Chapter 1: Vectors & Matrices

Learning Objectives:

1. *Recognize vectors and perform vector arithmetic*
2. *Identify matrices and perform matrix arithmetic*
3. *Identify square, symmetric, diagonal, identity, and null matrices*
4. *Compute the transpose, inverse, trace, and determinant of matrices*

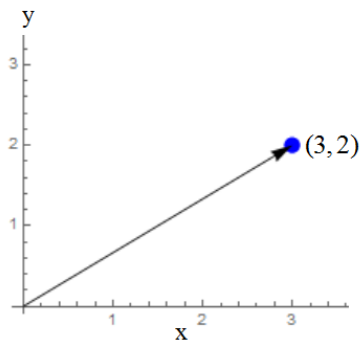
1.1 Vectors Review

A vector **v** in the plane is an ordered pair of real numbers which we can represent using a column matrix $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or a row matrix $\mathbf{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$.

A vector **v** has both a magnitude and direction. Geometrically **v** in $\mathbb{R}^2$ can be represented as a directed line segment in a rectangular coordinate system where the length is the **magnitude** (or **norm**) of the vector with an arrow indicating its direction. If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is positioned with its initial point at the origin of the coordinate system, then its terminal point will have coordinates (v_1, v_2).

Example 1.

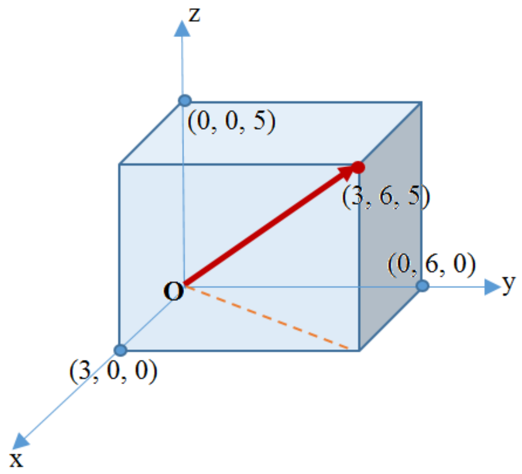
Represent the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in a plane.



The above can be extended to vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ or $\mathbf{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ in $\mathbb{R}^3$.

Example 2.

Represent the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}$ in 3-dimensional space.



Magnitude of a vector

The magnitude of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$. For a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in two dimensions, the magnitude is given by:

$$\boxed{\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2}}\,,}$$

and the magnitude of a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ in three dimensions is given by:

$$\boxed{\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}}$$

Unit vectors are vectors with a magnitude of one, that is, $\|\mathbf{v}\| = 1$.

Example 3.

Let $\mathbf{v} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$. Find $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$.

1.2 Introduction to Matrices

Matrices originated in mathematics more than a century ago, but its adaptation to statistics and computing occurred fairly recently in the mid-20th century, considerably due to the advent of technology. Multivariate data can be conveniently displayed as an array of numbers, thus the study of multivariate methods is greatly facilitated by the use of matrix algebra. The purpose of this chapter is to introduce, in a simplified manner, the fundamentals of matrix algebra that are useful in data analysis.

Definition

A **matrix** is a rectangular array of numbers, arranged in rows and columns.
Typically, uppercase letters in bold are used to denote a matrix, e.g. **A**, **C**, **R**.
We denote a matrix of n **rows** and p **columns** by:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2p} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{np} \end{bmatrix}$$

Either square or round brackets can be used to encase the array of numbers.
The **size** or **dimension** or **order** of the matrix is $n \times p$.
The size can be emphasized by writing in subscript, that is, $\mathbf{A}_{n\times p}$.
The entries in a matrix are called **elements**. Elements are typically denoted by lowercase letters, thus the element in the i^{th} row, j^{th} column of matrix **A** is denoted by a_{ij}.
Diagonal elements are of the form a_{ii}.

A matrix that has equal number of rows and columns is called a **square** matrix.
Try writing down a square matrix of order 2.

A matrix that has only one column is called a **column vector**, or simply, **vector**. It is typically denoted by lowercase letter in bold, e.g. **x**, or written as $\underset{\sim}{x}$.
In the same way, a matrix that has only one row is called a **row vector**.
Try writing down a column vector of order 3.

A single number such as 5, 1.8 or −13.2 is called a **scalar**. The elements of a matrix are usually scalars. Sometimes, a scalar can be thought of as a 1×1 matrix.

In statistics, data can be seen as a matrix. For example, the following data collected in a spreadsheet:

	A	B	C
1	Age	Salary($)	Years of Employment
2	22	2512.81	1.25
3	35	4580.10	10.5
4	41	4287.67	5
5	50	5985.50	18

can be written as the matrix: $\mathbf{X} =$

where number of rows = number of data points, i.e. $n =$ ___
number of columns = number of variables, i.e. $p =$ ___

Example 4.

Matrices **A**, **B**, **C**, **D** and **E** are given as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 3 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -5 \end{bmatrix}$$

(a) State the dimensions of the matrices.

(b) What are the values of a_{22}, b_{12}, c_{32} and e_{13}?

(c) Identify a square matrix, a column vector, a row vector and a scalar.

Example 5.

Write down the matrix **A** given that $\mathbf{A} = \left(a_{ij}\right)$ such that $a_{ij} = \left(-1\right)^{i+j} j$, for $i = 1, 2$ and $j = 1, 2, 3$.

1.3 Determinants

The **determinant** of a matrix is a scalar property of the matrix. In matrix algebra, it is the basis of many further calculations and procedures.

Note that determinant only exists for square matrices. It is typically denoted by 2 vertical lines, e.g. $|\mathbf{A}|$, or det(**A**).

The **determinant** of a **2×2** matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

This is the product of the diagonal elements subtracting the product of the non-diagonal elements.

Example 6.

Evaluate the determinant of the matrices $\mathbf{A}=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}6 & -4\\ 5 & 1\end{bmatrix}$.

The **determinant** of a **3×3** matrix $\mathbf{A}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ can be evaluated using a method called **Laplace expansion** (refer to Appendix 1). Similarly for determinants of higher order.

Finding a determinant, especially one of higher order, can be a tedious task, which is best left to a tool such as NumPy. Below we make use of NumPy to find the determinant of $\mathbf{A}=\begin{pmatrix}1 & 2 & -1\\ 3 & 5 & 0\\ -2 & -1 & 2\end{pmatrix}$.

```
import numpy as np
A = np.array([[1,2,-1],[3,5,0],[-2,-1,2]])
print(A)
detA = np.linalg.det(A)
print(detA)
```

```
[[ 1  2 -1]
 [ 3  5  0]
 [-2 -1  2]]
-9.000000000000002
```

1.4 Basic Matrix Operations

1.4.1 Transpose

The **transpose** operation of the $n\times p$ matrix **A** changes the columns into rows. The transposed matrix is denoted by $\mathbf{A}'$ or $\mathbf{A}^{\mathrm{T}}$, and has dimensions $p\times n$.

Example 7.

(a) If $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$, then $\mathbf{A}^T =$

(b) Transposing the column vector $\mathbf{x}=\begin{pmatrix}3\\ 0\\ -1\end{pmatrix}$ gives the row vector

Property: The transpose of a transposed matrix is the matrix itself. That is, $\left(\mathbf{A}^T\right)^T=\mathbf{A}$.

1.4.2 Trace

The **trace** of a square matrix is the sum of all its diagonal elements. Trace is not defined for a matrix that is not square.
So, if **A** is a square matrix of order n, then tr(**A**) = $a_{11} + a_{22} + \ldots + a_{nn}$.

Example 8.

(a) If $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$, then tr(**A**) =

(b) If $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, then tr(**B**)

Property: The trace of a transposed matrix is the same as the trace of the matrix itself.
That is, $\text{tr}(\mathbf{A}^T) = \text{tr}(\mathbf{A})$.

1.4.3 Equality

Two matrices are **equal** if and only if they have the same dimensions and the same corresponding elements.

Example 9.

(a) Given $\begin{pmatrix} 1 & p & 3 \\ -1 & q & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ r & 0 & r+s \end{pmatrix}$, what are the values of p, q, r and s?

(b) If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, then x_1 =_____, x_2 = _____ and x_3 = _____.

1.4.4 Addition & Subtraction

Suppose **A** and **B** are two matrices of the same dimensions.

- The sum, **A** + **B**, is obtained by adding the corresponding elements of the matrices.
- The difference, **A** – **B**, is obtained by subtracting the corresponding elements of the matrices.

Example 10.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 1 & 0 \\ 4 & 2 & -1 \end{pmatrix} =$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -3 & 1 & 0 \\ 4 & 2 & -1 \end{pmatrix} =$

1.4.5 Scalar Multiplication

Scalar multiplication of a matrix is when the matrix is multiplied by a scalar. In such operation, every element in the matrix will be multiplied by the scalar.

Example 11.

(a) $5\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} =$

(b) Find the values of x and y if $3\begin{pmatrix} x & 0 \\ 2 & 2y \end{pmatrix} - 4\begin{pmatrix} 3 & -1 \\ 1 & y \end{pmatrix} = 2\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

1.4.6 Matrix Multiplication

Before we take the product of matrices, let's first take the product of vectors.

The **inner (or dot) product** of vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}$ is defined as:

$$\mathbf{a}^T \mathbf{b} = \begin{pmatrix} a_1 & a_2 & \ldots & a_p \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} = a_1 b_1 + a_2 b_2 + \ldots + a_p b_p$$

Thus, inner product of two vectors exists only if they are of the same order, and the result is a scalar.

Example 12.

A research lab (Lab Alpha) in a university needs to buy supplies of rats, mice, and rabbits for experimental studies. The price per animal (from Supplier 1) of rats, mice and rabbits is \$2, \$1 and \$10 respectively, and the lab needs 30, 80 and 25 animals respectively.

(a) Write down **p**, the vector of price per animal.

(b) Write down **q**, the vector of quantity of animals needed.

(c) Hence, find the total cost of supplies using inner product of vectors.

Next, let's take product of a matrix $\mathbf{A}_{n\times p} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2p} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{np} \end{pmatrix}$ and a vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_p \end{pmatrix}$.

Steps:

1. Partition matrix **A** into its n rows.
2. Take the inner product of each row in **A** with vector **b**, each resulting in a scalar.
3. All resulting n scalars form a vector of order n.

$$\mathbf{Ab} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2p} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{np} \end{pmatrix}\mathbf{b} = \begin{pmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \boldsymbol{\alpha}_3^T \\ \vdots \\ \boldsymbol{\alpha}_n^T \end{pmatrix}\mathbf{b} = \begin{pmatrix} \boldsymbol{\alpha}_1^T\mathbf{b} \\ \boldsymbol{\alpha}_2^T\mathbf{b} \\ \boldsymbol{\alpha}_3^T\mathbf{b} \\ \vdots \\ \boldsymbol{\alpha}_n^T\mathbf{b} \end{pmatrix}$$

Here, $\boldsymbol{\alpha}_1^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} \end{bmatrix}$, $\boldsymbol{\alpha}_2^T = \begin{bmatrix} a_{21} & a_{22} & a_{23} & \cdots & a_{2p} \end{bmatrix}$, $\boldsymbol{\alpha}_3^T = \begin{bmatrix} a_{31} & a_{32} & a_{33} & \cdots & a_{3p} \end{bmatrix}$, … and $\boldsymbol{\alpha}_n^T = \begin{bmatrix} a_{n1} & a_{n2} & a_{n3} & \cdots & a_{np} \end{bmatrix}$ are all row vectors.

Observe that:

- For each inner product to exist, the order of the row vectors and vector **b** must be the same. Hence, the number of columns in matrix **A** must be the same as the order of vector **b**.
- When a matrix of order $n\times p$ is multiplied to a vector of order p, resultant vector is order n.

Example 13.

Extending from Example 12: Another supplier (supplier B) sells rats, mice and rabbits at \$2, \$2 and \$8 each animal, respectively. Write down **P**, the matrix of price per animal for suppliers A and B. Hence, find the vector of total cost of buying the supplies from both suppliers A and B by taking product of matrix **P** and vector **q**.

Finally, let's take product of matrices:

$$\mathbf{A}_{n\times p} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2p} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{np} \end{pmatrix} \text{ and } \mathbf{B}_{p\times m} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & b_{p3} & \cdots & b_{pm} \end{pmatrix}$$

Steps:

1. Partition matrix **A** into its n rows.
2. Partition matrix **B** into its m columns.
3. Take the inner product of each row in **A** with each column in **B**, each resulting in a scalar.
4. All resulting scalars form a matrix of order $n \times m$.

$$\mathbf{AB} = \begin{pmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \boldsymbol{\alpha}_3^T \\ \vdots \\ \boldsymbol{\alpha}_n^T \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \cdots & \mathbf{b}_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_1^T\mathbf{b}_1 & \boldsymbol{\alpha}_1^T\mathbf{b}_2 & \boldsymbol{\alpha}_1^T\mathbf{b}_3 & \cdots & \boldsymbol{\alpha}_1^T\mathbf{b}_m \\ \boldsymbol{\alpha}_2^T\mathbf{b}_1 & \boldsymbol{\alpha}_2^T\mathbf{b}_2 & \boldsymbol{\alpha}_2^T\mathbf{b}_3 & \cdots & \boldsymbol{\alpha}_2^T\mathbf{b}_m \\ \boldsymbol{\alpha}_3^T\mathbf{b}_1 & \boldsymbol{\alpha}_3^T\mathbf{b}_2 & \boldsymbol{\alpha}_3^T\mathbf{b}_3 & \cdots & \boldsymbol{\alpha}_3^T\mathbf{b}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\alpha}_n^T\mathbf{b}_1 & \boldsymbol{\alpha}_n^T\mathbf{b}_2 & \boldsymbol{\alpha}_n^T\mathbf{b}_3 & \cdots & \boldsymbol{\alpha}_n^T\mathbf{b}_m \end{pmatrix}$$

Observe that:

- For each inner product to exist, the order of the row vectors in **A** and column vectors in **B** must be the same. Hence, the number of columns in matrix **A** must be the same as the number of rows in matrix **B**.
- When a matrix of order $n \times p$ is multiplied to a matrix of order $p \times m$, the resultant matrix is order $n \times m$.
- The $(ij)^{\text{th}}$ element in the matrix product is simply the inner product of the i^{th} row of the first matrix and the j^{th} column of the second matrix.

Consequently, given two matrices **A** and **B**, the matrix product **AB** is **defined** provided the number of columns of **A** is equal to the number of rows of **B**. We can also say that the matrices **A** and **B** **conformable for multiplication** if the matrix product **AB** is defined.

In general, matrix multiplication is *not* commutative, that is, $\mathbf{AB} \neq \mathbf{BA}$.
Furthermore, assuming matrices **A**, **B** and **C** are all conformable for addition and multiplication, the following properties hold true:

- Matrix multiplication is *not* commutative, that is, $\mathbf{AB} \neq \mathbf{BA}$.
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- If k is a scalar, then $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$
- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Example 14.

Extending from Examples 12 & 13: Another 3 research labs in the university are also in need of experimental animals, hence the supplies for purchase is consolidated. Lab Beta needs 60 rats, 80 mice and 40 rabbits; Lab Chi needs 90 rats, 30 mice and 20 rabbits; Lab Delta needs 30 rats, 20 mice and 10 rabbits. Write down **Q**, the matrix of quantity of animals needed. Hence, find the matrix of total cost of supplies by taking product of matrices **P** and **Q**, if the product is defined.

Example 15.

Let $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$.

Check if the following matrix products are defined. If so, find the product.

(a) $\mathbf{Cd}$ (b) $\mathbf{AB}$ (c) $\mathbf{BA}$ (d) $\mathbf{CA}$ (e) $\mathbf{CB}$

(f) $\mathbf{d}^T\mathbf{d}$ (g) $\mathbf{A}^T\mathbf{A}$ (h) $\mathbf{B}^2$ (i) $\mathbf{C}^2$ (j) $\mathbf{BA}^T\mathbf{d}$

Many matrix operations can be implemented in Python using the NumPy module. Some examples are given below.

Define matrices **A**, **B**, **C**, and **d**:

```
import numpy as np
A = np.array([[2,1],[4,5],[3,6]])
B = np.array([[1,-2],[3,2]])
C = np.array([[1,3,4],[2,0,1]])
d = np.array([[-1],[3],[0]])
print(A)
print(B)
print(C)
print(d)
```

```
[[2 1]
 [4 5]
 [3 6]]
[[ 1 -2]
 [ 3  2]]
[[1 3 4]
 [2 0 1]]
[[-1]
 [ 3]
 [ 0]]
```

Find the products **Cd** in part (a) and **AB** in part (b):

```
part_a = C.dot(d)
print(part_a)
```

```
[[ 8]
 [-2]]
```

```
part_b = A.dot(B)
print(part_b)
```

```
[[ 5 -2]
 [19  2]
 [21  6]]
```

The transpose of **d** can be found using **d**.T. Hence, $\mathbf{d}^T\mathbf{d}$ in part (f) can be found by:

```
part_f = d.T.dot(d)
print(part_f)
```

```
[[10]]
```

In part (i), an error message results. Why?

```
part_h = B.dot(B)
print(part_h)
```

```
[[-5 -6]
 [ 9 -2]]
```

```
part_i = C.dot(C)
print(C)
```

```
---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
<ipython-input-14-d27e708c88f4> in <module>()
----> 1 part_i = C.dot(C)
      2 print(C)

ValueError: shapes (2,3) and (2,3) not aligned: 3 (dim 1) != 2 (dim 0)
```

The product of three matrices in part (j):

```
part_j = B.dot(A.T).dot(d)
print(part_j)
```

```
[[-18]
 [ 58]]
```

1.5 Special Matrices

There are several common types of matrices that frequently appear in applications. These special types of matrices are described below.

1.5.1 Zero or Null Matrix

A **zero matrix** or **null matrix** is a matrix where every element is zero.
That is, $a_{ij} = 0$ for all values of i and j.
Similarly, a **zero vector** or **null vector** is a vector where every element is zero.
Assuming that the matrices are conformable for multiplication: $\mathbf{A0} = \mathbf{0}$ and $\mathbf{0A} = \mathbf{0}$.

Example 16.

Write down zero matrices (or vectors) of the following dimensions:

(a) 2×2 (b) 3×2 (c) 3×1

1.5.2 Symmetric Matrix

A square matrix is a **symmetric matrix** if its transpose is equal to itself, that is, $\mathbf{A}^T = \mathbf{A}$.
That is, $a_{ij} = a_{ji}$ for all values of i and j.

Example 17.

Which of the following matrices are symmetric?

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \\ 4 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & -2 & 5 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix}$$

1.5.3 Diagonal Matrix

A square matrix is a **diagonal matrix** if all its non-diagonal elements are zero.
That is, $a_{ij} = 0$ if $i \neq j$.

Example 18.

Which of the following matrices are diagonal?

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

1.5.4 Identity Matrix

A diagonal matrix is an **identity matrix** if all its diagonal elements are 1.
That is, $a_{ij} = 0$ if $i \neq j$, and $a_{ij} = 1$ if $i = j$.
The n^{th} order identity matrix is usually denoted by $\mathbf{I}_n$.
Assuming that the matrices are conformable for multiplication: **AI** = **A** and **IA** = **A**.

Example 19.

Write down identity matrices $\mathbf{I}_2$ and $\mathbf{I}_3$.

1.5.5 Singular & Non-Singular Matrices

A square matrix is a **singular matrix** if its determinant is zero.
On the contrary, a square matrix is a **nonsingular matrix** if its determinant is not zero.

Example 20.

Which of the following matrices are singular?

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

1.5.6 Summing Vector

A **summing vector** is a vector in which all elements are 1.
The n^{th} order summing vector is usually denoted by $\mathbf{1}_n$.
It is called a summing vector because it can be used to express a sum of numbers in matrix notation as an inner product.

Example 21.

Given $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \end{pmatrix}$, find $\mathbf{1}_3^T \mathbf{a}$, $\mathbf{a}^T \mathbf{1}_3$, $\mathbf{1}_2^T \mathbf{B}$ and $\mathbf{B}^T \mathbf{1}_2$.

Example 22.

(a) Given that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & x+1 \\ 0 & 0 & 0 \end{bmatrix}$ is a **diagonal** matrix, what is the value of x?

(b) Find the value of k such that the matrix $\begin{bmatrix} 0 & 1 & k \\ 1 & 2 & 4 \\ 2k+1 & 4 & 3 \end{bmatrix}$ is a **symmetric** matrix.

(c) Find the value of m if $\begin{pmatrix} 2 & -6 \\ -4 & 3m \end{pmatrix}$ is a **singular** matrix.

1.6 The Inverse of a Matrix

The **inverse** of an $n \times n$ matrix **A** is the $n \times n$ matrix $\mathbf{A}^{-1}$ such that:

$$\boxed{\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n}$$

The inverse of **A** exists if and only if it is nonsingular, that is, $|\mathbf{A}| \neq 0$.
If $\mathbf{A}^{-1}$ exists, then **A** is said to be an **invertible** matrix.

Inverse of a 2×2 nonsingular matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is:

$$\boxed{\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$$

Inverse of a 3×3 nonsingular matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} |\mathbf{M}_{11}| & -|\mathbf{M}_{12}| & |\mathbf{M}_{13}| \\ -|\mathbf{M}_{21}| & |\mathbf{M}_{22}| & -|\mathbf{M}_{23}| \\ |\mathbf{M}_{31}| & -|\mathbf{M}_{32}| & |\mathbf{M}_{33}| \end{pmatrix}^{\mathrm{T}}$$

Recall that sub-matrix $\mathbf{M}_{ij}$ is obtained by deleting the i^{th} row, j^{th} column.

Some important properties of inverse matrix:
Let **A** and **B** be invertible matrices. Let k be a non-zero scalar and m be a positive integer.

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T}$
- $(\mathbf{A}^{-1})^{m} = (\mathbf{A}^{m})^{-1}$

Example 23.

Find the inverse of each of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Example 24.

Use NumPy to find the inverse of the matrices $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 7 & 1 \\ 2 & 5 & 0 \\ 4 & 0 & 8 \end{bmatrix}$.

References
Applied Multivariate Statistical Analysis (6th ed.) by Richard A. Johnson & Dean W. Wichern.
Matrix Algebra Useful for Statistics (2nd. Ed.) by Shayle R. Searle & Andre l. Khuri.

Tutorial 1

1. Write down the matrix **A** given that $\mathbf{A} = \left[a_{ij}\right]$ such that:

   (a) $a_{ij} = i + j$, for i = 1, 2, 3, 4 and j = 1, 2

   (b) $a_{ij} = (-1)^{i+1}\,2^{j-1}$, for i = 1, 2, 3 and j = 1, 2, 3

2. Evaluate the following determinants:

   (a) $\begin{vmatrix} 5 & -2 \\ 6 & -2 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & 5 \\ -1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix}$

3. Evaluate:

   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ (b) $2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} + 3\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

4. Find the matrix **A** such that $\mathbf{A} + \begin{bmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \end{bmatrix} = 2\begin{bmatrix} -1 & 4 & 3 \\ -2 & 0 & 4 \end{bmatrix}$.

5. Find the values of a and b such that $3\begin{bmatrix} a & 2 \\ -1 & 2b \end{bmatrix} + 2\begin{bmatrix} a+b & 2 \\ 3 & a-b \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 6 \end{bmatrix}$.

6. (a) If **A** is a 2×3 matrix and the matrix product **AB** is a 2×4 matrix, find the order of the matrix **B**.

   (b) If **A** is a 2×3 matrix and **C** is a 4×4 matrix, find the order of the matrix **B** such that the matrix product $\mathbf{AB}^T\mathbf{C}$ is well defined.

7. Evaluate the following matrix products, where possible:

   (a) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 6 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}$ (c) $\begin{bmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & 6 & 4 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix}$ (e) $\begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 & 8 \end{bmatrix}$ (f) $\begin{bmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{bmatrix}$

8. Find the inverse of each of the following matrices, if the inverse exists.

   (a) $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -4 \\ -1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & -2 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ -2 & 2 & 5 \end{bmatrix}$

9. Given $\mathbf{A}=\begin{bmatrix} 5 & -14 & 2 \\ -10 & -5 & -10 \\ 10 & 2 & -11 \end{bmatrix}$, find the matrix product $\mathbf{AA}^T$. Hence, deduce the matrix inverse of **A.**

10. If a square matrix **A** is <u>orthogonal</u>, then $\mathbf{AA}^T=\mathbf{I}=\mathbf{A}^T\mathbf{A}$. That is, its transpose equals its inverse.

Is $\mathbf{Q}=\begin{pmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{pmatrix}$ an orthogonal matrix? Show your workings.

<u>Answers</u>

1. (a) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$

2. (a) 2 (b) 5 (c) 0 (d) 5

3. (a) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 7 & 12 \\ 7 & 8 & 3 \end{bmatrix}$

4. $\begin{bmatrix} -4 & 7 & 9 \\ -5 & -5 & 8 \end{bmatrix}$

5. $a=1, b=1$

6. (a) 3×4 (b) 4×3

7. (a) $\begin{bmatrix} 5 & 4 \\ 27 & 26 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 4 & 8 \\ 27 & 26 & 42 \end{bmatrix}$ (c) not conformable for multiplication

(d) $[96]$ (e) $\begin{bmatrix} 6 & 18 & 12 & 24 \\ 10 & 30 & 20 & 40 \\ 2 & 6 & 4 & 8 \\ 14 & 42 & 28 & 56 \end{bmatrix}$ (f) $\begin{bmatrix} 11 & -22 & 29 \\ 9 & -27 & 32 \\ 13 & -17 & 26 \end{bmatrix}$

8. (a) no inverse (b) $-\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

(c) $-\frac{1}{28}\begin{bmatrix} -4 & -16 & 8 \\ -5 & 1 & 3 \\ -2 & 6 & -10 \end{bmatrix}$ (d) $\frac{1}{11}\begin{bmatrix} 15 & 2 & 3 \\ 5 & -3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

9. $225\mathbf{I}_3$; $\mathbf{A}^{-1}=\frac{1}{225}\begin{bmatrix} 5 & -10 & 10 \\ -14 & -5 & 2 \\ 2 & -10 & -11 \end{bmatrix}$

10. Yes

Practical 1

Task 1

Enter the matrix $\mathbf{A}=\begin{bmatrix}1 & 2 & 4 & 5\\ 0 & 1 & 5 & -2\\ 2 & -4 & 3 & 9\end{bmatrix}$ in Python.

Display the following:

(a) the size of the matrix $\mathbf{A}$

(b) a_{23}, a_{32}, and a_{34}

(c) row 1 and column 4 of $\mathbf{A}$

Task 2

Enter the following matrices in Python.

$$\mathbf{A}=\begin{bmatrix}7 & -3\\ 2 & 1\\ 9 & -6\\ -3 & 2\end{bmatrix}, \mathbf{B}=\begin{bmatrix}2\\ 5\end{bmatrix}, \mathbf{C}=\begin{bmatrix}5 & 1 & -8 & 4\\ 2 & 7 & -3 & 5\end{bmatrix}$$

Compute each of the following expressions if possible.

(a) $\mathbf{AC}$ (b) $\mathbf{AB}$ (c) $\mathbf{A}+\mathbf{C}^T$

(d) $\mathbf{BA}$ (e) $\mathbf{AA}^T+\mathbf{C}^T\mathbf{C}$ (f) $\mathbf{A}+2\mathbf{C}^T$

Task 3

Determine if each of the following matrices is singular.

(a) $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$ (b) $\mathbf{A}=\begin{bmatrix}0 & 1 & 3 & 4\\ -2 & 1 & 2 & 1\\ 2 & 0 & 1 & 0\\ 0 & 8 & 3 & 1\end{bmatrix}$

Task 4

Find the inverse of each matrix.

(a) $\mathbf{A}=\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$ (b) $\mathbf{A}=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 2 & 1\end{bmatrix}$

Task 5

Determine the norm of each of the following vectors.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (b) $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 3 \\ -1 \end{bmatrix}$ (c) $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 3 \end{bmatrix}$

Answers

1.

(a)
```
(3, 4)
```
(b)
```
5
-4
9
```
(c)
```
[1 2 4 5]
[ 5 -2  9]
```

2.

(a)
```
[[ 29 -14 -47  13]
 [ 12   9 -19  13]
 [ 33 -33 -54   6]
 [-11  11  18  -2]]
```
(b)
```
[[ -1]
 [  9]
 [-12]
 [  4]]
```
(c)
```
[[12 -1]
 [ 3  8]
 [ 1 -9]
 [ 1  7]]
```
(d)
```
ValueError: shapes (2,1) and (4,2) not aligned: 1 (dim 1) != 4 (dim 0)
```
(e)
```
[[ 87  30  35   3]
 [ 30  55 -17  35]
 [ 35 -17 190 -86]
 [  3  35 -86  54]]
```
(f)
```
[[ 17   1]
 [  4  15]
 [ -7 -12]
 [  5  12]]
```

3.

(a)
```
det(A) is:  -9.51619735392994e-16 , which is zero. Hence A is singular.
```
(b)
```
det(A) is:  -126.0 , which is nonzero. Hence A is non-singular.
```

4.

(a)
```
[[ 0.71428571 -0.14285714]
 [-0.42857143  0.28571429]]
```
(b)
```
[[-0.25  0.75 -0.25]
 [-0.25 -0.25  0.75]
 [ 0.75 -0.25 -0.25]]
```

5.

```
2.449489742783178 5.0990195135927845 4.242640687119285
```

Appendix 1: Evaluating Determinants using Laplace Expansion

The **determinant** of a **3×3** matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ can be evaluated using Laplace Expansion.

The steps are as follow:

- Select any one row (or column).
- For each element a_{ij} in the selected row (or column):
  - Delete the i^{th} row, j^{th} column to get sub-matrix $\mathbf{M}_{ij}$.
  - Find the product $a_{ij}(-1)^{i+j}\left|\mathbf{M}_{ij}\right|$.
- Sum up the products.

For example, to find a 3rd order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, we can expand along row 1:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left((-1)^{1+1}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\mathbf{M}_{11}}\right) + a_{12}\left((-1)^{1+2}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\mathbf{M}_{12}}\right) + a_{13}\left((-1)^{1+3}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\mathbf{M}_{13}}\right)$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Use Laplace expansion to evaluate the determinant of the matrices $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \\ -2 & -1 & 2 \end{pmatrix}$ and

$\mathbf{B} = \begin{pmatrix} 4 & 6 & -8 \\ 2 & 5 & -1 \\ 1 & -3 & 2 \end{pmatrix}$.

Using first row: $\left|\mathbf{A}\right| = 1\begin{vmatrix} 5 & 0 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 0 \\ -2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 5 \\ -2 & -1 \end{vmatrix} = 10 - 12 - 7 = -9$

Using second column: $\left|\mathbf{B}\right| = -6\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + 5\begin{vmatrix} 4 & -8 \\ 1 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 4 & -8 \\ 2 & -1 \end{vmatrix} = -30 + 80 + 36 = 86$