1 Simulating a BM in Python

We know by definition that for a Brownian motion W :

i) $W(0) = 0$;

ii) W has independent increments ;

iii) $W(t) - W(s) \sim N(0, t-s)$.

In general, given a partition $0 = t_0 < t_1 < \dots < t_m = T$ we have that

$$W(t_{i+1}) - W(t_i) \sim N(0, t_{i+1} - t_i)$$

Hence:

- We generate m iid unit normals $z_1, \dots, z_m$;
- We obtain the independent increments by

$$W(t_{i+1}) - W(t_i) = \sqrt{t_{i+1} - t_i}\, Z_{i+1}, \quad i = 0, \dots, m-1.$$

In other words, we have the recursion:

$$\begin{cases} W(0) = 0 \\ W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} \cdot Z_{i+1}, \quad i = 0, \dots, m-1. \end{cases}$$

2 Simulating a GBM in Python

We want to simulate $S(t) = S(0)\, e^{\mu t + \sigma B(t)}$.

Let $X(t) := \mu t + \sigma W(t)$.

Then $S(t) = S(0) e^{X(t)} = S(s)\, e^{X(t) - X(s)}$ for any $s \leq t$,

where $X(t) - X(s) = \mu(t-s) - \sigma(W(t) - W(s))$.

More generally, for a partition $0 = t_0 < t_1 < \dots < t_m = T$:

$$X(t_{i+1}) - X(t_i) = \mu(t_{i+1} - t_i) + \sigma(W(t_{i+1}) - W(t_i)).$$

<u>Note</u>: we know how to simulate the increments $W(t_{i+1}) - W(t_i)$.

Hence:

- We simulate X by the recursion

$$\begin{cases} X(0) = \log S(0) \\ X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma(W(t_{i+1}) - W(t_i)) \\ \qquad\quad \overset{d}{=} X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\, Z_{i+1}, \end{cases}$$

$i = 0, \dots m-1.$

- We get $S(t_{i+1}) = \exp(X(t_{i+1}))$, $i = 0, \dots, m-1$.

③ Simulating correlated (G)BMs in Python

In many applications, we may want to simulate several correlated assets that make up a portfolio or from which certain options / derivatives are created (e.g. basket options).

Assume we have two correlated GBMs

$$\begin{cases} S_1(t) = S_1(0)\, e^{X_1(t)}, & X_1(t) = \sigma_1 W_1(t) + \mu_1 t \\ S_2(t) = S_2(0)\, e^{X_2(t)}, & X_2(t) = \sigma_2 W_2(t) + \mu_2 t \end{cases}$$

where B_1 and B_2 are two correlated BMs such that

$\mathrm{corr}(W_1(t), W_2(t)) = \rho$ for every $t \geq 0$, with $-1 \leq \rho \leq 1$.

Let $W(t) := \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}$. Then to simulate W we can first simulate two independent BMs $\tilde{W}_1$ and $\tilde{W}_2$.

For $\tilde{W}(t) := \begin{pmatrix} \tilde{W}_1(t) \\ \tilde{W}_2(t) \end{pmatrix}$ we obtain W by

$$W(t) = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \tilde{W}(t) = \begin{pmatrix} \tilde{W}_1(t) \\ \rho \tilde{W}_1(t) + \sqrt{1-\rho^2}\, \tilde{W}_2(t) \end{pmatrix} = \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}.$$

Why? Let's check the covariance between $W_1(t)$ and $W_2(t)$:

$$\begin{aligned}
\mathbb{C}\mathrm{ov}[W_1(t)W_2(t)] &= \mathbb{E}[W_1(t)W_2(t)] - \mathbb{E}[W_1(t)]\mathbb{E}[W_2(t)] \\
&= \mathbb{E}\left[\widetilde{W}_1(t)\left(\rho \widetilde{W}_1(t) + \sqrt{1-\rho^2}\,\widetilde{W}_2(t)\right)\right] - 0 \\
&= \rho \mathbb{E}\left[(\widetilde{W}_1(t))^2\right] + \sqrt{1-\rho^2}\,\mathbb{E}[\widetilde{W}_1(t)]\mathbb{E}[\widetilde{W}_2(t)] \\
&= \rho \mathbb{E}[(\widetilde{W}_1(t))^2] = \rho \mathbb{V}\mathrm{ar}[\widetilde{W}_1(t)] = \rho t,
\end{aligned}$$

using $\mathbb{E}[W_i(t)] = \mathbb{E}[\widetilde{W}_i(t)] = 0$.

The correlation between $W_1(t)$ and $W_2(t)$ then equals ρ, as desired.

Hence :

- We generate m iid pairs of unit normal $Z_i \in \mathbb{R}^2$, $i = 1, \dots, m$.

- We obtain

$$\begin{cases} W(0) = 0 \in \mathbb{R}^2 \\ W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} Z_{i+1}, \quad i = 0, \dots, m-1. \end{cases}$$

Similarly, we can simulate correlated GBMs:

- We simulate a BM with drift $X \in \mathbb{R}^2$:

$$\begin{cases} X(0) = \log(S(0)) \\ X(t_{i+1}) = X(t_i) + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}(t_{i+1}-t_i) + \sqrt{t_{i+1}-t_i}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} Z_{i+1} \end{cases}$$

$i = 0, \dots, m-1.$

- We get $S(t_{i+1}) = \exp(X(t_{i+1}))$, $i = 0, \dots, m-1$

(here exp acts pointwise).

In practice we obtain:

$$S_1(t_{i+1}) = S_1(t_i)\exp\left\{\mu_1(t_{i+1}-t_i) + \sqrt{t_{i+1}-t_i}\cdot\sigma_1\cdot Z_{i+1,1}\right\}$$

$$S_2(t_{i+1}) = S_2(t_i)\exp\left\{\mu_2(t_{i+1}-t_i) + \right.$$

$$\left. + \sqrt{t_{i+1}-t_i}\cdot\sigma_2\cdot\left(\rho Z_{i+1,1} + \sqrt{1-\rho^2}\,Z_{i+1,2}\right)\right\}$$

What is the matrix $L := \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$?

The matrix L is the Cholesky decomposition of the correlation matrix, C:

Definition 7.1.2 (Cholesky decomposition) *Each symmetric positive definite matrix* $\mathbf{C}$ *has a unique factorization, the so-called Cholesky decomposition, of the form,*

$$\mathbf{C} = \mathbf{L}\mathbf{L}^{\mathrm{T}}, \tag{7.1}$$

where $\mathbf{L}$ *is a lower triangular matrix with positive diagonal entries.*

Example 7.1.2 (Example of Cholesky decomposition) For a given (2×2)-*correlation matrix* $\mathbf{C}$, we find the Cholesky decomposition (7.1), as

$$\mathbf{C} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & \rho_{\cancel{12}} \\ \rho_{\cancel{12}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho_{\cancel{12}} & \sqrt{1-\rho_{\cancel{12}}^2} \end{bmatrix} \begin{bmatrix} 1 & \rho_{\cancel{12}} \\ 0 & \sqrt{1-\rho_{\cancel{12}}^2} \end{bmatrix}. \tag{7.2}$$

♦

$$= \begin{pmatrix} \mathrm{corr}(X_1, X_1) & \mathrm{corr}(X_2, X_1) \\ \mathrm{corr}(X_1, X_2) & \mathrm{corr}(X_2, X_2) \end{pmatrix}$$

for two processes X_1, X_2.

We can generalize the concept to higher dimensions:
Let $W(t) = (W_1(t), \ldots, W_n(t))^T$ a vector of correlated BMs.
This means that:

- $\mathbb{E}[W_i(t) \cdot W_j(t)] = \rho_{i,j} \cdot t$ for $i \neq j$
- $\mathbb{E}[W_i(t) \cdot W_i(t)] = t$ for $i = j$

So:

- one first simulates $\tilde{W}(t) \sim N(0, I_n)$ obtaining a vector of independent BMs;
- Get $W(t) = L\tilde{W}(t)$, where L is the cholesky decomposition of the correlation matrix, $C = LL^T$.

Then $W(t) \sim N(0, C)$.