SUR MODEL   Assumption | Estimation | special cases | testing
(seemingly unrelated regressions)

in the previous lesson CAPM
M = equations N = number of obs

$$\begin{cases} \underset{(N\times 1)}{y_1} = \underset{(N\times k_1)}{X_1}\underset{(k_1\times 1)}{\theta_1} + \underset{(N\times 1)}{\varepsilon_1} \\ \underset{(N\times 1)}{y_2} = \underset{(N\times k_2)}{X_2}\underset{(k_2\times 1)}{\theta_2} + \underset{(N\times 1)}{\varepsilon_2} \\ \vdots \qquad \vdots \end{cases}$$

k_2 can be equal or not to k_1 and also x_1 can be different from x_2

$y = X\theta + \varepsilon$ $y = y_1 | y_2 | y_3 | \dots | y_M$ $NM\times 1$

$\varepsilon = \varepsilon_1 | \dots | \varepsilon_M$ $NM\times 1$

$$\underset{NM\times \sum_{i=1}^{M} k_i}{X} = \begin{bmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots \; x_M \end{bmatrix}$$

on the main diagonal there are vectors, not scalars, hence this is a block matrix

$$\underset{\sum_{i=1}^{M} k_i\times 1}{\theta} = [\theta_1 | \theta_2 | \dots | \theta_M]$$

$$\hat{\theta}_{OLS} = (X'X)^{-1}X'y$$

important assumption about the error terms

(1) EXOGENEITY $E[\varepsilon | X] = 0$ error term does not depend on any of the xs

② $E[\varepsilon_{i,s} \cdot \varepsilon_{j,t} \mid X] = \sigma_{ij}$

i, j: from 1 to M, they refer to the number of equations

s, t: $1 \dots N$ they refer to the observations

$$\begin{cases} \sigma_{ij} \neq 0 & \text{if } t = s \\ \sigma_{ij} = 0 & \text{if } t \neq s \end{cases}$$

For the same individual the covariance across equation is positive, while for different individuals the observations do not covary
↳ and the σ_{ij} is also constant

$N = 3 \quad M = 2$

$$\underset{(3\times2)\times1}{\varepsilon} = \begin{bmatrix} \varepsilon_{1,1} \\ \varepsilon_{1,2} \\ \varepsilon_{1,3} \\ \varepsilon_{2,1} \\ \varepsilon_{2,2} \\ \varepsilon_{2,3} \end{bmatrix}$$

$\varepsilon_{1,1}$ → error first equation first individual

$\varepsilon_{2,1}$ → second equation, first individual

$$\underset{\substack{(6\times6) \\ (NM\times NM)}}{\Omega} = E[\varepsilon\varepsilon' \mid X] = E\left[\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \end{bmatrix} \middle| X \right]$$

$$
= E\begin{bmatrix}
\varepsilon_{11}^2 & \varepsilon_{11}\varepsilon_{12} & \varepsilon_{11}\varepsilon_{13} & \varepsilon_{11}\varepsilon_{21} & \varepsilon_{11}\varepsilon_{22} & \varepsilon_{11}\varepsilon_{23} \\
\varepsilon_{12}\varepsilon_{11} & \varepsilon_{12}^2 & \varepsilon_{12}\varepsilon_{13} & \varepsilon_{12}\varepsilon_{21} & \varepsilon_{12}\varepsilon_{22} & \varepsilon_{12}\varepsilon_{23} \\
\varepsilon_{13}\varepsilon_{11} & \varepsilon_{13}\varepsilon_{12} & \varepsilon_{13}^2 & \varepsilon_{13}\varepsilon_{21} & \varepsilon_{13}\varepsilon_{22} & \varepsilon_{13}\varepsilon_{23} \\
\varepsilon_{21}\varepsilon_{11} & \varepsilon_{21}\varepsilon_{12} & \varepsilon_{21}\varepsilon_{13} & \varepsilon_{21}^2 & \varepsilon_{21}\varepsilon_{22} & \varepsilon_{21}\varepsilon_{23} \\
\varepsilon_{22}\varepsilon_{11} & \varepsilon_{22}\varepsilon_{12} & \varepsilon_{22}\varepsilon_{13} & \varepsilon_{22}\varepsilon_{21} & \varepsilon_{22}^2 & \varepsilon_{22}\varepsilon_{23} \\
\varepsilon_{23}\varepsilon_{11} & \varepsilon_{23}\varepsilon_{12} & \varepsilon_{23}\varepsilon_{13} & \varepsilon_{23}\varepsilon_{21} & \varepsilon_{23}\varepsilon_{22} & \varepsilon_{23}^2
\end{bmatrix}
$$

covariance two individuals in the same equation, we are working with homoskedasticity within each equation

errors associated to the same individual between different equations

equation by equation we assume homoskedasticity

$$
= \left[\begin{array}{ccc|ccc}
\sigma_1^2 & 0 & 0 & \sigma_{12} & 0 & 0 \\
0 & \sigma_1^2 & 0 & 0 & \sigma_{12} & 0 \\
0 & 0 & \sigma_1^2 & 0 & 0 & \sigma_{12} \\
\hline
\sigma_{12} & 0 & 0 & \sigma_2^2 & 0 & 0 \\
0 & \sigma_{12} & 0 & 0 & \sigma_2^2 & 0 \\
0 & 0 & \sigma_{12} & 0 & 0 & \sigma_2^2
\end{array}\right]
$$

kronecker product

$$
\underset{(NM\times NM)}{\Omega} = \underset{(M\times M)}{\Sigma} \otimes \underset{(N\times N)}{I_N}
$$

Consider

$$\underset{M\times M}{A} \otimes \underset{(N\times N)}{B} = \underset{(MN\times MN)}{C} = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1M}B \\ \vdots & \ddots & & \\ & & \ddots & \\ a_{1M}B & \cdots & \cdots & a_{NM}B \end{bmatrix}$$

not commutative

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

σ_1^2 → variance in the first equation, for N observation

σ_{12} → covariance for the same observation across equations

σ_2^2 → variance for the second equation

$$\Sigma \cdot I_3 = \left[\begin{array}{ccc|ccc} \sigma_1^2 & 0 & 0 & \sigma_{12} & 0 & 0 \\ 0 & \sigma_1^2 & 0 & 0 & \sigma_{12} & 0 \\ 0 & 0 & \sigma_1^2 & 0 & 0 & \sigma_{12} \\ \hline \sigma_{12} & 0 & 0 & \sigma_2^2 & 0 & 0 \\ 0 & \sigma_{12} & 0 & 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_{12} & 0 & 0 & \sigma_2^2 \end{array}\right]$$

$$Y = X\beta + \varepsilon \qquad E[\varepsilon\varepsilon' | X] = \underset{M\times M}{\Sigma} \otimes I_N$$

OLS should not be a good model, it ignores the correlation

$$\hat\beta_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

$$\hat\beta_{GLS} = (X'(\Sigma^{-1} \otimes I_N^{-1})X)^{-1}X'(\Sigma \otimes I_N)^{-1}y$$

I can take the inverse inside (because Σ is invertible)

$$\hat{\beta}_{FGLS} = (X'\hat{\Omega}^{-1}X)^{-1} X'\hat{\Omega}^{-1} y$$

$$\hat{\Omega} = \underset{(M\times M)}{\hat{\Sigma}} \otimes I_N$$

the problem is estimating Σ

1) $\hat{\beta}_{OLS} = (X'X)^{-1}X'y$ estimating equation by equation (limited information approach)

$$\underset{(NM\times 1)}{\hat{\varepsilon}} = y - X\hat{\beta}_{OLS}$$

$$\underset{(N\times M)}{\hat{E}} = \left[\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_M\right]$$

concatenated horizontally (reshape command)

$$\underset{(M\times M)}{\hat{E}'\hat{E}} = \underset{(M\times N)}{\begin{bmatrix} \underset{(1\times N)}{\hat{\varepsilon}_1'} \text{ it is a line} \\ \hat{\varepsilon}_2' \\ \vdots \\ \hat{\varepsilon}_M' \end{bmatrix}} \underset{(N\times M)}{\left[\hat{\varepsilon}_1 \,\middle|\, \hat{\varepsilon}_2 \,\middle|\, \dots \,\middle|\, \hat{\varepsilon}_M\right]}$$

$$= \begin{bmatrix} \sum_{i=1}^{N} \hat{\varepsilon}_{1i}^2 & \sum_{i=1}^{N} \hat{\varepsilon}_{1i}\hat{\varepsilon}_{2i} & \dots & \sum_{i=1}^{N} \hat{\varepsilon}_{1i}\hat{\varepsilon}_{Mi} \\ \sum_{i=1}^{N} \hat{\varepsilon}_{2i}\hat{\varepsilon}_{1i} & \sum_{i=1}^{N} \hat{\varepsilon}_{2i}^2 & \dots & \\ \vdots & & \ddots & \\ \sum_{i=1}^{N} \hat{\varepsilon}_{1i}\hat{\varepsilon}_{Mi} & \dots & & \sum \hat{\varepsilon}_{M,i}^2 \end{bmatrix}$$

variance errors last equation

$$\hat{\Sigma} = \frac{1}{N}\hat{E}'\hat{E} = \begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N}\hat{\varepsilon}_i & \frac{1}{N}\sum_{i=1}^{N}\hat{\varepsilon}_{1i}\hat{\varepsilon}_{2i} & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \cdots \end{bmatrix}$$

$\text{plim}\ \hat{\Sigma} = \Sigma$ it is an estimator of the variance covariance matrix term

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$Var(\hat{\beta}_{GLS}) = (X'\Omega^{-1}X)^{-1}$

two special cases where $\hat{\beta}_{OLS} = \hat{\beta}_{GLS}$

ZELLNER CONDITIONS (do not need to be satisfied jointly)

↳ i can estimate OLS equation by equation in this case

1) Σ is diagonal $\quad \Omega = \Sigma \otimes I_N$

if Σ is diagonal the correlation across equations is zero

2) Same regressors in each equation

$Y = X\beta + \varepsilon \qquad$ regressor x_0

$$\underset{(NM \times KM)}{X} = \begin{bmatrix} x_0 & 0 & \cdots & 0 \\ \vdots & x_0 & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & \cdots & x_0 \end{bmatrix}$$

when they repeat it is useful to use the kronecker product

x_0 is $(N \times 4)$

$$\underset{(N\times 4)}{I_M \otimes x_0} = X$$ → equivalent of scalar multiplied by the identity but for matrices

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

$$= \left[(I_M \otimes x_0)'(\Sigma^{-1} \otimes I_N)(I_M \otimes x_0)\right]^{-1}(I_M \otimes x_0)'(\Sigma^{-1} \otimes I_N)y$$

the transpose can go inside

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

$$\left(\Sigma^{-1} \otimes (x_0'x_0)\right)^{-1}(\Sigma^{-1} \otimes x_0')y$$

$$\left(\Sigma \otimes (x_0'x_0)^{-1}\right)(\Sigma^{-1} \otimes x_0')y$$

$$\left[I_M \otimes (x_0'x_0)^{-1}x_0'\right]y$$

we are repeating equation by equation the OLS

$$\hat{\beta}_{OLS} = \begin{bmatrix} (x_0'x_0)^{-1}x_0'y_1 \\ \vdots \\ (x_0'x_0)^{-1}x_0'y_M \end{bmatrix}$$