Appendix 1: Laws of Indices

A1.1 Exponential Form

Exponential Form

The expression $2\times2\times2\times2\times2$ may be written as 2^5.
The number **5** is called **exponent**, index, or power; and the number **2** is called the **base**.
We read 2^5 as "2 to the 5th power". It tells you how many 2's to multiply.

In general, if *a* is any real number and *m* is a positive integer, then

$$\underbrace{(a)(a)(a)...(a)}_{m\text{ factors}} = a^m$$

a is called the base; *m* is called the power, index or exponent.

For example, $3^7 = 3\times3\times3\times3\times3\times3\times3$

Example 1.

Evaluate the following **without the use of calculator**:

(a) $3^2 = 3\times3 = 9$

(b) $(-3)^2 = (-3)\times(-3) = 9$

(c) $-3^2 = -(3\times3) = -9$

(d) $\left(\frac{1}{3}\right)^2 = \frac{1}{3}\times\frac{1}{3} = \frac{1}{9}$

(e) $\left(-\frac{1}{3}\right)^2 = \left(-\frac{1}{3}\right)\times\left(-\frac{1}{3}\right) = \frac{1}{9}$

A1.2 Laws of Indices

Product Law

If a^2 is multiplied by a^3, we obtain:

$$a^2\times a^3 = (a\times a)\times(a\times a\times a) \quad = a^5$$

Notice that if we add the original exponents, we will get the same result:

$$a^2\times a^3 = a^{2+3} \quad = a^5$$

For any nonzero real number *a* and any real indices, *m* and *n*,

$$a^m\times a^n = a^{m+n}$$

Note:
$a^m\cdot a^n = a^m\times a^n$
$(a^m)(a^n) = a^m\times a^n$

Example 2.

Simplify the following:

(a) $(3^5)(3^2) = 3^{5+2} = 3^7 = 2187$

(b) $(3^5)(-3^2) = -(3^5 \times 3^2) = -3^7 = -2187$

(c) $2(2^x)(3^m)(3^{-n}) = (2^{1+x})(3^{m-n})$

(d) $(a^2 b)(b^3 ac) = (a^{2+1})(b^{1+3})c = a^3 b^4 c$

(e) $\left(\frac{1}{3}\right)^5 \left(\frac{1}{3}\right)^2 = \left(\frac{1}{3}\right)^7 = \frac{1^7}{3^7} = \frac{1}{2187}$

Quotient Law

If a^5 is divided by a^2, we obtain:

$$\frac{a^5}{a^2} = \frac{a \times a \times a \times a \times a}{a \times a} = \frac{\cancel{a} \times \cancel{a} \times a \times a \times a}{\cancel{a} \times \cancel{a}} = a^3$$

Notice that if we subtract the original exponents, we will get the same result:

$$\frac{a^5}{a^2} = a^{5-2} = a^3$$

For any nonzero real number a and any real indices, m and n,

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 3.

Simplify the following:

(a) $\frac{5^5}{5^2} = 5^{5-2} = 5^3 = 125$

(b) $\frac{2^a}{2^b} = 2^{a-b}$

(c) $\frac{a^3}{a^{-1}} = a^{3-(-1)} = a^4$

(d) $\frac{a^3 b^4}{ab^{-1}} = a^{3-1} b^{4-(-1)} = a^2 b^5$

Power Law

If a^2 is raised to power 4, we obtain:

$$\begin{aligned}\left(a^2\right)^4 &= a^2 \bullet a^2 \bullet a^2 \bullet a^2 \\ &= a^{2+2+2+2} = a^8\end{aligned}$$

Notice that if we multiply the original exponents, we will get the same result:

$$\left(a^2\right)^4 = a^{2\times 4} = a^8$$

For any nonzero real number a and any real indices, m and n,

$$(a^m)^n = a^{mn}$$

Example 4.

Simplify the following:

(a) $(2^{10})^3 = 2^{10\times 3} = 2^{30}$

(b) $(7^{-1})^{-2} = 7^{(-1)\times(-2)} = 7^2 = 49$

(c) $(x^{-1})^3 = x^{(-1)\times 3} = x^{-3} = \dfrac{1}{x^3}$

Product Raised to a Power

If a product *ab* is raised to power 3, we obtain:

$$\begin{aligned}(ab)^3 &= (ab)\times(ab)\times(ab)\\ &= (a\times b)\times(a\times b)\times(a\times b)\\ &= a\times b\times a\times b\times a\times b\\ &= a\times a\times a\times b\times b\times b\\ &= a^3b^3\end{aligned}$$

For any nonzero real number *a* and any real indices, *m* and *n*,

$$(ab)^m = a^m b^m$$

Example 5.

Simplify the following:

(a) $(3y^3)^2 = 3^{1\times 2}\cdot y^{3\times 2} = 9y^6$

(b) $(-2x^2y)^3 = (-2)^3 x^{2\times 3} y^{1\times 3} = -8x^6y^3$

(c) $-(xy^{-2})^3 = -(x^{1\times 3}y^{-2\times 3}) = -x^3y^{-6} = -\dfrac{x^3}{y^6}$

Quotient Raised to a Power

If a quotient $\frac{a}{b}$ is raised to power 3, we obtain:

$$\left(\frac{a}{b}\right)^3 = \left(\frac{a}{b}\right)\times\left(\frac{a}{b}\right)\times\left(\frac{a}{b}\right) = \frac{a\times a\times a}{b\times b\times b} = \frac{a^3}{b^3}$$

For any nonzero real number *a* and any real indices, *m* and *n*,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Example 6.

Simplify the following:

(a) $\left(\dfrac{2a}{3b^2}\right)^3 = \dfrac{2^{1\times 3}a^{1\times 3}}{3^{1\times 3}b^{2\times 3}} = \dfrac{8a^3}{27b^6}$

(b) $\left(-\dfrac{3x^3}{4y}\right)^2 = (-1)^{1\times 2}\dfrac{3^{1\times 2}x^{3\times 2}}{4^{1\times 2}y^{1\times 2}} = \dfrac{9x^6}{16y^2}$

A1.3 Zero and Negative Exponents

Zero Exponent

Any number divided by itself is 1.

For instance, $\dfrac{x^3}{x^3} = 1$

But using quotient law, $\dfrac{x^3}{x^3} = x^{3-3} = x^0$

Hence, $x^0 = 1$.

> For any nonzero real number *a*,
> $a^0 = 1$

Example 7.

Simplify the following:

(a) $3^0 = 1$

(b) $7x^0 = 7 \times 1 = 7$

(c) $-(x+2y)^0 = -1$

Negative Exponent

Using quotient law, $\dfrac{a^0}{a^3} = a^{0-3} = a^{-3}$

However, $\dfrac{a^0}{a^3} = \dfrac{1}{a^3}$

Therefore, $a^{-3} = \dfrac{1}{a^3}$

> For any nonzero real number a and any real index m,
> $a^{-m} = \dfrac{1}{a^m}$

Generally, we do not leave exponential expressions with negative exponents. To simplify an exponential expression is to express the answer in positive exponents.

Example 8.

Simplify the following:

(a) $-x^{-2} = -\dfrac{1}{x^2}$

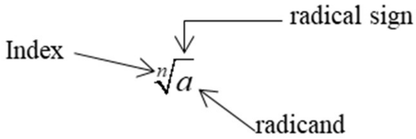
(b) $(-x)^{-2} = \dfrac{1}{(-x)^2} = \dfrac{1}{(-1)^2 x^2} = \dfrac{1}{x^2}$

(c) $\dfrac{1}{x^{-2}} = x^2$

A1.4 Relation between Fractional Exponents and Radicals

A quantity raised to a fractional exponent can also be written as a **radical**.

A radical is written as:



In general,

$$\boxed{\begin{aligned} a^{\frac{1}{m}} &= \sqrt[m]{a} \\ a^{\frac{n}{m}} &= \sqrt[m]{a^n} \end{aligned}}$$

Example 9.

Write the following in exponential form:

(a) $\sqrt{y} = y^{\frac{1}{2}}$

(b) $\sqrt[3]{y} = y^{\frac{1}{3}}$

(c) $\sqrt{y^3} = y^{\frac{3}{2}}$

(d) $\sqrt[3]{y^2} = y^{\frac{2}{3}}$

Example 10.

Express $x^{\frac{1}{2}}y^{-\frac{1}{3}}$ in radical form:

$$x^{\frac{1}{2}}y^{-\frac{1}{3}} = \sqrt{x} \cdot \frac{1}{y^{\frac{1}{3}}} = \frac{\sqrt{x}}{\sqrt[3]{y}}$$