System of linear equations

- augmented matrix

$$\begin{aligned} x - 3y + 4z &= -4 \\ 3x - 7y + 7z &= -8 \\ -4x + 6y - z &= 7 \end{aligned}$$

(linear eqns)

$$\rightarrow \begin{bmatrix} 1 & -3 & 4 \\ 3 & -7 & 7 \\ -4 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ 7 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array}\right]$$

(matrix form)

- no. of solutions

$Ax = b$ linear system

- $|A| \neq 0$ non-singular → unique or one solution
- $|A| = 0$ singular → ∞ly many solutions or no solution

$Ax = 0$ homogeneous linear system

- $|A| \neq 0$ non-singular → unique or one sol.
- $|A| = 0$ singular → ∞ly many solutions

$x = 0$ → trivial solution $x_1 = x_2 = \dots = x_n = 0$

the sol. set for HLS always contains the trivial solution

consistent linear system: one or ∞ many sols.

inconsistent —"— : no solution

proof that HLS always contains the trivial sol.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1p}x_p = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2p}x_p = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{np}x_p = 0$$

$x_1 = x_2 = \dots = x_p = 0$ always satisfy the n eqns above

Reduced Row Echelon.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{array}\right] \xrightarrow[2R_1-R_3]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -3 & 5 \end{array}\right] \xrightarrow{R_2-2R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 3 & -6 \end{array}\right] \xrightarrow{\frac{1}{3}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2+3R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] //$$

Tutorial 2

1) b, c, d, e

2a) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \end{bmatrix}$ $|A| = 6-6=0$

$\xrightarrow{2R_1+R_2} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Let $y = t$

$3x - y = 2$

$3x = 2+t$

$x = \frac{2+t}{3}$

2b) $|B| = 6-6=0$

$\begin{bmatrix} 2 & -3 & 5 \\ -2 & 3 & 2 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 2 & -3 & 5 \\ 0 & 0 & 7 \end{bmatrix}$

$0 \neq 7$

↳ no sol.

2c) $|C| = -4$

$\begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ 0 & 0 & 4 & 4 \end{bmatrix}$

$\xrightarrow{R_1-3R_2} \begin{bmatrix} 3 & -2 & 1 & -2 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 4 & 4 \end{bmatrix} \xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 3 & -2 & 1 & -2 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$z = 1$

$y - 8z = -17$

$y = -17 + 8$

$y = -9$

$3x - 2y + z = -2$

$3x = -2 + 2y - z$

$3x = -2 - 18 - 1$

$x = -7$

2d) $|D| = 0$

$\begin{bmatrix} 1 & 2 & -4 & 10 \\ 2 & -1 & 2 & 5 \\ 1 & 1 & -2 & 7 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 2 & -4 & 10 \\ 2 & -1 & 2 & 5 \\ 0 & 1 & -2 & 3 \end{bmatrix}$

$\xrightarrow{2R_1-R_2} \begin{bmatrix} 1 & 2 & -4 & 10 \\ 0 & 5 & -10 & 15 \\ 0 & 1 & -2 & 3 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & -4 & 10 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & -4 & 10 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $z = t$

$y - 2z = 3$

$y = 3 + 2t$

$x + 2y - 4z = 10$

$x = 10 - 2y + 4z$

$x = 10 - 6 - 8t + 4t$

$x = 4$

Tutorial 2

2e) $\left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow{3R_1-2R_3} \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 0 & -13 & -8 \end{array}\right] \xrightarrow{13R_2+8R_3} \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 5 \end{array}\right]$

$0 \neq 5 \rightarrow$ no sol.

2f) $\left[\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right] \xrightarrow{2R_1+5R_2} \left[\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ 0 & 1 & 27 & 5 \end{array}\right] \xrightarrow{2R_2+R_1}$

$\left[\begin{array}{ccc|c} 5 & 0 & 60 & 10 \\ 0 & 1 & 27 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array}\right]$

$2 \quad 15$

$\frac{1}{x} \quad \frac{1}{75}$

let $z = t$

$y + 27t = 5$

$y = 5 - 27t$

$x + 12z = 2$

$x = 2 - 12t$

2g) $\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{5R_1-3R_2} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$

29) $\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \xrightarrow{2R_1 - R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - R_3]{5R_1 - 3R_2} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-1/7 R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = 7$ //

$y - 11z = -75$

$y = -75 + 77$

$y = 2$ //

$3x + 2y - z = -15$

$3x = -15 - 2y + z$

$3x = -15 - 4 + 7$

$3x = -12$

$x = -4$ //

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24) $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & -1 & -1 & 0 \end{array}\right]$

$\xrightarrow{3R_1 - 3R_2} \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & 8 & 2 & 8 & 0 \end{array}\right] \xrightarrow{1/8 R_2} \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1/4 & 1 & 0 \end{array}\right]$

Let $x_3 = s$ $x_4 = t$

$x_2 + 0.25s + t = 0$

$x_2 = -0.25s - t$

$3x_1 = -x_2 - x_3 - x_4$

$3x_1 = 0.25s + t - s - t$

$3x_1 = -\frac{3}{4}s$

$x_1 = -\frac{1}{4}s$

3. consistent

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right] \xrightarrow[R_1-R_2]{2R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 1 & 1 & 2a-c \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{array}\right]$$

can be no sol or ∞ly many sol

∴ $c-a-b=0$ → consistent

$c=a+b$

4. $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & -2 & k \end{array}\right]$

no sol. → k=?

1 sol → k=?

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & -2 & k \end{array}\right] \xrightarrow{2R_1-R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 6-k \end{array}\right]$$

no sol: $6-k \neq 0$

$k \neq 6$ //

1 sol: impossible //

∞ # of sol: $6-k=0$

$k=6$ //

5) $A=\left[\begin{array}{cc} -1 & -4 \\ 3 & 12 \\ 2 & 8 \end{array}\right]$ find nontrivial solution to $Ax=0$

$$\left[\begin{array}{cc|c} -1 & -4 & 0 \\ 3 & 12 & 0 \\ 2 & 8 & 0 \end{array}\right] \xrightarrow[2R_1+R_3]{3R_1+R_2} \left[\begin{array}{cc|c} -1 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

let $y=t$

$-x-4y=0$

$x=-4t$

$x=\begin{bmatrix} -4t \\ t \end{bmatrix}$

$x=t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

a particular non trivial sol if $x_1=-4$, $x_2=1$

6) $2x + 3y + 4z = 250$

$4y + 3y + 4z = 250$

$7y + 4z = 250$

$x + y + z = 100$

$2y + y + z = 100$

$3y + z = 100$

$x = 2y$

$$\begin{bmatrix} 7 & 4 & 250 \\ 3 & 1 & 100 \end{bmatrix}$$

$3R_1 - 7R_2 \rightarrow$ $\begin{bmatrix} 7 & 4 & 250 \\ 0 & 5 & 50 \end{bmatrix}$ $\frac{1}{5}R_2 \rightarrow$ $\begin{bmatrix} 7 & 4 & 250 \\ 0 & 1 & 10 \end{bmatrix}$

$z = 10$

$x = 2y$

$x = 60$

$7y + 4z = 250$

$7y = 250 - 40$

$7y = 210$

$y = 30$