Mini-course:
Numerical Methods for Mathematical Finance
Projects list

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- You can choose one project from the list below, to be solved in Python.
- Whenever not specified, you can use the same model parameters as seen in class, or choose your own.
- For the course *Numerical Methods for Mathematical Finance* (6 credits): it is required to solve Problem 1, 3 and one between Problem 4 and Problem 5.

1. **The Milstein scheme for SDE simulations** (requires stochastic calculus)
   (a) Derive, via Itô-Taylor expansion, the Milstein Scheme;
   (b) Compose the code for the Milstein scheme to simulate an Ornstein-Uhlenbeck process of the form
   $$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t),$$
   for $\mu \in \mathbb{R}$, $\theta > 0$ and $\sigma > 0$;
   (c) For $\mu = 10$, $\theta = 1$ and $\sigma = 0.5$, apply the scheme for $\Delta t = 2^{-2}$ and $\Delta t = 2^{-4}$ and plot the results against the exact simulation with $\Delta t = 2^{-9}$;
   (d) For different values of Δt, compute the weak and the strong error for the scheme. Plot the results against Δt.

2. **Pricing vanilla options under the B&S model via PDE solver**

   Consider the PDE for pricing a European call option under the Black and Scholes model:
   (a) Compose the code for a finite difference method to solve the PDE;
   (b) Choose several strike values K, and apply the scheme with temporal grid size $\Delta t = 0.05$ and spatial grid size $\Delta S = 1$. Plot the result against the exact solution (Black and Scholes formula);
   (c) Repeat the experiments for different values of Δt (fixed ΔS) and plot the relative error at initial time against Δt. Repeat again by, this time, changing the spatial grid size ΔS (fixed Δt).

3. **Pricing vanilla options under the Ornstein-Uhlenbeck model with the COS method**
   (a) Consider an Ornstein-Uhlenbeck process of the form
   $$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t),$$
   for $\mu \in \mathbb{R}$, $\theta > 0$ and $\sigma > 0$, and derive its characteristic function;

(b) Compose the code for the COS method to price call options under the Ornstein-Uhlenbeck model;

(c) For $\mu = 30$, $\theta = 1$ and $\sigma = 0.5$, apply the COS method for $N = 16$ and $N = 32$ for different strike values, and plot the results against the "exact solution" obtained by the same method with $N = 5000$;

(d) Repeat the approximation for different truncation parameters N. Compute and plot the maximum absolute error against N.

4. **Pricing Asian option via Monte Carlo**

   Consider an arithmetic Asian option with call-type payoff function of the form

$$\max\left(\frac{1}{T}\int_0^T S(t)dt - K, 0\right),$$

   and consider a GMB for the underlying process S:

   (a) Use a Monte Carlo method with Euler scheme to approximate the option value;

   (b) Apply the scheme for $\Delta t = 2^{-2}$ and $\Delta t = 2^{-4}$ and plot the results against the "exact simulation" obtained by the same method with $\Delta t = 2^{-9}$ and $N = 10000$;

   (c) For different values of Δt, compute the relative error at initial time for the scheme. Plot the results against Δt.

5. **Pricing of options depending on two underlying processes**

   Let $\mathbf{X}(t) = (X(t), Y(t))^\top$ where X satisfies the following SDE under the pricing measure $\mathbb{Q}$

$$dX(t) = 0.06X(t)dt + 0.38X(t)dW^1(t),$$

   and Y satisfies under the pricing measure $\mathbb{Q}$

$$dY(t) = 0.06Y(t)dt + 0.15Y(t)dW^2(t),$$

   where W^1 and W^2 are two correlated Brownian motion with correlation $\rho = 0.7$. Consider the option with strike price K defined by

$$V(t, \mathbf{X}) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(\frac{1}{2}X(T) - \frac{1}{2}Y(T), K\right)\middle|\mathcal{F}_t\right].$$

   (a) Consider $r = 0.06$ and $T = 5$. Use a Monte Carlo method with Euler scheme to approximate the option value at time $t = 0$ for strike values $K = 0$ to $K = 10$, with interval $\Delta K = 0.5$. Plot the results againts K;

   (b) Apply the scheme for $\Delta t = 2^{-2}$ and $\Delta t = 2^{-4}$ and plot the results against the "exact simulation" obtained by the same method with $\Delta t = 2^{-9}$ and $N = 10000$;

   (c) For different values of Δt, compute the relative error at initial time for the scheme. Plot the results against Δt.