EXTRA LECTURE: MONDAY 1 Feb 2 - 3:15

MODEL THEORY 22/01/21

Recap: An L-structure N is
- **λ-saturated** if it realizes every type $p(x) \in L(A)$, where $p(x)$ is fin. sat. in N, $|x| = 1$ and $|A| < \lambda$
- **λ-universal** if for every $M \equiv N$ such that $|M| \leq \lambda$ there is an elem embedding $\beta: M \to N$
- **λ-homogeneous** if every elementary map $f: N \to N$ such that $|f| < \lambda$ extends to an automorphism of N

When $\lambda = |N|$ we say that N is saturated, universal, homogeneous.

Warning: "homogeneity" is used for several related properties in the literature - our is called "strong homogeneity" by some authors. There are also notions of weak homogeneity and ultrahomogeneity.

4.10 Theorem Let N be such that $|N| \geq |L|$. Then the following are equivalent:

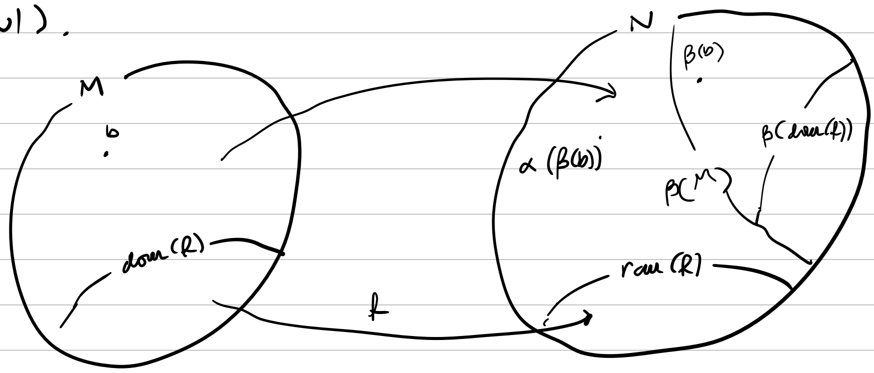
(i) N saturated

(ii) N is universal and homogeneous.

Proof (i) $\Rightarrow$ (ii) Universality: pick $M \equiv N$ such that $|M| \leq |N|$.
Let $a = \langle a_i : i < |M| \rangle$ be an enumeration of M, and let $p(x) = tp(a/\emptyset)$ (so $|x| = |M|$). Clearly $p(x)$ is satisfiable in M, and $M \equiv N$ implies that $p(x)$ is fin. sat. in N. But N is saturated and $|x| \leq |N|$, so N realizes $p(x)$ by Theorem 4.5. Suppose $N \models p(b)$. Then $a_i \mapsto b_i$ is an elementary embedding from M to N.

Homogeneity: follows from Corollary 4.6 with $M = N$.

(ii) $\Rightarrow$ (i) We show if $M \equiv N$, $b \in M$ and $f: M \to N$ is an elementary map s.t. $|f| < |N|$, there is $\hat{f}$ s.t. $\hat{f} \supseteq f$, $\hat{f}$ is elementary and $b \in dom(\hat{f})$ (by Theorem 4.5(ii) this shows that N is saturated).
We may assume $|M| \leq |N|$ (if not, use DLS to work $M' \preceq M$ with $dom(f) \cup \{b\} \subseteq M'$ and $|M'| \leq |N|$).



By universality, there is $\beta: M \to N$ elementary embedding.
Then $f \circ \beta^{-1} : \beta(dom(f)) \to ran(f)$ is elementary and such that $|f \circ \beta^{-1}| < |N|$.
By homogeneity, there is $\alpha \in Aut(N)$ such that $f \circ \beta^{-1} \subseteq \alpha$. Then $f \cup \{\langle b, \alpha(\beta(b))\rangle\}$ is elementary, as it is a restriction of α. $\square$

4.11 <u>Definition</u> Let a be a tuple in N, let $A \subseteq N$. Then
$$O_N(a/A) = \{\alpha(a) : \alpha \in \mathrm{Aut}(N/A)\}$$
is the <u>orbit of a over A</u>
(note: when $a = a_0, \dots, a_k$ then $\alpha(a) = (\alpha(a_0), \alpha(a_1), \dots, \alpha(a_k))$).

If $\phi(x) \in L(A)$ then $\phi(N) = \{a \in N^{|x|} : N \models \phi(a)\}$ is the <u>set defined by $\phi(x)$</u>.
A set is <u>definable over A</u> if it is defined by some $\phi(x) \in L(A)$.

Similarly, a set is type-definable over A if it has the form
$p(N) = \{a \in N^{|x|} : N \models p(a)\}$ for some type $p(x) \subseteq L(A)$.

Reminder: If $a, b \in N^{|a|}$ (so $|a| = |b|$) and $A \subseteq N$ then
$tp(a/A) = tp(b/A) \iff \{\langle a_i, b_i\rangle : i < |a|\} \cup id_A$ is an elementary map from N to N.

4.12 <u>Proposition</u> Let N be λ-homogeneous, $A \subseteq N$ s.t. $|A| < \lambda$. Let $a \in N^{|a|}$ be such that $|a| < \lambda$. Let $p(x) = tp(a/A)$.
Then $O_N(a/A) = p(N)$.

<u>Proof</u>: ($\subseteq$) Let $b \in O_N(a/A)$, i.e. $b = \alpha(a)$ for some $\alpha \in \mathrm{Aut}(N/A)$.
Then $tp_N(a/A) = tp_N(\alpha(a)/A) = tp_N(b/A)$
($\supseteq$) If $b \models p(x)$, then $tp(a/A) = tp(b/A)$ and so the map
$\{\langle a_i, b_i\rangle : i < |a|\} \cup id_A$ is elementary and has cardinality $< \lambda$. By λ-homogeneity, this map extends to $\alpha \in \mathrm{Aut}(N/A)$ and so $b \in O_N(a/A)$ (as $\alpha(a) = b$). $\square$

4.13 <u>Fact</u> If N is saturated, $p(x) \subseteq L(A)$ is such that $|x| < \omega$ and $|A| < |N|$, then
$$|p(N)| \geq \omega \iff |p(N)| = |N|.$$
In particular, if $\phi(x) \in L(A)$ and $|A| < |N|$ then $\phi(N)$ is either finite or has cardinality $|N|$.

<u>Proof</u>: ($\Leftarrow$) trivial
($\Rightarrow$) Let $q(x) = p(x) \cup \{x \neq a : a \in p(N)\}$. Then $q(x)$ has parameter set $A \cup p(N)$. Because $p(N)$ is infinite, $q(x)$ is fin. sat in N.
Suppose $\omega \leq |p(N)| < |N|$. Then by saturation there is $b \in N^{|x|}$ such that $N \models q(b)$.
But then $N \models p(b)$, and $b \neq a$ for all $a \in p(N)$ $\times$.
Therefore $|p(N)| = |N|$. $\square$

Remark: In 4.12, the containment $\subseteq$ holds in any N.
In the absence of homogeneity, the set of realizations of a type is a union of orbits.

(5) THE MONSTER MODEL

Let T be a complete theory with infinite model. Aim: work in a saturated model of T that is large enough to embed elementarily every other of T that we might be interested in. Such a model is called the <u>monster model</u> and it is usually denoted by $\mathcal{U}$, or $\mathbb{M}$.

(5.1) <u>Conventions, notation and terminology</u>

When working in the monster model $\mathcal{U}$ of a complete theory T, we say
- "$\phi(x)$ holds", or "$\models \phi(x)$", if $\mathcal{U} \models \forall x\, \phi(x)$
- "$\phi(x)$ is consistent" if $\mathcal{U} \models \exists x\, \phi(x)$
- a type $p(x)$ is consistent, or satisfiable, if there $a \in \mathcal{U}^{|x|}$ such that $\mathcal{U} \models p(a)$
- a cardinality is <u>small</u> if it is strictly less than $|\mathcal{U}|$ $\underbrace{\text{(usually denoted by } \kappa)}$
- a <u>model</u> M is $M \preceq \mathcal{U}$ such that $|M| < |\mathcal{U}|$

Notation: - $tp(a/A) = tp_{\mathcal{U}}(a/A)$ and $O(a/A) = O_{\mathcal{U}}(a/A)$
- symbols A, B, C denote <u>small subsets</u> of $\mathcal{U}$

Conventions: - tuples (of elements, parameters) have got small cardinality unless o/w specified
- formulas have parameters in $\mathcal{U}$ unless o/w specified
- definable set have the form $\phi(\mathcal{U})$ for some $\phi(x) \in L(\mathcal{U})$
- types have parameters from small sets
- type-definable sets have the form $p(\mathcal{U})$ for some type $p(x) \subseteq L(A)$ where A is small

Remark: If $p(x), q(x)$ are types then "$p(x) \to q(x)$" means $p(\mathcal{U}) \subseteq q(\mathcal{U})$ and "$p(x) \to \neg q(x)$" means $p(\mathcal{U}) \cap q(\mathcal{U}) = \emptyset$.

(5.2) <u>Fact</u> Let $p(x) \subseteq L(A)$, $q(x) \subseteq L(B)$ be consistent types such $p(x) \to \neg q(x)$. Then there are a conjunction $\phi(x)$ of fmlas in $p(x)$ and a conj. $\psi(x)$ of fmlas in $q(x)$ such that $\models \phi(x) \to \neg \psi(x)$ $\quad (\mathcal{U} \models \forall x\, [\phi(x) \to \neg \psi(x)])$.

<u>Proof</u> If $p(\mathcal{U}) \cap q(\mathcal{U}) = \emptyset$ then $p(x) \cup q(x)$ is not satisfiable in $\mathcal{U}$, hence it is not satisfiable (by the saturation of $\mathcal{U}$). By compactness, there are $\{\phi_1(x), \dots, \phi_n(x)\} \subseteq p(x)$ and $\{\psi_1(x), \dots, \psi_m(x)\} \subseteq q(x)$ such that $\bigwedge_i \phi_i(x) \wedge \bigwedge_j \psi_j(x)$ is not satisfiable, that is

$$\models \bigwedge_{i=1}^{n} \phi_i(x) \to \neg \bigwedge_{j=1}^{m} \psi_j(x), \quad \text{as required.}$$

(5.3) <u>Fact</u> If $\alpha \in Aut(\mathcal{U})$ and $\phi(\mathcal{U}, b)$ is a def-ble set where $\phi(x, z) \in L$ and $b \in \mathcal{U}^{|z|}$ then

$$\alpha[\phi(\mathcal{U}, b)] = \phi(\mathcal{U}, \alpha(b)) \qquad (*)$$

Similarly, if $p(x, z) \subseteq L$, $b \in \mathcal{U}^{|z|}$, then $\alpha[p(\mathcal{U}, b)] = p(\mathcal{U}, \alpha(b))$

For (*), let $c \in \alpha[\varphi(\mathcal{U}, b)]$. Then $c = \alpha(a)$ for some $a \in \varphi(\mathcal{U}, b)$, that is $\models \varphi(a, b)$. But then $\models \varphi(c, \alpha(b))$, i.e. $c \in \varphi(\mathcal{U}, \alpha(b))$.
The other containment is similar.

So $\mathrm{Aut}(\mathcal{U})$ has a natural (gp) action on the definable sets.

5.4 Definition A set $D \subseteq \mathcal{U}^n$ (not nec. small) is <u>invariant over</u> $A \subseteq \mathcal{U}$ (small) if for all $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ we have $\alpha[D] = D$ (α fixes D setwise)
or equivalently if for all $a \in D$, $O(a/A) \subseteq D$.

<u>Remark</u> If $a \in D$ and $q(x) = tp(a/A)$ and $b \models q(x)$ then by homogeneity there $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\alpha(a) = b$, and so $b \in O(a/A)$.
So invariance of D is also equivalent to the following condition:
$\forall a \in D \; \forall b \in \mathcal{U}$ if $tp(a/A) = tp(b/A)$ then $b \in D$

5.5 Proposition Let $A \subseteq \mathcal{U}$. For every $\varphi(x) \in L(\mathcal{U})$, tfae:
(i) There is $\psi(x) \in L(A)$ such that $\models \psi(x) \leftrightarrow \varphi(x)$
(ii) $\varphi(\mathcal{U})$ is invariant $/A$.

<u>Proof</u> (i) $\Rightarrow$ (ii) clear ($\psi(x) \in L(A) \Rightarrow \psi(\mathcal{U})$ is fixed setwise by $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$).

(ii) $\Rightarrow$ (i) Let $\varphi(x, z) \in L$ be such that $\varphi(\mathcal{U}, b)$ is invariant over A for some $b \in \mathcal{U}^{|z|}$.
Let $q(z) = tp(b/A)$ and $c \models q(x)$. By homogeneity, there is $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\alpha(b) = c$. Then

$$\alpha[\varphi(\mathcal{U}, b)] = \varphi(\mathcal{U}, c) \quad (\text{because } \alpha \in \mathrm{Aut}(\mathcal{U}/A))$$
$$\shortparallel$$
$$\varphi(\mathcal{U}, b)$$

so in fact $\varphi(\mathcal{U}, b) = \varphi(\mathcal{U}, c)$. So

$$q(z) \rightarrow \forall x \, [\varphi(x, b) \leftrightarrow \varphi(x, z)]$$

By an argument similar to that in 5.2, there is $\chi(z) \in q(z)$ such that
$$\models \chi(z) \rightarrow \forall x \, [\varphi(x, b) \leftrightarrow \varphi(x, z)]$$
But then $\exists z \, [\chi(z) \wedge \varphi(x, z)]$ defines $\varphi(\mathcal{U}, b)$, and it is a fmla in $L(A)$. $\square$