Lecture 18 The rank theorem

Recall from previous classes:

<u>Lemma</u> Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $\operatorname{rank} T = k$.

Then there is $\varepsilon_0(T) > 0$ s.t. if

$S \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with $\|S - T\| < \varepsilon_0(T)$

then $\operatorname{rank}(S) \geq k$.

Recall that a function $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$

is called a diffeomorphism (diffeo) if:

- f is differentiable
- f is bijective
- f^{-1} is differentiable

Moreover, f is a C^r-diffeo ($r \geq 1$) if

f is bijective, f & f^{-1} are of class C^r.

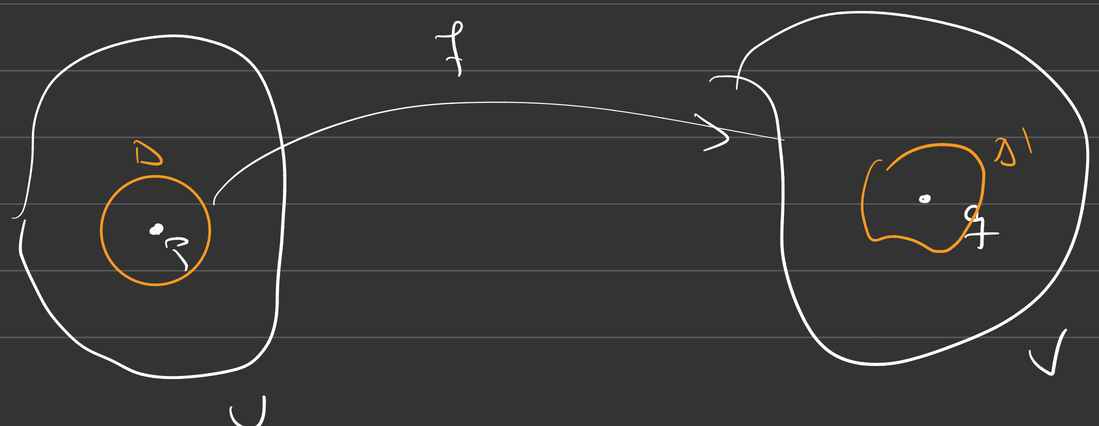
$$f : U \to V$$

- f is a local diffeo at $p \in U$ if

$q := f(p)$ there are open sets $p \in D \subset U$

$q \in D' \subset V$

s.t. $f : D \to D'$ is a diffeo

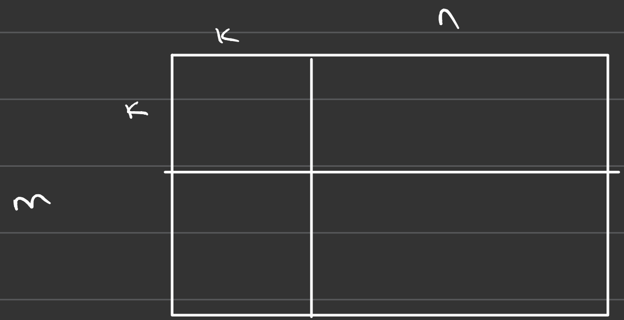


of class C^r $(r \geq 1)$

<u>Thm</u> (inverse function theorem) $f : U \to V$, $p \in U$

if $(Df)_p$ is invertible then f is a local diffeo at p of class C^r.

Obs Block matrices



$$
m\left(\begin{array}{c|cc}
 & k & n-k \\
\hline
k & \alpha_1 & \alpha_2 \\
m-k & \alpha_3 & \alpha_4
\end{array}\right) \cdot
\begin{array}{c|cc}
 & k & n-k \\
\hline
k & \beta_1 & \beta_2 \\
n-k & \beta_3 & \beta_4
\end{array}
$$

$$\qquad\quad A \qquad\qquad\qquad\qquad B$$

$$
= \begin{array}{|c|c|}
\hline
\alpha_1\beta_1 + \alpha_2\beta_3 & - \quad - \\
\hline
\alpha_3\beta_1 + \alpha_4\beta_3 & - \quad - \\
\hline
\end{array}
$$

$$n \quad \overset{n}{\begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \end{array}} = M$$

$$\det M = \det A \cdot \det C$$

<u>Definition</u> Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. The rank of f at $p \in U$ is by definition $\operatorname{rank}(Df)_p$.

A function f is of constant rank k if $\operatorname{rank}(Df)_p = k \quad \forall p \in U$.

Obs Let $f \in C^1(U)$ and let $p \in U$,

$\operatorname{rank}(Df)_p =: k$.

Then locally near p, $\operatorname{rank}(Df)_x \geq k$.

Indeed, $f \in C^1(U) \Rightarrow x \longmapsto (Df)_x$ is continuous.

This implies: $\exists\, \delta > 0$ s.t. if $\| p - x \| < \delta$

then $\| (Df)_p - (Df)_x \| < \varepsilon_0((Df)_p)$

By the previous lemma, $\operatorname{rank}(Df)_x \geq k$.

for all $x \in B(p, \delta)$.

Def Two functions $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$

are *equivalent* (or *conjugated*) if there

are bijections

α, β s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

$$g = \beta \circ f \circ \alpha^{-1}$$

Def Let f, g be two functions of class C^r.

f & g are **C^r-equivalent** : $f \cong_r g$

if there are **C^r-diffeos** α, β s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \qquad g = \beta \circ f \circ \alpha^{-1}$$

Think of α, β as "changes of variables".

Properties of the C_r-equivalence:

- $f \simeq_r f$

- $f \simeq_r g \;\&\; g \simeq_r h \Rightarrow f \simeq_r h$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dd, bend right] \arrow[dr, phantom, "\varnothing"] & B \arrow[d] \arrow[dd, bend left] \\
C \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\varnothing"] & D \arrow[d] \\
E \arrow[r, "h"'] & F
\end{tikzcd}$$

- $f \simeq_r g \Rightarrow g \simeq_r f$

- if $f : U \to V$ is a C^r - diffeo

  then $f \simeq_r id$

- if $f \simeq_r g$ then $\operatorname{rank}(f, p) = \operatorname{rank}(g, \alpha(p))$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

indeed, $g = \beta \circ f \circ \alpha^{-1}$

$$\Rightarrow (Dg)_{\alpha(p)} = \underbrace{(D\beta)_{f(p)}}_{\text{invertible}} \circ (Df)_p \circ \underbrace{(D\alpha^{-1})\big|_{\alpha(p)}}_{\text{invertible}}$$

α, β are diffeos $\Longrightarrow$ invertible

Then $(Dg)_{\alpha(p)} \sim (Df)_p \Rightarrow \operatorname{rank}(Dg)_{\alpha(p)} = \operatorname{rank}(Df)_p$ $\square$

Theorem (the rank theorem) Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function of class C^r $(r \geq 1)$ with constant rank k. Then locally f is C^r-equivalent to P_k, the linear projection in k coordinates.

More precisely, given $p \in U$ there are $\varepsilon > 0$, $\delta > 0$ and $V, W \subset \mathbb{R}^m$ open sets, and α, β C^r-diffeos such that:

$$\begin{tikzcd}
\mathbb{R}^n \supset U \supset B(p,\varepsilon) \arrow[r, "f"] \arrow[d, "\alpha"] & V \subset \mathbb{R}^m \arrow[d, "\beta"] \\
\mathbb{R}^n \supset B(0,\delta) \arrow[r, "P_K"'] & W \subset \mathbb{R}^m
\end{tikzcd}$$

- $f(B(p,\varepsilon)) \subset V$
- $P_K = \beta \circ f \circ \alpha^{-1}$.

$P_K(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$

proof We will perform a series of (4) changes of coordinates to locally turn f into P_k.

Fix $p \in U$, $q := f(p)$ $\quad f : U \subset \mathbb{R}^n \to \mathbb{R}^m$

Step 1 Define the translations $\tau : \mathbb{R}^n \to \mathbb{R}^n$

Clearly τ, τ' are C^∞ - diffeos $\quad \tau(a) = a + p$

Let

$f_1 := \tau' \circ f \circ \tau$ $\quad \tau' : \mathbb{R}^m \to \mathbb{R}^m$

$\tau'(b) = b - q$

$\Rightarrow f_1(0) = 0 \;\&\; f_1 \underset{r}{\sim} f$

Step 2 $\quad (Df_1)_0 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

$$\operatorname{rank}(Df_1)_0 = \operatorname{rank}(Df)_p = k$$

Then by the rank theorem for linear transformations we have that $(Df_1)_0 \sim P_k$. Therefore, there are isomorphisms

$$T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$$
$$T' \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$$

s.t.

$$P_k = T' \circ (Df_1)_0 \circ T$$

Let $f_2 := T^{-1} \circ f_1 \circ T$

T^{-1}, T are C^∞ diffeos, so

$$f_2 \simeq_r f_1$$

Moreover, $f_2(0) = 0$

& $(Df_2)_0 = (DT^{-1})_0 \circ (Df_1)_0 \circ (DT)_0$

$= T^{-1} \circ (Df_1)_0 \circ T = P_k$.

in step 2 we obtained $f_2 : U_2 \subset \mathbb{R}^n \to V_2 \subset \mathbb{R}^m$

$$f_2 \simeq_r f_1 \simeq_r f$$

s.t.

$$f_2(0) = 0$$

$$(Df_2)_0 = P_k = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

$$f_2 : U_2 \subset \underset{\substack{\Downarrow \\ (x,y)}}{\mathbb{R}^k \times \mathbb{R}^{n-k}} \longrightarrow V_2 \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$$

$$f_2(x,y) = \left(g_2(x,y), h_2(x,y) \right)$$

Step 3   We will conjugate f_2 to a function

f_3 s.t. $f_3(x,0) = (x,0) \quad \forall x$

$$f_2(x,y) = \left(g_2(x,y),\ h_2(x,y)\right)$$

$$f_2(x,0) = \left(\underbrace{g_2(x,0)}_{\rightsquigarrow\, x},\ \underbrace{h_2(x,0)}_{\rightsquigarrow\, 0}\right)$$

Consider the function $\mathbb{R}^k \ni x \xrightarrow{\sigma} g_2(x,0) \in \mathbb{R}^k$.

Then σ is a local diffeo (of class C^r) near 0

Indeed $(D\sigma)_0 = (\partial_x g_2)_{(0,0)} = I_k$ which is invertible.

because

$$f_2 = (g_2, h_2)$$

$$(Df_2)_{(0,0)} = \begin{pmatrix} \partial_x g_2 & \partial_y g_2 \\ \partial_x h_2 & \partial_y h_2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

By the inverse function theorem, $x \overset{\sigma}{\longmapsto} g_2(x,0)$ is a $\underbrace{\text{local}}_{\text{near } 0}$ C^r- diffeo. Then there are $0 \in D, D' \subset \mathbb{R}^k$ such that

$$\mathbb{R}^k \supset D \underset{\sigma^{-1}}{\overset{\sigma}{\rightleftarrows}} D' \subset \mathbb{R}^k$$

σ, σ^{-1} are C^r diffeos.

$x \in D$; $x' = \sigma(x) \in D'$

$$f_2(x,0) = \Big(\underset{\substack{\| \\ \sigma(x)=x' \\ x=\sigma^{-1}(x')}}{g_2(x,0)},\ \underbrace{h_2(x,0)}_{\substack{\| \\ \underbrace{h_2(\sigma^{-1}(x'),0)}_{\Phi(x')}}}\Big)$$

Let $\Phi : D' \subset \mathbb{R}^k \longrightarrow \mathbb{R}^{m-k}$

$$\Phi(x') = h_2(\sigma^{-1}(x'), 0)$$

Then

$$f_2(x,0) = \big(\sigma(x),\ \Phi(\sigma(x))\big)$$

Then define the function

$$\psi : D' \times \mathbb{R}^{m-r} \subset \mathbb{R}^m \longrightarrow D \times \mathbb{R}^{m-r} \subset \mathbb{R}^m$$

$$\psi(x', y') = \left(\sigma^{-1}(x'),\ y' - \Phi(x')\right)$$

We will verify later that ψ is a $\underbrace{C^r\text{-diffeo.}}_{\text{local}}$

Let $f_3 := \psi \circ f_2$ then $f_3 \simeq_r f_2$

& $f_3(x,0) = \psi(f_2(x,0)) =$

$= \psi(\sigma(x), \Phi(\sigma(x))) = (x, \Phi(\sigma(x)) - \Phi(\sigma(x))) = (x, 0)$

So $f_3(x,0) = (x,0)$ as desired.

Now let's check that

$$\tau(x',y') = \left(\sigma^{-1}(x'),\ y' - \phi(x')\right)$$

is a local C^r-diffeo.

As a composition of functions of class C^r, τ is of class C^r.

$$\psi(x', y') = \left(\sigma^{-1}(x'),\; y' - \phi(x')\right)$$

Then

$$(D\psi)_{(0,0)} = \left[\begin{array}{c|c} (D\sigma^{-1})_0 & \partial_y \sigma^{-1}(x') = 0 \\ \hline -(D\phi)_0 & I_{n-k} \end{array}\right]$$

$$(D\sigma)_0 = I_k \implies (D\sigma^{-1})_0 = (D\sigma)_0^{-1} = I_k^{-1} = I_k$$

So

$$(D\psi)_{(0,0)} = \left[\begin{array}{c|c} I_k & 0 \\ \hline -(D\phi)_0 & I_{n-k} \end{array}\right] \implies (D\psi)_{(0,0)}$$

has det. 1 so it is invertible.

We conclude that f is indeed a local C^r-diffeo (by the inverse function theorem).

Summarizing until now:

We started with f of class C^r, $\mathrm{rank}(f, x) = k$.

We C^r-conjugated f to a function f_3 s.t.

$$f_3(x, 0) = (x, 0) \quad \text{locally.}$$

$$f_3 = \tau \circ f_2$$

$$f_3 : U_3 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow V_3 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$$

- $f_3(x, 0) = (x, 0)$, $\quad f_3 \simeq_r f$

- $$(Df_3)_{(0,0)} = (D\tau)_{(0,0)} \circ (Df_2)_{(0,0)}$$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline -(D\varphi)_0 & I \end{array}\right] \cdot \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline -(D\varphi)_0 & 0 \end{array}\right]$$

Step 4 (next time)

Conjugate f_3 to P_k.