Conditional probability, quantum time and friends

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Considering a minimal number of assumptions and in the context of the timeless formalism, we derive conditional probabilities for subsequent measurements in the non-relativistic regime. The probability expressions are found unambiguously, also in puzzling cases like the Wigner's friend scenario, and they underline the relativity aspect of measurements. No paradoxical situations emerge and the roles of Wigner and the friend are completely interchangeable. In particular, Wigner can be seen as a superimposed of states from his/her friend. When only one clock system is considered, we demonstrate in addition that two-times probabilities cannot be defined.

The measurement problem is one of the most fundamental issues in Quantum Mechanics on which the numerous interpretations and foundations differ. Von Neumann was the first to try to formalize the problem of measurement assuming that there are basically two types of evolution in quantum mechanics [1]. The first is the unitary evolution, while the other is the collapse of the wavefunction during the measurement describing the passage from a superposition of states to one of the studied systems. Contrary to the unitary evolution, the second measurement process, formalized by the *projection postulate*, is irreversible, non-causal and, of course, non-unitary; it can be viewed as passage between the quantum to the classical realm. When such a process is not postulated, different mechanisms can be evoked to justify it, like the coupling to the environment (decoherence) [2], the addition of non-linearities in the evolution equation [3] and specific coupling to the gravitational fields [4], among many other approaches.

If an ideal Ockham's razor would be applied to the different interpretations of Quantum Mechanics, only the ones with a minimal number of hypotheses should be considered (and that agree with experimental measurements of course). Instead of postulating two type of natures, classical and quantum, the simplest approach is to consider only the second one; quantum phenomena cannot be explained by classical physics, when in opposite, classical behavior can emerge from quantum systems. With the same minimalist approach, only one evolutionary process–the unitary processes–should be taken into account. If only unitary transformations are considered, the measurement process has to be interpreted then as a unitary interaction between the detector system and the studied system. From such an interaction, positive-valued operators, or effects, can be built in the context of the Positive Valued Operator Measure (POVM) [5, 6], where imperfect detections can also be treated.

By treating the measuring apparatus as a quantum system, as well as the entire chain up to the observer's neurons, the measurement theory becomes part of the quantum theory of interacting compound systems. In this framework, the correlations created by interactions between systems play a central role, like in Relational Quantum Mechanics [7, 8], in the original Everett formulation [9] and its many-worlds theory derivation [10], in the Consistent Histories [11, 12] , in the History Projection Operator formalism [13] and in the QBism [14]. This simplicity has however a price. The entanglement between observer and observable leads to different measurement results of independent, i.e. non-interacting, observers. The relativity of what is normally considered universal rests however a big conceptual leap for beings living in a world where quantum behaviors are barely visible. This discomfort manifests itself in paradoxes like in the case of the Wigner's friend scenario [15].

Another thorny problem of the standard formulation of quantum mechanics is the special role occupied by the time coordinate t. In the unitary evolution, t is the evolutionary parameter; in the Schrödinger equation, time explicitly appears in the derivation with respect to it. In both cases, time plays a special role in the description of a system's evolution. Moreover, the formal description of a system before and after a measurement (via a short interaction) requires considering two distinct Hilbert spaces $H_{in} \otimes H_{out}$ [13, 16–18]. This complexification and the central role of the time coordinate can be eliminated at once by including the measurement of time itself in a more general description of the studied interaction together with a clock system. Originally designed to be compatible with general relativity, this formalism allows for considering dynamic processes without time as parameter. This leads to a simpler formulation of the measurement process acting in a unique Hilbert space, where time is an observable among others resulting in a timeless description of dynamics [19–23].

Starting from a minimalistic approach with the description of nature using only quantum systems and unitary interactions, in this letter we focus our interest on the formulation and properties of the conditional probability function in the quantum time formalism. Several works discussed on the conditional probability since the early formulations [19]. Here we take the advantage of the best both worlds of timeless and POVM formalisms to provide a coherent description on different measurement. More precisely, we present a derivation of conditional probability for subsequent measurements starting from first principles and we compare it with the previous works. In particular, we consider the Wigner's friend measurement scenario and we show that the associated probability expressions can be simply and unambiguously formulated.

As starting point, we consider the Wheeler-DeWitt equation for describing the dynamics of the global state $|\Psi\rangle$ in the time-space continuum [24]:

$$\hat{H}\,|\Psi\rangle\rangle = 0. \tag{1}$$

$\hat{H}$ is the total Hamiltonian acting in the kinematic Hilbert space $\mathcal{K}$ where space-time coordinates are measured. The notation with the double ket $|\cdot\rangle\rangle$ indicates the inclusion of a clock system in $|\Psi\rangle\rangle$ from which the time coordinate is measured. In the following paragraphs, we implicitly assume $c = \hbar = 1$.

In the non-relativistic approximation where the clock system C is considered non-interacting with the system of interest S, the kinematic Hilbert space can be decomposed in $\mathcal{K} = \mathcal{H}_T \otimes \mathcal{H}_S$ where $\mathcal{H}_T$ and $\mathcal{H}_S$ are the Hilbert sub-spaces of the clock and the studied system, respectively. The corresponding Hamiltonian is

$$\hat{H} = \hat{p}_T \otimes \mathbb{1}_S + \mathbb{1}_T \otimes \hat{H}_S \tag{2}$$

where $\hat{H}_T = \hat{p}_T$ consists of the conjugate operator of the time operator $\hat{T}$, with $[\hat{T}, \hat{p}_T] = i$. $\{|t\rangle_T\}$ are the corresponding time states that form the base of $\mathcal{H}_T$, with $\hat{T}\,|t\rangle_T = t\,|t\rangle_T$ and ${}_T\langle t|t'\rangle_T = \delta(t - t')$. The more general case where the clock can interact with the studied system or with another clock is treated in Refs. [25, 26]. The total wave-function can be then decomposed in $|\Psi\rangle\rangle = |t\rangle_T \otimes |\psi(t)\rangle_S$ where $|\psi(t)\rangle_S = {}_T\langle t|\Psi\rangle\rangle$ is the wave-function obtained by the condition of measuring the time t in the clock. Consequently, we can also write

$$|\Psi\rangle\rangle = \int dt\,|t\rangle_T \otimes |\psi(t)\rangle_S\,. \tag{3}$$

With this notation, considering the term ${}_T\langle t|\hat{H}|\Psi\rangle\rangle$ with Eqs. (1) and (2), we obtain the the standard form of the Schrödinger equation $i\frac{\partial}{\partial t}|\psi(t)\rangle_S = \hat{H}_S\,|\psi(t)\rangle_S$. If $\hat{H}_S$ does not depends on the time operator $\hat{T}$, a solution of the equation is the unitary operator

$$\hat{U}_S(t, t_0) = e^{-i\hat{H}_S(t-t_0)}. \tag{4}$$

Similarly to the procedure described in Refs. [20, 21], we consider here a measurement at a time t_M consisting of a unitary interaction of a negligible duration between the system S and a measuring system M. The total Hamiltonian describing the system is

$$\begin{aligned}\hat{H} &= \hat{p}_T \otimes \mathbb{1}_S \otimes \mathbb{1}_M + \mathbb{1}_T \otimes \hat{H}_S \otimes \mathbb{1}_M \\ &\quad + \hat{V}_{SM}\delta(\hat{T} - t_M) + \mathbb{1}_T \otimes \mathbb{1}_S \otimes \hat{H}_M. \end{aligned} \tag{5}$$

V_{SM} represents the interaction between $\mathcal{H}_S$ and $\mathcal{H}_M$ at the time t_M and where $\hat{H}_M$ is the Hamiltonian of the detector itself. With the detector space $\mathcal{H}_M$ we can associate the entire sub-system chain from the detector to the observer's brain. The changes in $\mathcal{H}_S$ of a measurement with outcome m is given by positive-valued operator $\hat{\Pi}^m$ linked to the detector ancillary state $|m\rangle_M \in \mathcal{H}_M$. The interaction between the detector and the system at time t_M results in the unitary mapping

$$|\psi(t_M)\rangle_S \otimes |r\rangle_M \to \sum_m K_M^m\,|\psi(t_M)\rangle_S \otimes |m\rangle_M\,, \tag{6}$$

where $|r\rangle_M$ is the "ready" detector state before the measurement, $\hat{K}_M^m$ are the different Kraus operators corresponding to the outcomes m with $\hat{\Pi}^m = (\hat{K}_M^m)^\dagger \hat{K}_M^m$. From Eq. (3), the total state $|\Psi\rangle\rangle$ can be written as

$$\begin{aligned}|\Psi\rangle\rangle &= \int_{-\infty}^{t_M} dt\,|t\rangle_T \otimes |\psi(t)\rangle_S \otimes |r\rangle_M \\ &+ \int_{t_M}^{\infty} dt\,|t\rangle_T \otimes \sum_i \hat{U}_S(t, t_M)\hat{K}_M^i\,|\psi(t_M)\rangle_S \otimes |i\rangle_M\,, \end{aligned} \tag{7}$$

where $\hat{U}_S$ is defined in Eq. (4).

We consider now the case of two subsequent measurements: one measurement M on S with output m operated at a time t_M and another N with output n done at the time $t_N > t_M$ with no other interaction between the different sub-systems. The corresponding Hamiltonian is

$$\begin{aligned}\hat{H} &= \hat{p}_T \otimes \mathbb{1}_S \otimes \mathbb{1}_M \otimes \mathbb{1}_N + \mathbb{1}_T \otimes \hat{H}_S \otimes \mathbb{1}_M \otimes \mathbb{1}_N \\ &\quad + \hat{V}_{SM}\delta(\hat{T} - t_M) \otimes \mathbb{1}_N + \hat{V}_{SN}\delta(\hat{T} - t_N) \otimes \mathbb{1}_M \\ &\quad + \mathbb{1}_T \otimes \mathbb{1}_S \otimes \hat{H}_M \otimes \mathbb{1}_N + \mathbb{1}_T \otimes \mathbb{1}_S \otimes \mathbb{1}_M \otimes \hat{H}_N \end{aligned} \tag{8}$$

and the associated wave-function is [21, 23],

$$\begin{aligned}|\Psi\rangle\rangle &= \int_{-\infty}^{t_M} dt\,|t\rangle_T \otimes |\psi(t)\rangle_S \otimes |r\rangle_M \otimes |r\rangle_N \\ &+ \int_{t_M}^{t_N} dt\,|t\rangle_T \otimes \sum_i \hat{U}_S(t, t_M)\hat{K}_M^i\,|\psi(t_M)\rangle_S \otimes |i\rangle_M \otimes |r\rangle_N \\ &+ \int_{t_M}^{\infty} dt\,|t\rangle_T \otimes \sum_{i,j} \hat{U}_S(t, t_N)\hat{K}_N^j \hat{U}_S(t_N, t_M)\hat{K}_M^i\,|\psi(t_M)\rangle_S \\ &\qquad \otimes |i\rangle_M \otimes |j\rangle_N\,. \end{aligned} \tag{9}$$

Note that this timeless representation of the total wave-function by a series of superimpositions of system/detector entangled states is *de facto* equivalent to the branched wave-functions in Everett's original paper [9], where the central concepts turn around the relativity of states of measured and measuring systems.

How can we now associate a probability to the measurement outputs? Different approaches have been developed in the past in the discussed context, as the definition via transition amplitudes in Quantum Gravity [20, 24] or postulating the validity of the Bohr rule [21]. Our approach consists in considering the general properties of Hilbert spaces and requiring only basic properties of the probability function $\wp$. For a Hilbert space with finite dimension $\mathcal{H}$, the Gleason-Busch theorem [27, 28] demonstrates that the probability function is in fact univocally defined by the trace rule

$$\wp(a) = tr(\hat{\rho}\,\hat{\Pi}^a), \tag{10}$$

where $\hat{\Pi}^a$ is a positive-valued operator in $\mathcal{H}$ and $\hat{\rho}$ is the density matrix of a given state. This is actually the same approach used by Pages and Wootters [19], where however only projector operators are considered. It is natural to extend this definition of probability to the infinite dimensional space $\mathcal{K}$. Such an extension is in fact equivalent to the standard definition in Quantum Gravity [29]. In our case, a generic measurement with output a must be represented by an operator $\hat{\Pi}_a$ in the kinematic Hilbert space $\mathcal{K} = \mathcal{H}_T \otimes \mathcal{H}_S \otimes \mathcal{H}_M \otimes \mathcal{H}_N$. Because there is only one clock subspace $\mathcal{H}_T$, it is impossible to construct a two-time probability for the event "m at the time t_1 and n at the time t_2", as discussed in Ref. [22]. On the contrary, it is possible to consider the measurement "$m \wedge n \wedge t$" $\equiv$ "m *AND* n *AND* t" corresponding to the "m and n results obtained at the time t", when the two measurements occur at the times t_M and t_N, respectively. Like in Refs. [20, 21], the formulation of such a probability implies that the two detector states $|m\rangle_M$ and $|n\rangle_N$ are stored in an internal memory that is read at the time $t > t_M, t_N$. It is important to note that, in contrast to a previous work on subsequent measurements without timeless notation [30], here the operator "$\wedge$" $\equiv$ "AND" is symmetric ($a \wedge b = b \wedge a$). The measurement order is in fact defined by the Hamiltonian itself. We are not simply discussing about $\wp(m \wedge n \wedge t)$, but are implicitly considering the probability $\wp(m \wedge n \wedge t | M_{t_M} \wedge N_{t_N})$, where we explicit the prior information on the Hamiltonian structure with the measurements M at the time t_M and N at t_N. More complex scenarios with an undefined measurement order in the Hamiltonian can be also considered [23, 31] but they are not discussed here.

We emphasize once again that there is no direct measurement on S but only via the ancillary detector states. The operator associated with this measurement is $\hat{\Pi}^{m,n,t}_{MN} = |t\rangle_T\, {}_T\langle t| \otimes \mathbb{1}_S \otimes |m\rangle_M\, {}_M\langle m| \otimes |n\rangle_N\, {}_N\langle n|$ that leads to

$$\begin{aligned} \wp(m \wedge n \wedge t) &= tr(\hat{\rho}\,\hat{\Pi}^{m,n,t}_{MN}) = ||({}_T\langle t| \otimes {}_M\langle m| \otimes {}_N\langle n|)\,|\Psi\rangle\,||^2 \\ &= \begin{cases} ||{}_M\langle m|r\rangle_M \;\; {}_N\langle n|r\rangle_N\,|\psi(t_M)\rangle_S\,||^2 & \text{if } t < t_M, \\ ||{}_N\langle n|r\rangle_N\, K^m_M\,|\psi(t_M)\rangle_S\,||^2 & \text{if } t_M \le t < t_N, \\ ||K^n_N \hat{U}_S(t_N, t_M) K^m_M\,|\psi(t_M)\rangle_S\,||^2 & \text{if } t \ge t_N. \end{cases} \end{aligned} \tag{11}$$

With two separate detections, we can now discuss the conditional probabilities of the different outcomes. The conditional probabilities and $\wp(n|m \wedge t)$ can be calculated with the standard procedure from the Bayes' rule and the probabilities $\wp(n \wedge t)$ and $\wp(m \wedge t)$ by applying Eq. (10) with the corresponding operators. We obtain in particular

$$\wp(n|m \wedge t) = \frac{\wp(m \wedge n \wedge t)}{\wp(m \wedge t)} = \frac{tr(\hat{\rho}\,\hat{\Pi}^{m,n,t}_{MN})}{tr(\hat{\rho}\,\hat{\Pi}^{m,t}_{M})} = \wp(n \wedge t | m \wedge t), \tag{12}$$

where $\Pi^{m,t}_M = |t\rangle_T\, {}_T\langle t| \otimes \mathbb{1}_S \otimes |m\rangle_M\, {}_M\langle m| \otimes \mathbb{1}_N$. For $\wp(m|n \wedge t)$ a similar equation can be derived. The validity of the last equivalence in the above formula is justified by the presence of the time measurement operator in both numerator and denominator [32].

With the notation and formulas presented above, we can now consider the case of Wigner's friend. This measurement scenario, introduced for the first time by Wigner [15], is constituted by an observer W (Wigner) that observes another observer F (his/her friend) who performs a quantum measurement on a physical system S. The friend makes a measurement of the system and records a well-defined result. During this measurement and just after, the system/friend ensemble is isolated from Wigner. For W, the measurement of F on S becomes entangled and both are described by a superposition of the different possible final states. Paradoxically, W and F have at the same time a different description of S. The Wigner's friend scenario with the timeless formalism has been extensively discussed in a recent work of Baumann, Brukner and collaborators [22], where several forms of probabilities are considered. The notation presented here leads in contrast to a unique form of probability. Its derivation is presented in detail in the following paragraphs applying the results presented above.

Without losing in generality and to keep simple expressions, we consider no-free dynamics for three systems: the studied system S, Wigner's friend F, and Wigner W. The corresponding Hamiltonian, relative to a measurement M at the time t_M between the system and the friend, and a measurement N at the time t_N between Wigner and the ensemble friend-system, is

$$\begin{aligned} H = \hat{p}_T \otimes \mathbb{1}_S \otimes \mathbb{1}_F \otimes \mathbb{1}_W + \mathbb{1}_T \otimes \hat{H}_S \otimes \mathbb{1}_F \otimes \mathbb{1}_W \\ + \hat{V}_{SF}\,\delta(\hat{T} - t_M) \otimes \mathbb{1}_W + \hat{V}_{SFW}\,\delta(\hat{T} - t_M) \\ + \mathbb{1}_T \otimes \mathbb{1}_S \otimes \hat{H}_F \otimes \mathbb{1}_W + \mathbb{1}_T \otimes \mathbb{1}_S \otimes \mathbb{1}_F \otimes \hat{H}_W. \end{aligned} \tag{13}$$

We consider a system initial condition described by $|\psi\rangle_S = a\,|\uparrow\rangle_S + b\,|\downarrow\rangle_S$ with $|a|^2 + |b|^2 = 1$. For Wigner, after the friend's measurement, S and F are described by

$$|\psi\rangle_S \otimes |\varphi\rangle_F = a\,|\uparrow\rangle_S\,|\uparrow\rangle_F + b\,|\downarrow\rangle_S\,|\downarrow\rangle_F\,. \tag{14}$$

We consider also that Wigner's measurement on F and S consists in the detection of the ancillary state $|yes\rangle_W$,

TABLE I. $\wp(f \wedge w \wedge t)$ values for $t > t_N$.

f \ w	yes	no
$\uparrow$	$\lvert a\rvert^2\lvert\alpha\rvert^2$	$\lvert a\rvert^2\lvert\beta\rvert^2$
$\downarrow$	$\lvert b\rvert^2\lvert\beta\rvert^2$	$\lvert b\rvert^2\lvert\alpha\rvert^2$

TABLE II. Values of $\wp(w \wedge t)$ (left) and $\wp(f \wedge t)$ (right) for $t > t_N$.

w	$\wp(w \wedge t)$
yes	$\lvert a\rvert^2\lvert\alpha\rvert^2 + \lvert b\rvert^2\lvert\beta\rvert^2$
no	$\lvert b\rvert^2\lvert\alpha\rvert^2 + \lvert a\rvert^2\lvert\beta\rvert^2$

f	$\wp(f \wedge t)$
$\uparrow$	$\lvert a\rvert^2$
$\downarrow$	$\lvert b\rvert^2$

which corresponds to the operator on the system-friend space $\hat{\Pi}_N^{yes} = |yes\rangle_{SF\ SF}\langle yes| \in \mathcal{H}_S \otimes \mathcal{H}_F$ with

$$|yes\rangle_{SF} = \alpha\,|\uparrow\rangle_S\,|\uparrow\rangle_F + \beta\,|\downarrow\rangle_S\,|\downarrow\rangle_F\,, \tag{15}$$

with $|\alpha|^2 + |\beta|^2 = 1$, and its complementary operator $\hat{\Pi}_N^{no} = \mathbb{1}_S \otimes \mathbb{1}_F - \hat{\Pi}_N^{yes}$

The global wave-function with these two measurements has a similar form as Eq. (9) [33] and the probability of having the measurement results w and f at the time t is associated to the operators $\hat{\Pi}_{MN}^{f,w,t} = |t\rangle_{T\ T}\langle t| \otimes \mathbb{1}_S \otimes |f\rangle_{F\ F}\langle f| \otimes |w\rangle_{W\ W}\langle w|$. Its values are

$$\wp(f\wedge w\wedge t) = tr(\hat{\rho}\,\hat{\Pi}_{MN}^{f,w,t}) = ||({}_T\langle t|\otimes_F\langle f|\otimes_W\langle w|)\,|\Psi\rangle\,||^2 = \begin{cases} ||{}_F\langle f|r\rangle_F\ {}_W\langle w|r\rangle_W\,|\psi\rangle_S\,||^2 & \text{if } t < t_M, \\ ||{}_W\langle w|r\rangle_W\,K_M^f\,|\psi\rangle_S\,||^2 & \text{if } t_M \leq t < t_N, \\ ||K_N^w K_M^f\,|\psi\rangle_S\,||^2 & \text{if } t \geq t_N, \end{cases} \tag{16}$$

where $|r\rangle_F$ and $|r\rangle_F$ are the "ready" state of the friend and of Wigner. The associated Kraus operators are $\hat{K}_M^{\uparrow} = |\uparrow\rangle_{S\ S}\langle\uparrow|\,, \hat{K}_M^{\downarrow} = |\downarrow\rangle_{S\ S}\langle\downarrow|\,, \hat{K}_N^{yes} = |yes\rangle_{S\ S}\langle yes| = (\alpha\,|\uparrow\rangle_S + \beta\,|\downarrow\rangle_S)(\alpha^*{}_S\langle\uparrow| + \beta^*{}_S\langle\downarrow|), \hat{K}_N^{no} = |no\rangle_{S\ S}\langle no| = (-\beta^*\,|\uparrow\rangle_S + \alpha^*\,|\downarrow\rangle_S)(-\beta_S\langle\uparrow| + \alpha_S\langle\downarrow|)$. The different possible values of $\wp(f \wedge w \wedge t)$ are summarized in Tab. I.

Similarly to Eq. (12), the conditional probabilities $\wp(w|f \wedge t)$ and $\wp(f|w \wedge t)$ can be calculated from

$$\wp(w|f \wedge t) = \frac{\wp(f \wedge w \wedge t)}{\wp(f \wedge t)} = \frac{tr(\hat{\rho}\,\hat{\Pi}_{MN}^{f,w,t})}{tr(\hat{\rho}\,\hat{\Pi}_M^{f,t})} \tag{17}$$

and

$$\wp(f|w \wedge t) = \frac{\wp(f \wedge w \wedge t)}{\wp(w \wedge t)} = \frac{tr(\hat{\rho}\,\hat{\Pi}_{MN}^{f,w,t})}{tr(\hat{\rho}\,\hat{\Pi}_N^{w,t})} \tag{18}$$

where $\hat{\Pi}_M^{f,t} = |t\rangle_{T\ T}\langle t| \otimes \mathbb{1}_S \otimes |f\rangle_{F\ F}\langle f| \otimes \mathbb{1}_W$ and $\hat{\Pi}_N^{w,t} = |t\rangle_{T\ T}\langle t| \otimes \mathbb{1}_S \otimes \mathbb{1}_F \otimes |w\rangle_{W\ W}\langle w|$. To deduce their values, $\wp(w \wedge t)$ and $\wp(f \wedge t)$ have to be evaluated first. For $\wp(w \wedge t)$ we have

$$\wp(w \wedge t) = tr(\hat{\rho}\,\hat{\Pi}_N^{w,t}) = ||({}_T\langle t| \otimes_W\langle w|)\,|\Psi\rangle\,||^2 = \begin{cases} ||{}_W\langle w|r\rangle_W\,|\psi\rangle_S \otimes |r\rangle_F\,||^2 & \text{if } t < t_M, \\ ||{}_W\langle w|r\rangle_W \sum_f K_M^f\,|\psi\rangle_S \otimes |f\rangle_F\,||^2 & \text{if } t_M \leq t < t_N, \\ ||K_N^w \sum_f K_M^f\,|\psi\rangle_S \otimes |f\rangle_F\,||^2 & \text{if } t \geq t_N, \end{cases} \tag{19}$$

Similarly, for $\wp(f \wedge t)$ we have

$$\wp(f \wedge t) = tr(\hat{\rho}\,\hat{\Pi}_M^{w,t}) = ||({}_T\langle t| \otimes_F\langle f|)\,|\Psi\rangle\,||^2 = \begin{cases} ||{}_F\langle f|r\rangle_F\,|\psi\rangle_S\,||^2 \otimes |r\rangle_W & \text{if } t < t_M, \\ ||K_M^f\,|\psi\rangle_S \otimes |r\rangle_W\,||^2 & \text{if } t_M \leq t < t_N, \\ ||\sum_w K_N^w K_M^f\,|\psi\rangle_S \otimes |w\rangle_W\,||^2 & \text{if } t \geq t_N, \end{cases} \tag{20}$$

TABLE III. Values of $\wp(w|f \wedge t)$ (left) and $\wp(f|w \wedge t)$ (right) for $t > t_N$.

f \ w	yes	no
$\uparrow$	$\lvert\alpha\rvert^2$	$\lvert\beta\rvert^2$
$\downarrow$	$\lvert\beta\rvert^2$	$\lvert\alpha\rvert^2$

f \ w	yes	no
$\uparrow$	$\frac{\lvert a\rvert^2\lvert\alpha\rvert^2}{\lvert a\rvert^2\lvert\alpha\rvert^2 + \lvert b\rvert^2\lvert\beta\rvert^2}$	$\frac{\lvert a\rvert^2\lvert\beta\rvert^2}{\lvert b\rvert^2\lvert\alpha\rvert^2 + \lvert a\rvert^2\lvert\beta\rvert^2}$
$\downarrow$	$\frac{\lvert b\rvert^2\lvert\beta\rvert^2}{\lvert a\rvert^2\lvert\alpha\rvert^2 + \lvert b\rvert^2\lvert\beta\rvert^2}$	$\frac{\lvert b\rvert^2\lvert\alpha\rvert^2}{\lvert b\rvert^2\lvert\alpha\rvert^2 + \lvert a\rvert^2\lvert\beta\rvert^2}$

The different values of $\wp(w\wedge t)$ and $\wp(f\wedge t)$ are summarized in Tab. II. The conditional probabilities $\wp(w|f \wedge t)$ and $\wp(f|w \wedge t)$ can be deduced and are presented in Tab. III. As we can see, the probability function is defined without ambiguities and is automatically normalized. The relativity of the states is visible in particular in the last equations corresponding to $\wp(w\wedge t)$ and $\wp(f\wedge t)$. From the Wigner point of view, the superimposition of the friend and the system is indeed present and represented by the sum of terms $\hat{K}_M^f\,|\psi\rangle_S \otimes |f\rangle_F$ in Eq. (19). Similarly, the friend sees the state of Wigner as a superposition represented the sum of the terms $\hat{K}_M^f\,|\psi\rangle_S \otimes |f\rangle_F$ in Eq. (20). Such a superimposition of states is due to the non-compatiblity of basis between Wigner and the friend, essential for the calculation of $\wp(w|f \wedge t)$ via the term $\wp(f \wedge t)$. Neither Wigner nor the friend have a privileged observation position and the two associated probabilities lead to the same type of expression but different forms and, more importantly, with no universal description of the system S. The probabilities depend only on the relations $S - W$, $S - F$ and $F - W$. Recently, no-go theorems evoking situations similar to the Wigner's friend have been formulated and experimentally tested [34–36]. They demonstrated that one of the following assumptions has to be violated: i) the universal validity of quantum mechanics, ii) the locality, iii) the freedom of choice on the measurement settings and iv) the observer-independent experimental outcomes. The above probability expressions indicate a clear violation of the measurement output universality. If we assume that the experimental outputs are relative to each pair of observer and observed, and we can have a different description of a same system from different obervers, there is no real paradox in the Wigner's friend scenario.

The key feature of the Wigner's friend scenario is the isolation of W from F and S. The experimental realizations of scenarios similar to the Wigner's friend case are based on photons for both the system S and the friend F [35, 36], where the coupling and decoupling between different subsystems can be easily controlled. If we consider a real human friend that operates a measurement in a real laboratory, the situation would be much different because it requires screeneing any mechanical and electromagnetic interaction between macroscopic systems [37]. The coupling between the subsystems F and S with the laboratory environment $\mathcal{E}_{FS}$ can also have an influence [38]. Because of the high number of degrees of freedom of $\mathcal{E}_{FS}$, decoherence processes can bring to a destruction of state superimposition for F and S [2, 39] producing a universal measurement output, a stable fact in the language of Ref. [38].

In conclusion, we presented here a derivation of the conditional probability $\wp$ for subsequent measurements in the context of the quantum time approach. This is obtained with a minimal number of assumptions, considering only systems subject to the rules of Quantum Mechanics, without classical systems, and interacting with each other by unitary operators with the implementation of POVM. Applying the Gleason-Busch theorem to the global kinetic Hilbert space, $\wp$ is simply built from first principles. In particular for the non-relativistic case presented here with a one clock system, it emerges that a two-time probability function cannot be defined. The causal order of the subsequent measurements are encoded in the Hamiltonian, whose structure should be considered as prior knowledge in the probability function.

When the Wigner's friend scenario is considered, the relativity of measurements naturally emerges, without contradictions or ambiguities. The roles of Wigner and the friend are completely interchangeable, and Wigner can be seen as a superimposed state from the friend. This is particularly evident in the marginal probabilities $\wp(w \wedge t)$ and $\wp(f \wedge t)$ and, as consequence, the conditional probabilities $\wp(w|f \wedge t)$ and $\wp(f|w \wedge t)$. In the future, it would be very interesting to experimentally investigate on the identification of the important parameter responsible of the coherences between Wigner W and the friend F. Is it the degree of complexity the friend and/or her/his environment? Or is it related to the mass and/or energy levels associated with the friend? Or maybe is it just related to the coupling/interaction between W and F? If it is the case, could we modulate such an interaction, like in interference experiments with a tunable which-path detection sensitivity [30, 40]?

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