Aula 23 O espaço de funções integráveis

Recall that if $B \subset \mathbb{R}^n$ box, if we denote by

$$\mathcal{R}(B) := \{ f : B \to \mathbb{R} \mid f \text{ R-I integrable} \}$$

then $\mathcal{R}(B)$ is a vector space, and

$$f \longmapsto \int_B f$$ is a linear transformation

Given $E \subset \mathbb{R}^n$ a bounded set with negligeable boundary

"reasonable set"

we can define $\int_E f$ for a function $f : E \to \mathbb{R}$

by extending f to a function on the box B by zero. Then $\mathcal{R}(E) = \{ f : E \to \mathbb{R} \mid f \text{ integrable} \}$ is a vector space and $f \mapsto \int_E f$ is linear. and monotonic.

reasonable set

Exercise if $f : E \to \mathbb{R}$ is R-D integ.

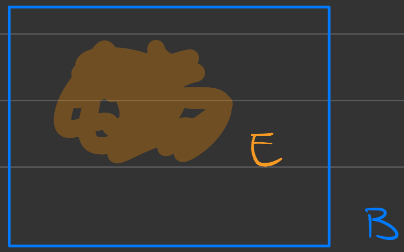
and $|f| \le M$

then $\left| \int_E f \right| \le M \cdot m(E)$,

where $m(E) \overset{\text{def}}{=} \int_B 1_E$

Jordan measure of E

where $B \supset E$ box.



Lemma Let $E \subset \mathbb{R}^n$ be a reasonable set and let $f : E \to \mathbb{R}$ be an integrable function.

Then $|f|$ is also integrable and

$$\left| \int_E f \right| \leq \int_E |f| \quad \text{(triangle inequality)}$$

Prove . f bounded $\Rightarrow$ $|f|$ is bounded

$\mathcal{D}(f)$ = set of discontinuity points of f.

if f is continuous at p

then $|f|$ is continuous at p also.

$$\left(x_n \to p \Rightarrow f(x_n) \to f(p) \right.$$

$$\left. \Rightarrow |f(x_n)| \to |f(p)| \right)$$

Then $\mathcal{D}(|f|) \subset \mathcal{D}(f)$

$\hookrightarrow$ negligeable (by R-L theorem)

$\Rightarrow \mathcal{D}(|f|)$ is negligeable $\Rightarrow |f|$ is integrable (by R-L theorem)

To prove the Δ-inequality, just note that

$$-|f| \le f \le |f|$$

By the monotonicity of the integral,

$$-\int_E |f| \le \int_E f \le \int_E |f|$$

(when $-a \le x \le a$, $|x| \le a$)

So $$\left|\int_E f\right| \le \int_E |f| \qquad \blacksquare$$

Proposition Let $E \subset \mathbb{R}^n$ be a reasonable set, let $\{f_n\}_{n \geq 1}$ be a sequence of integrable functions from E to $\mathbb{R}$ and let $f: E \to \mathbb{R}$ be another function.

Assume that $f_n \longrightarrow f$ uniformly

Then f is integrable and

$$(*) \qquad \int_E f = \lim_{n \to \infty} \int_E f_n$$

Proof

Step 1 We 1st prove that the sequence

$$\left\{ \int_E f_n \right\}_{n\geq 1} \subset \mathbb{R}$$

is Cauchy.

Indeed, since $f_n \longrightarrow f$ uniformly, given $\varepsilon > 0$

there is N_ε s.t. $|f_n(x) - f(x)| \leq \varepsilon$ $\forall n \geq N_\varepsilon$
$\forall x \in E$

$$|f_n - f| < \varepsilon \qquad \forall n \geq N_\varepsilon$$

$$\Rightarrow \quad |f_n - f_m| < 2\varepsilon \qquad \forall n, m \geq N_\varepsilon$$

By the previous lemma,

$$\left| \int_E f_n - \int_E f_m \right| = \left| \int_E (f_n - f_m) \right|$$

$$\leq \int_E \underbrace{|f_n - f_m|}_{\leq 2\varepsilon} \leq \int_E (2\varepsilon)$$

$$= 2\varepsilon \int_E 1$$

$$= 2\varepsilon \, m(E).$$

We proved that

$$\left| \int_E f_n - \int_E f_m \right| \leq \big(2m(E)\big)\,\varepsilon$$

$$\forall\, n, m \geq N_\varepsilon,$$

which shows that indeed, the sequence of real numbers $\left\{ \int_E f_n \right\}_{n \geq 1}$ is Cauchy.

Let $I := \lim_{n \to \infty} \int_E f_n$.

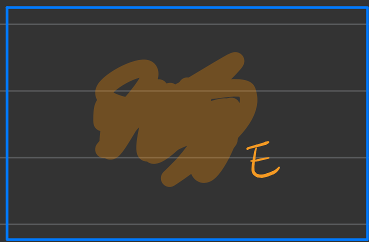
Step 2

Let $I := \lim_{n\to\infty} \int_E f_n$.

We show that f is integrable and $\int_E f = I$.

Let $B \supset E$ be a box.
Denote by $\tilde{f}_n$, $\tilde{f}$ the extensions by zero of f_n, f to B. Fix $\varepsilon > 0$; there is N'_ε s.t. $\left| \int_E f_n - I \right| < \varepsilon \quad \forall n \geq N'_\varepsilon$

$f_n \longrightarrow f$ uniformly on E,
therefore also on B



Then there is N_ε^2 s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N_\varepsilon^2$$
$$\forall x \in B$$

In particular, fixing $N := \max(N_\varepsilon^1, N_\varepsilon^2)$

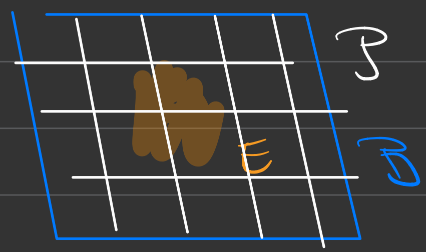
$$|f_N(x) - f(x)| < \varepsilon \quad \forall x \in B$$

f_N is Riemann integrable.

Also $\left| \int_E f_N - I \right| < \varepsilon$

f_N is Riemann integrable, so there is $\delta > 0$

s.t. given any partition $\mathcal{P}$ of B with



$\Delta(\mathcal{P}) < \delta$ and given any

sample points

$\mathcal{X} = \{x_P\}_{P \in \mathcal{P}}$

we have that

$$\left| \sum_{P \in \mathcal{P}} f_N(x_P) \cdot |P| - \int_B f_N \right| < \varepsilon$$

$$\left| \sum_{P \in \mathcal{P}} f_N(x_P) \cdot |P| - \int_B f_N \right| < \varepsilon \qquad (1)$$

Since $|f_N(x_P) - f(x_P)| < \varepsilon \quad \forall P \in \mathcal{P}$

we get that

$$\left| \sum_{P \in \mathcal{P}} f_N(x_P) \cdot |P| - \sum_{P \in \mathcal{P}} f(x_P) \cdot |P| \right|$$

$$\leq \sum_{P \in \mathcal{P}} \underbrace{|f_N(x_P) - f(x_P)|}_{< \varepsilon} \cdot |P| \leq \varepsilon \sum_{P \in \mathcal{P}} |P| = \varepsilon |B|$$

(2)

(1) & (2) imply by the Δ-ineq :

$$(3) \quad \left| \sum_{P \in \mathcal{P}} f(x_P)\,|P| - \int_E f_N \right| \leq \varepsilon + \varepsilon |B|$$

$$= \varepsilon (1 + |B|)$$

Moreover, by step 1, $\int_E f_n \longrightarrow I$

So $\left| \int_E f_N - I \right| < \varepsilon$ (4)

(3) & (4) $\Rightarrow$ via the Δ-inequality that

$$\left| \sum_{P \in \mathcal{P}} f(x_P) \cdot |P| - I \right| \le \varepsilon (1 + |B|) + \varepsilon$$

$$= \varepsilon (2 + |B|)$$

for all collections $\{x_P\}_{P \in \mathcal{P}}$ of sample points

and all partitions $\mathcal{P}$ of B.

This shows that f is Riemann integ &

$$\int_B f = I. \qquad \square$$

ex $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$

$f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

Then $f_n(x) \longrightarrow f(x)$

for all $x \in [0,1]$

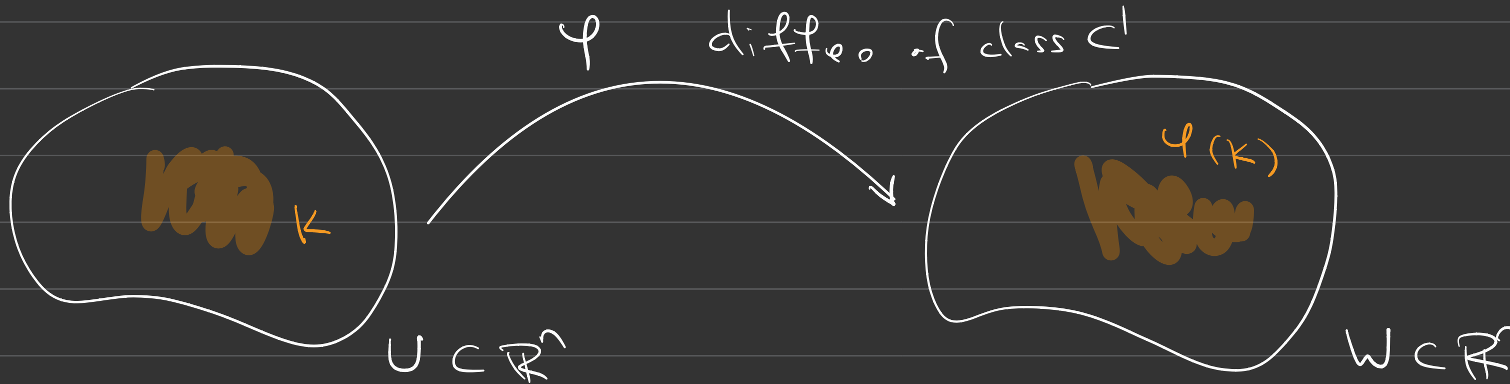
but the convergence is NOT uniform.

Note that $\int_0^1 f_n = \int_0^1 x^n = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} \to 0$

$= \int_0^1 f$

This example shows that the assumption of uniform convergence is too strong.

The change of variables formula



<u>Claim</u> If K is "very reasonable" then $\varphi(K)$ is "very reasonable" as well

Lemma Let $\varphi : Z \subset \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function.

if Z is negligeable

then $\varphi(Z) \subset \mathbb{R}^n$ is also negligeable.

Proof Fix $\varepsilon \in (0,1)$.

Since φ is Lipschitz, there is $L < \infty$ s.t.

$$\|\varphi(x) - \varphi(x')\| \leq L \|x - x'\|$$

$$\forall x, x' \in Z.$$

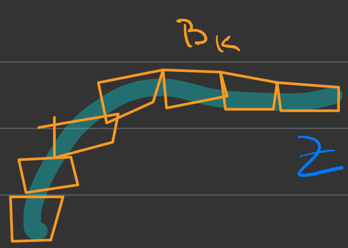
Since Z is negligeable, there is a cover

$Z \subset \bigcup_{k\geq 1} B_k$ by cubes B_k s.t.

$$\sum_{k=1}^{\infty} |B_k| < \varepsilon.$$

$$\Downarrow$$

$$Z = \bigcup_{k\geq 1} (B_k \cap Z)$$



$$\Rightarrow \varphi(Z) = \bigcup_{k\geq 1} \varphi(B_k \cap Z)$$

Since φ is L-Lipschitz,

$$\operatorname{diam} \varphi(A) \leq L \operatorname{diam}(A)$$

So $\operatorname{diam} \varphi(B_K \cap Z) \leq L \cdot \operatorname{diam}(B_K)$

S_K  $\longleftarrow \varphi(B_K \cap Z)$

$\Longrightarrow$ there exists a cube S_K of side length $L \cdot \operatorname{diam}(B_K)$

s.t. $\varphi(B_K \cap Z) \subset S_K$

Then $\underbrace{\bigcup \varphi(B_K \cap Z)}_{\varphi(Z)} \subset \bigcup_{K \geq 1} S_K$

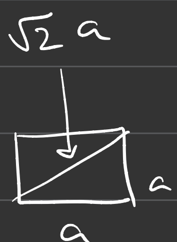
We then have

$$\varphi(Z) \subset \bigcup_{k\geq 1} S_k$$

$\leftarrow$ cubes of side length $L \cdot \operatorname{diam} B_k$

Then

$$\sum_{k=1}^{\infty} |S_k| = \sum_{k=1}^{\infty} (L \operatorname{diam} B_k)^n$$



$$= L^n \sum_{k=1}^{\infty} (\operatorname{diam} B_k)^n$$

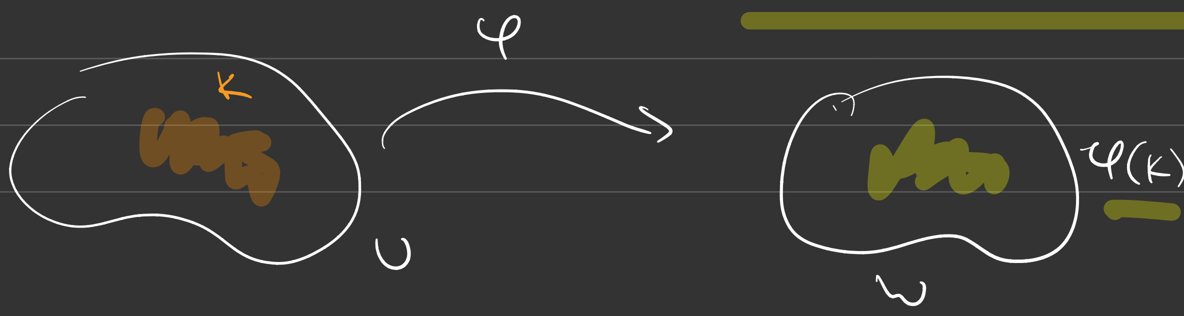
$$= L^n \, c \sum_{k=1}^{\infty} |B_k| < (L^n c)\,\varepsilon$$

$< \varepsilon$

Proposition Let $\varphi : U \to W$ be a C^1-diffeo where $U, W \subset \mathbb{R}^n$ are open sets.

Let $K \subset U$ be a compact set with ∂K negligeable.

Then $\varphi(K)$ is compact and $\partial \varphi(K)$ is negligeable



__proof__ φ is a C^1-diffeo. Then in particular,

φ is a homeomorphism (continuous,
bijective
φ^{-1} continuous)

Then K compact $\Rightarrow \varphi(K)$ compact

& $\partial\varphi(K) = \varphi(\partial K)$

Therefore we need to prove that

∂K negligeable $\Rightarrow \varphi(\partial K)$ negligeable

Since K is compact, ∂K is also compact.

$\forall x \in \partial K \subset U$ $\exists r_x > 0$ s.t. $\overline{B(x, r_x)} \subset U$
open

Then $\partial K \subset \bigcup_{x \in \partial K} B(x, r_x)$

compact

open cover $\Rightarrow \exists$ finite subcover.

Therefore, we can write $\partial K \subset B_1 \cup \dots \cup B_n$
where $B_1, \dots, B_n$ are balls s.t. $\overline{B_j} \subset U$ $\forall j$.

In particular, $\partial K = \bigcup_{j=1}^{n} (\partial K \cap B_j) \Rightarrow \varphi(\partial K) = \bigcup_{j=1}^{n} \varphi(\partial K \cap B_j)$

$$\varphi(\partial K) = \bigcup_{j=1}^{N} \varphi(\partial K \cap B_j)$$

It is enough to prove that $\varphi(\partial K \cap B)$ negligeable for any ball B s.t. $\overline{B} \subset U$

But this is obvious. Since φ is C^1-differentiable, $\varphi|_{\overline{B}}$ is Lipschitz by the MVT:

$$\|\varphi(x) - \varphi(y)\| \le \underbrace{\sup_{p \in \overline{B}} \|(Df)_p\|}_{=: L < \infty} \|x - y\| \quad \forall x, y \in \overline{B}$$

($\overline{B}$ is compact! & convex)

Since φ is Lipschitz on $\overline{B}$, and since

∂K is negligeable, so $\partial K \cap B$ is negligeable,

by the previous lemma, $\varphi(\partial K \cap B)$ is negligleable.

Then $\varphi(\partial K) = \bigcup_{j=1}^{n} \varphi(\partial K \cap B_j)$ is negligeable.

$\square$