6.15 Theorem Let $N \models T$ be a model, $|N| \geq |L|$. The following are equivalent:

(i) N is saturated

(ii) $\dim(N) = |N|$

Proof: (i) $\Rightarrow$ (ii) Let B be a basis, so $N = \mathrm{acl}(B)$.

Let $p(x) \subseteq B$, where $|x| = 1$, be type

$$p(x) = \{ \neg\varphi(x) \in L(B) : \varphi(x) \text{ is algebraic} \} \quad |\varphi(N)| = |\varphi(U)| < \omega.$$

Let $\varphi(x) \in p(x)$. Then $|\varphi(U)| < \omega$ implies $N \cap |\neg\varphi(U)| \neq \emptyset$.

But $p(x)$ is closed under $\wedge$, so $p(x)$ is fin. sat. in N. But $p(x)$ is not satisfiable in N (i.e. not realized) because no real. of $p(x)$ is in $\mathrm{acl}(B) = N$.

By saturation of N, we must have $|B| = |N|$.

(ii) $\Rightarrow$ (i) Let $f : M \to N$ be elementary and such that $|f| < |N|$. Let $b \in M$.

Want: $\hat{f} \supseteq f$ elem. s.t. $b \in \mathrm{dom}(\hat{f})$ (by Thm 4.5, this proves N is saturated).

If $b \in \mathrm{acl}(\mathrm{dom}(f))$, let $\alpha \in \mathrm{Aut}(U)$ extend f. Then

$$\alpha[\mathrm{acl}(\mathrm{dom}(f))] = \mathrm{acl}(\alpha[\mathrm{dom}(f)]) = \mathrm{acl}(\mathrm{ran}(f)) \subseteq N$$

so $f \cup \{\langle b, \alpha(b)\rangle\}$ is the required extension.

If $b \notin \mathrm{acl}(\mathrm{dom}(f))$, note $|f| < |N| = \dim(N)$ implies $\mathrm{acl}(\mathrm{ran}(f)) \subsetneq N$.

Then $\exists\, c \in N \setminus \mathrm{acl}(\mathrm{ran}(f))$. By Prop. 6.10, $f \cup \{\langle b, c\rangle\}$ is elementary. $\square$

6.16 Remark When $|K| = \omega$, T_{VS_K} is λ-categorical for every $\lambda > \omega$.

However, T_{VS_K} is NOT ω-categorical: it has a ctble model of dim n for each $n \in \omega \setminus \{0\}$ as well a model of dim ω.

If $|K| < \omega$, add to T_{VS_K} axioms that say that U has infinite cardinality. Call this theory $T^{\infty}_{VS_K}$ - it is s.m. and it has ctble model of dim ω - this model is saturated by 6.15 and unique up to isom.

7 MORE ON $\mathfrak{U}$

When do large (uncountable!) saturated models of a complete theory exist?

Focus: saturated models vs λ-saturated models ← not hard to get

↓ why?

We proved that $N \models T$ is saturated $\iff$ (i) universal (ii) homogeneity

Why is λ-saturation not enough?

Def A model $M \models T$ is <u>weakly λ-homogeneous</u> if for all $f: M \to M$ elementary and such that $|f| < \lambda$ and $b \in M$, there is $c \in M$ s.t. $f \cup \{\langle b, c\rangle\}$ is elementary.

In general: M is λ-satur $\Rightarrow$ M is weakly λ-homogeneous (Thm 4.5)

However, $\Leftarrow$ does not hold!

Only if $|M| = \lambda$ λ-homogeneity equivalent to weak λ-homogeneity. (provide example!)

Some facts about cardinals

Def Let α be a limit ordinal. The cofinality of α is the least cardinal λ such that $f: \lambda \to \alpha$ with $\operatorname{ran}(f)$ unbounded in α.

Ex: $\operatorname{cof}(\omega) = \aleph_0$ (no finite function is unbdd in ω)

$\operatorname{cof}(\omega_\omega) = \omega$ ($n \to \omega_n$ has unbounded range in ω_ω)

If $\kappa \geq \aleph_0$ is a cardinal, then κ is <u>regular</u> if $\operatorname{cof}(\kappa) = \kappa$. Otherwise, κ is s.t.b. <u>singular</u>.

7.1 <u>Notation</u> For $n \in \omega$ and $A \subseteq M \models T$ $S_n(A)$ is the set of complete consistent types with n free variables and parameters in A. When $A = \emptyset$, we write $S_n(T)$

Assume T ctble

7.2 <u>Lemma</u> Let $|M| \geq \aleph_0$, let $\kappa > \aleph_0$ be a cardinal. Then there is $M' \succeq M$ such that for every $A \subseteq M$ with $|A| \leq \kappa$ every type in $S_1^M(A)$ is realized in M' and $|M'| \leq |M|^\kappa$.

Proof: Note $|\{A \subseteq M : |A| = \kappa\}| = |M|^\kappa$ and for $A \subseteq M$ s.t. $|A| \leq \kappa$, $\underbrace{|S_1^M(A)| \leq 2^\kappa}$

Enumerate types over A as $\langle p_\alpha : \alpha < |M|^\kappa \rangle$. We build an elementary chain $\langle M_\alpha : \alpha < |M|^\kappa \rangle$ of $M_\alpha \preceq M_{\alpha+1}$ such that:

- $M_0 = M$
- $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ at limit stage
- $M_\alpha \preceq M_{\alpha+1}$ such that $p_\alpha(x)$ is realized in $M_{\alpha+1}$ (Theor 3.6) and $|M_{\alpha+1}| = |M_\alpha|$

By induction $|M_\alpha| \leq |M|^\kappa$ and so $M' = \bigcup_{\alpha < |M|^\kappa} M_\alpha$ is the required extension.

(7.3) **Theorem** Let $\kappa > \aleph_0$, $M \models T$. Then there is a κ^+-saturated model of T, $N \succcurlyeq M$, such that $|N| \leq |M|^\kappa$.

Proof Build an elementary chain $\langle N_\alpha : \alpha < \kappa^+ \rangle$ such that
- $N_0 = M$
- $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ if α is a limit
- $N_\alpha \preceq N_{\alpha+1}$, $|N_\alpha| \leq |M|^\kappa$ and if $A \subseteq N_\alpha$ with $|A| \leq \kappa$, $p(x) \in S_1^{N_\alpha}(A)$, then $p(x)$ is realized in $N_{\alpha+1}$. We can do this by Lemma 7.2, because

$$|N_{\alpha+1}| \leq (|M|^\kappa)^\kappa = |M|^\kappa.$$

Let $N = \bigcup_{\alpha<\kappa^+} N_\alpha$. Since $\kappa^+ \leq |M|^\kappa$, N is the union of at most $|M|^\kappa$ sets of size $\leq |M|^\kappa$, hence $|N| \leq |M|^\kappa$.
Let $A \subseteq N$, $|A| \leq \kappa$. Since κ^+ is regular, there is $\alpha < \kappa^+$ such that $A \subseteq N_\alpha$ (o/w $|A|$ is cofinal in κ^+) and so every $p(x) \in S_1^N(A)$ is realized in $N_{\alpha+1} \preceq N$. □

We have proved: for arbitrarily large κ, there is a κ^+-saturated $N \succcurlyeq M$ such that $|N| \leq |M|^\kappa$.

Set-theoretic assumptions that imply the existence of saturated models of size κ^+
- GHC ($\kappa^+ = 2^\kappa = |M|^\kappa$ if $|M| \leq 2^\kappa$).
- existence of strongly inaccessible cardinals. A cardinal is inaccessible if it is a regular limit cardinal. An inacc. cardinal κ is strongly inaccessible if $2^\lambda < \kappa$ for $\lambda < \kappa$.

A corollary of 7.3: if κ is uncountable and strongly inaccessible, then there is a saturated model of size κ.

(7.4) **Definition** Let κ be an inf. cardinal, T a complete theory s.t. $|T| \leq \kappa$. Then T is said to be κ-stable if for every $M \models T$, $A \subseteq M$ s.t. $|A| = \kappa$ $|S_n^M(A)| \leq \kappa$ for all $n \in \omega$.

(7.5) **Theorem** Let κ a regular cardinal, T a κ-stable theory. Then there is a saturated model of T of size κ.

Proof Build an elementary chain $\langle M_\alpha : \alpha < \kappa \rangle$ where:
(i) $M_0 \models T$ s.t. $|M_0| = \kappa$
(ii) we take unions at limit stages
(iii) $M_\alpha \preceq M_{\alpha+1}$ s.t. every type $p(x) \in S_1^{M_\alpha}(M_\alpha)$ is realized in $M_{\alpha+1}$
Since T is κ-stable and $|M_\alpha| = \kappa$, $|S_1^{M_\alpha}(M_\alpha)| = \kappa$, so we can find $M_{\alpha+1}$ s.t. $|M_{\alpha+1}| = \kappa$ and every type is realized in $M_{\alpha+1}$

Let $M = \bigcup_{\alpha<\kappa} M_\alpha$ - then $|M| = \kappa$. Let $A \subseteq M$, $|A| < \kappa$.

Use regularity of κ to show M realizes all types in $S_1^M(A)$. □