Numerical Methods for Mathematical Finance

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Lecture 1

University of Verona
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ABOUT ME: SILVIA LAVAGNINI

- Assistant Professor at the **Department of Data Science and Analytics** at the **BI Norwegian Business School** (Oslo);
- PhD in Mathematical Finance from the University of Oslo (2021) with thesis *Stochastic modeling in energy markets*;
- One-and-a-half-year post-doc at the Economics Department (University of Verona);
- Before that: Bachelor's degree in Matematica Applicata and Master's degree in Applied Mathematics from University of Verona;
- 2x3 months visiting University of California (Santa Barbara);
- Research interests:
  - Mathematical finance, energy markets;
  - HJM models and SPDEs;
  - Machine learning/Deep learning.
- Links: Google Scholar, GitHub.

ABOUT THIS MINI-COURSE

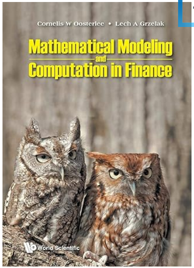
- **Timetable:** 12 hours in class:
  - Monday 13/05: 14:30-16:30
  - Tuesday 14/05: 10:30-13:30
  - Thursday 16/05: 10:30-11:30
  - Monday 20/05: 14:30-17:30
  - Wednesday 22/04: 13:30-16:30

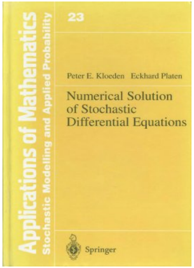
- **Prerequisites:** Stochastic Calculus and Mathematical Finance;

- We shall use **Python** for simulations and in particular **Jupyter notebook**: for installing Jupyter notebook see `https://docs.jupyter.org/en/latest/install/notebook-classic.html`;

- **Exam:** individual project to be chosen from a list (tbd).

SYLLABUS AND MATERIAL





- MAIN: *Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes* by C. W. Oosterlee and L. A. Grzelak
  - Codes both in Python and in Matlab available in my GitHub page
- *Numerical Solution of Stochastic Differential Equations* by P. E. Kloeden and E. Platen

Material (mostly notebooks) for the course is published daily on GitHub.

STOCHASTIC PROCESSES IN FINANCE

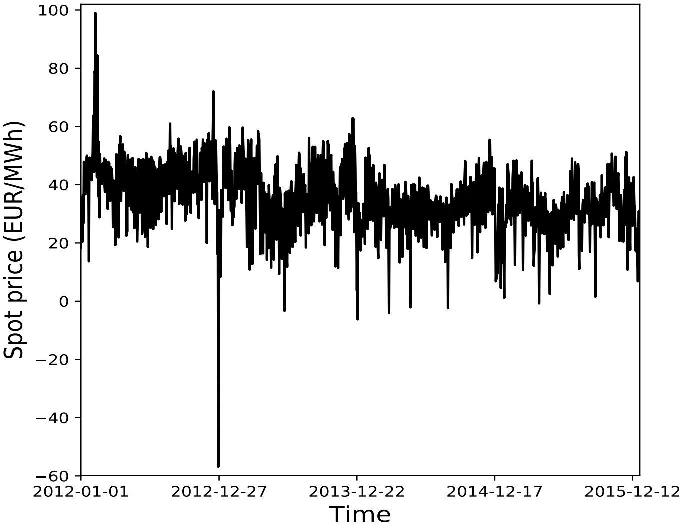
- Stochastic processes can be used to model uncertain or random events that fluctuate through time.

- We are interested in constructing stochastic processes whose dynamics resembles the one that we observe in finance and economics.

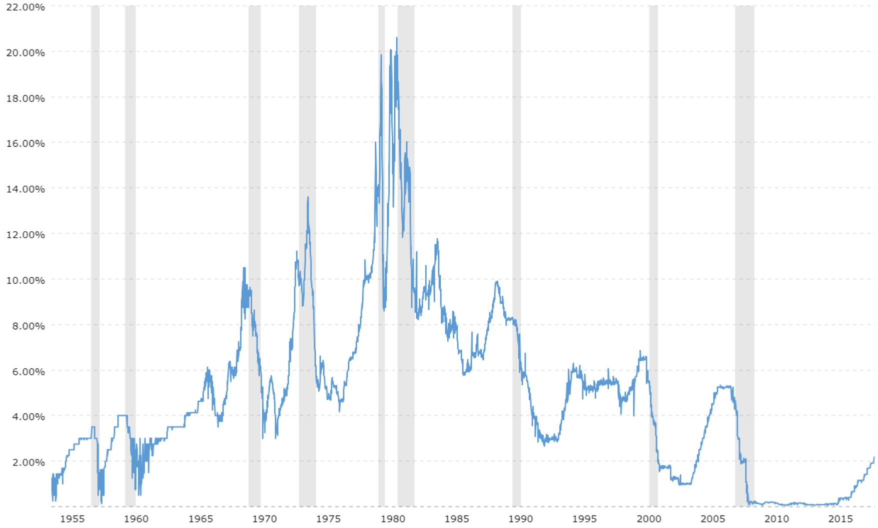
- If we can "recreate" a function which "behaves like" a specific phenomenon, we can use this function to answer questions related to the potential behavior of this phenomenon, and gain insight into what we may expect to happen.

EXAMPLE: STOCK PRICES

EXAMPLE: SPOT PRICE OF ELECTRICITY



EXAMPLE: INTEREST RATES



STOCHASTIC PROCESSES IN FINANCE

- Simulations
- Pricing
- Calibration
- Hedging

Questions?

PLAN FOR THE COURSE

- Simulations of BM, GBM, correlated (G)BMs
- Simulations of processes starting from the SDE
  - The Euler-Maruyama scheme
  - The Milstein scheme
- Pricing of European options
  - The Monte Carlo approach
  - The Fourier approach
-

OUTLINE

Definition

A **stochastic process** $\{X(t),\ t \in \mathcal{T}\}$ is a collection of random variables indexed by t taking values in some index set $\mathcal{T}$:

- If the stochastic process is discrete in time, then $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$ (or $\mathcal{T} = \{t_0, t_1, ...\}$). For discrete processes, we often write $\{X_n = X(t_n),\ t_n \in \mathcal{T}\}$.
- If the stochastic process continuous in time, then $\mathcal{T} = [a, b]$ or $\mathcal{T} = \mathbb{R}_+$.

We usually think of the index t as time, hence we assume $t_0 < t_1 < t_2 < \ldots$.
Then $X(t)$ is the value of the process at time t.

THE FLOW OF INFORMATION: FILTRATION

- Suppose we have a set of calendar dates/days, $t_1, t_2, \ldots, t_m$;
- Up to today, we have observed certain state values of the stochastic process $X(t)$, hence the past is known;
- For the future, we do not know the precise path but we may simulate the future according to some asset price distribution.

The information available at time t_i is described by a filtration:

Definition

We say that $\{\mathcal{F}_t,\ t \in \mathcal{T}\}$ is the **filtration** associated to a stochastic process $\{X(t),\ t \in \mathcal{T}\}$, if for all $t_i \in \mathcal{T}$, $\mathcal{F}_{t_i}$ is the sigma algebra $\mathcal{F}_{t_i} = \sigma(X_{t_j}, 1 \leq j \leq i)$ generated by the sequence X_{t_j} for $1 \leq j \leq i$.

Note that for $s < t$, we have that $\mathcal{F}_s \subset \mathcal{F}_t$.

- We call $\mathbb{F} = \{\mathcal{F}_t,\ t \in \mathcal{T}\}$ the **natural filtration** generated by the process X.
- In words: $\{\mathcal{F}_t,\ t \in \mathcal{T}\}$ is the **filtration** associated to a stochastic process $\{X(t),\ t \in \mathcal{T}\}$, if for all $t_i \in \mathcal{T}$, $\mathcal{F}_{t_i}$ contains all the information about the history of the stochastic process X up to time t_i;
- A stochastic process X is **adapted to the filtration** $\mathbb{F}$ if $\sigma(X_{t_j}, 1 \leq j \leq i) \subseteq \mathcal{F}_{t_i}$;
- In words: the process X can not "look into the future".

PROBABILITY SPACE

We shall work with the notion of probability space:

Definition

We call the tuple $(\Omega, \mathbb{F}, \mathbb{P})$ a **probability space** where:

- Ω is the set of all possible outcomes;
- $\mathbb{F}$ is a filtration;
- $\mathbb{P}$ is a probability measure.

BROWNIAN MOTION

A fundamental stochastic process, which is also commonly used in the construction of stochastic differential equations (SDEs) to describe asset price movements, is the **Wiener process**, also called **Brownian motion**:

Definition

We say that a stochastic process $\{B(t),\ t \in [0, \infty)\}$ is a **Brownian motion** if the following conditions holds:

1. $B(0) = 0$.
2. $B(t) - B(s)$ is independent from $B(t') - B(s')$ whenever $[s, t] \cap [s', t'] = \emptyset$.
3. $B(t) - B(s) \sim \mathcal{N}(0, t - s)$.
4. $B(t)$ is almost surely continuous.

MARTINGALES

Definition

Given a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, we say that a (right-continuous with left limit) stochastic process $\{M(t),\ t \in \mathcal{T}\}$ is a **martingale** associated to the filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \in \mathcal{T}\}$ if the following properties holds:

1) M is adapted to the filtration $\mathbb{F}$;
2) $\mathbb{E}[|M(t)|] < \infty$, for all $t \in \mathcal{T}$;
3) For all $s < t \in \mathcal{T}$, we have

$$\mathbb{E}[M(t)|\mathcal{F}_s] = M(s).$$

In words: the best prediction of the expectation of a martingale's future value is its present value.

OUTLINE

Introduction to Stochastic processes and Brownian motion

Introduction to Financial Asset Dynamics

Simulations in Python

GEOMETRIC BROWNIAN MOTION ASSET PRICE PROCESS

The most commonly used asset price process in finance is the **geometric Brownian motion** (GBM), where the logarithm of the asset price follows a Brownian motion with drift:

Definition

If $\{B(t),\ t \in [0, \infty)\}$ is a Brownian motion, then the process $\{Y(t),\ t \in [0, \infty)\}$ defined by

$$Y(t) = Y(0)e^{\sigma B(t) + \mu t}, \text{ for } t \geq 0,$$

is called **a geometric Brownian motion**, with $\sigma > 0$ and $\mu \in \mathbb{R}$. More generally:

$$Y(t) = Y(s)e^{\sigma(B(t) - B(s)) + \mu(t-s)}, \text{ for } t \geq s \geq 0.$$

Geometric Brownian motion is useful in the modeling of stock prices overtime when you assume that the percentage changes are independent and identically distributed.

GBM: DERIVATION

- If we estimate the daily returns of a stock $S(t)$, we find that returns are (approximately) normally distributed. In fact, we have for all t and Δt

$$\frac{S(t+\Delta t) - S(t)}{S(t)} \sim \mathcal{N}(\mu \Delta t, \sigma \Delta t).$$

- We therefore use a Brownian motion to model the returns, i.e. let $\Delta S(t) = S(t+\Delta t) - S(t)$ for small $\Delta t > 0$ denote a small change in the stock price. Then we assume that

$$\frac{\Delta S(t)}{S(t)} = \mu \Delta t + \sigma \Delta B(t),$$

where $\Delta B(t) = B(t+\Delta t) - B(t)$.

GBM: DERIVATION - CONTINUES

We have

$$\frac{\Delta S(t)}{S(t)} = \mu \Delta t + \sigma \Delta B(t),$$

where $\Delta B(t) = B(t + \Delta t) - B(t)$. For Δt small, we can expect that $\frac{\Delta S(t)}{S(t)}$ is also small. Then we can approximate

$$\frac{\Delta S(t)}{S(t)} \approx \log\left(1 + \frac{\Delta S(t)}{S(t)}\right) = \log\left(\frac{S(t) + \Delta S(t)}{S(t)}\right) = \log\left(\frac{S(t + \Delta t)}{S(t)}\right).$$

Then

$$\begin{aligned}
\log\left(\frac{S(t + \Delta t)}{S(t)}\right) &\approx \mu \Delta t + \sigma \Delta B(t) \\
\frac{S(t + \Delta t)}{S(t)} &\approx \exp\left(\mu \Delta t + \sigma \Delta B(t)\right) \\
S(t + \Delta t) &\approx S(t) \exp\left(\mu \Delta t + \sigma \Delta B(t)\right).
\end{aligned}$$

GBM: DERIVATION - CONTINUES

We have

$$S(t+\Delta t) \approx S(t)\exp\left(\mu\Delta t + \sigma(B(t+\Delta t) - B(t))\right).$$

Then also

$$\begin{aligned} S(t+2\Delta t) &\approx S(t+\Delta t)\exp\left(\mu\Delta t + \sigma(B(t+2\Delta t) - B(t+\Delta t))\right) \\ &\approx S(t)\exp\left(2\mu\Delta t + \sigma(B(t+2\Delta t) - B(t))\right), \end{aligned}$$

and, more generally,

$$\begin{aligned} S(t+n\Delta t) &\approx S(t+\Delta t)\exp\left(\mu\Delta t + \sigma(B(t+2\Delta t) - B(t+\Delta t))\right) \\ &\approx S(t)\exp\left(n\mu\Delta t + \sigma(B(t+n\Delta t) - B(t))\right). \end{aligned}$$

If we define $s := t + n\Delta t$, then $s - t = n\Delta t$ and we then find

$$S(s) \approx S(t)\exp\left(\mu(s-t) + \sigma(B(s) - B(t))\right).$$

OUTLINE

NEXT STEPS

- Simulate a BM in Python;
- Simulate a GBM in Python;
- Simulate correlated (G)BMs in Python.