Chapter 4: Eigenvalues & Eigenvectors

Learning Objectives:
1. *Describe and interpret eigenvalues and eigenvectors*
2. *Solve for the eigenvalues and eigenvectors of a square matrix*
3. *Use Python to find eigenvalues and eigenvectors*
4. *Identify linearly independent vectors*
5. *Compute diagonalization of square matrix*

4.1 Introduction

In machine learning performing computations on a large matrix is a very slow process. Eigenvalues and eigenvectors can be used in some algorithms such as Principal Component Analysis to reduce the number of dimensions. In this topic we will learn the technique of computing eigenvalues and eigenvectors for square matrices up to order 3. Computations for square matrices of higher order can be implemented in the *numpy.linalg* package in Python.

First a definition.

Definition

Let **A** be a square matrix of order *n*. A real number λ is an **eigenvalue** of **A** if there exists a non-zero vector **v** such that

$$\boxed{\mathbf{A}\mathbf{v} = \lambda\mathbf{v}}$$

In this case, vector **v** is called an **eigenvector of A corresponding to λ.**

Eigenvalues are also called **characteristic values** and eigenvectors are called **characteristic vectors**.

If **v** is an eigenvector associated with an eigenvalue λ for a square matrix **A** of order *n*, then by the preceding definition, the matrix product $\mathbf{A}\mathbf{v}$ is equivalent to the scalar product $\lambda\mathbf{v}$. Thus, $\mathbf{A}\mathbf{v}$ is parallel to the vector **v**. Geometrically, the action of **A** on an eigenvector **v** causes **v** to stretch (or shrink) and/or to reverse direction.

Notice also that an eigenvalue can be zero. However, by definition, an eigenvector is *never* the zero vector.

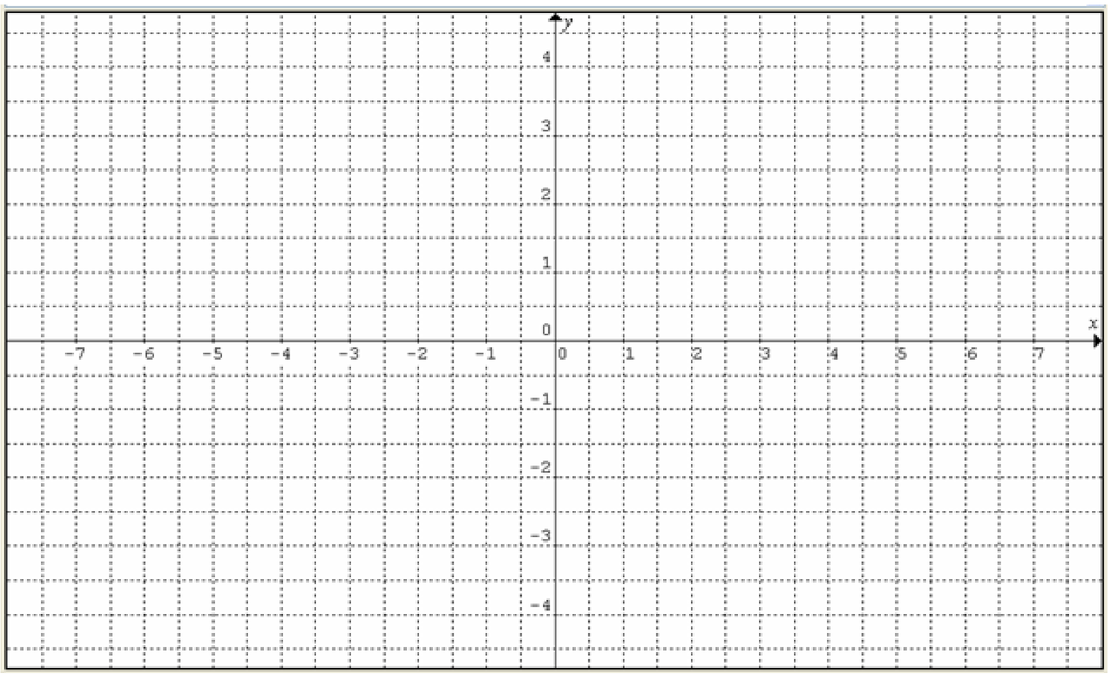
Example 1.

Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1.5 \\ -1 \end{bmatrix}$.

(a) Find **Au**, **Av**, **Aw** and **Ax**.

(b) Plot the vectors **u**, **v**, **w**, **x**, **Au**, **Av**, **Aw** and **Ax** in the grid provided below.

(c) Based on your results in (a)-(b), what can you say about the eigenvalues and eigenvectors of **A**?



Example 2.

Let $\mathbf{A} = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$. Show that $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

Note that if necessary, an eigenvector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ can be **normalized** (i.e. converted into a vector of magnitude one) as follows:

$$\boxed{\hat{\mathbf{v}} = \frac{\mathbf{v}}{\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}}}$$

4.2 Finding Eigenvalues and Eigenvectors in NumPy

NumPy provides us with a function that returns the eigenvalues and eigenvectors of a matrix **A**. The Python code for the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ in Example 1 above is given below:

```python
import numpy as np
A=np.array([[1,1],[0.5,1.5]])
print(A)
eigvals, eigvecs = np.linalg.eig(A)
print("Eigenvalues are: ", eigvals)
print("Eigenvectors are: ", eigvecs)
```

```
[[1.  1. ]
 [0.5 1.5]]
Eigenvalues are:  [0.5 2. ]
Eigenvectors are:  [[-0.89442719 -0.70710678]
 [ 0.4472136  -0.70710678]]
```

Notice that the eigenvalues are the same as those we have in Example 1 above, that is, $\lambda = 0.5$ and $\lambda = 2.0$.

We can read the eigenvectors from the columns of the second array. This means that the eigenvector corresponding to $\lambda = 0.5$ is the first column:

$$\begin{bmatrix} -0.89442719 \\ 0.4472136 \end{bmatrix}$$

and the eigenvector corresponding to $\lambda = 2.0$ is the second column:

$$\begin{bmatrix} -0.70710678 \\ -0.70710678 \end{bmatrix}$$

Notice that the eigenvectors returned by NumPy look different from what we have in Example 1. This is because the eigenvectors returned by NumPy are normalized. We can check as follows.

Exercise

Find the magnitude of each eigenvector $\begin{bmatrix} -0.89442719 \\ 0.4472136 \end{bmatrix}$ and $\begin{bmatrix} -0.70710678 \\ -0.70710678 \end{bmatrix}$.

(Ans: 1, 1)

Notice that any nonzero, scalar multiple of an eigenvector is also an eigenvector. If we let nonzero c represent a scalar, then

$$\mathbf{A}(c\mathbf{v}) = c(\mathbf{A}\mathbf{v}) = c(\lambda\mathbf{v}) = \lambda(c\mathbf{v})$$

Put another way, if **v** is an eigenvector of **A**, then $\frac{1}{2}\mathbf{v}, 0.123\mathbf{v}, -100\mathbf{v}, ...,$ etc , are also eigenvectors of **A** corresponding to the same eigenvalue λ. So there are infinitely many eigenvectors corresponding to λ.

4.3 Computing Eigenvalues & Eigenvectors Manually

We can rewrite the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ as

$$\boxed{(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}}$$

where **I** is the identity matrix of order n. Now, in order for $\mathbf{v} \neq \mathbf{0}$ to satisfy this equation, $\mathbf{A} - \lambda\mathbf{I}$ must **not** be invertible. Thus the determinant $|\mathbf{A} - \lambda\mathbf{I}| = 0$. We call the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ the **characteristic polynomial** of **A**. The equation

$$\boxed{|\mathbf{A} - \lambda\mathbf{I}| = 0}$$

is called the **characteristic equation** of **A**.

Example 3.

Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ and their corresponding eigenvectors.

Hence the eigenvalues of **A** are $\lambda = 3$ and $\lambda = 2$.

To find the corresponding eigenvectors:

For $\lambda = 3$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ becomes:

Using Gaussian elimination, we have:

Hence eigenvectors are $\mathbf{v} = \begin{bmatrix} t \\ t \end{bmatrix}$ or $t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $t \neq 0$.

For $\lambda = 2$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ becomes:

Hence eigenvectors are $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$ or $\dfrac{t}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 4.

If $\lambda = 1$ is an eigenvalue of $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$, find a corresponding eigenvector. Then check with NumPy. Is your result consistent with that given by NumPy?

Example 5.

Use NumPy to find the eigenvalues and normalized eigenvectors of the covariance structure of the data in Chapter 3 Example 6. Explain the directions of the eigenvectors.

4.4 Linear Independence

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors of n^{th} order and $a_1, a_2, \ldots, a_p$ be scalars. The expression $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_p\mathbf{v}_p$ is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, where $a_1, a_2, \ldots, a_p$ are the coefficients or weights.

If there exist scalars not all equal to zero such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_p\mathbf{v}_p = \mathbf{0}$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are said to be **linearly dependent**.

For example, $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ are linearly dependent because $1\begin{pmatrix} 2 \\ -4 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

However, if $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_p\mathbf{v}_p = \mathbf{0}$ if only if all of $a_1, a_2, \ldots, a_p$ are zero, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are said to be **linearly independent**.

For example, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ are linearly independent because $a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only if $a = b = 0$.

In other words, a set of vectors is linearly dependent if and only if at least one of the vectors in the set is a linear combination of the others.

Example 6.

Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Are the following statements true or false?

(a) $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent

(b) $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly dependent

(c) $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent

Note that if the number of vectors (p) is bigger than the order of the vectors (n), then the vectors are surely linearly dependent. That is:

- If $p > n$, then the vectors are linearly dependent.
- If the vectors are linearly independent, then $p \leq n$.

Example 7.

Let $\mathbf{x}_1 = \begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{x}_4 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$. Are the following statements true or false?

(a) $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent.

(b) $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_4$ are linearly independent.

(c) $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ are linearly independent.

(d) $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ are linearly dependent.

4.5 Eigendecomposition

Now that we have learned what eigenvalues and eigenvectors are, we can make use of them to diagonalize a square matrix **A**, using the result below.

The Diagonalization Theorem

Suppose a square matrix **A** of order n has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.
Put the eigenvectors in the columns of an **eigenvector matrix V**. Then

$$\mathbf{A} = \mathbf{V\Lambda V}^{-1}$$

where (capital lambda) $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ is a diagonal matrix of corresponding eigenvalues.

We say that the matrix **A** is diagonalized, and **A** is diagonalizable if and only if it has n linearly independent eigenvectors.

Example 8.

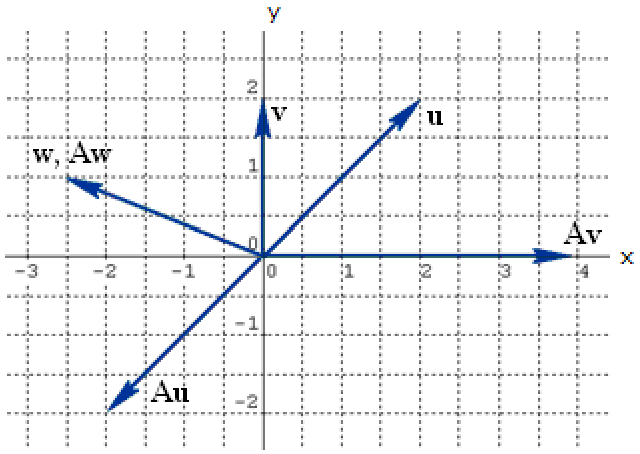
Refer to $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ in Example 1. Diagonalize **A**, if possible.

Example 9.

Refer to $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ again. Compute $\mathbf{A}^3$.

Tutorial 4

1. The diagram below shows the vectors **u**, **v** and **w** and their corresponding image vectors **Au**, **Av** and **Aw**, where **A** is a square matrix of order 2. What can you say about the eigenvalues and eigenvectors of **A**?



2. Let $\mathbf{A} = \begin{bmatrix} 0 & -2 & -5 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$. Are the vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ eigenvectors of **A**? State the corresponding eigenvalue if it is.

3. Determine the characteristic polynomial, eigenvalues and associated eigenvectors for each matrix below.

(a) $\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 0 & 6 \\ 4 & 5 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

4. Check if the following set of vectors are linearly independent or dependent.

(a) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

5. Diagonalize $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Hence find $\mathbf{A}^4$.

6. If $\mathbf{A}$ has $\lambda_1 = 2$ with eigenvector $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $\mathbf{V\Lambda V}^{-1}$ to find $\mathbf{A}$.

7. If $\mathbf{A} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 5$, diagonalize $\mathbf{A}$.

<u>Answers</u>

1. $\mathbf{Au} = -\mathbf{u}$, so $\mathbf{u}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda = -1$
   $\mathbf{Av}$ and $\mathbf{v}$ are not parallel, so $\mathbf{v}$ is not an eigenvetcor of $\mathbf{A}$
   $\mathbf{Aw} = \mathbf{w}$, so $\mathbf{w}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda = 1$

2. $\mathbf{v}$ is not an eigenvector; $\mathbf{w}$ is an eigenvector corresponding to $\lambda = 2$

3. (a) $\lambda_1 = 2, \lambda_2 = 7$; $\mathbf{v_1} = t\begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{v_2} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (b) $\lambda_1 = -1, \lambda_2 = -3, \lambda_3 = 7$; $\mathbf{v_1} = t\begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}$, $\mathbf{v_2} = t\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v_3} = t\begin{bmatrix} 6 \\ 27 \\ 10 \end{bmatrix}$

4. (a) dependent (b) independent (c) dependent (d) independent

5. $\mathbf{A}^4 = \begin{bmatrix} 1 & 80 \\ 0 & 81 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

7. $\mathbf{V} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{\Lambda} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Alternatively, using Python:

   $\mathbf{V} =$
```
[[-0.90453403  0.57735027 -0.04093652]
 [ 0.30151134  0.57735027  0.46313831]
 [-0.30151134 -0.57735027  0.88534011]]
```
   and $\mathbf{\Lambda} =$
```
[[1 0 0]
 [0 5 0]
 [0 0 1]]
```

Practical 4

Task 1

Some properties of eigenvalues include:
- The sum of eigenvalues of a matrix equals its trace.
- The product of eigenvalues equals its determinant.

Verify the properties above using the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$.

Task 2

Symmetric Matrix: The eigenvalues of a symmetrical matrix are always real and the eigenvectors are always orthogonal. Verify this in Python for the matrix **A** below.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Answers

1. Eigenvalues = -1, 4, 2
   tr(**A**) = 5
   det(**A**) = -8

2. Eigenvalues = 3, 1, 0
   $\mathbf{v}_1^T \mathbf{v}_2 = 0$, $\mathbf{v}_1^T \mathbf{v}_3 = 0$, $\mathbf{v}_2^T \mathbf{v}_3 = 0$