# Bound on measurement based on the no-signaling condition

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We obtain a tight bound on the measurement which discriminates without error between two nonorthogonal states. The derivation is based on the no-signaling condition, without any reference to generalized measurements.

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It is very interesting to note that the fact that superluminal communication is forbidden [1] restricts, in a general way, the allowed operations which can be carried out on a quantum system [2,3], thus linking special relativity and quantum mechanics. One should therefore expect to be able to obtain specific bounds on quantum operations, starting from the no-signaling condition. An example is the bound obtained on symmetric cloning [4]. In this Report, we will obtain a tight bound on the unambiguous discrimination between two nonorthogonal states. This problem was originally addressed by Ivanovic, Dieks, and Peres [5–7], who considered the case of two equally probable nonorthogonal states, and subsequently generalized to states with unequal prior probabilities by Jaeger and Shimony [8]. These authors derived their results by considering the unitary interaction between the system of interest and an ancilla, followed by measurement of both systems. The measurement result is then obtained by measuring the system together with the ancilla. A recent review of unambiguous state discrimination can be found in [9]. Usually generalized measurements of this kind are treated within the formalism of POM measurements [10–12]. Essentially, POM measurements are equivalent to restricting the transformations of the measured system to be described by completely positive maps. Here, however, we will obtain a tight bound on optimal unambiguous discrimination between two nonorthogonal states completely without reference to the measurement transformation. This demonstrates, in a remarkable way, the power of the no-signaling condition in limiting quantum operations.

The no-signaling condition implies that entanglement cannot be used for faster-than-light communication. Suppose two spacelike separated particles are prepared in the entangled quantum state

$$|\psi\rangle=\sqrt{q}|+\rangle_R|a\rangle_L+\sqrt{1-q}|-\rangle_R|b\rangle_L, \tag{1}$$

where

$$|a\rangle_L=\cos\theta|+\rangle_L+\sin\theta|-\rangle_L,$$
$$|b\rangle_L=\cos\theta|+\rangle_L-\sin\theta|-\rangle_L, \tag{2}$$

$|+\rangle_{L,R}$ and $|-\rangle_{L,R}$ are orthogonal basis states for the two particles, referred to as "left" and "right," and $0\leqslant q\leqslant 1$. Without loss of generality, we choose $0\leqslant\theta\leqslant\pi/2$. The no-signaling condition then requires that no operation performed on the right particle can be detected by observation of the left particle alone [1,12,13].

The reduced density matrix of the left system is

$$\rho_L=\mathrm{Tr}_R|\psi\rangle\langle\psi|=q|a\rangle_{LL}\langle a|+(1-q)|b\rangle_{LL}\langle b|, \tag{3}$$

corresponding to a statistical mixture of the nonorthogonal states $|a\rangle_L$ and $|b\rangle_L$ with respective prior probabilities $q$ and $1-q$. On this left system, we intend to carry out a measurement to unambiguously distinguish between these two states. The measurement on the left system has three possible outcomes: "$a$," "$b$," or "inconclusive," occurring with probabilities $P_a$, $P_b$, and $P_?=1-P_a-P_b$, respectively. We will obtain a bound on $P_?$, without any further reference to the measurement procedure.

According to the no-signaling condition, measurement of the left particle is not allowed to change the reduced density matrix of the right system,

$$\rho_R=\mathrm{Tr}_L|\psi\rangle\langle\psi|$$
$$=\begin{pmatrix} q & \sqrt{q(1-q)}\cos 2\theta \\ \sqrt{q(1-q)}\cos 2\theta & 1-q \end{pmatrix}. \tag{4}$$

Any arbitrarily small, but finite, change in the reduced density matrix of the right system could be detected by preparing a large number of entangled pairs. Measurements on the pair halves located to the right would, with some nonzero probability, reveal a change in their reduced density matrices. This would immediately lead to the possibility of superluminal communication.

If the result of the measurement performed on the left particle is "$a$" or "$b$," it is clear from Eq. (1) that the state of the right system after the measurement will be $|+\rangle_R$ or $|-\rangle_R$, respectively. If the result of the measurement is inconclusive, the right system is in an as yet undetermined state with density matrix

$$\rho_?=\begin{pmatrix} \rho_?^{++} & \rho_?^{+-} \\ \rho_?^{-+} & \rho_?^{--} \end{pmatrix}. \tag{5}$$

Putting these together, we find that, after the measurement, the density matrix of the right particle is

$$\rho_R=\begin{pmatrix} P_a & 0 \\ 0 & P_b \end{pmatrix}+P_?\begin{pmatrix} \rho_?^{++} & \rho_?^{+-} \\ \rho_?^{-+} & \rho_?^{--} \end{pmatrix}. \tag{6}$$

By the no-signaling condition, this has to be equal to Eq. (4). This immediately results in the equations

$$q=P_a+P_?\rho_?^{++},$$

 

$$1-q=P_b+P_?\rho_?^{--},$$

$$\sqrt{q(1-q)}\cos 2\theta=P_?\rho_?^{+-}=P_?\rho_?^{-+}. \tag{7}$$

Rewriting the third of these equations, we obtain

$$P_?^2=\frac{q(1-q)\cos^2 2\theta}{\rho_?^{+-}\rho_?^{-+}}. \tag{8}$$

The best possible unambiguous measurement will correspond to the minimum value of $P_?$ subject to the conditions

$$0\leqslant P_a=q-P_?\rho_?^{++},$$

$$0\leqslant P_b=1-q-P_?\rho_?^{--}, \tag{9}$$

obtained from the first two equations in Eq. (7).

The global minimum of $P_?$ in Eq. (8) is reached when $\rho_?$ corresponds to the pure state $1/\sqrt{2}(|+\rangle+|-\rangle)$. This implies that

$$P_?\geqslant 2\sqrt{q(1-q)}\cos 2\theta. \tag{10}$$

It will not always be possible to attain this bound, due to violation of the conditions in Eq. (9). We find that this is the case whenever

$$\sqrt{(1-q)/q}\cos 2\theta\geqslant 1 \quad\text{or}\quad \sqrt{q/(1-q)}\cos 2\theta\geqslant 1. \tag{11}$$

In this case, since the global minimum of Eq. (8) could not be reached, the minimum value of $P_?$ will lie on the boundary of the parameter space allowed. This means that either $P_a$ or $P_b$ will be equal to zero. If $P_a=0$, we obtain $\rho_?^{++}=q/P_?$ and $\rho_?^{--}=1-q/P_?$. Using this, Eq. (8), and the fact that $\rho^{++}\rho^{--}\geqslant\rho^{+-}\rho^{-+}$ for any density matrix, we obtain

$$P_?\geqslant 1-(1-q)\sin^2 2\theta. \tag{12}$$

In a similar way, $P_b=0$ gives

$$P_?\geqslant 1-q\sin^2 2\theta. \tag{13}$$

Depending on whether $q$ is smaller or greater than $1-q$, one or the other will be optimal. These bounds agree with those obtained by Jaeger and Shimony [8] by considering unitary interaction between the system to be measured and an ancilla, followed by measurement of both systems.

Knowing the form (6) of the density-matrix $\rho_R$ of the right system, it is possible to reconstruct the measurement on the left system which produces this particular decomposition of $\rho_R$ [3]. When $P_a$, $P_b$, and $P_?$ are all nonzero, the measurement will be described by a generalized measurement strategy, which can be implemented by coupling the system to be measured to an ancilla as considered in [5–8]. Whenever $P_a$ or $P_b$ is equal to zero, a projective von Neumann measurement will be enough: For example, if $P_a$ is equal to zero, the measurement on the left system is a projection in the basis $\{|a\rangle_L,|a^\perp\rangle_L\}$. Obtaining the result "$a^\perp$" will correspond to the result "$b$," since in this case, the state cannot have been $|a\rangle_L$. The result "$a$" will be the inconclusive result, since in this case, it is impossible to rule out the possibility that the state was either $|a\rangle_L$ or $|b\rangle_L$, due to the nonorthogonality of these.

We should also note that unambiguous state discrimination is more than simply a theoretical construction. Unambiguous discrimination between two equiprobable states has indeed been realized in experiments with photon polarization [14,15].

It has long been appreciated that entangled quantum systems display correlations, but that these do not allow for communication by choice of measurement [1] or other operations [12,13] on one of the systems. Recently, is has been suggested that this idea might be given the status of a postulate and used to restrict quantum theory and possible generalizations of it [3]. In this Paper, we have shown that it can also be employed as a physical law and used to obtain a tight bound on quantum measurement. The example given in this paper is by no means unique. Using a very similar argument, it is possible to use the no-signaling condition to derive a bound on the success of probabilistic cloning. This bound was originally derived by Duan and Guo by assuming that the system to be cloned is subject to a unitary evolution and a subsequent measurement [16]. We will discuss this and other examples elsewhere, including using Bell's inequality to limit simultaneous measurements on a pair of spin components.

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