$Y = X\beta + \varepsilon$ is our model

β is the unknown we want to estimate
↓
the focus is there now

$\hat{\beta}_{OLS}$ is the estimator
↳ $(X'X)^{-1}X'Y$ solution to the problem of the least squares

$$\underset{k\times k}{(X'X)}\,\underset{(k\times 1)}{\beta} = \underset{(k\times 1)}{X'Y}$$ is a NORMAL EQUATION

$$\hat{\beta} = (X'X)^{-1}X'Y$$

→ these are all variables that i already know

forget about the error term for now

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1k} \\ \vdots & \vdots & & \vdots \\ 1 & x_{N1} & \cdots & x_{Nk} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$$

→ it is a system of equation, one for each "individual"
↓
N equations, k variables unknown, and $k \ll N$
↓
into vector β

it is a system $Ax = b$

A has to be square and has to have full rank, hence there is no solution, for this reason there is an error term

X isn't square

$$\underset{(N\times k)}{X'}\underset{(k\times 1)}{X\beta} = \underset{(N\times 1)}{X'Y}$$

we premultiply by X' to make the matrix square
↓
and we get the same equation of before

the OLS estimator is what makes the closest possible the left hand side and the right hand side of

$Y = X\beta$ (pseudo-solution)

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

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by taking expectation

- $\beta_0 = E(Y) - \beta_1 E(X)$
- $\beta_1 = \dfrac{Cov(X,Y)}{Var(X)}$

these are the parameters but they have an interpretation

these parameters are unknown

$$\underset{N\times 1}{i} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \underset{N\times 1}{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$\underset{(N\times 2)}{X} = [\, i \, , x \,]$$

little x

iota, vector of ones $(N\times 1)$

we have to compute $\underset{(2\times N)(N\times 2)}{X'X} = \underset{2\times N}{\begin{bmatrix} i' \\ x' \end{bmatrix}} [\, i \; x \,]$

the output is going to be a 2x2

$$\begin{bmatrix} i'i = N & i'x = \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \underbrace{\sum_{i=1}^{N} x_i^2}_{x'x} \end{bmatrix}$$

gives me N

the matrix is symmetric

$$= \begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$\underset{2\times N}{X'}\underset{N\times 1}{Y}$ → 2×1 is the dimension

$$X'Y = \begin{bmatrix} i' \\ x' \end{bmatrix} [\, Y \,] = \begin{bmatrix} i'Y = \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i y_i \end{bmatrix}$$

NORMAL EQUATION $\dfrac{(X'X)}{N}\beta = \dfrac{X'Y}{N}$ divide both members by N

$$\frac{X'X}{N} = \begin{bmatrix} 1 & \bar{x} \\ \bar{x} & (\bar{x^2}) \end{bmatrix}$$

notation for the second moment

$$\frac{X'y}{N} = \begin{bmatrix} \bar{y} \\ \bar{xy} \end{bmatrix} \qquad \bar{xy} = \frac{1}{N}\sum_{i=1}^{N} x_i y_i$$

$$\hat{\beta} = \left(\frac{X'X}{N}\right)^{-1}\left(\frac{X'y}{N}\right) \qquad \left(\frac{X'X}{N}\right)^{-1} = \frac{1}{(\bar{x^2})-(\bar{x})^2}\begin{pmatrix} (\bar{x^2}) & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}$$

sample covariance $\hat{VAR}(x)$ and it is the determinant of the matrix

formula for the inverse of a 2x2 matrix

because β is a column vector $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$

$$\hat{\beta}_1 = \frac{1}{\hat{var}(x)}\begin{pmatrix} \_\_\_ & \_\_\_ \\ -\bar{x} & 1 \end{pmatrix}\begin{pmatrix} \bar{y} \\ \bar{xy} \end{pmatrix}$$

because this equation is for β_0

$$\hat{\beta}_1 = \frac{-\bar{x}\bar{y} + \bar{xy}}{Var(x)}$$

SAMPLE COVARIANCE = $\hat{Cov}(x, y)$

$$\hat{\beta}_1 = \frac{\hat{Cov}(x,y)}{\hat{Var}(x)}$$

what about $\hat{\beta}_0$ $\longrightarrow$ $\begin{pmatrix} 1 & \bar{x} \\ \_\_\_ & \_\_\_ \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \bar{y}$ ← this is the normal equation

the second equation has been already solved

$$\hat{\beta}_0 + \bar{x}\,\beta_1 = \bar{y}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

MATLAB PART

sqrt(sigma_2) is to give the epsilon the same variance that we want it to take

select the line and select F9 to run a single line

GEOMETRY OF OLS

we look at some algebraic properties of the OLS

$$\hat{\beta} = (X'X)^{-1} X'y$$

$$\hat{y} = X\hat{\beta}$$

are the fitted values, that lie on the fitted line

are the values predicted by the model

$\hat{\varepsilon} \rightarrow$ residuals

the error term is a stochastic variable that cannot be observed

the error is different from ε, we do not know the true beta, we only can know the residuals

$$\hat{\varepsilon} = y - x\hat{\beta}$$

$$\hat{y} = \underbrace{X (X'X)^{-1} X'}_{(N \times k)\ (k \times k)(k \times N)} y = P_x y$$

$\rightarrow$ projection matrix

$N \times N$ is the result

we call it P_x

other important matrix is Residual maker matrix

$$\hat{\varepsilon} = y - \hat{y} = y - P_x y = (I_N - P_x) y = M_x \cdot y$$

M_x is the residual maker matrix

$$E[z_i(y_i - x_i'\beta)] = 0$$

$L \times 1$ $\qquad$ $L \times 1$ $\qquad$ β

$$J(\beta) = \bar{m}(\beta)' W_N \bar{m}(\beta)$$

$1 \times L$ $\qquad$ $L \times L$ $\qquad$ $L \times 1$

$$\frac{\partial J(\beta)}{\partial \beta} = 2 \frac{\partial m(\beta)'}{\partial \beta} W_N \bar{m}(\beta)$$

$2 \quad (k \times L) \quad L \times L \quad L \times 1$

$k \times 1$