Chapter 6: Differentiation

Learning Objectives:

1. *State the derivative of the function* $y = x^n$ *for all real n.*
2. *State the rules:*

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x),$$

$$\frac{d}{dx}\left[k\,f(x)\right] = k\frac{d}{dx}f(x)$$ *, where k is a constant.*

3. *State the derivatives of trigonometric functions.*
4. *Explain derivative of f(x) as the gradient of the tangent to the graph of y = f(x) at a point.*
5. *Define the derivative of a function as a limit given by* $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.
6. *State product rule.*
7. *State quotient rule.*
8. *State chain rule.*
9. *Apply the rules to find derivatives of algebraic & trigonometric functions.*
10. *Evaluate the derivatives at given points.*
11. *Use Python to evaluate derivatives of algebraic and trigonometric functions.*

6.1 Derivative of the Power Function

The derivative of the power function, $y = x^n$, where n is a real constant is:

$$\boxed{\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}}$$

In words, to differentiate a power of the variable, we "bring down the power and decrease the power by 1."

Note: The derivative can be denoted by $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}\left[f(x)\right]$ and y' .

The derivative symbol $\frac{d}{dx}(\;\;)$ can be considered as an operator. The $\boldsymbol{\frac{d}{dx}}$ in front of an expression indicates that the expression is to be differentiated.

Example 1.

Find each of the following derivatives:

(a) $\frac{d}{dx}\left(x^{10}\right)$ (b) $f(x) = \sqrt{x}$.

(c) $y = \dfrac{1}{x^3}$

(d) $\dfrac{d}{dt}\left(\dfrac{1}{\sqrt[3]{t}}\right)$

[Ans: (a) $10x^9$ (b) $\dfrac{1}{2\sqrt{x}}$ (c) $-\dfrac{3}{x^4}$ (d) $-\dfrac{1}{3t^{\frac{4}{3}}}$]

6.2 Constant Rule, Constant Multiple Rule, and Linearity Rule

Constant Rule

$\dfrac{d}{dx}(k) = 0$, where k is a constant.

Example 2.

(a) Given that $y = 9$, find $\dfrac{dy}{dx}$.

(b) Given that $f(x) = \pi^3$, find $\dfrac{d}{dx}\big(f(x)\big)$.

Constant Multiple Rule

$\dfrac{d}{dx}\big[k\,f(x)\big] = k\dfrac{d}{dx}\big[f(x)\big]$ where k is a constant.

Example 3.

(a) $\dfrac{d}{dx}(5x^2)$

(b) Given that $f(x) = 3x$, find $f'(x)$.

[Ans: e.g. 2(a) 0 (b) 0 e.g. 3(a) $10x$ (b) 3]

Linearity Rule

If $u = f(x)$ and $v = g(x)$, then

$$\frac{d}{dx}(au + bv) = a\frac{du}{dx} + b\frac{dv}{dx}$$

Example 4.

(a) If $f(x) = 3x^2 + 4x + 5$, find $f'(x)$.

(b) Find $\frac{d}{dx}\left(3x^5 + 4\sqrt{x} - \pi x\right)$.

[Ans: (a) $f'(x) = 6x + 4$ (b) $15x^4 + \frac{2}{\sqrt{x}} - \pi$]

6.3 Differentiation of Trigonometric Functions

In the previous sections, we learnt how to differentiate algebraic expressions and also the rules of differentiation. We will now look at differentiating trigonometric functions.

$$\frac{d}{dx}(\sin x) = \cos x\,; \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x\,; \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x\,; \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Notice that if the trigonometric function starts with 'c', for example 'cosine', its derivative will have a negative sign.

We remind readers that any numerical computation involving the derivatives of the trigonometric functions must be done in **radians**. This is because their derivative formulae were essentially obtained based on the use of the unit radians.

Example 5.

Differentiate the following functions with respect to their respective variables.

(a) $x = t + 3\cos(t) + \sin(t)$

(b) $f(x) = 2\tan(x) + 3\sin(\pi) + \pi$

(c) $u = 2\sin(x) - 5\csc(x) + \dfrac{1}{x^2}$

(d) $g(t) = \sqrt{t^3} - 5\cot(t)$

[Ans: (a) $\dfrac{dx}{dt} = 1 - 3\sin t + \cos t$ (b) $f'(x) = 2\sec^2 x$

(c) $\dfrac{du}{dx} = 2\cos x + 5\csc x \cot x - \dfrac{2}{x^3}$ (d) $g'(t) = \dfrac{3}{2}\sqrt{t} + 5\csc^2 t$]

6.4 The Derivative and the Gradient of a Graph

In *Figure 6.1*, a secant line intersects the curve $y = f(x)$ at $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Gradient of secant line: $m_s = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

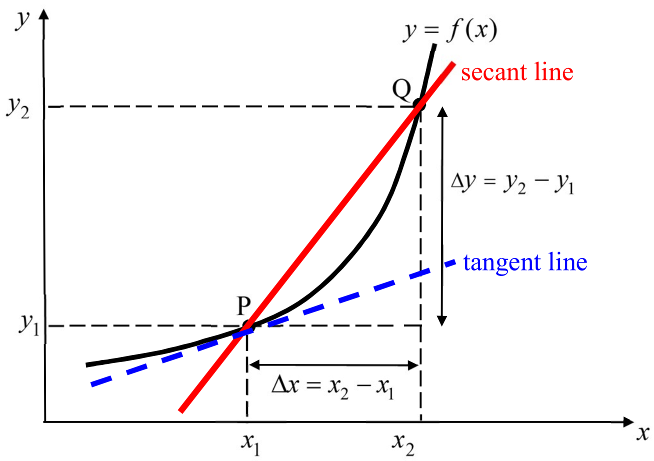


Figure 6.1

As Q approaches P (i.e. as x_2 approaches x_1, written as $x_2 \to x_1$), the gradient of the secant line approaches the gradient of the tangent line. As the two points P and Q meet together, the tangent line touches the curve at one point.

The gradient of the tangent line at P is also the gradient of the graph at P.

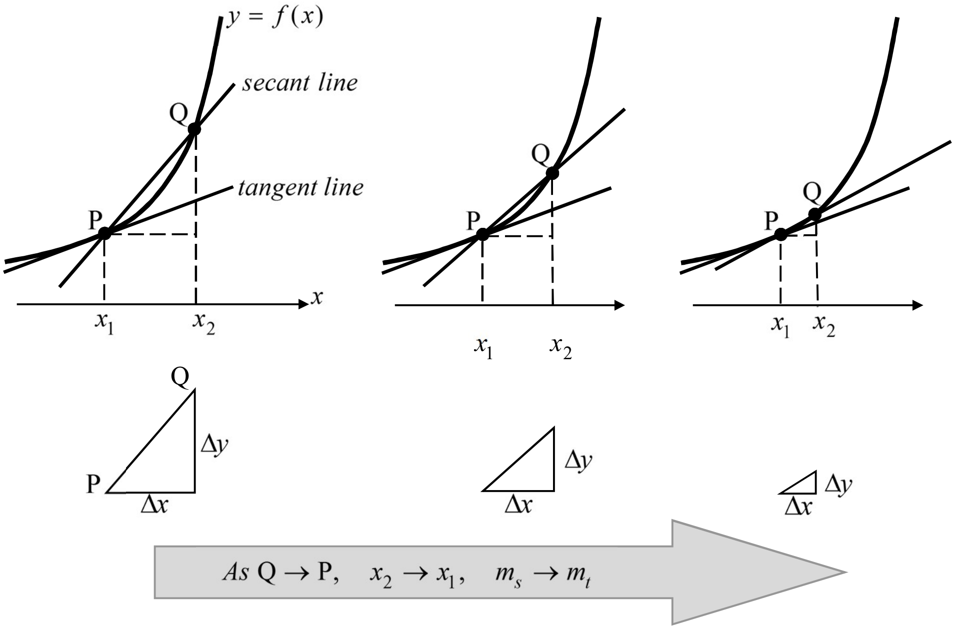


Figure 6.2. Secant line approaches tangent line

If $y = x^2$, the gradient of the tangent line at $x = 2$ can be estimated by $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Taking $x_1 = 2$, $x_2 = 2.5, 2.1, 2.01, 2.001$, $\dfrac{y_2 - y_1}{x_2 - x_1}$ approaches the limiting value 4.

x_1	x_2	y_1	y_2	$\dfrac{y_2 - y_1}{x_2 - x_1}$
2.0	2.5	4.0	6.25	4.5
2.0	2.1	4.0	4.41	4.1
2.0	2.01	4.0	4.0401	4.01
2.0	2.001	4.0	4.004001	4.001

Similarly, when $x_1 = 2$, $x_2 = 1.5, 1.9, 1.99, 1.999$, $\dfrac{y_2 - y_1}{x_2 - x_1}$ approaches the limiting value 4.

x_1	x_2	y_1	y_2	$\dfrac{y_2 - y_1}{x_2 - x_1}$
2.0	1.5	4.0	2.25	3.5
2.0	1.9	4.0	3.61	3.9
2.0	1.99	4.0	3.9601	3.99
2.0	1.999	4.0	3.996001	3.999

In general, the gradient of tangent line at point $\mathrm{P}(x_1, y_1)$ is the limiting value of $\dfrac{y_2 - y_1}{x_2 - x_1}$ as $x_2 \to x_1$.

Let the change in x and y-coordinates from P to Q be δx and δy respectively, then

$$\delta x = x_2 - x_1 \quad \text{and} \quad \delta y = y_2 - y_1$$

Thus,
$$\frac{\delta y}{\delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As $x_2 \to x_1$, $\delta x = x_2 - x_1 \to 0$.

Hence, the gradient of the tangent at P can be written as $\lim\limits_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$.

The limiting value of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$ is known as the **derivative** of the function.

$$\boxed{\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)}$$

The notation $\frac{dy}{dx}$ is read as 'the derivative of y with respect to x'.

If $y = f(x)$, then the derivative may also be denoted by $f'(x)$, $\frac{d}{dx}\big(f(x)\big)$ or y'.

The process of finding $\frac{dy}{dx}$ is called differentiation.

We will first find the derivative of the function $f(x) = x^n$, where $-\infty < x < \infty$ and n is a positive integer. The derivative is:

$$f'(x) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}\frac{(x+h)^n - x^n}{h}$$

Using Binomial theorem: $(x+h)^n = x^n + {}_nC_1 x^{n-1}h + {}_nC_2 x^{n-2}h^2 + {}_nC_3 x^{n-3}h^3 + \ldots + h^n$

$$\Rightarrow \quad (x+h)^n - x^n = {}_nC_1 x^{n-1}h + {}_nC_2 x^{n-2}h^2 + {}_nC_3 x^{n-3}h^3 + \ldots + h^n$$

Next, divide throughout by h, we get:

$$\frac{(x+h)^n - x^n}{h} = {}_nC_1 x^{n-1} + {}_nC_2 x^{n-2}h + {}_nC_3 x^{n-2}h^2 + \ldots + h^{n-1}$$

After dividing by h, observe that only one term ${}_nC_1 x^{n-1}$ does not have the factor h, all the other terms contain positive powers of h.

As $h \to 0$, the only term that remains is ${}_nC_1 x^{n-1}$ and all the other terms vanish.

Since ${}_nC_1 = n$, then: $\quad f'(x) = {}_nC_1 x^{n-1} = nx^{n-1}$

Therefore, the derivative of the power function $y = x^n$, where n is a real constant is:

$$\boxed{\frac{d}{dx}\left[x^n\right] = n\,x^{n-1}}$$

Example 6.

Find the gradient of the curve $y = x^2 + 1$ at the point $x = 1$.

6.5 Product Rule

When y, a function of x, is a **product** of two functions u and v, both of which are also functions of x, then the derivative of y with respect to x can be evaluated using the Product Rule as follows:

Product Rule

If $y = u\,v$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Example 7.

(a) If $g(x) = (2x^3 - 3x^2 + 7)(3x^3 + 4x)$, find $g'(x)$.

(b) Find $\dfrac{d}{dt}\left(\pi t^2 \sin(t)\right)$.

[Ans: (a) $g'(x) = 6x^2\left(3x^2+4\right)\left(x-1\right)+\left(2x^3-3x^2+7\right)\left(9x^2+4\right)$ (b) $2\pi t\sin t + \pi t^2\cos t$]

6.6 Quotient Rule

On the other hand, if y is a **quotient** of two functions u and v, then the derivative of y with respect to x can be evaluated using the Quotient Rule as follows:

Quotient Rule

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Example 8.

(a) If $y = \dfrac{3x+1}{2x-1}$, find $\dfrac{dy}{dx}$.

(b) If $f(x) = \dfrac{\tan(x)}{\sqrt{x}}$, find $\dfrac{d}{dx}\left(f(x)\right)$.

[Ans: (a) $\dfrac{dy}{dx} = \dfrac{-5}{(2x-1)^2}$ (b) $\dfrac{d}{dx}\left(f(x)\right) = \dfrac{1}{\sqrt{x}}\left(\sec^2 x - \dfrac{\tan x}{2x}\right)$]

6.7 Chain Rule

If $y = f(u)$ and $u = g(x)$, then $y = f[\,g(x)\,]$.

We say that y is a **composite function** of x.

Example 9.

(a) If $y = \sqrt{u}$ and $u = 3x^2 - 2$,

then $y = \sqrt{3x^2 - 2}$ is a composite function.

(b) If $y = u^2$ and $u = \sin x$,

then $y = \left(\sin x\right)^2 = \sin^2 x$ is a composite function.

For $y = f(u)$, we can find $\dfrac{dy}{du}$.

For $u = g(x)$, we can find $\dfrac{du}{dx}$.

So, to find $\dfrac{dy}{dx}$, we apply the Chain Rule, a rule which is essential to understanding backpropagation in a neural network.

Chain Rule

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{or}$$

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Example 10.

Differentiate the following functions with respect to their respective variables:

(a) $y = \left(x^3 - 5\right)^2$

(b) $u = \sqrt{3t^2 - 2}$

(c) $s = \sin(3x)$

(d) $y = 5\cot(x^2)$

(e) $y = \sin^3(4x)$

[Ans: (a) $\dfrac{dy}{dx} = 6x^2(x^3 - 5)$ (b) $\dfrac{du}{dt} = \dfrac{3t}{\sqrt{3t^2 - 2}}$ (c) $\dfrac{ds}{dx} = 3\cos(3x)$
(d) $\dfrac{dy}{dx} = -10x\csc^2(x^2)$ (e) $\dfrac{dy}{dx} = 12\cos(4x)\sin^2(4x)$]

Example 11.

Suppose that $f(2) = 3$, $f'(2) = 4$, $g(3) = 6$, and $g'(3) = -5$. Evaluate:

(a) $h'(2)$, where $h(x) = g(f(x))$

(b) $k'(3)$, where $k(x) = f\left(\dfrac{1}{3}g(x)\right)$

[Ans: (a) -20 (b) $-\dfrac{20}{3}$]

The chain rule can be extended to an arbitrarily long composite function.
We can then use a combination of the rules to differentiate more complex functions.

Example 12.

(a) If $y = (t-1)^2(5t+3)$, find $\dfrac{dy}{dt}$.

(b) If $y = 2x^3 \csc(4x)$, find y'.

(c) If $y = \dfrac{3x^2+1}{(5x-4)^2}$, find y'.

(d) If $h(x) = \dfrac{\cos(x)}{\sin\left(3x-\pi\right)}$, find $\dfrac{d}{dx}\left(h(x)\right)$.

[Ans: (a) $\dfrac{dy}{dt} = (t-1)(15t+1)$ (b) $y' = 2x^2\csc\left(4x\right)\left[3-4x\cot(4x)\right]$ (c) $y' = \dfrac{-2\left(12x+5\right)}{(5x-4)^3}$

(d) $\dfrac{d}{dx}\left(h(x)\right) = \dfrac{-\left[\sin(x)\sin\left(3x-\pi\right)+3\cos(x)\cos\left(3x-\pi\right)\right]}{\sin^2\left(3x-\pi\right)}$]

6.8 Differentiating using Python

6.8.1 Differentiation using SymPy

SymPy is a Python library that allows us to do algebraic manipulation.
Firstly, the symbols of the variables need to be declared:

```
import sympy as sp

# Define the symbol x
x=sp.Symbol('x')
```

Figure 6.3 Python code for declaring symbols for variables

We can then differentiate any function using *diff(func,var)*, where *func* is the expression of the function to be differentiated and *var* is the variable we are differentiating with respect to.

The following Python code differentiates $y = x^3 - 2x + 1$ to get $\dfrac{dy}{dx}$:

```
sp.diff(x**3-2*x+1,x)
```

Figure 6.4 Python code for differentiation

Example 13.

Find the derivative of the following functions using SymPy.

(a) $y = \sqrt{x} - \csc x$

(b) $w = \dfrac{4}{t^2} + 3t^2 - 3$

(c) $x = s\tan(2s+1)^3$

(d) $z = \dfrac{(3y+1)^3}{(2y+5)^2}$

[Ans: (a) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} + \csc x \cot x$ (b) $\dfrac{dw}{dt} = -\dfrac{8}{t^3} + 6t$

(c) $\dfrac{dx}{ds} = \tan(2s+1)^3 + 6s(2s+1)^2(\tan^2(2s+1)^3 + 1)$ (d) $\dfrac{dz}{dy} = \dfrac{(3y+1)^2(6y+41)}{(2y+5)^3}$]

6.8.2 Numerical differentiation

Recall in Section 6.4 when differentiation was first introduced, the value of $\frac{dy}{dx}$ of $y = x^2$ at $x = 2$ was estimated by $\frac{y_2 - y_1}{x_2 - x_1}$, where $y_1 = x_1^2$, $y_2 = x_2^2$, and x_1 and x_2 were values very near 2 or equal to 2.

In more general terms, for a differentiable function $y = f(x)$, the value of $f'(a)$ can be approximated by:

$$f'(a) \approx \frac{f(a+\epsilon)-f(a-\epsilon)}{2\epsilon} \approx \frac{f(a+\epsilon)-f(a)}{\epsilon} \approx \frac{f(a)-f(a-\epsilon)}{\epsilon},$$

where $\epsilon > 0$ is a very small number.

In practice, it may be more important to get the numerical value of $f'(a)$ than to get the symbolic representation of $f'(x)$. Thus, an alternative to finding the expression of $f'(x)$ and substituting $x = a$ into the expression will be to make use of the approximation method above, which is called **numerical differentiation**. In particular, the first approximation above, called the **symmetric difference quotient**, provides a more accurate estimation than the other two approximations.

Example 14.

Use numerical differentiation to estimate the value of $f'(3)$ for $f(x) = x^3 - 2\sin x$.
Compare the estimated value with the actual value.

For the same example, we can compute the approximation in Python as follows:

```
import numpy as np

epsilon = 0.001 # small constant
func = lambda x: x**3-2*np.sin(x) # function of which derivative is estimated

a=3 # point at which derivative is estimated
sym_diff_quo = (func(a+epsilon)-func(a-epsilon))/(2*epsilon) # estimate using symmetric difference quotient
sym_diff_quo
```

```
28.979985663198704
```

Figure 6.5 Python code for numerical differentiation

Example 15.

Given $f(x) = (2x+1)^3 \cos 4x$.

(a) Find $f'(x)$, and hence compute $f'(0.1)$.

(b) Using Python or otherwise, estimate the value of $f'(0.1)$ using numerical differentiation and compare with your computed answer in part (a).

SUMMARY

1. Definition

If $y = f(x)$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[f(x)\right] = f'(x) = y'$ is known as:

- the derivative of $y = f(x)$ with respect to the variable x
- the gradient of a tangent line to the curve $y = f(x)$

2. Standard Derivatives

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\dfrac{d}{dx}\left(\sin x\right) = \cos x$	$\dfrac{d}{dx}\left(\tan x\right) = \sec^2 x$	$\dfrac{d}{dx}\left(\sec x\right) = \tan x \sec x$
$\dfrac{d}{dx}\left(k\right) = 0$, k is a constant	$\dfrac{d}{dx}\left(\cos x\right) = -\sin x$	$\dfrac{d}{dx}\left(\cot x\right) = -\csc^2 x$	$\dfrac{d}{dx}\left(\csc x\right) = -\cot x \csc x$

3. General Rules

Suppose $u = u(x)$, $v = v(x)$, $y = y(u)$, $k =$ constant	
Constant multiple rule	$\dfrac{d}{dx}\left(ku\right) = k\dfrac{du}{dx}$
Sum rule	$\dfrac{d}{dx}\left(u+v\right) = \dfrac{du}{dx} + \dfrac{dv}{dx}$
Product rule	$\dfrac{d}{dx}\left(u \cdot v\right) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
Quotient rule	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{\left(v\right)^2}$
Chain rule	$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

4. Using Python for Differentiation

- SymPy library for algebraic expression of derivative
- Numerical differentiation to approximate value of derivative at a point

Tutorial 6

Section A: Multiple Choice Questions

1. If $y = x^3 \sec(x)$, then $\dfrac{dy}{dx}$ is equal to

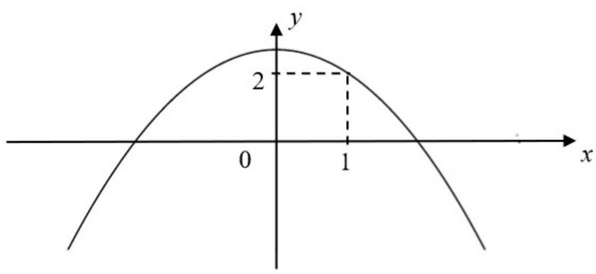
(a) $3x^2 \sec(x)$ (b) $x^3 \sec(x)\tan(x)$

(c) $3x^2 \sec(x)\tan(x)$ (d) $x^2 \sec(x)(3 + x\tan(x))$

2. If $y = \sin\left(t^3\right)$, then $\dfrac{dy}{dt}$ is equal to

(a) $3\sin(t^2)$ (b) $\cos(t^3)$ (c) $3t^2\cos(t^3)$ (d) $\sin(3t^2)$

3. Given the graph of $y = f(x)$ shown below, $f'(1)$ is _______.



(a) zero (b) positive (c) negative (d) undefined

4. If $f(x) = k_1 g(x)h(x) + k_2$, where k_1 and k_2 are constants, find $f'(x)$.

(a) $g'(x)h'(x)$ (b) $k_1 g'(x)h'(x) + k_2$

(c) $k_1\left[g'(x)h(x) + g(x)h'(x)\right]$ (d) none of the above

5. Given that $f'(x)$ exist and $f(x) > 0$ for all values of x, which of the following is true?

(a) $\dfrac{d}{dx}\left[f\left(x^2\right)\right] = 2xf'(x)$ (b) $\dfrac{d}{dx}\left[\sqrt{f(x)}\right] = \dfrac{f'(x)}{2\sqrt{f(x)}}$

(c) $\dfrac{d}{dx}\left[f(x)\right]^2 = 2f'(x)$ (d) $\dfrac{d}{dx}\left[f\left(\sqrt{x}\right)\right] = \dfrac{f'(x)}{2\sqrt{x}}$

Section B

1. Find the derivative of each of the following functions:

   (a) $f(x) = x^4$ (b) $g(x) = 1/x^5$

   (c) $h(x) = \sqrt[3]{x}$ (d) $k(x) = 1/\sqrt{x}$

2. Find each of the following derivatives:

   (a) $\frac{d}{dx}\left(x^9\right)$ (b) $\frac{d}{dx}\left(\sqrt{x^3}\right)$

   (c) $\frac{d}{dx}\left(\frac{1}{x^{-4}}\right)$ (d) $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2}}\right)$

   (e) $\frac{d}{dx}\left(\frac{\sqrt[5]{x^3}+\sqrt[3]{x}-\sqrt[4]{x}}{\sqrt{x}}\right)$

3. (a) Find $\frac{dy}{dx}$, if $y = 4x^2 + \frac{3}{x^3} - 6\sqrt{x}$.

   (b) Find $\frac{ds}{dr}$, if $s = 2\sqrt{r} + \frac{5}{r^2} - \pi$.

   (c) Find $\frac{d}{dr}(ar^3 + br)$ where a, b are real constants.

   (d) Find $\frac{d}{dt}(\pi t^3 + \sqrt{2}\,t)$.

4. For $f(x) = 2x^5 - 4x^3 + 3x^2 + 6x - 9$, find $f'(x)$ and hence the value $f'(1)$.

5. If $f(x) = 2.75x^2 - 5.02x$, find $f'\left(3.36\right)$.

6. For $f(x) = 2\sqrt{x} + \tan(x)$, find $f'(x)$.

7. If $g(t) = 2\sin(t) + 3\cos(t)$, find $g'(t)$.

8. Given $f(u) = a\cos\left(u\right) + b\sin(u)$. If $f\left(0\right) = 6$ and $f'\left(\frac{\pi}{3}\right) = -1$, find the values of a and b.

   Express your answer correct to 2 decimal places.

9. For each of the following functions, find $f'(x)$ using the product rule:

(a) $f(x)=(2x^3-3x)(x^2+1)$
(b) $f(x)=(x+2)\left(1-\dfrac{1}{x^2}\right)$
(c) $f(x)=(x^2+1)(\sqrt{x}+2)$
(d) $f(x)=5x\left(\sqrt{x}-\dfrac{1}{x}\right)$
(e) $f(x)=x\sin(x)$
(f) $f(x)=(x+5)\cos(x)$
(g) $f(x)=x^2\tan(x)$
(h) $f(x)=\cos(x)\sin(x)$

10. By using the quotient rule, find the derivative of the following functions:

(a) $y=\dfrac{x+3}{x-3}$
(b) $y=\dfrac{x^2}{2+3x^5}$
(c) $f(x)=\dfrac{x^2}{\sin(x)}$
(d) $f(x)=\dfrac{1+\cos(x)}{1-\cos(x)}$

11. Find $\dfrac{dy}{dx}$ by using chain rule:

(a) $y=(2x+1)^8$
(b) $y=(x^2+1)^{15}$
(c) $y=\sqrt{1-x}$
(d) $y=\sqrt{x^2+x+1}$
(e) $y=\csc(5x)$
(f) $y=\sec(3x)$
(g) $y=6\tan\left(\dfrac{1}{2}x\right)$
(h) $y=2\cot(0.1x)$
(i) $y=2\sin\left(3x-\dfrac{\pi}{4}\right)$
(j) $y=\pi\cos(6x+0.71)$
(k) $y=\sin\left(3x^2\right)$
(l) $y=\cos^3 x$
(m) $y=\sin^3(3x)$
(n) $y=\cos^4\left(\dfrac{x}{2}\right)$

12. Differentiate the following functions with respect to its variables. Simplify your answers whenever possible.

(a) $s=0.5\sin(4x)+2x$
(b) $h=\dfrac{2}{3}\cos(6t)-5t$
(c) $y=4\cos(7x)-2\sec(5x)$
(d) $w=300\sin\left(\dfrac{r}{150}\right)+\tan(2r)$

13. Differentiate the following with respect to its variables and simplify your answers whenever possible.

(a) $u=(2r+1)\sqrt{r^2-3}$ (b) $s=\sin(t)\cos(3t)$

(c) $y=\left(x^3+1\right)\tan(2x)$ (d) $x=\dfrac{1-u}{(1+u)^2}$

(e) $y=\dfrac{x}{\sin(2x)}$ (f) $w=\dfrac{\sin(u)}{\cos(3u)}$

14. Given $f(x)=x(1-4x)^3$, find $f'(0)$.

15. Suppose $y=\dfrac{2x}{(1-x)^3}$, find $\dfrac{dy}{dx}$. Hence, find $\left.\dfrac{dy}{dx}\right|_{x=0}$.

16. Find the gradient of the curve $y=\sqrt{x^2+4x-7}$ at the point (4, 5).

*17. Given $h(x)=g(x)\sin(2x)$, $g\left(\dfrac{\pi}{6}\right)=\dfrac{1}{2}$ and $g'\left(\dfrac{\pi}{6}\right)=\sqrt{3}$. Find the value of $h'\left(\dfrac{\pi}{6}\right)$.

*18. Given $h(x)=f(g(x))$, $f\left(x\right)=\cos(x)$ and $g\left(x\right)=\dfrac{1}{x}$, find $h'(x)$.

19. Differentiate the following with respect to their variables using the SymPy library in Python.

(a) $y=\sin^2(2x+3)^3$ (b) $w=s^4(3s-2)^5$

(c) $x=\dfrac{t\cos t}{(1-t)^2}$ (d) $z=\dfrac{(3t+2)^4}{t^3(4t-1)^2}$

20. For the following, find the exact value of the derivative at the point stated with the help of SymPy library in Python and also estimate it using numerical differentiation in Python.

(a) $f(x)=x^2\sin(x+\pi)$; $f'(\pi)$

(b) $f(x)=4x^5-5x^3+1$; $f'(1)$

(c) $g(x)=2x(x-3)(2x+5)$; $g'(0)$

(d) $g(x)=\dfrac{6x\sin x}{(6x+1)^2}$; $g'(0)$

Answers

Section A

1. (d) 2. (c) 3. (c) 4. (c) 5. (b)

Section B

1. (a) $4x^3$ (b) $-\dfrac{5}{x^6}$ (c) $\dfrac{1}{3x^{2/3}}$ (d) $-\dfrac{1}{2x^{3/2}}$

2. (a) $9x^8$ (b) $\dfrac{3}{2}\sqrt{x}$ (c) $4x^3$ (d) $-\dfrac{2}{3x^{5/3}}$ (e) $\dfrac{1}{10x^{9/10}}-\dfrac{1}{6x^{7/6}}+\dfrac{1}{4x^{5/4}}$

3. (a) $8x-\dfrac{9}{x^4}-\dfrac{3}{\sqrt{x}}$ (b) $\dfrac{1}{\sqrt{r}}-\dfrac{10}{r^3}$ (c) $3ar^2+b$ (d) $3\pi t^2+\sqrt{2}$

4. $f'(x)=10x^4-12x^2+6x+6$; $f'(1)=10$

5. 13.46

6. $\dfrac{1}{\sqrt{x}}+\sec^2 x$

7. $2\cos t-3\sin t$

8. $a = 6.00$, $b = 8.392$

9. (a) $10x^4-3x^2-3$ (b) $\dfrac{1}{x^2}+\dfrac{4}{x^3}+1$ (c) $\dfrac{5}{2}x^{3/2}+4x+\dfrac{1}{2\sqrt{x}}$

   (d) $5x\cdot\left(\dfrac{1}{2\sqrt{x}}+\dfrac{1}{x^2}\right)+\left(\sqrt{x}-\dfrac{1}{x}\right)\cdot 5=\dfrac{15\sqrt{x}}{2}$ (e) $\sin x+x\cos x$

   (f) $-(x+5)\sin x+\cos x$ (g) $x^2\sec^2 x+2x\tan x$ (h) $\cos^2 x-\sin^2 x$

10. (a) $\dfrac{-6}{(x-3)^2}$ (b) $\dfrac{(2+3x^5)(2x)-x^2(15x^4)}{(2+3x^5)^2}=\dfrac{4x-9x^6}{(2+3x^5)^2}$

   (c) $\dfrac{2x\sin x-x^2\cos x}{\sin^2 x}$ (d) $\dfrac{-2\sin x}{(1-\cos x)^2}$

11. (a) $16(2x+1)^7$ (b) $30x(x^2+1)^{14}$ (c) $-\dfrac{1}{2\sqrt{1-x}}$

   (d) $\dfrac{2x+1}{2\sqrt{x^2+x+1}}$ (e) $-5\csc 5x\cot 5x$ (f) $3\sec 3x\tan 3x$

   (g) $3\sec^2\left(\dfrac{x}{2}\right)$ (h) $-0.2\csc^2 0.1x$ (i) $6\cos\left(3x-\dfrac{\pi}{4}\right)$

   (j) $-6\pi\sin(6x+0.71)$ (k) $6x\cos\left(3x^2\right)$ (l) $-3\cos^2 x\sin x$

   (m) $6\sin(3x)\cos(3x)$ (n) $-2\cos^3\left(\dfrac{x}{2}\right)\sin\left(\dfrac{x}{2}\right)$

12. (a) $\dfrac{ds}{dx} = 2(\cos 4x + 1)$ (b) $\dfrac{dh}{dt} = -4\sin 6t - 5$

(c) $\dfrac{dy}{dx} = -28\sin 7x - 10\sec 5x \tan 5x$ (d) $\dfrac{dw}{dr} = 2\cos\dfrac{r}{150} + 2\sec^2 2r$

13. (a) $\dfrac{du}{dr} = 2\sqrt{r^2-3} + \dfrac{r(2r+1)}{\sqrt{r^2-3}}$ (b) $\dfrac{ds}{dt} = -3\sin t(\sin 3t) + \cos 3t(\cos t)$

(c) $\dfrac{dy}{dx} = 2(x^3+1)\sec^2 2x + 3x^2\tan 2x$ (d) $\dfrac{dx}{du} = \dfrac{u^2-2u-3}{(1+u)^4} = \dfrac{u-3}{(1+u)^3}$

(e) $\dfrac{dy}{dx} = \dfrac{\sin 2x - 2x\cos 2x}{\sin^2 2x}$ (f) $\dfrac{dw}{du} = \dfrac{\cos 3u\cos u + 3\sin u(\sin 3u)}{\cos^2 3u}$

14. $-12x(1-4x)^2 + (1-4x)^3$; 1

15. $\dfrac{2(1-x)^3 - 2x(3)(1-x)^2(-1)}{(1-x)^6} = \dfrac{2(1-x)^3 + 6x(1-x)^2}{(1-x)^6} = \dfrac{2(1-x)+6x}{(1-x)^4} = \dfrac{2+4x}{(1-x)^4}$; 2

16. 1.2 17. 2 18. $\dfrac{\sin\left(\frac{1}{x}\right)}{x^2}$

19. (a) $12(2x+3)^2\sin(2x+3)^3\cos(2x+3)^3$ (b) $15s^4(3s-2)^4 + 4s^3(3s-2)^5$

(c) $\dfrac{\cos t - t\sin t}{(1-t)^2} + \dfrac{2t\cos t}{(1-t)^3}$ (d) $-\dfrac{8(3t+2)^4}{t^3(4t-1)^3} + \dfrac{12(3t+2)^3}{t^3(4t-1)^2} - \dfrac{3(3t+2)^4}{t^4(4t-1)^2}$

20. (a) $f'(x) = x^2\cos(x+\pi) + 2x\sin(x+\pi)$; $f'(\pi) = 9.8696$; $f'(\pi) \approx 9.8696$

(b) $f'(x) = 20x^4 - 15x^2$; $f'(1) = 5$; $f'(1) \approx 5.000$

(c) $g'(x) = 12x^2 - 4x - 30$; $g'(0) = -30$; $g'(0) \approx -30.000$

(d) $g'(x) = \dfrac{6x\cos x + 6\sin x}{(6x+1)^2} - \dfrac{72x\sin x}{(6x+1)^3}$; $g'(0) = 0$; $g'(0) \approx 0.000$