F-test → ① : test the model unrestrictedly at first

$$\hat{\beta}_U | X \sim N\left(\beta_0, (X'X)^{-1}\sigma^2_\varepsilon\right)$$

↳ unrestricted

we could have impose the restriction before the estimate

$$R\hat{\beta} - c | X \sim N\left(R\beta_0 - c; \sigma^2_\varepsilon R(X'X)^{-1}R'\right)$$

↳ wald test statistic

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Restricted least squares: Find a set of coefficients $\tilde{\beta}$ (RLS) that satisfies the following problem

Restricted least squares

$$\boxed{\tilde{\beta} = \arg\min \left(\varepsilon'(\beta)\,\varepsilon(\beta)\right) \quad \boxed{\text{over } R\tilde{\beta} = c}}$$

we solve this problem with the lagrangian

$$\frac{\delta Ax}{\delta x} = A'$$

$$L(\beta, \lambda) = \underbrace{\frac{1}{2}(y - X\beta)'(y - X\beta)} + \underbrace{\lambda'(R\tilde{\beta} - c)}_{(1\times r)(r\times k)(k\times 1)(r\times 1)}$$

$$\frac{\delta L}{\delta \beta} = (X'X)\beta - X'y + R'\lambda = 0$$

First order condition

$$\frac{\delta L}{\delta \lambda} = R\tilde{\beta} - c = 0$$

$$(X'X)\tilde{\beta} - X'y + R'\lambda = 0$$

it is on this equality on which we have to work

if $\lambda = 0$ we are back to the unrestricted model

. Multiply by $(X'X)^{-1}$

$$\cancel{(X'X)^{-1}}\cancel{(X'X)}\tilde{\beta} - \overbrace{(X'X)^{-1}X'y}^{\hat{\beta}} + (X'X)^{-1}R'\lambda = 0$$

$$\tilde{\beta} - \hat{\beta} = -\underbrace{(X'X)^{-1}R'}_{k\times k}\lambda$$

↳ not a square, not invertible, ∴ can pre-multiply by R

$$R\tilde{\beta} - R\hat{\beta} = -[R(X'X)^{-1}R']\lambda$$
$$\underset{c}{\parallel}$$

Flip the signs

$$R\hat{\beta} - c = [R(X'X)^{-1}R']\lambda$$

$$\tilde{\lambda} = [R(X'X)R']^{-1}(R\hat{\beta} - c)$$

$\hookrightarrow$ if i move far the constraint i am going to pay a price that is represented from $\tilde{\lambda}$

$$\boxed{\tilde{\beta} = \hat{\beta} - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - c)}$$

$\tilde{\beta} = \hat{\beta}$ when $(R\hat{\beta} - c) = 0$ hence when $\hat{\beta}$ is already satisfying the constraints

$$R\tilde{\beta} = \cancel{R\hat{\beta}} - \cancel{R(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}}(\cancel{R\hat{\beta}} - c)$$

$R\tilde{\beta} = c$ i obtain exactly the constraints, the computations were right so

Restricted residuals

$$\tilde{\varepsilon} = y - X\tilde{\beta} \qquad \tilde{\varepsilon} = y - X\left[\hat{\beta} - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - c)\right]$$

$$= \hat{\varepsilon} + X(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - c)$$

i would like to compute $\tilde{\varepsilon}'\tilde{\varepsilon} = RSS_R$ Residual sum of squares for the restricted model

$$\tilde{\varepsilon}'\tilde{\varepsilon} = \left[\hat{\varepsilon} + X(X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta}-c)\right]' \cdot$$

$$\left[\hat{\varepsilon} + X(X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta}-c)\right]$$

$$(a+b)(a+b)$$

$$\tilde{\varepsilon}'\tilde{\varepsilon} = \hat{\varepsilon}'\hat{\varepsilon} + \underbrace{\hat{\varepsilon}'X}_{0 \text{ they are orthogonal}}(X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta}-c)$$

$$+(R\hat{\beta}-c)'\left[R(X'X)^{-1}R'\right]^{-1}R(X'X)^{-1}\underbrace{X'\hat{\varepsilon}}_{\text{is same, also this is zero}}$$

$$+(R\hat{\beta}-c)'\left[R(X'X)^{-1}R'\right]^{-1}\underbrace{R\cancel{(X'X)}^{-1}}\cancel{X'X}\cancel{(X'X)^{-1}}\underbrace{R'\left[R(X'X)^{-1}R'\right]}^{-1}(R\hat{\beta}-c)$$

$$= \hat{\varepsilon}'\hat{\varepsilon} + \underbrace{(R\hat{\beta}-c)'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta}-c)}_{\text{numerator of the F-test}} \rightarrow \text{and it is a PD matrix}$$

Hence the $RSS_R$ is $\geq RSS_U$

↓

equal when $R\hat{\beta}=c$

$$\text{F-test} = \frac{\overbrace{(R\hat{\beta}-c)'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta}-c)}^{\tilde{\varepsilon}'\tilde{\varepsilon}-\hat{\varepsilon}'\hat{\varepsilon}}/r}{\frac{\hat{\varepsilon}'\hat{\varepsilon}}{N-k} = s^2_{\varepsilon}} \sim F(k, N-k)$$

$$\text{F-test} = \frac{\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon}/r}{\frac{\hat{\varepsilon}'\hat{\varepsilon}}{N-k}} = \frac{\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon}}{\hat{\varepsilon}'\hat{\varepsilon}} \cdot \frac{N-k}{r} \sim F(r, N-k)$$

the difference is that now we are estimating the model two times instead of one (like in wald)

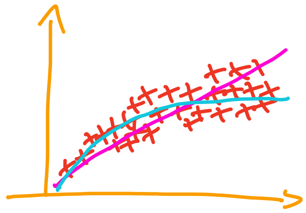
if the estimation is time consuming → this may make a difference

How much the fit worsen in relative term by adding constraints

# DIAGNOSTIC TESTS:

LINEARITY: $H_0$: $y = X\beta + \varepsilon$

$$H_1: \; y = X\beta + \hat{y}^2 \gamma_1 + \hat{y}^3 \gamma_2 + \eta$$



the pink line isn't a good model, the light blue one may be better and is quadratic

we run an F-test on the significance of these parameters

Reset test

it is a way to challenge the model that we would like to be correct

$\gamma_1 = \gamma_2 = 0$ we are back to $H_0$

the test is not informative under the alternative

↳ we can just know to have a problem but not how to solve it

$$H_1: \; y = W\psi + \eta \qquad W = [X, \hat{y}^2, \hat{y}^3] \qquad \boxed{\gamma_1 = \gamma_2 = 0}$$

the hypothesis is that $\gamma_1$ and $\gamma_2$ are equal to zero

$$\underset{(k+2)\times 1}{\psi} = [\beta', \gamma_1, \gamma_2]'$$

$$\hat{\psi} = (W'W)^{-1} W' y$$

$$\underset{2\times(k+2)}{R} = \left[ \underbrace{O_{r \times k}}_{r=2} \quad I_2 \right] \qquad \underset{(2\times 1)}{C} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

βs are excluded from the constraint

$$\underset{(\text{Reset})}{F\text{-test}} = \frac{(R\psi' - c)' \left[ R (W'W)^{-1} R' \right]^{-1} (R\psi' - c) / 2}{\dfrac{\tilde{\eta}' \tilde{\eta}}{(N-k-2)}}$$

→ $s^2_\eta$ estimator of the variance of the β in the auxiliary model

↳ W has now k+2 elements