05/02/2021

Recall that an L-theory T has quantifier elimination (q.e.) if for every formula $\varphi(x) \in L$ there is $\psi(x) \in L$ that is quantifier-free and such that

$$T \vdash \forall x\, [\varphi(x) \leftrightarrow \psi(x)]$$

From now on, work in monster model $\mathcal{U}$ of a complete theory T.

7.6 Proposition Let $\varphi(x) \in L$. Tfae

(i) there is $\psi(x) \in L$, $\psi(x)$ quantifier-free, such that
$$T \vdash \forall x\, [\varphi(x) \leftrightarrow \psi(x)]$$

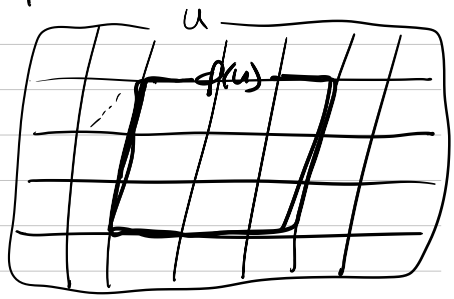
(ii) for all partial embedding $p: \mathcal{U} \to \mathcal{U}$, $a \in \mathrm{dom}(p)^{|x|}$
$$\models \varphi(a) \leftrightarrow \varphi(p(a)).$$

Proof: (i) $\Rightarrow$ (ii) clear $\qquad\qquad$ $\varphi(x) \leftrightarrow$ q.f. fmla

(ii) $\Rightarrow$ (i) For $a \in \mathcal{U}^{|x|}$, let
$$\mathrm{qftp}(a) = \{\psi(x) \in \mathrm{tp}(a) : \psi(x) \text{ is q.f.}\}.$$

Let $D = \{ q(x) : q(x) = \mathrm{qftp}(b) \text{ for some } b \in \varphi(\mathcal{U})\}$.

Claim $\varphi(\mathcal{U}) = \bigcup_{q(x)\in D} q(\mathcal{U})$



($\subseteq$) clear (by choice of D)

($\supseteq$) For $b \models q(x)$ for $q(x) = \mathrm{qftp}(a)$ for $a \in \varphi(D)$
Then a, b have same q.f. type so $a_i \mapsto b_i$ is a partial embedding. Therefore by (ii)
$$\models \varphi(a) \leftrightarrow \varphi(b) \quad \text{so} \quad b \in \varphi(\mathcal{U}).$$
Therefore $q(x) \subseteq \varphi(\mathcal{U})$, and so $\bigcup_{q(x)\in D} q(\mathcal{U}) \subseteq \varphi(\mathcal{U})$. $\square$ claim

In particular $q(x) \to \varphi(x)$ for every $q(x) \in D$. By a compactness/saturation argument (cf. 5.2) there is $\psi_q(x) \in q(x)$ s.t.
$$\psi_q(x) \to \varphi(x).$$
Then by the claim $\varphi(x) \leftrightarrow \bigvee_{q(x)\in D} \{\psi_q(x) : \psi_q(x) \to \varphi(x) \text{ and } \psi_q(x) \in q(x)\}$

Again by compactness / saturation, there are $q_1(x), \dots, q_n(x)$ s.t.
$$(\mathcal{U} \models) \forall x [\varphi(x) \leftrightarrow \bigvee_{i=1}^{n} \psi_{q_i}(x)]$$

Then $\bigvee_{i=1}^{n} \psi_{q_i}(x)$ is the required q.f. fmla. $\square$

7.7 Theorem For complete theory T with monster $\mathcal{U}$, tfae:

(i) T has q.e.

(ii) every partial embedding is elementary

(iii) for every partial embedding $p: \mathcal{U} \to \mathcal{U}$ such $|p| < |\mathcal{U}|$ and $b \in \mathcal{U}$ there is $\hat{p} \supseteq p$ partial emb. s.t. $b \in \mathrm{dom}(\hat{p})$.

[(iv) for every finite partial emb $p: \mathcal{U} \to \mathcal{U}$ and $b \in \mathcal{U}$, there is $c \in \mathcal{U}$

s.t. $p \cup \{\langle b, c\rangle\}$ is a partial embedding.]

Proof (i) $\Rightarrow$ (ii) clear (every fmla is eq. to a q.f. fmla and partial embeddings preserve q.f. fmlas)

(ii) $\Rightarrow$ (i) Since all partial embeddings are elementary, any $\phi(x) \in L$ is preserved by all partial embedding. By Prop 7.6, $\phi(x)$ is equivalent to a q.f. fmla.

(ii) $\Rightarrow$ (iii) Let $p: \mathcal{U} \to \mathcal{U}$ with $|p| < |\mathcal{U}|$, let $b \in \mathcal{U}$. Then by (ii) p is elementary, so there is $\alpha \in \mathrm{Aut}(\mathcal{U})$ such that $p \subseteq \alpha$. Then $p \cup \{\langle b, \alpha(b)\rangle\}$ is the required extension

(iii) $\Rightarrow$ (ii) Let $p: \mathcal{U} \to \mathcal{U}$ be a partial emb. Consider $p_0 \subseteq p$, $|p_0| < \omega$. Extend p_0 to $\alpha \in \mathrm{Aut}(\mathcal{U})$ by back and forth using (ii). Then p_0 is elementary. This can be done for all finite $p_0 \subseteq p$, so p is elementary. $\square$

(Proof of the equivalence of (iv) will be in the Comment file).

In Chapter 2 we proved T_{dlo} and T_{rg} have q.e. (we established completeness via categoricity).

(8) COUNTABLE MODELS

Let T a complete theory with monster $\mathcal{U}$, and assume $|x|, |a| < \omega$ unless o/w specified.

(8.1) <u>Def</u> A fmla $\varphi(x) \in L$ isolates the type $p(x) \subseteq L$ if
(i) $\varphi(x)$ is consistent
(ii) $\varphi(x) \to p(x)$ (i.e. $\models p(a)$ whenever $\models \varphi(a)$)

If $A \subseteq \mathcal{U}$, A is said to isolate $p(x)$ if there is $\varphi(x) \in L(A)$ that isolates $p(x)$. When A is clear from the context, we say that $p(x)$ is <u>isolated</u>, or <u>principal</u>.

The model M omits $p(x)$ if $p(x)$ is not realized in M.

(8.2) <u>Remark</u> If M is a model and $p(x) \subseteq L(M)$ then
M realizes $p(x) \iff M$ isolates $p(x)$
($\Rightarrow$) If $M \models p(a)$, then $x = a$ isolates $p(x)$
($\Leftarrow$) Let $\varphi(x) \in L(M)$ isolate $p(x)$. Then $M \models \varphi(a)$ for some a, and a is a realization of $p(x)$.

(8.3) <u>Lemma</u> Let $|L(A)| = \omega$ and let $p(x) \subseteq L(A)$. Suppose A does not isolate $p(x)$. Let $\underbrace{\psi(z)}_{\text{consistent}} \in L(A)$, $|z| = 1$. Then there is $a \in \mathcal{U}$ such that
(i) $\models \psi(a)$
(ii) $A \cup \{a\}$ does not isolate $p(x)$.

<u>Proof</u> We build a sequence $\langle \psi_i(z) : i < \omega \rangle$ of fmlas such that:
(i) $\psi_i(z) \in L(A)$, $\psi_i(z)$ consistent
(ii) $\psi_{i+1}(z) \to \psi_i(z)$
(iii) a realization of the type $\{\psi_i(z) : i < \omega\}$ is the required $a \in \mathcal{U}$.

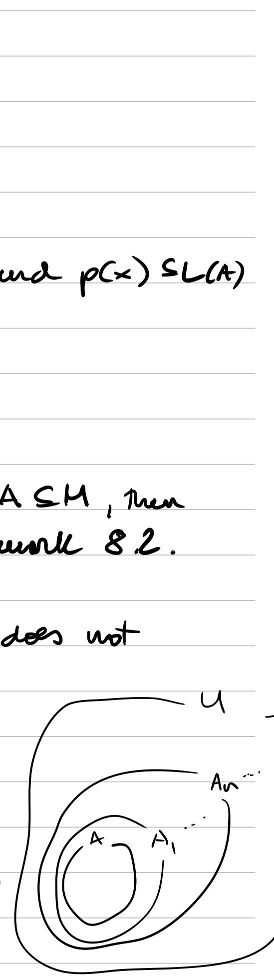
Let $\psi_0(z) = \psi(z)$. Let $\langle \chi_i(x, z) : i < \omega \rangle$ be an enumer. of $L(A)$-fmlas with variables in $x \cup z$.
At stage $i+1$:
- if $\chi_i(x, z)$ is inconsistent, let $\psi_{i+1}(z) = \psi_i(z)$.
- otherwise, let $\varphi(x) \in p(x)$ be such that
$$\psi_i(z) \wedge \exists x\, [\chi(x, z) \wedge \neg\varphi(x)]$$
is consistent. Let $\psi_{i+1}(z)$ be this conjunction.

Then $\psi_{i+1}(z)$ does not isolate $p(x)$.
We show that such a $\varphi(x)$ can always be found. If no $\varphi(x) \in p(x)$ makes $\psi_{i+1}(z)$ consistent, then
$$[\chi_i(x, z) \wedge \psi_i(z)] \to \varphi'(x) \quad \text{for all } \varphi'(x) \in p(x)$$
But then the fmla
$$\exists z\, [\chi_i(x, z) \wedge \psi_i(z)] \text{ isolates } p(x) \quad \text{※} \qquad \square$$

8.4 Theorem (Omitting Types) [OTT]. Let $|L(A)| = \omega$ and $p(x) \subseteq L(A)$ be a consistent type. Then tfae:
(i) all models containing A realize $p(x)$
(ii) A isolates $p(x)$

<u>Proof</u> (ii) $\Rightarrow$ (i) If $A \subseteq \mathcal{U}$ and A isolates $p(x)$. If $A \subseteq M$, then clearly M isolates $p(x)$, and so M realizes $p(x)$ by Remark 8.2.

(i) $\Rightarrow$ (ii) We argue by contraposition. Suppose $A \subseteq \mathcal{U}$ does not isolate $p(x)$. We build a chain $\langle A_i : i < \omega \rangle$ of subsets of $\mathcal{U}$ such that
(a) $A_0 = A$
(b) $|A_i| \leq \omega$ for all i
(c) A_i does not isolate $p(x)$
[At the end: we show $\bigcup A_i$ is a model that omits $p(x)$]



At stage $i+1$, enumerate consistent $L(A_i)$-fmlas with one free variable, say $\langle \varphi_k^i(z) : k < \omega \rangle$. For $k < \omega$ find $a_k^i \in \mathcal{U}$ s.t.
(i) $\models \varphi_k^i(a_k^i)$ (ii) $A \cup \{a_0^i, \ldots, a_k^i\}$ does not isolate $p(x)$.
By Lemma 8.3, it is always possible to find a_k^i.
Let $A_{i+1} := A_i \cup \{a_k^i : k < \omega\}$.

Let $M = \bigcup_{i<\omega} A_i$. Then (i) $M \preccurlyeq \mathcal{U}$ (ii) M omits $p(x)$

For (i), use Tarski-Vaught Test: if $\varphi(z)$ is a consistent fmla in $L(M)$,

$|z|=1$, then $\psi(z) \in L(A_i)$ for some i, and so A_{i+1} contains a realiz. of $\psi(z)$. $\square$ ($M \preceq \mathfrak{U}$)

By construction, M does not isolate $p(x)$. By Remark 8.2, M omits $p(x)$. $\square$

8.5 <u>Definition</u> Let M be a model and $A \subseteq M$. Then:

(i) M is <u>prime over A</u> if for every $N \supseteq A$ there is $f: M \to N$ elementary embedding that fixes A pointwise.
When $A = \emptyset$, we say M is <u>prime</u>

(ii) M is <u>atomic</u> over A if for all $n \in \omega$ and $a \in M^n$, the type $tp(a/A)$ is isolated. If $A = \emptyset$, M is said to be <u>atomic</u>

8.6 <u>Remark</u> Let $a, b \in \mathfrak{U}^n$ ($|a| = |b| = n < \omega$) and suppose $tp(ba/A)$ is isolated, then $tp(b/Aa)$ is isolated.
This is because if $\varphi(x, z)$ that isolates $tp(ba/A) = p(x, z)$, then $\varphi(x, a)$ isolates $tp(b/Aa)$.

8.7 <u>Fact</u> If M is atomic over A, then for every finite tuple $a \in M^n$, M is atomic over $A \cup \{a\}$.
<u>Proof</u>: Let $b \in M^k$ ($k < \omega$). Then $tp(ba/A)$ is isolated, so $tp(b/Aa)$ is isolated by Remark 8.6.

8.8 <u>Proposition</u> Let $f: M \to N$ be elementary map, where M is atomic over $dom(f)$. Then for every $b \in M$, there $c \in N$ such that $f \cup \{\langle b, c\rangle\}$ is elementary.

<u>Proof</u> Let a enumerate $dom(f)$ and let $p(x, z) = tp(ba/\emptyset)$ (so $|x| = 1$ and $|z| = |a| = |dom(f)|$).
Then $p(x, a)$ is isolated by some $\varphi(x, a)$. By elementarity of f, $\varphi(x, f(a))$ isolates $p(x, f(a))$. Then any realization of $p(x, f(a))$ in N is the required c.

8.9 <u>Proposition</u> Any two countable models that are atomic over A are isomorphic.
<u>Proof</u>: Use a back and forth argument and Fact 8.7 and Prop. 8.8. $\square$