1) LOGLIKELIHOOD OF THE MODEL

$$f(\varepsilon_i,\theta) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(y_i - x_i'\beta)^2}$$

$$\ell(\varepsilon_i,\theta) = -\frac{1}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}(y_i - x_i'\beta)^2$$

$$L(\theta|\varepsilon_i) = -\frac{N}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\Sigma(y_i - x_i'\beta)^2$$

2) Score function

$$\frac{\partial \ell(\varepsilon_i,\theta)}{\partial\theta} = +\frac{1}{2\sigma^2}\cdot 2 \quad (x_i y_i - x_i x_i'\beta)$$

$$E\left[\frac{\partial(\varepsilon_i,\theta)}{\partial\theta}\right] = \frac{1}{\sigma^2}\cdot\frac{1}{N}\Sigma(x_i y_i - x_i x_i'\beta)$$

$$= \frac{1}{\sigma^2}E(X'y - X'X\beta|X)$$

$$= \frac{1}{\sigma^2}E[X'(X\beta^0 + \varepsilon) - X'X\beta|X]$$

$$= \frac{1}{\sigma^2}E[X'X\beta^0 + X'\varepsilon - X'X\beta|X]$$

$$= \frac{1}{\sigma^2}(X'X\beta^0 + 0 - X'X\beta) = 0 \text{ if } \beta = \beta^0$$

3) $\tilde{\beta}_{ML}$ = OLS estimator

$$L(\theta|\varepsilon_i) = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(y_i - x_i'\beta)^2$$

$$\frac{\partial L(\theta|\varepsilon_i)}{\partial\beta} = \frac{1}{2\sigma^2}(X'Y - X'X\beta) = 0$$

$$X'Y = X'X\beta \quad \tilde{\beta}_{ML} = \tilde{\beta}_{OLS} = (X'X)^{-1}X'Y$$

(4) Asymptotic covariance matrix of $\hat{\beta}_{ML}$

$$\sqrt{N}(\tilde{\beta}_{ML} - \beta^0) \sim N\left(0; -E\left[\frac{\partial^2 \ell(y_i;\theta)}{\partial\theta\partial\theta'}\Big|_{\theta=\theta_0}\right]^{-1}\right)$$

$$E\left[\frac{\partial \ell^2(y_i;\theta)}{\partial\theta\partial\theta'}\Big|_{\theta=\theta_0}\right] = \frac{\partial}{\partial\beta}\left(\frac{1}{\sigma^2}(X'X\beta^0 - X'X\beta)\right)$$

$$= -\frac{1}{\sigma^2}(X'X)$$

Hence $-E\left[\frac{\partial \ell(y_i;\theta)}{\partial\theta\partial\theta'}\Big|_{\theta=\theta_0}\right]^{-1} = \frac{(X'X)^{-1}}{\sigma^2}$

$$\tilde{\beta}_{ML} \sim N\left(\beta^0; \sigma^2\frac{(X'X)^{-1}}{N}\right)$$

$$\sqrt{N}\left(\hat{\beta}_{ML} - \beta^0\right) \sim N\left(0; -H^{-1}\right)$$

$$l_i = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y_i - x_i\beta)^2 \qquad H = E\left[\frac{\partial l_i(y_i,\theta)}{\partial\theta\,\partial\theta'}\Big|_{\theta_0}\right]$$

$$\frac{\partial l_i}{\partial\beta} = +\frac{1}{2\sigma^2}\cdot 2\left(x_i y_i - x_i x_i'\beta\right)$$

$$E\left[\frac{\partial l_i}{\partial\beta}\right] = \frac{1}{\sigma^2}E\left[X'y - X'X\beta\right]$$