

# NUMERICAL METHODS FOR BM and GBM SIMULATION

by SILVIA LAVAGNINI

BI Norwegian Business School

13 May 2024

# ① Simulating a BM in Python

We know by definition that for a Brownian motion  $W$ :

- i)  $W(0) = 0$ ;
- ii)  $W$  has independent increments;
- iii)  $W(t) - W(s) \sim N(0, t-s)$ .

In general, given a partition  $0 = t_0 < t_1 < \dots < t_m = T$  we have that

$$W(t_{i+1}) - W(t_i) \sim N(0, t_{i+1} - t_i)$$

Hence:

- We generate  $m$  iid unit normals  $z_1, \dots, z_m$ ;
- We obtain the independent increments by

$$W(t_{i+1}) - W(t_i) = \sqrt{t_{i+1} - t_i} \cdot Z_{i+1}, \quad i = 0, \dots, m-1.$$

In other words, we have the recursion:

$$\begin{cases} W(0) = 0 \\ W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} \cdot Z_{i+1}, \quad i = 0, \dots, m-1. \end{cases}$$

## ② Simulating a GBM in Python

We want to simulate  $S(t) = S(0) e^{\mu t + \sigma B(t)}$ .

Let  $X(t) := \mu t + \sigma W(t)$ .

Then  $S(t) = S(0) e^{X(t)} = S(s) e^{X(t) - X(s)}$  for any  $s \leq t$ ,

where  $X(t) - X(s) = \mu(t-s) + \sigma(W(t) - W(s))$ .

More generally, for a partition  $\emptyset = t_0 < t_1 < \dots < t_m = T$ :

$$X(t_{i+1}) - X(t_i) = \mu(t_{i+1} - t_i) + \sigma(W(t_{i+1}) - W(t_i)).$$

Note: we know how to simulate the increments  $W(t_{i+1}) - W(t_i)$ .

Hence:

- We simulate  $X$  by the recursion

$$\begin{cases} X(0) = \log S(0) \\ X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma(W(t_{i+1}) - W(t_i)) \\ \quad \downarrow_d \\ = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z_{i+1}, \end{cases}$$

$$i = 0, \dots, m-1.$$

- We get  $S(t_{i+1}) = \exp(X(t_{i+1}))$ ,  $i = 0, \dots, m-1$ .

3

## Simulating correlated (G)BMs in Python

In many applications, we may want to simulate several correlated assets that make up a portfolio or from which certain options / derivatives are created (e.g. basket options).

Assume we have two correlated GBMs

$$\begin{cases} S_1(t) = S_1(0) e^{X_1(t)}, & X_1(t) = \sigma_1 W_1(t) + \mu_1 t \\ S_2(t) = S_2(0) e^{X_2(t)}, & X_2(t) = \sigma_2 W_2(t) + \mu_2 t \end{cases}$$

where  $B_1$  and  $B_2$  are two correlated BMs such that  
 $\text{corr}(W_1(t), W_2(t)) = \rho$  for every  $t \geq 0$ , with  $-1 \leq \rho \leq 1$ .

Let  $W(t) := \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}$ . Then to simulate  $W$  we can

first simulate two independent BMs  $\tilde{W}_1$  and  $\tilde{W}_2$ .

For  $\tilde{W}(t) := \begin{pmatrix} \tilde{W}_1(t) \\ \tilde{W}_2(t) \end{pmatrix}$  we obtain  $W$  by

$$W(t) = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \tilde{W}(t) = \begin{pmatrix} \tilde{W}_1(t) \\ \rho \tilde{W}_1(t) + \sqrt{1-\rho^2} \tilde{W}_2(t) \end{pmatrix} = \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}.$$

Why? Let's check the covariance between  $W_1(t)$  and  $W_2(t)$ :

$$\begin{aligned} \text{Cov}[W_1(t)W_2(t)] &= \mathbb{E}[W_1(t)W_2(t)] - \mathbb{E}[W_1(t)]\mathbb{E}[W_2(t)] \\ &= \mathbb{E}\left[\tilde{W}_1(t)\left(\rho \tilde{W}_1(t) + \sqrt{1-\rho^2} \tilde{W}_2(t)\right)\right] - 0 \\ &= \rho \mathbb{E}\left[(\tilde{W}_1(t))^2\right] + \sqrt{1-\rho^2} \mathbb{E}[\tilde{W}_1(t)]\mathbb{E}[\tilde{W}_2(t)] \\ &= \rho \mathbb{E}[(\tilde{W}_1(t))^2] = \rho \text{Var}[\tilde{W}_1(t)] = \rho t, \end{aligned}$$

using  $\mathbb{E}[W_i(t)] = \mathbb{E}[\tilde{W}_i(t)] = 0$ .

The correlation between  $W_1(t)$  and  $W_2(t)$  then equals  $\rho_{\text{***}}$ , as desired.

Hence :

- We generate  $m$  iid pairs of unit normal  $z_i \in \mathbb{R}^2$ ,  
 $i = 1, \dots, m$ .
- We obtain

$$\left\{ \begin{array}{l} W(0) = \mathbf{0} \in \mathbb{R}^2 \\ W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1}-t_i} \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} z_{i+1}, \\ i = 0, \dots, m-1. \end{array} \right.$$

Similarly, we can simulate correlated GBMs:

- We simulate a BM with drift  $X \in \mathbb{R}^2$ :

$$\begin{cases} X(0) = \log(S(0)) \\ X(t_{i+1}) = X(t_i) + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} (t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p & \sqrt{1-p^2} \end{pmatrix} z_{i+1} \end{cases}$$

$i=0, \dots, m-1.$

- We get  $S(t_{i+1}) = \exp(X(t_{i+1}))$ ,  $i=0, \dots, m-1$

(here  $\exp$  acts pointwise).

In practice we obtain:

$$S_1(t_{i+1}) = S_1(t_i) \exp \left\{ \mu_1(t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i} \cdot \sigma_1 \cdot Z_{i+1,1} \right\}$$

$$S_2(t_{i+1}) = S_2(t_i) \exp \left\{ \mu_2(t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i} \cdot \sigma_2 \cdot \left( p Z_{i+1,1} + \sqrt{1-p^2} Z_{i+1,2} \right) \right\}$$

What is the matrix  $L := \begin{pmatrix} 1 & 0 \\ p & \sqrt{1-p^2} \end{pmatrix}$ ?

The matrix  $L$  is the Cholesky decomposition of the correlation matrix  $C$ :

**Definition 7.1.2 (Cholesky decomposition)** Each symmetric positive definite matrix  $C$  has a unique factorization, the so-called Cholesky decomposition, of the form,

$$C = LL^T, \quad (7.1)$$

where  $L$  is a lower triangular matrix with positive diagonal entries.

**Example 7.1.2 (Example of Cholesky decomposition)** For a given  $(2 \times 2)$ -correlation matrix  $\mathbf{C}$ , we find the Cholesky decomposition (7.1), as

$$\begin{aligned} \mathbf{C} &\stackrel{\text{def}}{=} \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} \end{bmatrix} \begin{bmatrix} 1 & \frac{\rho_{12}}{\sqrt{1 - \rho_{12}^2}} \\ 0 & \sqrt{1 - \rho_{12}^2} \end{bmatrix}. \quad (7.2) \\ &= \begin{pmatrix} \text{corr}(X_1, X_1) & \text{corr}(X_2, X_1) \\ \text{corr}(X_1, X_2) & \text{corr}(X_2, X_2) \end{pmatrix} \end{aligned}$$

for two processes  $X_1, X_2$ .

We can generalize the concept to higher dimensions:

Let  $W(t) = (W_1(t), \dots, W_n(t))^T$  a vector of correlated BMs.

This means that :

- $\mathbb{E}[W_i(t) \cdot W_j(t)] = \rho_{i,j} \cdot t \quad \text{for } i \neq j$
- $\mathbb{E}[W_i(t) \cdot W_i(t)] = t \quad \text{for } i=j$

So :

- one first simulates  $\tilde{W}(t) \sim N(0, I_n)$  obtaining a vector of independent BMs;
- Get  $W(t) = L \tilde{W}(t)$ , where  $L$  is the cholesky decomposition of the correlation matrix,  $C = L L^T$ .  
Then  $W(t) \sim N(0, C)$ .