Check "Comments and errata" file on course website! 29/01/2021

"Extra" lecture on Mon 1st Feb 2 - 3:15

ACF_p and ACF_0 → alg closed fields of char 0

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algebr closed fields of char p

ACF_p and ACF_0 are complete and have quantifier elimination (proving this requires some work). Then:

- atomic formulas are polynomial equations
- if $A \subseteq M \models ACF_p$ or ACF_0, then an atomic fmla in $L_{ring}(A)$ with one free variable is equivalent to a polynomial eqn of the form $p(x) = 0$, where $p(x) \in F[x]$ where F is the subfield generated by A
- atomic fmlas with one variable and parameters define finite sets
- quantifier-free fmlas with one free var. and param. define finite or cofinite sets.

By q.e., ACF_p and ACF_0 are strongly minimal.

From now on, T is a complete strongly minimal theory with monster model $\mathcal{U}$.

Notation: we write aA for $\{a\} \cup A$ and $B \setminus a$ for $B \setminus \{a\}$

Remember: $a \in A$ is independent from $B \subseteq \mathcal{U}$ if $a \notin acl(B)$
B is indep. if for all $b \in B$, $b \notin acl(B \setminus b)$

6.4 Theorem Let $B \subseteq \mathcal{U}$, $a, b \in \mathcal{U} \setminus acl(B)$. Then
$$b \in acl(aB) \iff a \in acl(Bb)$$

Proof: Assume for ※ that $a \in acl(bB)$ and $b \notin acl(aB)$

Let $\phi(x, y) \in L(B)$ be such that $\models \phi(a, b) \wedge \exists^{\leq n} x\, \phi(x, b)$ for some $n \in \omega \setminus \{0\}$. Consider the following $L(aB)$-fmla:
$$\psi(a, y) \equiv \phi(a, y) \wedge \exists^{\leq n} x\, \phi(x, y).$$
Since $b \notin acl(aB)$, we must have $|\psi(a, \mathcal{U})| \geq \omega$, and so $|\psi(a, \mathcal{U})| = |\mathcal{U}|$. By strong minimality $|\neg\psi(a, \mathcal{U})| < \omega$, and so any model M containing B also contains $c \in \psi(a, \mathcal{U})$. Then $a \in acl(cB)$ and so $a \in M$. Therefore $a \in M$ whenever $B \subseteq M$. But this implies $a \in acl(B)$ by Theorem 5.8 ※.

6.5 Definition Let $B \subseteq C \subseteq \mathcal{U}$. Then B is a basis of C if B is an independent set and $C \subseteq acl(B)$ - or, equivalently, if $acl(B) = acl(C)$.

6.6 <u>Lemma</u> If B is an indep. set and $a \notin acl(B)$, then $\{a\} \cup B$ is independent.

<u>Proof</u> Assume $a \notin acl(B)$ but $\{a\} \cup B$ is not independent.
Then there $b \in B$ such that $b \in acl(aB \setminus b)$.
By indep. of B, $b \notin acl(B \setminus b)$.
$a \notin acl(B) \Rightarrow a \notin acl(B \setminus b)$

$\Rightarrow$ by Thm 6.4

$\Rightarrow a \in acl(bB \setminus b) = acl(B)$ X. $\square$

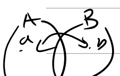
6.7 <u>Corollary</u> If $B \subseteq C \subseteq \mathcal{U}$, the following are equivalent:
(i) B is a basis of C
(ii) B is a maximal indep. subset of C (i.e. if $B \subseteq B' \subseteq C$, and B' is indep. then $B' = B$).

6.8 <u>Theorem</u> Let $C \subseteq \mathcal{U}$. Then
(i) every indep $B \subseteq C$ can be extended to a basis of C
(ii) if A, B are bases of C then $|A| = |B|$.

<u>Proof</u> (i) Claim: if $\langle B_i : i < k \rangle$ is a chain of indep. subsets of C such that $B \subseteq B_i$ for all i, then $\bigcup_{i<k} B_i$ is indep.
Suppose not. Then there is $b \in \bigcup B_i$ such that $b \in acl(\bigcup B_i \setminus b)$
Then $b \in acl(A \setminus b)$ for finite $A \subseteq \bigcup B_i$. But then $A \subseteq B_i$ for some i, so $b \in acl(B_i \setminus b)$ which contradicts indep. of B_i.
By Zorn's lemma, there is a maximal indep. subset of C. By Corollary 6.7, this maximal subset is the required basis of C.

(ii) Suppose A, B are bases of C and $|B| \geq \omega$. Assume $|A| < |B|$.
For $a \in A$, let $D_a \subseteq B$ be finite and such $a \in acl(D_a)$.
Then $D = \bigcup_{a \in A} D_a$ is such that $|D| < |B|$ and $a \in acl(D)$.

A is a basis of C, therefore $C \subseteq acl(D)$, so $B \subseteq acl(D)$.
But then any $b \in B \setminus D$ is such that $b \in acl(B \setminus b)$, contradicting the independence of B.



If A and B are both finite, we can show $|A| \leq |B|$ by using the following fact: if $a \in A \setminus B$ then there is $b \in B \setminus A$ such that $\{b\} \cup (A \setminus \{a\})$ is independent. This is because if $a \in A \setminus B$, then $a \in acl(B)$ (as B basis). Then $B \nsubseteq acl(A \setminus a)$, o/w B is not independent. Let $b \in B \setminus acl(A \setminus a)$. Then $\{b\} \cup (A \setminus \{a\})$ is independent by Lemma 6.6.

6.9 <u>Definition</u> Let $C \subseteq \mathcal{U}$ is algeb. closed, i.e. $C = acl(C)$, then the <u>dimension</u> $\dim(C)$ of C is $|A|$ for any basis A of C.

6.10 Proposition Let $f: \mathcal{U} \to \mathcal{U}$ be an elementary map. Let $b \notin \mathrm{acl}(\mathrm{dom}(f))$ and $c \notin \mathrm{acl}(\mathrm{ran}(f))$. Then $f \cup \{\langle b, c\rangle\}$ is elementary

Proof: Let a enumerate $\mathrm{dom}(f)$, and $\varphi(x, a) \in L(\mathrm{dom}(f)) = L(a)$ with $|x| = 1$. Claim: $\vdash \varphi(b, a) \Leftrightarrow \varphi(c, f(a))$.

Case 1 $|\varphi(\mathcal{U}, a)| < \omega$. Then $\models \neg\varphi(b, a)$. Also f elementary implies $|\varphi(\mathcal{U}, f(a))| < \omega$ and so $\models \neg\varphi(c, f(a))$ (because $c \notin \mathrm{acl}(f(a))$). Hence $\vdash \neg\varphi(b,a) \wedge \neg\varphi(c, f(a))$.

Case 2 $|\varphi(\mathcal{U}, a)| \geq \omega$; Then $|\neg\varphi(\mathcal{U}, a)| < \omega$ by strong minimality. The argument in case 1 applies to $\neg\varphi(x, a)$ and gives $\models \varphi(b, a) \wedge \varphi(c, f(a))$. □

Remark: an element $a \notin \mathrm{acl}(A)$ is said to be transcendental over A. So the proposition says that given A there is a unique transcendental type $/A$: if $a, b \notin \mathrm{acl}(A)$ $\mathrm{id}_A \cup \{\langle a, b\rangle\}$ is elementary.

6.11 Corollary Every bijection between indep. subsets of $\mathcal{U}$ is elementary.

Proof: Let $A, B \subseteq \mathcal{U}$ be independent, let $f: A \to B$ be a bijection. Let $a = \langle a_i : i < |A|\rangle$ and let $f(a_i) = b_i$ (so b enumerates B).

Then: $a_0 \notin \mathrm{acl}(\emptyset)$ and $b_0 \notin \mathrm{acl}(\emptyset)$ (o/w A, B not indep).

But by Prop 6.10 applied to $\emptyset$, $\{\langle a_0, b_0\rangle\}$ is elementary. Successor and limit stages are similar, e.g. at stage $i+1$, $a_{i+1} \notin \mathrm{acl}(a_0, \dots a_i)$ and $b_{i+1} \notin \mathrm{acl}(b_0, \dots, b_i)$ and $a_i \mapsto b_i$ is elem, and so $\{\langle a_j, b_j\rangle : j \leq i+1\}$ is elementary.

6.12 Remark: In any theory T, a model M is such that $\mathrm{acl}(M) \subseteq M$, i.e. a model is algebraically closed. If T is str. minimal, a model has a dimension.

6.13 Theorem Suppose $M, N \subseteq \mathcal{U}$ are models such that $\dim(M) = \dim(N)$. Then $M \cong N$.

Proof: Let A, B be bases of M, N respectively. Then $|A| = |B|$ and there is $f: A \to B$ bijection. By Coroll. 6.11, f is elementary. By homogeneity, there is $\alpha \in \mathrm{Aut}(\mathcal{U})$ that extends f. Then $\alpha[M] = \alpha[\mathrm{acl}(A)] = \mathrm{acl}(\alpha[A]) = \mathrm{acl}(B) = N$. □

$\hookrightarrow$ Prop. 5.10.

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(6.14) **Corollary** Let T be strongly minimal and let $\lambda > |L|$. Then T is λ-categorical.

Proof If $A \subseteq \mathcal{U}$, then $|acl(A)| \leq |L(A)|$ because:
- there are at most $|L(A)|$ formulas
- there are finitely many elements in $acl(A)$ that are solutions of a given algebraic formula in $L(A)$

If $|M| = \lambda > |L|$, then a basis of M must have cardinality λ.
By Theorem 8.14, any two models of cardinality λ are isomorphic □