NUMERICAL METHODS FOR OPTION PRICING

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PRICING OF OPTIONS

Financial derivatives are product that are based on the performance of some underlying asset, like a stock, an interest rate or a commodity price.

Important question in quantitative finance:
what is a fair value for an option at the time of selling the option?

Based on the assumption of a GBM process for the asset price $S(t)$ (along with other assumptions, such as no dividends, no transaction costs, ...), namely

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t), \quad // \text{ under } \mathbb{P}$$

Black and Scholes derived their famous PDE for the valuation of European options, $V = V(t, S)$:

$$\begin{cases} \dfrac{\partial V}{\partial t} + rS\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} - rV = 0 \\ V(T, S) = H(T, S) \end{cases}$$

where $H(T, S)$ is the payoff function.

Hence to find the price of the option, we need to solve a parabolic PDE.

Different approaches:

- PDE methods

- Numerical integration (Fourier methods)
- Monte Carlo methods

based on the Feynman-Kac formula:

Theorem 3.2.1 (Feynman-Kac theorem) *Given the money-savings account, modeled by* $\mathrm{d}M(t) = rM(t)\mathrm{d}t$*, with constant interest rate* r*, let* $V(t,S)$ *be a sufficiently differentiable function of time* t *and stock price* $S = S(t)$*. Suppose that* $V(t,S)$ *satisfies the following partial differential equation, with general drift term,* $\bar{\mu}(t,S)$*, and volatility term,* $\bar{\sigma}(t,S)$*:*

$$\frac{\partial V}{\partial t} + \bar{\mu}(t,S)\frac{\partial V}{\partial S} + \frac{1}{2}\bar{\sigma}^2(t,S)\frac{\partial^2 V}{\partial S^2} - rV = 0, \tag{3.18}$$

with a final condition given by $V(T,S) = H(T,S)$*. The solution* $V(t,S)$ *at any time* $t < T$ *is then given by:*

$$V(t,S) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[H(T,S)\big|\mathcal{F}(t)\right] =: M(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{H(T,S)}{M(T)}\big|\mathcal{F}(t)\right]$$

where the expectation is taken under the measure $\mathbb{Q}$, with respect to a process S, which is defined by:

$$\mathrm{d}S(t) = \bar{\mu}(t,S)\mathrm{d}t + \bar{\sigma}(t,S)\mathrm{d}W^{\mathbb{Q}}(t), \quad t > t_0. \tag{3.19}$$

Thus we translate the problem of solving the PDE to the calculation of an expectation of a discounted payoff under the $\mathbb{Q}$ measure.

The Feynman-Kac formula is the basis for a closed-form expression for the option value for some payoff functions under the B&S dynamics.

Monte Carlo methods

By the Feynman-Kac formula, we know that for an option with payoff $H(T,S)$, the price is given by

$$V(t,S) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left[H(T,S) \mid \mathcal{F}_t \right]$$

where the underlying process S has dynamics under $\mathbb{Q}$

$$dS(t) = \bar{\mu}(t,S)\,dt + \bar{\sigma}(t,S)\,dW^{\mathbb{Q}}(t).$$

Monte Carlo methods are sampling methods based on probability theory:

- stochastic asset paths are generated under $\mathbb{Q}$;

- for each random path, the corresponding value of the payoff function is calculated;

- an approximated value for the option is obtained as the discounted average of these very many simulated payoff values.

Why does this work? Mathematically, MC methods are based on

- the central limit theorem;
- the law of large numbers.

Since the result of an experiment is a random number, we can also compute the standard error / standard deviation for the obtained prices.

A **Monte Carlo algorithm** to approximate an option value can be summarized as follows:

1. Partition the time interval $[0, T]$, so that the following time points, $0 = t_0 < t_1 < \cdots < t_m = T$, are obtained. The time points are defined by $t_i = \frac{i \cdot T}{m}$, $i = 0, \ldots, m$, where $m+1$ represents the number of time steps; The time step is denoted by $\Delta t = t_{i+1} - t_i$, where this time partitioning does not need to be uniform.

2. Generate asset values, $s_{i,j}$, taking the risk-neutral dynamics of the underlying model into account. Note that asset path value $s_{i,j}$ has two indices, $i = 1, \ldots, m$ (the time steps) and $j = 0, \ldots, N$, with N being the number of generated Monte Carlo asset paths.

3. Compute the N *payoff values*, H_j, and store these results. In the case of European options, we have $H_j = H(T, s_{m,j})$, whereas in the case of path-dependent options we may have $H_j = H(T, s_{i,j})$, $i = 1, \ldots m$.

4. Compute the average:

$$\mathbb{E}^{\mathbb{Q}}\left[H(T,S)\middle|\mathcal{F}(t_0)\right] \approx \frac{1}{N}\sum_{j=1}^{N} H_j.$$

   The right-hand side is known as the *Monte Carlo estimate*.

5. Calculate the option value as

$$V(t_0, S) \approx \mathrm{e}^{-r(T-t_0)}\frac{1}{N}\sum_{j=1}^{N} H_j.$$

6. Determine the standard error (standard deviation) related to the obtained prices in Step 5.

ERROR ANALYSIS

[redacted] *With a Monte Carlo method, we approximate a solution by a sequence of random realizations, i.e., with N Monte Carlo paths we obtain the following approximation:*

$$\mathbb{E}^{\mathbb{Q}}\left[H(T,S)\right] \approx \bar{H}_N(T,S) := \frac{1}{N}\sum_{j=1}^{N} H_j.$$

By the strong law of large numbers, we know that, for $N \to \infty$,

$$\lim_{N\to\infty} \bar{H}_N(T,S) = \mathbb{E}^{\mathbb{Q}}[H(T,S)], \quad \textit{with probability} \quad 1.$$

Actually,

we have two sources of error:

- estimate the expectation from finitely many samples;
- approximate $S(T)$ by a numerical method.

Both errors are measured by the mean-squared error:

DEFINITION (MSE)

Let $\hat{\theta}$ be an estimator for an unknown (deterministic) quantity θ. Then the MSE of $\hat{\theta}$ is

$$MSE(\hat{\theta}) = \mathbb{E}[(\theta - \hat{\theta})^2] \overset{(*)}{=} \underbrace{\mathbb{V}(\hat{\theta})}_{\text{variance}} + \underbrace{\left(\mathbb{E}[\hat{\theta} - \theta]\right)^2}_{\text{bias}}$$

Proof. $(*)$

$$\begin{aligned}\mathbb{V}(\hat{\theta}) + \mathbb{E}(\hat{\theta} - \theta)^2 &= \left(\mathbb{E}(\hat{\theta}^2) - \mathbb{E}(\hat{\theta})^2\right) + \left(\mathbb{E}(\hat{\theta})^2 - 2\mathbb{E}(\hat{\theta})\theta + \theta^2\right) \\ &= \mathbb{E}\left(\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2\right) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right].\end{aligned}$$

$\square$

In our setting:

$$\begin{cases} \theta = \mathbb{E}^{Q}[H(T, S) \mid \mathcal{F}_{t_0}] \\ \hat{\theta} = \overline{H}_N(T, S) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} H(T, s_{m,j}) \end{cases}$$

Hence,

In order to estimate the error which occurs due to a finite number of Monte Carlo paths, we need to compute the variance of the estimator [redacted]), *as follows,*

$\bar{H}_N(T,S)$

$$\begin{aligned}
\mathbb{V}ar^{\mathbb{Q}}\left[\bar{H}_N(T,S)\right] &= \mathbb{V}ar^{\mathbb{Q}}\left[\frac{1}{N}\sum_{j=1}^{N} H_j\right] \\
&= \frac{1}{N^2}\sum_{j=1}^{N}\mathbb{V}ar^{\mathbb{Q}}[H_j] \\
&\approx \frac{1}{N}\mathbb{V}ar^{\mathbb{Q}}[H(T,S)], \qquad (9.3)
\end{aligned}$$

given that samples $s_{i,j}$ are drawn independently.

In practice, the variance $\mathbb{V}ar^{\mathbb{Q}}[H(T,S)]$ is unknown. However, it can be approximated by the sample variance, $\bar{\upsilon}_N^2$, which is an unbiased estimator, i.e.

$$\bar{\upsilon}_N^2 := \frac{1}{N-1}\sum_{j=1}^{N}\left(H_j - \bar{H}_N(T,S)\right)^2.$$

Commonly, the standard error, ϵ_N, is defined as:

$$\epsilon_N := \frac{\bar{\upsilon}_N}{\sqrt{N}}. \qquad (9.4)$$

From (9.4) we see that when the number of samples increases by a factor of 4, the error reduces only by a factor of 2. ▲

$$\mathrm{MSE}[\hat{\theta}] = \frac{1}{N^2}\mathbb{V}\left(\sum_{j=1}^{m} H(T, S_{m,j})\right) + \left(\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}[H(T,S_{m,j})] - \mathbb{E}[H(T,S)]\right)^2$$

for iid samples

$$= \frac{1}{N}\mathbb{V}(H(T,S_m)) + \left(\mathbb{E}[H(T,S_m)] - \mathbb{E}[H(T,S)]\right)^2$$

$$\leq \frac{\bar{v}_N}{N} + C\,\Delta t^{2\gamma}$$ for a γ-weakly method.

Hence $\sqrt{\mathrm{MSE}(\hat{\theta})} \simeq C\sqrt{N^{-1} + \Delta t^{2\gamma}}$ (slow convergence w.r.t. γ).

<u>Example</u> (Euler-Maruyama method)

If $\varepsilon > 0$ is a given error tolerance, then

$$\mathrm{MSE}[\hat{\theta}] = \varepsilon^2 \leq \frac{C}{N} + C\Delta t^2 \iff N = \Theta(\varepsilon^{-2}) \text{ and } \Delta t = \Theta(\varepsilon).$$

(hence $m = \Theta(\varepsilon^{-1})$ time steps per simulation)

Example: analytic vs MC for B&S model (9.2.3)

We can use one of the numerical methods for SDE discussed before for simulating the underlying trajectories.

We shall compare the Euler scheme and the Milstein scheme for European call options and digital call options (payoff function for a cash-or-nothing option is $H(T,S) = K \mathbb{1}_{\{S(T) > K\}}$).

CODE → Both methods require a similar number of paths to achieve satisfactory pricing results: this is because they have the same order of weak convergence.

Example : Heston model (9.4.1)

We shall compare the Euler discretization scheme with truncated variance and the "Almost exact simulation scheme" ;

$$\begin{cases} x_{i+1} \approx x_i + k_0 + k_1 v_i + k_2 v_{i+1} + \sqrt{k_3 v_i} Z_x, \\ v_{i+1} = \bar{c}(t_{i+1}, t_i)\chi^2(\delta, \bar{\kappa}(t_{i+1}, t_i)), \end{cases} \quad Z_x \sim \mathcal{N}(0,1)$$

as in (9.30), with

$$\bar{c}(t_{i+1}, t_i) = \frac{\gamma^2}{4\kappa}(1 - e^{-\kappa(t_{i+1}-t_i)}), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2},$$

$$\bar{\kappa}(t_{i+1}, t_i) = \frac{4\kappa e^{-\kappa\Delta t}}{\gamma^2(1 - e^{-\kappa\Delta t})} v_i,$$

and $\chi^2(\delta, \bar{\kappa}(\cdot,\cdot))$ the noncentral chi-squared distribution with δ degrees of freedom and noncentrality parameter $\bar{\kappa}(t_{i+1}, t_i)$. The remaining constants are known as

$$k_0 = \left(r - \frac{\rho_{x,v}}{\gamma}\kappa\bar{v}\right)\Delta t, \quad k_1 = \left(\frac{\rho_{x,v}\kappa}{\gamma} - \frac{1}{2}\right)\Delta t - \frac{\rho_{x,v}}{\gamma},$$

$$k_2 = \frac{\rho_{x,v}}{\gamma}, \quad k_3 = (1 - \rho_{x,v}^2)\Delta t.$$

$\boxed{\text{CODE}}$ $\hookrightarrow$ With smaller time steps, the numerical results improve with AES, while for Euler we don't see this clear convergence.

REMARK

Many techniques can be used to improve the convergence of MC methods:

- Importance sampling;
- Quasi MC;
- control variates;
- Multi-level MC;
- ...

All of these accelerate the convergence of MC by reducing the variance of the sampling.

THE COS METHOD FOR PRICING

The COS method is an efficient option pricing method based on the discounted expected payoff integral representation for European options.
The key idea is to approximate the probability density function (which appears in the risk-neutral valuation formula) by a Fourier cosine series expansion.

Fourier cosine series coefficients have a closed-form relation with the characteristic function. Hence, the COS method can be applied to a quite general class of asset price processes, namely for those for which the characteristic function is available. These include:

- exponential Lévy processes,
- affine jump diffusion processes;
- stochastic volatility models,
- ...

Density approximation via Fourier cosine expansion

We consider European options with

- initial date t_0 (at which the asset value $x = X(t_0)$ is known)
- maturity date T, with $y = X(T)$ unknown.

We denote by $f_X(y) := f_X(T, y; t_0, x)$ the density function.

The density and its characteristic function, $f_X(y)$ and $\phi_X(u)$, form an example of a Fourier pair,[a]

$$\phi_X(u) = \int_{\mathbb{R}} \mathrm{e}^{iyu} f_X(y) \mathrm{d}y, \tag{6.1}$$

and,

$$f_X(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-iuy} \phi_X(u) \mathrm{d}u. \tag{6.2}$$

[a]Here we use the convention of the Fourier transform as often seen in the financial literature. Other conventions can also be used, and modifications to the methods are then straightforward.

[redacted] *The general definition of the Fourier expansion of a function $g(x)$ on an interval $[-1, 1]$ is as follows,*

$$g(\theta) = {\sum_{k=0}^{\infty}}' \bar{A}_k \cos(k\pi\theta) + \sum_{k=1}^{\infty} \bar{B}_k \sin(k\pi\theta), \tag{6.3}$$

where the prime at the sum, $\sum'$, indicates that the first term in the summation is weighted by one-half, and the coefficients are given by

$$\bar{A}_k = \int_{-1}^{1} g(\theta) \cos(k\pi\theta) d\theta, \quad \bar{B}_k = \int_{-1}^{1} g(\theta) \sin(k\pi\theta) d\theta. \tag{6.4}$$

By setting $\bar{B}_k = 0$, we obtain the classical Fourier cosine expansion, by which we can represent even functions around $\theta = 0$ exactly. ▲

We can extend any function $g : [0, \pi] \to \mathbb{R}$ to become an *even function on* $[-\pi, \pi]$, as follows,

$$\bar{g}(\theta) = \begin{cases} g(\theta), & \theta \geq 0 \\ g(-\theta), & \theta < 0. \end{cases} \tag{6.5}$$

Even functions can be expressed as Fourier cosine series. For a function $\bar{g}(\theta)$ supported on $[-\pi, \pi]$, the cosine expansion reads

$$\bar{g}(\theta) = {\sum_{k=0}^{\infty}}' \bar{A}_k \cdot \cos(k\theta), \tag{6.6}$$

with

$$\bar{A}_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \bar{g}(\theta) \cos(k\theta) \mathrm{d}\theta = \frac{2}{\pi} \int_0^{\pi} g(\theta) \cos(k\theta) \mathrm{d}\theta, \tag{6.7}$$

This works for functions defined on the interval $[-\pi, \pi]$.

For functions supported on any other finite interval, say $[a, b]$, the Fourier cosine series expansion is obtained via a change of variables:

$$\theta := \frac{y-a}{b-a}\pi, \quad y = \frac{b-a}{\pi}\theta + a.$$

It then reads

$$g(y) = \sum_{k=0}^{\infty}{}' \bar{A}_k \cdot \cos\left(k\pi \frac{y-a}{b-a}\right), \tag{6.8}$$

with

$$\bar{A}_k = \frac{2}{b-a} \int_a^b g(y) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y. \tag{6.9}$$

Since any real function has a cosine expansion when it has finite support, we need first to truncate the infinite integration range in (6.2). Due to the conditions for the existence of a Fourier transform, the integrand in (6.2) have to decay to 0 at $\pm\infty$ and we can truncate without losing significant accuracy.

Suppose $[a, b] \in \mathbb{R}$ is chosen such that the *truncated integral* approximates the infinite counterpart very well, i.e.,

$$\hat{\phi}_X(u) := \int_a^b \mathrm{e}^{iuy} f_X(y)\mathrm{d}y \approx \int_{\mathbb{R}} \mathrm{e}^{iuy} f_X(y)\mathrm{d}y = \phi_X(u). \tag{6.10}$$

We relate equation (6.10) to (6.9), by recalling the well-known Euler formula:

$$\mathrm{e}^{iu} = \cos(u) + i\sin(u),$$

which implies that $\mathrm{Re}\{\mathrm{e}^{iu}\} = \cos(u)$, where $\mathrm{Re}\{\cdot\}$ denotes the real part of the argument. Based on this, for any random variable, X, and constant, $a \in \mathbb{R}$, the following equality holds:

$$\phi_X(u)\mathrm{e}^{ia} = \mathbb{E}[\mathrm{e}^{iuX+ia}] = \int_{-\infty}^{\infty} \mathrm{e}^{i(uy+a)} f_X(y)\mathrm{d}y. \tag{6.11}$$

By taking the real parts in (6.11), we find:

$$\mathrm{Re}\left\{\phi_X(u)\mathrm{e}^{ia}\right\} = \mathrm{Re}\left\{\int_{-\infty}^{\infty} \mathrm{e}^{i(uy+a)} f_X(y)\mathrm{d}y\right\} = \int_{-\infty}^{\infty} \cos\left(uy+a\right) f_X(y)\mathrm{d}y.$$

We substitute the Fourier argument $u = \frac{k\pi}{b-a}$, and multiply the characteristic function in (6.10), by $\exp\left(-i\frac{ka\pi}{b-a}\right)$, i.e.

$$\hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot \exp\left(-i\frac{ka\pi}{b-a}\right) = \int_a^b \exp\left(iy\frac{k\pi}{b-a} - i\frac{ka\pi}{b-a}\right) f_X(y)\mathrm{d}y. \tag{6.12}$$

By taking the real part at both sides of the equation, we find:

$$\mathrm{Re}\left\{\hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot \exp\left(-i\frac{ka\pi}{b-a}\right)\right\} = \int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right) f_X(y)\mathrm{d}y. \tag{6.13}$$

At the right-hand side of (6.13), we have the definition of $\bar{A}_k$ in (6.9), so

$$\bar{A}_k \equiv \frac{2}{b-a}\mathrm{Re}\left\{\hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot \exp\left(-i\frac{ka\pi}{b-a}\right)\right\}. \tag{6.14}$$

It follows from (6.10) that $\bar{A}_k \approx \bar{F}_k$ with

$$\bar{F}_k := \frac{2}{b-a}\mathrm{Re}\left\{\phi_X\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ka\pi}{b-a}\right)\right\}. \tag{6.15}$$

We now *replace* $\bar{A}_k$ by $\bar{F}_k$ in the series expansion of $f_X(y)$ on $[a,b]$, i.e.,

$$\hat{f}_X(y) \approx {\sum_{k=0}^{\infty}}' \bar{F}_k \cos\left(k\pi\frac{y-a}{b-a}\right), \tag{6.16}$$

and *truncate the series summation*, so that

$$\hat{f}_X(y) \approx {\sum_{k=0}^{N-1}}' \bar{F}_k \cos\left(k\pi\frac{y-a}{b-a}\right). \tag{6.17}$$

Remember that the first term in the summation should be multiplied by one-half (indicated by the $\sum'$ symbol). *Forgetting to multiply the first term by one-half is an often made computer implementation mistake!*

The resulting error in the approximation $\hat{f}_X(y)$ consists of two parts: a series truncation error from (6.16) to (6.17) and an error originating from the approximation of $\bar{A}_k$ by $\bar{F}_k$.

Since the cosine series expansions of so-called *entire functions* (i.e. functions without any singularities[2] anywhere in the complex plane, except at ∞) exhibit an *exponential convergence* [Boyd, 1989], we can expect (6.17) to give highly accurate approximations, with a small value for N, to density functions that have no singularities on $[a,b]$.

$\boxed{\text{CODE}}$

- normal dist.
- lognormal dist.

[2] By "singularity" we mean, as in [Boyd, 1989], poles, fractional powers, logarithms, other branch points, and discontinuities in a function or in any of its derivatives.

Pricing European Options by the COS method

The COS formula for pricing European options is obtained by substituting for the density function its Fourier cosine series expansion.

Since probability densities are usually smooth, only a few terms in the expansion may already give a highly accurate approximation.

We set $X(t) := \log S(t)$, with $X(t_0) = x$ and $X(T) = y$.

Then the value of the plain vanilla option is

$$V(t_0, x) = e^{-r(T-t_0)}\, \mathbb{E}^{\mathbb{Q}}\left[V(T,y) \mid \mathcal{F}_{t_0} \right] = e^{-r(T-t_0)} \int_{\mathbb{R}} V(T,y)\, f_X(y)\, dy ,$$

where

- $f_X(y) = f_X(T, y; t_0, x)$ is the transition probability density of $X(T)$;
- r is the interest rate.

$\boxed{\text{STEP 1: truncation}}$

Since $f_X(y)$ rapidly decays to zero as $y \to \pm\infty$, we truncate the infinite integration range to $[a, b] \subset \mathbb{R}$, without losing significant accuracy, and for $S(t) = \mathrm{e}^{X(t)}$, with $x := X(t_0)$ and $y := X(T)$, we obtain approximation V_{I}:

$$V(t_0, x) \approx V_{\mathrm{I}}(t_0, x) = \mathrm{e}^{-r\tau} \int_a^b V(T, y) f_X(y) \mathrm{d}y, \tag{6.21}$$

with $\tau = T - t_0$.

STEP 2 : Approximation by Fourier cosine expansion

[redacted] Since $f_X(y)$ is usually not known whereas the characteristic function is, the density is approximated by its Fourier cosine expansion in y, as in (6.8),

$$\hat{f}_X(y) = {\sum_{k=0}^{+\infty}}' \bar{A}_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right), \tag{6.22}$$

with

$$\bar{A}_k(x) := \frac{2}{b-a} \int_a^b \hat{f}_X(y) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y, \tag{6.23}$$

so that,

$$V_{\mathrm{I}}(t_0, x) = \mathrm{e}^{-r\tau} \int_a^b V(T,y) {\sum_{k=0}^{+\infty}}' \bar{A}_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y. \tag{6.24}$$

We interchange the summation and integration (application of Fubini's Theorem), and insert the following definition

$$H_k := \frac{2}{b-a} \int_a^b V(T,y) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y, \tag{6.25}$$

resulting in

$$V_{\mathrm{I}}(t_0, x) = \frac{b-a}{2} \mathrm{e}^{-r\tau} \cdot {\sum_{k=0}^{+\infty}}' \bar{A}_k(x) \cdot H_k. \tag{6.26}$$

Here, the H_k are the *cosine series coefficients of the payoff function*, $V(T,y)$. Thus, from (6.21) to (6.26) we have transformed an integral of the product of two real functions, $f_X(y)$ and $V(T,y)$, into a product of their Fourier cosine series coefficients.

Due to the rapid decay rate of these coefficients, we may truncate the series summation :

$$V_{\mathrm{II}}(t_0, x) = \frac{b-a}{2} \mathrm{e}^{-r\tau} \cdot {\sum_{k=0}^{N-1}}' \bar{A}_k(x) H_k. \tag{6.27}$$

Step 3: derive the COS formula

[redacted] The coefficients $\bar{A}_k(x)$, as defined in (6.23), can be approximated by $\bar{F}_k(x)$, as defined in (6.15):

By replacing $\bar{A}_k(x)$ in (6.27) by $\bar{F}_k(x)$, we obtain

$$V(t_0,x) \approx V_{\mathrm{III}}(t_0,x) = \mathrm{e}^{-r\tau} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ \phi_X\left(\frac{k\pi}{b-a}\right) \exp\left(-ik\pi\frac{a}{b-a}\right)\right\} \cdot H_k, \tag{6.28}$$

with $\tau = T - t_0$ and x a function of $S(t_0)$ (like $x = \log(S(t_0))$ or $x = \log(S(t_0)/K)$), and the characteristic function

$$\boxed{\phi_X(u) = \phi_X(u; t_0, T) \equiv \phi_X(u, x; t_0, T),} \tag{6.29}$$

as the characteristic function also depends on variable x.
This is the *COS formula* for general underlying processes. [redacted]

<u>Remark</u>: the H_k-coefficients can be obtained analytically for plain vanilla and digital options.

Payoff coefficients

For $y(T) = \log\left(\frac{S(T)}{K}\right)$ (adjusted log-asset price), we rewrite the payoff of European options as

$$V(T,y) := [\bar{\alpha} \cdot K(\mathrm{e}^y - 1)]^+ \quad \text{with} \quad \bar{\alpha} = \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put.} \end{cases}$$

We need now two basic results:

Result 6.2.1 *The cosine series coefficients, χ_k, of $g(y) = \mathrm{e}^y$ on an integration interval $[c,d] \subset [a,b]$,*

$$\chi_k(c,d) := \int_c^d \mathrm{e}^y \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y, \tag{6.30}$$

and the cosine series coefficients, ψ_k, of $g(y) = 1$ on an integration interval $[c,d] \subset [a,b]$,

$$\psi_k(c,d) := \int_c^d \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y, \tag{6.31}$$

are known analytically. Basic calculus shows that

$$\chi_k(c,d) := \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)\mathrm{e}^d - \cos\left(k\pi\frac{c-a}{b-a}\right)\mathrm{e}^c + \frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)\mathrm{e}^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)\mathrm{e}^c\right], \tag{6.32}$$

and

$$\psi_k(c,d) := \begin{cases} \left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi}, & k \neq 0, \\ (d-c), & k = 0. \end{cases} \tag{6.33}$$

Focusing on a call option, in the case that $a < 0 < b$, we obtain

$$\begin{aligned} H_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b) - \psi_k(0,b)\right), \end{aligned} \tag{6.34}$$

where χ_k and ψ_k are given by (6.32) and (6.33), respectively. Similarly, for a vanilla put, we find

$$H_k^{put} = \frac{2}{b-a}K\left(-\chi_k(a,0) + \psi_k(a,0)\right). \tag{6.35}$$

In the situation, where we deal with $a < b < 0$, we have, of course,

$$H_k^{call} = 0,$$

while for $0 < a < b$, the payoff coefficients H_k^{call} are defined by $c \equiv a$, $d \equiv b$ in (6.25) and in Result 6.2.1. For put options the relations are reversed.

Remark: for options contracts for which the H_k-coefficients can be obtained only numerically, the error convergence will be dominated by the numerical quadrature rule employed.

(Exercise: derive the coefficients H_k for cash-or-nothing call options with payoff $K \mathbb{1}_{\{S(T) > K\}}$.)

ERROR ANALYSIS for the COS METHOD

In the derivation of the COS formula, there were three stages in which approximation errors were introduced:

1. The *integration range truncation error*:

$$\epsilon_{\mathrm{I}} := V(t_0, x) - V_{\mathrm{I}}(t_0, x) = \int_{\mathbb{R}\setminus[a,b]} V(T, y) f_X(y) \mathrm{d}y. \tag{6.38}$$

The truncation of the integration range

2. The *series truncation error on* $[a, b]$:

$$\epsilon_{\mathrm{II}} := V_{\mathrm{I}}(t_0, x) - V_{\mathrm{II}}(t_0, x) = \frac{1}{2}(b-a)\mathrm{e}^{-r\tau} \sum_{k=N}^{+\infty} \bar{A}_k(x) \cdot H_k, \tag{6.39}$$

where $\bar{A}_k(x)$ and H_k are defined by (6.23) and (6.25), respectively.

The substitution of the density by its truncated cosine series expansion.

3. The *error related to approximating* $\bar{A}_k(x)$ *by* $\bar{F}_k(x)$ in (6.15):

$$\epsilon_{\mathrm{III}} := V_{\mathrm{III}}(t_0, x) - V_{\mathrm{II}}(t_0, x) \tag{6.40}$$

$$= \mathrm{e}^{-r\tau} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ \int_{\mathbb{R}\setminus[a,b]} \exp\left(ik\pi \frac{y-a}{b-a} \right) f_X(y) \mathrm{d}y \right\} \cdot H_k.$$

The substitution of the series coefficients by the characteristic function approximation.

COMMENTS:

1) Notice that we do not take any error in the coefficients H_k into account.

2) The key to bound the errors lies in the decay rate of the cosine series coefficients, and this depends on the properties of the functions on the expansion interval.

3) With a properly chosen truncation of the integration range, the overall error converges:

- exponentially for density functions in $C^\infty([a,b])$;
- algebraically for density functions with a discontinuity in one of the derivatives.

4) The COS method replaces the density by a partial sum of the Fourier cosine series of the density. Therefore, when the integration-range truncation error is not dominant, the overall error depends heavily on the properties of the density.

5) Why are we using only cosine expansions and not the full Fourier series?

The Fourier series representation of a function of a continuous variable x in an interval, defines a periodic extension *of that function for all $x \in \mathbb{R}$. If the function values at the interval's end points are not equal, the periodic extension is not continuous, which will give rise to convergence problems near the interval's endpoints (known as the Gibb's phenomenon).*

One way to overcome such problems is by considering the function to be an even function on an interval of twice the size. *The periodic extension of this function will then be* continuous at the interval's end points. *With* an even periodic extension *of a data set, the sine functions,* [redacted] *will have weight 0,* [redacted]

> The choice of numerical method for option valuation is [redacted] often based on whether the computation is meant for *model calibration*, for which the speed of a pricing method for European options is essential, or for the *pricing of a specific involved derivatives contract*, for which robustness of the numerical method is of importance.

- For pricing, MC and PDE methods are usually preferred.
- Fourier methods are fast and fit well for calibration tasks. However, it is more challenging to use them for exotic payoff contracts.

Other efficient techniques are:

- Carr-Madan FFT-based method;
- fast Gaussian transform
- B-splines methods, ...

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