METHOD OF MOMENTS

THE NOVELTY IS THAT WE DO NOT NEED TO MAKE ASSUMPTIONS ON THE DISTRIBUTION!

$\{x_i\}_{i=1}^{N}$ from a distribution we do not know $\sim D(\mu; \sigma^2)$ How can we estimate μ and σ^2

we use sample moments to estimate parameters for the population

$$\begin{cases} E[x] - \mu = 0 \\ E[x_i^2] - (\mu^2 + \sigma^2) = 0 \end{cases}$$

$$\begin{cases} \frac{1}{N}\sum_{i=1}^{N} x_i - \mu = 0 \\ \frac{1}{N}\sum_{i=1}^{N} x_i^2 - (\mu^2 + \sigma^2) = 0 \end{cases}$$

$$\frac{1}{N}\sum_{i=1}^{N}(\cdot) \approx E[\cdot]$$

this is true at infinity

it is an approximation true for large samples

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

plug in the second eq.

$$\frac{1}{N}\sum_{i=1}^{N} x_i^2 - \hat{\mu}^2 = \hat{\sigma}^2$$

we did not make any assumption on the distribution

$y = X\beta + \varepsilon$ typical regression model

$$\begin{cases} E(\varepsilon | X) = 0 \\ E(\varepsilon\varepsilon' | X) = \sigma_\varepsilon^2 I_N \end{cases}$$

assumption of homoskedasticity and orthogonality of errors with x

$$E[\varepsilon x | x] = 0$$

$$y_i = x_i'\beta + \varepsilon_i \quad i = 1 \ldots N$$

$$E\left[\underset{k\times 1}{x_i} \cdot \varepsilon_i\right] = 0$$ MOMENT CONDITION

exogeneity assumption imposes K moment conditions

$$E[x_i (y_i - x_i'\beta)] = 0$$

$$\frac{1}{N} \sum_{i=1}^{N} \left(x_i (y_i - x_i'\beta) \right) = 0$$

$X'X = x_i x_i'$

$$\frac{1}{N} \left(\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i x_i' \beta \right) = 0$$

$$\frac{1}{N} \left(X'Y - (X'X)\beta \right) = 0$$

$$\hat{\beta}_{MM} = (X'X)^{-1} X'Y$$

$y = X\beta + \varepsilon$

$E[\varepsilon | x_1] \neq 0$ x_1 are endogenous while x_2 are exogenous

$y = x_1\beta_1 + x_2\beta_2 + \varepsilon$

$E[\varepsilon | x_2] = 0$

we use $\underset{(N\times L)}{Z} = [z_1 \quad x_2]$

$L = K$ Let's make this assumption

$E[\varepsilon | z_i] = 0$

$\Downarrow$

$E[\varepsilon | Z] = 0$

$E[z_i \; \varepsilon_i] = 0 \quad i = 1 \ldots N$

MOMENT CONDITION

$$E[z_i \varepsilon_i] = E[z_i (y_i - x_i'\beta)]$$
$4 = L \times 1$

Now we do the same as before

$$\frac{1}{N}\left(\Sigma z_i y_i - \Sigma z_i x_i'\beta\right) = 0$$

$$\frac{1}{N}\left(Z'Y - Z'X\beta\right) = 0$$

$$\hat{\beta}_N = \underset{(K\times 4)}{(Z'X)^{-1}} Z'y = \hat{\beta}_{MM}$$

IDEA BEHIND THE MOM:

in random sampling, a sample statistic will converge in probability to some constant et:

$$\frac{1}{N}\sum_{i=1}^{N} y_i^2 \longrightarrow \sigma^2 + \mu^2$$

the goal is that to estimate k parameters, we can compute k of such sample statistic, to have a solvable system

How to solve cases where $L > 4$?

Until now we have seen cases of exactly identification, k equation for k parameters

GENERALIZED METHOD OF MOMENTS (GMM)

$$y = x\beta + \varepsilon \qquad Y = x_1\beta_1 + x_2\beta_2 + \varepsilon$$

$$E[\varepsilon | x_1] \neq 0$$
$$E[\varepsilon | x_2] = 0$$

$$\underset{(N\times L)}{Z} = [z_1 \; x_2]$$

$$E[\varepsilon | z_1] = 0$$

but now $L > K$

$$\to E[\varepsilon | z] = 0$$
$$\Downarrow$$
$$E[z_i \varepsilon_i] = 0$$

$$\underbrace{E\left[z_i(y_i - x_i'\beta)\right]}_{m_i(\beta,\, DATA)} = 0$$

moment condition for a function of β and DATA

it is just a relabeling

$$E(m_i(\beta)) = 0$$

(to squeeze the notation we only write β)

$$\underset{L\times 1}{\bar{m}(\beta)} = \frac{1}{N}\sum_{i=1}^{N} m_i(\beta)$$

$L > K$

↳ until here is the same as before

L equations and K unknown cannot be solved

↓

we use the J formula

$$J(\beta) = \left(\hat{\bar{m}}(\beta)' \; W_N \; \bar{m}(\beta)\right)$$

W_N weighting matrix symmetric and PD (can also be the identity)

$$\hat{\beta}_{GMM} = \underset{\beta \in B \subset \mathbb{R}^K}{\operatorname{argmin}} J(\beta)$$

FOC

$$\frac{\partial J(\beta)}{\partial \beta} = 0$$

$$= 2 \underset{(K\times L)}{\frac{\partial \bar{m}(\beta)}{\partial \beta}}' \underset{(L\times L)}{W_N} \underset{(L\times 1)}{\bar{m}(\beta)} = 0$$

is a Jacobian

in a linear regression model with linear restriction we can continue

the results will be not general!

$$\bar{m}(\beta) = \frac{1}{N}\left(\sum_{i=1}^{N} z_i (y_i - x_i'\beta)\right)$$

avg. moment. condition in the special case of IV

$$\frac{1}{N}\left(Z'Y - (Z'X)\beta\right)$$

$$\left(\frac{\partial \bar{m}(\beta)}{\partial \beta}\right)' = -\frac{1}{N}(X'Z)$$

$$FOC = -\frac{2}{N}(X'Z)\, W_N \frac{1}{N}(Z'Y - (Z'X)\beta)$$

$$(X'Z)\, W_N\, (Z'Y - (Z'X)\beta) = 0 \rightarrow$$ this is the foc

$$X'Z W_N Z'Y - X'Z W_N (Z'X)\beta = 0$$

$$X'Z W_N (Z'X)\beta = X'Z W_N Z'Y$$

$$\tilde{\beta}_{GMM} = \left[X'Z\, W_N (Z'X)\right]^{-1} X'Z\, W_N Z' Y$$

$$E(\tilde{\beta}_{GMM} | X, Z)$$
$$= E\left[\left[X'Z\, W_N (Z'X)\right]^{-1} X'Z\, W_N Z'(X\beta^0 + \varepsilon)\right)$$

$$E(\hat{\beta}_{GMM}|X,Z) = \beta^0$$

$$Var(\hat{\beta}_{GMM}|X,Z) =$$

$$E\left[\left[(X'Z)W_N(Z'X)\right]^{-1}(X'Z)W_N Z'\varepsilon\varepsilon' Z W_N (Z'X)\left[(X'Z)W_N Z'X\right]^{-1}\right]$$

$$\left[(X'Z)W_N(Z'X)\right]^{-1}X'Z W_N Z' E[\varepsilon\varepsilon'|X,Z] Z W_N (Z'X)\left[X'Z W_N Z'X\right]^{-1}$$

$$\sigma^2_\varepsilon\left[X'Z W_N Z'X\right]^{-1}X'Z W_N Z' Z W_N (Z'X)\left[X'Z W_N Z'X\right]$$

$$(Z'Z)^{-1} = W_N$$

$$\sigma^2_\varepsilon\left[X'Z(Z'Z)^{-1}(Z'X)\right]^{-1}$$

$$= \sigma^2_\varepsilon(X'P_Z X)^{-1} \longrightarrow \text{VAR COVAR MAT}$$

also $\sigma^2_\varepsilon(\hat{X}'\hat{X})^{-1}$

$$\hat{\beta}_{GMM} = \left((X'Z)W_N(Z'X)\right)^{-1}X'Z W_N Z'y$$

$$W_N = (Z'Z)^{-1}$$

$$\left((X'Z)(Z'Z)^{-1}(Z'X)\right)^{-1}X'Z(Z'Z)^{-1}Z'y$$

$$= (X'P_Z X)^{-1}X'P_Z y = (\hat{X}'\hat{X})^{-1}\hat{X}'y$$

$y = X\beta$

$(N \times 4)$ $\quad g = \frac{1}{N}(Z'y - Z'X\beta)$

$\neq 0$

$H = -Z'X$

$$X'Z W_N Z'y - X'Z W_N (Z'X)\beta = 0$$

$H = (X'Z) W_N (Z'X)$

$H^{-1} = (Z'X)^{-1} W_N^{-1} (X'Z)^{-1}$

$G'G = (\beta'(X'Z) W_N Z'X - y'Z W_N Z'X)$

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