$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right] \xrightarrow[R_3+R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x + y - z = -2$

$y - z = -3$

$x - 3 = -2$

$\boxed{x = 1}$

$\boxed{z = 2}$

$y - z = -3$

$y = -3 + z$

$y = -3 + 2$

$\boxed{y = -1}$

$\therefore x = 1,\ y = -1,\ z = 2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & 10 \\ 2 & -1 & 2 & 6 \end{array}\right] \xrightarrow[R_3-R_1]{} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & 10 \\ 0 & -2 & 3 & 5 \end{array}\right] \xrightarrow{-3R_1+2R_2} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & 5 & 17 \\ 0 & -2 & 3 & 5 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 13 & 39 \end{array}\right] \xrightarrow[\frac{1}{13}R_3]{} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

● : turn into 0

subtract/add

● : turn into 1

divide

$2x + y - z = 1$

$2x = 1 - y + z$

$2x = 1 - 2 + 3$

$2x = 2$

$x = 1$

$z = 3$

$y + 5z = 17$

$y + 15 = 17$

$y = 2$

$\therefore x, y, z = 1, 2, 3$

Gaussian elimination

special matrices

- zero or null matrices
- symmetric matrix
- diagonal matrix
- identity matrix
- singular & non singular matrix
  - ↳ $|A| = 0$ ↳ $|A| \neq 0$
- summing vector
  - ↳ vector in which all elements are 1

inverse

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\begin{bmatrix} |M_{11}| & -|M_{12}| & |M_{13}| \\ -|M_{21}| & |M_{22}| & -|M_{23}| \\ |M_{31}| & -|M_{32}| & |M_{33}| \end{bmatrix}^T}$$

invertible matrix

↳ $|A| \neq 0$ (non-singular)

properties of inverse matrix

- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^{-1})^m = (A^m)^{-1}$

Laplace expansion

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left((-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right) + a_{12}\left((-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right)$$

$$+ a_{13}\left((-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* can use any row.
in this case, we're
using row 1.
↳ if possible, choose
a row w/ a '0' element

1. a. $a_{ij} = i+j$ for $i = 1,2,3,4$ & $j = 1,2$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{bmatrix}$$

(b) $a_{ij} = (-1)^{i+1} 2^{j-1}$ for $i = 1,2,3,\ j = 1,2,3$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{bmatrix}$$

2. a. $\begin{vmatrix} 5 & -2 \\ 6 & -2 \end{vmatrix} = -10+12$
$= 2$

b. $\begin{vmatrix} 3 & 5 \\ -1 & 0 \end{vmatrix} = 5$

c. $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1\left((-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right)$
$= -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$
$= -(1-1)$
$= 0$

d. $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$
$= 2\cdot(-1) - 3\cdot 1$
$= -5$

3. a. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$
$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 6 \end{bmatrix} + \begin{bmatrix} -3 & 3 & 6 \\ 3 & 6 & -3 \end{bmatrix}$
$= \begin{bmatrix} -1 & 7 & 12 \\ 7 & 8 & 3 \end{bmatrix}$

4. $A = 2\begin{bmatrix} -1 & 4 & 3 \\ -2 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \end{bmatrix}$

$A = \begin{bmatrix} -4 & 7 & 9 \\ -5 & -5 & 8 \end{bmatrix}$

5. $\begin{pmatrix} 3a & 6 \\ -3 & 6b \end{pmatrix} + \begin{bmatrix} 2a+2b & 4 \\ 6 & 2a-2b \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 6 \end{bmatrix}$

$15+12$
$12+14$
$24+18$
42

$5a+2b=7$

$2a+4b=6 \rightarrow a+2b=3$

$$\left[\begin{array}{cc|c} 5 & 2 & 7 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{R_1-5R_2} \left[\begin{array}{cc|c} 5 & 2 & 7 \\ 0 & -8 & 8 \end{array}\right] \xrightarrow[-1/8R_2]{} \left[\begin{array}{cc|c} 5 & 2 & 7 \\ 0 & 1 & 1 \end{array}\right]$$

$\boxed{b=1}$

$5a+2b=7$

$5a=7-2b$

$5a=7-2$

$\boxed{a=1}$

$\therefore a=1$

$b=1$

6. (a) $A=2\times3$

$B=?$

$AB=2\times4$

$\boxed{B=3\times4}$

$\begin{matrix} A & B \\ 2\times3 & 3\times4 \end{matrix}$

(b) $A:2\times3$

$C=4\times4$

$B=?$

AB^TC valid

$\begin{matrix} A & B^T & C \\ 2\times3 & 3\times4 & 4\times4 \end{matrix}$

$\boxed{B=4\times3}$

7a) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 6 & 7 \end{pmatrix}$

$\begin{pmatrix} 5 & 4 \\ 27 & 26 \end{pmatrix}$

7b) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}$

$\begin{pmatrix} 5 & 4 & 8 \\ 27 & 26 & 42 \end{pmatrix}$

7c) $\begin{pmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ X

7d) $\begin{bmatrix} 2 & 6 & 4 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix}$

$6+30+4+56$

96//

7e) $\begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 & 8 \end{bmatrix}$

$4 \times 1 \quad 1 \times 4$

$$\begin{bmatrix} 6 & 18 & 12 & 24 \\ 10 & 30 & 20 & 40 \\ 2 & 6 & 4 & 8 \\ 14 & 42 & 28 & 56 \end{bmatrix}$$

7f) $\begin{bmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{bmatrix}$

$$\begin{bmatrix} 11 & -22 & 29 \\ 9 & -27 & 32 \\ 13 & -17 & 26 \end{bmatrix}$$

8a) $\begin{vmatrix} 2 & 6 \\ -1 & -3 \end{vmatrix} = -6 + 6 = 0$

not invertible

8b) $\begin{vmatrix} 1 & -4 \\ -1 & 2 \end{vmatrix} = 2 - 4 = -2$

$B^{-1} = -\frac{1}{2} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

8c) $\begin{vmatrix} 1 & 4 & 2 \\ 2 & -2 & 1 \\ 1 & -2 & 3 \end{vmatrix} = -2 \begin{vmatrix} 4 & 2 \\ -2 & 3 \end{vmatrix} + (-2) \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix}$

$i + j = 2 + 1$
$= 3$

$= -2 \cdot 16 - 2 \cdot 1 - 1 \cdot -6$

$= -32 - 2 + 6$

$= -28$

$$C^{-1} = -\frac{1}{28} \begin{bmatrix} -4 & -(5) & -2 \\ -(16) & 1 & -(-6) \\ 8 & -(-3) & -10 \end{bmatrix}^T$$

$$= -\frac{1}{28} \begin{bmatrix} -4 & -5 & -2 \\ -16 & 1 & 6 \\ 8 & 3 & -10 \end{bmatrix}^T$$

$$C^{-1} = -\frac{1}{28} \begin{bmatrix} -4 & -16 & 8 \\ -5 & 1 & 3 \\ -2 & 6 & -10 \end{bmatrix}$$

8d) $\begin{vmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ -2 & 2 & 5 \end{vmatrix} = 1\begin{vmatrix} -3 & 0 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix}$ $\quad 100 + 4 + 121$

$= -15 + 4$

$= -11$

$$D^{-1} = -\frac{1}{11}\begin{bmatrix} -15 & -(5) & -4 \\ -(2) & 3 & -(2) \\ -3 & -(1) & -3 \end{bmatrix}^T$$

$$= -\frac{1}{11}\begin{bmatrix} -15 & -5 & -4 \\ -2 & 3 & -2 \\ -3 & -1 & -3 \end{bmatrix}^T$$

$$= \frac{1}{11}\begin{bmatrix} 15 & 2 & 3 \\ 5 & -3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

9) $\begin{bmatrix} 5 & -14 & 2 \\ -10 & -5 & -10 \\ 10 & 2 & -11 \end{bmatrix}\begin{bmatrix} 5 & -10 & 10 \\ -14 & -5 & 2 \\ 2 & -10 & -11 \end{bmatrix}$

$= \begin{bmatrix} 225 & 0 & 0 \\ 0 & 225 & 0 \\ 0 & 0 & 225 \end{bmatrix}$

$= 225 I_3$

$AA^T = 225 I_3$

$AA^T = 225 AA^{-1}$

$\frac{1}{225} AA^T = AA^{-1}$

$\frac{1}{225} \overset{I}{\cancel{A^{-1}A}}A^T = \overset{I}{\cancel{A^{-1}A}}A^{-1}$

$\frac{1}{225} A^T = A^{-1}$

$A^{-1} = \frac{1}{225}\begin{bmatrix} 5 & -10 & 10 \\ -14 & -5 & 2 \\ 2 & -10 & -11 \end{bmatrix}$

10. orthogonal matrix $\longrightarrow$ $\boxed{AA^T = I = A^TA}$

$\boxed{A^T = A^{-1}}$

is $Q = \begin{bmatrix} 5/13 & 12/13 \\ -12/13 & 5/13 \end{bmatrix}$ orthogonal?

$|Q| = \frac{25}{169} + \frac{144}{169}$

$= \frac{169}{169}$

$= 1$

$Q^{-1} = \frac{1}{|Q|} \begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$

$Q^{-1} = \begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$

$Q^T = \begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$

$\boxed{Q^{-1} = Q^T}$

Q is an orthogonal matrix.