POSITIVITY OF THE LYAPUNOV EXPONENT OF QUASI-PERIODIC COCYCLES VIA SUBHARMONICITY

Exercise 1. Use Herman's trick to establish the positivity of the Lyapunov exponent for a quasi-periodic Schrödinger operator with potential function given by a trigonometric polynomial. Determine an explicit lower bound, similar to the case of the almost Mathieu operator.

Exercise 2. Let $n \in \mathbb{N}$ and for every index $0 \leq j \leq n-1$ consider the $\mathrm{SL}_2(\mathbb{R})$ matrix

$$g_j := \begin{pmatrix} a_j & -1 \\ 1 & 0 \end{pmatrix} .$$

Assume that for all $0 \leq j \leq n-1$,

$$|a_j| \geq \lambda > 2 .$$

Consider the product of these matrices $g^n := g_{n-1} \, \dots \, g_1 \, g_0$.
Prove that $\|g^n\| \geq (\lambda - 1)^n$.

Exercise 3. Consider the Schrödinger linear cocycle over the torus translation (or any other base dynamics) $A_E \colon \mathbb{T} \to \mathrm{SL}_2(\mathbb{R})$,

$$A_E(\theta) := \begin{pmatrix} \lambda v(\theta) - E & -1 \\ 1 & 0 \end{pmatrix} .$$

Assume that $|\lambda| > 2$ and that $|E| \geq |\lambda| \, (\|v\|_\infty + 1)$. Prove that

$$L(A_{\lambda,E}) \geq \log\left(|\lambda| - 1\right) .$$

Exercise 4. Let $v \in C^\omega_\rho(\mathbb{T}, \mathbb{R})$, let $0 < \delta < \rho$ and let $\mathcal{I} \subset \mathbb{R}$ be a compact interval. Define

$$\epsilon_0(v) := \inf_{s \in \mathcal{I}} \sup_{r \in [1, 1+\delta]} \inf_{\theta \in [0,1]} |v(re(\theta)) - s| .$$

Prove that if v is non-constant then $\epsilon_0(v) > 0$.