Lecture 17 The rank theorem (continuation)

Recall from last lecture:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$\ker T = \{ x \in \mathbb{R}^n : Tx = 0 \} \subset \mathbb{R}^n$$

$$\operatorname{im} T = \{ Tx : x \in \mathbb{R}^n \} \subset \mathbb{R}^m$$

$$\dim \ker T = \operatorname{null}(T), \quad \dim \operatorname{im} T = \operatorname{rank} T$$

$$\boxed{\operatorname{rank}(T) + \operatorname{null}(T) = n}$$

Two linear transformations $T, S : \mathbb{R}^n \to \mathbb{R}^m$

are equivalent $T \sim S$ if

there are isomorphisms $\alpha : \mathbb{R}^n \to \mathbb{R}^n$,
(think of α, β $\beta : \mathbb{R}^m \to \mathbb{R}^m$
as linear changes of
coordinates) such that

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "1"] & \mathbb{R}^m \arrow[d, "\beta"] \\ \mathbb{R}^n \arrow[r, "S"'] & \mathbb{R}^m \end{tikzcd}$$

$$\boxed{S = \beta \circ T \circ \alpha^{-1}}$$

<u>Obs</u> if $T \sim S$ then $\operatorname{rank}(T) = \operatorname{rank}(S)$, $\operatorname{null}(T) = \operatorname{null}(S)$.

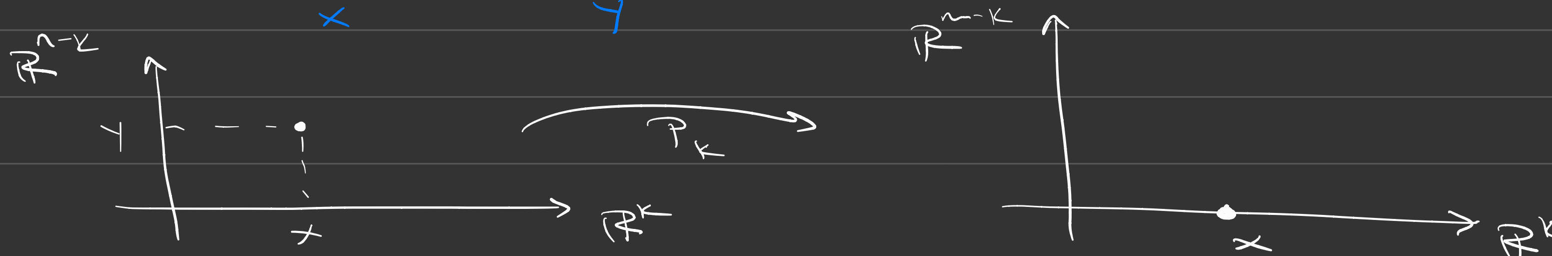
Theorem (the rank theorem for linear transformations)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

and let $k := \operatorname{rank}(T)$.

Then $T \sim P_k$ where $P_k : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{m-k}$

$\mathbb{R}^n \ni (\underbrace{x_1 - x_k}_{x}, \underbrace{x_{k+1} - x_n}_{y})$ $\qquad P_k(x, y) = (x, 0)$

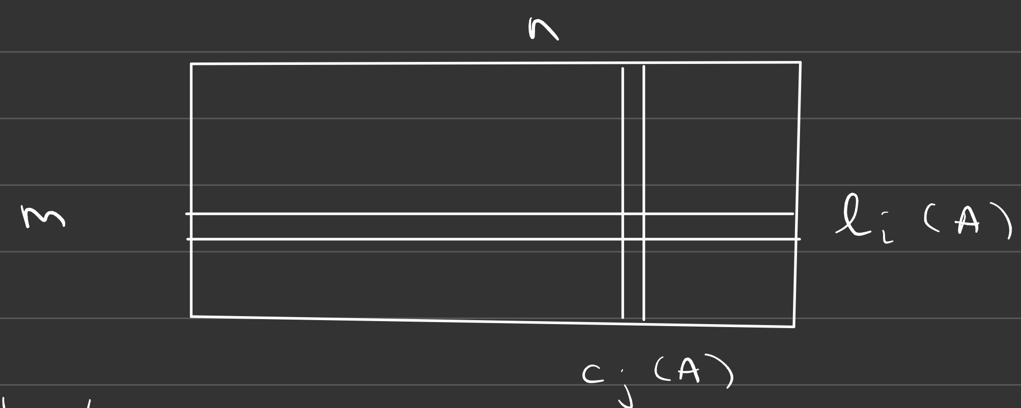


Obs The matrix of P_k (relative to the canonical bases) is

$$
\begin{array}{c|c|c|}
 & k & n-k \\
\hline
k & I_k & 0 \\
\hline
m-k & 0 & 0 \\
\hline
\end{array}
$$

More linear algebra

Let $A \in \mathrm{Mat}(m, n)$



We denote by

$l_1(A), \_\_\_, l_m(A) \in \mathbb{R}^n$ the rows of A

$c_1(A), \_\_\_, c_n(A) \in \mathbb{R}^m$ the columns of A.

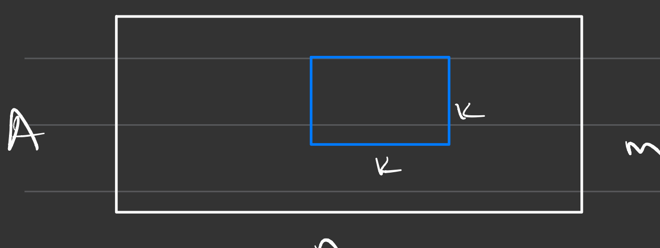
$$\mathcal{L}(A) = \text{Span}\{ l_1(A), \ldots, l_m(A) \} \subset \mathbb{R}^n$$

the lines space

$$\mathcal{C}(A) = \text{Span}\{ c_1(A), \ldots, c_n(A) \} \subset \mathbb{R}^m$$

the columns space

<u>Proposition</u> $\dim \mathcal{L}(A) = \dim \mathcal{C}(A)$ = the dim. of the largest invertible (non singular) submatrix of A.



<u>Def</u> We call the rank(A) any of these equal numbers.

proof ① We first prove that

$$\dim \mathcal{L}(A) \leq \dim \mathcal{C}(A)$$

Let $k := \dim \mathcal{C}(A)$. Then there is a base $\{v_1, \ldots, v_k\} \subset \mathbb{R}^m$ for the space of columns $\mathcal{C}(A)$.

Then every column $c_j(A)$ can be written as

$$c_j(A) = c_{1j} v_1 + c_{2j} v_2 + \ldots + c_{kj} v_k$$

$\forall\, j = 1, \ldots, n$

$$c_j(A) = c_{1j}\, v_1 + c_{2j}\, v_2 + \dots + c_{kj}\, v_k \qquad (*)$$

Denote by

$$B = \begin{array}{c c} m & \left[\begin{array}{c|c|c|c} v_1 & v_2 & \dots & v_k \end{array}\right] \\ & k \end{array} \in \mathrm{Mat}(m,k)$$

$$C = (c_{ij})_{\substack{1\le i\le k \\ 1\le j\le n}} \in \mathrm{Mat}(k,n)$$

$$(*) \Rightarrow \quad A = B \cdot C$$

$$A = B \cdot C$$

$$A = (a_{ij}) \qquad B = (b_{ij}) \qquad C = (c_{ij})$$

So $\quad a_{ij} = \sum_{t=1}^{k} b_{it} c_{tj}$

For all $\quad i = 1, \ldots, m,$

$$l_i(A) = (a_{i1}, \ldots, a_{in})$$

$$= \left(\sum_{t=1}^{k} b_{it} c_{t1}, \ldots, \sum_{t=1}^{k} b_{it} c_{tn} \right)$$

$$= \sum_{t=1}^{k} b_{it} (c_{t1}, \ldots, c_{tn}) = l_t(C)$$

So $\forall\, i = 1, \ldots, m,$

$$l_i(A) = \underbrace{\sum_{t=1}^{k} b_{it}\, l_t(C)}_{\text{linear combination of lines of } C} \in \mathcal{L}(C)$$

We conclude that $\mathcal{L}(A) \subset \mathcal{L}(C)$

$$\Rightarrow \dim \mathcal{L}(A) \leq \dim \mathcal{L}(C) \leq k = \dim \mathcal{C}(A)$$

But $C \in \mathrm{Mat}(k, n)$

(2) $\dim \mathcal{L}(A) \leq \dim \mathcal{C}(A)$ holds for

any matrix A. In particular it holds for the transpose A^t of (any) matrix A;

so $\dim \mathcal{L}(A^t) \leq \dim \mathcal{C}(A^t)$

$$\begin{array}{ccc} \dim \mathcal{L}(A^t) & \leq & \dim \mathcal{C}(A^t) \\ \| & & \| \\ \dim \mathcal{C}(A) & \leq & \dim \mathcal{L}(A) \end{array}$$

We conclude that $\dim \mathcal{C}(A) = \dim \mathcal{L}(A)$

$=: k.$

Let l be the dim of the largest nonsingular submatrix of A.

We will prove that $\ell = k$.
For this we need the following

Lemma Let $M \in \mathrm{Mat}(d,d)$.

Then $c_1(M), \ldots, c_d(M)$ are linearly independent

iff M is nonsingular
$(\det M \neq 0)$

Proof (of the lemma)

$c_1(M), \ldots, c_d(M)$ are not lin. indep

$\Leftrightarrow \exists\, x_1, \ldots, x_d \in \mathbb{R}$, not all zero

s.t. $x_1 \cdot c_1(M) + \ldots + x_d \cdot c_d(M) = 0$

$\Updownarrow$

$x = (x_1, \ldots, x_d)$ $\quad M \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = 0$

M not invertible

$\Updownarrow$

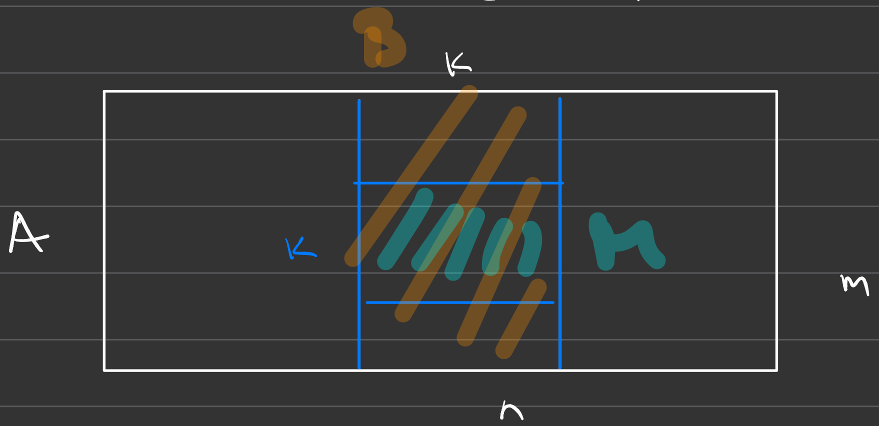
$\Leftrightarrow \exists\, x \in \mathbb{R}^d,\ x \neq 0$ s.t. $M \cdot x = 0$

$\square$

$\dim \mathcal{C}(A)$ $\rightarrow$ dim of largest invertible submatrix

Back to the proof of $k = \ell$

- $\dim \mathcal{C}(A) = k \Rightarrow A$ has k columns that are lin. indep.



Let B be the submatrix of A containing these k lin. indep. columns.
So $B \in Mat(m, k)$

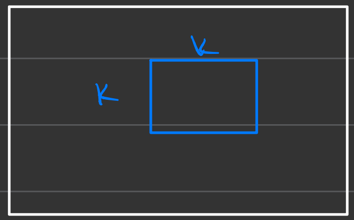
$k = \dim \mathcal{C}(B) = \dim \mathcal{L}(B) \Rightarrow$ there are k lines in B that are lin. indep.

Let M be the submatrix of of B with these lin. indep. lines.

Then M is a submatrix of A

$$M \in \mathrm{Mat}(k,k)$$

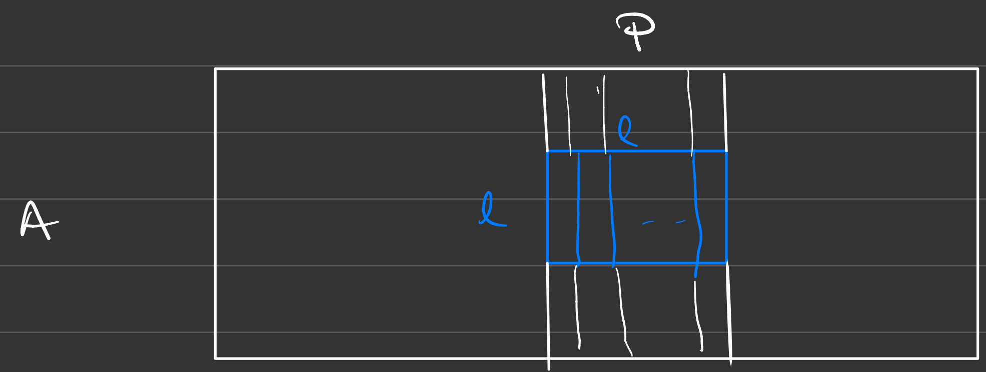
& (by the previous lemma)



M is nonsingular.

Then $k \le \ell$.

- Let L be an $\ell \times \ell$ submatrix of A which is non singular.



The columns of L are lin. indep. Then the corresponding columns of A are also lin. indep.

$$\Rightarrow \quad \dim \mathcal{C}(A) \geq \ell \quad \Rightarrow k \geq \ell .$$

<u>Corollary</u> Let $A \in \mathrm{Mat}(m,n)$ and let

$T_A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $T_A(x) := A \cdot x$

be the corresponding linear transformation.

Then $\operatorname{rank}(A) = \operatorname{rank}(T_A)$

<u>proof</u> $\operatorname{rank}(T_A) = \dim \operatorname{im} T_A$

$$\operatorname{im} T_A = \{T_A(x) : x \in \mathbb{R}^n\} = \{A \cdot x : x \in \mathbb{R}^n\}$$

$$= \operatorname{span}\{\underbrace{Ae_1}_{c_1(A)}, \ldots, \underbrace{Ae_n}_{c_n(A)}\} = \mathcal{C}(A)$$

$\blacksquare$

Lemma Let $A \in Mat(d, d)$ s.t. $\|A\| < 1$.

Then $\sum_{n \geq 0} A^n$ converges, $I - A$ is invertible

and $$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}.$$

$0 \leq r < 1 \Rightarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

Obs

Here we choose the identification

$$\mathrm{Mat}(d, d) \simeq \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$$

So $\|A\| := \|T_A\|$
operator norm

In other words, $\|A\| = \sup_{\|x\| \leq 1} \|A \cdot x\|$.

Exercise

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|$$

Let $(V, \|\cdot\|)$ be a normed space

- $x_n \longrightarrow x$ means $\|x_n - x\| \longrightarrow 0$

- $\sum_{n \geq 0} x_n$ converges to v means:

the sequence $S_n := x_0 + \dots + x_{n-1}$
of partial sums converges to v:

$$\| S_n - v \| \longrightarrow 0 .$$

proof of the lemma

$\|A\| < 1 \implies$ ① $\sum_{n \geq 0} A^n$ converges

② $I - A$ is invertible

& $\sum_{n=0}^{\infty} A^n = (I - A)^{-1}$

① Let $r := \|A\| < 1$ and let

$$f_k : \overline{B}(0, r) \subset Mat(d, d) \to Mat(d, d)$$

$$f_k(M) := M^k$$

$$f_k(M) = M^k$$

$$\Rightarrow \|f_k(M)\| = \|M^k\| \leq \|M\|^k \leq r^k$$

$$\sum_{k \geq 0} r^k < \infty$$

$(r < 1)$

By the Weierstrass M-test, $\sum_{k \geq 0} f_k$

converges uniformly in $\overline{B}(0, r)$.

In particular, since $A \in \overline{B}(0, r)$,

$$\sum_{k \geq 0} A^k = \sum_{k \geq 0} f_k(A) \text{ converges.}$$

2 Let $B := \sum_{k=0}^{\infty} A^k$. Then

$$B \cdot (I - A) = \left(\sum_{k=0}^{\infty} A^k \right) \cdot (I - A)$$

$$= \lim_{N \to \infty} \left(I + A + \dots + A^{N-1} \right) (I - A)$$

$$= \lim_{N\to\infty} \left(I + A + \cdots + A^{N-1}\right)(I - A)$$

$$= \lim_{N\to\infty} \left(I + A + \cdots + A^{N-1} - A - A^2 - \cdots - A^{N-1} - A^N\right)$$

$$= \lim_{N\to\infty} \left(I - A^N\right) = I - 0 = I.$$

$\lim_{N\to\infty} A^N = 0$ since $\|A^N\| \le \|A\|^N \le r^N \to 0$

$0 \le r < 1$

We proved that $B \cdot (I - A) = I$.

Similarly, $(I - A) \cdot B = I$.

This shows that $I - A$ is invertible

and $(I - A)^{-1} = B = \sum_{n=0}^{\infty} A^n$.

$\blacksquare$

Corollary Let $B \in Mat(d, d)$ s.t.

$$\| I - B \| < 1$$

Then B is inventible.

De fato, $B = I - \underbrace{(I - B)}_{A}$

Then $\| A \| = \| I - B \| < 1 \Rightarrow B = I - A$ is inventible (by the previous lemma)

Exercise Let A be an invertible matrix.

Prove that there is $\varepsilon_0 = \varepsilon_0(A) > 0$

s.t. if $\| A - B \| < \varepsilon_0(A)$

then B is invertible.

This shows that $GL_d(\mathbb{R}) = \{ A \in Mat(d,d) : \det A \neq 0 \}$

is an open set.

<u>Lemma</u> Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and let $k = \operatorname{rank}(T)$. Then there is $\varepsilon_0 = \varepsilon_0(T) > 0$ s.t. for any $S \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with $\|T - S\| < \varepsilon_0$, we have

$$\operatorname{rank}(S) \geq k.$$

<u>proof</u> By the rank theorem for lin. transf.,

$T \sim P_K$ where the matrix of P_K

is
$$\left[\begin{array}{c|c} I_K & 0 \\ \hline 0 & 0 \end{array}\right]$$

Then there are isomorphisms α, β
(that only depend on T)
such that

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\oslash"] & \mathbb{R}^m \arrow[d, "\beta"] \\
\mathbb{R}^n \arrow[r, "P_K"'] & \mathbb{R}^m
\end{tikzcd}$$

$$P_K = \beta \circ T \circ \alpha^{-1}$$

Let $S \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, and let

$$S' := \beta \circ S \circ \alpha^{-1}$$

Then $\| S' - P_K \| = \| \beta \circ (S - T) \circ \alpha^{-1} \|$

$$\leq \|\beta\| \cdot \| S - T \| \cdot \| \alpha^{-1} \|$$

Let $\varepsilon_0(T) := \dfrac{1}{\|\beta\| \cdot \|\alpha^{-1}\|}$

If $\| S - T \| < \varepsilon_0(T)$ we have

$$\| S' - I_k \| < 1 .$$

$$S' = \begin{array}{cc} & \begin{array}{cc} k & \end{array} \\ k & \left[\begin{array}{c|c} M & \\ \hline & \end{array}\right] n \\ & m \end{array}$$

Let $M \in \mathrm{Mat}(k,k)$ be the upper left $k \times k$ block of S'

$$S' \quad \begin{array}{cc} & k \\ k & \left[\begin{array}{c|c} M & * \\ \hline * & * \end{array}\right] \end{array} \quad \text{vs} \quad \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] \quad P_k$$

$$\| S' - P_k \| < 1$$

then $\| M - I_k \| \leq \| S' - P_k \| < 1$

$\hookrightarrow$ why? (exercise)

Thus $\| M - I_k \| < 1 \xRightarrow[\text{lemma}]{\text{previous}}$ M is invertible

$\Rightarrow \operatorname{rank}(S') \geq k$

Since $S' = \beta \circ S \circ \alpha^{-1}$

we have $S' \sim S$

$\Rightarrow \operatorname{rank}(S') = \operatorname{rank}(S)$

$\operatorname{rank}(S') \geq k$

$\Rightarrow \operatorname{rank}(S) \geq k$.

$\blacksquare$