1 PRELIMINARIES

Let $L = \{\{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K}\}$ be a language

relation, function, constant symbols

(Abuses of) notation

- We do not distinguish between an L-structure $\langle M, \{R_i^M\}, \{f_j^M\}, \{c_k\}\rangle$ and its domain, nor between symbols in L and their interpretations.
- In particular, if $A \subseteq M$ then $L(A)$ denotes the expansion of L to $L \cup \{a : a \in A\}$

  ↳ constant symbol interpreted as a
- We write "$\phi \in L$" for "ϕ is an L-formula"
- Tuples of variables / constants are denoted by x / a (occasionally $\bar{x}, \bar{a}$), and $|x|$ denotes the length of the tuple x – for ex: $a \in M^{|a|}$
- M, N, U, V usually denote structures; A, B, C usually denote subsets of (domains of structures)

Recall that if M, N are L-structures then:

(i) $f : M \to N$ is an embedding $\Leftrightarrow$ for all atomic $\phi(x) \in L$, $a \in M^{|x|}$ $\quad M \models \phi(a) \Leftrightarrow N \models \phi(f(a))$

(ii) $f : M \to N$ is an elementary embedding iff for all $\phi(x) \in L$ $a \in M^{|x|}$ $\quad M \models \phi(a) \Leftrightarrow N \models \phi(f(a))$

If f is the inclusion map, then in

(i) M is a substr of N, written $M \subseteq N$

(ii) M is an elementary substructure of N, written $M \preccurlyeq N$

If f is a bijective embedding, then it is an isomorphism ($M \cong N$) and, in particular, an elementary map.

Example: $L_{lo} = \{<\}$, binary relation symbol. Then

$[0,1]$, $[0,2] \subseteq \mathbb{R}$ are L_{lo} structures

Then: $[0,1] \cong [0,2]$, but the inclusion map is not elementary: let $\phi(x) \equiv \forall y\,(y < x \vee y = x)$. Then

$$[0,1] \models \phi(1) \quad \text{but} \quad [0,2] \not\models \phi(2)$$

So $[0,1] \subseteq [0,2]$, but $[0,1] \not\preccurlyeq [0,2]$.

Recall that $Th(M) = \{\phi \in L : \phi \text{ is a sentence and } M \models \phi\}$ ↗ a Theory is a set of L-sentences

Then if $Th(M) = Th(N)$ we write $M \equiv N$.

In part, $M \equiv N$ whenever $M \cong N$ (so $[0,1] \equiv [0,2]$)

Typically we work in a very large model of a theory T which is such that every other model of T that we are interested in occurs as an elementary substructure of $\mathcal{U}$.

Results that we use often

<u>Tarski-Vaught test</u> (TVT)
Let N be an L-structure, $A \subseteq N$ a subset. Then tfae:
(i) A is the domain of an elementary substructure of N
(ii) for every $\phi(x) \in L(A)$ s.t. $|x| = 1$,
$$N \models \exists x\, \phi(x) \Rightarrow N \models \phi(b) \text{ for some } b \in A.$$

<u>Downward Löwenheim - Skolem theorem (DLS)</u>

Let N be an L-structure such that $|N| \geq |L|$ and let $A \subseteq N$
Then for every cardinal λ such that
$$|L| + |A| \leq \lambda \leq |N|$$
there is an elementary substructure $M \preccurlyeq N$ such that
(i) $A \subseteq M$ (ii) $|M| = \lambda$

Recall that a theory (set of L-sentences) is said to be
(i) <u>consistent</u>, or <u>satisfiable</u>, if it has a model
(ii) <u>finitely consistent</u>, or <u>finitely satisfiable</u>, if every finite subset of T has a model.

<u>Compactness Theorem</u>
An L-theory is consistent iff it is finitely consistent

If T is an L-theory and ϕ is an L-sentence, we write
$T \vdash \phi$ if for all L-structures M
$M \models T \Rightarrow M \models \phi$.
Then T is a <u>complete</u> theory if for all L-sentences ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$.

(2) TWO RELATIONAL STRUCTURES

(2.1) Definition Let $L_{lo} = \{<\}$, binary relation.

An L_{lo}-structure is a linear order if it satisfies

1. $\forall x \quad \neg(x = x)$ irreflexivity
2. $\forall xyz \quad [(x<y \ \wedge \ y<z) \rightarrow x<z]$ transitivity
3. $\forall xy \quad [x<y \ \vee \ y<x \ \vee \ x=y]$ linearity

A linear order is dense if it satisfies

4. $\exists xy \quad [x<y]$ non-triviality
5. $\forall xy \quad [x<y \rightarrow \exists z \ (x<z \wedge z<y)]$ density

A linear order has no endpoints if

6. $\forall x \quad [\exists y \ (y<x) \ \wedge \ \exists z \ (x<z)]$

T_{lo} : The theory (set of L_{lo}-sentences) that includes axioms 1, 2, 3

T_{dlo} : The theory formed by axioms 1 to 6.

$(\mathbb{Q},<) \models T_{lo}$ and $(\mathbb{Q},<) \models T_{dlo}$

Clearly any model of T_{dlo} is infinite.

(2.2) Definition If M, N are L-structures, then a partial embedding is an injective map $p : \mathrm{dom}(p) \subseteq M \rightarrow N$ such that

(i) for all n-ary relations R, $a \in \mathrm{dom}(p)^n$,
$$a \in R^M \Leftrightarrow p(a) \in R^N$$
(ii) for all n-ary functions f, $a \in \mathrm{dom}(p)^n$,
$$p(f^M(a)) = f^N(p(a))$$
(iii) for all constants c, $p(c^M) = c^N$ whenever $c^M \in \mathrm{dom}(p)$

M, N are said to be partially isomorphic if there is a nonempty collection I of partial embeddings such that

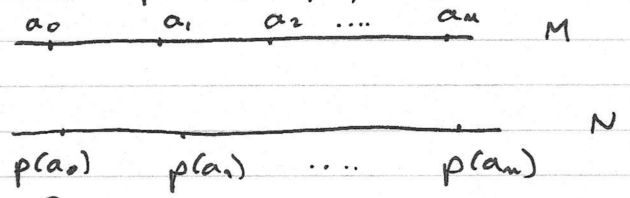
1. if $p \in I$ and $m \in M$, then there is $\hat{p} \in I$ s.t. $p \subseteq \hat{p}$ and $m \in \mathrm{dom}(\hat{p})$
2. if $p \in I$ and $n \in N$, there is $\hat{p} \in I$ s.t. $p \subseteq \hat{p}$ and $n \in \mathrm{ran}(\hat{p})$

(2.3) Lemma (Back and forth) If $|M| = |N| = \omega$ and M, N are partially isomorphic via I, then $M \cong N$.

Proof Enumerate M and N, say $M = \langle a_i : i<\omega \rangle$ and $N = \langle b_i : i<\omega \rangle$. Define inductively a chain $\langle p_i : i<\omega \rangle$ where for all i, $p_i \in I$ and $a_i \in \mathrm{dom}(p_{i+1})$, $b_i \in \mathrm{ran}(p_{i+1})$.
Let p_0 be any map in I. At stage $i+1$, p_i is given.
Use property 1. in Def 2.2 to find $\hat{p} \supseteq p_i$ s.t. $a_i \in \mathrm{dom}(\hat{p})$, and property 2. to find $p_{i+1} \supseteq \hat{p}$ s.t. $b_i \in \mathrm{ran}(p_{i+1})$.
Then $\bigcup p_i$ is the required isomorphism

(2.4) <u>Lemma (extension)</u> Let $M \models T_{lo}$, $N \models T_{dlo}$. Let $p : M \to N$ be a partial embedding such that $|\mathrm{dom}(p)| < \omega$ (finite). Then if $c \in M$ there is a finite partial embedding $\hat{p} \supseteq p$ such that $c \in \mathrm{dom}(\hat{p})$.

Proof Let $\mathrm{dom}(p) = \{a_0, \dots, a_n\}$ where $a_i < a_j$ if $i < j$.



Let $c \in M$. Then:

(i) if $c < a_0$, use Axiom 6 to find $d < p(a_0)$, $d \in N$.

(ii) if $a_i < c < a_j$, use density to find d s.t. $p(a_i) < d < p(a_j)$

(iii) if $a_n < c$, use Axiom 6 again to find d s.t. $p(a_n) < d$.

Put $\hat{p} := p \cup \{\langle c, d\rangle\}$

(2.5) <u>Theorem</u> Let $M, N \models T_{dlo}$, $|M| = |N| = \omega$. Then $M \cong N$.

Proof: Let $I = \{p : M \to N : p \text{ finite partial embedding}\}$.
Then: $I \neq \emptyset$ because $\{\langle m, n\rangle\} \in I$ for any $m \in M$, $n \in N$.
By Lemma 2.4, I satisfies 1 and 2 in Def 2.2, so Lemma 2.3 applies.

(2.6) <u>Definition</u> If $|L| = \omega$, an L-theory T is ω-categorical if T has an infinite model, and any two countable models are isomorphic.

So Theorem 2.5 says that T_{dlo} is ω-categorical.

(2.7) <u>Fact</u> If T is an ω-categorical theory with no finite models, then T is complete.

Proof: Suppose $M, N \models T$ and $\varphi \in L$ sentence s.t. $M \models \varphi$.
By DLS, there are $M', N' \models T$ such that $|M'| = |N'| = \omega$ and $M' \preccurlyeq M$, $N' \preccurlyeq N$. By hypothesis, $M' \cong N'$ and so $N' \models \varphi$.
By elementarity, $N \models \varphi$. Hence $Th(M) = Th(N)$.

(2.8) <u>Corollary</u> T_{dlo} is complete

(2.9) <u>Definition</u> Let M, N be L-structures. A map $f : \mathrm{dom}(f) \subseteq M \to N$ is elementary if for all $\varphi(x) \in L$, $a \in \mathrm{dom}(f)^{|x|}$

$$M \models \varphi(a) \iff N \models \varphi(f(a))$$

(2.10) <u>Remark</u> A map f is elementary iff every finite restriction of f is elementary

($\Rightarrow$) clear

($\Leftarrow$) Suppose f is not elementary, say $\phi(x) \in L$ and $a \in M$ are s.t.

$$M \models \phi(a) \quad \text{and} \quad N \not\models \phi(f(a))$$

Then $f\restriction a$ is finite and not elementary.

2.11 Proposition Let $M, N \models T_{dlo}$ and suppose $p : M \to N$ is a partial embedding. Then p is an elementary map.

Proof By Remark 2.10, enough to consider p finite.
By DLS, there are $M' \preccurlyeq M$, $N' \preccurlyeq N$ such that $|M'| = |N'| = \omega$, $\mathrm{dom}(p) \subseteq M'$, $\mathrm{ran}(p) \subseteq N'$. By a back-and-forth argument (proof of Thm 2.3 with $p_0 = p$, $I = \{\text{partial emb.} : M \to N\}$, p can be extended to an isomorphism $\pi : M' \to N'$. In particular, p is an elementary map.

2.12 Corollary $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$

Pf: the inclusion map is a partial embedding, so it is elementary.

2.13 Definiton (Graphs and random graphs)

Let $L_{gph} = \{R\}$, where R is a binary relation symbol

A graph is an L_{gph}-structure which satisfies

1. $\forall x \; \neg R(x,x)$ irreflexivity
2. $\forall x y \; [R(x,y) \to R(y,x)]$ symmetry

If G is a graph, the elements of G are called vertices, and 2-subsets $\{a,b\}$ such that $R(a,b)$ are called edges.

A graph is a random graph if it satisfies 1, 2 and

3. $\exists x y \; x \neq y$ non triviality

together with axioms r_n for $n \in \omega$

(r_n) $\forall x_0 \cdots x_n y_0 \cdots y_n \left[\bigwedge_{i,j=0}^{n} x_i \neq y_j \to \right.$

$$\left. \exists z \left[\bigwedge_{i=0}^{n} [(z \neq x_i) \wedge (z \neq x_i) \wedge R(z, x_i) \wedge \neg R(z, y_i)] \right] \right]$$

T_{gph} : theory incl 1, 2

T_{rg} : theory including 1 → 3 and r_n $\forall n \in \omega$.

Axiom 3 and the r_n imply that a model of T_{rg} is infinite

2.14 Fact T_{rg} is consistent

Proof: Define a graph on ω as follows: for $i, j \in \omega$ with $i < j$, $\{i, j\}$ is an edge if the ith digit in the binary expansion of j is a 1 (count digits from right to left).
Check this structure is a model of T_{rg}!