F-test : Chow-test → Diagnostic test, carried out after having estimated the model, based on an auxiliary regression

$$Y = X\beta + \varepsilon$$

i split the regression in two groups $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $(N_1 \times 1)$ $(N_2 \times 1)$

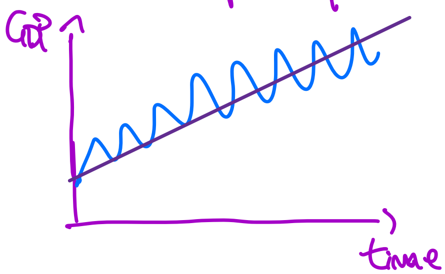
(vertical concatenation in this case)

$N_1 + N_2 = N$

$$\begin{cases} Y_1 = X_1\beta_1 + \varepsilon_1 \\ Y_2 = X_2\beta_2 + \varepsilon_2 \end{cases}$$

we want to test
$H_0: \beta_1 = \beta_2$ $H_1: \beta_1 \neq \beta_2$

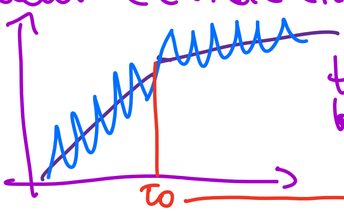
Imagine a time series model
$\Delta GDP = \beta_0 + \beta_1 T + \varepsilon_t$



in the model under the null the coefficients are the same with regard to time they do not change
↓
i have to construct another model

white noise variation around the trend

- i can believe that there is a big change in the parameters after a given event (ex. oil crisis)



→ only the slope changes, the variance of the error remains the same

there is a trend, but it changes

→ the break date is known in this case

$H_0: \quad y = \beta_0 + \beta_1 \tau + \varepsilon$

maybe this is like another intercept

$H_1: \quad y = \beta_0 + \beta_1 \tau + \beta_2 D(\tau \geq \tau_0) + \underbrace{\beta_3 \tau D(\tau \geq t_0)}_{\text{interaction dummy}} \quad \begin{cases} D(t \geq \tau_0) \\ D(t < \tau_0) = 0 \end{cases}$

τ_0: break date

$$H_1: \begin{cases} y = \beta_0 + \beta_1 \tau + \varepsilon \quad \text{if } \tau < \tau_0 \\ y = (\beta_0 + \beta_2) + (\beta_1 + \beta_3)\tau + \varepsilon \quad \text{if } \tau \geq t_0 \end{cases}$$

can be related to time or to other things that can be reconducted to a dummy variable

if $\beta_2 = 0$ and $\beta_3 = 0$ we are under the null hypothesis H_0

important that there is variability in the dummy otherwise there is multicollinearity

in Matlab trend = [1:T];
and to create the Dummy:
D=(trend ≥ 100)
↳ add "real" in front to transform the result in numbers and not logical values

i test H_1

$$y = \beta_0 + \beta_1 \tau + \beta_2 D(t \geq t_0) + \beta_3 \tau \cdot D(t \geq t_0)$$

$$W = \left[\text{iota} \;\; \text{Trend} \;\; \underbrace{D(t \geq \tau_0) \;\; \tau \cdot D(t \geq t_0)}_{\text{we want to test if those two are } = 0} \right]$$

$$y = W\psi + \varepsilon$$

$$\hat{\psi} = (W'W)^{-1} W' y$$

$$R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we are putting two parameters of restriction (2×4)

$$F\text{-test} = \frac{\dfrac{(R\hat{\psi} - c)'\left[R(W'W)^{-1}R'\right]^{-1}(R\hat{\psi} - c)}{2}}{\dfrac{\hat{\varepsilon}'\hat{\varepsilon}}{N-4-2}} \sim F(2, N-4)$$

$(N-4)$ in this case

General
$$\frac{\underset{(k \times 2k)}{(R}\,\underset{(2k \times 1)}{\hat{\psi}} - \underset{(k \times 1)}{c)'}\left[R(W'W)^{-1}R'\right]^{-1}(R\hat{\psi} - c)/k}{\dfrac{\hat{\varepsilon}'\hat{\varepsilon}}{N-2k}} \sim F(k, N-2k)$$

$H_0: \; y = X\beta + \varepsilon$

$H_1: \; y = \underset{(N \times k)}{X}\beta + \underset{(N \times k)}{Z}\gamma + \varepsilon$

$\underset{(N \times k)}{Z} = X \odot D \longrightarrow X.^{*}\ \text{repmat}(D, 1, k)$

each column of x multiplied by D

i just want to repeats it along the columns

$\underset{(2k \times 1)}{\psi} = [\beta', \gamma']'$

is $\hat{\beta}_{OLS}$ consistent for β^0

$\hat{\beta}_{OLS} \xrightarrow{P} \beta^0$ this is the consistency

$$\lim_{n \to \infty} P\left(\left|\hat{\beta}_N - \beta^0\right| > \varepsilon\right) = 0$$

Now the estimator is a function depending on N

WLLN

$\bar{X}_N \xrightarrow{P} E[X]$

$\{x_1 \dots x_N\}$ iid

$E(x_i) = E[X]$

$Var(x_i) = \sigma^2$

$$\hat{\beta}_N = (X'X)^{-1}X'y = \beta^0 + (X'X)^{-1}X'\varepsilon$$

$\operatorname{plim}_{N \to \infty} (X'X)^{-1}X'\varepsilon = 0$ $\qquad (X'X)^{-1}X'\varepsilon \xrightarrow{P} 0$

is the same just change of notation

probability limit

$$\left(\frac{X'X}{N}\right)^{-1} \frac{X'\varepsilon}{N}$$

i am dividing an multiplying by N

$X = [i \;\; x]$

$$\frac{X'X}{N} = \begin{bmatrix} \frac{N}{N} & \frac{\sum x_i}{N} \\ \frac{\sum x_i}{N} & \frac{\sum x_i^2}{N} \end{bmatrix}$$

$$\operatorname{plim}_{N \to \infty} \left(\frac{X'X}{N}\right)^{-1} = Q^{-1}$$

(continuous mapping theorem or Slutsky theorem)

$$\operatorname{plim} \frac{X'X}{N} = \begin{bmatrix} 1 & E(x) \\ E(x) & E(x^2) \end{bmatrix}$$

$\Downarrow$

Q matrix containing moments of the xs

we assume that is not a matrix of zero (it has to be invertible)

continuous transformation theorem
to a sequence of random v. converging to a limit
the transformation apply also to the limit

Where does $\frac{X'\varepsilon}{N}$ converge to?

$$X = [i \; x]$$

$$\underset{N \to \infty}{\text{Plim}} \frac{X'\varepsilon}{N} = \begin{bmatrix} \frac{i'\varepsilon}{N} \\ \frac{x'\varepsilon}{N} \end{bmatrix} = \begin{bmatrix} \frac{\Sigma \varepsilon_i}{N} \\ \frac{\sum_{i=1}^{N} x_i \varepsilon_i}{N} \end{bmatrix}$$

$$\bar{\varepsilon} \xrightarrow{P} E(\varepsilon) = 0$$

$$\bar{w} = \overline{x\varepsilon} = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\varepsilon_i x_i}_{= w_i}$$

$$\underset{N \to \infty}{\text{Plim}} \frac{X'\varepsilon}{N} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{k \times 1}$$

$$\bar{w} \xrightarrow{P} E[w] = E[\varepsilon w] = 0$$

Exogeneity

$$E(w) = E[E[w \mid x]]$$
$$= E[E[\varepsilon x \mid x]] = E[x E(\varepsilon \mid x)] = 0$$

$$\underset{N \to \infty}{\text{Plim}} \left(\frac{X'X}{N}\right)^{-1} \frac{X'\varepsilon}{N} = Q^{-1} 0 = 0$$

$$\text{Since } \underset{N \to \infty}{\text{Plim}} \hat{\beta}_N = \underset{N \to \infty}{\text{Plim}} (\beta^0 + (X'X)^{-1} X'\varepsilon)$$

$$= \beta_0 + \text{Plim} (X'X)^{-1} X'\varepsilon = \beta^0$$

CLT $(x_1 \ldots x_N)$ $\underline{iid}$

$$E(x) = \mu$$
$$Var(x) = \sigma^2$$

No need for the x to be a gaussian, they can be of any distribution

$$\sqrt{N}(\bar{x}_N - \mu) \xrightarrow{d} N(0; \sigma^2)$$

this is sample size

this is normal distribution, the two Ns are different

$$\hat{\beta} = \beta_0 + (X'X)^{-1} X'\varepsilon$$

$$\sqrt{N}\hat{\beta}_N = \sqrt{N}\beta^0 + \left(\frac{X'X}{N}\right)^{-1} \sqrt{N} \left(\frac{X'\varepsilon}{N}\right)$$

otherwise they will converge to a degenerate random variable

$$\sqrt{N}(\hat{\beta}_N - \beta^0) = \left(\frac{X'X}{N}\right)^{-1} \boxed{\sqrt{N}\left(\frac{X'\varepsilon}{N}\right)}$$

converge to Q^{-1} in probability

where does this converge

$$\sqrt{N}\left(\frac{X'\varepsilon}{N}\right) = \sqrt{N}\,\overline{w} \xrightarrow{\delta} N(0; \sigma_\varepsilon^2 Q)$$

$(k \times 1)$ vector of sample averages

$$\frac{X'\varepsilon}{N} = \begin{bmatrix} \frac{i'\varepsilon}{N} \\ \frac{x'\varepsilon}{N} \end{bmatrix}$$

$$E(\overline{w}) = 0$$

$$Var(\overline{w}) = \frac{E[x_i \varepsilon_i \varepsilon_i' x_i']}{N} = \frac{\sigma_\varepsilon^2 E[x_i x_i']}{N} = \frac{\sigma_\varepsilon^2 Q}{N}$$

$$\sqrt{N}(\hat{\beta}_N - \beta^0) \xrightarrow{\delta} Q^{-1} N(0, \sigma_\varepsilon^2 Q)$$

$$= N(0; \sigma_\varepsilon^2 Q^{-1} \cancel{Q} \cancel{Q^{-1}})$$

$$= N(0; \sigma_\varepsilon^2 Q^{-1})$$

$$Var(\overline{w}) = E[x'\varepsilon\varepsilon' x] = \sigma_\varepsilon^2 E[x'x] = \sigma_\varepsilon^2 Q$$

at infinity

$$\lim_{N\to\infty}(\hat{\beta}_{OLS}-\beta^0)$$ we analyze this

$$\hat{\beta}_{OLS}-\beta^0 = (X'X)^{-1}X'y - \beta^0 = (X'X)^{-1}X'(X\beta^0+\varepsilon)-\beta^0$$
$$=(X'X)^{-1}X'X\beta^0+(X'X)^{-1}X'\varepsilon-\beta^0$$
$$=\beta^0+(X'X)^{-1}X'\varepsilon-\beta^0=\underline{(X'X)^{-1}X'\varepsilon}$$

Hence we study
$$\lim_{n\to\infty}(X'X)^{-1}X'\varepsilon \qquad \frac{(X'X)^{-1}}{N} \longrightarrow \text{converges to } Q^{-1}$$

$$\frac{X'\varepsilon}{N}=\begin{bmatrix} i' \\ x' \end{bmatrix}\varepsilon = \begin{bmatrix} \frac{i'\varepsilon}{N} \\ \frac{x'\varepsilon}{N} \end{bmatrix} = \begin{bmatrix} \bar{\varepsilon} \\ \overline{x\varepsilon} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1N} \\ x_{21} & & & \\ x_{k1} & - & - & x_{kN} \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

$$\bar{w}=\overline{x\varepsilon}$$
$$E[\bar{w}]=E[E[\bar{w}|x]]=E[E[x\varepsilon|x]]$$
$$=E[xE(\varepsilon|x)]=0$$

$$x_{11}\varepsilon_1+x_{12}\varepsilon_2+x_{13}\varepsilon_3$$

$$\frac{1}{n}\sum x_{1i}\varepsilon_i$$

CLT

$$\hat{\beta}_N=\beta^0+\frac{(X'X)^{-1}}{N}\frac{X'\varepsilon}{N}$$

$$\sqrt{N}(\hat{\beta}_N-\beta^0)=\sqrt{N}\frac{X'\varepsilon}{N}\left(\frac{X'X}{N}\right)^{-1}\longrightarrow Q^{-1}$$

$$\bar{w}$$

$$\sqrt{N}\frac{X'\varepsilon}{N}=\sqrt{N}\,\bar{w} \qquad E(\bar{w})=0$$
$$\text{Var}(\bar{w})=\frac{E[x_i\varepsilon_i\varepsilon_i'x_i']}{N}=\frac{\sigma_\varepsilon^2 Q}{N}$$

$$\mathrm{Var}\left(\sqrt{N}\,\overline{W}\right) = N\,\frac{\sigma_\varepsilon^2 Q}{N} = \sigma_\varepsilon^2 Q$$

Hence

$$\sqrt{N}\left(\hat{\beta}_N - \beta^0\right) = Q^{-1} N\left(0; \sigma_\varepsilon^2 Q\right) = N\left(0; Q^{-1}\sigma_\varepsilon^2 \cancel{Q}\,\cancel{Q^{-1}}\right)$$

$$= N\left(0; \sigma_\varepsilon^2 Q^{-1}\right)$$

$$E\left[x_i x_i'\right] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix} =$$

$$\sqrt{N}\left(\hat{\beta}_{GLS} - \beta^0\right) \sim N\left(0; \Omega Q^{*-1}\right) \qquad Q^* = \left(\frac{X'X}{N}\right)$$

$$R\sqrt{N}\left(\hat{\beta}_{GLS} - \beta^0\right) = \sqrt{N}\left(R\hat{\beta}_{GLS} - c\right) \sim N\left(0; R'\Omega Q^{-1} R\right)$$

$$R'\Omega\,\frac{X^{*\prime} X^*}{n}\,R = R'\Omega\left(\frac{X'C'CX}{N}\right)R$$

$$= R'\Omega\left(\frac{X'\Omega^{-1}X}{N}\right)R$$

$$\frac{1}{\sqrt{N}} X'\varepsilon = \sqrt{N}\left(\overline{w} - E[\overline{w}]\right)$$

$$\downarrow$$

$$\sqrt{N}\left(\frac{X'\varepsilon}{N}\right) = \sqrt{N}\begin{bmatrix} i'\varepsilon \\ x'\varepsilon \end{bmatrix} = \sqrt{N}\left[\frac{1}{N}\sum_{i=1}^{N} x_i\varepsilon_i\right]$$

$$\frac{1}{\sqrt{N}}(X'\varepsilon) = \sqrt{N}\left(\frac{X'\varepsilon}{N}\right) = \sqrt{N}\,\overline{w} = \sqrt{N}\left(\overline{w} - \underset{\overset{\parallel}{0}}{E(\overline{w})}\right)$$

$\sqrt{N}(\overline{w})$ is a normal

$\overline{w}$ is the average of N independent random vectors $w_i = x_i\varepsilon_i$
$E(w_i) = 0$ and variance
$Var[w_i] = Var(x_i\varepsilon_i) = E(x_i\varepsilon_i\varepsilon_i' x_i')$

$$= \sigma_\varepsilon^2 E[x_i x_i'] = \sigma_\varepsilon^2 Q$$

$$Var\left(\frac{X'\varepsilon}{N}\right) \Rightarrow E\left[\left(\frac{X'\varepsilon}{N}\right)\left(\frac{X'\varepsilon}{N}\right)'\right]$$

$$\frac{X'\varepsilon}{N} = \begin{bmatrix} \iota'\varepsilon \\ x'\varepsilon \end{bmatrix}\begin{bmatrix} \varepsilon'\iota & \varepsilon' x \end{bmatrix} \qquad - \sigma_\varepsilon^2 Q$$

$$= E\begin{bmatrix} \frac{\iota'\varepsilon\varepsilon'\iota}{N} & \frac{\iota'\varepsilon\varepsilon' x}{N} \\ \frac{x'\varepsilon\varepsilon'\iota}{N} & \frac{x'\varepsilon\varepsilon' x}{N} \end{bmatrix} = \sigma_\varepsilon^2\begin{bmatrix} 1 & \overline{x} \\ \overline{x} & \overline{x^2} \end{bmatrix}$$