Please **hand in** solutions to **Questions 2** and **5** by 2 pm on Wed 26 February, for discussion in class on Thursday 27 February. You are welcome to email a pdf (typeset or scanned) if you prefer.

If you spot any typos or problems, please email `silvia.barbina@open.ac.uk`.

Question 1. Let N be a countable random graph, that is, $N \models T_{\mathrm{rg}}$. If x is a vertex in N, a vertex y is a *neighbour* of x if $R(x, y)$. Suppose that N' is obtained from N by erasing finitely many vertices from N and all their neighbours. Is N' a random graph?

Question 2. Let N_1 and N_2 be two countable random graphs. Let N be the *free amalgamation* of N_1 and N_2, that is, the graph with domain $N = N_1 \sqcup N_2$ and let $R^N = R^{N_1} \sqcup R^{N_2}$, where $\sqcup$ denotes the disjoint union.

(i) Is N a random graph?

(ii) Find an L_{gph}-formula $\psi(x, y) \in L$ which is true in N if either both x and y belong to N_1, or both belong to N_2.

(iii) Let $M \equiv N$. Is M the free amalgamation of two random graphs?

Question 3. Let $L = \{<, E\}$, where $<$ and E are binary relations. Let T_0 be the theory that says that $<$ is a strict linear order and that E is an equivalence relation.

Find a theory $T_1 \supseteq T_0$ such that every $N \models T_1$ has the following property: for every $M \models T_0$, every $b \in M$, every finite partial embedding $p : M \to N$ has an extension to a partial embedding defined in b.

Question 4. Prove that the theory T_1 in Exercise 3 is ω-categorical.

Question 5. Let N be a saturated L-structure, and let $p(x)$ be a type (with one free variable) in $L(A)$, where $A \subset N$ and $|A| < |N|$. Let

$$p(N) = \{a \in N : N \models p(a)\}.$$

Are the following conditions equivalent?

(i) $p(N)$ is infinite;

(ii) $|p(N)| = |N|$.

Are (i) and (ii) equivalent for a type $q(\bar{x})$, where $|\bar{x}|$ is possibly infinite?

Question 6. Let N be saturated and let $p(x)$ be a type with parameters in $A \subseteq N$ such that $|A| < |N|$. Suppose that $p(x)$ is closed under conjunction (that is, if $\varphi(x), \psi(x) \in p(x)$, then $\varphi(x) \wedge \psi(x) \in p(x)$). Prove that the following are equivalent:

(i) $p(x)$ has finitely many realizations;

(ii) $p(x)$ contains a formula with finitely many realizations.

Question 7. Let $N \models T_{\mathrm{lo}}$ be saturated. Prove that the following are equivalent:

(i) there is a sequence $\langle a_i : i < \omega \rangle$ such that $a_i < a_{i+1}$ for every $i < \omega$;

(ii) there is a sequence $\langle a_i : i < \omega \rangle$ such that $a_i > a_{i+1}$ for every $i < \omega$.

Question 8.

(a) Let I be an infinite set. An ultrafilter F on I is said to be *principal* if there is $x \in I$ such that

$$F = \{A \subseteq I : x \in A\}.$$

Show that if the ultrafilter F is nonprincipal, then F contains the filter $G = \{A \subseteq I : I \setminus A \text{ is finite}\}$.

(b) Let F be a nonprincipal ultrafilter on ω. Let $p(x) = \{\varphi_i(x) : i \in \omega\}$ be a type in L, and let $\langle M_i : i \in \omega \rangle$ be a collection of L-structures such that for all $n \in \omega$

$$M_n \models \exists x \bigwedge_{i=0}^{n} \varphi_i(x).$$

Prove that the ultraproduct $\prod_{i \in \omega} M_i / \sim_F$ realizes the type $p(x)$.