LINEAR COCYCLES AND LYAPUNOV EXPONENTS

Exercise 1. Let M be compact metric space, let $f\colon M \to M$ be a homeomorphism and let $A\colon M \to \mathrm{SL}_2(\mathbb{R})$ be a continuous map. Consider the corresponding linear cocycle

$$M \times \mathbb{R}^2 \ni (x, v) \mapsto (f(x), A(x)v) \in M \times \mathbb{R}^2 \,.$$

Assume that for all $x \in M$ we have an F-invariant line $E^u_x \subset \mathbb{R}^2$. Prove that the following statements are equivalent.

(i) For all $x \in M$, $v^u \in E^u_x$, $n \geq 1$,

$$\|A^n(x)v^u\| \geq C^{-1}\lambda^{-n}\|v^u\| \,.$$

(ii) For all $x \in M$, $v^u \in E^u_x$, $n \geq 1$,

$$\|A^{-n}(x)v^u\| \leq C\lambda^{n}\|v^u\| \,.$$

Exercise 2. Let M be compact metric space, let $f\colon M \to M$ be a homeomorphism and let $A\colon M \to \mathrm{SL}_2(\mathbb{R})$ be a continuous map. Prove that the corresponding linear cocycle

$$M \times \mathbb{R}^2 \ni (x, v) \mapsto (f(x), A(x)v) \in M \times \mathbb{R}^2$$

is uniformly hyperbolic if and only if there is $n_0 \in \mathbb{N}$ and for all $x \in M$ there is an F-invariant splitting $\mathbb{R}^2 = E^s_x \oplus E^u_x$ such that

$$\|A^{n_0}(x)v^u\| > 1 > \|A^{n_0}(x)v^u\|$$

for all unit vectors $v^s \in E^s_x$ and $v^u \in E^u_x$.

Exercise 3. Let (M, f, μ) be an ergodic dynamical system and let $v\colon M \to \mathbb{R}$ be a bounded, measurable potential function. For $E \in \mathbb{R}$ consider the Schrödinger cocycle $A_E\colon M \to \mathrm{SL}_2(\mathbb{R})$,

$$A_E(x) := \begin{pmatrix} v(x) - E & -1 \\ 1 & 0 \end{pmatrix} .$$

Show that if $\left|E\right| > 2 + \|v\|_\infty$ then A_E is uniformly hyperbolic.

Exercise 4. Let $g \in \mathrm{SL}_2(\mathbb{R})$ with $\|g\| > 1$. Show that there are two orthogonal unit vectors $u = u(g), s = s(g) \in \mathbb{R}^2$ such that gu and gs are orthogonal and $\|gu\| = \|g\|$ while $\|gs\| = \|g\|^{-1}$.

Exercise 5. Complete the proof of Oseledets' theorem for $\mathrm{SL}_2(\mathbb{R})$-valued cocycles.

That is, prove that if $v \in E^s_x \setminus \{0\}$, then

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| = -L \,.$$

Recall that $E^s_x = \mathbb{R}s(x)$, where $s(x) := \lim_{n\to\infty} s_n(x)$, while $s_n(x) := s(A^n(x))$.

Furthermore, derive the two sided version of the Oseledets theorem by applying the one sided version (also) to the inverse cocycle F^{-1}.

Exercise 6. Let (M, f, μ) be an ergodic dynamical system and let $A \colon M \to \mathrm{GL_d}(\mathbb{R})$ be a measurable map such that $\log^+\|A^{\pm}\| \in L_1(\mu)$. Define $c(x) := \left|\det A(x)\right|^{1/d}$ and then let $B \colon M \to \mathrm{SL_d}(\mathbb{R})$ be given by $A(x) = c(x)B(x)$.

Show that $\log c$, $\log^+\|B^{\pm}\| \in L_1(\mu)$. Check that for μ-almost every $x \in M$ and $v \neq 0$,

$$\lim_{n\to+\infty} \frac{1}{n} \log\|A^n(x)v\| = \int_M \log c(x) d\mu(x) + \lim_{n\to+\infty} \frac{1}{n} \log\|B^n(x)v\| .$$

Deduce that the associated cocycles $F(x, v) = (f(x), A(x)v)$ and $G(x, v) = (f(x), B(x)v)$ have the same Oseledets filtration and decomposition at almost every point. Moreover, the Lyapunov spectrum of the former cocycle is the translate of the Lyapunov spectrum of the latter by $\int \log c\, d\mu$.