LISTA 5: CONSTRUÇÃO ABSTRATA DE MEDIDAS, OUTROS TÓPICOS

Exercício 1. Let $(\Omega, \mathcal{F}, \mathrm{m})$ be an honest measure space and let $f \in L^1(\mathrm{m})$. Define the map $\mathrm{m}_f \colon \mathcal{F} \to [-\infty, \infty]$ by

$$\mathrm{m}_f(E) := \int_E f \, d\mathrm{m} = \int_\Omega f \, \mathbf{1}_E \, d\mathrm{m} \quad \text{for all } E \in \mathcal{F}.$$

(a) Verify that m_f is a finite signed measure on $(\Omega, \mathcal{F})$.
(b) Prove that for any $\mathcal{F}$-measurable function $g \colon \Omega \to \mathbb{R}$ we have

$$\int_\Omega g \, d\mathrm{m}_f = \int_\Omega g \, f \, d\mathrm{m}.$$

(c) Find two sets P and N representing a Hahn decomposition of the signed measure m_f.

Exercício 2. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ be a *signed* measure on it.
For every $\mathcal{F}$-measurable set E show that:
(a) $\big|\mu(E)\big| \le \big|\mu\big|(E)$. This of course implies that $\mu(E)$ and $-\mu(E)$ are both $\le \big|\mu\big|(E)$.
(b) If $\Omega = P \sqcup N$ is a Hahn decomposition of μ, then $\big|\mu\big|(E) = \mu(E \cap P) - \mu(E \cap N)$.
In particular, $\big|\mu\big|(\Omega) = \mu(P) - \mu(N)$.
(c) $\big|\mu\big|(E) = \max\left\{ \sum_{k=1}^{\infty} \big|\mu(E_k)\big| \; : \; E_1, E_2, \ldots \text{ are disjoint and } E = \bigcup_{k=1}^{\infty} E_k \right\}$.

Exercício 3. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ be a *signed* measure on it.
Denote by $\mathcal{M}$ the space of all finite signed measures on $(\Omega, \mathcal{F})$. For every $\mu \in \mathcal{M}$ let

$$\|\mu\| := \big|\mu\big|(\Omega).$$

(a) Prove that $(\mathcal{M}, \| \, \|)$ is a normed space.
In fact, this is a Banach space (you may try proving this as well).
Hint: The trickier part is verifying the triangle inequality $\|\mu_1 + \mu_2\| \le \|\mu_1\| + \|\mu_2\|$.
Begin by considering a Hahn decomposition $\Omega = P \sqcup N$ for the signed measure $\mu := \mu_1 + \mu_2$. Use Problem **??** part (b) for μ and then part (a) for μ_1 and μ_2.
(b) Given $f \in L^1(\mathrm{m})$, consider the associated signed measure m_f and prove that its (total variation) norm is simply the norm of f in $L^1(\mathrm{m})$, that is,

$$\|\mathrm{m}_f\| = \int_\Omega \big|f\big| \, d\mathrm{m}.$$

Exercício 4. Assume that Ω is *countable* and that $\mathcal{F} = \mathcal{P}(\Omega)$, meaning that $\mathcal{F}$ is the collection of all subsets of Ω. Show that every measure on $(\Omega, \mathcal{F})$ is absolutely continuous w.r.t. the counting measure $\#$.

Hint: Recall that the counting measure $\#(E)$ is defined as the number of elements of E if E is finite, and $+\infty$ otherwise. It helps to know (verify this first), that if $f \colon \Omega \to [0, \infty)$, then

$$\int_\Omega f \, d\# = \sum_{k \in \Omega} f(k).$$

Exercício 5. (Vitali-type covering lemma) Let $B_1, B_2, \ldots, B_n$ be a finite collection of open balls in $\mathbb{R}^d$ which are *not* necessarily disjoint. Then there exists a sub-collection $B_{m_1}, B_{m_2}, \ldots, B_{m_k}$ of *disjoint* balls in this collection, such that

$$\bigcup_{i=1}^{n} B_i \subset \bigcup_{j=1}^{k} 3\, B_{m_j},$$

where for a ball B, we denote by $3\,B$ the ball with the same center as B but with radius 3 times the radius of B. In particular, by finite sub-additivity, we have:

$$\lambda\Big(\bigcup_{i=1}^{n} B_i\Big) \le 3^d \sum_{j=1}^{k} \lambda(B_{m_j}).$$

Hint: Use a "greedy" algorithm of selecting balls of maximal radius amongst the ones that are disjoint from the previously selected ones.

The setup for the following problems is the following. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ be the measure space on $\mathbb{R}$ where $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets of $\mathbb{R}$ and λ denotes the Lebesgue measure. Let μ be a *finite* Borel measure on $\mathbb{R}$ and denote by F_μ its distribution function.
The Fundamental Theorem of Calculus II, the Hahn-Kolmogorov extension theorem and the Lebesgue-Radon-Nikodym decomposition theorem will be useful in solving these problems.

Exercício 6. Prove that F_μ is right continuous and that $\lim_{x\to-\infty} F_\mu(x) = 0$.

Exercício 7. Prove that F_μ is continuous at a if and only if $\mu(\{a\}) = 0$. Then conclude that μ is a continuous measure if and only if its distribution function F_μ is continuous.
Hint: Use the monotone convergence theorem for sets.

Exercício 8. (a) Prove that $\mu \ll \lambda$ if and only if F_μ is an absolutely continuous function.

(b) Assuming that $\mu \ll \lambda$, and since we know that an absolutely continuous function is differentiable almost everywhere, it is natural to ask what is the derivative of F_μ.

Prove that it must be, almost everywhere, the Radon-Nikodym derivative of μ w.r.t. λ:

$$\frac{dF_\mu}{dx} = \frac{d\mu}{d\lambda}.$$

In other words, if $\mu = \lambda_f$, prove that $F'_\mu(x) = f(x)$ for λ a.e. x.

Hint for part (b): First off, realize that what you actually have to prove is that

$$\mu(E) = \int_E F'_\mu(x)\, d\lambda(x)$$

holds for every Boolean set E.

Verify that this holds: when E is an interval (that is when you need FTC II); then when E is an elementary set (i.e. a finite union of intervals); and finally when E is the complement of an elementary set. In other words, verify that this holds for the Boolean algebra of all elementary sets and their complements.
Conclude by using the uniqueness in the Kolmogorov's extension theorem.

Exercício 9. Prove that $\mu \perp \lambda$ if and only if $F'_\mu(x) = 0$ for λ a.e. $x \in \mathbb{R}$.
Hint: First prove the " $\Longrightarrow$ " direction. Then use this implication and also the L-R-N theorem (applied to μ) to prove " $\Longleftarrow$ ".