$y = X\beta$ $\qquad X = [x_1 \quad x_2]$

$T \times k$

endogenous

exogenous

$Z = [z_1, x_2]$ $\qquad \boxed{L > k\,!}$

1) Show that $L = k$ TSLS = IV

$$\begin{cases} y = X\beta + \varepsilon \\ X = Z\Gamma + \eta \end{cases}$$

Reduced form

$$y = (Z\Gamma + \eta)\beta + \varepsilon$$

$$y = Z\Gamma\beta + \eta\beta + \varepsilon \longrightarrow u$$

$$y = Z\lambda + u \qquad \lambda = \Gamma\beta \qquad \beta = \Gamma^{-1}\lambda$$

in this case Γ is square

and invertible if

Γ_{11} is full rank

$$\hat{\Gamma} = (Z'Z)^{-1}Z'X$$

$$\Gamma^{-1} = (Z'X)^{-1}Z'Z \qquad \text{while } \hat{\lambda} = (Z'Z)^{-1}Z'y$$

hence $\hat{\beta}_{IV} = (Z'X)^{-1}\cancel{Z'Z}\cancel{(Z'Z)^{-1}}Z'y = (Z'X)^{-1}Z'y$

Derive asymptotic distribution of $\hat{\beta}_{IV}$

$$(\hat{\beta}-\beta^0) = \left(\frac{\hat{X}'\hat{X}}{N}\right)^{-1}\frac{\hat{X}'\varepsilon}{N}$$

this will converge to a matrix Q_z^{-1}

while

$$\frac{\hat{X}'\varepsilon}{N} = Z(Z'Z)^{-1}Z'X\varepsilon$$

$$\text{plim}\,\hat{\beta}_{TSLS} = \beta_0 \qquad X'P_Z \qquad P_Z X$$

$$\text{plim}\left(\frac{\hat{X}'\hat{X}}{N}\right)^{-1}\frac{\hat{X}'\varepsilon}{N} \longrightarrow \frac{1}{N}X'Z(Z'Z)^{-1}Z'\varepsilon \to 0$$

$$\to Q_z^{-1}\cdot 0$$

$$\hat{X}'\hat{X} \qquad (X'P_Z X)^{-1}$$

Hence $\beta_{IV} \longrightarrow \beta_0$

$$\text{plim}\,\hat{\beta}_{TSLS} \longrightarrow \beta_0$$

For the asymptotic

$$\sqrt{N}\,(\hat{\beta}_{TSLS} - \beta_0) = N(0;?)$$

I have to look at

$$\text{plim}\left(\frac{\hat{X}'\hat{X}}{N}\right)^{-1}\sqrt{N}\,\frac{\hat{X}'\varepsilon}{N} \to \text{plim}\,\frac{X'Z}{N}\left(\frac{Z'Z}{N}\right)^{-1}\frac{Z'\varepsilon}{N}$$

$$= Q_{xz}\,Q_{zz}\,0$$

$$\left(\frac{X'Z}{N}\left(\frac{Z'Z}{N}\right)^{-1}\cancel{Z'Z(Z'Z)^{-1}}\frac{Z'X}{N}\right)$$

$$\left(Q_{xz}\,Q_{zz}^{-1}\,Q_{zx}\right)^{-1}$$

$$\sqrt{N}\,\frac{\hat{X}'\varepsilon}{N} \Rightarrow \sqrt{N}\,\frac{Z'\varepsilon}{N} \longrightarrow N(0;H)$$

hence

$$\sqrt{N}\,(\hat{\beta}_{TSLS} - \beta_0) = \left(Q_{xz}\,Q_{zz}^{-1}\,Q_{zx}\right)^{-1} Q_{xz}\,Q_{zz}^{-1}\,N(0;H)$$

$$= N\left(0;\left(Q_{xz}\,Q_{zz}^{-1}\,Q_{zx}\right)^{-1} Q_{xz}\,Q_{zz}^{-1}\,H\,Q_{zz}^{-1}\,Q_{zx}\left(Q_{xz}\,Q_{zz}^{-1}\,Q_{zx}\right)^{-1}\right)$$

$$\text{Var}(Z'\varepsilon) = E[Z'\varepsilon\varepsilon'Z] = \sigma^2_\varepsilon\,(Z'Z)$$

$$\Sigma = \sigma^2_\varepsilon\left(Q_{xz}\,Q_{zz}^{-1}\,Q_{zx}\right)^{-1}\cancel{Q_{xz}\,Q_{zz}^{-1}}\,\cancel{Q_{zz}\,Q_{zz}^{-1}}\,\cancel{Q_{zx}}\left(Q_{xz}\,Q_{zz}^{-1}\,Q_{zx}\right)^{-1}$$

$$= \sigma^2_\varepsilon\left(Q_{xz}\,Q_{zz}^{-1}\,Q_{zx}\right)^{-1}$$