$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right] \xrightarrow[R_3+R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_3]{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x + y - z = -2$

$y - z = -3$

$x - 3 = -2$

$\boxed{x = 1}$

$\boxed{z = 2}$

$y - z = -3$

$y = -3 + z$

$y = -3 + 2$

$\boxed{y = -1}$

$\therefore x = 1,\ y = -1,\ z = 2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & 10 \\ 2 & -1 & 2 & 6 \end{array}\right] \xrightarrow[R_3-R_1]{} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & 10 \\ 0 & -2 & 3 & 5 \end{array}\right] \xrightarrow{-3R_1+2R_2} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & 5 & 17 \\ 0 & -2 & 3 & 5 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 13 & 39 \end{array}\right] \xrightarrow[\frac{1}{13}R_3]{} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

●: turn into 0

subtract/add

●: turn into 1

divide

$2x + y - z = 1$

$2x = 1 - y + z$

$2x = 1 - 2 + 3$

$2x = 2$

$x = 1$

$z = 3$

$y + 5z = 17$

$y + 15 = 17$

$y = 2$

$\therefore x, y, z = 1, 2, 3$

Gaussian elimination

special matrices

- zero or null matrices
- symmetric matrix
- diagonal matrix
- identity matrix
- singular & non singular matrix
  ↳ $|A| = 0$ ↳ $|A| \neq 0$
- summing vector
  ↳ vector in which all elements are 1

inverse

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\begin{bmatrix} |M_{11}| & -|M_{12}| & |M_{13}| \\ -|M_{21}| & |M_{22}| & -|M_{23}| \\ |M_{31}| & -|M_{32}| & |M_{33}| \end{bmatrix}^T}$$

invertible matrix

↳ $|A| \neq 0$ (non-singular)

properties of inverse matrix

- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^{-1})^m = (A^m)^{-1}$

Laplace expansion

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left((-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right) + a_{12}\left((-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right)$$

$$+ a_{13}\left((-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* can use any row.
  in this case, we're
  using row 1.
  ↳ if possible, choose
  a row w/ a '0' element

1. a. $a_{ij} = i+j$ for $i = 1,2,3,4$ & $j = 1,2$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{bmatrix}$$

(b) $a_{ij} = (-1)^{i+1} 2^{j-1}$ for $i = 1,2,3$, $j = 1,2,3$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{bmatrix}$$

2. a. $\begin{vmatrix} 5 & -2 \\ 6 & -2 \end{vmatrix} = -10 + 12$
$= 2$

b. $\begin{vmatrix} 3 & 5 \\ -1 & 0 \end{vmatrix} = 5$

c. $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1 \left((-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right)$
$= -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$
$= -(1-1)$
$= 0$

d. $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$
$= 2 \cdot (-1) - 3 \cdot 1$
$= -5$

3. a. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$
$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 6 \end{bmatrix} + \begin{bmatrix} -3 & 3 & 6 \\ 3 & 6 & -3 \end{bmatrix}$
$= \begin{bmatrix} -1 & 7 & 12 \\ 7 & 8 & 3 \end{bmatrix}$

4. $A = 2\begin{bmatrix} -1 & 4 & 3 \\ -2 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \end{bmatrix}$

$A = \begin{bmatrix} -4 & 7 & 9 \\ -5 & -5 & 8 \end{bmatrix}$

5. $\begin{bmatrix} 3a & 6 \\ -3 & 6b \end{bmatrix} + \begin{bmatrix} 2a+2b & 4 \\ 6 & 2a-2b \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 6 \end{bmatrix}$

15+12
12+14
24+18
42

$5a+2b=7$
$2a+4b=6 \rightarrow a+2b=3$

$\left[\begin{array}{cc|c} 5 & 2 & 7 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{R_1-5R_2} \left[\begin{array}{cc|c} 5 & 2 & 7 \\ 0 & -8 & 8 \end{array}\right] \xrightarrow{-1/8 R_2} \left[\begin{array}{cc|c} 5 & 2 & 7 \\ 0 & 1 & 1 \end{array}\right]$

$\boxed{b=1}$

$5a+2b=7$
$5a=7-2b$
$5a=7-2$
$\boxed{a=1}$

$\therefore a=1$
$b=1$

6. (a) $A=2\times3$
$B=?$
$AB=2\times4$

$\boxed{B=3\times4}$

A B
$2\times3\ 3\times4$

(b) $A:2\times3$
$C=4\times4$
$B=?$

$AB'C$
valid

A B' C
2×3 3×4 4×4

$\boxed{B=4\times3}$

7a) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 6 & 7 \end{pmatrix}$

$\begin{pmatrix} 5 & 4 \\ 27 & 26 \end{pmatrix}$

7b) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}$

$\begin{pmatrix} 5 & 4 & 8 \\ 27 & 26 & 42 \end{pmatrix}$

7c) $\begin{pmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ X

7d) $\begin{bmatrix} 2 & 6 & 4 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix}$

$6+30+4+56$
96

7e) $\begin{bmatrix}3\\5\\1\\7\end{bmatrix}\begin{bmatrix}2&6&4&8\end{bmatrix}$

4x1 1x4

$$\begin{bmatrix}6&18&12&24\\10&30&20&40\\2&6&4&8\\14&42&28&56\end{bmatrix}$$

7f) $\begin{bmatrix}5&8&-4\\6&9&-5\\4&7&-3\end{bmatrix}\begin{bmatrix}3&2&5\\4&-1&3\\9&6&5\end{bmatrix}$

$$\begin{bmatrix}11&-22&29\\9&-27&82\\13&-17&26\end{bmatrix}$$

8a) $\begin{vmatrix}2&6\\-1&-3\end{vmatrix} = -6+6=0$

not invertible

8b) $\begin{vmatrix}1&-4\\-1&2\end{vmatrix} = 2-4=-2$

$B^{-1} = -\frac{1}{2}\begin{bmatrix}2&4\\1&1\end{bmatrix}$

8c) $\begin{vmatrix}1&4&2\\2&-2&1\\1&-2&3\end{vmatrix} = -2\begin{vmatrix}4&2\\-2&3\end{vmatrix} + (-2)\begin{vmatrix}1&2\\1&3\end{vmatrix} - 1\begin{vmatrix}1&4\\1&-2\end{vmatrix}$

$i+j = 2+1$
$= 3$

$= -2 \cdot 16 - 2 \cdot 1 - 1 \cdot -6$

$= -32 - 2 + 6$

$= -28$

$$C^{-1} = -\frac{1}{28}\begin{bmatrix}-4&-(5)&-2\\-(16)&1&-(-6)\\8&-(-3)&-10\end{bmatrix}^T$$

$$= -\frac{1}{28}\begin{bmatrix}-4&-5&-2\\-16&1&6\\8&3&-10\end{bmatrix}$$

$$C^{-1} = -\frac{1}{28}\begin{bmatrix}-4&-16&8\\-5&1&3\\-2&6&-10\end{bmatrix}$$

8d) $\begin{vmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ -2 & 2 & 5 \end{vmatrix} = 1 \begin{vmatrix} -3 & 0 \\ 2 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix}$ $100 + 4 + 121$

$= -15 + 4$

$= -11$

$D^{-1} = -\frac{1}{11} \begin{bmatrix} -15 & -(5) & -4 \\ -(2) & 3 & -(2) \\ -3 & -(1) & -3 \end{bmatrix}^T$

$= -\frac{1}{11} \begin{bmatrix} -15 & -5 & -4 \\ -2 & 3 & -2 \\ -3 & -1 & -3 \end{bmatrix}^T$

$= \frac{1}{11} \begin{bmatrix} 15 & 2 & 3 \\ 5 & -3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

9) $\begin{bmatrix} 5 & -14 & 2 \\ -10 & -5 & -10 \\ 10 & 2 & -11 \end{bmatrix} \begin{bmatrix} 5 & -10 & 10 \\ -14 & -5 & 2 \\ 2 & -10 & -11 \end{bmatrix}$

$= \begin{bmatrix} 225 & 0 & 0 \\ 0 & 225 & 0 \\ 0 & 0 & 225 \end{bmatrix}$

$= 225 I_3$

$AA^T = 225 I_3$

$AA^T = 225 AA^{-1}$

$\frac{1}{225} AA^T = AA^{-1}$

$\frac{1}{225} \cancel{A^{-1}A} A^T = \cancel{A^{-1}A} A^{-1}$

$\frac{1}{225} A^T = A^{-1}$

$A^{-1} = \frac{1}{225} \begin{bmatrix} 5 & -10 & 10 \\ -14 & -5 & 2 \\ 2 & -10 & -11 \end{bmatrix}$

10. orthogonal matrix $\longrightarrow$ $\boxed{AA^T = I = A^TA}$

$\boxed{A^T = A^{-1}}$

is $Q = \begin{bmatrix} 5/13 & 12/13 \\ -12/13 & 5/13 \end{bmatrix}$ orthogonal?

$|Q| = \frac{25}{169} + \frac{144}{169}$

$= \frac{169}{169}$

$= 1$

$Q^{-1} = \frac{1}{|Q|}\begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$

$Q^{-1} = \begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$

$Q^T = \begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$

$\boxed{Q^{-1} = Q^T}$

Q is an orthogonal matrix.

System of linear equations

- augmented matrix

$$\begin{aligned} x - 3y + 4z &= -4 \\ 3x - 7y + 7z &= -8 \\ -4x + 6y - z &= 7 \end{aligned} \rightarrow \begin{bmatrix} 1 & -3 & 4 \\ 3 & -7 & 7 \\ -4 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ 7 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array}\right]$$

(linear eqns) (matrix form)

- no. of solutions

$Ax = b$
linear system

- $|A| \neq 0$ non-singular: unique or one solution
- $|A| = 0$ singular:
  - ∞ly many solutions or
  - no solution

$Ax = 0$
homogeneous linear system

- $|A| \neq 0$ non-singular: unique or one sol. $x = 0$ → trivial solution $x_1 = x_2 = \ldots = x_p = 0$
- $|A| = 0$ singular: ∞ly many solutions

the sol. set for HLS always contains the trivial solution

consistent linear system: one or ∞ many sols.

inconsistent ——— " ———: no solution

proof that HLS always contains the trivial sol.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1p} x_p &= 0 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2p} x_p &= 0 \\ &\vdots \\ a_{n1} x_1 + a_{n2} x_2 + \dots + a_{np} x_p &= 0 \end{aligned}$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$x_1 = x_2 = \dots = x_p = 0$ always satisfy the n eqns above

Reduced Row Echelon.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{array}\right] \xrightarrow[2R_1-R_3]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -3 & 5 \end{array}\right] \xrightarrow[R_2-2R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 3 & -6 \end{array}\right] \xrightarrow[\frac{1}{3}R_3]{}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_2+3R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

//

Tutorial 2

1) b, c, d, e

2a) $\left[\begin{array}{cc|c} 3 & -1 & 2 \\ -6 & 2 & -4 \end{array}\right]$ $|A| = 6-6=0$

$\xrightarrow{2R_1+R_2} \left[\begin{array}{cc|c} 3 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

let $y = t$

$3x - y = 2$

$3x = 2 + t$

$x = \dfrac{2+t}{3}$

2b) $|B| = 6-6=0$

$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ -2 & 3 & 2 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} 2 & -3 & 5 \\ 0 & 0 & 7 \end{array}\right]$

$0 \neq 7$

↳ no sol.

$|C| = -4$

2c) $\left[\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ 0 & 0 & 4 & 4 \end{array}\right]$

$\xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 4 & 4 \end{array}\right] \xrightarrow{1/4 R_3} \left[\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$z = 1$

$y - 8z = -17$

$y = -17 + 8$

$y = -9$

$3x - 2y + z = -2$

$3x = -2 + 2y - z$

$3x = -2 - 18 - 1$

$x = -7$

2d) $|D| = 0$

$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 2 & -1 & 2 & 5 \\ 1 & 1 & -2 & 7 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 2 & -1 & 2 & 5 \\ 0 & 1 & -2 & 3 \end{array}\right]$

$\xrightarrow{2R_1-R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 5 & -10 & 15 \\ 0 & 1 & -2 & 3 \end{array}\right] \xrightarrow{1/5 R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

let $z = t$

$y - 2z = 3$

$y = 3 + 2t$

$x + 2y - 4z = 10$

$x = 10 - 2y + 4z$

$x = 10 - 6 - \cancel{4t} + \cancel{4t}$

$x = 4$

Tutorial 2

2e) $\left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow{3R_1-2R_3} \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 0 & -13 & -8 \end{array}\right] \xrightarrow{13R_2+R_3} \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 5 \end{array}\right]$

$0 \neq 5 \rightarrow$ no sol.

2f) $\left[\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right] \xrightarrow{2R_1+5R_2} \left[\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ 0 & 1 & 27 & 5 \end{array}\right] \xrightarrow{2R_2+R_1}$

$\left[\begin{array}{ccc|c} 5 & 0 & 60 & 10 \\ 0 & 1 & 27 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array}\right]$

$\begin{array}{r} {}^{2}15 \\ \times\ 5 \\ \hline 75 \end{array}$

let $z = t$

$y + 27t = 5$

$y = 5 - 27t$

$x + 12z = 2$

$x = 2 - 12t$

2g) $\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{5R_1-3R_2} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$

2g) $\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \xrightarrow{2R_1 - R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - R_3]{5R_1 - 3R_2} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-1/7 R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = 7$

$y - 11z = -75$

$y = -75 + 77$

$y = 2$

$3x + 2y - z = -15$

$3x = -15 - 2y + z$

$3x = -15 - 4 + 7$

$3x = -12$

$x = -4$

2h) $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$

$\xrightarrow{5R_1 - 3R_2} \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & 8 & 2 & 8 & 0 \end{array}\right] \xrightarrow{1/8 R_2} \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1/4 & 1 & 0 \end{array}\right]$

Let $\boxed{x_3 = s}$ $\boxed{x_4 = t}$

$x_2 + 0.25s + t = 0$

$\boxed{x_2 = -0.25s - t}$

$3x_1 = -x_2 - x_3 - x_4$

$3x_1 = 0.25s + \cancel{t} - s - \cancel{t}$

$3x_1 = -\frac{3}{4}s$

$\boxed{x_1 = -\frac{1}{4}s}$

3. consistent

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right] \xrightarrow[R_1-R_2]{2R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 1 & 1 & 2a-c \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{array}\right]$$

$c = a+b$

can be no sol

or ∞ly many sol → consistent

$\therefore c-a-b=0$

$\boxed{c=a+b}$

4. $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & -2 & k \end{array}\right]$

no sol. → k=?

1 sol → k=?

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & -2 & k \end{array}\right] \xrightarrow{2R_1-R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 6-k \end{array}\right]$$

no sol: $6-k \neq 0$

$k \neq 6$

1 sol: impossible

∞ # of sol: $6-k=0$

$k=6$

5) $A = \begin{bmatrix} -1 & -4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}$ find nontrivial solution to $Ax=0$

$$\left[\begin{array}{cc|c} -1 & -4 & 0 \\ 3 & 12 & 0 \\ 2 & 8 & 0 \end{array}\right] \xrightarrow[2R_1+R_3]{3R_1+R_2} \left[\begin{array}{cc|c} -1 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

let $y = t$

$-x-4y=0$

$x=-4t$

$x = \begin{bmatrix} -4t \\ t \end{bmatrix}$

$x = t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

a particular non trivial sol is $x_1=-4$, $x_2=1$

6) $2x + 3y + 4z = 250$ $\quad x = 2y$

$4y + 3y + 4z = 250$

$7y + 4z = 250$

$x + y + z = 100$

$2y + y + z = 100$

$3y + z = 100$

$$\left[\begin{array}{cc|c} 7 & 4 & 250 \\ 3 & 1 & 100 \end{array}\right] \xrightarrow{3R_1 - 7R_2} \left[\begin{array}{cc|c} 7 & 4 & 250 \\ 0 & 5 & 50 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{cc|c} 7 & 4 & 250 \\ 0 & 1 & 10 \end{array}\right]$$

$z = \underline{\underline{10}}$ $\quad 7y + 4z = 250$

$7y = 250 - 40$

$7y = 210$

$y = \underline{\underline{30}}$

$x = 2y$

$x = \underline{\underline{60}}$

Chapt 3.

sample mean vector

$$\boxed{\bar{\underset{\sim}{x}} = \frac{1}{n} X' 1_n}$$

sample cov. matrix

$$\boxed{C = \frac{1}{n-1} (X - 1_n \bar{\underset{\sim}{x}}')' (X - 1_n \bar{\underset{\sim}{x}}')}$$

generalised sample variance

↳ $|C|$

total sample variance

↳ $tr(C)$

interpreting sample cov matrix

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

= variance of each variable

↳ explains variability

$var(X_1) = C_{11}$

$var(X_2) = C_{22}$

= covariance

↳ explains relationship between different variables

$cov(X_1, X_2) = C_{12} = C_{21}$

Correlation matrix

$$R = \begin{bmatrix} 1 & r_{12} & \cdots & r_{1p} \\ r_{12} & 1 & \cdots & r_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1p} & r_{2p} & \cdots & 1 \end{bmatrix} \text{ where } \boxed{r_{ik} = \frac{C_{ik}}{\sqrt{C_{ii}}\sqrt{C_{kk}}}}$$

$$\boxed{R = S^{-1} C S^{-1}}$$

interpreting corr. matrix

$$R = \begin{bmatrix} 1 & r_{12} \\ r_{21} & 1 \end{bmatrix}$$

$$r_{12} = r_{21} = \text{corr}(X_1, X_2)$$

$$\text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{SD(X) \cdot SD(Y)}$$

sample SD matrix

$$S = \begin{bmatrix} \sqrt{C_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{C_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \sqrt{C_{pp}} \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{1}{\sqrt{C_{11}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{C_{22}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \frac{1}{\sqrt{C_{pp}}} \end{bmatrix}$$

1) a. I. $r_{12} = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \underline{\underline{0}}$ II. $r_{12} = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \frac{4}{\sqrt{5}\sqrt{5}} = \underline{\underline{0.8}}$

III $r_{12} = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \frac{-4}{\sqrt{5}\sqrt{5}} = \underline{\underline{-0.8}}$

b. I → B, II → A, III → C

c. I. $|C| = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = \underline{\underline{9}}$ II. $|C| = \begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} = 25 - 16 = \underline{\underline{9}}$

III. $|C| = \begin{vmatrix} 5 & -4 \\ -4 & 5 \end{vmatrix} = 25 - 16 = \underline{\underline{9}}$

2). a. $X = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$

$\bar{\underset{\sim}{x}} = \frac{1}{n}(X^T 1_n)$

$\bar{\underset{\sim}{x}} = \frac{1}{3}\begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\bar{\underset{\sim}{x}} = \frac{1}{3}\begin{bmatrix} 15 \\ 6 \end{bmatrix}$

$\bar{\underset{\sim}{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$X - 1_n \bar{\underset{\sim}{x}}^T \rightarrow \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \end{bmatrix}$

$\begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 5 & 2 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix}$

$C = \frac{1}{n-1}(X - 1_n \bar{\underset{\sim}{x}}')'(X - 1_n \bar{\underset{\sim}{x}}')$

$= \frac{1}{2}\begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix}'\begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 16 & -2 \\ -2 & 2 \end{bmatrix}$

contd...

Chapt 4

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

v can be normalized

$$\boxed{\hat{v} = \frac{\vec{v}}{\sqrt{v_1^2+v_2^2+\dots+v_n^2}}}$$

characteristic polynomial of A

$$|A-\lambda I|$$

characteristic equation of A

$$|A-\lambda I| = 0$$

Diagonalization theorem

$$A = V\Lambda V^{-1}$$

diagonalizable: has n linearly independent eigenvectors

linear independence

$a_1v_1 + a_2v_2 = 0$ if $a_1 \wedge a_2 \neq 0$

↳ linearly dependent

ex $\begin{pmatrix}2\\-4\end{pmatrix}$ & $\begin{pmatrix}-1\\2\end{pmatrix}$ are ↰ because

$$\begin{bmatrix}2\\-4\end{bmatrix} + 2\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$a_1v_1 + a_2v_2 = 0$

iff $a_1 = a_2 = 0$

↳ linearly independent

$$a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

if & only if $\boxed{a = b = 0}$

1) $Au = -u$. $\therefore u =$ an eigenvector of A corresponding to $\lambda = -1$
$Av \neq \lambda v$, $\therefore v \neq$ an ——— " ———
$Aw = w$, $\therefore w =$ ——— " ——— " ——— " ——— $\lambda = 1$

2) $$Av = \begin{bmatrix} 0 & -2 & -5 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ -1 \end{bmatrix}$$ v is not an evector of A

$$Aw = \begin{bmatrix} 0 & -2 & -5 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$ w is an evector of A corresponding to $\lambda = 2$

3) a) $$cp: \left|\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = \left|\begin{bmatrix} 4-\lambda & 3 \\ 2 & 5-\lambda \end{bmatrix}\right| = (4-\lambda)(5-\lambda) - 6$$
$$= \lambda^2 - 9\lambda + 14$$

λ: $\lambda^2 - 9\lambda + 14 = 0$
$(\lambda - 7)(\lambda - 2) = 0$
$\lambda_1 = 7 \;\vee\; \lambda_2 = 2$

$\boxed{(A-\lambda I)v = 0}$ v_1: $$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{2R_1 - 3R_2} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-3v_1 + 3v_2 = 0$
$v_1 = v_2$
$v_1 = t$

Let $v_2 = t$
$$\vec{v_1} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ where $t \neq 0$

$\lambda_2 = 2$

$$\begin{bmatrix} 4-2 & 3 \\ 2 & 5-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

let $v_2 = t$

$2v_1 + 3v_2 = 0$

$v_1 = -\frac{3}{2}t$

$$\vec{v}_2 = \begin{bmatrix} -\frac{3}{2}t \\ t \end{bmatrix}$$

$$\vec{v}_2 = \frac{t}{2}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\vec{v}_2 = -\frac{t}{2}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

1) c. $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 3 & 4 & 2 & 0 \\ 5 & 9 & 1 & 0 \end{array}\right] \xrightarrow[5R_1 - R_2]{3R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right] \xrightarrow{6R_1 - 5R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$5y - 5z = 0$

$y - z = 0$

independent

4) a. $\left[\begin{array}{c|c} 4 & 6 \\ 2 & \end{array}\right]$

$= 3\begin{bmatrix} 4 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 12 \\ 6 \end{bmatrix} - \begin{bmatrix} 12 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

dependent

b. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

independent

4d. $\left[\begin{array}{ccc|c} 3 & 9 & 0 & 0 \\ 2 & 6 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{array}\right]$

$\boxed{\begin{array}{c} \text{independent: } |A| \neq 0 \\ \text{dependent: } |A| = 0 \end{array}}$

$$\begin{vmatrix} 3 & 9 & 0 \\ 2 & 6 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 3 \begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix} - 9 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 3 \cdot 2 - 9 \cdot 1$$

$$= 6 - 9$$

$$= -3$$

independent

$$\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} = 12 - 12 = 0$$

dependent

7) $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ $\lambda = 1$, $\lambda = 5$ diagonalize A.

1. find v_1, v_2, v_3

$$\lambda = 1 \left[\begin{array}{ccc|c} 2-1 & 2 & -1 & 0 \\ 1 & 3-1 & -1 & 0 \\ -1 & -2 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_1+R_3]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let $v_2 = s$ $v_3 = t$

$v_1 + 2v_2 - v_3 = 0$

$v_1 + 2s - t = 0$

$v_1 = t - 2s$

$$\vec{v} = \begin{bmatrix} t-2s \\ s \\ t \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} // \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} //$$

$$\lambda = 5 \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & -2 & -3 & 0 \end{array}\right] \xrightarrow[R_1-3R_3]{R_1+3R_2}$$

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 8 & 8 & 0 \end{array}\right] \xrightarrow{2R_2+R_3}$$

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ let $v_3 = t$

$v_2 + v_3 = 0$ $\quad -3v_1 = v_3 - 2v_2$

$v_2 = -t$ $\quad -3v_1 = t + 2 \cdot t$

$v_1 = -t$

$$\vec{v}_3 = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} //$$

7). $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ $\lambda = 1$ $\lambda = 5$ diagonalize A.

$$V = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$