Review for the second exam

I. 1 The implicit function theorem

2. The inverse function theorem

I.1 **Theorem** Let

$$f : \underset{\text{open}}{U} \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m,$$

be a function of class C^1, let $(x_0, y_0) \in U$

and let $z_0 = f(x_0, y_0)$

Assume that

$\partial_y f(x_0, y_0)$ is invertible

Then, locally near (x_0, y_0), the solution of the equation

$$f(x,y) = z_0$$

is the graph of a function $\varphi : D \subset \mathbb{R}^n \to \mathbb{R}^m$

$$(x, \varphi(x)).$$

More precisely,

- there are $\varepsilon_1, \varepsilon_2 > 0$ and there is a function

$$\varphi : B(x_0, \varepsilon_2) \subset \mathbb{R}^n \to B(y_0, \varepsilon_1) \subset \mathbb{R}^m$$

such that

$$\begin{cases} f(x,y) = z_0 \\ (x,y) \in B(x_0, \varepsilon_2) \times B(y_0, \varepsilon_1) \end{cases} \quad \text{if and only if} \quad \begin{matrix} y = \varphi(x) \\ x \in B(x_0, \varepsilon_2) \end{matrix}$$

This function φ is called the implicit function.

- The implicit function φ is of class C^1 and

$$(D\varphi)_x = -\left[(\partial_y f)_{(x,\varphi(x))}\right]^{-1} \circ (\partial_x f)_{(x,\varphi(x))}$$

$$\partial_x f(x,\varphi(x)) = \partial_x(z_0) = 0$$

$$(\partial_x f)\Big|_{(x,\varphi(x))} \cdot \mathrm{id} + (\partial_y f)\Big|_{(x,\varphi(x))} \cdot (D\varphi)_x = 0$$

- if f is of class C^k $(1 \le k \le \infty)$ then φ is of class C^k.

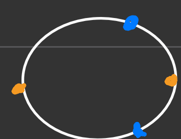
Recall that

$$\partial_y f(x_0, y_0) = \big(D f(x_0, \cdot)\big)_{y_0} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$$

$$f : U \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$$

$$f(x_0, \cdot) : D \subset \mathbb{R}^m \to \mathbb{R}^m$$

$x^2 + y^2 = 1$ $\qquad$ $y = \sqrt{1 - x^2}$ $\qquad$ if $y \neq 0$



$y = -\sqrt{1-x^2}$

$$f(x,y) = x^2 + y^2$$

$$f_y = 2y \neq 0$$

I.2 <u>Theorem</u> (the inverse function theorem)

Let $f : \underset{\text{open}}{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and let $p \in U$, be a function of class C^1

if $(Df)_p$ is invertible (isomorphism)

then f is a local C^1-diffeo near p.

More precisely, there are $\varepsilon > 0$ and $\underset{\text{open set}}{V \subset \mathbb{R}}$

Such that $f : B(p, \varepsilon) \longrightarrow V$ is C^1-diff.
bijective
and f^{-1} is C^1.

Moreover, if

f is of class C^k $(1 \le k \le \infty)$

then locally near p,

f is a C^k-diffeo.

II. The rank theorem and its consequences

1. Linear algebra concepts
2. The rank theorem
3. immersions and submersions
4. Local immersion & submersion theorems.

I.1 <u>Linear algebra</u>

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

$\ker T \subset \mathbb{R}^n$, $\operatorname{im} T \subset \mathbb{R}^m$

$\operatorname{null} T$ $\qquad$ $\operatorname{rank} T = \dim \operatorname{im} T$

$$\operatorname{rank} T + \operatorname{null} T = n$$

Equivalent lin. transf. $T \sim S$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R}^m \arrow[d, "\beta"] \\ \mathbb{R}^n \arrow[r, "S"] & \mathbb{R}^m \end{tikzcd}$$

α, β (linear) isomorphisms

$$S \circ \alpha = \beta \circ T$$

$$\Leftrightarrow \quad S = \beta \circ T \circ \alpha^{-1}$$

<u>obs</u> $T \sim S \implies \mathrm{null}(T) = \mathrm{null}(S)$

$\mathrm{rank}(T) = \mathrm{rank}(S)$

Lemma $A \in \mathrm{Mat}(m,n)$

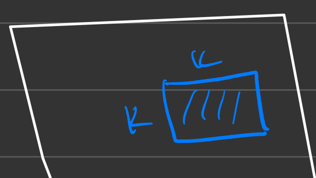
$T_A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ corresponding (lin. transf.

$$\operatorname{rank}(A) = \operatorname{rank}(T_A)$$

$$\|$$

$$\dim \mathcal{C}(A) = \dim \mathcal{L}(A)$$

A



$=$ dim of largest invertible submatrix of A.

<u>Lemma</u> Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

$$k := \operatorname{rank}(T)$$

Then $\exists\, \varepsilon_0(T) > 0$ s.t. $\forall\, S \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

if $\|S - T\| < \varepsilon_0(T)$

then $\operatorname{rank}(S) \geq k$

Theorem (the rank thm for lin. transf)

Given $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $\operatorname{rank} T = k$,

Then $T \sim P_k$ where

$$P_k : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{m-k}$$

$$P_k(x_1 - x_k, x_{k+1} - x_n) = (x_1 - x_k, 0 \,\text{---}\, 0)$$

$$P_k = \begin{array}{ccc} & k & \\ k & \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] & m \\ & n & \end{array}$$

1.2 General rank theorem

Def $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ differentiable

$p \in U$

$\operatorname{rank} (Df)_p$ is called the rank of f at p.

We say that f has constant rank k in U

if $\operatorname{rank} (Df)_x = k \quad \forall x \in U$.

Def Two functions $f : A \to B$
of class C^r $\quad g : C \to D$

$(1 \leq r \leq \infty)$

are C^k-equivalent $\quad f \simeq_r g$

if there are α, β C^r-diffeos

s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \qquad g = \beta \circ f \circ \alpha^{-1}$$

<u>Theorem</u> Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function of class C^r, with constant rank k.

Then, locally, $f \simeq_r P_k$. More precisely,

$\forall p \in U \ \exists \ \varepsilon > 0, \ \delta > 0 \ \exists \ V, W \subset \mathbb{R}^m$ open s.t.

$$
\begin{tikzcd}
\mathbb{R}^n \supset U \supset B(p,\varepsilon) \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & V \subset \mathbb{R}^m \arrow[d, "\beta"] \\
\mathbb{R}^n \supset B(0,\delta) \arrow[r, "P_k"'] & W \subset \mathbb{R}^m
\end{tikzcd}
$$

$\exists \ \alpha, \beta$ C^r-diffeos.

$$P_k(x_1, \ldots, x_k, x_{k+1}, \ldots x_n) = (x_1, \ldots x_k, 0 \ldots 0)$$

II.3 Immersions and submersions

$f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Def • immersion : $(Df)_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ injective $\forall x \in U$

$(\Rightarrow n \leq m)$

• submersion : $(Df)_x$ surjective $\forall x \in U$

$(n \geq m)$

Example

- $i : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+\ell}$

$i(x) = (x, 0)$ lin $\Rightarrow$ diff

$(Di)_p = i$ injective

immersion

- $\pi : \mathbb{R}^{n+\ell} \longrightarrow \mathbb{R}^n$ lin

$\pi(x, y) = x$ $(D\pi)_p = \pi$ surjective

Submersion

Lemma $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

- f immersion $\iff$ f has constant rank n
- f submersion $\iff$ f has constant rank m

Lemma $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ C^1, constant rank k.

Then

- f immersion $\iff$ f is locally injective
- f submersion $\iff$ f is an open map.

II.4 The local immersion and submersion theorems

Thm Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^{n+l}$

be a function of class C^r $(1 \leq r \leq \infty)$

Let $p \in U$, $q = f(p)$.

Assume that $(Df)_p$ is injective

Then locally near p, f is a immersion and

$f \underset{r}{\simeq} i$.

More precisely, there are $Z \subset \mathbb{R}^{n+\ell}$

$V \subset \mathbb{R}^n$, $p \in V$

$W \subset \mathbb{R}^\ell$ open

$\exists\ g$ diffeo of class C^r

s.t.

$$\begin{tikzcd}
\mathbb{R}^n \supseteq V \arrow[r, "f"] \arrow[dr, bend right, "i"'] & Z \subset \mathbb{R}^{n+\ell} \arrow[d, "g"] \\
& V \times W \subset \mathbb{R}^{n+\ell}
\end{tikzcd}$$

$$g \circ f = i$$

Theorem $f : U \subset \mathbb{R}^{m+l} \longrightarrow \mathbb{R}^m$ class C^r

$p \in U$, $f(p) = q$

If $(Df)_p$ is surjective

then locally near p f is a submersion

and $f \simeq_r \pi$

More precisely,

$\exists \quad V \subset \mathbb{R}^m$

$Z \in \mathbb{R}^{m+\ell}$ $\qquad z \in Z$

$W \subset \mathbb{R}^\ell$ $\qquad \gamma \quad C^r$-diffeo s.t.

$$
\begin{tikzcd}
\mathbb{R}^{m+\ell} \supseteq Z \arrow[r, "f"] & V \subset \mathbb{R}^m \\
& V \times W \subset \mathbb{R}^{m+\ell} \arrow[ul, "\gamma"] \arrow[u, "\pi"'] \arrow[ul, phantom, "\mathrm{I}", shift left=4]
\end{tikzcd}
$$

$f \circ \gamma = \pi$.

III. Multiple Riemann - Darboux integration

1. The concepts of Riemann & Darboux integrability and their equivalence (in a box)

2. Integrability and continuity

(the Riemann - Lebesgue theorem and its consequences)

3. Jordan measurable sets ; the extension of the concept of integral to more general domain.

4. Iterated integrals : Fubini

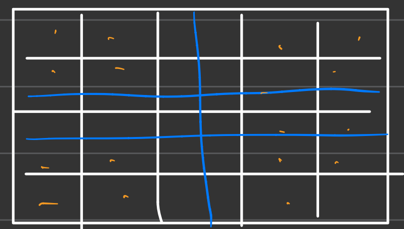
Cavalieri

5. The space of integrable functions

6. Change of variables formula.

III.1 Concept of R-I integ.

box $B \subset \mathbb{R}^n$ partition $\mathcal{P}$



$$\Delta(\mathcal{P}) = \max_{P \in \mathcal{P}} \operatorname{diam}(P)$$

$$\mathcal{P}_1 < \mathcal{P}_2$$

- Sample points $\{x_P\}_{P \in \mathcal{P}}$

- Riemann Sum $\displaystyle\sum_{P \in \mathcal{P}} f(x_P) \cdot |P|$

Def $f : B \to \mathbb{R}$ is R-integ. and

its integral $\int_B f = I$ if:

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $\forall \mathcal{P}$ partition

with $\Delta(\mathcal{P}) < \delta$

$\forall \{x_P\}_{P \in \mathcal{P}}$ sample points

$$\left| \sum_{P \in \mathcal{P}} f(x_P)\,|P| - I \right| < \varepsilon .$$

Obs f R-integ $\Rightarrow$ f bounded.

Darboux' approach: $f : B \subset \mathbb{R}^n \to \mathbb{R}$ bounded

$\mathcal{P}$ partition

upper / lower Darboux sums

$$\overline{S}(f, \mathcal{P}) := \sum_{P \in \mathcal{P}} \sup_{P} f \cdot |P|$$

$$\underline{S}(f, \mathcal{P}) := \sum_{P \in \mathcal{P}} \inf_{P} f \cdot |P|$$

$$\overline{\int_B} f = \inf_{\mathcal{P}} \overline{S}(f, \mathcal{P}) \qquad \Rightarrow \underline{\int_B} f \leq \overline{\int_B} f$$

$$\underline{\int_B} f = \sup_{\mathcal{P}} \underline{S}(f, \mathcal{P})$$

upper / lower Darboux integrals

Def If $\underline{\int_B} f = \overline{\int_B} f$ then f is called Darboux integ.

Lemma Let $f : B \subset \mathbb{R}^n \to \mathbb{R}$ bounded

f is D-integ $\iff \forall \varepsilon > 0 \ \exists P$ t.q.

$$\overline{S}(f, P) - \underline{S}(f, P) < \varepsilon$$

Theorem Let $f : B \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function.

Then f is R-integ. $\iff$ f is D-integ.

In this case,

$$\int_B f = \overline{\int_B} f = \underline{\int_B} f$$

III.2 Integrability vs. continuity

Thm f continuous $\Rightarrow$ f R-D integ.

$$\underset{\text{box}}{f : B} \subset \mathbb{R}^n \to \mathbb{R}$$

Def A set $E \subset \mathbb{R}^n$ is negligeable (or of measure zero)

if $\forall \varepsilon > 0 \quad \exists \{B_n\}_{n=1}^{\infty}$ boxes s.t.

$$E \subset \bigcup_{n=1}^{\infty} B_n$$

$$\sum_{n=1}^{\infty} |B_n| < \varepsilon .$$

$P(x)$ holds for almost every point x

a.e.

if $\{x : P(x) \text{ does NOT hold}\}$ is negligeable.

Theorem (Riemann - Lebesgue)

Let $f : B \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function.
Then

f is R-I integ $\iff$ f is continuous a.e.

III.3 More general domains

Let $E \subset \mathbb{R}^n$ be a bounded set.

Let B box, $E \subset B$

if $f : E \to \mathbb{R}$ is a bounded function,

consider its extension to B

$$\tilde{f}(x) = \begin{cases} f(x), & x \in E \\ 0 & x \in B \setminus E \end{cases}$$

<u>Def</u> f is integ on E if its extension

$\tilde{f}$ to the box B is integrable.

In this case,

$$\int_E f \underset{=}{\overset{\text{def}}{=}} \int_B \tilde{f}.$$

Obs If $f : E \to \mathbb{R}$ is continuous bounded and

then $\tilde{f} : B \to \mathbb{R}$ could be discontinuous

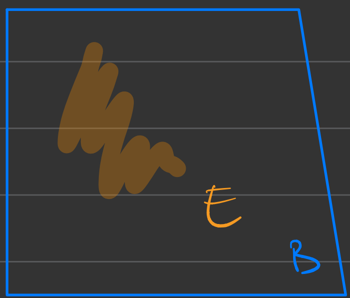
but only on ∂E.

So if ∂E is negligeable, then

f is integrable on E.

(by Riemann - Lebesgue)

Def A bounded set $E \subset \mathbb{R}^n$ is (Jordan) measurable if



1_E is integrable, moreover,

$$m(E) \overset{\text{def}}{=} \int 1_E$$

Therefore, E is Jordan meas. iff ∂E is negligeable.

Lemma Let $h : Z \subset \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function.

If Z is negligeable then $h(Z)$ is negligeable

Corollary If $\varphi : U \to V$ C^1-diffeo

$K \subset U$ compact

∂K negligeable

Then $\varphi(\partial K)$ is negligeable.

III.4. Iterated integrals

Thm (Fubini) $n=2$

Let $f : \underbrace{[a,b] \times [c,d]}_{B} \to \mathbb{R}$ be an integrable function.

Let

$$\underline{F}(y) = \underline{\int_a^b} f(x,y)\, dx$$

$$\overline{F}(y) = \overline{\int_a^b} f(x,y)\, dx$$

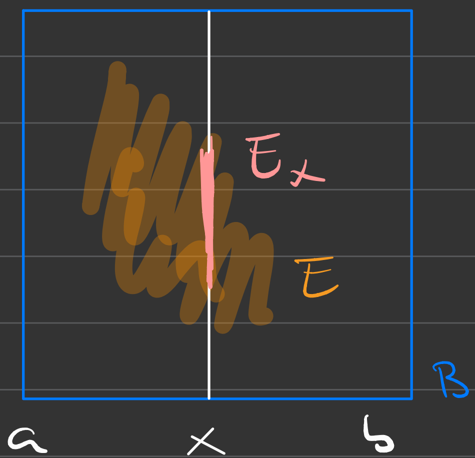
Then $\underline{F}$, $\overline{F}$ are integrable and

$$\int_B f = \int_c^d \underline{F}(y)\, dy = \int_c^d \overline{F}(y)\, dy.$$

Moreover, $\underline{F}(y) = \overline{F}(y) = \int_a^b f(x,y)\, dx$

for a.e. $y \in [c,d]$

Principle of Cavalieri



Let $E \subset \mathbb{R}^n$ be Jordan measurable

Then

$$m(E) = \int m(E_x)\, dx$$