1. **Cautionary tale about importance sampling (10 points)**

   Suppose that we wish to estimate the normalizing constant Z_p for an un-normalized Gaussian $\tilde{p}(x) = \exp(-\frac{1}{2\sigma_p^2}x^2)$. In other words, we have $p(\cdot) \sim \mathcal{N}(\cdot|0, \sigma_p^2)$ with $p(x) = \tilde{p}(x)/Z_p$.

   Given N i.i.d. samples $x^{(1)}, \ldots, x^{(N)}$ from a standard normal $q(\cdot) \sim \mathcal{N}(0, 1)$, consider the importance sampling estimate:

   $$\hat{Z} = \frac{1}{N} \sum_{i=1}^{N} \frac{\tilde{p}(x^{(i)})}{q(x^{(i)})}.$$

   (a) Show that $\hat{Z}$ is an unbiased estimator of Z_p.

   (b) Letting $f(x) := \tilde{p}(x)/q(x)$, show that $\text{var}(\hat{Z}) = \frac{1}{N}\text{Var}(f(X))$ whenever $\text{Var}(f(X)) < \infty$.

   (c) For what values of σ_p^2 is this variance finite?
   Hint: Keep your variances non-negative!

2. **Gibbs sampling and mean field variational inference (10 points)**

   Consider the Ising model with binary variables $X_s \in \{0, 1\}$ and a factorization of the form:

   $$p(x; \eta) = \frac{1}{Z_p} \exp\left(\sum_{s \in V} \eta_s x_s + \sum_{\{s,t\} \in E} \eta_{st} x_s x_t\right).$$

   (a) Derive the Gibbs sampling updates for this model.

   (b) Consider fully factorized approximation q. Use the notation $q(X_s = 1) = \tau_s$. Derive an expression for $KL(q||p) - \log(Z_p)$.
   Note: We can't directly compute the value of the KL divergence, as it would depend on the intractable quantity $\log(Z_p)$. *When we subtract it, the remainder* $KL(q||p) - \log(Z_p)$ *can be computed efficiently.*

   (c) Derive the naive mean field updates, based on a fully factorized approximation. More specifically, we do cyclic coordinate descent on $KL(q||p)$, sequentially updating the parameter $\tau_s \in [0, 1]$. Note that Z_p is a constant with respect to the parameters τ_s.

3. **Implementation of Gibbs sampling and mean field variational inference (20 points)**

   Follow the instructions in this Colab notebook:

   https://colab.research.google.com/drive/181XP5w5xEE2UCJUObMEEDrpR77RYcz-U