Questions 1 and 3 ask you for short proofs that were left as exercises in the first two lectures.

Please **hand in** solutions to **Questions 6** and **8** by 2 pm on Wed 5 February, for discussion in class on 6 February. You are welcome to email a pdf (typeset or scanned) if you prefer.

If you spot any typos or problems, please email `silvia.barbina@open.ac.uk`.

Question 1 Let M and N be L-structures, and let $\beta : M \to N$ be an embedding. Let $|\bar{x}|$ denote the length of the tuple $\bar{x}$, and recall that if $\bar{a} \in M^{|\bar{x}|}$ then $\beta(\bar{a}) := (\beta(a_1), \ldots, \beta(a_{|\bar{x}|}))$.

(i) A formula is said to be *quantifier-free* if it does not use the symbol $\exists$ (or the abbreviation $\forall$). Prove that if $\varphi(\bar{x})$ is a quantifier-free formula and $\bar{a} \in M^{|\bar{x}|}$, then

$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\beta(\bar{a})).$$

(You may want to give a recursive definition of the set of quantifier-free formulas first.)

(ii) An *existential formula* is a formula that has the form $\exists \bar{y}\, \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is quantifier-free. Suppose that M and N are L-structures and let $\beta : M \to N$ be an embedding. Prove that if $\varphi(\bar{x})$ is an existential formula and $\bar{a} \in M^{|\bar{x}|}$, then

$$M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(\beta(\bar{a})).$$

(iii) Suppose that β is surjective, so $\beta : M \simeq N$. Show that if $\varphi(\bar{x})$ is an L-formula and $\bar{a} \in M^{|\bar{x}|}$, then

$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\beta(\bar{a})).$$

Question 2 Let L be the language that only contains a binary function symbol $+$. Regard $\mathbb{Z}$ as an L-structure by interpreting $+$ as the usual addition of integers. Prove that there is no existential formula $\varphi(x)$ such that

$$\mathbb{Z} \models \forall x \left(\varphi(x) \leftrightarrow \forall y \left(y + y \neq x\right)\right).$$

(The formula $\forall y \left(y + y \neq x\right)$ says that x is odd.)

Question 3 Let N be an L-structure and let A be a subset of N. Prove that the following are equivalent.

(i) A is the domain of a substructure of N;

(ii) for every constant c of L, $c^N \in A$; and for every function symbol f of L and $\bar{a} \in A^{n_f}$, we have $f^N(\bar{a}) \in A$.

If (i) and (ii) hold, then the structure on A is uniquely determined.

Question 4 Let $L = \{+, -, 0\}$ be the language of additive groups (so in particular $-$ is a unary function). Let $\mathbb{Z}^n$ denote the direct product $\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}}$, regarded as an L-structure.

Show that if $m \neq n$ then $\mathbb{Z}^m \not\equiv \mathbb{Z}^n$.

Question 5 Let M and N be L-structures and let $A \subseteq M \cap N$. Show that

$$M \equiv_A N \quad \Leftrightarrow \quad M \equiv_B N \text{ for all finite } B \subseteq A.$$

Question 6 Let $M_1 = [0,1)$, $M_2 = [0,1) \cup (2,3)$, $M_3 = [0,1) \cup [2,3)$ and $M_4 = [0,3]$ be the usual intervals in $\mathbb{Q}$, regarded as L_{lo}-structures. For all $i, j \in \{1,2,3,4\}$ with $i < j$ determine whether $M_i \preceq M_j$. When $M_i \preceq M_j$, give a proof without using Proposition 4.12.

Question 7 Let $\langle M_i : i \in \lambda \rangle$ be an *elementary chain*, that is, a sequence of L-structures indexed by an ordinal λ and such that, for all $i < j < \lambda$, $M_i \preceq M_j$.

Let N be the union of the chain as defined in Definition 3.10.

Prove that, for every $i < \lambda$, we have $M_i \preceq N$.

Question 8 Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Question 9 If λ is an infinite cardinal and T is a theory with models of size λ, we say that T is *λ-categorical* if any two models of T of cardinality λ are isomorphic. Prove that T_{dlo}, the theory of dense linear orders without endpoints, is not λ-categorical for any uncountable λ.