If you spot any typos or problems, please email `silvia.barbina@open.ac.uk`.

Question 4 Let $L = \{+, -, 0\}$ be the language of additive groups (so in particular $-$ is a unary function). Let $\mathbb{Z}^n$ denote the direct product $\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}}$, regarded as an L-structure.

Show that if $m \neq n$ then $\mathbb{Z}^m \not\equiv \mathbb{Z}^n$.

Solution In $\mathbb{Z}^k$ an element x is said to be divisible by 2 if $\exists z\,(z + z = x)$.

Then in $\mathbb{Z}^k$ there is a set of 2^k distinct elements $x_1, \ldots, x_k$ such that for all $y \in \mathbb{Z}^k$, there is $i \in \{1, \ldots, k\}$ such that $x_i + y$ is divisible by 2. For example, in $\mathbb{Z}^3$

$$\{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

is such a set.

This property can be expressed via a first-order sentence. If $m < n$, this sentence is true in $\mathbb{Z}^m$ but not in $\mathbb{Z}^n$, hence $\mathbb{Z}^m \not\equiv \mathbb{Z}^n$.

Alternatively: two elements x and y are equal modulo 2 if the difference $x - y$ is divisible by 2. In $\mathbb{Z}^k$ there is a set of $2^k - 1$ elements that are not divisible by 2 and are pairwise distinct modulo 2, but not a set of 2^k such elements. Again, this property can be expressed via a first-order sentence which is true in $\mathbb{Z}^m$ but not in $\mathbb{Z}^n$ if $m < n$.

Question 8 Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Solution

Method 1 Use the statement in Q7 (often called the Elementary Chain Lemma in the literature). If M is countable, take $K = M$. If not, build chains $\langle k_i : i < \omega \rangle$ of countable K_i and $\langle M_i : i < \omega \rangle$ such that

(i) $A \cap M \subseteq M_i$ for all i

(ii) $A \subseteq K_0$ (and hence $A \subseteq K_i$ for all i)

(iii) $A \cup M_i \subseteq K_{i+1} \preceq N$.

Start with $K_0 \supseteq A$, $K_0 \preceq N$ countable. At stage $i + 1$, let

- $M \cap K_i \subseteq M_i \preceq M$, and
- $K_i \cup M_i \subseteq K_{i+1} \preceq N$.

Then take $M = \bigcup_{i<\omega} M_i$ and $K = \bigcup_{i<\omega} K_i$. Then $K \cap M = M \preceq N$ and $A \subseteq K \preceq N$.

Method 2 Mimic the proof of the Downward Löwenhein-Skolem Theorem. Build inductively a chain $\langle K_i : i < \omega \rangle$ such that $A \subseteq K_i$ and $|K_i| = \omega$ for all i.

Let $K_0 = A$. At stage $i+1$ enumerate the $L(K_i)$-formulas $\varphi(x)$ with one free variable such that $N \models \exists x\, \varphi(x)$. If $\varphi(x)$ has parameters in $M \cap K_i$, then by elementarity of M there is $a_\varphi \in M$ such that $M \models \varphi(a_\varphi)$. Otherwise, pick $a_\varphi \in M$ such that $N \models \varphi(a_\varphi)$. Let

$$K_{i+1} = K_i \cup \{a_\varphi : \varphi(x) \text{ is as described}\}.$$

Let $K = \bigcup_{i<\omega} K_i$. Use the Tarski-Vaught Test to show $K \preceq N$.

Now let $\varphi(x)$ be in $L(M \cap K)$. Then in particular the parameters in φ are in $M \cap K_i$ for some i, so there is $a_\varphi \in K_{i+1}$ such that $N \models \varphi(a_\varphi)$.

Question 9 If λ is an infinite cardinal and T is a theory with models of size λ, we say that T is λ-*categorical* if any two models of T of cardinality λ are isomorphic. Prove that T_{dlo}, the theory of dense linear orders without endpoints, is not λ-categorical for any uncountable λ.

Solution Let λ be an uncountable cardinal, and let $\lambda^\star$ be λ with the reverse order. Then $\lambda \times \mathbb{Q}$ (with the lexicographic order, that is, λ copies of $\mathbb{Q}$), and $\lambda^\star \times \mathbb{Q}$ are both models of T_{dlo} of cardinality λ.

These two models are not isomorphic: pick any $(0,q) \in \lambda \times \mathbb{Q}$. Then $(0,q)$ has countably many predecessors. However, the image $f(0,q)$ under any map $f : \lambda \times \mathbb{Q} \mapsto \lambda^\star \times \mathbb{Q}$ has uncountably many predecessors. Therefore f is not an isomorphism.

Hence T_{dlo} is not λ-categorical.