Chapter 7: Differentiation of Exponential & Logarithmic Functions and Higher-Order Derivatives

Learning Objectives:

1. *Define logarithm of any number to any base.*
2. *Define common logarithm and natural logarithm.*
3. *Convert logarithmic form to exponential form.*
4. *State the laws of logarithms.*
5. *Use the laws to simplify expressions.*
6. *Apply rules of differentiation to find derivatives of functions involving algebraic, trigonometric, logarithmic and exponential functions.*
7. *Define higher-order derivatives.*
8. *Evaluate higher-order derivatives of algebraic, exponential, logarithmic and trigonometric functions.*
9. *Use Python to differentiate exponential and logarithmic functions, as well as carry out higher-order differentiation.*

7.1 Exponential & Logarithmic Functions

DEFINITION

If $y = a^x$

then $\log_a y = x$

i.e. x is the logarithm of number y to the base a.

$y = a^x$ is said to be the **exponential form**, and $\log_a y = x$ is said to be in the **logarithmic form**.

For $\log_a y = x$ to be defined:

(i) $y > 0$

(ii) $a > 0$ but $a \neq 1$

Example 1.

(a) $2^3 = 8$

=> $3 = \log_2 8$

i.e. 3 is the logarithm of 8 to base 2.

(b) $316.22 = 10^{2.5}$

=> $\log_{10} 316.22 = 2.5$

Example 2.

Write each of the following in exponential form:

(a) $\log_6 216 = 3$

(b) $\log_3 \dfrac{1}{27} = -3$

7.2 The Common Logarithmic & Natural Logarithm

Logarithms may be expressed in terms of any “base”.

- Logarithms to **base 10** are called **common logarithms**.
  When the base is 10, this number is conventionally omitted from writing.

  log N or **lg N** denotes the logarithm of N to the **base 10**.

  Common logarithm
  $\log x = \log_{10} x$
  $\lg x = \log_{10} x$
  The expression $\log_{10} x$ is commonly denoted by log x or **lg *x***, where the base 10 is omitted.

- Logarithms to **base *e*** are called **natural logarithms**. e has approximately the value **2.718**.

  log *e* N is written as **ln N**.

  But, when Python outputs *log x*, it actually means **ln *x***. We should write the answer as **ln *x***.

Example 3.

Write each of the following in exponential form:

(a) $\ln x = 5$ (b) $\log 100 = 2$

7.3 Properties of Logarithm

In general, for $a > 0$ but $a \neq 1$,

$\boxed{\log_a a = 1}$ since $a^1 = a$;

$\boxed{\log_a 1 = 0}$ since $a^0 = 1$.

Example 4.

Evaluate each of the following:

(a) $\log_3 3$ (b) $\log_2 1$

(c) log 10 (d) log 1

(e) ln 1 (f) $\ln e$

Example 5.

Find each of the following using a calculator:

(a) $\log 2$ (b) $\log(-2)$

(c) $\ln 11.18$ (d) $\ln 0.0037$

[Ans: (a) 0.301 (b) not defined (c) 2.414 (d) -5.599]

7.4 Laws of Logarithm

The following laws of logarithms are useful for simplifying logarithmic expressions. These laws can be derived, but we will just make use of the results.

(A) Product Law

$$\boxed{\log_a(XY) = \log_a X + \log_a Y}$$

(B) Quotient Law

$$\boxed{\log_a\left(\frac{X}{Y}\right) = \log_a X - \log_a Y}$$

<Caution> $\log_a(x-y) \neq \log_a x - \log_a y$ and $\log_a(x+y) \neq \log_a x + \log_a y$

Example 6.

Express $\log_a 3x + \log_a 7x^3$ as a single logarithm.

Example 7.

Express $\ln xy$ as a sum of logarithms.

Example 8.

Express $\log_2\left(\frac{b}{a}\right)$ as a difference of logarithms.

Example 9.

Express $\log_a 32 - \log_a 16$ as a single logarithm.

[Ans: Eg6. $\log_a\left(21x^4\right)$ Eg7. $\ln x - \ln y$ Eg8. $\log_2 b - \log_2 a$ Eg9. $\log_a 2$]

(C) Power Law

$$\boxed{\log_a X^n = n\log_a X}$$

Example 10.

Evaluate each of the following without using calculator:

(a) $\log_2 8$ (b) $\log 100$

Example 11.

Express $\log_b\left(\dfrac{x^3 y}{z^5}\right)$ in terms of sum and difference of logarithms.

Example 12.

Express $\log_2\left(\dfrac{\sqrt{y}}{4x^3}\right)$ in terms of sum and difference of logarithms.

Example 13.

Simplify $\dfrac{1}{2}\ln x + 3\ln(x-1)$ to a single logarithm.

Example 14.

Simplify $\log\left(x^2 - xy\right) - \log(x-y)$ to a single logarithm.

[Ans: Eg10. (a) 3 (b) 2 Eg11. $3\log_b x + \log_b y - 5\log_b z$

Eg12. $\frac{1}{2}\log_2 y - 2 - 3\log_2 x$ Eg13. $\ln\left[\sqrt{x}\,(x-1)^3\right]$ Eg14. $\log x$]

7.5 Change of Base

A general formula for changing from base a to base b can be derived as follows:

Let	$\log_a N = x$
Change to exponential form	$N = a^x$
Take logarithm to the base b:	$\log_b N = \log_b a^x$
Apply power law of logarithm:	$\log_b N = x \log_b a$
	$\dfrac{\log_b N}{\log_b a} = x$

Hence,

$$\boxed{\log_a N = \frac{\log_b N}{\log_b a}}$$

Example 15.

Find $\log_3 140$.

[Ans: 4.498]

7.6 Differentiation of Logarithm Functions

In this section we will state the formula for differentiating a logarithmic function. Our focus is on the natural logarithmic function, $\ln x$, as the formula for this function is more simplified than the general logarithmic function, $\log_a x$ ($a > 0$, $a \neq 1$). Next, we will show you how to differentiate $\log_a x$ using a change of base.

7.6.1 Differentiating $y = \ln x$

If $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

In general, if $y = \ln\left(f(x)\right)$, where $f(x)$ is a function of x, to differentiate y, we will have to use Chain Rule as follows:

$$\boxed{\frac{d}{dx}\left[\ln f(x)\right] = \frac{1}{f(x)} \bullet \frac{df}{dx}}$$

Example 16.

Find the derivatives of the following functions with respect to their respective variables.

(a) $y = 2 + \ln(x)$ (b) $y = 2\ln(2x+3)$

(c) $u = \ln(x^3 - 3x^2 + 6)$ (d) $u = 4\sin x \ln\left(x^2 - 3\right)$

(e) $x = \dfrac{\ln t}{3 + \tan t}$ (f) $f(x) = \left(\ln 2x\right)^4$

[Ans: (a) $\dfrac{1}{x}$ (b) $\dfrac{4}{2x+3}$ (c) $\dfrac{3x(x-2)}{x^3 - 3x^2 + 6}$ (d) $\dfrac{8x\sin x}{x^2 - 3} + 4\cos x \ln(x^2 - 3)$

(e) $\dfrac{(3+\tan t)\left(\dfrac{1}{t}\right) - \sec^2 t \cdot \ln t}{(3+\tan t)^2}$ (f) $\dfrac{4}{x}(\ln 2x)^3$]

7.6.2 Differentiating $y = \log_a x$

In this section, we will discuss how to differentiate the general logarithmic function. Recall the change of base formula below ($a > 0$, $a \neq 1$):

$$\log_a x = \frac{\ln x}{\ln a}$$

We will use it to first change the '$\log_a$' to the natural log 'ln', then perform the differentiation, since we know how to differentiate the natural logarithmic function.

Example 17.

Find $\dfrac{dy}{dx}$ given that:

(a) $y = \log_2(2x+1)$ (b) $y = 4\lg(x+2)$

[Ans: (a) $\dfrac{2}{(2x+1)\ln 2}$ (b) $\dfrac{4}{(x+2)\ln 10}$]

When differentiating logarithmic functions, we should break down, whenever possible, the given logarithmic expression into simpler ones by using the laws of logarithms. This will make the expression easier to differentiate as the next set of examples will show.

Example 18.

Differentiate the following with respect to x.

(a) $s = 6\ln\sqrt[3]{5x^2+1}$ (b) $u = 7\ln\dfrac{6e^x}{\left(x^3+1\right)^2}$

[Ans: (a) $\dfrac{20x}{5x^2+1}$ (b) $7\left(1-\dfrac{6x^2}{x^3+1}\right)$]

7.7 Differentiation of Exponential Functions

In this section, we will first look at the derivative of the basic exponential form $y = a^x$ ($a > 0$, $a \neq 0$). After which, we will look at the derivative of $y = e^x$ and its general form.

7.7.1 Differentiating $y = a^x$

The derivative of the basic exponential form $y = a^x$ is given by:

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

If $y = a^u$, where u is a function of x, its derivative is given by:

$$\boxed{\frac{d}{dx}(a^u) = a^u \frac{du}{dx} \ln a}$$

Example 19.

Differentiate the following with respect to *x*:

(a) $y = 2^x$ (b) $y = 5^{-2x}$

[Ans: (a) $2^x \ln 2$ (b) $-\frac{2\ln 5}{5^{2x}}$]

7.7.2 Differentiating $y = e^x$

If we replace the base a with Euler's constant e in $y = a^x$ and apply the rule we just derived above, we will get, $\frac{d}{dx}\left(e^x\right) = e^x \ln e$. Since $\ln e = 1$, this simplifies to:

$$\frac{d}{dx}(e^x) = e^x$$

In general,

$$\boxed{\frac{d}{dx}(e^u) = e^u \frac{du}{dx}}$$

Example 20.

Differentiate the following expressions with respect to x:

(a) $t = 2e^{x}$

(b) $y = e^{3x}$

(c) $R = 5x^2 e^{-3x^2}$

(d) $y = \dfrac{5e^{2x}}{1-6x}$

[Ans: (a) $2e^{x}$ (b) $3e^{3x}$ (c) $10xe^{-3x^2}(1-3x^2)$ (d) $\dfrac{20e^{2x}(2-3x)}{(1-6x)^2}$]

7.8 Higher-Order Derivatives

Higher-order derivatives are essential in the study of series – Taylor's series, Maclaurin's series and Fourier series, just to name a few. These series are important in the fields of science and engineering as many processes can be mathematically approximated by them. Higher-order derivatives also enable us to find maximum and minimum points.

The process of differentiation can be carried out as many times as we want. Consider the following:

$$y = x^3$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2$$ looking at $3x^2$, we know we can differentiate it to get $6x$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 6x$$ similarly, looking at $6x$, we can differentiate it to get 6

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(6x) = 6$$ finally, the constant 6 can be differentiated to get 0

$$\frac{d^4y}{dx^4} = \frac{d}{dx}(6) = 0$$

In summary: If $y = f(x)$

$\boldsymbol{\frac{dy}{dx} = f'(x)}$	is the notation for the first order derivative of y with respect to x. Another way of writing it is $\boldsymbol{y'}$ (read as "y prime").
$\boldsymbol{\frac{d^2y}{dx^2} = f''(x)}$	is the notation for the second order derivative of y with respect to x. We read it as "dee two y by dee x squared". Another way of writing it is $\boldsymbol{y''}$. The superscript '2' on the '*d*' and '*x*' gives the order of the derivative and tells us that we have differentiated y twice. What we really did was to differentiate the expression for $\frac{dy}{dx}$.
$\boldsymbol{\frac{d^3y}{dx^3} = f'''(x)}$	is the notation of the third order derivative of y with respect to x, which is read as "dee three y by dee x cubed". It tells us that we have differentiated thrice.

Thus if we were to carry out the process of differentiation *n* times, the n^{th} order derivative of y with respect to x would be written as $\boldsymbol{\frac{d^ny}{dx^n}}$ **or** $\boldsymbol{f^{(n)}(x)}$ (which reads as "dee '*n*' y by dee x '*n*' ") or as $\boldsymbol{y^{(n)}}$.

Example 21.

Given that $y = x^4 + 2x^3 - 3x^2 + x - 5$, obtain the first, second and third order derivatives of y with respect to x.

[Ans: $4x^3 + 6x^2 - 6x + 1$; $12x^2 + 12x - 6$; $24x + 12$]

Tutorial 7

1. Express each of the following in logarithmic form:

   (a) $7^3 = 343$ (b) $5^{-2} = \frac{1}{25}$ (c) $10^x = \sqrt{2}$

2. Express each of the following in exponential form:

   (a) $\log_6 216 = 3$ (b) $\lg 0.01 = -2$ (c) $\ln a = c$

3. Simplify and express each of the following as a single logarithm.

   (a) $3\log_2 5 - 2\log_2 7$ (b) $\frac{1}{2}\log_5 64 + \frac{1}{3}\log_5 27 - \log_5\left(x^2 + 4\right)$

   (c) $\ln x^3 + \ln y^2 - 2\ln x - \ln y$

4. Differentiate the following functions with respect to x. Simplify your answers whenever possible.

   (a) $y = \ln\left(x^2 + 5\right)$ (b) $y = 4\ln x + 3\ln\left(1 - x\right)$

   (c) $y = 5 + 4\ln\left(3x^2 - 1\right)$ (d) $y = \lg(3x)$

   (e) $y = 5\log_3(x + 2)$ (f) $y = (2\ln x)^3$

   (g) $y = x\ln(3x)$ (h) $y = \dfrac{\ln\left(2x + 5\right)}{x^2}$

   (i) $y = \sqrt{\ln(2x + 1)}$

5. Differentiate the following functions with respect to x. Simplify your answers whenever possible.

   (a) $s = \ln\sqrt{5 + x^3}$ (b) $s = \frac{1}{2}\ln\left(x^2 + 7\right)^6$

   (c) $y = \ln\left[\left(5x - 4\right)\sqrt{x^2 + 3}\right]$ (d) $s = 2\ln\left(\dfrac{2x + 1}{2x - 1}\right)$

6. Differentiate the following functions with respect to the given variables. Simplify your answers, where possible.

   (a) $y = 4^{2x}$ (b) $y = 2^{4x+1}$

   (c) $w = 2e^{2x} + 4e^{-2x}$ (d) $w = \sqrt{e^x} + \dfrac{1}{e^{2x}} - e^{\frac{1}{2}}$

   (e) $w = 3e^{2x+7} + 4e^{x^2} + e^{\frac{1}{x}}$ (f) $y = \left(e^{2x} - 4\right)^3$

7. Use Product or Quotient Rule to differentiate the following with respect to the given variables and simplify your answers whenever possible.

   (a) $y = xe^{2x}$ (b) $y = 3e^{2x}\ln(x+2)$ (c) $y = \dfrac{\sin 2x}{e^x}$

8. The impressed voltage in a circuit is given by $E(t) = 6\ln(1+0.25t^2)$ volts. Find $E'(t)$ as a function of time *t*.

9. The model for a certain population over time *t* years is given as $P(t) = \dfrac{578}{1+4e^{-t/3}}$.
   Find $P'(6)$ when $t = 6$ years.

10. The mass *m* (in grams) of a certain substance is believed to dissolve in one litre of water at temperature T °C according to the law: $m(T) = 2(10^{0.02T})$. Differentiate *m* with respect to *T*.

11. The amount of radioactive radium after t years is given by $N = N_o e^{-4.279\times10^{-4}t}$, where N_o is the original amount of radioactive radium. Find $\dfrac{dN}{dt}$ if $N_o = 10$g.

12. If $f(t) = 2e^{-t}\cos 2t$, find $f'(0)$.

13. Obtain the first and second order derivatives of the following with respect to their respective variables. Simplify your answers whenever possible.

   (a) $y = x^2 + \dfrac{4}{x^2} - 5$ (b) $z = \dfrac{1}{t} + 3\sin(4t+1)$

14. If $p = 3s^6 + 4s^3 - 5$, find $\dfrac{dp}{ds}$, $\dfrac{d^2p}{ds^2}$ and $\dfrac{d^3p}{ds^3}$.

15. If $h(t) = at^3 - 2t^2 + 1$ and $h'''(t) = 12$, find the value of $h(-1)$.

16. If $f(x) = 6\sin 4x$, find $f'(0.3)$ and $f''(0.3)$.

17. The impressed voltage in a circuit, at time t seconds, is given by $E(t) = 6\ln(1+0.25t^2)$ volts. Find the time t when $E'(t) = 0$. Hence find the value of $E''(t)$ at this point of time.

18. Show that the function $y = A\sin(\omega t + \alpha) + B\cos(\omega t + \alpha)$ satisfies the equation $\dfrac{d^2y}{dt^2} + \omega^2 y = 0$. Note that *A*, *B*, ω, and α are constants.

Answers

1. (a) $3 = \log_7 343$ (b) $-2 = \log_5 \dfrac{1}{25}$ (c) $x = \log_{10} \sqrt{2}$

2. (a) $216 = 6^3$ (b) $0.01 = 10^{-2}$ (c) $a = e^c$

3. (a) $\log_2 \dfrac{125}{49}$ (b) $\log_5 \dfrac{24}{x^2 + 4}$ (c) $\ln xy$

4. (a) $\dfrac{2x}{x^2+5}$ (b) $\dfrac{4}{x} - \dfrac{3}{1-x}$ (c) $\dfrac{24x}{3x^2-1}$

   (d) $\dfrac{1}{x \ln 10}$ (e) $\dfrac{5}{(x+2)\ln 3}$ (f) $\dfrac{24(\ln x)^2}{x}$

   (g) $1 + \ln(3x)$ (h) $\dfrac{2}{x^3}\left[\dfrac{x}{2x+5} - \ln(2x+5)\right]$

   (i) $\dfrac{1}{(2x+1)\sqrt{\ln(2x+1)}}$

5. (a) $\dfrac{3x^2}{2(5+x^3)}$ (b) $\dfrac{6x}{x^2+7}$ (c) $\dfrac{5}{5x-4} + \dfrac{x}{x^2+3}$

   (d) $\dfrac{4}{2x+1} - \dfrac{4}{2x-1}$

6. (a) $4^{2x}\, 2\ln 4$ (b) $2^{4x+1}\, 4\ln 2$ (c) $4(e^{2x} - 2e^{-2x})$

   (d) $\dfrac{1}{2}e^{\frac{x}{2}} - \dfrac{2}{e^{2x}}$ (e) $6e^{2x+7} + 8xe^{x^2} - \dfrac{e^{\frac{1}{x}}}{x^2}$ (f) $6e^{2x}(e^{2x}-4)^2$

7. (a) $e^{2x}(2x+1)$ (b) $3e^{2x}\left[\dfrac{1}{x+2} + 2\ln(x+2)\right]$

   (c) $\dfrac{2\cos(2x) - \sin(2x)}{e^x}$

8. $E'(t) = \dfrac{dE}{dt} = \dfrac{3t}{1+0.25t^2}$

9. 43.9

10. $(0.04 \ln 10)\ 10^{0.02T}$ g/°C

11. $-4.279 \times 10^{-3} e^{-4.279\times 10^{-4} t}$

12. $f'(0) = -2$

13. (a) $2\left(x - \dfrac{4}{x^3}\right)$; $2\left(1 + \dfrac{12}{x^4}\right)$ (b) $12\cos(4t+1) - \dfrac{1}{t^2}$; $\dfrac{2}{t^3} - 48\sin(4t+1)$

14. $18s^5 + 12s^2$; $90s^4 + 24s$; $360s^3 + 24$

15. $h = -3$

16. 8.697; − 89.476

17. 0; 3

Practical 7

We can also use SymPy to differentiate exponential and logarithmic functions, as well as carry out higher order differentiation.

Task 1

Find the following:

(a) $\dfrac{d}{dx}[2\ln(2x+3)]$

(b) $\dfrac{d}{dx}[\log_2(2x+1)]$

(c) $\dfrac{d}{dx}[4\lg(x+2)]$

(d) $\dfrac{d}{dx}(2^x)$

(e) $\dfrac{d}{dx}(5^{-2x})$

(f) $\dfrac{d}{dx}(e^{3x})$

(g) $\dfrac{d}{dx}(5x^2e^{-3x^2})$

Task 2

(a) Given $y = x^4 + 2x^3 - 3x^2 + x - 5$, obtain the first, second and third order derivatives of y with respect to x. Also, find $y'(1.5)$.

(b) Find the first, second and third order derivatives of $f(y) = \sin(3y) + e^{-2y} + \ln(7y)$. Also, find $f''(2)$.

Selected Answers

1. (c) $\dfrac{4}{(x+2)\ln 10}$ (e) $-2\cdot 5^{-2x}\ln 5$ (g) $-10x(3x^2-1)e^{-3x^2}$

2. (b) $3\cos 3y - 2e^{-2y} + \dfrac{1}{y}$, $-9\sin 3y + 4e^{-2y} - \dfrac{1}{y^2}$, $-27\cos 3y - 8e^{-2y} + \dfrac{2}{y^3}$, 2.338