Probabilistic properties of $\hat{\beta}_{OLS}$ → it is an unbiased estimator?

$$\boxed{E[\hat{\beta}_{OLS}|X] = \beta^0}$$

it is unbiased if the mean of the estimator is equal to the value he is trying to estimate

$$E[(X'X)^{-1}X'y \,|\, X]$$

$$y = X\beta^0 + \varepsilon$$

$$E[(X'X)^{-1}X'(X\beta^0 + \varepsilon)|X]$$

$$\cancel{(X'X)^{-1}X'X}\beta^0 + (X'X)^{-1}X' \underbrace{E[\varepsilon|X]}_{=0} = \beta^0$$

→ true parameter value

Exogeneity condition

is the estimator also unconditionally unbiased?

Yes, because of the LIE and also because β^0 is a constant, does not depend on X $(E(\beta^0) = \beta^0)$

1. $\boxed{\text{UNBIASED}}$ → on average we are estimating the right parameter
2. Efficient → $\boxed{\text{it has to have the smallest possible variance between all the unbiased estimators}}$

efficiency is always relative to a benchmark, all the other possible unbiased estimator (GAUSS MARKOV THEOREM)

What is the variance of $\hat{\beta}_{OLS}$

$$\boxed{(\hat{\beta} - \beta^0) = (X'X)^{-1}X'\varepsilon}$$

$$VAR(\hat{\beta}_{OLS}|X) = E[\underset{(k\times1)}{(\hat{\beta} - \beta^0)}\underset{(1\times k)}{(\hat{\beta} - \beta^0)'} \,|\, X] = E[(X'X)^{-1}X'\varepsilon\,\varepsilon'X(X'X)^{-1} \,|\, X]$$

$E[\hat{\beta}]$

$\hat{\beta}_{OLS}$ $(k\times1)$ will be a variance covariance matrix of dim $k\times k$

outer product we do not get a scalar but a $k\times k$ matrix

$$(X'X)^{-1}X'\underbrace{E[\varepsilon\varepsilon'|X]}_{N\times N}X(X'X)^{-1}$$

SANDWICH ESTIMATOR

covariance matrix of the error

$$\underset{(N\times N)}{I_N\sigma^2_\varepsilon} = \Sigma \text{ or } \Omega$$

$\hat{\beta} - \beta_0 = (X'X)^{-1}X'(X\beta^0 + \varepsilon) - \beta_0$

$(X'X)^{-1}X' \sigma^2_\varepsilon I_N X (X'X)^{-1}$

Remembering that the error is $\sim N(0; I_N \sigma^2_\varepsilon)$

$$= \sigma^2_\varepsilon \cancel{(X'X)^{-1}} \cancel{(X'X)} (X'X)^{-1} = \sigma^2_\varepsilon \underbrace{(X'X)^{-1}} = Var(\hat{\beta}_{OLS}|X)$$

the variance of the OLS depends on

σ^2_ε

more uncertainty of the error means more uncertainty in the estimator

if is like correlation between the X

if it is higher the variability in the β is lower (it is an inverse so it is in the denominator)

Hence the estimator $\hat{\beta}_{OLS}$

$E[\hat{\beta}_{OLS}] = \beta^0 \longrightarrow$ GOOD, the estimator is unbiased

$Var[\hat{\beta}_{OLS}] = \sigma^2_\varepsilon \underbrace{(X'X)^{-1}}$

the variance of the estimator depends on the variance of the error term

this term capture something like the covariance of the Xs, and it is in the denominator

I want $\hat{\beta}_{OLS}$ to be efficient

smallest possible var-covar matrix compared to other estimators

$$\hat{\beta}_{OLS} = \underbrace{(X'X)^{-1}X'}_{\text{this is the } L \text{ operator for } \hat{\beta}_{OLS}} y$$

$\tilde{\beta} = Ly$

i want also $\tilde{\beta}$ to be unbiased to compare two similar things

$$E[\tilde{\beta}|X,L] = E[Ly|X,L] = E[L(X\beta^0+\varepsilon)|X,L] = LX\beta^0 + E[\varepsilon|X,L]$$

$$= LX\beta^0 = \beta_0 \text{ if } \underset{(K\times N)(N\times K)}{LX} = \underset{(K\times K)}{I_K}$$

this has to hold in order to have $\tilde{\beta}$ unbiased

$Var(\hat{\beta}|X) \leq Var(\tilde{\beta}|X)$?

$Var(\tilde{\beta}|X) - Var(\hat{\beta}|X) = C$ if C is positive definite, then the inequality is true

proving this we prove that $\hat{\beta}_{OLS}$ is BLUE

BEST LINEAR UNBIASED ESTIMATOR

Most efficient

GAUSS MARKOV THEOREM TO PROVE THIS

$Var(\hat{\beta}_{OLS}|X) = \sigma^2_\varepsilon(X'X)^{-1}$ we proved this before

$$Var(\tilde{\beta}_{OLS}|X,L) \underset{\text{if } LX=I_K}{=} E[(\underbrace{\tilde{\beta}-\beta^0}_{L\varepsilon})(\tilde{\beta}-\beta^0)'|X,L]$$

$$= E[L\varepsilon\varepsilon'L'|X,L] = LE[\varepsilon\varepsilon'|X,L]L' = L\sigma^2_\varepsilon I_N L' = L\sigma^2_\varepsilon L'$$

$$= \sigma^2_\varepsilon LL'$$

Hence:

in choosing or not L does not change anything
because $\hat{\beta}$ does not depend on it

$$\mathrm{Var}(\tilde{\beta}\,|X,L) - \mathrm{Var}(\hat{\beta}\,|X)$$

$$= \sigma_\varepsilon^2 LL' - \sigma_\varepsilon^2 (X'X)^{-1} = \boxed{\sigma_\varepsilon^2 (LL' - (X'X)^{-1})}$$

If this is PD I have proved that $\hat{\beta}_{OLS}$ is BLUE

Define D as the difference between the two estimator operators

$$\underset{(k\times N)}{D} = L - (X'X)^{-1}X' \qquad DX = 0 \qquad D \perp X$$

$$LL' = (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')'$$

$$= DD' + \cancel{(X'X)^{-1}X'D'} + \cancel{DX(X'X)^{-1}} + \cancel{(X'X)^{-1}}\cancel{X'X}(X'X)^{-1}$$

$$= DD' + (X'X)^{-1}$$

$$LL' - (X'X)^{-1} = DD'$$

positive because outer product of a matrix by itself, hence we proved that $\hat{\beta}_{OLS}$ is BLUE

$\mathrm{Var}(\hat{\beta}_{OLS}|X)$ is the CRAMER RAO LOWER BOUND smallest variance covariance matrix (in matrix sense) of all the unbiased estimators $\tilde{\beta}$

we based our calculation on the assumptions

$$\varepsilon | X \sim N(0, \sigma_\varepsilon^2 I_N)$$

exogeneity homoskedasticity

$$\hat{\beta}|X \sim \underbrace{\beta^0 + (X'X)^{-1}X'\underbrace{\varepsilon | X}_{\text{Normal}}}$$

also this is normal as a consequence

$$\hat{\beta}|X \sim N(\beta^0; \sigma_\varepsilon^2 (X'X)^{-1})$$

$$\mathrm{Var}\left(\left(X'X\right)^{-1}X'\varepsilon \mid X\right) = \left(X'X\right)^{-1}X' I_N \sigma_\varepsilon^2 X\left(X'X\right)^{-1}$$

$$= \sigma_\varepsilon^2 \left(X'X\right)^{-1}$$

$$\hat{\sigma}_\varepsilon^2 = S^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{N-4}$$

Error variance estimator