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On the End-Fire Super Directivity of Arrays of Two Elementary Dipoles and Isotropic Radiators

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Abstract: The concept of source currents of a radiating source can be employed to express directivity in some particular cases analytically. For an antenna array, this concept can be extended to the concept of generalized directivity based on mutual radiation intensity, mutual power and excitation current of array elements. We apply these approaches to examples of the array of two elementary dipoles and the array of two isotropic radiators. Particularly, we treat the evaluation of the directivity of the end-fire arrays, which are of special attention due to superdirective properties. Novel closed-form expressions for the directivity of these arrays with out-of-phase excitation are derived. It is observed that end-fire directivity can be further enhanced by optimizing the excitation currents of the arrays. Their optimal relative phase and corresponding increased directivity are also found analytically. The results are validated by a full-wave simulator.

1 Introduction

It is well known that an end-fire antenna array of closely-spaced elements is able to show a significant increase in directivity (termed superdirectivity) compared to a sole element [1], [2]. Uzkov derived the end-fire directivity limit for the case of N isotropic radiators, when directivity approaches N^2 as the distance between them reaches zero [3].

Recently, the design of closely spaced array elements (when the distance is less than $\lambda/4$, where λ is wavelength) attracted both theoretical and practical interest [4], [5], [6], [7], [8], [9]. We should also mention the first realization of such an array, the Kraus W8JK antenna[10].

In this paper, we firstly derive a generalized directivity of a radiating source, the antenna array, based on its current distributions and excitation currents. This results in a framework of self- and mutual intensities and self- and mutual radiated powers of array elements which is similar to the approach developed by Hansen who used mutual radiation resistances in his derivations of array directivity [11].

The concept of the generalized directivity is applied to examples of the end-fire arrays of two elementary dipoles and two isotropic radiators. In these cases, the integrals contained in the relation for the directivity are easy to work out in the closed-form. Novel expressions for the directivity of the arrays of two out-of-phase excited elementary dipoles and two isotropic radiators spaced less than $\lambda/2$ are obtained. Furthermore, the quadratic form of the excitation currents involved allows, by means of the generalized eigenvalue problem, the optimum to be found, thus producing a maximal directivity of this configuration. The optimum is also derived in the closed-form by following the approach of Uzsoky and Solymar [12]. In this manner, the "superdirective factor" of $21/15$, accounting for the increased directivity between the optimal and out-of-phase excitation, is found. A similar factor of $4/3$ is discovered for an array of two isotropic radiators.

The results are verified by a full-wave simulator Computer Simulation Technology Microwave Studio (CST MWS) [13] using its time-domain solver and show good agreement.

2 Directivity in Terms of Source Currents

A concept of the generalized directivity of a radiating source, the antenna array, based on its current distributions and excitation currents, is reviewed in this section.

A directivity of a radiating source in an angular direction (θ, ϕ) in the spherical coordinates is defined as [14]

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{U_0} = 4\pi \frac{U(\theta, \phi)}{P_{\mathrm{r}}} \tag{1}$$

where U is radiation intensity in the direction (θ, ϕ), $U_0 = P_{\mathrm{r}}/4\pi$ is average radiation intensity and P_{r} is radiated power. Intensity U is related to a far electric field $\mathbf{E}_{\mathrm{far}}$ of the source as

$$U(\theta, \phi) = r^2 S_{\mathrm{r}} = r^2 \frac{|\mathbf{E}_{\mathrm{far}}(r, \theta, \phi)|^2}{2Z_0} \tag{2}$$

where r is distance from the origin of the coordinates, S_{r} is radial power density and $Z_0 = 120\pi$ is an impedance of free space. The far electric field $\mathbf{E}_{\mathrm{far}}$ may be expressed as

$$\mathbf{E}_{\mathrm{far}}(r, \theta, \phi) = \mathrm{j}\omega \mathbf{r}_0 \times (\mathbf{r}_0 \times \mathbf{A}_{\mathrm{far}}(r, \theta, \phi)) \tag{3}$$

where $\mathbf{A}_{\mathrm{far}}$ is a vector potential of the source given by

$$\mathbf{A}_{\mathrm{far}}(r, \theta, \phi) = \frac{\mu_0}{4\pi} \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \int\limits_V \mathbf{J}(\mathbf{r}') \mathrm{e}^{\mathrm{j}k\Delta'} \,\mathrm{d}\mathbf{r}'. \tag{4}$$

In the above equation, the integration is performed over a (finite) volume V of a current density $\mathbf{J}$ of the source. Furthermore, μ_0 is a permeability of vacuum, $k = 2\pi/\lambda$ is a wavenumber and $\Delta' = \mathbf{r}_0 \cdot \mathbf{r}'$ where a unit vector $\mathbf{r}_0 = [\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)]$ determines the direction of radiation and the radius vector $\mathbf{r}' = [x', y', z']$ describes the location of current $\mathbf{J}$.

In the case of the source represented by an array of N elements, current $\mathbf{J}$ can be written as

$$\mathbf{J}(\mathbf{r}) = \sum_{n=1}^{N} \mathbf{J}_n(\mathbf{r}) = \sum_{n=1}^{N} I_n \mathbf{j}_n(\mathbf{r}) \tag{5}$$

where $\mathbf{J}_n$ is a current density existing in a volume V_n of the n-th element and $\mathbf{j}_n$ is a current density normalized to its excitation current I_n. By inserting (5) through (4) and (3) into (2) and using $|\mathbf{J}|^2 = \mathbf{J}^* \cdot \mathbf{J}$, we arrive at the expression

$$U(\theta,\phi) = \sum_{m=1}^{N}\sum_{n=1}^{N} U_{mn}(\theta,\phi) \tag{6}$$

where

$$\begin{aligned} U_{mn}(\theta,\phi) &= I_m^* I_n \frac{15k^2}{4\pi} \int\limits_{V_m}\int\limits_{V_n} \Lambda(\mathbf{r},\mathbf{r}')\mathrm{e}^{-\mathrm{j}k(\Delta-\Delta')}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}' \\ &= I_m^* I_n u_{mn}(\theta,\phi) \end{aligned} \tag{7}$$

is a mutual radiation intensity that accounts for the interaction of the m-th and n-th elements and u_{mn} is its normalization to the currents I_m and I_n. Furthermore,

$$\begin{aligned} \Delta - \Delta' = \mathbf{r_0}\cdot(\mathbf{r}-\mathbf{r}') =& (x-x')\sin\theta\cos\phi \\ &+ (y-y')\sin\theta\sin\phi \\ &+ (z-z')\cos\theta \end{aligned} \tag{8}$$

$$\Lambda(\mathbf{r},\mathbf{r}') = \mathbf{j}_m^*(\mathbf{r})\cdot\mathbf{j}_n(\mathbf{r}') - \mathbf{r}_0\cdot\mathbf{j}_m^*(\mathbf{r})\mathbf{r}_0\cdot\mathbf{j}_n(\mathbf{r}') \tag{9}$$

see Appendix A*.

The radiated power P_r required for calculating directivity D (1) can be obtained through the EMF method [15], or by integrating intensity U (6) over the complete solid angle

$$P_\mathrm{r} = \int\limits_0^{2\pi}\int\limits_0^{\pi} U(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \sum_{m=1}^{N}\sum_{n=1}^{N} P_{mn} \tag{10}$$

where

$$P_{mn} = I_m^* I_n \int\limits_0^{2\pi}\int\limits_0^{\pi} u_{mn}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = I_m^* I_n p_{mn} \tag{11}$$

is a mutual power of the m-th and n-th elements and p_{mn} is its normalization to the currents I_m and I_n.

Consequently, directivity D (1) takes a compact matrix form using (6), (7), (10), (11) [12], [16], [17]

$$\begin{aligned} D(\theta,\phi) &= 4\pi \frac{\mathbf{I}^\mathrm{H}\begin{bmatrix} u_{11}(\theta,\phi) & \cdots & u_{1N}(\theta,\phi) \\ \vdots & \ddots & \vdots \\ u_{N1}(\theta,\phi) & \cdots & u_{NN}(\theta,\phi)\end{bmatrix}\mathbf{I}}{\mathbf{I}^\mathrm{H}\begin{bmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & \ddots & \vdots \\ p_{N1} & \cdots & p_{NN}\end{bmatrix}\mathbf{I}} \\ &= 4\pi\frac{\mathbf{I}^\mathrm{H}\mathbf{u}(\theta,\phi)\mathbf{I}}{\mathbf{I}^\mathrm{H}\mathbf{p}\mathbf{I}} \end{aligned} \tag{12}$$

where H stands for Hermitian transpose, $\mathbf{I} = [I_1, \cdots, I_N]^\mathrm{T}$ is a vector of excitation currents and $\mathbf{u}$ and $\mathbf{p}$ are matrices of the normalized mutual radiation intensities u_{mn} and powers p_{mn} respectively. For some simple current distributions $\mathbf{J}$, integrals in (7) and (11) may be evaluated analytically as is shown later.

3 End-Fire Array of Two Elementary Dipoles

Directivity D, according to (12), is further calculated for an array of two elementary dipoles with end-fire radiation. This necessitates the finding of entries u_{mn} and p_{mn} of matrices $\mathbf{u}$ and $\mathbf{p}$.

For the sake of simplicity, we do not treat intensity U_{mn} for vertical and horizontal polarization separately in this paper.

3.1 Elementary Dipole

Firstly, let us consider an elementary dipole of length $L \to 0$ in the origin of the coordinates oriented in the z-axis with constant current density $\mathbf{J}_1 = I_1\delta(x)\delta(y)\mathbf{z}_0 = I_1 j_{1z}\mathbf{z}_0$, see Fig. 1 a).

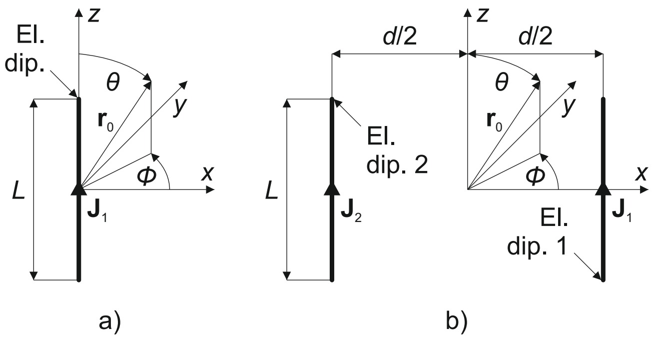

Fig. 1: Geometry: a) elementary dipole, b) array of two elementary dipoles.

In this case, for intensity u_{11}, (7) becomes

$$\begin{aligned} u_{11}(\theta,\phi) &= \frac{15k^2}{4\pi}\int\limits_{-L/2}^{L/2}\int\limits_{-L/2}^{L/2} \sin^2\theta\mathrm{e}^{-\mathrm{j}k(z-z')\cos\theta}\,\mathrm{d}z\,\mathrm{d}z' \\ &\approx \frac{15k^2}{4\pi}\int\limits_{-L/2}^{L/2}\int\limits_{-L/2}^{L/2} \sin^2\theta\,\mathrm{d}z\,\mathrm{d}z' \\ &= \frac{15k^2L^2}{4\pi}\sin^2\theta. \end{aligned} \tag{13}$$

In the above equation, the Dirac δ-functions reduce 3D volume integrals from (7) to the 1D line integrals and effectively simplify (8), (9) to

$$\Delta - \Delta' = (z-z')\cos\theta \tag{14}$$

$$\begin{aligned} \Lambda(\mathbf{r},\mathbf{r}') = \Lambda_{1z,1z}(\mathbf{r},\mathbf{r}') &= j_{1z}^*(\mathbf{r})j_{1z}(\mathbf{r}')\sin^2\theta \\ &= \sin^2\theta \end{aligned} \tag{15}$$

and, finally, approximation $z - z' \approx 0$ is used for $L \to 0$.

Then, (11) for power p_{11} using (13) reads

$$p_{11} = \frac{15k^2L^2}{2}\int\limits_0^{\pi}\sin^3\theta\,\mathrm{d}\theta = 10k^2L^2. \tag{16}$$

For the elementary dipole, considering the excitation current $\mathbf{I} = [I_1]$ and using the above found entries of matrices $\mathbf{u}$ and $\mathbf{p}$, directivity D (12) becomes

$$D(\theta) = \frac{15}{10}\sin^2\theta. \tag{17}$$

The value of current I_1 is insignificant when calculating directivity D since it is ultimately canceled in (12).

3.2 Array of Two Elementary Dipoles

Now, consider an array of two elementary dipoles of length $L \to 0$ oriented in the z-axis and spaced in the x-coordinate by a distance d with constant current densities $\mathbf{J}_1 = I_1\delta(x-d/2)\delta(y)\mathbf{z}_0$ and $\mathbf{J}_2 = I_2\delta(x+d/2)\delta(y)\mathbf{z}_0$, see Fig. 1 b). It is well known that this arrangement produces end-fire radiation if the dipoles are closely-spaced ($d < \lambda/2$) and excited by out-of-phase currents, i.e., $I_1 = -I_2 = I$ [10].

In this case, the self-intensities u_{11} and u_{22} are equal to (13), i.e., $u_{11} = u_{22}$, since they cannot depend either on the placement in the coordinates, nor on the mutual placement of the dipoles, and due to the dipoles being identical. From (7)–(9), it follows for mutual intensities u_{12} and u_{21} that $u_{12} = u_{21}^*$ and

$$\begin{aligned} u_{12}(\theta,\phi,s) &= \frac{15k^2}{4\pi}\int\limits_{-L/2}^{L/2}\int\limits_{-L/2}^{L/2}\sin^2\theta \mathrm{e}^{-\mathrm{j}k(\Delta-\Delta')}\,\mathrm{d}z\,\mathrm{d}z' \\ &\approx \frac{15k^2}{4\pi}\int\limits_{-L/2}^{L/2}\int\limits_{-L/2}^{L/2}\sin^2\theta \mathrm{e}^{-\mathrm{j}kd\sin\theta\cos\phi}\,\mathrm{d}z\,\mathrm{d}z' \\ &= \frac{15k^2L^2}{4\pi}\sin^2\theta\mathrm{e}^{-\mathrm{j}kd\sin\theta\cos\phi} \\ &= u_{11}(\theta,\phi)\mathrm{e}^{-\mathrm{j}s\sin\theta\cos\phi}. \end{aligned} \tag{18}$$

In the above equation, the Dirac δ-functions reduce 3D volume integrals from (7) to the 1D line integrals and effectively simplify (8), (9) to

$$\Delta - \Delta' = d\sin\theta\cos\phi + (z - z')\cos\theta \tag{19}$$

$$\begin{aligned}\Lambda(\mathbf{r},\mathbf{r}') = \Lambda_{1z,2z}(\mathbf{r},\mathbf{r}') &= j_{1z}^*(\mathbf{r})j_{2z}(\mathbf{r}')\sin^2\theta \\ &= \sin^2\theta\end{aligned} \tag{20}$$

and, finally, the approximation $z - z' \approx 0$ for $L \to 0$ and a comparison with (13) are used with normalized spacing $s = kd$ being defined.

Powers p_{11} and p_{22} are equal to (16), i.e., $p_{11} = p_{22}$, since intensities u_{11} and u_{22} are equal to (13). From (11), it follows for powers p_{12} and p_{21} that $p_{12} = p_{21}^*$ since it holds true that $u_{12} = u_{21}^*$, for intensities u_{12} and u_{21}. Additionally, the power p_{12} is real, thus, $p_{12} = p_{21}$, see Appendix B. Further, (11) for a power p_{12} using (18) reads

$$p_{12}(s) = \frac{15k^2L^2}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}\sin^3\theta\mathrm{e}^{-\mathrm{j}s\sin\theta\cos\phi}\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{21}$$

The above integral was evaluated elsewhere [16], [18]. It is noted that exactly the same result can be obtained by the EMF method [19], where all terms containing $z - z'$ are discarded. It yields*

$$p_{12}(s) = 15k^2L^2\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right). \tag{22}$$

For the given array, considering the out-of-phase excitation currents $\mathbf{I} = [I, -I]^{\mathrm{T}}$ with magnitude I and using the above found entries of matrices $\mathbf{u}$ and $\mathbf{p}$, directivity D (12) becomes

$$D(\theta,\phi,s) = \frac{3\sin^2\theta(1-\cos(s\sin\theta\cos\phi))}{2 - 3\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right)}. \tag{23}$$

The value of magnitude I is insignificant when calculating directivity D since it is ultimately canceled in (12). Further, considering the spacing $d < \lambda/2$, the maximal (end-fire) radiation occurs for the direction ($\theta = 90^\circ, \phi = 0^\circ$) and the corresponding directivity D is

$$D(90^\circ, 0^\circ, s) = \frac{3(1-\cos s)}{2 - 3\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right)} \tag{24}$$

with the limit $15/4 = 3.75$ (5.74 dBi) for the spacing $d \to 0$ (i.e., $s \to 0$).

The term in brackets in (22) can be written with the help of spherical Bessel functions as $\mathrm{j}_0(s) - \mathrm{j}_1(s)/s$.

The denominator of (23) and (24) represents the interaction of the self- and mutual powers p_{11} and p_{12} and is the leading function describing the behavior of quality factor Q of this array [16], [20], which behaves as

$$Q(s) \propto \frac{1}{p_{11} - p_{12}(s)}. \tag{25}$$

4 Optimal Excitation and Superdirectivity

In directivity D (23), the currents $\mathbf{I} = [I, -I]^{\mathrm{T}}$ are considered. However, the general expression of directivity D (12) is a quadratic form in terms of the currents $\mathbf{I}$ and can be used to find their optimum $\mathbf{I}_{\mathrm{opt}}$ which maximizes directivity D for a given direction (θ, ϕ) and spacing s by solving the related weighted eigenvalue equation [21]

$$4\pi\mathbf{u}\mathbf{I}_{\mathrm{opt}} = D\mathbf{p}\mathbf{I}_{\mathrm{opt}}. \tag{26}$$

In this particular case, the currents $\mathbf{I}_{\mathrm{opt}} = \left[I_{1,\mathrm{opt}}, I_{2,\mathrm{opt}}\right]^{\mathrm{T}}$ can be found analytically by following the procedure in [12], [17]. They are given by solution

$$\mathbf{I}_{\mathrm{opt}}(\theta,\phi,s) = \frac{1}{4\pi}\mathbf{p}^{-1}(s)\mathbf{V}(\theta,\phi,s) \tag{27}$$

where

$$\mathbf{V}(\theta,\phi,s) = \begin{bmatrix}\mathrm{e}^{-\mathrm{j}s/2\sin\theta\cos\phi}f(\theta,\phi) \\ \mathrm{e}^{\mathrm{j}s/2\sin\theta\cos\phi}f(\theta,\phi)\end{bmatrix} \tag{28}$$

$$f(\theta,\phi) = \sin\theta \tag{29}$$

and f is a normalized far field pattern of the elementary dipole. Thus, the currents $\mathbf{I}_{\mathrm{opt}}$ (27) can be written with the help of the previously found matrix $\mathbf{p}$

$$\mathbf{I}_{\mathrm{opt}}(\theta,\phi,s) = \begin{bmatrix} I\mathrm{e}^{-\mathrm{j}\alpha(\theta,\phi,s)/2} \\ I\mathrm{e}^{\mathrm{j}\alpha(\theta,\phi,s)/2}\end{bmatrix} \tag{30}$$

where magnitude I is the same for currents $I_{1,\mathrm{opt}}$ and $I_{2,\mathrm{opt}}$ and α is their phase difference, which reads

$$\alpha(\theta,\phi,s) = -s\sin\theta\cos\phi + 2\arg\left(\rho_{\mathrm{Re}}(\theta,\phi,s) + \mathrm{j}\rho_{\mathrm{Im}}(\theta,\phi,s)\right) \tag{31}$$

where

$$\begin{aligned}\rho_{\mathrm{Re}}(\theta,\phi,s) =& 2\cos(s\sin\theta\cos\phi) \\ &- 3\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right)\end{aligned} \tag{32}$$

$$\rho_{\mathrm{Im}}(\theta,\phi,s) = 2\sin(s\sin\theta\cos\phi). \tag{33}$$

For the given array, considering the optimal excitation currents $\mathbf{I}_{\mathrm{opt}}$ (30) and using the above found entries of matrices $\mathbf{u}$ and $\mathbf{p}$, directivity D (12) becomes

$$D(\theta,\phi,s) = \frac{3\sin^2\theta(\cos\alpha + \cos(s\sin\theta\cos\phi))}{2\cos\alpha + 3\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right)}. \tag{34}$$

This relation express the maximum directivity D for the given direction (θ, ϕ) and spacing s which is achieved by the excitation of the given array by the currents $\mathbf{I}_{\mathrm{opt}}$ set for the direction (θ, ϕ) and spacing s according to (30). Further, considering spacing $d < \lambda/2$, the

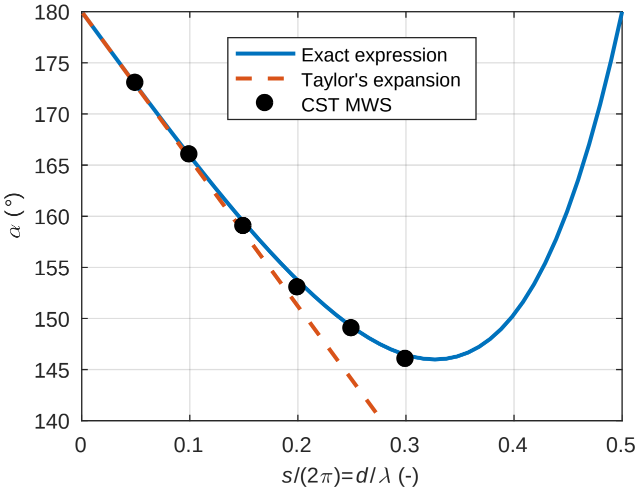

Fig. 2: Phase difference of optimal excitation currents for maximal directivity of the end-fire radiation of an array of two elementary dipoles: exact expression (blue-solid), Taylor's expansion (red-dashed), CST MWS simulation (black-dot).

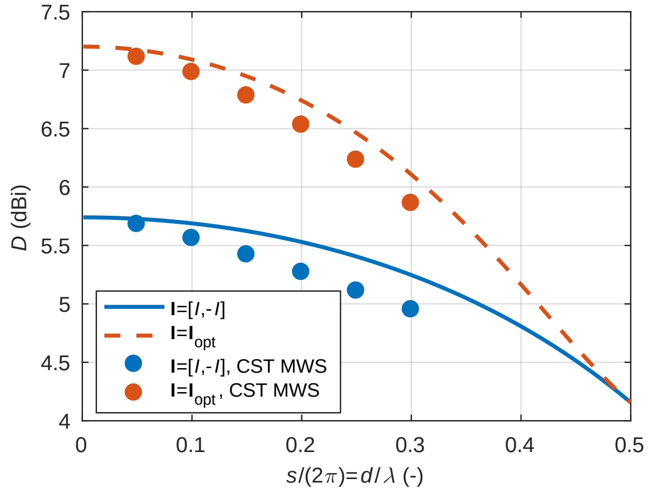

Fig. 3: Directivity of end-fire radiation of an array of two elementary dipoles with out-of-phase (blue-solid) and optimal for maximal directivity (red-dashed) excitation. CST MWS simulation (dot).

maximal (end-fire) radiation occurs for the direction $(\theta = 90^\circ, \phi = 0^\circ)$ and the corresponding directivity D is

$$D(90^\circ, 0^\circ, s) = \frac{3(\cos\alpha + \cos s)}{2\cos\alpha + 3\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right)} \tag{35}$$

where the phase difference α (31) is now

$$\alpha(90^\circ, 0^\circ, s) = 2\pi - s + 2\arctan\left(\frac{2\tan s}{2 - 3\left(\frac{\tan s}{s} + \frac{1}{s^2} - \frac{\tan s}{s^3}\right)}\right). \tag{36}$$

Directivity D (35) has a limit of $21/4 = 5.25$ (7.20 dBi) when spacing $d \to 0$ (i.e., $s \to 0$). Compared to the out-of-phase excitation, this represents an increase by the "superdirective factor" of $21/15 = 1.4$ (1.46 dB). As seen from Fig. 2, phase difference α is almost linear for a close spacing s. This motivates its Taylor's expansion, which, by taking the first terms, gives a simple relation

$$\alpha(90^\circ, 0^\circ, s) \approx \pi - \frac{2}{5}s. \tag{37}$$

Phase difference α (36) is notably similar to that obtained numerically by Yaghjian [4] and Altshuler [6].

The calculated directivities D (24) and (35) for both out-of-phase and optimal excitation are shown in Fig. 3. The results are also validated by the time-domain simulation in CST MWS [13], in which the given array is modeled by two thin dipoles of length $L = \lambda/30$ and spacing of $d = 0.1\ \lambda$. The optimal phase difference of their excitation currents is found from (37) to be $\alpha = 166^\circ$ and verified manually by varying the phase of the currents in the postprocessing stage and checking the end-fire radiation for maximal directivity. It is seen from Fig. 4 that the radiation patterns of the array for the out-of-phase and optimal excitation are quite distinct. Streamlines of the Poynting vector [22], [23] are also shown. The interaction between the two dipoles is much stronger for the superdirective case and the power density is more closely bound to the dipoles. Indeed, the fine structure of the power flow is remarkable.

5 The Uzkov's Limit for Two Isotropic Radiators

Following proposed approach, Uzkov's limit N^2 for the end-fire directivity of N isotropic radiators [3] can be verified for $N = 2$.

Let us consider an array of two isotropic radiators spaced in the x-coordinate by a distance d in the same manner as the elementary dipoles in Fig. 1 b).

Similarly, as for the case of the array of two elementary dipoles, it holds true for the intensities $u_{11} = u_{22}$, $u_{12} = u_{21}^*$ and

$$u_{12}(\theta, \phi, s) = u_{11}(\theta, \phi)\mathrm{e}^{-\mathrm{j}s\sin\theta\cos\phi}. \tag{38}$$

However, in comparison to (13), the intensity u_{11} cannot depend on the direction (θ, ϕ) of radiation for the isotropic radiator. From (11), the relation of the intensity u_{11} and the power p_{11} can be found

$$p_{11} = u_{11}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi u_{11} \tag{39}$$

$$u_{11} = \frac{p_{11}}{4\pi}. \tag{40}$$

It holds true for the powers $p_{11} = p_{22}$ and $p_{12} = p_{21}^* = p_{21}$ since the intensities u_{11} and u_{22} are equal, $u_{12} = u_{21}^*$ and the power

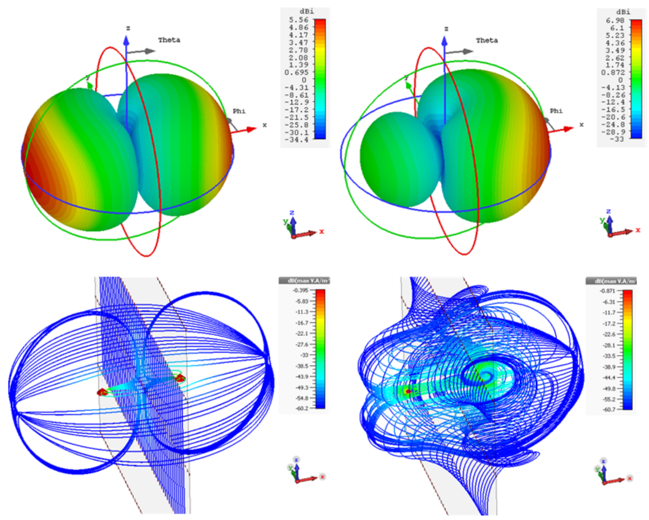

Fig. 4: Radiation pattern for out-of-phase (top-left) and optimal for maximal directivity (top-right) excitation. Streamlines of the Poynting vector are shown below.

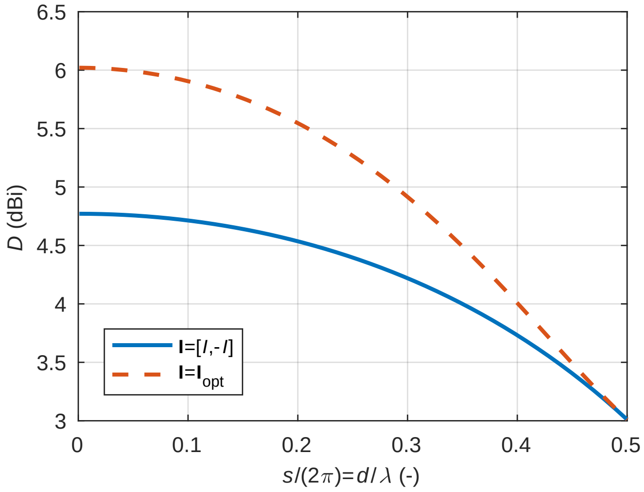

Fig. 5: Directivity of end-fire radiation of an array of two isotropic radiators with out-of-phase (blue-solid) and optimal for maximal directivity (red-dashed) excitation.

p_{12} is real, see Appendix B. Further, (11) for the power p_{12} using (38) and (40) reads

$$p_{12}(s) = \frac{p_{11}}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\theta \mathrm{e}^{-\mathrm{j}s\sin\theta\cos\phi}\, \mathrm{d}\theta\, \mathrm{d}\phi = p_{11}\frac{\sin s}{s}. \quad (41)$$

For the given array, considering the out-of-phase excitation currents $\mathbf{I} = [I, -I]^{\mathrm{T}}$ and using the above found entries of the matrices $\mathbf{u}$ and $\mathbf{p}$, directivity D (12) becomes

$$D(\theta, \phi, s) = \frac{1 - \cos(s\sin\theta\cos\phi)}{1 - \frac{\sin s}{s}}. \quad (42)$$

Further, considering the spacing $d < \lambda/2$, the maximal (end-fire) radiation occurs for the direction $(\theta = 90^\circ, \phi = 0^\circ)$ and the corresponding directivity D is

$$D(90^\circ, 0^\circ, s) = \frac{1 - \cos s}{1 - \frac{\sin s}{s}} \quad (43)$$

with limit 3 (4.77 dBi) for the spacing $d \to 0$ (i.e., $s \to 0$).

In this case, the optimal currents $\mathbf{I}_{\mathrm{opt}}$ for the maximal directivity D for a given direction (θ, ϕ) and spacing s can be also found in the manner given by (27) and (28) but the normalized far field pattern f of the isotropic radiator is

$$f(\theta, \phi) = 1. \quad (44)$$

The currents $\mathbf{I}_{\mathrm{opt}}$ have the same form as (30) but the phase difference α is now

$$\alpha(\theta, \phi, s) = -s\sin\theta\cos\phi + 2\arg\left(\rho_{\mathrm{Re}}(\theta, \phi, s) + \mathrm{j}\rho_{\mathrm{Im}}(\theta, \phi, s)\right) \quad (45)$$

where

$$\rho_{\mathrm{Re}}(\theta, \phi, s) = \cos(s\sin\theta\cos\phi) - \frac{\sin s}{s} \quad (46)$$

$$\rho_{\mathrm{Im}}(\theta, \phi, s) = \sin(s\sin\theta\cos\phi). \quad (47)$$

For the given array, considering the optimal excitation currents $\mathbf{I}_{\mathrm{opt}}$ (30) and using the above found entries of the matrices $\mathbf{u}$ and $\mathbf{p}$,

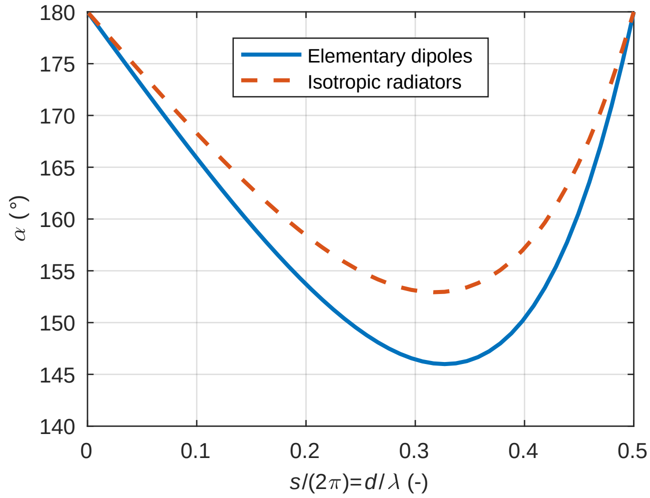

Fig. 6: Comparison of phase difference of optimal excitation currents for maximal directivity of end-fire radiation of arrays of two elementary dipoles (blue-solid) and two isotropic radiators (red-dashed).

directivity D (12) becomes

$$D(\theta, \phi, s) = \frac{\cos\alpha + \cos(s\sin\theta\cos\phi)}{\cos\alpha + \frac{\sin s}{s}}. \quad (48)$$

Further, considering the spacing $d < \lambda/2$, the maximal (end-fire) radiation occurs for the direction $(\theta = 90^\circ, \phi = 0^\circ)$ and the corresponding directivity D is

$$D(90^\circ, 0^\circ, s) = \frac{\cos\alpha + \cos s}{\cos\alpha + \frac{\sin s}{s}} \quad (49)$$

where the phase difference α (45) is now

$$\alpha(90^\circ, 0^\circ, s) = -s + 2\arctan\left(\frac{\tan s}{1 - \frac{\tan s}{s}}\right) \quad (50)$$

with first terms of Taylor's expansion for close spacing s

$$\alpha(90^\circ, 0^\circ, s) \approx \pi - \frac{1}{3}s. \quad (51)$$

Directivity D (49) has a limit 4 (6.02 dBi) for the spacing $d \to 0$ (i.e., $s \to 0$) corresponding with Uzkov's limit [3].

The calculated directivities D (43) and (49) for both out-of-phase and optimal excitation are shown in Fig. 5. A comparison of the phase difference α for the arrays of two elementary dipoles and two isotropic radiators is given in Fig. 6.

6 Conclusion

By using the generalized concept of directivity, analytical expressions for the directivity of out-of-phase excited arrays of two closely-spaced elementary dipoles and two isotropic radiators were derived. Further, the optimal excitations to maximize the directivity of the arrays for a given direction and spacing were also found analytically with an emphasis on end-fire radiation. Although this is rather an academic case, it illustrates the interesting properties of end-fire radiating arrays. Namely, it shows the dependence of their maximal directivity on the excitation currents, particularly, on their phase difference and the limit of the directivity for the spacing of the array elements approaching zero.

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7 References

1 Hansen, W.W., Woodyard, J.R.: 'A new principle in directional antenna design', *Proceedings of the Institute of Radio Engineers*, 1938, **26**, (3), pp. 333–345
2 Bloch, A., Medhurst, R.G., Pool, S.D.: 'A new approach to the design of superdirective aerial arrays', *Proc IEE*, 1953, **100**, pp. 303–314
3 Uzkov, A.I.: 'An approach to the problem of optimum directive antennae design', *Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS*, 1946, **53**, pp. 35–38
4 Yaghjian, A.D., O'Donnell, T.H., Altshuler, E.E., Best, S.R.: 'Electrically small supergain end-fire arrays', *Radio Science*, 2008, **43**, (3), pp. 1–13
5 Haskou, A., Sharaiha, A., Collardey, S.: 'Design of small parasitic loaded superdirective end-fire antenna arrays', *IEEE Trans Antennas Propag*, 2015, **63**, (12), pp. 5456–5464
6 Altshuler, E.E., O'Donnell, T.H., Yaghjian, A.D., Best, S.R.: 'A monopole superdirective array', *IEEE Transactions on Antennas and Propagation*, 2005, **53**, (8), pp. 2653–2661
7 Noguchi, A., Arai, H. '3-element super-directive endfire array with decoupling network'. In: 2014 International Symposium on Antennas and Propagation Conference Proceedings. (, 2014. pp. 455–456
8 Best, S.R., Altshuler, E.E., Yaghjian, A.D., McGinthy, J.M., O'Donnell, T.H.: 'An impedance-matched 2-element superdirective array', *IEEE Antennas and Wireless Propagation Letters*, 2008, **7**, pp. 302–305
9 Clemente, A., Pigeon, M., Rudant, L., Delaveaud, C.: 'Design of a super directive four-element compact antenna array using spherical wave expansion', *IEEE Transactions on Antennas and Propagation*, 2015, **63**, (11), pp. 4715–4722
10 Kraus, J.D.: 'Antennas'. (McGraw-Hill, 1988)
11 Rudge, A.W.: 'The Handbook of Antenna Design, Vol. 2'. (Institution Of Engineering And Technology, 1983)
12 Uzsoky, M., Solymár, L.: 'Theory of super-directive linear arrays', *Acta Physica Academiae Scientiarum Hungaricae*, 1956, **6**, (2), pp. 185–205
13 Computer Simulation Technology. 'Microwave Studio', 2018. Available from: http://www.cst.com
14 Orfanidis, S.J. 'Electromagnetic waves & antennas', 2016. Available from: www.ece.rutgers.edu/˜orfanidi/ewa
15 Jordan, E.C., Balmain, K.G.: 'Electromagnetic Waves and Radiating Systems'. (Pearson Education, 2nd ed., 2015)
16 Lo, Y.T., Lee, S.W., Lee, Q.H.: 'Optimization of directivity and signal-to-noise ratio of an arbitrary antenna array', *Proceedings of the IEEE*, 1966, **54**, (8), pp. 1033–1045
17 Shamonina, E., Solymar, L.: 'Maximum directivity of arbitrary dipole arrays', *IET Microw Antenna P*, 2015, **9**, pp. 101–107
18 Margetis, D., Fikioris, G., Myers, J.M., Wu, T.T.: 'Highly directive current distributions: General theory', *Phys Rev E*, 1998, **58**, pp. 2531
19 Polivka, M., Vrba, D.: 'Input resistance of electrically short not-too-closely spaced multielement monopoles with uniform current distribution', *IEEE Antennas and Wireless Propagation Letters*, 2012, **11**, pp. 1576–1579
20 Hazdra, P., Capek, M., Eichler, J., Mazanek, M.: 'The radiation Q-factor of a horizontal $\lambda/2$ dipole above ground plane', *IEEE Antennas Wireless Propag Lett*, 2014, **13**, pp. 1073–1075
21 Harrington, R.F.: 'Field Computation by Moment Methods'. (Wiley – IEEE Press, 1993)
22 Shamonina, E., Kalinin, V.A., Ringhofer, K.H., Solymar, L.: 'Short dipole as a receiver: effective aperture shapes and streamlines of the poynting vector', *IEE Proceedings - Microwaves, Antennas and Propagation*, 2002, **149**, (3), pp. 153–159
23 Diao, J., Warnick, K.F.: 'Poynting streamlines, effective area shape, and the design of superdirective antennas', *IEEE Transactions on Antennas and Propagation*, 2017, **65**, (2), pp. 861–866

Appendix A

The normalized current densities $\mathbf{j}_m$ and $\mathbf{j}_n$ are usually expressed as vectors $\mathbf{j}_m = [j_{mx}, j_{my}, j_{mz}]$ and $\mathbf{j}_n = [j_{nx}, j_{ny}, j_{nz}]$ in Cartesian coordinates similarly as the unit vector $\mathbf{r}_0 = [\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)]$, which determines the direction of radiation. This leads to the expression of (9) as

$$\begin{aligned}\Lambda =& \Lambda_{mx,nx} + \Lambda_{mx,ny} + \Lambda_{mx,nz} \\ &+ \Lambda_{my,nx} + \Lambda_{my,ny} + \Lambda_{my,nz} \\ &+ \Lambda_{mz,nx} + \Lambda_{mz,ny} + \Lambda_{mz,nz}\end{aligned} \tag{52}$$

where

$$\Lambda_{mx,nx} = j^*_{mx} j_{nx} (\cos^2\theta\cos^2\phi + \sin^2\phi) \tag{53}$$
$$\Lambda_{mx,ny} = j^*_{mx} j_{ny} (-\sin^2\theta\cos\phi\sin\phi) \tag{54}$$
$$\Lambda_{mx,nz} = j^*_{mx} j_{nz} (-\cos\theta\sin\theta\cos\phi) \tag{55}$$
$$\Lambda_{my,nx} = j^*_{my} j_{nx} (-\sin^2\theta\cos\phi\sin\phi) \tag{56}$$
$$\Lambda_{my,ny} = j^*_{my} j_{ny} (\cos^2\theta\sin^2\phi + \cos^2\phi) \tag{57}$$
$$\Lambda_{my,nz} = j^*_{my} j_{nz} (-\cos\theta\sin\theta\sin\phi) \tag{58}$$
$$\Lambda_{mz,nx} = j^*_{mz} j_{nx} (-\cos\theta\sin\theta\cos\phi) \tag{59}$$
$$\Lambda_{mz,ny} = j^*_{mz} j_{ny} (-\cos\theta\sin\theta\sin\phi) \tag{60}$$
$$\Lambda_{mz,nz} = j^*_{mz} j_{nz} \sin^2\theta. \tag{61}$$

Appendix B

Power p_{12} (21) and (41) can be written as

$$\begin{aligned}p_{12}(s) =& \frac{15k^2L^2}{4\pi}\int_0^{\pi} |f(\theta)|^2 \sin\theta \int_0^{2\pi} \cos(s\sin\theta\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &- \mathrm{j}\,\frac{15k^2L^2}{4\pi}\int_0^{\pi} |f(\theta)|^2 \sin\theta \underbrace{\int_0^{2\pi} \sin(s\sin\theta\cos\phi)\,\mathrm{d}\phi}_{\Phi(\theta,\phi,s)}\,\mathrm{d}\theta\end{aligned} \tag{62}$$

where

$$f(\theta) = \begin{cases} \sin(\theta), & \text{for (21)} \\ 1, & \text{for (41).}\end{cases} \tag{63}$$

The function Φ can be further modified with the help of the properties of the sin and cos functions

$$\begin{aligned}\Phi(\theta,\phi,s) &= \int_0^{\pi} \sin(s\sin\theta\cos\phi)\,\mathrm{d}\phi + \int_{\pi}^{2\pi} \sin(s\sin\theta\cos\phi)\,\mathrm{d}\phi \\ &= \int_0^{\pi} \sin(s\sin\theta\cos\phi)\,\mathrm{d}\phi + \int_0^{\pi} \sin(-s\sin\theta\cos\phi)\,\mathrm{d}\phi \\ &= \int_0^{\pi} \sin(s\sin\theta\cos\phi)\,\mathrm{d}\phi - \int_0^{\pi} \sin(s\sin\theta\cos\phi)\,\mathrm{d}\phi \\ &= 0.\end{aligned} \tag{64}$$

Thus power p_{12} (62) can be simplified to a formula

$$p_{12}(s) = \frac{15k^2L^2}{4\pi}\int_0^{\pi} |f(\theta)|^2 \sin\theta \int_0^{2\pi} \cos(s\sin\theta\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{65}$$

which produces only real values. Note that this result holds true for all radiators whose normalized far field pattern f is rotationally symmetrical, i.e., it is a function of the coordinate θ only.