(2.15) Lemma (extension)

Let $M \models T_{gph}$, $N \models T_{rg}$, $p : \mathrm{dom}(p) \subseteq M \to N$ a finite partial embedding. Let $c \in M$. Then there is a finite partial embedding $\hat{p} \supseteq p$ such that $c \in \mathrm{dom}(\hat{p})$.

Proof assume $c \notin \mathrm{dom}(p)$ and let
$X = \{a \in \mathrm{dom}(p) : R(a,c)\}$
$Y = \{b \in \mathrm{dom}(p) : \neg R(b,c)\}$.
Then X and Y are finite, because p is. Let $n = \max\{|X|, |Y|\}$ and, by padding X or Y if necessary, find $d \in N \setminus [p(X) \cup p(Y)]$ such that use r_n

(i) $R(d, p(a))$ if $a \in X$
(ii) $\neg R(d, p(b))$ if $b \in Y$.

The element d exists because of (r_n), which holds in N

(2.16) Theorem Let $M, N \models T_{rg}$ be such that $|M| = |N| = \omega$. Then $M \cong N$

Proof The same as the proof of Thm 2.5 with Lemma 2.15 in place of Lemma 2.4

(2.17) Corollary T_{rg} is ω-categorical and complete.
Every partial embedding between models of T_{rg} is elementary.

(2.18) Remarks The unique (up to isom) ctble model Γ of T_{rg} is called Rado's graph, or the random graph. It embeds every finite and ctble graph, and it is ultrahomogeneous (i.e. every finite partial isomorphism between subsets of Γ extends to an automorphism).

Fun questions on Γ:
(i) If $x \notin \Gamma$, is $\Gamma \cup \{x\}$ the random graph?
(ii) If $X \subseteq \Gamma$ is a finite subset, is $\Gamma \setminus X$ the random graph?

3 TYPES

3.1 Definition A *type* $p(x)$ is a set of L-formulas whose free variables are among $x = \langle x_i : i < \lambda \rangle$ (λ a cardinal, possibly infinite). We write $p(x) \subseteq L$.

A type $p(x)$ is

- *satisfiable in an L-structure M* if there is $a \in M^{|x|}$ such that $M \models \phi(a)$ for all $\phi(x) \in p(x)$. We write
$$M \models p(a) \quad \text{or} \quad M, a \models p(x)$$
and we say that a *realizes* $p(x)$ in M
- *satisfiable* if it is satisfiable in some M
- *finitely satisfiable in M* if all its finite subsets are satisfiable in M
- *finitely satisfiable* if all its finite subsets are satisfiable.

We also say "consistent" in place of satisfiable throughout the definition.

Remark: a type may be finitely satisfiable in M but not satisfiable in M. For example let $M = \langle \mathbb{N}, < \rangle$ and let $\phi_n(x) \equiv$ "there are at least n elements less tha x".
Then $p(x) = \{\phi_n(x) : n < \omega\}$ is finitely satisfiable in M, but not satisfiable in M.

3.2 Theorem (Compactness for types)
Every finitely satisfiable type is satisfiable.

Proof Let $p(x) \subseteq L$ with $x = \langle x_i : i < \lambda \rangle$. Expand L to $L' = L \cup \{c_i : i < \lambda\}$, where the c_i are distinct constants not already in L. Then $p(c)$ is a finitely consistent theory in L'. By the Compactness Theorem, there is an L'-structure M' such that $M' \models p(c)$. But M' can be reduced to an L-structure M (simply forget the interpretations of the c_i). Then
$$M, c^{M'} \models p(x)$$
as required.

3.4 Lemma Let M be an L-structure, $a = \langle a_i : i < \lambda \rangle$ an enumeration of M. Let $q(x) = \{\phi(x) \in L : M \models \phi(a)\}$ where $|x| = \lambda$.
Then $q(x)$ is satisfiable in an L-structure N if and only if there is an elementary embedding $\beta : M \to N$.

Proof: ($\Rightarrow$) Let $b \in N^{|x|}$ be a tuple that realizes p in N, that is,
$$N \models \phi(b) \quad \text{for all } \phi(x) \in q(x).$$
Then the map $\beta : a_i \mapsto b_i$ is an elem embedding of M into N: if $\phi(x) \in L$ then
$$M \models \phi(a) \iff \phi(x) \in q(x) \iff N \models \phi(b), \text{ and } b = \beta(a)$$
($\Leftarrow$) Clear, since $N, \beta(a) \models q(x)$.

3.5 Remarks If M is an L-structure and $A \subseteq M$, we can work with types in $L(A)$ - such types are said to have parameters in A, or to be over A.
In particular, we can have $A = M$. If $p(x) \subseteq L(M)$, then there are an enumeration a of M and a type $q(x, z) \subseteq L$ such that $p(x) = q(x, a)$
Lemma 3.4 is also called Diagram Lemma, and it is also stated as follows:
Let $Th(M_M)$ be the theory of M in $L(M)$. Suppose $N \models Th(M_M)$. Then $M \preccurlyeq N$.

3.6 Theorem If M is an L-structure and $p(x) \subseteq L(M)$ is a type that is finitely satisfiable in M, then $p(x)$ is realized in some $N \succcurlyeq M$.

Example: Let $(0,1) \subseteq \mathbb{Q}$ be the L_{ω}-structure M.
Let $a_n = 1 - \frac{1}{n} \in M$ for $n \in \omega \setminus \{0\}$.

Let $\varphi_n(x) \equiv a_n < x$. Then $p(x) = \{\varphi_n(x) : n \in \omega \setminus \{0\}\}$ is a type in $L(M)$ that is finitely satisfiable in M, but not satisfiable. However, $p(x)$ is realized in $\mathbb{Q} \succcurlyeq (0,1)$.

Proof of Theorem 3.6 Let $a = \langle a_i : i < \lambda \rangle$ be an enumeration of M, and let

$$q(z) = \{\varphi(z) : M \models \varphi(a)\}$$

where $|z| = \lambda$ and the z_i are new variables, i.e. in part $z \cap x = \emptyset$
Write $p(x)$ as $p'(x, a)$, where $p'(x, z) \subseteq L$.
Then $p'(x, z) \cup q(z)$ is finitely satisfiable in M, because $p(x, a)$ is fin sat and $q(z)$ is realized by a.
By compactness for types (Thm 3.2), there are an L-structure N and tuples $c \in N^{|x|}$, $d \in N^{|z|}$ such that

$$N \models p'(c, d) \cup q(d).$$

In particular, $N \models q(d)$. By Lemma 3.4, $\beta : a_i \mapsto d_i$ is an elem embedding. We may assume $M \preccurlyeq N$.

3.7 Theorem (Upward Löwenheim Skolem)
Let M be an infinite L-structure. Then for any $\lambda \geq |M| + |L|$, there is $N \succcurlyeq M$ such that $|N| = \lambda$

Proof Let $x = \langle x_i : i < \lambda \rangle$ be a tuple of distinct variables.
Let $p(x) = \{x_i \neq x_j : i < j < \lambda\}$.
Clearly, $p(x)$ is fin. sat. in M. By Theorem 3.6, $p(x)$ is realized in some $N \succcurlyeq M$, and so $|N| \geq \lambda$. By DLS, we may assume $|N| = \lambda$.