Chapter 2: Systems of Linear Equations

Learning Objectives:

1. *Determine what is a system of linear equations.*
2. *Recognize the matrix representations of a system of linear equations.*
3. *Determine the number of solutions a system of linear equations has.*
4. *Solve a system of linear equations using Gaussian elimination.*
5. *Code the Gaussian elimination algorithm to solve a system of linear equations with unique solution.*
6. *Use Python to solve a system of linear equations.*

2.1 Introduction

A **linear equation** in the variables $x_1, x_2, ..., x_p$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + ... + a_px_p = b$$

where the **coefficients** $a_1, a_2, ..., a_p$ and b are known numbers.

Examples of linear equations are:

$$x_1 - 2x_2 = 0$$
$$5x_1 - 8x_2 + 7x_3 = 1$$

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations. An example is:

$$x_2 + 4x_3 = -5$$
$$3x_1 + 7x_2 + 7x_3 = 6$$

A **solution** of the linear system is a sequence of numbers $s_1, s_2, ..., s_p$ which has the property that each equation in the linear system is satisfied when $x_1 = s_1,\ x_2 = s_2,\ ...,\ x_p = s_p$ are substituted. Thus $x = 1$ and $y = 1$ is a solution of the system:

$$3x + 2y = 5$$
$$x + y = 2$$

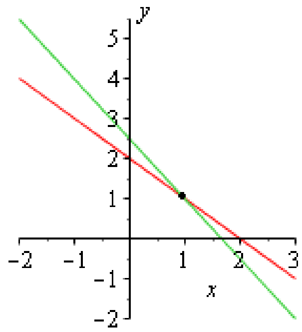
The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are said to be **equivalent** if they have the same solution set.

In general, a system of *n* linear equations with *p* variables $x_1, x_2, ..., x_p$ can be written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{np}x_p &= b_n \end{aligned}$$

Example 1.

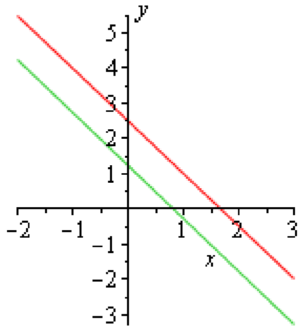
(a)
$$\begin{aligned} 3x+2y&=5 \\ x+y&=2 \end{aligned}$$



The two lines intersect at the point (1, 1). The system has exactly one solution.

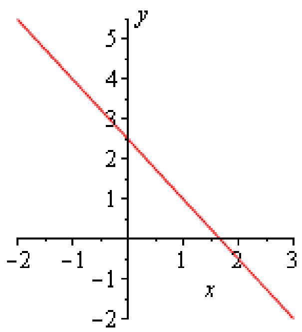
$x =$ ____________ $y =$ _____________

(b)
$$\begin{aligned} 3x+2y&=5 \\ 6x+4y&=5 \end{aligned}$$



The two lines are parallel and do not intersect. The linear system has no solution.

(c)
$$\begin{aligned} 3x+2y&=5 \\ 6x+4y&=10 \end{aligned}$$



The two lines intersect completely. The system has infinitely many solutions.

Parametrized solution(s): _________________________________

Alternative parametrization: ______________________________

2.2 Matrix Notation

A system of *n* linear equations with *p* variables $x_1, x_2, \ldots, x_p$ written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{np}x_p &= b_n \end{aligned}$$

can be expressed in matrix form $\mathbf{Ax} = \mathbf{b}$, where:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The system of linear equations can also be represented as an **augmented matrix**:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1p} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2p} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} & b_n \end{array}\right]$$

Each row in the augmented matrix represents a linear equation in the system.

Example 2.

Given the linear system:

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}$$

Write down its augmented matrix.

2.3 Number of Solutions

In any linear system, only one of the following is true:

- If **A** is <u>nonsingular</u>, then the system has a <u>unique</u> (i.e. exactly one) solution.
- Else,
  - o the system has <u>infinitely many</u> solutions, or
  - o the system has <u>no</u> solution.

Specifically, a linear system is called **homogeneous** if all the constant terms on the right-hand side are zero. That is, **Ax** = **0**.

In any homogeneous linear system, only one of the following is true:

- If **A** is <u>nonsingular</u>, then the system has a unique solution, which is <u>**x** = **0**</u>.
- Else, the system has an <u>infinitely</u> many solutions.

A system of linear equations is said to be **consistent** if it has one solution or infinitely many solutions; a system is **inconsistent** if it has no solutions.

The solution $x_1 = x_2 = \ldots = x_p = 0$ of a linear system is called a **trivial** solution. Not all linear systems have trivial solutions.

Example 3.

Show that the solution set to a homogenous linear system always contain the trivial solution.

2.4 Solving a Linear System

To find a solution to a linear system, we shall learn a technique called **Gaussian elimination**. Gaussian elimination uses **elementary row operations** to obtain a new linear system that is equivalent to the original system, but is much easier to solve.

2.4.1 Elementary Row Operations

The following are elementary row operations:

- Interchange two rows
- Multiply a row by a nonzero constant
- Add a multiple of a row to another row

We use the notations below to denote the above operations. Let R_i denote the *i*th row of a matrix.

- $R_i \leftrightarrow R_j$ to mean: Interchange row *i* with row *j*.
- cR_i to mean: Replace row *i* with c times of row *i*.
- $R_i + aR_j$ to mean: Replace row i with the sum of row *i* and *a* times of row *j*.

Example 4.

Write down the row operations corresponding in the following sequence of Gaussian eliminations.

$$\begin{aligned} -2x_1 - 7x_2 &= -5 \\ x_1 + 5x_2 &= 7 \end{aligned} \qquad \left[\begin{array}{cc|c} -2 & -7 & -5 \\ 1 & 5 & 7 \end{array}\right]$$

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned} \qquad \to \left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right]$$

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ 2x_1 + 7x_2 &= 5 \end{aligned} \qquad \to \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 2 & 7 & 5 \end{array}\right]$$

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -3x_2 &= -9 \end{aligned} \qquad \to \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & -3 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned} \qquad \to \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right]$$

2.4.2 Row Echelon Form

A rectangular matrix is in **row echelon form** if it has the following three properties:

- All nonzero rows are above any rows of all zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

Example 5.

The following matrices are in row echelon form.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 5 & 7 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Why aren't the following matrices in row echelon form?

(d) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2.4.3 Solutions of Linear Systems

Gaussian elimination consists of two steps:

Step 1. Transform the augmented matrix $\left[\mathbf{A} \mid \mathbf{b}\right]$ to the matrix $\left[\mathbf{C} \mid \mathbf{d}\right]$ in row echelon form using elementary row operations.

Step 2. Solve the linear system corresponding to the augmented matrix $\left[\mathbf{C} \mid \mathbf{d}\right]$ using **back substitution**.

Example 6.

Solve the given linear system by Gaussian elimination:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 8 \\ 3x_1 - x_3 &= 3 \end{aligned}$$

Example 7.

(a) Solve the given linear system by Gaussian elimination:

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(b) Determine the solution to the same system if the constants on the RHS of the equations are all zero. (i.e. solve the homogenous linear system).

Example 8.

Solve the following linear system if possible:

$$\begin{aligned} x-4y+2z &= -2 \\ x+2y-2z &= -3 \\ x-y &= 4 \end{aligned}$$

Example 9.

Determine the entire solution set of the following linear system:

$$y_1 + 3y_2 - y_4 = 0$$
$$2y_1 - y_2 + y_3 + 4y_4 = 5$$

Example 10.

Ace Novelty wishes to produce three types of souvenirs: types A, B, and C. To manufacture a type-A souvenir requires 2 minutes on machine I, 1 minute on machine II, and 2 minutes on machine III. A type-B souvenir requires 1 minute on machine I, 3 minutes on machine II, and 1 minute on machine III. A type-C souvenir requires 1 minute on machine I and 2 minutes each on machines II and III. There are 3 hours available on machine I, 5 hours available on machine II, and 4 hours available on machine III for processing the order. How many souvenirs of each type should Ace Novelty make in order to use all of the available time?

2.4.4 Reduced Row Echelon Form

An augmented matrix $\left[\mathbf{C} \mid \mathbf{d}\right]$ is in **reduced row echelon form** if **C** has the following three properties:

- It is in row echelon form.
- The leading entry in each nonzero row is a 1.
- Each column containing a leading 1 has zeros everywhere else.

Examples of augmented matrices in reduced row echelon form are

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right] \text{ and } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Tutorial 2

1. Which of the following matrices are in row-echelon form?

(a) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2. Use Gaussian elimination to find all solutions (if any) to each of the following system of linear equations.

(a) $3x - y = 2$
$2y - 6x = -4$

(b) $2x - 3y = 5$
$3y - 2x = 2$

(c) $3x - 2y + z = -2$
$x - y + 3z = 5$
$-x + y + z = -1$

(d) $x + 2y - 4z = 10$
$2x - y + 2z = 5$
$x + y - 2z = 7$

(e) $2x - 3y = -2$
$2x + y = 1$
$3x + 2y = 1$

(f) $5x - 2y + 6z = 0$
$-2x + y + 3z = 1$

(g) $3x + 2y - z = -15$
$5x + 3y + 2z = 0$
$3x + y + 3z = 11$
$-6x - 4y + 2z = 30$

(h) $3x_1 + x_2 + x_3 + x_4 = 0$
$5x_1 - x_2 + x_3 - x_4 = 0$

3. Consider the system of linear equations:

$$\begin{aligned} x + y + 2z &= a \\ x + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

Show that for this system to be consistent, the constants a, b and c must satisfy $c = a + b$.

4. Consider the system of linear equations:

$$x - y = 3$$
$$2x - 2y = k$$

For which value(s) of the constant k does the system have no solutions? Exactly one solution? Infinitely many solutions? Explain.

5. Let $\mathbf{A} = \begin{bmatrix} -1 & -4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}$. Find a nontrivial solution to $\mathbf{Ax} = \mathbf{0}$.

6. The management of Hartman Rent-A-Car has allocated \$1.5 million to buy a fleet of new automobiles consisting of small, intermediate, and full-size cars. Small-size cars cost \$12,000 each, intermediate-size cars cost \$18,000 each, and full-size cars cost \$24,000 each. If Hartman purchases twice as many small-size cars as intermediate-size cars and the total number of cars to be purchased is 100, determine how many cars of each type will be purchased. (Assume that the entire budget will be used.)

7. An executive recently travelled to London, Paris, and Rome. He paid \$180, \$230, and \$160 per night for lodging in London, Paris, and Rome, respectively, and his hotel bills totalled \$2660. He spent \$110, \$120, and \$90 per day for his meals in London, Paris, and Rome, respectively, and his expenses for meals totalled \$1520. If he spent as many days in London as he did in Paris and Rome combined, how many days did he stay in each city?

Answers

1. (a) No (b) Yes (c) Yes (d) Yes
(e) Yes (f) No (g) No (h) No

2. (a) $x = \frac{1}{3}(t+2),\ y = t$ (b) No solutions
(c) $x = -7,\ y = -9,\ z = 1$ (d) $x = 4,\ y = 3 + 2t,\ z = t$
(e) No solutions (f) $x = 2 - 12t,\ y = 5 - 27t,\ z = t$
(g) $x = -4,\ y = 2,\ z = 7$ (h) $x_1 = -\frac{1}{4}s,\ x_2 = -\frac{1}{4}(s + 4t),\ x_3 = s,\ x_4 = t$

4. No solution: $k \neq 6$; Cannot have exactly one solution; Infinite number of solutions: $k = 6$

5. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4t \\ t \end{bmatrix}$, where t is free.

A particular nontrivial solution is $x_1 = -4$ and $x_2 = 1$

6. 60 small-size, 30 intermediate-size and 10 full-size

7. 7 days in London, 4 days in Paris, and 3 days in Rome

Practical 2

In this practical, we will code the Gaussian elimination algorithm to solve a system of linear equations with unique solution.

Task 1

Solve the following system of linear equations

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 22 \\ 3x_1 + 8x_2 + 5x_3 &= 27 \\ -x_1 + x_2 + 2x_3 &= 2 \end{aligned}$$

given that it has unique solution.

Task 2

Modify the code from task 1 and use it to solve the following system of linear equations

$$\begin{aligned} 2x_2 + 3x_3 &= 7 \\ 3x_1 + 6x_2 - 12x_3 &= -3 \\ 5x_1 - 2x_2 + 2x_3 &= -7 \end{aligned}$$

given that it has unique solution.

(Hint: You will need to make an improvement to the section of the code that converts the augmented matrix to row echelon form.)

Task 3

Modify the code from task 2 and use it to solve the following system of linear equations

$$\begin{aligned} x_1 - x_2 \qquad &= 3 \\ x_1 \qquad - x_3 &= -2 \\ -6x_1 + 2x_2 + 3x_3 &= 0 \end{aligned}$$

given that it has unique solution.

(Hint: You will need to make an improvement to the section of the code that obtains the row-echelon form.)

Task 4

Given a system of 10 linear equations (refer to the downloaded jupyter notebook) that has a unique solution, compare the time taken to solve the system of linear equations using your code from task 3 vs. using numpy.linalg.solve.

Task 5

If we do not know the number of solutions our system of linear equations has, we can use sympy.linsolve.

For example, we can use sympy.linsolve to solve the following system of linear equations which has many solutions:

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

Answers

1. $x_1 = 3$, $x_2 = 1$, $x_3 = 2$

2. $x_1 = -1$, $x_2 = 2$, $x_3 = 1$

3. $x_1 = 0$, $x_2 = -3$, $x_3 = 2$

5. $x_1 = 2t + 1$, $x_2 = 3t + 2$, $x_3 = t$