we extend id_A to an elementary map $f: M \to N$ such that $\bar{a} \in \mathrm{dom}(f)$. Then $N \models p(f(\bar{a}))$.
We build f in stages. Let $f_0 = id_A$
At stage $i+1$, use (ii) to put a_i in $\mathrm{dom}(f_{i+1})$.
At limit stages $\mu < \lambda$, let $f_\mu = \bigcup_{i<\mu} f_i$

Then $\bigcup_{i<|\bar{a}|} f_i$ is the required elementary map.

(iii) $\Rightarrow$ (i) trivial

6.6 Corollary If M and N are saturated, $M \equiv N$ and $|M| = |N|$, then any elementary $f: M \to N$, $|f| < |M|$, extends to an isomorphism.
In particular, if $M \equiv N$ and $|M| = |N|$ and M, N are saturated, then $M \cong N$.
Proof: Use Theorem 6.5 (ii) to extend $f: M \to N$ to an isomorphism by back and forth. This can also be done with $f = \emptyset$ (since $M \equiv N$, $\emptyset$ is elementary).

6.7 Corollary Models of T_{dlo} and T_{rg} are ω-saturated.
Proof By Theorem 6.5, Lemma 4.4 and Proposition 4.12 (linear orders) and Lemma 4.16 and 4.18 (graphs).

So $(\mathbb{Q}, <)$ is saturated. On the other hand, $\langle \mathbb{R}, < \rangle$ is not ω_1-saturated: the type $\{x > q : q \in \mathbb{Q}\}$ in $L(\mathbb{Q})$ is finitely satisfiable in $\mathbb{R}$ but not satisfiable.

(6.8) __Definition__ An isomorphism $\alpha : N \to N$ is called an __automorphism.__ The automorphisms of N form a group denoted by $Aut(N)$.
If $A \subseteq N$, then

$$Aut(N/A) = \{\alpha \in Aut(N) : \alpha = id_A\}$$

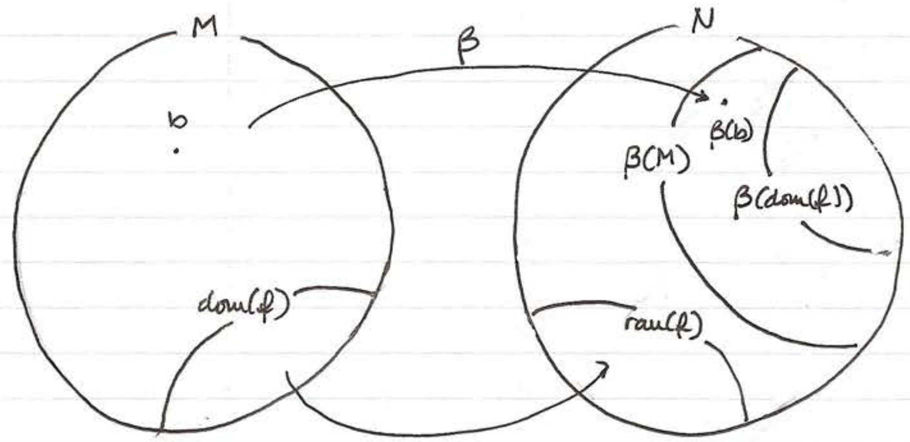
(automorphisms that fix A pointwise

(6.9) __Definition__ Let N be an L-structure
(i) N is __λ-universal__ if for every $M \equiv N$ such that $|M| \leq \lambda$ there is an elementary embedding $\beta : M \to N$.
N is __universal__ if it is $|N|$-universal
(ii) N is __λ-homogeneous__ if every elementary map $f : N \to N$ such that $|f| < \lambda$ extends to an automorphism of N.
N is __homogeneous__ if it is $|N|$-homogeneous
Warning: in some books / papers, λ-universality according to (i) is called λ^+-universality
Homogeneity according to (ii) is called strong homogeneity (as opposed to weak homogeneity, a property similar to the one in Theorem 6.5 (ii) with $N = M$).
There is also a notion of "ultrahomogeneity" defined for partial embeddings, rather than elementary maps, as in (ii). In Chapter 4 we actually saw that $(\mathbb{Q}, <)$ and the countable random graph are ultrahomogeneous.
Countable models of T_{dlo} and T_{rg} are universal and homogeneous.

(6.10) __Theorem__ Let N be such that $|N| \geq |L|$. Then the following are equivalent:
(i) N is saturated
(ii) N is universal and homogeneous.

Proof: (i) $\Rightarrow$ (ii) Assume N is saturated, and let $M \equiv N$ be such that $|M| \leq |N|$. Let $\bar{a} = \langle a_i : i < |M| \rangle$ be an enumeration of M, and let $p(\bar{x}) = tp(\bar{a}/\emptyset)$ (so $|\bar{x}| = |\bar{a}| = |M|$).
Since $p(\bar{x})$ is finitely satisfiable in M, $p(\bar{x})$ is finitely satisfiable in N (if $\phi_1(\bar{x}), \ldots, \phi_n(\bar{x}) \in p(\bar{x})$ then $M \models \exists \bar{x} \bigwedge_{i=1}^{n} \phi_i(\bar{x})$. Since $M \equiv N$, $N \models \exists \bar{x} \bigwedge_{i=1}^{n} \phi_i(\bar{x})$).
Since $|\bar{x}| \leq |N|$, N realizes $p(\bar{x})$ by saturation (Theorem 6.5 (iii)).
Then if $N \models p(\bar{b})$, $a_i \mapsto b_i$ is an elementary embedding from M to N.
Homogeneity follows from Corollary 6.6 with $M = N$.

(ii) $\Rightarrow$ (i) We show that if $M \equiv N$, $b \in M$ and $f : M \to N$ is an elementary map such that $|f| < |N|$, then there is $\hat{f} \supseteq f$ which is elementary and defined on b. By Theorem 6.5, this shows that N is saturated.
By working in $M' \preccurlyeq M$ such that $\mathrm{dom}(f) \cup \{b\} \subseteq M'$ if necessary (use Downward Löwenheim-Skolem), we may assume $|M| \leq |N|$.
Since $M \equiv N$, by universality there is an elementary embedding $\beta : M \to N$, so $\beta(M) \preccurlyeq N$.



The map $f \circ \beta^{-1} : \beta(\mathrm{dom}(f)) \to \mathrm{ran}(f)$ is elementary, and $|f \circ \beta^{-1}| < |N|$. By homogeneity, there is $\alpha \in \mathrm{Aut}(N)$ such that $f \circ \beta^{-1} \subseteq \alpha$. Then $f \cup \{\langle b, \alpha(\beta(b))\rangle\}$ is elementary, as it is a restriction of $\alpha \circ \beta$.

6.11 Definition Let $\bar{a}$ be a tuple in N and $A \subseteq N$.
The orbit of $\bar{a}$ over the set A is

$$O_N(\bar{a}/A) := \{\alpha(\bar{a}) : \alpha \in \mathrm{Aut}(N/A)\}$$

If $\varphi(\bar{x})$ is an $L(A)$-formula, then

$$\varphi(N) := \{\bar{a} \in N^{|\bar{x}|} : N \models \varphi(\bar{a})\}$$

is the set defined by $\varphi(\bar{x})$.
A set is definable over A if it is defined by some $L(A)$-formula.

There are analogous notions of sets defined by types / type-definable sets.

Recall that if $\bar{a}, \bar{b}$ are tuples in N, $A \subseteq N$ and $|\bar{a}| = |\bar{b}|$, the following are equivalent
(i) $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$
(ii) $\{a_i \mapsto b_i : i < |\bar{a}|\} \cup \mathrm{id}_A$ is an elementary map from N to N.

6.12 <u>Proposition</u> Let N be λ-homogeneous, $A \subseteq N$, $|A| < \lambda$, and let $\bar{a}$ be a tuple in N such that $|\bar{a}| < \lambda$. Then

$$O_N(\bar{a}/A) = p(N)$$

where $p(\bar{x}) = tp_N(\bar{a}/A)$, and $p(N) = \{\bar{b} \in N^{|\bar{a}|} : N \vDash p(\bar{b})\}$

Proof If $\bar{b} = \alpha(\bar{a})$ with $\alpha \in Aut(N/A)$, then $tp_N(\bar{a}/A) = tp_N(\bar{b}/A)$. Conversely, if $N \vDash p(\bar{b})$, then $tp_N(\bar{a}/A) = tp_N(\bar{b}/A)$ and so the map $\{\langle a_i, b_i\rangle : i < |\bar{a}|\} \cup id_A$ is elementary. ~~By~~ λ-homogeneity, this elementary map extends to $\alpha \in Aut(N)$, and in particular $\alpha \in Aut(N/A)$, so $\bar{b} \in O_N(\bar{a}/A)$.

(7) THE MONSTER MODEL

Given a complete theory T with an infinite model, it is often convenient to work in a saturated model of T large enough to embed elementarily any other model of T we might be interested in. We call such a large saturated model of T the monster model. The monster model is usually denoted by $\mathfrak{U}$, or $\mathbb{M}$, and its cardinality is denoted by κ.

The existence of arbitrarily large saturated models for general theories may depend on extra set-theoretic assumptions on the desired cardinality (e.g. the cardinality is inaccessible). In specific cases, such assumptions can be avoided by ad-hoc arguments. The monster model is an expository device that allows us to prove results about all the models of a theory by proving results about elementary substructures of saturated models of T.

(7.1) Terminology and conventions

When working in the monster model $\mathfrak{U}$ of a theory T, we say that

- $\phi(\bar{x})$ holds if $\mathfrak{U} \models \forall \bar{x}\ \phi(\bar{x})$
- $\phi(\bar{x})$ is consistent if $\mathfrak{U} \models \exists \bar{x}\ \phi(\bar{x})$
- a type $p(\bar{x})$ is consistent / satisfiable if there is $\bar{a} \in \mathfrak{U}^{|\bar{x}|}$ such that $\mathfrak{U} \models p(\bar{a})$
- a cardinality is <u>small</u> if it is strictly less than $\kappa = |\mathfrak{U}|$
- a <u>model</u> is $M \preccurlyeq \mathfrak{U}$ such that $|M|$ is small

<u>Conventions</u>

- tuples have length $< \kappa$ unless otherwise specified
- formulas have parameters in $\mathfrak{U}$, unless otherwise specified
- definable sets have the form $\phi(\mathfrak{U})$ for some $L(\mathfrak{U})$-fmla $\phi(\bar{x})$
- type-definable sets have the form $p(\mathfrak{U})$ for some type $p(\bar{x})$ in $L(A)$, where $|A| < \kappa$

<u>Notation</u>

- the symbols $A, B, C, \ldots$ denote sets of parameters of small cardinality
- $tp(\bar{a}/A) = tp_{\mathfrak{U}}(\bar{a}/A)$
- $O(\bar{a}/A)$ is the orbit of $\bar{a}$ in $\mathfrak{U}$ under $Aut(\mathfrak{U}/A)$

If $p(\bar{x})$ and $q(\bar{x})$ are types, we write $p(\bar{x}) \rightarrow q(\bar{x})$ to mean that every realization of $p(\bar{x})$ is also a realization of $q(\bar{x})$.
We extend the semantics of formulas to types, where a type is thought of as an infinite conjunction of formulas. However, such infinite conjunctions are not formulas, and their "truth" is not preserved by elementary maps

(7.2) Fact Let $p(\bar{x})$ be a satisfiable type in $L(A)$ and $q(\bar{x})$ a satisfiable type in $L(B)$. Suppose $p(\bar{x}) \to \neg q(\bar{x})$ (that is, $p(\bar{x})$ and $q(\bar{x})$ have no realizations in common). Then there are a conjunction $\phi(\bar{x})$ of formulas in $p(\bar{x})$ and a conjunction $\psi(\bar{x})$ of formulas in $q(\bar{x})$ such that

$$\models \phi(\bar{x}) \to \neg\psi(\bar{x})$$

Proof If $p(\bar{x}) \to \neg q(\bar{x})$ then $p(\bar{x}) \cup q(\bar{x})$ is not satisfiable in $\mathcal{U}$, hence it is not finitely satisfiable (by saturation of $\mathcal{U}$).
Let $\{\phi_1(\bar{x}), \dots, \phi_n(\bar{x})\} \subseteq p(\bar{x})$
$\{\psi_1(\bar{x}), \dots, \psi_m(\bar{x})\} \subset q(\bar{x})$
be such that $\bigwedge_{i=1}^{n} \phi_i(\bar{x}) \wedge \bigwedge_{i=1}^{m} \psi_i(\bar{x})$ is not satisfiable.

Then $\models \bigwedge_{i=1}^{n} \phi_i(x) \to \neg \bigwedge_{i=1}^{m} \psi_i(\bar{x})$, as required

(7.3) Remark Let $\phi(\mathcal{U}, \bar{b})$ be a definable set, where $\phi(\bar{x}, \bar{z})$ is an L-formula and $\bar{b} \in \mathcal{U}^{|\bar{z}|}$. Let $\alpha \in Aut(\mathcal{U})$. Then

$$\alpha[\phi(\mathcal{U}, \bar{b})] = \phi(\mathcal{U}, \alpha(\bar{b}))$$

If $\bar{c} \in \alpha[\phi(\mathcal{U}, \bar{b})]$, then $\bar{c} = \alpha(\bar{a})$ for some $\bar{a}$ s.t. $\models \phi(\bar{a}, \bar{b})$.
Then $\models \phi(\alpha(\bar{a}), \bar{\alpha}(\bar{b}))$, that is, $\models \phi(\bar{c}, \alpha(\bar{b}))$ and so $\bar{c} \in \phi(\mathcal{U}, \alpha(\bar{b}))$.
The other containment is similar.
So $Aut(\mathcal{U})$ acts on the definable sets in a natural way
Similarly, if $p(\bar{x}, \bar{z})$ is a type in L and $\bar{b} \in \mathcal{U}^{|\bar{z}|}$, then

$$\alpha[p(\mathcal{U}, \bar{b})] = p(\mathcal{U}, \alpha(\bar{b}))$$

(7.4) Definition A set $D \subseteq \mathcal{U}^{\lambda}$ ($\lambda < \kappa$) is invariant over A if for all $\alpha \in Aut(\mathcal{U}/A)$

$$\alpha(D) = D$$

or, equivalently, if for all $\bar{a} \in D$ $O(\bar{a}/A) \subseteq D$.

Remark if $\bar{a} \in D$, $q(\bar{x}) = tp(\bar{a}/A)$ and $b \models q(\bar{x})$, then by homogeneity there is $\alpha \in Aut(\mathcal{U}/A)$ such that $\alpha(\bar{a}) = \bar{b}$, and so $\bar{b} \in O(\bar{a}/A)$. So the invariance of D is equivalent to the following condition.

for all $\bar{a} \in D$, if $tp(\bar{b}/A) = tp(\bar{a}/A)$ then $\bar{b} \in D$.

Definability and invariance are closely related in $\mathcal{U}$.

(7.5) Proposition Let $A \subseteq \mathcal{U}$. For every $L(\mathcal{U})$-formula $\phi(\bar{x})$ the following are equivalent.
(i) there is a fmla $\psi(\bar{x})$ in $L(A)$ such that

$$\models \forall \bar{x}\, [\psi(\bar{x}) \leftrightarrow \phi(\bar{x})]$$

(ii) $\phi(\mathcal{U})$ is invariant over A

Proof (i) ⇒ (ii) Clear, since $\psi(\mathcal{U})$ is fixed setwise by $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ (by Remark 7.3)

(ii) ⇒ (i) Suppose $\phi(\bar{x}, \bar{z})$ is an L-formula such that $\phi(\mathcal{U}, \bar{b})$ is invariant over A for some $\bar{b} \in \mathcal{U}^{|\bar{z}|}$.

Let $q(\bar{z}) = tp(\bar{b}/A)$. Then if $\bar{c} \models q(\bar{z})$, by homogeneity there is $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\alpha(\bar{b}) = \bar{c}$. Then

$$\begin{array}{ll} \alpha[\phi(\mathcal{U}, \bar{b})] = \phi(\mathcal{U}, \bar{c}) & \text{by Remark 7.3} \\ \quad \parallel & \\ \phi(\mathcal{U}, \bar{b}) \text{ by invariance} & \end{array}$$

and so $\phi(\mathcal{U}, \bar{b}) = \phi(\mathcal{U}, \bar{c})$. Therefore

$$q(\bar{z}) \rightarrow \forall \bar{x} \, [\phi(\bar{x}, \bar{z}) \leftrightarrow \phi(\bar{x}, \bar{b})]$$

By (an argument similar to the one in) Fact 7.2, there is $\theta(\bar{z}) \in q(\bar{z})$ such that $\theta(\bar{z}) \rightarrow \forall \bar{x} \, [\phi(\bar{x}, \bar{z}) \leftrightarrow \phi(\bar{x}, \bar{b})]$

Then the formula

$$\exists \bar{z} \, [\theta(\bar{z}) \wedge \phi(\bar{x}, \bar{z})]$$

defines $\phi(\mathcal{U}, \bar{b})$ and it is in $L(A)$, as required. □

Recall that a partial embedding preserves the truth of quantifier-free fmlas (see Definition 4.2)

7.6 Proposition Let $\phi(\bar{x})$ be an L-formula. The following are equivalent:

(i) there is a quantifier-free L-fmla $\psi(\bar{x})$ such that

$$\models \forall \bar{x} \, [\phi(\bar{x}) \leftrightarrow \psi(\bar{x})]$$

(ii) for all partial embeddings $g : \mathcal{U} \rightarrow \mathcal{U}$, for all tuples $\bar{a}$ in $\mathrm{dom}(g)$

$$\models \phi(\bar{a}) \leftrightarrow \phi(g(\bar{a}))$$

Proof (i) ⇒ (ii) clear, since partial embeddings preserve q.f. formulas.

(ii) ⇒ (i) For a tuple $\bar{a}$ in $\mathcal{U}$, let

$$qftp(\bar{a}) = \{\psi(\bar{x}) \in tp(\bar{a}) : \psi(\bar{x}) \text{ is quantifier-free}\}$$

Let $D = \{q(\bar{x}) : q(\bar{x}) = qftp(\bar{a}) \text{ for } \bar{a} \in \phi(\mathcal{U})\}$.

Claim: $\displaystyle \phi(\mathcal{U}) = \bigcup_{q(\bar{x}) \in D} q(\mathcal{U})$

$\subseteq$ is clear (by the way D is defined)

For $\supseteq$, let $q(\bar{x}) \in D$, say $q(\bar{x}) = qftp(\bar{a})$, and let $b \models q(\bar{x})$. Then $a_i \mapsto b_i$ is a partial embedding, and so, by (ii), $\phi(\bar{b})$ holds. Hence $q(\mathcal{U}) \subseteq \phi(\mathcal{U})$, and therefore

$$\bigcup_{q \in D} q(\mathcal{U}) \subseteq \phi(\mathcal{U})$$

Then in particular $q(\bar{x}) \to \varphi(\bar{x})$. By (an argument similar to that in) Fact 7.2, there is $\psi_q(\bar{x}) \in q(\bar{x})$ such that
$\models \psi_q(\bar{x}) \to \varphi(\bar{x})$.

Then $\varphi(\bar{x}) \leftrightarrow \bigvee_{q(\bar{x}) \in D} \{\psi_q(\bar{x}) : \psi_q(\bar{x}) \to \varphi(\bar{x}) \text{ and } \psi_q(\bar{x}) \in q(\bar{x})\}$

Again by a version of Fact 7.2, there are $\psi_{q_1}(\bar{x}), \dots, \psi_{q_n}(\bar{x})$ such that

$$\varphi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{n} \psi_{q_i}(\bar{x})$$

and $\bigvee_{i=1}^{n} \psi_{q_i}(\bar{x})$ is the required quantifier-free formula. $\square$

7.7 Definition An L-theory T has quantifier elimination (q.e.) if for every L-formula $\varphi(\bar{x})$ there is a quantifier-free L-formula $\psi(\bar{x})$ such that

$$T \vdash \forall \bar{x}\ [\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})]$$

Remark: if a theory has q.e., then the type of a tuple in $\mathcal{U}$ is determined by the quantifier-free type, and so types of tuples are easier to describe and understand – and so are definable sets.

7.8 Theorem Let T be a complete theory with an infinite model. The following are equivalent:

(i) T has quantifier elimination

(ii) every partial embedding $p : \mathcal{U} \to \mathcal{U}$ is elementary

(iii) for every partial embedding $p : \mathcal{U} \to \mathcal{U}$ such that $|p| < |\mathcal{U}|$ and for every $b \in \mathcal{U}$ there is a partial embedding $\hat{p} \supseteq p$ such that $b \in \mathrm{dom}(\hat{p})$.

Proof: (i) $\Rightarrow$ (ii) Clear, since every formula is equivalent to a quantifier-free one and partial embeddings preserve q.f. formulas
(ii) $\Rightarrow$ (i) By (ii) and Proposition 7.6 (since all partial embeddings are elementary, any formula is preserved by all partial embeddings, and so it is equivalent to a q.f. fmla by Proposition 7.6)

(ii) $\Rightarrow$ (iii) Let $p : \mathcal{U} \to \mathcal{U}$ be such that $|p| < \mathcal{U}$. Then by (ii) p is elementary, so it extends to $\alpha \in \mathrm{Aut}(\mathcal{U})$, and $p \cup \{\langle b, \alpha(b) \rangle\}$ is the required extension of p.
(iii) $\Rightarrow$ (ii) Let $p : \mathcal{U} \to \mathcal{U}$ be a partial embedding. Consider $p_0 \subseteq p$ finite. Extend p_0 to $\alpha \in \mathrm{Aut}(\mathcal{U})$ by back-and-forth and saturation of $\mathcal{U}$. Then p_0 is elementary. This can be done for all finite restrictions of p, so p is elementary.

Remark: a fourth condition equivalent to (i), (ii) and (iii) is
(iv) for every finite partial embedding $p: \mathcal{U} \to \mathcal{U}$ and $b \in \mathcal{U}$, there is $\hat{p} \supseteq p$ such that $\hat{p}$ is a partial embedding and $b \in \mathrm{dom}(\hat{p})$.
Proof: exercise, later.

This condition and the results in Chapter 4 show that T_{rg} and T_{dlo} have quantifier elimination.

Remark: if a theory T has q.e. then all embeddings between models of T are elementary (that is, T is model-complete)

(7.9) Definition An element $a \in \mathcal{U}$ is definable over $A \subseteq \mathcal{U}$ if the singleton $\{a\}$ is a definable set (i.e. there is an $L(A)$-formula $\phi(x)$ such that $\phi(\mathcal{U}) = \{a\}$).
An element $a \in \mathcal{U}$ is algebraic over $A \subseteq \mathcal{U}$ if there is an $L(A)$-formula $\phi(x)$ such that $a \in \phi(\mathcal{U})$ and $|\phi(\mathcal{U})| < \omega$
(such a formula is said to be algebraic)

The definable closure of A is the set
$$\mathrm{dcl}(A) = \{a \in \mathcal{U} : a \text{ is definable over } A\}$$
The algebraic closure of A is the set
$$\mathrm{acl}(A) = \{a \in \mathcal{U} : a \text{ is algebraic over } A\}$$

If $A = \mathrm{acl}(A)$, A is said to be algebraically closed.

(7.10) Proposition For $a \in \mathcal{U}$ and $A \subseteq \mathcal{U}$ the following are equivalent:
(i) $a \in \mathrm{dcl}(A)$
(ii) $O(a/A) = \{a\}$ (i.e. a is fixed by all $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$).
Proof Let $\phi(x)$ define a over A. Then $\phi(\mathcal{U})$ is invariant under $\mathrm{Aut}(\mathcal{U}/A)$, and so $O(a/A) \subseteq \{a\}$.
For the other direction, use that $O(a/A)$ is invariant over A and so by Proposition 7.5 there is an $L(A)$-fmla that defines $O(a/A)$.

(7.11) Theorem Let $A \subseteq \mathcal{U}$ and $a \in \mathcal{U}$. The following are equivalent:
(i) $a \in \mathrm{acl}(A)$
(ii) $O(a/A)$ is finite
(iii) $a \in M$ for any model M such that $A \subseteq M$

Proof (i) $\Rightarrow$ (ii) If $a \in \mathrm{acl}(A)$ then there is $\phi(x)$ in $L(A)$ such that $\phi(a)$ holds and $|\phi(\mathcal{U})| < \omega$.
Since $\phi(\mathcal{U})$ is invariant over A, $O(a/A) \subseteq \phi(\mathcal{U})$, and so $|O(a/A)| < \omega$

(ii) $\Rightarrow$ (i) Since $O(a/A)$ is finite, it is definable (by the formula $\bigvee_{i=1}^{n} x = a_i$, where $\{a_1, \dots, a_n\} = O(a/A)$. But $O(a/A)$ is invariant over A, so by Proposition 7.5 there is an $L(A)$-formula $\phi(x)$ that defines $O(a/A)$. Then $|\phi(\mathcal{U})| < \omega$ and $a \in \phi(\mathcal{U})$, so $a \in acl(A)$.

(i) $\Rightarrow$ (iii) If $a \in acl(A)$ then there is an $L(A)$-formula $\phi(x)$ such that $\models \phi(a) \wedge \exists^{=n} x\, \phi(x)$. By elementarity, $\exists^{=n} x\, \phi(x)$ holds in any model containing A, and the n solutions in M are also the solutions in $\mathcal{U}$, therefore $a \in M$.

(iii) $\Rightarrow$ (i) Suppose $a \notin acl(A)$ and let $p(x) = tp(a/A)$. Then for all $\phi(x) \in p(x)$ we have $|\phi(\mathcal{U})| \geq \omega$. Then $|p(\mathcal{U})| \geq \omega$ (see Q6 on Example Sheet 2), and we can show $|p(\mathcal{U})| = |\mathcal{U}|$ (see Example Sheet 2). If M is a model such that $A \subseteq M$, by cardinality considerations $p(\mathcal{U}) \setminus M \neq \emptyset$. Let $b \in p(\mathcal{U}) \setminus M$. Then by homogeneity there is $\alpha \in Aut(\mathcal{U}/A)$ such that $\alpha(b) = a$. Then $\alpha[M]$ is a model that contains A but does not contain $a = \alpha(b)$.

(7.12) Proposition Let $a \in \mathcal{U}$ and $A \subseteq \mathcal{U}$. Then
(i) if $a \in acl(A)$ then $a \in acl(A_0)$ for $A_0 \subseteq A$, A_0 finite
(ii) $A \subseteq acl(A)$
(iii) if $A \subseteq B$ then $acl(A) \subseteq acl(B)$
(iv) $acl(A) = acl(acl(A))$
(v) $acl(A) = \bigcap_{A \subseteq M} M$ (where M ranges over models containing A).

Proof (ii) if $a \in A$ then $a \in dcl(A) \subseteq acl(A)$
(iv) By (ii) and (iii), $acl(A) \subseteq acl(acl(A))$. For the converse inclusion: by Theorem 7.11, $A \subseteq M \Leftrightarrow acl(A) \subseteq M$.
Moreover, $a \in acl(acl(A)) \Rightarrow a \in M$ for every $M \supseteq acl(A)$.
Therefore $a \in acl(acl(A)) \Rightarrow a \in M$ for every $M \supseteq A$, so $a \in acl(A)$.
(v) follows from Theorem 7.11.

Properties (i) to (iv) say that acl is a closure operator with finite character.

(7.13) Proposition Let $\beta \in Aut(\mathcal{U})$ and $A \subseteq \mathcal{U}$. Then
$$\beta[acl(A)] = acl(\beta[A]).$$
Proof: ($\subseteq$) For $a \in acl(A)$ let $\phi(x, \bar{z})$ be an L-fmla and $\bar{b} \in A^{|\bar{z}|}$ such that $a \in \phi(\mathcal{U}, \bar{b})$ and $|\phi(\mathcal{U}, \bar{b})| < \omega$. Then $\models \phi(\beta(a), \beta(\bar{b}))$ and $|\phi(\mathcal{U}, \beta(\bar{b}))| < \omega$, so $\beta(a)$ is algebraic over $\beta(\bar{b})$.
This proves $\beta[acl(A)] \subseteq acl(\beta[A])$.
The same proof with β^{-1} in place of β and $\beta[A]$ in place of A gives the converse inclusion

8 STRONGLY MINIMAL THEORIES

8.1 Definition Let M be an infinite L-structure. A subset $A \subseteq M$ is cofinite if $M \setminus A$ is finite.

8.2 Remark Finite and cofinite sets are definable in every structure. In this chapter, we'll look at structures where these are the only definable sets.

8.3 Definition An L-structure M is said to be minimal if all its definable subsets are finite or cofinite.
M is strongly minimal if it is minimal and all its elementary extensions are minimal.
If T is a consistent theory without finite models, then T is strongly minimal if for every L-formula $\phi(x, \bar{z})$ there is $n \in \omega$ such that

$$T \vdash \forall \bar{z} \left[\exists^{\leq n} x \ \phi(x, \bar{z}) \vee \exists^{\leq n} x \ \neg\phi(x, \bar{z}) \right].$$

Example: let $L = \{E\}$ where E is a binary relation, and let M be an L-structure where E is interpreted as an equivalence relation with exactly one equivalence class of size n $\forall n \in \omega \setminus \{0\}$ and no infinite classes. It can be shown that M is minimal. However, there is an elementary extension $N \succcurlyeq M$ where E has an infinite class. Let a be in this infinite class. Then $E(a, x)$ defines an infinite / coinfinite subset of the domain. Hence M is not strongly minimal.

From now on, T is a complete strongly minimal theory with monster model $\mathcal{U}$.

8.4 Definition Let $a \in \mathcal{U}$, $B \subseteq \mathcal{U}$. Then a is independent from B if $a \notin acl(B)$.
The set B is independent if every $a \in B$ is independent from $B \setminus \{a\}$

Notation: we write aB for $\{a\} \cup B$ and $B \setminus a$ for $B \setminus \{a\}$.

8.5 Theorem Let $B \subseteq \mathcal{U}$, $a, b \in \mathcal{U} \setminus acl(B)$. Then
$$b \in acl(aB) \iff a \in acl(bB).$$

Proof Assume for a contradiction that $a \in acl(bB)$ and $b \notin acl(aB)$.
Let $\phi(x, y)$ be an $L(B)$-formula such that
$$\models \phi(a, b) \wedge \exists^{\leq n} x \ \phi(x, b)$$
for some $n \in \omega \setminus \{0\}$

$\in L(aB)$

Consider the formula $\psi(a,y) \equiv \phi(a,y) \wedge \exists^{\leq n} x\ \phi(x,y)$.
Since $b \notin acl(aB)$, $|\psi(a,\mathbb{U})| \geq \omega$, so $|\psi(a,\mathbb{U})| = |\mathbb{U}|$.
By strong minimality, $|\neg\psi(a,\mathbb{U})| < \omega$, so any model M containing B also contains $c \in \psi(a,\mathbb{U})$.
Then $a \in acl(cB)$, and so $a \in M$. So $a \in M$ whenever $M \supseteq B$. By Theorem 7.12 (v), $a \in acl(B)$. $\ddot{X}$

Main examples of strongly minimal theories.

(1) Let K be a field. The language $L_K = \{+, -, \underline{0}, \{\lambda\}_{\lambda \in K}\}$ can be used to axiomatize vector spaces over K, where for $\lambda \in K$ the symbol λ is a unary function symbol that is interpreted as multiplication by the scalar λ. We write λv for $\lambda(v)$.
The theory T_{VSK} includes the following axioms / axiom schemata:
- axioms for abelian groups (using $+, -, \underline{0}$)
- axioms for scalar product:
  - for each $\lambda \in K$, $\forall xy\ [\lambda(x+y) = \lambda x + \lambda y]$
  - for $\lambda_1, \lambda_2, \mu \in K$ such that $\lambda_1\lambda_2 = \mu$ $\forall x\ [\lambda_1(\lambda_2 x) = \mu x]$
  - for $\lambda_1, \lambda_2, \nu \in K$ s.t. $\lambda_1 + \lambda_2 = \nu$ $\forall x\ [\lambda_1 x + \lambda_2 x = \nu x]$
- $\forall x\ [1x = x]$
- $\exists x\ [x \neq \underline{0}]$ → it is categorical in every $\lambda > |K|$

If K is an infinite field, it can be shown (with some work!) that T_{VSK} is complete and has q.e. Then
- linear combinations $\lambda_1 x_1 + \dots + \lambda_n x_n$ are terms
- atomic formulas are equalities between linear combinations
- atomic formulas with one variable and parameters are equivalent to formulas of the form $\lambda x = a$. Therefore such formulas define singletons.
- therefore, quantifier-free formulas with one variable and parameters define finite or cofinite sets.

By q.e., T_{VSK} is strongly minimal. Moreover one can show
- $acl(A) = \langle A \rangle$, the linear span
- $a \notin acl(A) \iff a$ is linearly independent from A
- A is independent $\iff$ it is linearly independent

(If K is finite, there is a theory T^{∞}_{VSK} of infinite dimensional v. spaces)

(2) Let $L_{ring} = \{+, \cdot, -, 0, 1\}$. Then ACF is the L_{ring}-theory that includes:
- axioms for abelian groups (using $+, -, 0$)
- axioms for commutative monoids (using $\cdot, 1$)
- field axioms:
  - $\forall xyz\ [x \cdot (y+z) = x \cdot y + x \cdot z]$
  - $\forall x\ [x = 0 \vee \exists y\ (x \cdot y) = 1]$
- axioms for algebraic closure: for all $n < \omega$
  - $\forall x_0 \dots x_n \exists y\ [x_0 + x_1 y + \dots + x_n y^n = 0]$.

Let $\chi_p = \underbrace{1+1+\cdots+1}_{p \text{ times}} = 0$. Then $ACF_p = ACF \cup \{\chi_p\}$ (p prime)

and $ACF_0 = ACF \cup \{\chi_n : n < \omega\}$

ACF_0 and ACF_p for a given p are both complete and have q.e. (showing this requires some work). Then.

- atomic formulas are polynomial equations;
- if $A \subseteq M \models ACF_p$ or ACF_0, an atomic fmla in $L_{ring}(A)$ with one free variable is equivalent to a polynomial equation $p(x) = 0$, where $p(x) \in F[x]$ and F is the subfield of K generated by A;
- therefore, atomic fmlas with parameters and one free variable define finite sets.
- quantifier-free formulas with parameters and one free variable define finite or cofinite sets
- by q.e., ACF_0 and ACF_p are strongly minimal.

(8.6) **Definition** Let $B \subseteq C \subseteq \mathcal{U}$. Then B is a basis of C if B is an independent set and $C \subseteq acl(B)$, or, equivalently, if $acl(B) = acl(C)$.

(8.7) **Lemma** If B is an independent set and $a \notin acl(B)$, then $\{a\} \cup B$ is an independent set.

Proof: Suppose $a \notin acl(B)$ and that $\{a\} \cup B$ is not independent. Then there is $b \in B$ such that $b \in acl(a, B \setminus b)$. Since B is independent, $b \notin acl(B \setminus b)$. Also, $a \notin acl(B \setminus b)$. So by Theorem 8.5 we have $a \in acl(b, B \setminus b) = acl(B)$ ※.

(8.8) **Corollary** If $B \subseteq C \subseteq \mathcal{U}$, the following are equivalent:

(i) B is a basis of C

(ii) B is a maximal independent subset of C (i.e. if $B \subseteq B' \subseteq C$ and B' is independent, then $B = B'$).

(8.9) Theorem Let $C \subseteq \mathcal{U}$. Then

(i) every independent subset $B \subseteq C$ can be extended to a basis of C

(ii) if A, B are bases of C, then $|A| = |B|$

Proof (i) If $\langle B_i : i < \lambda \rangle$ is a chain of independent subsets of C such that for all i $B \subseteq B_i$, then $\bigcup_{i<\lambda} B_i$ is independent.

If not, let $b \in \bigcup B_i$ be such that $b \in acl(\bigcup B_i \setminus b)$. Then there is finite $A \subseteq B_i \setminus b$ for some i such that $b \in acl(A \setminus b)$. But this implies $b \in acl(B_i \setminus b)$ for some i. ※

Therefore, by Zorn's Lemma, there is a maximal independent subset of C. By Corollary 8.8, this maximal subset is the required basis.

(ii) Suppose A and B are bases of C and B is infinite. Assume for a contradiction that $|A| < |B|$. For $a \in A$, let $D_a \subseteq B$ be finite and such that $a \in acl(D_a)$. Let $D = \bigcup_{a \in A} D_a$. Then $A \subseteq acl(D_a)$, and $|D| < |B|$.
Since $A \subseteq acl(D)$ and A is a basis, $C \subseteq acl(D)$. But then $B \subseteq acl(D)$, which contradicts the independence of B.

If A and B are finite, show that $|A| \leq |B|$ by using the following result: if there is $a \in A \setminus B$, then there is $b \in B \setminus A$ such that the set $\{b\} \cup (A \setminus \{a\})$ is independent. This is because if $a \in A \setminus B$, then $a \in acl(B)$, as B is a basis. Then $B \nsubseteq acl(A \setminus a)$, otherwise A is not independent. Let $b \in B \setminus acl(A \setminus a)$. Then $\{b\} \cup A \setminus \{a\}$ is independent by Lemma 8.7.

8.10 Definition Let $C \subseteq \mathcal{U}$ be algebraically closed. Then the dimension of C, $\dim(C)$, is $|A|$ for any basis A of C.

8.11 Proposition Let $f : \mathcal{U} \to \mathcal{U}$ be a partial elementary map. Let $b \notin acl(dom(f))$, $c \notin acl(ran(f))$. Then $f \cup \{\langle b, c \rangle\}$ is elementary.
Proof Let $\bar{a}$ be an enumeration of $dom(f)$, and let $\varphi(x, \bar{a})$ be an $L(\bar{a})$-formula.
Claim: $\models \varphi(b, \bar{a}) \leftrightarrow \varphi(c, f(\bar{a}))$.
Case 1: $|\varphi(\mathcal{U}, \bar{a})| < \omega$. Then $|\varphi(\mathcal{U}, f(\bar{a}))| < \omega$ by elementarity of f. Since $b \notin acl(\bar{a})$ and $c \notin acl(f(\bar{a}))$, we have
$$\models \neg\varphi(b, \bar{a}) \wedge \neg\varphi(c, f(\bar{a}))$$
Case 2: $|\varphi(\mathcal{U}, \bar{a})| \geq \omega$. By strong minimality, $|\neg\varphi(\mathcal{U}, \bar{a})| < \omega$ and the argument in Case 1 applies to give
$$\models \varphi(b, \bar{a}) \wedge \varphi(c, f(\bar{a})).$$

8.12 Corollary Every bijection between independent subsets of $\mathcal{U}$ is elementary
Proof: Let $A, B \subseteq \mathcal{U}$ be independent and let $f : A \to B$ be a bijection. Let $\bar{a}$ enumerate A and let $\bar{b} = f(\bar{a})$. Then $a_0 \notin acl(\emptyset)$ and $b_0 \notin acl(\emptyset)$ (otherwise neither A nor B is independent). By Proposition 8.11, $\{\langle a_0, b_0 \rangle\}$ is an elementary map. The step $i+1$ is similar, since $a_{i+1} \notin acl(a_0, \ldots, a_i)$, and similarly for b_{i+1}.

8.13 Remark If $M \subseteq \mathcal{U}$ is a model, then by Proposition 7.12 (v) M is algebraically closed.

8.14 Theorem Suppose $M, N \subseteq \mathcal{U}$ are such that $\dim(M) = \dim(N)$. Then $M \cong N$.
Proof: let A, B be bases of M and N respectively.

Let $f : A \to B$ be a bijection. By Corollary 8.12, f is elementary.
By homogeneity, there is $\alpha \in Aut(U)$ that extends f. Then
$$\alpha(M) = \alpha(acl(A)) = acl(\alpha(A)) = acl(B) = N$$
↳ by Proposition 7.13.

8.15 __Corollary__ Let T be strongly minimal and let $\lambda > |L|$.
Then T is λ-categorical.

Proof: if $A \subseteq U$, then $|acl(A)| \leq |L(A)|$ because:
- there are at most $|L(A)|$ formulas
- there are finitely many elements in $acl(A)$ that are solutions of a given formula in $L(A)$.

If $|M| = \lambda > |L|$, then a basis of M must have cardinality λ.
By Theorem 8.14, any two models of cardinality λ are isomorphic.

Models of a strongly minimal theory are determined up to isomorphism by their dimension, and hence by their cardinality if the cardinality is greater than the cardinality of the language.

9 COUNTABLE MODELS

Let T be a complete L-theory with a monster model $\mathcal{U}$, and $|\bar{x}| < \omega$.

9.1 Definition An L-formula $\phi(\bar{x})$ is said to isolate a type $p(\bar{x})$ in L if: (i) $\phi(\bar{x})$ is consistent
(ii) $\phi(\bar{x}) \to p(\bar{x})$ (that is, $\bar{a} \models p(\bar{x})$ whenever $\models \phi(\bar{a})$).

A set Δ of L-formulas isolates $p(\bar{x})$ if there is $\phi(\bar{x}) \in \Delta$ which isolates $p(\bar{x})$. When Δ is the set of $L(A)$ formulas with free variables in $\bar{x}$, we say that the set A isolates $p(\bar{x})$ or, if A is clear from the context, that $p(\bar{x})$ is isolated.
Isolated types are sometimes called principal
A model M is said to omit the type $p(\bar{x})$ if $p(\bar{x})$ is not realized in M.

9.2 Remark If M is a model and $p(\bar{x})$ is a type in $L(M)$, then M realizes $p(\bar{x}) \Leftrightarrow M$ isolates $p(\bar{x})$.
($\Leftarrow$) Let $\phi(\bar{x})$ be a consistent $L(M)$-fmla that isolates $p(\bar{x})$. Then $M \models \phi(\bar{a})$ for some $\bar{a}$, and $\bar{a}$ is a realization of $p(\bar{x})$.
($\Rightarrow$) Let $\bar{a} \models p(\bar{x})$. Then $p(\bar{x})$ is isolated by $\bar{x} = \bar{a}$.

9.3 Lemma Let $|L(A)| = \omega$. Let $p(\bar{x})$ be a type in $L(A)$ and suppose that A does not isolate $p(\bar{x})$. Let $\psi(z)$ be a consistent $L(A)$-formula with one free variable.
Then there is $a \in \mathcal{U}$ such that
(i) $\psi(a)$
(ii) $A \cup \{a\}$ does not isolate $p(\bar{x})$.

Proof We build a sequence $\langle \psi_i(z) : i < \omega \rangle$ of $L(A)$-formulas such that:
(i) $\psi_i(z)$ is consistent
(ii) $\psi_{i+1}(z) \to \psi_i(z)$
(iii) a realization of the type $\{\psi_i(z) : i < \omega\}$ is the required solution of $\psi(z)$.

Let $\langle \chi_i(\bar{x}, z) : i < \omega \rangle$ be an enumeration of formulas with free variables in $\bar{x} \cup \{z\}$ and parameters in A, and let $\psi_0(z) = \psi(z)$.
At stage $i+1$:
- if $\chi_i(\bar{x}, z)$ is inconsistent, let $\psi_{i+1}(x) = \psi_i(x)$
- otherwise let $\phi(\bar{x}) \in p(\bar{x})$ be such that
$$\psi_i(z) \wedge \exists \bar{x} (\chi_i(\bar{x}, z) \wedge \neg \phi(\bar{x}))$$
is consistent. Let $\psi_{i+1}(z)$ be this conjunction. Then at stage $i+1$

we have guaranteed that a realization of ψ_{i+1} does not isolate the type.
We must show that $\phi(\bar{x}) \in p(\bar{x})$ can always be found.
Suppose no $\phi(\bar{x}) \in p(\bar{x})$ makes $\psi_{i+1}(z)$ consistent. Then

$$\chi_i(\bar{x}, z) \wedge \psi_i(z) \rightarrow \phi(\bar{x})$$

for all $\phi(\bar{x})$ in $p(\bar{x})$. But this implies that

$$\exists z\ [\chi_i(\bar{x}, z) \wedge \psi_i(z)] \rightarrow p(\bar{x})$$

that is, $p(\bar{x})$ is isolated by an $L(A)$-formula with free variables in $\bar{x}$. ⨳

9.4 Theorem (Omitting types) Let $L(A)$ be countable and let $p(\bar{x})$ be a consistent type in $L(A)$. Then the following are equivalent:
(i) all models containing A realize $p(\bar{x})$
(ii) A isolates $p(\bar{x})$.

Proof (ii) ⇒ (i) Let $\phi(\bar{x})$ be an $L(A)$-formula that isolates $p(\bar{x})$. Then $\models \exists \bar{x}\, \phi(\bar{x})$, so if M is a model containing A then there is $\bar{a} \in M^{|\bar{x}|}$ such that $M \models \phi(\bar{a})$, and $\bar{a}$ is a realization of $p(\bar{x})$.

(i) ⇒ (ii) We argue by contraposition. Suppose that A does not isolate $p(\bar{x})$. We build a chain $\langle A_i : i < \omega \rangle$ of sets such that
(i) $A_0 = A$
(ii) $|A_i| \leq \omega$ for all i
(iii) A_i does not isolate $p(\bar{x})$.
We eventually take $M = \bigcup_{i \in \omega} A_i$ and show that M is a model that omits $p(\bar{x})$.
At stage $i+1$, enumerate consistent $L(A_i)$-formulas with one free variable, say $\langle \psi_k(z) : k < \omega \rangle$. For $k \in \omega$, find $a^i_k \in \mathfrak{U}$ such that $A_i \cup \{a^i_0, \ldots, a^i_k\}$ does not isolate $p(\bar{x})$. This is possible by Lemma 9.3. We let

$$A_{i+1} := A_i \cup \{a^i_k : k < \omega\}.$$

Let $M = \bigcup_{i<\omega} A_i$. Claim: (i) $M \preccurlyeq \mathfrak{U}$ (ii) M omits $p(\bar{x})$.
For (i), we use the Tarski-Vaught Test: let $\psi(z)$ be a consistent $L(M)$-formula. Then $\psi(z)$ is an $L(A_i)$-fmla for some i, and so it is realized at stage $i+1$.
By construction, M does not isolate $p(\bar{x})$. By Remark 9.2, M does not realize $p(\bar{x})$.

Remark: the hypothesis that $|L(A)| = \omega$ is necessary. There are counterexamples to the OTT where a model M is the disjoint union of two uncountable sets and the set of all bijections between these two sets

9.5 <u>Definition</u> Let M be a model and $A \subseteq M$. Then

(i) M is <u>prime over A</u> if for every $N \supseteq A$ there is an elementary embedding $f : M \to N$ that fixes A pointwise. When $A = \emptyset$, M is said to be <u>prime</u>

(ii) M is <u>atomic over A</u> if for all n and all $\bar{a} \in M^n$, the type $tp(\bar{a}/A)$ is isolated. When $A = \emptyset$, M is said to be <u>atomic</u>.

9.6 <u>Fact</u> Let $\bar{a}, \bar{b} \in \mathfrak{U}^n$, $A \subseteq \mathfrak{U}$, and suppose that $tp(\bar{b}\bar{a}/A)$ is isolated. Then $tp(\bar{b}/A\bar{a})$ and $tp(\bar{a}/A)$ are isolated.

Proof Let $p(\bar{x}, \bar{z}) = tp(\bar{b}\bar{a}/A)$. Then $tp(\bar{b}/A\bar{a}) = p(\bar{x}, \bar{a})$, and

$tp(\bar{a}/A) = \{\exists \bar{x}\, \varphi(\bar{x}, \bar{z}) : \varphi(\bar{x}, \bar{z})$ is a conjunction of fmlas in $p(\bar{x}, \bar{z})\}$

$= \{\exists \bar{x}\, \varphi(\bar{x}, \bar{z}) : \varphi(\bar{x}, \bar{z}) \in p(\bar{x}, \bar{z})\}$ (check)

Suppose $\varphi(\bar{x}, \bar{z})$ isolates $p(\bar{x}, \bar{z})$. Then

(i) $\varphi(\bar{x}, \bar{a})$ isolates $tp(\bar{b}/A\bar{a})$

(ii) $\exists \bar{x}\, \varphi(\bar{x}, \bar{z})$ isolates $tp(\bar{a}/A)$.

9.7 <u>Proposition</u> If M is atomic over A, then for every finite tuple $\bar{a} \in M^n$, M is atomic over $A\bar{a}$.

Proof Let $\bar{b}$ be a finite tuple. Then $tp(\bar{b}\bar{a}/A)$ is isolated, and so $tp(\bar{b}/A\bar{a})$ is isolated by Fact 9.6.

9.8 <u>Proposition</u> Let $f : M \to N$ be an elementary map, and let M be atomic over $\operatorname{dom} f$. Then for every $b \in M$ there is $c \in N$ such that $f \cup \{\langle b, c\rangle\}$ is elementary.

Proof Let $\bar{a}$ enumerate $\operatorname{dom}(f)$ and let $p(x, \bar{z}) = tp(b\bar{a}/\emptyset)$. Let $\varphi(x, \bar{a})$ isolate $p(x, \bar{a})$. By elementarity, $\varphi(x, f(\bar{a}))$ isolates $p(x, f(\bar{a}))$. Then any solution of $\varphi(x, f(\bar{a}))$ is the required c.

9.9 <u>Proposition</u> Any two countable models that are atomic over A are isomorphic.

Proof By a back-and-forth argument using Propositions 9.7 and 9.8.

9.10 <u>Theorem</u> Assume that $L(A)$ is countable. Then for any model $M \supseteq A$ the following are equivalent:

(i) M is countable and atomic over A

(ii) M is prime over A.

Proof (i) $\Rightarrow$ (ii) Let N be a model containing A. Then $id_A : M \to N$ is elementary, and it can be extended to an embedding $g : M \to N$ by Propositions 9.8 and 9.9.

(ii) $\Rightarrow$ (i) By Downward Löwenheim - Skolem, we know there is a countable model containing A. Since M embeds in this model, M is itself countable. Suppose that there is $\bar{b} \in M^n$ such that $tp(\bar{b}/A)$ is not isolated. By Theorem 9.4 (Omitting Types), there is a countable model N that contains A and omits $tp(\bar{b}/A)$. But then

M does not embed in N. $\times$.

(9.11) Definition For $n \in \omega$ and $A \subseteq \mathcal{U}$, $S_n(A)$ denotes the collection of complete, consistent types with n variables and parameters in A. When $A = \emptyset$, we write $S_n(T)$ for complete consistent types with no parameters and n free variables.

(9.12) Theorem (Ryll-Nardzewski, Engeler, Svenonius) For a countable theory T, the following are equivalent:
(i) T is ω-categorical
(ii) for all n, every type $p(\bar{x}) \in S_n(T)$ is isolated
(iii) for all n, $|S_n(T)| < \omega$
(iv) $\mathrm{Aut}(\mathcal{U})$ has finitely many orbits on $\mathcal{U}^n$ for all $n \in \omega$.

Proof (Sketch) (i) $\Rightarrow$ (ii) Suppose $p(\bar{x}) \in S_n(T)$ is not isolated. Then there is a countable model M that realizes $p(\bar{x})$ (by Downward Löwenheim-Skolem) and there is a countable model N that omits $p(\bar{x})$ (by Omitting Types Theorem). Then $M \not\cong N$.
(ii) $\Rightarrow$ (i) If all types are isolated, then all countable models are atomic. By Proposition 9.9, all countable models are isomorphic.

(ii) $\Rightarrow$ (iii) We have $\mathcal{U}^n = \bigcup p(\mathcal{U})$. If all types in $S_n(T)$ are isolated, then $\mathcal{U}^n = \bigcup_{p \in S_n(T)} \phi_p(\mathcal{U})$, ${}^{p \in S_n(T)}$ where $\phi_p(\bar{x})$ isolates $p(\bar{x})$.
By compactness
$$\mathcal{U}^n = \bigcup_{i=1}^{k} \phi_{p_i}(\bar{x}) \quad \text{(check this!)}$$
for some $k \in \omega$, hence $|S_n(T)| < \omega$.
(iii) $\Rightarrow$ (ii) Let $p(\bar{x}) \in S_n(T)$. If $|S_n(T)| < \omega$, then $\mathcal{U} \setminus p(\mathcal{U})$ is the union of finitely many type-definable sets, hence it is type-definable. By an argument similar to that in Fact 7.2, there are formulas $\phi(\bar{x})$ and $\chi(\bar{x})$ such that $\phi(\mathcal{U}) = p(\mathcal{U})$ and $\chi(\bar{x}) = \mathcal{U} \setminus p(\mathcal{U})$. It follows that $\phi(\bar{x})$ isolates $p(\bar{x})$.

(iii) $\Leftrightarrow$ (iv) This follows from the fact that for $\bar{a}, \bar{b} \in \mathcal{U}^n$
$tp(\bar{a}) = tp(\bar{b}) \Leftrightarrow O(\bar{a}/\emptyset) = O(\bar{b}/\emptyset)$.

Remark We can prove that if M is a countable model of an ω-categorical theory, then $\mathrm{Aut}(M)$ is oligomorphic (i.e. it has finitely many orbits on M^n for all $n \in \omega$). The converse also holds.

(9.13) Definition A complete theory T is small if for all $n \in \omega$ $|S_n(T)| \leq \omega$.

(9.14) <u>Proposition</u> If T is countable, complete and has no finite models, the following are equivalent:

(i) T is small

(ii) For all $n < \omega$ and finite $A \subseteq \mathfrak{U}$ we have $|S_n(A)| \leq \omega$

(iii) For all finite $A \subseteq \mathfrak{U}$, $|S_1(A)| \leq \omega$

(iv) T has a countable saturated model.

Proof (sketch) (i) $\Rightarrow$ (ii) Let A be finite and let $\bar{a}$ enumerate A. Let $q(\bar{y}) = tp(\bar{a})$, and

$$X = \{p(\bar{x}, \bar{y}) \in S_{n+m}(T) : q(\bar{y}) \subseteq p(\bar{x}, \bar{y})\},$$

where $|\bar{y}| = |\bar{a}| = m$ and $|\bar{x}| = n$. Then X is countable, and the map $p(\bar{x}, \bar{y}) \mapsto p(\bar{x}, \bar{a})$ from $S_{n+m}(T)$ to $S_n(A)$ is surjective. Hence $|S_n(A)| \leq |X| \leq \omega$

(ii) $\Rightarrow$ (iii) clear

(iii) $\Rightarrow$ (iv) Build a chain $\langle A_n : n < \omega \rangle$ where each A_i is such that all types over $A \subseteq A_{i-1}$, $|A| < \omega$, are realized. Since $|S_1(A)| \leq \omega$ for all finite A, each A_i can be chosen countable. Then $M = \bigcup_{i<\omega} A_i$ is the required countable saturated model.

(iv) $\Rightarrow$ (i) Since all $p(\bar{x}) \in S_n(T)$ are realized in the countable saturated model, there are at most countably many realizations.

(9.15) <u>Corollary</u> Countable models of ω-categorical theories are saturated.