Impedance, Bandwidth, and Q of Antennas

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***Abstract*—To address the need for fundamental universally valid definitions of exact bandwidth and quality factor (Q) of tuned antennas, as well as the need for efficient accurate approximate formulas for computing this bandwidth and Q, exact and approximate expressions are found for the bandwidth and Q of a general single-feed (one-port) lossy or lossless linear antenna tuned to resonance or antiresonance. The approximate expression derived for the exact bandwidth of a tuned antenna differs from previous approximate expressions in that it is inversely proportional to the magnitude $|Z_0'(\omega_0)|$ of the frequency derivative of the input impedance and, for not too large a bandwidth, it is nearly equal to the exact bandwidth of the tuned antenna at every frequency ω_0, that is, throughout antiresonant as well as resonant frequency bands. It is also shown that an appropriately defined exact Q of a tuned lossy or lossless antenna is approximately proportional to $|Z_0'(\omega_0)|$ and thus this Q is approximately inversely proportional to the bandwidth (for not too large a bandwidth) of a simply tuned antenna at all frequencies. The exact Q of a tuned antenna is defined in terms of average internal energies that emerge naturally from Maxwell's equations applied to the tuned antenna. These internal energies, which are similar but not identical to previously defined quality-factor energies, and the associated Q are proven to increase without bound as the size of an antenna is decreased. Numerical solutions to thin straight-wire and wire-loop lossy and lossless antennas, as well as to a Yagi antenna and a straight-wire antenna embedded in a lossy dispersive dielectric, confirm the accuracy of the approximate expressions and the inverse relationship between the defined bandwidth and the defined Q over frequency ranges that cover several resonant and antiresonant frequency bands.**

***Index Terms*—Antennas, antiresonance, bandwidth, impedance, quality factor, resonance.**

I. Introduction

The primary purpose of this paper is twofold: first, to define a fundamental, universally applicable measure of bandwidth of a tuned antenna and to derive a useful approximate expression for this bandwidth in terms of the antenna's input impedance that holds at every frequency, that is, throughout the entire antiresonant as well as resonant frequency ranges of the antenna; and second, to define an exact antenna quality factor Q independently of bandwidth, to derive an approximate expression for this exact Q, and to show that this Q is approximately inversely proportional to the defined bandwidth.

The average "internal" electric, magnetic, and magnetoelectric energies that we use to define the exact Q of a linear antenna are similar though not identical to those of previous authors [1]–[8]. The approximate expression for the bandwidth and its relationship to Q are both more generally applicable and more accurate than previous formulas. As part of the derivation of the relationship between bandwidth and Q, exact expressions for the input impedance of the antenna and its derivative with respect to frequency are found in terms of the fields of the antenna. The exact Q of a general lossy or lossless antenna is also re-expressed in terms of two dispersion energies and the frequency derivative of the input reactance of the antenna. The value of the total internal energy, as well as one of these dispersion energies, for an antenna with an asymmetric far-field magnitude pattern, and thus the value of Q for such an antenna, is shown to depend on the chosen position of the origin of the coordinate system to which the fields of the antenna are referenced. A practical method is found to emerge naturally from the derivations that removes this ambiguity from the definition of Q for a general antenna.[1] The validity and accuracy of the expressions are confirmed by the numerical solutions to straight-wire and wire-loop, lossy and lossless tuned antennas, as well as to a Yagi antenna and a straight-wire antenna embedded in a frequency dependent dielectric material, over a wide enough range of frequencies to cover several resonant and antiresonant frequency bands. The remainder of the paper, many of the results of which were first presented in [9], is organized as follows.

Preliminary definitions required for the derivations of the expressions for impedance, bandwidth, and Q of an antenna are given in Section II.

In Section III, the fractional conductance bandwidth and the fractional matched voltage-standing-wave-ratio (VSWR) bandwidth are defined and determined approximately for a general tuned antenna in terms of the input resistance and magnitude of the frequency derivative of the input impedance of the antenna. It is shown that the matched VSWR bandwidth is the more fundamental measure of bandwidth because, unlike the conductance bandwidth, it exists in general for all frequencies at which an antenna is tuned. (Throughout this paper, we are considering only the bandwidth relative to a change in the accepted power and not to any additional loss of bandwidth caused, for example, by a degradation of the far-field pattern of the antenna.)

In Section IV, the input impedance, its frequency derivative, the internal energies, and the Q of a tuned antenna are given in terms of the antenna fields, and the relationship between bandwidth and Q is determined. In particular, the frequency derivative of the input reactance is expressed in terms of integrals of the electric and magnetic fields of the tuned antenna. These integrals of the fields are then re-expressed in terms of internal

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[1]This ambiguity in the values of internal energy and Q engendered by subtracting the radiation-field energy of an antenna with an asymmetric far-field magnitude pattern is not mentioned or addressed in [1]–[8], probably because these references concentrate on defining the Q of individual spherical multipoles which have far-field magnitude patterns that are symmetric about the origin.

energies used to define the Q of the antenna and two dispersion energies: the first dispersion energy determined by an integral involving the far field and the frequency derivative of the far field of the antenna; and the second determined by an integral involving the fields and the frequency derivative of the fields within the antenna material. The dependence in the value of the far-field dispersion energy on the origin of the coordinate system, and thus the ambiguity in Q (mentioned above), is removed by the procedure derived in Section IV-E. An apparently new energy theorem proven in Appendix B is used to derive a number of inequalities that the constitutive parameters must satisfy in lossless antenna material. We find that the Foster reactance theorem, which states that the frequency derivative of the reactance of a one-port linear, lossless, passive network is always positive, does not hold for antennas (whether or not the antenna *is* lossless because the radiation from the antenna acts as a loss) [10, Sec. 8-4]. Although the general formula we derive for the bandwidth of an antenna involves the frequency derivative of resistance as well as the frequency derivative of reactance, it is found that the half-power matched VSWR bandwidth of a simply tuned lossy or lossless antenna is approximately equal to $2/Q$ for all frequencies if the bandwidth of the antenna is not too large. It is proven in Appendix C that the Q of an antenna increases extremely rapidly as the maximum dimension of the source region is decreased while maintaining the frequency, efficiency, and far-field pattern—making supergain above a few dB impractical. It is also shown in Appendix C that the quality factors determined by previous authors [1]–[3] are lower bounds for our defined Q applied to electrically small antennas with nondispersive $\mu_r \geq 0$ and $\epsilon_r \geq 0$.

In Section V, we discuss how the internal energy, Q, and bandwidth of an antenna would be affected by the presence of material with negative values of μ_r or ϵ_r.

In Section VI, exact VSWR bandwidths are computed from the magnitude of the reflection coefficient versus frequency curves obtained from the numerical solutions to tuned, thin straight-wire and wire-loop lossy and lossless antennas ranging in length from a small fraction of a wavelength to many wavelengths, as well as to a tuned Yagi antenna and a straight-wire antenna embedded in a frequency dependent dielectric material. The exact values of Q for these antennas are computed from the general expression (80) derived for the Q of tuned antennas. The exact values of VSWR bandwidth and Q are compared to the approximate values obtained from the derived approximate formulas in (87) for VSWR bandwidth and Q. These numerical comparisons confirm that the approximate formulas in (87) for VSWR bandwidth and Q of a tuned antenna give much more accurate values in antiresonant frequency ranges than the conventional formula (81) (or its absolute value) commonly used to determine bandwidth and quality factor.

Before leaving this Introduction, a few remarks about the usefulness of antenna Q may be appropriate. We can ask why the concept of antenna Q is introduced when it is the bandwidth of an antenna that has practical importance. One advantage of Q is that the inverse of the matched VSWR bandwidth of an antenna tuned at the frequency ω_0 is approximated by the value of the Q of the antenna at the single frequency ω_0. The bandwidth of some antennas may be much more difficult to directly compute, measure, or estimate than the Q, which is fundamentally defined in terms of the fields of the antenna, is independent of the characteristic impedance of the antenna's feed line, and has a number of lower-bound formulas derived in the published literature [1]–[3] (see Appendix C). The simple approximate, yet accurate formulas for exact bandwidth and Q that are derived in the present paper can be evaluated for an antenna and compared to the lower bounds for Q to decide if the antenna is nearly optimized with respect to Q and bandwidth. It is often possible to increase the bandwidth of electrically small antennas by simply restructuring the antenna to reduce its interior fields and therefore its Q [11]. Moreover, because the Q of an antenna is determined by the fields of the antenna, Maxwell's equations can be used, as we do in Appendix C, to obtain fundamental limitations on the bandwidth of antennas. Finally, regardless of the utility of the concepts of Q and bandwidth, it seems quite remarkable that at any frequency of most antennas, the Q, which is defined in terms of the fields of a simply tuned one-port linear passive antenna, and the bandwidth, which is defined in terms of the input reflection coefficient of the same antenna, are approximately inversely proportional (provided the bandwidth is narrow enough) and that this approximate inverse relationship is given by the simple formulas in (87) below.

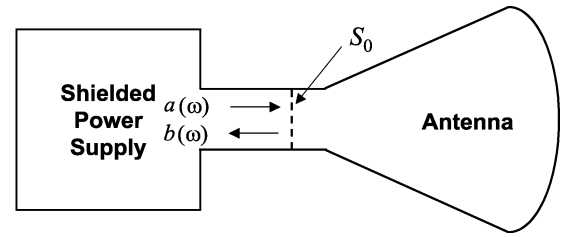

Fig. 1. Schematic of a general transmitting antenna, its feed line, and its shielded power supply.

II. Preliminary Definitions

Consider a general transmitting antenna (shown schematically in Fig. 1) composed of electromagnetically linear materials and fed by a waveguide or transmission line (hereinafter referred to as the "feed line") that carries just one propagating mode at the time-harmonic ($e^{j\omega t}$) frequency $\omega > 0$. (The feed line is assumed to be composed of perfect conductors separated by a linear, homogeneous, isotropic medium.) The propagating mode in the feed line can be characterized at a reference plane S_0 (which separates the antenna from its shielded power supply) by a complex voltage $V(\omega)$, complex current $I(\omega)$, and complex input impedance $Z(\omega)$ defined as

$$Z(\omega) = R(\omega) + jX(\omega) = V(\omega)/I(\omega) \tag{1}$$

where the real number $R(\omega)$ is the input resistance and the real number $X(\omega)$ is the input reactance of the antenna. The voltage and current can also be decomposed into complex coefficients $a(\omega)$ and $b(\omega)$ of the propagating mode traveling toward (incident) and away (emergent) from the antenna, respectively, such that

$$V(\omega) = a(\omega) + b(\omega), \quad I(\omega) = [a(\omega) - b(\omega)]/Z_{\rm ch} \tag{2}$$

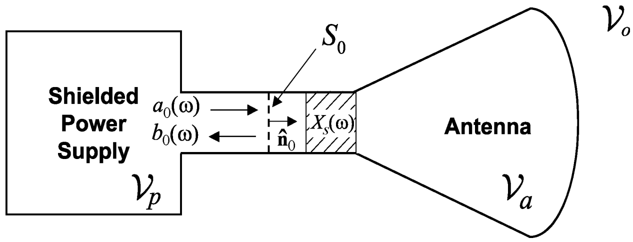

Fig. 2. Schematic of a general transmitting antenna, its feed line, its shielded power supply, and a series reactance X_s.

with $Z_{\rm ch}$ equal to the feed-line characteristic impedance, which can be chosen to be independent of frequency [12, pp. 255–256]. Alternatively, a and b can be defined in terms of V and I as

$$\begin{aligned} a(\omega) &= [V(\omega) + Z_{\rm ch} I(\omega)]/2 \\ b(\omega) &= [V(\omega) - Z_{\rm ch} I(\omega)]/2. \end{aligned} \tag{3}$$

The reflection coefficient of the antenna is defined as

$$\Gamma(\omega) = \frac{b(\omega)}{a(\omega)} = \frac{Z(\omega) - Z_{\rm ch}}{Z(\omega) + Z_{\rm ch}}. \tag{4}$$

As indicated, the parameters Z, R, X, and Γ, as well as V, I, a, and b, are in general functions of ω.

Assume the antenna is tuned at a frequency ω_0 with a series reactance $X_s(\omega)$ (as shown in Fig. 2) comprised of either a positive series inductance L_s or a positive series capacitance C_s, where L_s and C_s are independent of frequency, to make the total reactance

$$\begin{aligned} X_0(\omega) &= X(\omega) + X_s(\omega) \\ X_s(\omega) &= \begin{cases} \omega L_s, & X(\omega_0) < 0 \\ -1/(\omega C_s), & X(\omega_0) > 0 \end{cases} \end{aligned} \tag{5}$$

equal to zero at $\omega = \omega_0$, that is

$$X_0(\omega_0) = X(\omega_0) + X_s(\omega_0) = 0. \tag{6}$$

Then the derivative of $X_0(\omega)$ with respect to ω can be written as

$$X_0'(\omega) = \begin{cases} X'(\omega) + L_s, & X(\omega_0) < 0 \\ X'(\omega) + 1/(\omega^2 C_s), & X(\omega_0) > 0 \end{cases} \tag{7}$$

or simply as

$$X_0'(\omega_0) = X'(\omega_0) + |X(\omega_0)|/\omega_0 \tag{8}$$

at the frequency ω_0. The equations corresponding to (1)–(4) for the tuned antenna can be written as

$$Z_0(\omega) = R_0(\omega) + jX_0(\omega) = V_0(\omega)/I_0(\omega), \quad X_0(\omega_0) = 0 \tag{9}$$

$$V_0(\omega) = a_0(\omega) + b_0(\omega), \quad I_0(\omega) = [a_0(\omega) - b_0(\omega)]/Z_{\rm ch} \tag{10}$$

$$\begin{aligned} a_0(\omega) &= [V_0(\omega) + Z_{\rm ch} I_0(\omega)]/2, \\ b_0(\omega) &= [V_0(\omega) - Z_{\rm ch} I_0(\omega)]/2 \end{aligned} \tag{11}$$

$$\Gamma_0(\omega) = \frac{b_0(\omega)}{a_0(\omega)} = \frac{Z_0(\omega) - Z_{\rm ch}}{Z_0(\omega) + Z_{\rm ch}}. \tag{12}$$

Because the tuning inductor or capacitor is assumed lossless and in series with the antenna, $R_0(\omega) = R(\omega)$. The frequency ω_0, at which $X_0(\omega_0) = 0$, defines a *resonant frequency* of the antenna if $X_0'(\omega_0) > 0$ and an *antiresonant frequency* of the antenna if $X_0'(\omega_0) < 0$.[2] For the sake of brevity, we shall sometimes refer to the resonant or antiresonant frequency ω_0 as simply the "tuned frequency". Note that we are defining a "tuned antenna" at the frequency ω_0 as an antenna that has a total input reactance equal to zero at ω_0. Therefore, a tuned antenna will not have a reflection coefficient $\Gamma_0(\omega_0)$ equal to zero unless the characteristic impedance of its feed line is matched to the antenna ($Z_{\rm ch} = R_0(\omega_0)$) at the frequency ω_0. If an *untuned* antenna has $X(\omega_0) = 0$, it is said to have a *natural resonant frequency* at ω_0 if $X'(\omega_0) > 0$ and a *natural antiresonant frequency* at ω_0 if $X'(\omega_0) < 0$.

The tangential electric and magnetic fields $[\mathbf{E}_0(\boldsymbol{\rho}), \mathbf{H}_0(\boldsymbol{\rho})]$ on the reference plane S_0 of the feed line can be written in terms of real electric and magnetic basis fields $[\mathbf{e}_0(\boldsymbol{\rho}), \mathbf{h}_0(\boldsymbol{\rho})]$ of the single propagating feed-line mode with voltage $V_0(\omega)$ and current $I_0(\omega)$; specifically

$$\mathbf{E}_0(\boldsymbol{\rho}) = V_0 \mathbf{e}_0(\boldsymbol{\rho}) \quad \mathbf{H}_0(\boldsymbol{\rho}) = I_0 \mathbf{h}_0(\boldsymbol{\rho}). \tag{13}$$

There may be evanescent modes on the feed line, but the fields of these evanescent modes are assumed to be negligible on the reference plane S_0. If the dimensional units of $\mathbf{e}_0$ and $\mathbf{h}_0$ are chosen as (meter)$^{-1}$ and they are consistent with Maxwell's equations in the International System of mksA units, then $V_0(\omega)$ has units of Volts, $I_0(\omega)$ has units of Amperes, and the characteristic impedance $Z_{\rm ch}$ of the feed line can be chosen as a real positive constant independent of frequency with units of Ohms. It then follows that the normalization of the basis fields may be expressed as a nondimensional number equal to one, that is

$$\int_{S_0} [\mathbf{e}_0(\boldsymbol{\rho}) \times \mathbf{h}_0(\boldsymbol{\rho})] \cdot \hat{\mathbf{n}}_0 \, dS = 1 \tag{14}$$

where $\hat{\mathbf{n}}_0$ is the unit normal (pointing toward the antenna) on the plane S_0. If the plane S_0 simply cuts two wire leads from a generator at quasistatic frequencies, V_0 and I_0 refer to conventional circuit voltages and currents that do not serve as genuine modal coefficients. In that case, the equations in (11) become definitions of a_0 and b_0 with $Z_{\rm ch}$ equal to the internal resistance of the generator whose internal reactance is tuned to zero. For the TEM mode on a coaxial cable, the basis fields $[\mathbf{e}_0(\boldsymbol{\rho}), \mathbf{h}_0(\boldsymbol{\rho})]$, as well as the characteristic impedance $Z_{\rm ch}$, are independent of frequency. Also, one of the basis fields, either $\mathbf{e}_0(\boldsymbol{\rho})$ or $\mathbf{h}_0(\boldsymbol{\rho})$, in addition to $Z_{\rm ch}$, can always be made independent of frequency for feed lines composed of perfect conductors separated by linear, homogeneous, isotropic materials [12, pp. 255–256]. We shall use this fact in deriving (64) below.

[2]These definitions of resonance and antiresonance come from the behavior of the reactance of series and parallel *RLC* circuits, respectively, at their natural frequencies of oscillation. At the "resonant frequency" of a series *RLC* circuit with positive L and $C, X' > 0$ and at the "antiresonant frequency" of a parallel *RLC* circuit with positive L and $C, X' < 0$.

With the help of (14) we can determine various expressions for the total power P_A accepted by the antenna

$$P_A(\omega) = \frac{1}{2}\mathrm{Re}\int_{S_0}(\mathbf{E}_0 \times \mathbf{H}_0^*)\cdot\hat{\mathbf{n}}_0\, dS = \frac{1}{2}\mathrm{Re}[V_0(\omega)I_0^*(\omega)] = \frac{1}{2Z_{\rm ch}}[|a_0(\omega)|^2 - |b_0(\omega)|^2] \quad (15)$$

or

$$P_A(\omega) = \frac{1}{2}|I_0(\omega)|^2 R_0(\omega) = \frac{1}{2}|V_0(\omega)|^2 G_0(\omega) = \frac{1}{2Z_{\rm ch}}|a_0(\omega)|^2[1 - |\Gamma_0(\omega)|^2]. \quad (16)$$

The superscript * in (15) denotes the complex conjugate and $G_0(\omega) = \mathrm{Re}[1/Z_0(\omega)]$ in (16) is the input conductance of the antenna. The power accepted by the antenna equals the power dissipated by the antenna in the form of power radiated $P_{\mathcal{R}}$ by the antenna plus the power loss $P_{\mathcal{L}}$ in the material of the antenna. Defining the radiation resistance $R_{\mathcal{R}}$ of the antenna as $2P_{\mathcal{R}}/|I_0|^2$ and the loss resistance $R_{\mathcal{L}}$ of the antenna as $2P_{\mathcal{L}}/|I_0|^2$, we have

$$P_A(\omega) = P_{\mathcal{R}}(\omega) + P_{\mathcal{L}}(\omega) = \frac{1}{2}|I_0(\omega)|^2 R_0(\omega) = \frac{1}{2}|I_0(\omega)|^2[R_{\mathcal{R}}(\omega) + R_{\mathcal{L}}(\omega)] \quad (17)$$

so that

$$R_0(\omega) = R_{\mathcal{R}}(\omega) + R_{\mathcal{L}}(\omega). \quad (18)$$

The power radiated can also be expressed in terms of the far fields of the antenna

$$P_{\mathcal{R}}(\omega) = \frac{1}{2}|I_0(\omega)|^2 R_{\mathcal{R}}(\omega) = \frac{1}{2}\mathrm{Re}\int_S[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})]\cdot\hat{\mathbf{n}}\, dS = \frac{1}{2Z_f}\int_{4\pi}|\mathbf{F}(\theta,\phi)|^2\, d\Omega \quad (19)$$

where S is a surface in free space surrounding the antenna and its power supply, the solid angle integration element equals $d\Omega = dS/r^2 = \sin\theta\, d\theta\, d\phi$ with (r,θ,ϕ) being the usual spherical coordinates of the position vector $\mathbf{r}$, and the complex far electric field pattern $\mathbf{F}(\theta,\phi)$ is defined by

$$\mathbf{F}(\theta,\phi) = \lim_{r\to\infty} r e^{jkr}\mathbf{E}(\mathbf{r}) \quad (20)$$

with $k = \omega/c, c$ being the speed of light in free space. The impedance of free space is denoted by Z_f in (19) and $\hat{\mathbf{n}}$ is the unit normal out of S. The radiation resistance is always equal to or greater than zero ($R_{\mathcal{R}}(\omega) \geq 0$) because the power radiated by the antenna is always equal to or greater than zero ($P_{\mathcal{R}}(\omega) \geq 0$). Also, the loss resistance is equal to or greater than zero ($R_{\mathcal{L}}(\omega) \geq 0$) if the material of the antenna is passive ($P_{\mathcal{L}}(\omega) = P_A(\omega) - P_{\mathcal{R}}(\omega) \geq 0$).

III. Formulas for the Bandwidth of Antennas

The bandwidth of an antenna tuned to zero reactance is often defined in one of two ways. The first way defines what is commonly called the *conductance bandwidth* and the second way defines what is commonly called the *matched VSWR bandwidth*. We shall show that the matched VSWR bandwidth, unlike the conductance bandwidth, is well-defined for all frequencies ω_0 at which the antenna is tuned to zero reactance.

A. Conductance Bandwidth

The conductance bandwidth for an antenna tuned at a frequency ω_0 is defined as the difference between the two frequencies at which the power accepted by the antenna, excited by a constant value of voltage V_0, is a given fraction of the power accepted at the frequency ω_0. With the help of (9), the conductance at a frequency ω of an antenna tuned at the frequency ω_0 can be written as

$$G_0(\omega) = \mathrm{Re}[1/Z_0(\omega)] = \frac{R_0(\omega)}{R_0^2(\omega) + X_0^2(\omega)}. \quad (21)$$

We can immediately see from (21) that there is a problem with using conductance bandwidth, namely, that the derivative of $G_0(\omega)$ evaluated at ω_0 equals

$$G_0'(\omega_0) = -R_0'(\omega_0)\,/R_0^2(\omega_0) \quad (22)$$

and thus it is not zero at ω_0 unless $R_0'(\omega_0) = 0$. This means that in general the conductance will not reach a maximum at the frequency ω_0. Moreover, in antiresonant frequency ranges where both the resistance and reactance of the antenna are changing rapidly with frequency, the conductance may not possess a maximum and consequently the conductance bandwidth may not exist in these antiresonant frequency ranges. (As we shall show in Section III-B, the matched VSWR bandwidth does not suffer from these limitations.)

Well away from the antiresonant frequency ranges of most antennas, we have $X_0'(\omega_0) > 0, |R_0'(\omega_0)|$ is much smaller than $X_0'(\omega_0)$, the conductance will peak at a frequency much closer to ω_0 than the bandwidth, and a simple approximate expression for the conductance bandwidth can be found as follows.

Having tuned the antenna at ω_0 so that $X_0(\omega_0) = 0$, we can find the frequency $\omega_{\rm cd}$ where $G_0'(\omega_{\rm cd}) = 0$ by taking the frequency derivative of the expression for $G_0(\omega)$ in (21) and setting it equal to zero to get

$$R_0'(\omega_{\rm cd})\left[X_0^2(\omega_{\rm cd}) - R_0^2(\omega_{\rm cd})\right] = 2R_0(\omega_{\rm cd})X_0(\omega_{\rm cd})X_0'(\omega_{\rm cd}). \quad (23)$$

With $\Delta\omega_0 \equiv \omega_{\rm cd} - \omega_0$, the functions $R_0(\omega_{\rm cd}), X_0(\omega_{\rm cd})$ and their derivatives can be expanded in Taylor series about ω_0

$$R_0(\omega_{\rm cd}) = R_0(\omega_0) + R_0'(\omega_0)\Delta\omega_0 + O[(\Delta\omega_0)^2] \quad (24a)$$
$$R_0'(\omega_{\rm cd}) = R_0'(\omega_0) + R_0''(\omega_0)\Delta\omega_0 + O[(\Delta\omega_0)^2] \quad (24b)$$
$$X_0(\omega_{\rm cd}) = X_0'(\omega_0)\Delta\omega_0 + O[(\Delta\omega_0)^2] \quad (24c)$$
$$X_0'(\omega_{\rm cd}) = X_0'(\omega_0) + X_0''(\omega_0)\Delta\omega_0 + O[(\Delta\omega_0)^2] \quad (24d)$$

which can be substituted into (23) to obtain for small $\Delta\omega_0$

$$[R_0(\omega_0) + R_0'(\omega_0)\Delta\omega_0][R_0'(\omega_0) + R_0''(\omega_0)\Delta\omega_0] \approx -2[X_0'(\omega_0)]^2\Delta\omega_0 \quad (25)$$

or

$$\Delta\omega_0 \approx -\frac{R_0(\omega_0)R_0'(\omega_0)}{2[X_0'(\omega_0)]^2 + [R_0'(\omega_0)]^2 + R_0(\omega_0)R_0''(\omega_0)}. \quad (26)$$

In resonant frequency ranges well away from antiresonant ranges, we can assume $2[X_0'(\omega_0)]^2 \gg |[R_0'(\omega_0)]^2 + R_0(\omega_0)R_0''(\omega_0)|$ so that (26) reduces to

$$\Delta\omega_0 \approx -\frac{R_0(\omega_0)R_0'(\omega_0)}{2[X_0'(\omega_0)]^2}. \tag{27}$$

That is, the frequency shift $\Delta\omega_0$ in the peak of the conductance $G_0(\omega)$ of an antenna tuned at the frequency ω_0 in a resonant frequency range $(X_0'(\omega_0) > 0)$ is given by the simple relationship in (27) involving only the input resistance and the first frequency derivatives of the input resistance and reactance of the antenna at the tuned frequency ω_0. In other words, $G_0(\omega_0)$ peaks at a frequency $\omega_{\rm cd}$ given by

$$\omega_{\rm cd} = \omega_0 + \Delta\omega_0 \approx \omega_0 - \frac{R_0(\omega_0)R_0'(\omega_0)}{2[X_0'(\omega_0)]^2}. \tag{28}$$

To determine the conductance bandwidth about the shifted frequency $\omega_{\rm cd}$ at which $G_0(\omega)$ peaks when the antenna is tuned at ω_0, we find the two frequencies $\omega_\pm = \omega_{\rm cd} + \Delta\omega_\pm$ at which the accepted power is $(1-\alpha)$ times its value at $\omega_{\rm cd}$ is given from (21) as

$$\frac{R_0(\omega_\pm)}{R_0^2(\omega_\pm) + X_0^2(\omega_\pm)} = (1-\alpha)\frac{R_0(\omega_{\rm cd})}{R_0^2(\omega_{\rm cd}) + X_0^2(\omega_{\rm cd})} \tag{29}$$

provided, as discussed above, we are well within the resonant frequency ranges where $|R_0'(\omega_0)| \ll X_0'(\omega_0)$. The value of the constant α, which lies in the range $0 \le \alpha \le 1$, is assumed chosen $\le 1/2$. We can re-express (29) as in (30), shown at the bottom of the page, whose left-hand side is more suitable to a power series expansion about $\omega_{\rm cd}$ than the left-hand side of (29) because the function $X_0^2(\omega)$, which rapidly varies from its value of zero at ω_0, is not contained in the denominator of (30).

Since the conductance on the left-hand side of (30), and its first derivative, are zero at $\omega_\pm = \omega_{\rm cd}$, a Taylor series expansion of the left-hand side of (30) about $\omega_{\rm cd}$ recasts (30) in the form of (31), shown at the bottom of the page, in which $\omega_{\rm cd}$ has been replaced by ω_0 because $\omega_{\rm cd} - \omega_0 = \Delta\omega_0 < \Delta\omega_\pm$ for ω_0 well within resonant frequency ranges. Evaluating the second derivative in (31), we find

$$\begin{aligned}[R_0(\omega_0)R_0''(\omega_0) + [R_0'(\omega_0)]^2 + 2[X_0'(\omega_0)]^2] \\ \times(\Delta\omega_\pm)^2 + O[(\Delta\omega_\pm)^3] = 2\beta R_0^2(\omega_0)\end{aligned} \tag{32}$$

where use has been made of $X_0(\omega_0) = 0$. Again, in resonant frequency ranges we can assume that $2[X_0'(\omega_0)]^2 \gg |[R_0'(\omega_0)]^2 + R_0(\omega_0)R_0''(\omega_0)|$ and, therefore, (32) yields

$$\Delta\omega_\pm \approx \pm\frac{\sqrt{\beta}R_0(\omega_0)}{X_0'(\omega_0)} \tag{33}$$

under the additional assumption that the $O[(\Delta\omega_\pm)^3]$ terms are negligible, an assumption that is generally satisfied if $|\Delta\omega_\pm/\omega_0| \ll 1$.

The fractional conductance bandwidth $\mathrm{FBW}_{\rm cd}$ is therefore given approximately by

$$\begin{aligned}\mathrm{FBW}_{\rm cd}(\omega_0) &= \frac{\omega_+ - \omega_-}{\omega_0} = \frac{\Delta\omega_+ - \Delta\omega_-}{\omega_0} \\ &\approx \frac{2\sqrt{\beta}R_0(\omega_0)}{\omega_0|X_0'(\omega_0)|}, \quad \beta = \frac{\alpha}{1-\alpha} \le 1\end{aligned} \tag{34}$$

under the assumptions that we are well within resonant frequency ranges $(X_0' > 0)$ where $|R_0'(\omega_0)| \ll X_0'(\omega_0)$ and $X_0'(\omega)$ and $R_0'(\omega)$ do not change greatly over the bandwidth (an assumption that holds if $|\Delta\omega_\pm/\omega_0| \ll 1$ or, equivalently, $\mathrm{FBW}_{\rm cd}(\omega_0) \ll 1$, which can always be satisfied if β is chosen small enough). The expression (34) for the fractional conductance bandwidth of tuned antennas was derived previously by Fante [3] for the half-power bandwidth $(\beta = 1)$, assuming $R_0'(\omega_0) = 0$.

Rhodes [13] postulates the "half-power bandwidth" B of an "electromagnetic system" as

$$B = \frac{2R_0}{\omega_0|X_0'|}. \tag{35}$$

He then *defines* $1/B$ as the Q of the electromagnetic system and finds "stored electric and magnetic energies" that are consistent with this $Q \equiv 1/B$ and (59) below. The shortcomings of this method are that (35) is postulated as the half-power bandwidth of a general antenna and that Q is *defined* as $1/B$ rather than as a physical quantity determined independently of B from the fields of the antenna. Moreover, (35) as well as (34) does not accurately approximate the bandwidth of tuned antennas in antiresonant frequency ranges (except at antiresonant frequencies with $R_0'(\omega_0) \approx 0$).

B. Matched VSWR Bandwidth

The matched voltage-standing-wave-ratio (VSWR) bandwidth for an antenna tuned at a frequency ω_0 is defined as the

$$\frac{\left[R_0^2(\omega_\pm) + X_0^2(\omega_\pm)\right]R_0(\omega_{\rm cd}) - \left[R_0^2(\omega_{\rm cd}) + X_0^2(\omega_{\rm cd})\right]R_0(\omega_\pm)}{R_0(\omega_\pm)\left[R_0^2(\omega_{\rm cd}) + X_0^2(\omega_{\rm cd})\right]} = \beta, \quad \beta = \frac{\alpha}{1-\alpha} \le 1 \tag{30}$$

$$\frac{d^2}{d\omega^2}\left[\frac{\left[R_0^2(\omega) + X_0^2(\omega)\right]R_0(\omega_0) - \left[R_0^2(\omega_0) + X_0^2(\omega_0)\right]R_0(\omega)}{R_0(\omega)\left[R_0^2(\omega_0) + X_0^2(\omega_0)\right]}\right]_{\omega=\omega_0}(\Delta\omega_\pm)^2 + O[(\Delta\omega_\pm)^3] = 2\beta \tag{31}$$

difference between the two frequencies on either side of ω_0 at which the VSWR equals a constant s, or, equivalently, at which the magnitude squared of the reflection coefficient $|\Gamma_0(\omega)|^2$ equals $\alpha = (s-1)^2/(s+1)^2$ (the constant α is assumed chosen $\leq 1/2$), provided the characteristic impedance $Z_{\rm ch}$ of the feed line equals $Z_0(\omega_0) = R_0(\omega_0)$. Then the magnitude squared of the reflection coefficient can be found from (12) as

$$|\Gamma_0(\omega)|^2 = \frac{X_0^2(\omega) + [R_0(\omega) - R_0(\omega_0)]^2}{X_0^2(\omega) + [R_0(\omega) + R_0(\omega_0)]^2}. \tag{36}$$

Both $|\Gamma_0(\omega)|^2$ and its derivative with respect to ω are zero at ω_0. Consequently, $|\Gamma_0(\omega)|^2$ has a minimum at ω_0 for all values of the frequency ω_0 at which the antenna is tuned $(X_0(\omega_0) = 0)$ and matched to the feed line $(Z_{\rm ch} = R_0(\omega_0))$. This means that the matched VSWR bandwidth, $(\omega_+ - \omega_-)$, determined by

$$\frac{X_0^2(\omega_\pm) + [R_0(\omega_\pm) - R_0(\omega_0)]^2}{X_0^2(\omega_\pm) + [R_0(\omega_\pm) + R_0(\omega_0)]^2} = \alpha \tag{37}$$

unlike the conductance bandwidth, exists at all frequencies (for small enough α), that is, throughout both the antiresonant $(X_0'(\omega_0) < 0)$ and resonant $(X_0'(\omega_0) > 0)$ frequency ranges. Therefore, the matched VSWR bandwidth is a more fundamental, universally applicable definition of bandwidth for a general antenna than conductance bandwidth.

Bringing the denominator from the left-hand side of (37) to the right-hand side and rearranging terms to remove the rapidly varying function $X_0(\omega)$ from the denominator on the left-hand side of (37) produces

$$\frac{X_0^2(\omega_\pm) + [R_0(\omega_\pm) - R_0(\omega_0)]^2}{R_0(\omega_\pm)} = 4\beta R_0(\omega_0),$$
$$\beta = \frac{\alpha}{1-\alpha} \leq 1. \tag{38}$$

Expanding the left-hand side of (38) in a Taylor series about ω_0, we find

$$|Z_0'(\omega_0)|^2(\Delta\omega_\pm)^2 \approx 4\beta R_0^2(\omega_0) \tag{39}$$

under the assumption that the $O[(\Delta\omega_\pm)^3]$ terms are negligible. This assumption is generally satisfied if $|\Delta\omega_\pm/\omega_0| \ll 1$. The solutions to (39) for $\Delta\omega_\pm$ are

$$\Delta\omega_\pm \approx \pm\frac{2\sqrt{\beta}R_0(\omega_0)}{|Z_0'(\omega_0)|} \tag{40}$$

so that the fractional matched VSWR bandwidth $\mathrm{FBW}_V(\omega_0)$ takes the simple form

$$\mathrm{FBW}_V(\omega_0) = \frac{\omega_+ - \omega_-}{\omega_0} = \frac{\Delta\omega_+ - \Delta\omega_-}{\omega_0}$$
$$\approx \frac{4\sqrt{\beta}R_0(\omega_0)}{\omega_0|Z_0'(\omega_0)|}, \quad \sqrt{\beta} = \sqrt{\frac{\alpha}{1-\alpha}} = \frac{s-1}{2\sqrt{s}} \leq 1 \tag{41}$$

which holds for tuned antennas under the sufficient conditions that $X_0'(\omega)$ and $R_0'(\omega)$ do not change greatly over the bandwidth (conditions that hold if $|\Delta\omega_\pm/\omega_0| \ll 1$ or, equivalently, $\mathrm{FBW}_V(\omega_0) \ll 1$, which can always be satisfied if β is chosen small enough). For half-power VSWR bandwidth, $\alpha = 1/2$ $(s = 5.828)$ and $\sqrt{\beta} = 1$.

A comparison of (41) with (34) reveals that under their stated conditions of validity

$$\mathrm{FBW}_V(\omega_0) \approx 2\mathrm{FBW}_{\rm cd}(\omega_0) \tag{42}$$

wherever the conductance bandwidth $\mathrm{FBW}_{\rm cd}(\omega_0)$ exists, namely outside the antiresonant frequency ranges. The matched VSWR bandwidth $\mathrm{FBW}_V(\omega_0)$ has the distinct advantage over the conductance bandwidth $\mathrm{FBW}_{\rm cd}(\omega_0)$ of existing at every tuned frequency ω_0 (for small enough β), that is, within both resonant $(X_0'(\omega_0) > 0)$ and antiresonant $(X_0'(\omega_0) < 0)$ frequency ranges. Moreover, if $X_0'(\omega)$ and $R_0'(\omega)$ do not change greatly over the bandwidth (which can always be satisfied if β is chosen small enough), $\mathrm{FBW}_V(\omega_0)$ is reasonably well approximated at all tuned frequencies by the simple expression (41) even in frequency bands where $X_0'(\omega_0)$ or $R_0'(\omega_0)$ are zero, close to zero, or negative (but not both $X_0'(\omega_0)$ and $R_0'(\omega_0)$ too close to zero). As far as we know, (41) is a general result for antennas that has not been established previously.

The approximate formula for bandwidth in (41) should be applied judiciously to antennas that are designed to have a combination of two or more natural resonances and antiresonances at $\omega_1, \omega_2, \ldots, \omega_n, \ldots, \omega_N$ $(N \geq 2)$ that are so close together that the curve of $1 - |\Gamma(\omega)|^2$ versus ω has N closely spaced peaks equal to unity $(|\Gamma(\omega_n)|^2 = 0, n = 1, 2, \ldots N)$ at these frequencies. Then the half-power bandwidth $(\beta = 1)$, for example, may extend over all these natural resonant and antiresonant frequencies even though there will be a resonant peak in $1 - |\Gamma(\omega_n)|^2$ at each natural resonant or antiresonant frequency ω_n that has its own bandwidth for some β such that $\beta < \beta_0 < 1/2$ (say $\beta_0 = 1/10$). The formula in (41) approximates the bandwidth of each of these individual minor resonant and antiresonant peaks with some β that is less than β_0.

IV. Formulas for Impedance and Q and Its Relationship to Bandwidth

The formula for matched VSWR bandwidth given in (41) requires, in addition to $R_0(\omega_0)/\omega_0$, the derivative of impedance with respect to frequency evaluated at $\omega = \omega_0$, that is, $Z_0'(\omega_0) = R_0'(\omega_0) + jX_0'(\omega_0)$. As we shall see, an explicit expression for $R_0'(\omega_0)$ in terms of the electromagnetic fields is not needed in the derivation of Q and its relationship to the bandwidth of a tuned antenna. On the other hand, the evaluation of the frequency derivative of the reactance, $X_0'(\omega_0)$, in terms of the electromagnetic fields of the antenna is crucial to the derivation of Q and its relationship to bandwidth. As a lead-in to the desired expression for $X_0'(\omega_0)$, we begin by deriving general expressions for the input impedance $Z_0(\omega)$ of an antenna tuned at ω_0 in terms of the fields of the antenna.

A. Field Expressions for Accepted Power and Input Impedance

To obtain expressions for the input impedance $Z_0(\omega)$ of the antenna shown in Fig. 2 tuned at the frequency ω_0, apply the complex Poynting's theorem [10, (1-54)] to the infinite volume

$\mathcal{V}_o$ outside the volume $\mathcal{V}_p$ of the shielded power supply. The volume $\mathcal{V}_o$ includes the volume $\mathcal{V}_a$ of the antenna material that lies to the right of the feed-line reference plane S_0. The closed surfaces of the volumes $\mathcal{V}_p$ and $\mathcal{V}_a$ have the feed-line reference plane S_0 in common. Therefore, the volume $\mathcal{V}_a$ includes the volume of the series tuning reactance $X_s(\omega)$. Assuming the integral of the Poynting's vector is zero over the shielded surface of the power supply and using (13)–(14), we find

$$\frac{1}{2}V_0(\omega)I_0^*(\omega) = \frac{1}{2}|I_0(\omega)|^2 Z_0(\omega) = P_A(\omega) + \frac{1}{2}j\omega\,\mathrm{Re}\int_{\mathcal{V}_o}(\mathbf{B}^*\cdot\mathbf{H} - \boldsymbol{D}^*\cdot\mathbf{E})\,d\mathcal{V} \tag{43}$$

where, as in (17)

$$P_A(\omega) = P_{\mathcal{R}}(\omega) + P_{\mathcal{L}}(\omega). \tag{44}$$

The power radiated $P_{\mathcal{R}}$ is given in terms of the fields by (19) and the power loss is given as

$$P_{\mathcal{L}}(\omega) = \frac{\omega}{2}\,\mathrm{Im}\int_{\mathcal{V}_a}(\mathbf{B}^*\cdot\mathbf{H} + \boldsymbol{D}^*\cdot\mathbf{E})\,d\mathcal{V} \tag{45}$$

where $P_{\mathcal{L}}(\omega) \geq 0$ in passive material. Then the power accepted, which equals the total power dissipated by the antenna, can be written as

$$P_A(\omega) = \frac{1}{2Z_f}\int_{4\pi}|\mathbf{F}(\theta,\phi)|^2\,d\Omega + \frac{\omega}{2}\,\mathrm{Im}\int_{\mathcal{V}_a}(\mathbf{B}^*\cdot\mathbf{H} + \boldsymbol{D}^*\cdot\mathbf{E})\,d\mathcal{V}. \tag{46}$$

The efficiency η of the antenna is defined as

$$\eta(\omega) = \frac{P_{\mathcal{R}}(\omega)}{P_A(\omega)} = \frac{P_{\mathcal{R}}(\omega)}{P_{\mathcal{R}}(\omega) + P_{\mathcal{L}}(\omega)} \tag{47}$$

which has a value equal to or less than unity. The usual electric and magnetic vectors are denoted by $(\mathbf{E},\mathbf{D})$ and $(\mathbf{B},\mathbf{H})$, respectively, with $\boldsymbol{D} = \mathbf{D} + \mathbf{J}/(j\omega)$, the vector $\mathbf{J}$ being the current density. Since $Z_0 = R_0 + jX_0$, we find from (43) that

$$R_0(\omega) = \frac{2P_A(\omega)}{|I_0(\omega)|^2} \tag{48}$$

and

$$X_0(\omega) = \frac{\omega}{|I_0(\omega)|^2}\,\mathrm{Re}\int_{\mathcal{V}_o}(\mathbf{B}^*\cdot\mathbf{H} - \boldsymbol{D}^*\cdot\mathbf{E})\,d\mathcal{V}. \tag{49}$$

Of course, the reactance of the antenna is equal to zero at the tuned frequency ($\omega = \omega_0$), that is

$$X_0(\omega_0) = \frac{\omega_0}{|I_0(\omega_0)|^2}\,\mathrm{Re}\int_{\mathcal{V}_o}(\mathbf{B}^*\cdot\mathbf{H} - \boldsymbol{D}^*\cdot\mathbf{E})\,d\mathcal{V} = 0. \tag{50}$$

Throughout the derivations in Sections II and III, it is assumed that the antennas are linear, that is, composed of materials governed by linear constitutive relations that relate $\mathbf{B}$ and $\boldsymbol{D}$ to $\mathbf{H}$ and $\mathbf{E}$. With the most general linear, spatially nondispersive constitutive relations

$$\mathbf{B} = \bar{\boldsymbol{\mu}}\cdot\mathbf{H} + \bar{\boldsymbol{\nu}}\cdot\mathbf{E}, \quad \boldsymbol{D} = \bar{\boldsymbol{\epsilon}}\cdot\mathbf{E} + \bar{\boldsymbol{\tau}}\cdot\mathbf{H} \tag{51}$$

where $\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\epsilon}}$, and $(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{\tau}})$ are the permeability dyadic, the permittivity dyadic, and the magnetoelectric dyadics, respectively, the reactance in (49) and (50) of the antenna tuned at the frequency ω_0 can be written as

$$X_0(\omega) = \frac{\omega}{|I_0(\omega)|^2}\mathrm{Re}\int_{\mathcal{V}_o}[\mathbf{H}\cdot\bar{\boldsymbol{\mu}}^*\cdot\mathbf{H}^* - \mathbf{E}\cdot\bar{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^* + \mathbf{H}\cdot(\bar{\boldsymbol{\nu}}^* - \bar{\boldsymbol{\tau}}_t)\cdot\mathbf{E}^*]\,d\mathcal{V} \tag{52}$$

and

$$X_0(\omega_0) = \frac{\omega_0}{|I_0(\omega_0)|^2}\mathrm{Re}\int_{\mathcal{V}_o}[\mathbf{H}\cdot\bar{\boldsymbol{\mu}}^*\cdot\mathbf{H}^* - \mathbf{E}\cdot\bar{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^* + \mathbf{H}\cdot(\bar{\boldsymbol{\nu}}^* - \bar{\boldsymbol{\tau}}_t)\cdot\mathbf{E}^*]\,d\mathcal{V} = 0 \tag{53}$$

in which the subscript "t" on a dyadic denotes its transpose. *All the field vectors as well as the real and imaginary parts of $\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\nu}}$, and $\bar{\boldsymbol{\tau}}$ are, in general, functions of both frequency ω and the spatial position vector* $\mathbf{r}$. Outside the volume $\mathcal{V}_a$ of the antenna material, $\bar{\boldsymbol{\mu}} = \mu_0\bar{\mathbf{I}}, \bar{\boldsymbol{\epsilon}} = \epsilon_0\bar{\mathbf{I}}$, and $\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\tau}} = 0$ in $\mathcal{V}_o$, where μ_0 and ϵ_0 are the permeability and permittivity of free space and $\bar{\mathbf{I}}$ is the unit dyadic.

With the constitutive relations in (51), the power loss and power accepted in (45) and (46) become

$$P_{\mathcal{L}}(\omega) = \frac{\omega}{2}\mathrm{Im}\int_{\mathcal{V}_a}[\mathbf{H}\cdot\bar{\boldsymbol{\mu}}^*\cdot\mathbf{H}^* + \mathbf{E}\cdot\bar{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^* + \mathbf{H}\cdot(\bar{\boldsymbol{\nu}}^* - \bar{\boldsymbol{\tau}}_t)\cdot\mathbf{E}^*]\,d\mathcal{V} \tag{54}$$

$$P_A(\omega) = \frac{1}{2Z_f}\int_{4\pi}|\mathbf{F}(\theta,\phi)|^2\,d\Omega + \frac{\omega}{2}\,\mathrm{Im}\int_{\mathcal{V}_a}[\mathbf{H}\cdot\bar{\boldsymbol{\mu}}^*\cdot\mathbf{H}^* + \mathbf{E}\cdot\bar{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^* + \mathbf{H}\cdot(\bar{\boldsymbol{\nu}}^* - \bar{\boldsymbol{\tau}}_t)\cdot\mathbf{E}^*]\,d\mathcal{V}. \tag{55}$$

Since $P_{\mathcal{L}}(\omega) \geq 0$ for all values of $\mathbf{E}$ and $\mathbf{H}$ in passive material, (54) implies that a material is passive (lossy or lossless) if and only if its associated Hermitian "loss" matrix is positive semidefinite [14], [15], [16, Sec. 5.2], a property that can be expressed symbolically as

$$\begin{bmatrix} j(\bar{\boldsymbol{\epsilon}}_t^* - \bar{\boldsymbol{\epsilon}}) & (\bar{\boldsymbol{\nu}}_t^* - \bar{\boldsymbol{\tau}}) \\ (\bar{\boldsymbol{\nu}} - \bar{\boldsymbol{\tau}}_t^*) & j(\bar{\boldsymbol{\mu}}_t^* - \bar{\boldsymbol{\mu}}) \end{bmatrix} \geq 0. \tag{56}$$

In lossless material $P_{\mathcal{L}}(\omega) = 0$ and the loss matrix is zero, that is

$$\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}_t^*, \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}_t^*, \quad \bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\tau}}_t^*. \tag{57}$$

If the material is reciprocal, $\bar{\boldsymbol{\mu}}_t = \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\epsilon}}_t = \bar{\boldsymbol{\epsilon}}$, and $\bar{\boldsymbol{\tau}}_t = -\bar{\boldsymbol{\nu}}$ [16, Sec. 5.1].

For the simple isotropic constitutive relations

$$\mathbf{B} = \mu \mathbf{H}, \quad \boldsymbol{D} = \epsilon \mathbf{E} \tag{58a}$$

with complex permeability and permittivity given by [10, p. 451]

$$\mu = \mu_r - j\mu_i, \quad \epsilon = \epsilon_r - j\epsilon_i \tag{58b}$$

these equations for $X_0, P_{\mathcal{L}}$, and P_A become

$$X_0(\omega) = \frac{\omega}{|I_0(\omega)|^2} \int\limits_{\mathcal{V}_o} (\mu_r |\mathbf{H}|^2 - \epsilon_r |\mathbf{E}|^2) \, d\mathcal{V} \tag{59}$$

$$X_0(\omega_0) = \frac{\omega_0}{|I_0(\omega_0)|^2} \int\limits_{\mathcal{V}_o} (\mu_r |\mathbf{H}|^2 - \epsilon_r |\mathbf{E}|^2) \, d\mathcal{V} = 0. \tag{60}$$

$$P_{\mathcal{L}}(\omega) = \frac{\omega}{2} \int\limits_{\mathcal{V}_a} (\mu_i |\mathbf{H}|^2 + \epsilon_i |\mathbf{E}|^2) \, d\mathcal{V} \tag{61}$$

and

$$P_A(\omega) = \frac{1}{2Z_f} \int\limits_{4\pi} |\mathbf{F}(\theta, \phi)|^2 \, d\Omega + \frac{\omega}{2} \int\limits_{\mathcal{V}_a} (\mu_i |\mathbf{H}|^2 + \epsilon_i |\mathbf{E}|^2) \, d\mathcal{V} \tag{62}$$

wherein it can be noted that μ_i and ϵ_i must both be positive (or zero) in passive material to ensure that $P_{\mathcal{L}} \geq 0$—a result that also follows from (57) and (58).

B. Field Expressions for the Frequency Derivative of Impedance and for Internal Energies

The formulas for conductance bandwidth and matched VSWR bandwidth given in (34) and (41), respectively, require the derivative of impedance with respect to frequency evaluated at $\omega = \omega_0$, that is, $Z_0'(\omega_0) = R_0'(\omega_0) + jX_0'(\omega_0)$.

The derivative of the resistance, $R_0'(\omega_0)$, can be written from (48) as

$$R_0'(\omega_0) = \frac{2}{|I_0|^2} (P_A')_{I_0}(\omega_0) \tag{63}$$

in which $(P_A')_{I_0}(\omega_0) = [dP_A(\omega)/d\omega]_{I_0, \omega=\omega_0}$, that is, the frequency derivative of $P_A(\omega)$ holding the feed-line current I_0 constant with frequency and evaluated at the tuned frequency ω_0. The expression (55) or (62) can be inserted for P_A in (63) to get $R_0'(\omega_0)$ in terms of the electromagnetic fields of the tuned antenna. However, as we shall see below, the expression for $R_0'(\omega_0)$ given in (63) is not needed in the derivation of the quality factor Q and its relationship to the bandwidth of the tuned antenna and thus such an exercise proves unnecessary. On the other hand, the evaluation of the frequency derivative of the reactance, $X_0'(\omega_0)$, in terms of the electromagnetic fields of the antenna is crucial to the derivation of Q and its relationship to the bandwidth.

Taking the frequency derivative of (52) and setting $\omega = \omega_0$ obviously produces an expression for $X_0'(\omega_0)$ in terms of the antenna fields. Unfortunately, however, this expression cannot be used directly in the derivation of Q. A more useful expression for $X_0'(\omega_0)$ is derived in Appendix A (see (A.15)) by combining Maxwell's equations with the frequency derivative of Maxwell's equations to get

$$\begin{aligned} |I_0|^2 X_0'(\omega_0) &= \lim_{r\to\infty} \left[\mathrm{Re} \int\limits_{\mathcal{V}_o(r)} (\mathbf{B}^* \cdot \mathbf{H} + \boldsymbol{D}^* \cdot \mathbf{E}) \, d\mathcal{V} - 2\epsilon_0 r \int\limits_{4\pi} |\mathbf{F}|^2 \, d\Omega \right] \\ &+ \omega_0 \mathrm{Re} \int\limits_{\mathcal{V}_a} \big(\mathbf{B}_{I_0}'^* \cdot \mathbf{H} - \mathbf{B}^* \cdot \mathbf{H}_{I_0}' \\ &+ \boldsymbol{D}_{I_0}'^* \cdot \mathbf{E} - \boldsymbol{D}^* \cdot \mathbf{E}_{I_0}' \big) \, d\mathcal{V} \\ &+ \frac{2}{Z_f} \mathrm{Im} \int\limits_{4\pi} \mathbf{F}_{I_0}' \cdot \mathbf{F}^* \, d\Omega. \end{aligned} \tag{64}$$

The primes indicate derivatives with respect to ω evaluated at the tuned frequency ω_0, and the subscripts "I_0" indicate that the input current I_0 at the reference plane S_0 in the feed line of the antenna is held constant with frequency during the indicated differentiations. The volume $\mathcal{V}_o(r)$ is $\mathcal{V}_o$ capped by a sphere of radius r surrounding the antenna system. Each of the two integral terms inside the square brackets of (64) approaches a positive infinite value as $r \to \infty$, but together they approach a finite value because all the other terms in (64) are finite. As $r \to \infty$, the second integral term inside the square brackets, the one involving $|\mathbf{F}|^2$, subtracts the infinite energy in the radiation fields from the infinite energy in the total fields to leave a finite average "reactive energy" involving "static" and "induction" fields. The expression in (64) was derived by Rhodes [13], [17] in the less general form given with the isotropic permeability μ and permittivity ϵ in the constitutive relations (58). The expression corresponding to (64) for perfectly conducting lossless antennas was derived by Levis [18] and Fante [3].

For antennas with asymmetric far-field magnitude patterns, we shall show in Section IV-E that the square-bracketed energy ("reactive energy") and the last integral in (64) are each dependent on the position of the origin of the coordinate system in which the integrals are evaluated. The values of $|I_0|^2 X_0'(\omega_0)$ and the integral over $\mathcal{V}_a$ in (64) are both independent of the position of the origin of the coordinates, and thus the sum of the square-bracketed energy plus the last integral in (64) is independent of the position of the origin of the coordinates.

We could have combined the entire last integral in (64) with the square brackets of (64) to get an alternative "reactive energy" that is independent of the chosen origin for all antennas. We do not want to do this, however, for two reasons. First, there is little physical justification for including this integral as part of the average reactive energy. Second, in all of our numerical work (see Section VI) we have found that the inverse relationship between the exact Q defined with this alternative reactive energy and the exact bandwidth does not hold accurately in the antiresonant bands of tuned antennas. The ideal choice of the origin of the coordinate system is discussed in Section IV-E.

Using the general constitutive relations in (51), (64) can be rewritten as

$$|I_0|^2 X_0'(\omega_0) = 4[W(\omega_0) + W_{\mathcal{L}}(\omega_0) + W_{\mathcal{R}}(\omega_0)] \tag{65}$$

with

$$W(\omega_0) = W_m(\omega_0) + W_e(\omega_0) + W_{\rm me}(\omega_0) \tag{66}$$

and

$$W_m(\omega_0) = \frac{1}{4}\lim_{r\to\infty}\left[\mathrm{Re}\int_{\mathcal{V}_o(r)} \mathbf{H}^*\cdot(\omega_0\bar{\boldsymbol{\mu}})'\cdot\mathbf{H}\,d\mathcal{V} - \epsilon_0 r\int_{4\pi}|\mathbf{F}|^2\,d\Omega\right] \tag{67a}$$

$$W_e(\omega_0) = \frac{1}{4}\lim_{r\to\infty}\left[\mathrm{Re}\int_{\mathcal{V}_o(r)} \mathbf{E}^*\cdot(\omega_0\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E}\,d\mathcal{V} - \epsilon_0 r\int_{4\pi}|\mathbf{F}|^2\,d\Omega\right] \tag{67b}$$

$$W_{\rm me}(\omega_0) = \frac{1}{4}\mathrm{Re}\int_{\mathcal{V}_a}\mathbf{E}\cdot[(\omega_0(\bar{\boldsymbol{\nu}}_t+\bar{\boldsymbol{\tau}}^*)]'\cdot\mathbf{H}^*\,d\mathcal{V} \tag{67c}$$

$$W_{\mathcal{L}}(\omega_0) = \frac{\omega_0}{4}\mathrm{Re}\int_{\mathcal{V}_a}\left[\mathbf{H}'_{I_0}\cdot(\bar{\boldsymbol{\mu}}_t-\bar{\boldsymbol{\mu}}^*)\cdot\mathbf{H}^* + \mathbf{E}'_{I_0}\cdot(\bar{\boldsymbol{\epsilon}}_t-\bar{\boldsymbol{\epsilon}}^*)\cdot\mathbf{E}^* + \mathbf{H}'_{I_0}\cdot(\bar{\boldsymbol{\tau}}_t-\bar{\boldsymbol{\nu}}^*)\cdot\mathbf{E}^* + \mathbf{E}'_{I_0}\cdot(\bar{\boldsymbol{\nu}}_t-\bar{\boldsymbol{\tau}}^*)\cdot\mathbf{H}^*\right]d\mathcal{V} \tag{67d}$$

$$W_{\mathcal{R}}(\omega_0) = \frac{1}{2Z_f}\mathrm{Im}\int_{4\pi}\mathbf{F}'_{I_0}\cdot\mathbf{F}^* d\Omega. \tag{67e}$$

Note that the finite magnetic, electric, and magnetoelectric energies $[W_m(\omega), W_e(\omega)$, and $W_{\rm me}(\omega)]$ at any frequency ω can be defined by the formulas (67a)–(67c) evaluated at any frequency ω instead of ω_0. However, the formula in (52) for the reactance at any frequency ω can generally only be rewritten in terms of these finite energies as

$$X_0(\omega) = \frac{4\omega}{|I_0(\omega)|^2}[W_m(\omega) - W_e(\omega)] \tag{68}$$

if $\bar{\boldsymbol{\nu}}^* = \bar{\boldsymbol{\tau}}_t$ and the contributions to $W_m(\omega)$ and $W_e(\omega)$ from $\bar{\boldsymbol{\mu}}'$ and $\bar{\boldsymbol{\epsilon}}'$ are negligible (so that $(\omega\bar{\boldsymbol{\mu}})' = \bar{\boldsymbol{\mu}}$ and $(\omega\bar{\boldsymbol{\epsilon}})' = \bar{\boldsymbol{\epsilon}}$).

For lossless antenna material, (57) shows that $W_{\mathcal{L}} = 0$. Moreover, energy relations are used in Appendix B to prove that if the antenna material is lossless in a frequency window about ω, then

$$\frac{1}{4}\,\mathrm{Re}\{\mathbf{H}^*\cdot(\omega\bar{\boldsymbol{\mu}})'\cdot\mathbf{H} + \mathbf{E}^*\cdot(\omega\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E} + \mathbf{E}\cdot[(\omega(\bar{\boldsymbol{\nu}}_t+\bar{\boldsymbol{\tau}}^*)]'\cdot\mathbf{H}^*\} - [\mu_0|\mathbf{H}|^2 + \epsilon_0|\mathbf{E}|^2] \geq 0 \tag{69}$$

which is equivalent to the associated Hermitian susceptibility energy matrix being positive semidefinite [14], a property expressible symbolically as

$$\begin{bmatrix} \left[\frac{(\omega\bar{\boldsymbol{\epsilon}}_t^*+\omega\bar{\boldsymbol{\epsilon}})'}{2} - \epsilon_0\bar{\mathbf{I}}\right] & -j\frac{(\omega\bar{\boldsymbol{\nu}}_t^*+\omega\bar{\boldsymbol{\tau}})'}{2} \\ j\frac{(\omega\bar{\boldsymbol{\nu}}+\omega\bar{\boldsymbol{\tau}}_t^*)'}{2} & \left[\frac{(\omega\bar{\boldsymbol{\mu}}_t^*+\omega\bar{\boldsymbol{\mu}})'}{2} - \mu_0\bar{\mathbf{I}}\right] \end{bmatrix} = \begin{bmatrix} [(\omega\bar{\boldsymbol{\epsilon}})' - \epsilon_0\bar{\mathbf{I}}] & -j(\omega\bar{\boldsymbol{\tau}})' \\ j(\omega\bar{\boldsymbol{\tau}}_t^*)' & [(\omega\bar{\boldsymbol{\mu}})' - \mu_0\bar{\mathbf{I}}] \end{bmatrix} \geq 0. \tag{70a}$$

The energy relations in Appendix B also reveal that the real parts of the elements of lossless (in a frequency window about ω) constitutive parameters obey the inequalities

$$\begin{aligned} &[(\omega\epsilon_{kl,r})' - \delta_{kl}\epsilon_0] \geq \omega\epsilon'_{kl,r}/2 \geq 0 \\ &[(\omega\mu_{kl,r})' - \delta_{kl}\mu_0] \geq \omega\mu'_{kl,r}/2 \geq 0, \\ &(\omega\nu_{kl,r})' \geq \omega\nu'_{kl,r}/2 \geq 0 \quad (\omega\tau_{kl,r})' \geq \omega\tau'_{kl,r}/2 \geq 0. \end{aligned} \tag{70b}$$

The inequalities in (70) and thus (69) can also be proven from the Kramers-Kronig dispersion relations in a manner analogously to the proofs in [15] and [19, Sec. 84]. The last part of Appendix B proves that in a lossless medium, the left-hand side of (69) equals the average reversible kinetic plus potential energy of the charge carriers in a final time-harmonic field that is built up gradually from a zero magnitude at $t = -\infty$.

Using the terminology of Brillouin [20, p. 88] and Landau *et al.* [19, p. 275] for our purposes, we shall refer to $W_m(\omega_0), W_e(\omega_0)$, and $W_{\rm me}(\omega_0)$ as the average "internal" magnetic, electric, and magnetoelectric energies of the tuned antenna, respectively. They are finite and have dimensions of energy, W_m and W_e have the far-field radiation energies subtracted from them, and if they were the energies in quasistatic fields in free space or nondispersive media, they would equal the amounts of energy one could quasistatically extract from these magnetic and electric fields. In reality, however, they are not just quasistatic energies and, in addition, the antenna may contain dispersive materials, that is, constitutive parameters that are strongly frequency dependent. Nonetheless, treating W_m, W_e, and $W_{\rm me}$ as internal energies of the antenna in order to define a quality factor Q for the antenna, we shall find the satisfying result that $\mathrm{FBW}_V(\omega_0) \approx 2\sqrt{\beta}/Q(\omega_0)$ provided the bandwidth of the antenna is narrow enough.

If in addition to the antenna being lossless in a frequency window about ω_0, it is also nonradiating, (69) reduces (65)–(67) to

$$\begin{aligned} |I_0|^2 X_0'(\omega_0) &= \mathrm{Re}\int_{\mathcal{V}_o(r)}\{\mathbf{H}^*\cdot(\omega_0\bar{\boldsymbol{\mu}})'\cdot\mathbf{H} + \mathbf{E}^*\cdot(\omega_0\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E} + \mathbf{E}\cdot[(\omega_0(\bar{\boldsymbol{\nu}}_t+\bar{\boldsymbol{\tau}}^*)]'\cdot\mathbf{H}^*\}\,d\mathcal{V} \\ &\geq \int_{\mathcal{V}_a}[\mu_0|\mathbf{H}|^2 + \epsilon_0|\mathbf{E}|^2]\,d\mathcal{V} \geq 0. \end{aligned} \tag{71}$$

This equation implies that the frequency derivative of the reactance (actually $|I_0|^2X_0'(\omega_0)/4$) of a lossless and nonradiating antenna is equal to the "internal energy," which is greater than or equal to $(1/4)\int_{\mathcal{V}_a}[\mu_0|\mathbf{H}|^2 + \epsilon_0|\mathbf{E}|^2]\,d\mathcal{V}$ and thus always greater than or equal to zero (Foster reactance theorem for lossless nonradiating antennas or purely reactive one-port passive terminations [21, Sec. 4.3]. Equation (71) remains valid if the tuned frequency ω_0 is replaced by any frequency ω at which the antenna is lossless and nonradiating but untuned.

In lossy media, it is possible to have negative values of the first integral in (71) and, thus, it is conceivable that $W(\omega_0)$ could be negative for certain lossy antennas.

The energies in (65) and (67d)–(67e) denoted by $W_{\mathcal{L}}$ and $W_{\mathcal{R}}$ are dispersive quantities (in that they depend on the frequency derivative of the fields) associated with the power dissipated by the antenna as power loss ($P_{\mathcal{L}}$) and power radiated ($P_{\mathcal{R}}$), respectively. Unlike the power loss and power radiated (each of which cannot be negative), however, the sum of these "dispersion energies" can be negative as well as positive or zero and $X_0'(\omega_0)$ in (65) can be negative as well as positive or zero. *Therefore, the Foster reactance theorem, which says that $X_0'(\omega)$ for a one-port linear, lossless, passive network is always positive, does not hold for antennas* even if $W(\omega) > 0$ unless the antenna is not only lossless but does not radiate, in which case (71) holds. (Because both $W_{\mathcal{L}}$ and $W_{\mathcal{R}}$ are missing from the expression for $X_0'(\omega)$ in [5, eq. (43)], it is mistakenly concluded in [5] that the Foster reactance theorem holds at all frequencies for antennas with $W(\omega) > 0$.)

With the simple isotropic constitutive relations in (58), the energy expressions in (67a)–(67d) become

$$W_m(\omega_0) = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)} (\omega_0\mu_r)'|\mathbf{H}|^2\, d\mathcal{V} - \epsilon_0 r \int\limits_{4\pi} |\mathbf{F}|^2\, d\Omega\right] \tag{72a}$$

$$W_e(\omega_0) = \frac{1}{4}\lim_{r\to\infty}\left[\int\limits_{\mathcal{V}_o(r)} (\omega_0\epsilon_r)'|\mathbf{E}|^2\, d\mathcal{V} - \epsilon_0 r \int\limits_{4\pi} |\mathbf{F}|^2\, d\Omega\right] \tag{72b}$$

$$W_{\rm me}(\omega_0) = 0 \tag{72c}$$

$$W_{\mathcal{L}}(\omega_0) = \frac{\omega_0}{2}\,\mathrm{Im}\int\limits_{\mathcal{V}_a}\left(\mu_i \mathbf{H}'_{I_0}\cdot\mathbf{H}^* + \epsilon_i\mathbf{E}'_{I_0}\cdot\mathbf{E}^*\right) d\mathcal{V}. \tag{72d}$$

The inequalities in (70), which hold in material that is lossless ($\mu_i = 0, \epsilon_i = 0$) in a frequency window about ω, reduce to

$$(\omega\mu_r)' - \mu_0 \geq \omega\mu_r'/2 \geq 0, \quad (\omega\epsilon_r)' - \epsilon_0 \geq \omega\epsilon_r'/2 \geq 0 \tag{73}$$

regardless, incidentally, of whether the values of (μ_r, ϵ_r) are positive or negative.

The far-field dispersion energy $W_{\mathcal{R}}$ given by (67e) can be evaluated from the antenna's complex far electric field pattern $\mathbf{F}(\theta,\phi)$ defined in (20). The material-loss dispersion energy $W_{\mathcal{L}}$ given by (67d) or (72d) requires a knowledge of the electric and magnetic fields in the material of the antenna. For thin-wire lossy antennas with $\mu_i = 0, \epsilon_i = \sigma/\omega$, where $\sigma(\omega)$ is the conductivity of the wire material and $\mathbf{J} = \sigma\mathbf{E}$, the dispersion energy $W_{\mathcal{L}}$ in (72d) reduces to

$$W_{\mathcal{L}} = \frac{1}{2}\mathrm{Im}\int\limits_{\rm wire}\left(\mathbf{J}'_{I_0}\cdot\mathbf{J}^*/\sigma\right) d\mathcal{V}. \tag{74}$$

If the cross section of the wire is circular and the "skin depth" δ of the current density $\mathbf{J}$ is much smaller than the diameter d of the wire, (74) further reduces to

$$W_{\mathcal{L}} \approx \mathrm{Im}\int\limits_{\rm wirelength} \mathrm{R}(\ell) I'_{I_0}(\ell) I^*(\ell)\, d\ell, \quad \delta/d \ll 1 \tag{75}$$

under the approximation $\mathbf{J}(\rho,\ell) \approx \mathbf{J}_0\exp[-(d/2-\rho)/\delta]$, where $\mathbf{J}_0$ is the current density at the surface of the wire and ρ is the radial distance from the center of the wire. In (75), $\mathrm{R}(\ell) = 1/(\pi\sigma\, d\delta)$ is the resistance per unit length of wire and $I(\ell)$ is the total current flowing in the wire at the position ℓ along the wire. As usual, the primes indicate differentiation with respect to frequency ω and the subscript I_0 means that the frequency derivative is taken with the input current I_0 that feeds the antenna held constant (independent of frequency). If the diameter or resistivity of the wire varies along the wire, $\mathrm{R}(\ell)$ will be a function of ℓ. If the current were uniform across the wire as in a lumped circuit resistor $R_c(\omega)$ carrying a current I_c, (75) is replaced by

$$W_{\mathcal{L}} = \frac{R_c}{2}\,\mathrm{Im}\left[(I_c)'_{I_0} I_c^*\right]. \tag{76}$$

The formula in (75) is used in Section VI to numerically evaluate $W_{\mathcal{L}}$ for lossy wire antennas, and (76) is applied in Appendix D to lumped resistors in series and parallel *RLC* circuits. Within resonant frequency ranges ($X_0'(\omega_0) > 0$), the tuned antenna can usually be approximated by a series *RLC* circuit. For that approximation, $I_c = I_0$ in (76) and since $(I_0)'_{I_0} = 0$, it follows that $W_{\mathcal{L}} \approx 0$ within resonant frequency ranges.

Because $(\omega_0\bar{\boldsymbol{\mu}})' = \bar{\boldsymbol{\mu}} + \omega_0\bar{\boldsymbol{\mu}}'$, $(\omega_0\bar{\boldsymbol{\epsilon}})' = \bar{\boldsymbol{\epsilon}} + \omega_0\bar{\boldsymbol{\epsilon}}'$, $(\omega_0\bar{\boldsymbol{\nu}})' = \bar{\boldsymbol{\nu}} + \omega_0\bar{\boldsymbol{\nu}}'$, and $(\omega_0\bar{\boldsymbol{\mu}})' = \bar{\boldsymbol{\tau}} + \omega_0\bar{\boldsymbol{\tau}}'$, the energies $W_m(\omega_0), W_e(\omega_0)$, and $W_{\rm me}$ include the derivative terms with $\omega_0\bar{\boldsymbol{\mu}}', \omega_0\bar{\boldsymbol{\epsilon}}', \omega_0\bar{\boldsymbol{\nu}}'$, and $\omega_0\bar{\boldsymbol{\tau}}'$. In [9] we defined $W(\omega_0) = W_m(\omega_0) + W_e(\omega_0) + W_{\rm me}(\omega_0)$ and the associated $Q(\omega_0)$ in (78) below without these derivative terms. However, in order to define a $Q(\omega_0)$ in (78) that is proportional to the inverse of the matched VSWR fractional bandwidth given approximately in (41), the energy $W(\omega_0)$ must equal $|I_0|^2 X_0'(\omega_0)/4$ when $R_0'(\omega_0), W_{\mathcal{L}}(\omega_0)$, and $W_{\mathcal{R}}(\omega_0)$ are negligible. It then follows from (64) that $W(\omega_0) = W_m(\omega_0) + W_e(\omega_0) + W_{\rm me}(\omega_0)$ must be defined as shown in (67a)–(67c) with the derivative terms included. For example, at a natural resonant frequency ω_0 of an antenna that can be modeled by an *RLC* series circuit with $R'(\omega_0)$ negligible but $L'(\omega_0)$ and $C'(\omega_0)$ nonnegligible (because the inductor and capacitor is filled with a material that has a nonnegligible $\mu'(\omega_0)$ and $\epsilon_r'(\omega_0)$, respectively)

$$\begin{aligned} Q(\omega_0) &= \frac{\omega_0[(\omega_0 L)' + (\omega_0 C)'/(\omega_0 C)^2]}{2R} \\ &= \frac{\omega_0 X_0'(\omega_0)}{2R} = \frac{\omega_0|Z_0'(\omega_0)|}{2R} \approx \frac{2\sqrt{\beta}}{\mathrm{FBW}_V(\omega_0)}. \end{aligned} \tag{77a}$$

Similarly, at a natural antiresonant frequency ω_0 of an antenna that can be modeled by an *RLC* parallel circuit with $R'(\omega_0)$ negligible but $L'(\omega_0)$ and $C'(\omega_0)$ nonnegligible

$$\begin{aligned} Q(\omega_0) &= \frac{\omega_0 R[(\omega_0 C)' + (\omega_0 L)'/(\omega_0 L)^2]}{2} \\ &= -\frac{\omega_0 X_0'(\omega_0)}{2R} = \frac{\omega_0|Z_0'(\omega_0)|}{2R} \approx \frac{2\sqrt{\beta}}{\mathrm{FBW}_V(\omega_0)}. \end{aligned} \tag{77b}$$

For the parallel *RLC* circuit, $W(\omega_0) = -|I_0|^2 X_0'(\omega_0)/4$ and $W_{\mathcal{L}}(\omega_0) = -2W(\omega_0)$. If $Q(\omega_0)$ were defined without the $\omega_0\mu'$ and $\omega_0\epsilon'$ terms included in $W(\omega_0)$, then $Q(\omega_0)$ for these *RLC* antennas would not include the derivatives of $L(\omega_0)$ and $C(\omega_0)$, and the $Q(\omega_0)$ would not closely approximate the inverse of $\mathrm{FBW}_V(\omega_0)$. The need to include the derivative terms in the $W(\omega_0)$ used to define $Q(\omega_0)$ is confirmed in the Numerical Results Section VI-D for a straight-wire antenna embedded in a frequency dependent dielectric material (see Figs. 18 and 19).

C. Definition and Exact Expressions of Q

The quality factor $Q(\omega_0)$ for an antenna tuned to have zero reactance at the frequency ω_0 $(X_0(\omega_0)=0)$ can now be defined as

$$Q(\omega_0)=\frac{\omega_0|W(\omega_0)|}{P_A(\omega_0)}. \tag{78}$$

Absolute value signs are placed about $W(\omega_0)$ in the definition of $Q(\omega_0)$ in (78) to allow for hypothetical antennas (mentioned in the previous subsection) with $W(\omega_0)<0$; see Q in Fig. 19. Formulas for $W(\omega_0)$ in terms of fields are given by means of (66)–(67) and formulas for the power accepted by the antenna are given by means of (15)–(19) and (55). In particular, $P_A=|I_0|^2R_0/2$ and $W(\omega_0)$ can be written from (65) as

$$W(\omega_0)=\frac{|I_0|^2}{4}X_0'(\omega_0)-[W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)] \tag{79}$$

so that $Q(\omega_0)$ can be expressed as

$$Q(\omega_0)=\left|\frac{\omega_0}{2R_0(\omega_0)}X_0'(\omega_0)\right.\\ \left.-\frac{2\omega_0}{|I_0|^2R_0(\omega_0)}[W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)]\right|. \tag{80}$$

The expressions on the right-hand sides of (78) and (80) are very different in form, yet they are exact and thus produce the same value of $Q(\omega_0)$. In Section VI, the formula (80) rather than (78) is used to compute the exact values of Q for various antennas because it is easier to numerically compute $X_0'(\omega_0)$ for these antennas than to numerically evaluate the integrals in (67a)–(67c) that define $W(\omega_0)$ used in (78).

Especially note that the $Q(\omega_0)$ in (80) differs from both the conventional formula for the quality factor [1]

$$Q_{\rm cv}(\omega_0)=\frac{\omega_0}{2R_0(\omega_0)}X_0'(\omega_0) \tag{81}$$

and from Rhodes's formula in (35) above, namely, $1/B=|Q_{\rm cv}(\omega_0)|$, because of the term $2\omega_0[W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)]/[|I_0|^2R_0(\omega_0)]$. (Rhodes [13] assumes (mistakenly) that the right-hand side of (80) is not a valid expression for Q because it does not, in general, equal $\omega_0|X_0'|/(2R_0)$. Fante [3] assumes that (80) is a valid expression for Q if $\mu=\mu_0, \epsilon=\epsilon_0$, and $W_{\mathcal{R}}(\omega_0)$ as well as $W_{\mathcal{L}}(\omega_0)$ are negligible.) The formula in (81) is commonly used to determine the quality factor and the bandwidth ($1/Q_{\rm cv}$ for half-power conductance bandwidth, as in (34), and $2/Q_{\rm cv}$ for half-power matched VSWR bandwidth) of tuned antennas. In general, neither $Q_{\rm cv}(\omega_0)$ in (81) nor $|Q_{\rm cv}(\omega_0)|$ accurately approximates the exact Q in (78) and (80) of tuned antennas in antiresonant frequency ranges.

It is proven in Appendix C that the $Q(\omega_0)$ of an antenna increases extremely rapidly as the maximum dimension of the effective source region is decreased while maintaining the frequency, efficiency, and far-field pattern. This implies that supergain above a few dB is impractical. It is also shown in Appendix C that the quality factors determined by previous authors [1]–[3] are lower bounds for our defined Q applied to electrically small antennas with nondispersive $\mu_r\geq 0$ and $\epsilon_r\geq 0$.

D. Approximate Expression for Q and Its Relationship to Bandwidth

We can estimate the total dispersion energy, $W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)$, in (80) to get an approximate expression for $Q(\omega_0)$ that can be immediately related to the bandwidth of the tuned antenna. Away from antiresonant frequency ranges of tuned antennas, $X_0'(\omega_0)>0$ and usually $|R_0'(\omega_0)|\ll X_0'(\omega_0)$. Furthermore, for the sake of evaluating Q, we assume the power loss and power radiated can both be approximated by ohmic loss in a resistor of a series *RLC* circuit, where R can be a function of ω. Evaluating $W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)$ in Appendix D for such a series *RLC* circuit reveals that it's value is small enough to make the second term on the right-hand side of (80) negligible compared to the first. Therefore, away from antiresonant frequency ranges, that is, within resonant frequency ranges

$$Q(\omega_0)\approx\frac{\omega_0}{2R_0(\omega_0)}X_0'(\omega_0) \tag{82}$$

or, since $|R_0'(\omega_0)|\ll X_0'(\omega_0)$ away from antiresonant frequency ranges

$$Q(\omega_0)\approx\frac{\omega_0}{2R_0(\omega_0)}|Z_0'(\omega_0)|. \tag{83}$$

At an antiresonant frequency ω_0, we assume that tuned antennas can be approximated by a tuning inductor (L_s) or capacitor (C_s) in series with a parallel *RLC* circuit. An evaluation of $W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)$ in Appendix D for such a tuned parallel *RLC* circuit reveals that

$$X_0'(\omega_0)-\frac{4}{|I_0|^2}[W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)]\\ \approx\sqrt{X_0'^2(\omega_0)+R_0'^2(\omega_0)} \tag{84}$$

so that

$$X_0'(\omega_0)-\frac{4}{|I_0|^2}[W_{\mathcal{L}}(\omega_0)+W_{\mathcal{R}}(\omega_0)]\approx|Z_0'(\omega_0)|. \tag{85}$$

Inserting (85) into (80) yields

$$Q(\omega_0)\approx\frac{\omega_0}{2R_0(\omega_0)}|Z_0'(\omega_0)| \tag{86}$$

which, combined with (83), holds for all ω_0.

Comparing the approximate formula for the quality factor $Q(\omega_0)$ in (86) with the approximate formula for the matched VSWR fractional bandwidth $\mathrm{FBW}_V(\omega_0)$ in (41), one finds

$$Q(\omega_0)\approx\frac{2\sqrt{\beta}}{\mathrm{FBW}_V(\omega_0)}\approx\frac{\omega_0}{2R_0(\omega_0)}|Z_0'(\omega_0)|,\\ \sqrt{\beta}=\frac{s-1}{2\sqrt{s}}\leq 1 \tag{87}$$

provided $X_0'(\omega)$ and $R_0'(\omega)$ do not change greatly over the bandwidth of the antenna (assumptions that hold if the bandwidth is narrow enough.) As noted in (35), Rhodes [13] *defines* Q by the expression in (87) with $|Z_0'(\omega_0)|$ replaced by $|X_0'(\omega_0)|$ (and $\beta=1$). Such an expression does not produce an accurate approximation to Q and bandwidth in antiresonant frequency bands (except at antiresonant frequencies with $R_0'(\omega_0)\approx 0$).

In concluding this section, it is emphasized that not every tuned antenna has to obey the inverse relationship between bandwidth and Q given in (87). The derivation of (41) and (86) assumes that the antenna is linear, passive, and tuned by a linear passive circuit. If the antenna contains nonlinear or active materials and tuning elements, the bandwidth could conceivably be appreciably widened without decreasing commensurately the internal energy and Q (as defined in (78)) of the antenna. Then, of course, (41), (86), and the inverse relationship between bandwidth and Q would not necessarily hold.

Also, the derivation of (86) approximates $|Z_0'(\omega_0)|$ by $X_0'(\omega_0)$ in resonant frequency ranges. Consequently, the approximation (86) could be inaccurate in a resonant frequency range if the resistivity of the antenna changed rapidly enough with frequency to make $|R_0'(\omega_0)| > X_0'(\omega_0)$. In general, the derivation of (86) breaks down if $R_{\rm sc}'(\omega)$ or $R_{\rm pc}'(\omega)$ in the *RLC* series and parallel circuit antenna models of Appendix D become too large and one would not expect the exact Q to be a highly accurate approximation to the inverse of the exact bandwidth. In all our numerical simulations to date with practical antenna models, however, the approximations in (86) and (87) have exhibited high accuracy throughout both resonant and antiresonant frequency ranges.

Nonetheless, the derivation of (86) in Appendix D that uses series and parallel *RLC* circuits to model antennas in their resonant and antiresonant frequency ranges, respectively, has a serious limitation. In this derivation, the radiation resistance of the antenna is lumped into the antenna's resistive loss so that the $W_{\mathcal{R}}$ term is replaced by a contribution to $W_{\mathcal{L}}$. As discussed in Section IV-B, the value of $W_{\mathcal{R}}$ for antennas with asymmetric far-field magnitude patterns depends on the position of the origin of the coordinate system. Therefore, in replacing $W_{\mathcal{R}}$ with a contribution to $W_{\mathcal{L}}$, which is independent of the origin of the coordinates, it is implicitly assumed that the antenna's $W_{\mathcal{R}}$ is either independent of the origin or that the origin is chosen to make $W_{\mathcal{R}}$ of the antenna approximately equal to the $W_{\mathcal{L}}$ of the *RLC* circuit that is used in Appendix D to model the antenna. In the following Section IV-E, we shall give a practical method for determining approximately such an ideal location for the origin of the coordinates at each tuned frequency ω_0. Moreover, a simpler alternative method is given in the last paragraph of Section IV-E for obtaining an approximate value of the $Q(\omega_0)$ associated with the ideal origin at each tuned frequency ω_0.

Kuester [22] has pointed out that an *RLC* circuit can be constructed with an arbitrary value of Q by separating the resistor from the inductor and capacitor by a length of transmission line whose characteristic impedance is equal to the resistance of the resistor that terminates this line. The input impedance and bandwidth of such an *RLC* circuit is independent of the length of this transmission line, whereas the internal energy and Q as defined by (78) or (80) will increase with the length of this transmission line. Increases in internal energy and Q without a change in the input impedance can also occur using "surplus" capacitors and inductors [23, p. 176]. These spurious contributions to the exact Q that create discrepancies between the exact value of Q in (78) or (80) and the approximate value in (86), as well as the ambiguity in Q with respect to the chosen origin for the far-field pattern of the antenna, can be removed by the simple procedure given in the last paragraph of Section IV-E.

E. Determination of the Ideal Location for the Origin of the Coordinates and the Associated Q

The values of W_m, W_e, and $W_{\mathcal{R}}$ may depend on the choice of the origin of the coordinates to which the far-field pattern is referenced. To prove this, let the origin of the coordinate system be displaced by an amount $\Delta\mathbf{r}$ with respect to the antenna. Then the far-field pattern (at frequency ω) with respect to this new coordinate system is given by

$$\mathbf{F}_{\Delta}(\theta,\phi) = e^{-jk\Delta\mathbf{r}\cdot\hat{\mathbf{r}}}\mathbf{F}(\theta,\phi) \tag{88}$$

and, thus

$$\mathbf{F}_{\Delta}'(\theta,\phi) = e^{-jk\Delta\mathbf{r}\cdot\hat{\mathbf{r}}}[\mathbf{F}'(\theta,\phi) - (j\Delta\mathbf{r}\cdot\hat{\mathbf{r}}/c)\mathbf{F}(\theta,\phi)]. \tag{89}$$

Inserting $\mathbf{F}_{\Delta}$ and $\mathbf{F}_{\Delta}'$ into the last integral in (64), that is, into $W_{\mathcal{R}}$, shows that the change in this integral caused by a displacement of the origin of the coordinates is given by

$$-2\epsilon_0\Delta\mathbf{r}\cdot\int\limits_{4\pi}\hat{\mathbf{r}}|\mathbf{F}|^2d\Omega \tag{90}$$

which has a magnitude that is less than or equal to $2\epsilon_0\Delta r\int_{4\pi}|\mathbf{F}|^2d\Omega$. If the magnitude of the far-field pattern is symmetric about the origin, then the change given in (90) is zero, that is, the value of the last integral in (64), and thus the square-bracketed energy in (64), is independent of the origin of the coordinate system.

If $\int_{4\pi}\hat{\mathbf{r}}|\mathbf{F}|^2d\Omega = 0$ (for example, if $|\mathbf{F}|$ is symmetric about the origin), the choice of the origin of the coordinate system is irrelevant. If $\int_{4\pi}\hat{\mathbf{r}}|\mathbf{F}|^2\,d\Omega \neq 0$, then the radiation-field energy (second integral in the square brackets of (64)) that subtracts from the total-field energy (first integral in the square brackets of (64)) may either overcompensate or undercompensate for the radiation energy if the origin is too far from the center of the source region of the antenna. Thus, it is reasonable, though not necessarily ideal, to choose the origin of the coordinates at the center of the imaginary spherical surface that circumscribes the source region of the antenna. Nonetheless, we ultimately have to live with the fact that our defined reactive and internal energies of an antenna (like that of previous authors [1]–[8]) and thus its Q defined in (78) depends to some degree on the choice of the origin of the coordinate system relative to the antenna (unless $\int_{4\pi}\hat{\mathbf{r}}|\mathbf{F}|^2d\Omega = 0$). This nonuniqueness in reactive energy and Q of an antenna arises because of the need to subtract the infinite energy in the radiation fields from the infinite energy in the total fields of the antenna to obtain finite values of reactive and internal energies, which turn out to depend on the point to which the far field is referenced if $\int_{4\pi}\hat{\mathbf{r}}|\mathbf{F}|^2d\Omega \neq 0$. The above derivation shows that the amount that $Q(\omega_0)$ changes with a shift $\Delta\mathbf{r}$ in the origin is less than $\eta k_0\Delta r$, where $\eta \leq 1$ is the efficiency of the antenna [see (47)] and $k_0 = \omega_0/c$.

The quality factor Q is most often determined for antennas whose maximum linear dimensions are on the order of a wavelength or less because it is these relatively small antennas that usually determine the bandwidth of a one-port antenna system. For example, the bandwidth of a reflector antenna or an array fed by one element is usually determined mainly by the bandwidth of the feed element. Choosing the origin near the center of the dominant radiating sources of an antenna that is not much larger than a wavelength across involves an ambiguity Δr of no more than about a wavelength and, thus, an ambiguity in Q of no more than about $\eta k_0 \Delta r \approx \eta k_0 \lambda = 2\pi\eta$. Nonetheless, it would be desirable to determine an ideal criterion for choosing the origin of the coordinate system. Fortunately, the results of Sections IV-C and IV-D reveal such a criterion that we can be used to specify a practical way to choose a reasonable position of the origin for each tuned frequency ω_0.

As discussed in the previous subsection, it is assumed in the derivation of (86) and thus in the derivation of the first equation in (87), namely

$$Q(\omega_0) \approx \frac{2\sqrt{\beta}}{\mathrm{FBW}_V(\omega_0)} \tag{91}$$

that either $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega = 0$, so that $W_{\mathcal{R}}$ is independent of the location of the origin, or if $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega \neq 0$ then the location of the origin is chosen to produce a $W_{\mathcal{R}}$ that maintains the relationship (86) and thus (91). If the location of the origin is chosen such that (86) remains valid when $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega \neq 0$, then (86) and (80) imply

$$\left|\frac{\omega_0}{2R_0(\omega_0)}X_0'(\omega_0) - \frac{2\omega_0}{|I_0|^2 R_0(\omega_0)}[W_{\mathcal{L}}(\omega_0) + W_{\mathcal{R}}(\omega_0)]\right| \approx \frac{\omega_0}{2R_0(\omega_0)}|Z_0'(\omega_0)|. \tag{92}$$

At a natural resonant frequency ω_{res} of an untuned antenna where $X'(\omega_{\mathrm{res}}) > 0$ and $X_0(\omega_{\mathrm{res}}) = X(\omega_{\mathrm{res}}) = 0$, we have shown (see Sections IV-B and IV-D) that $|Z_0'(\omega_{\mathrm{res}})| \approx X'(\omega_{\mathrm{res}})$. Therefore, in order for (86) and (91) to hold at a natural resonant frequency when $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega \neq 0$, (92) implies that $W_{\mathcal{L}} + W_{\mathcal{R}}$ should be ≈ 0. From (67e) it is seen that this means that at a natural resonant frequency one should choose the position of the origin of the coordinate system to make

$$W_{\mathcal{R}}(\omega_{\mathrm{res}}) = \frac{1}{2Z_f}\mathrm{Im}\int_{4\pi} \mathbf{F}_{I_0}' \cdot \mathbf{F}^*\, d\Omega \approx -W_{\mathcal{L}}(\omega_{\mathrm{res}}). \tag{93}$$

If we know the far field pattern $\mathbf{F}$ of the antenna, it is straightforward to evaluate the integral in (93) for different positions of the origin to find an origin O_{res} that makes $W_{\mathcal{R}}(\omega_{\mathrm{res}}) = -W_{\mathcal{L}}(\omega_{\mathrm{res}})$ at a natural resonant frequency ω_{res}. (Note from (90) that any vector perpendicular to $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega$ can be used to shift O_{res} without changing the value of $W_{\mathcal{R}}$. Also, the value of $W_{\mathcal{L}}(\omega_{\mathrm{res}})$ at a natural resonant frequency ω_{res} is usually negligible.)

At a natural antiresonant frequency ω_{ares} of the untuned antenna where $X'(\omega_{\mathrm{ares}}) < 0$ and $X_0(\omega_{\mathrm{ares}}) = X(\omega_{\mathrm{ares}}) = 0$, we also have $R_0'(\omega_{\mathrm{ares}}) \approx 0$ (so that $|Z_0'(\omega_{\mathrm{ares}})| \approx -X'(\omega_{\mathrm{ares}})$) and (92) along with (67e) imply

$$W_{\mathcal{R}}(\omega_{\mathrm{ares}}) = \frac{1}{2Z_f}\mathrm{Im}\int_{4\pi} \mathbf{F}_{I_0}' \cdot \mathbf{F}^*\, d\Omega \approx \frac{|I_0|^2}{2}X'(\omega_{\mathrm{ares}}) - W_{\mathcal{L}}(\omega_{\mathrm{ares}}). \tag{94}$$

Thus, one can find the position of the origin O_{ares} that makes $W_{\mathcal{R}}(\omega_{\mathrm{ares}})$ in (94) equal to $|I_0|^2 X'/2 - W_{\mathcal{L}}$ in (94). To find O_{ares} numerically from (94), the value of $X'(\omega_{\mathrm{ares}})$ must be determined. This can be done either by directly computing the derivative of the input reactance of the antenna or by indirectly computing the derivative of $X(\omega)$ from the fields of the antenna in expressions (52) or (59).

Once the origins O_{res} and O_{ares} are found at the natural resonant and antiresonant frequencies ω_{res} and ω_{ares} of the untuned antenna, one can linearly extrapolate between the positions of these origins to obtain approximate values of the ideal position of the origin at every frequency. [For tuned frequencies ω_0 between 0 and the lowest natural resonant or antiresonant frequency ω_l, that is, for $0 < \omega_0 < \omega_l$ where ω_l is the smallest frequency (either a resonant or antiresonant frequency) that satisfies $X(\omega_l) = 0$, one can use O_l (with O_l equal to O_{res} if ω_l is a natural resonant frequency or O_{ares} if ω_l is a natural antiresonant frequency).] In Section VI-C, (93) and (94) are used to numerically evaluate the ideal origin positions O_{res} and O_{ares} for a Yagi antenna at two natural resonant and two natural antiresonant frequencies. The numerical results show that with these origins, the approximation in (86) and (91) hold with considerable accuracy throughout the resonant and antiresonant frequency bands.

We emphasize that this procedure for finding the ideal location of the origin for determining an unambiguous exact Q of antennas with $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega \neq 0$ is given for the sake of academic completeness and for comparing the approximate formulas in (87) with an exact Q. The formulas in (87) are the ones that are convenient and useful in numerical practice provided it is possible to directly compute $Z_0'(\omega_0)$. Even if an unambiguous exact Q is desired, it can be found from (78) or (80) using any position of the origin if $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2 d\Omega = 0$. If $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega \neq 0$, a reasonable exact Q can be found from (78) or (80) by choosing the origin as the center of the sphere that circumscribes the dominant sources of the antenna.

Once this origin of the circumscribing sphere is chosen, an even better exact Q can be obtained by adjusting the values of $Q(\omega_0)$ to equal $\omega_0|X_0'(\omega_0)|/[2R_0(\omega_0)]$ at the natural resonant and antiresonant frequencies (that is, at $\omega_0 = \omega_{\mathrm{res}}$ or ω_{ares}). At other frequencies, the values of Q can be adjusted by an amount equal to the linear extrapolation of the adjustments at the adjacent natural resonant and antiresonant frequencies. For $0 < \omega_0 < \omega_l$, the linear extrapolation can be formed between an adjustment of zero at $\omega_0 = 0$ and the adjustment at $\omega_0 = \omega_l$. This simple procedure can be used independently of the value of $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega$ to define an exact Q that will reasonably compensate for both a nonideal origin and spurious contributions to Q mentioned in the last paragraph of Section IV-D. In Section VI-C, this simple procedure is applied to the Yagi antenna mentioned in the previous paragraph to obtain an alternative exact Q curve that agrees reasonably well with the exact Q curve obtained by shifting the origin of the coordinates.

V. NEGATIVE VALUES OF μ_r AND ϵ_r

Our definitions of the internal energy W and quality factor Q of antennas are quite general and, in particular, in (72) it is not assumed that the values of the real parts of the permeability and permittivity $[\mu_r(\omega), \epsilon_r(\omega)]$ of the antenna material are greater than or equal to zero. For low-loss materials (73) shows that $(\omega_0\mu_r)' \geq \mu_0$ and $(\omega_0\epsilon_r)' \geq \epsilon_0$ even if $\mu_r(\omega_0)$ and $\epsilon_r(\omega_0)$ are negative [25]–[28]. Thus, it seems reasonable to assume that the $Q(\omega_0)$ defined by (78) with the $W(\omega_0)$ given in (66) and (72a)–(72c) remains valid for low-loss materials with negative μ_r and ϵ_r. The approximate formulas for $Q(\omega_0)$ and $\mathrm{FBW}_V(\omega_0)$ in (87) and the inverse relationship between them may become less accurate the faster $\mu_r(\omega)$ and $\epsilon_r(\omega)$ change over the bandwidth of the antenna, regardless of whether the values of μ_r and ϵ_r are positive or negative (assuming the amount of this (μ_r, ϵ_r) material is large enough to significantly affect the bandwidth of the antenna).

For lossy materials, $(\omega_0\mu_r)'$ and $(\omega_0\epsilon_r)'$ can be less than zero near antiresonances of the material where the loss is very large, and it is conceivable that $(\omega_0\mu_r)'$ or $(\omega_0\epsilon_r)'$ could be negative enough in the antenna material to produce a negative value of $W(\omega_0)$ (but not a negative value of $Q(\omega_0)$, which is defined in terms of $|W(\omega_0)|$). However, the antiresonances in lossy media that produce negative values of $(\omega_0\mu_r)'$ or $(\omega_0\epsilon_r)'$ would likely have such narrow bandwidths or high loss that they would make the antenna impractical if they contributed significantly to $W(\omega_0)$ and $Q(\omega_0)$; see Section VI-D.

Finally, consider tuning an electrically small capacitive or inductive antenna, that is, an antenna with $X'(\omega) > 0$, with a low-loss series inductor or capacitor (having reactance X_s) filled with either a μ_r or ϵ_r material that can be positive or negative. Since (73) implies from (71) that $X_s' \geq 0$, the tuned antenna has a reactance derivative $X_0'(\omega_0) = X'(\omega_0) + X_s'(\omega_0) > X'(\omega_0)$. It follows from (41), therefore, that the bandwidth of an electrically small capacitive or inductive antenna cannot be dramatically increased by tuning with a negative capacitance or inductance instead of a positive inductance or capacitance, respectively, as long as the capacitors and inductors are linear, passive, low-loss circuit elements. Tretyakov *et al.* [28] conclude that the bandwidths of radiating electric or magnetic line currents cannot be increased by covering them with electrically thin lossless dispersive materials having negative permeability or negative permittivity, respectively.

VI. Numerical Results

In this section, the expressions for the exact bandwidth and quality factor as well as for the approximate bandwidth and quality factor derived in Sections III–V are evaluated numerically for representative lossless and lossy tuned antennas. These numerical solutions are determined over a wide enough range of frequencies to allow the antenna to vary in size from a small fraction of a wavelength to several wavelengths across. The antennas considered here are the thin straight-wire antenna, the circular wire-loop antenna, a three-element directive Yagi antenna, and a straight-wire antenna embedded in a frequency dependent dielectric material. The numerical analysis of all but the last of these antennas is performed using the Numerical Electromagnetics Code, Version 4 (NEC) [29], which is capable of determining the current, input impedance, and far-fields of these antennas over a wide range of operating frequencies. For each of these tuned antennas, close agreement is found between the numerically computed exact and approximate formulas for the bandwidth and quality factor over the full range of frequencies. Moreover, the inverse relationship (87) between bandwidth and quality factor is confirmed for each of the tuned antennas at every frequency.

Using the computed impedance data from NEC, the exact matched VSWR bandwidth is obtained by tuning the antenna at the desired operating frequency ω_0 with a lossless series inductor or capacitor. A lossless inductor (L_s) is used to tune the antenna's initial reactance $X(\omega_0)$ to zero $(X_0(\omega_0) = X(\omega_0) + \omega_0 L_s = 0)$ if $X(\omega_0)$ is less than zero, and a lossless capacitor (C_s) is used to tune the antenna's initial reactance to zero $(X_0(\omega_0) = X(\omega_0) - 1/(\omega_0 C_s) = 0)$ if $X(\omega_0)$ is greater than zero. Once the antenna is tuned at the desired operating frequency, the VSWR of the tuned antenna is determined for all frequencies ω under the condition that the characteristic feed-line impedance Z_{ch} is equal to the antenna's tuned impedance at ω_0, that is, $Z_{\mathrm{ch}} = Z_0(\omega_0) = R_0(\omega_0)$. The exact matched VSWR bandwidth about the tuned frequency ω_0 is computed for a specific value s of the VSWR by finding the frequency range (ω_-, ω_+) in which the VSWR is less than or equal to s. As defined in Section III-B, the fractional matched VSWR bandwidth is then $\mathrm{FBW}_V(\omega_0) = (\omega_+ - \omega_-)/\omega_0 = (\Delta\omega_+ - \Delta\omega_-)/\omega_0$ with $\omega_\pm = \omega_0 + \Delta\omega_\pm$. To compare the inverse of the exact matched VSWR bandwidth with the antenna's exact and approximate quality factor, the exact matched VSWR bandwidth is converted to an equivalent quality factor $Q_B(\omega_0)$ defined by [see (87)]

$$Q_B(\omega_0) \equiv \frac{2\sqrt{\beta}}{\mathrm{FBW}_V(\omega_0)}, \quad \sqrt{\beta} = \frac{s-1}{2\sqrt{s}}. \tag{95}$$

In our numerical examples, the bandwidth VSWR is given by $s = 1.5\,(\sqrt{\beta} = .2041)$.

In addition to determining the inverse of the exact matched VSWR bandwidth (Q_B), the exact Q of the antenna is found using (80). The first term on the right-hand side of (80), $\omega_0 X_0'(\omega_0)/[2R_0(\omega_0)]$, is evaluated directly from the antenna's feed-point impedance with $X_0'(\omega_0)$ evaluated from (8). The second term on the right-hand side of (80), $2\omega_0[W_{\mathcal{L}}(\omega_0) + W_{\mathcal{R}}(\omega_0)]/[|I_0|^2 R_0(\omega_0)]$, is evaluated numerically from the antenna's current, conductivity, and complex far-field pattern.

The far-field dispersion energy $W_{\mathcal{R}}(\omega_0)$ is evaluated directly from (67e). The frequency derivative and integral in (67e) are evaluated numerically for each observation angle and frequency as necessary to accurately compute the frequency derivative with a finite difference. These quantities are calculated with the input (feed) current held at a constant value independent of frequency. This is accomplished in NEC by feeding the antenna with a voltage source having a voltage equal to the antenna's input impedance at each frequency, thereby setting the current to 1 A at all frequencies. For the lossy antennas, the material-loss dispersion energy $W_{\mathcal{L}}(\omega_0)$ was evaluated from (75).

The approximate conventional quality factor Q_{cv} is given in (81) and we shall designate the newly derived approximate

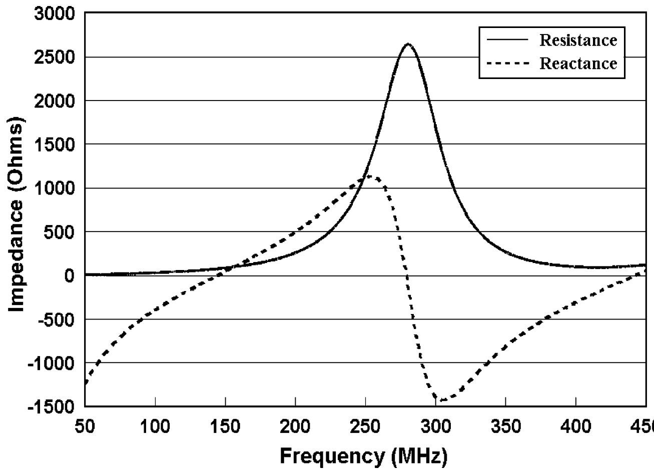

Fig. 3. Input impedance of the center-fed, untuned, lossless straight-wire antenna having a total length of 1 m and a wire diameter of 1 mm.

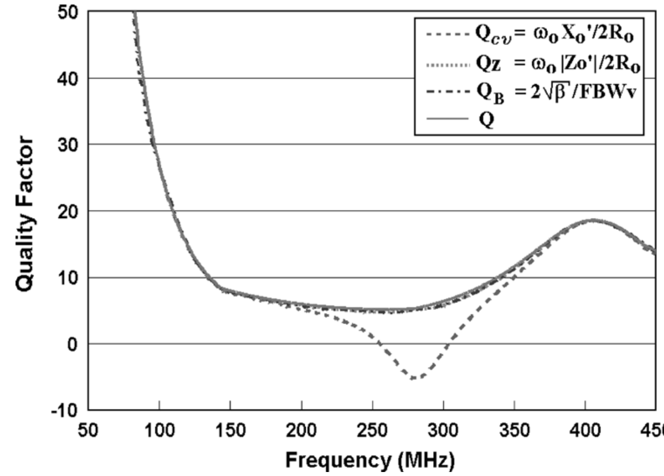

Fig. 4. Comparison of the exact Q, $Q_{\rm cv}$, Q_Z, and Q_B (1.5:1 matched VSWR bandwidth) for the center-fed, tuned, lossless straight-wire antenna.

quality factor in (86) by Q_Z. We can rewrite $Q_Z(\omega_0)$ from (86) and (8) as

$$\begin{aligned} Q_Z(\omega_0) &= \frac{\omega_0}{2R_0(\omega_0)} |Z_0'(\omega_0)| \\ &= \frac{\omega_0}{2R(\omega_0)} \sqrt{[R'(\omega_0)]^2 + [X'(\omega_0) + |X(\omega_0)|/\omega_0]^2} \end{aligned} \tag{96}$$

where $R(\omega) = R_0(\omega)$ and $X(\omega)$ are the resistance and reactance of the untuned antenna. These approximate expressions for the quality factors given by the conventional formula (81) (and/or its absolute value) and by our newly derived formula (86), which is re-expressed in (96), are evaluated using the antenna's impedance and compared to the exact values of Q and the inverse of the exact VSWR bandwidth (Q_B).

A. Bandwidth and Quality Factor of the Lossless Straight-Wire Antenna

The first antenna we consider is the lossless, center-fed, straight-wire antenna that has an overall length of 1 m and a wire diameter of 1 mm. The NEC-calculated impedance of this untuned antenna is given in Fig. 3 for a frequency range covering the first natural resonance and antiresonance. Using the calculated feed-point impedance of the corresponding tuned antenna, the exact matched VSWR bandwidth was calculated for a bandwidth VSWR of $s = 1.5\,(\sqrt{\beta} = .2041)$.

A comparison of the exact $Q, Q_B, Q_{\rm cv}$, and Q_Z for the tuned lossless straight-wire antenna is shown in Fig. 4, which demonstrates excellent agreement between the exact Q, the equivalent Q_B obtained from the exact bandwidth, and the approximate quality factor Q_Z obtained from the frequency derivative of the antenna's input impedance. These latter three quality factors are determined in significantly different manners, yet they remain in excellent agreement throughout the entire frequency range. Fig. 4 also reveals that the conventional approximation $Q_{\rm cv}$ to the quality factor, determined from the frequency derivative of the antenna's reactance, does not provide a reasonable estimate of the exact Q or the inverse of the antenna's exact matched VSWR bandwidth (Q_B) for frequencies about the natural antiresonance.

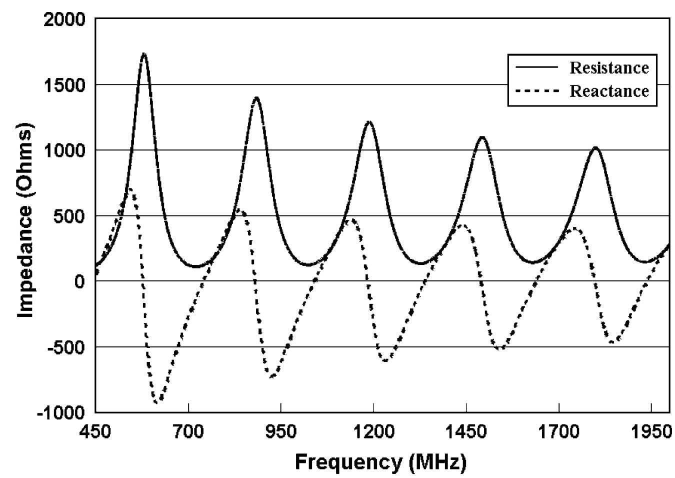

Fig. 5. Input impedance to higher frequencies of the center-fed, untuned, lossless straight-wire antenna having a total length of 1 m and a wire diameter of 1 mm.

Beyond the antenna's first natural resonant and antiresonant frequency ranges, the antenna's input impedance will undergo successive alternating regions of natural resonances and antiresonances, as seen in Fig. 5 for a frequency range of 450 MHz through 2000 MHz. At frequencies near the natural resonances, the antenna's input resistance is relatively low in value, while at frequencies near the natural antiresonances, it is relatively high in value. A comparison of the exact $Q, Q_B, Q_{\rm cv}$, and Q_Z for the tuned antenna over this frequency range is given in Fig. 6, where it can be seen that the values of exact Q, Q_B, and Q_Z remain in excellent agreement over the full frequency range. Fig. 6 reveals again, however, that the conventional approximate quality factor $Q_{\rm cv}$ does not provide an accurate estimate of the exact Q or inverse bandwidth Q_B in antiresonant frequency ranges.

Considering the form of the exact Q expression in (80), the reasonable agreement that exists between the exact Q and the conventional approximation $Q_{\rm cv}$ at low frequencies and in resonant frequency ranges, and the disagreement between the exact Q and $Q_{\rm cv}$ in antiresonant frequency ranges, we can conclude the following. At very low frequencies, where the antenna is

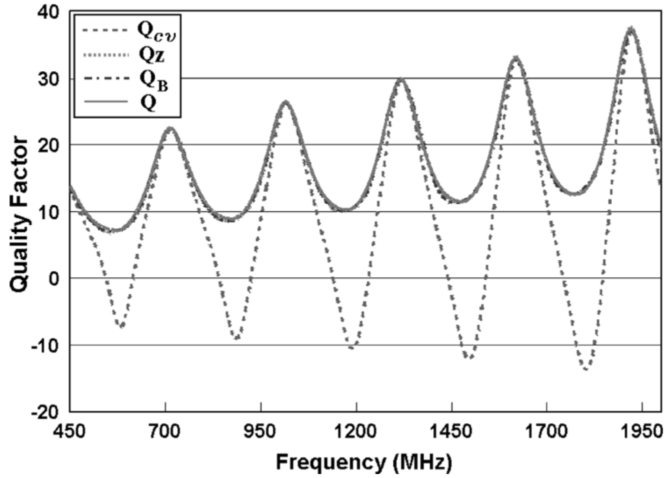

Fig. 6. Comparison of the $Q, Q_{\rm cv}, Q_Z$, and Q_B (1.5:1 matched VSWR bandwidth) for the center-fed, tuned, lossless straight-wire antenna over a wider range of frequencies.

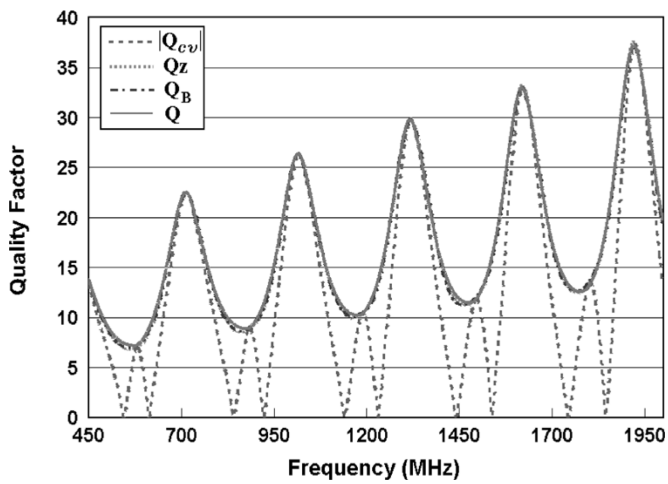

Fig. 7. Comparison of the $Q, |Q_{\rm cv}|, Q_Z$, and Q_B (1.5:1 matched VSWR bandwidth) for the center-fed, tuned, lossless straight-wire antenna.

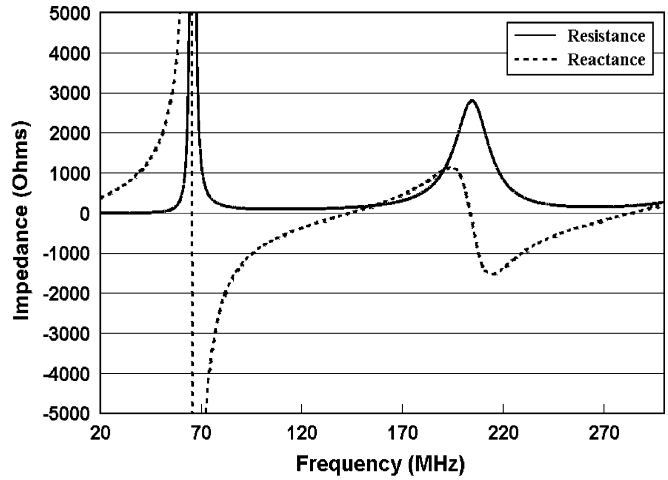

Fig. 8. Input impedance of the untuned, lossless circular-loop antenna with a radius of .348 m and a wire diameter of 1 mm.

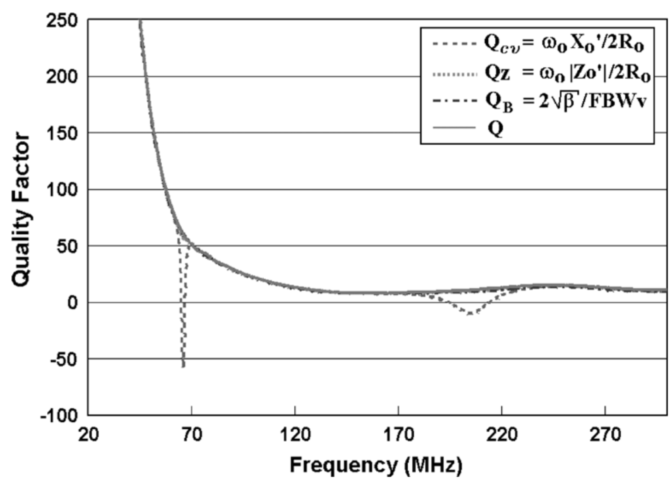

Fig. 9. Comparison of the $Q, Q_{\rm cv}, Q_Z$, and Q_B (1.5:1 matched VSWR bandwidth) for the tuned, lossless circular-loop antenna.

electrically small, and in resonant frequency ranges, the dominant factor in determining the quality factor of the tuned antenna is the frequency derivative of its input reactance $X_0'(\omega_0)$. This implies from (80) that the values of the dispersion energies $W_{\mathcal{L}}$ and $W_{\mathcal{R}}$ are close to zero in these resonant frequency regions. Also, in these resonant frequency regions, the value of $R_0'(\omega_0)$ is relatively small. In antiresonant frequency regions, the frequency derivative of the antenna's input reactance is less dominant, the magnitude of $R_0'(\omega_0)$ can be significant, and the values of the dispersion energies $W_{\mathcal{L}}$ and $W_{\mathcal{R}}$ become important in determining the quality factor of the antenna.

As mentioned in Section IV-C, Rhodes [13] has appropriately suggested that $|X_0'(\omega_0)|$ should be used instead of $X_0'(\omega_0)$ in the conventional approximation (81) for the quality factor in order to increase its accuracy in antiresonant frequency ranges. The approximate quality factor over the full range of frequencies was calculated using (81) with $X_0'(\omega_0)$ replaced by $|X_0'(\omega_0)|$. This approximate quality factor, which equals $|Q_{\rm cv}|$, is compared with the exact Q, Q_Z, and Q_B in Fig. 7. Other than right at the natural antiresonant frequencies of the antenna, $|Q_{\rm cv}|$ does not provide an accurate estimate of the exact Q or inverse bandwidth Q_B in antiresonant frequency ranges. Using $|X_0'(\omega_0)|$ in (81) provides an accurate approximation to the exact Q and inverse bandwidth right at the natural antiresonant frequencies of the antenna because at these frequencies, $R_0'(\omega_0) = 0$ and $|X_0'(\omega_0)| = |Z_0'(\omega_0)|$.

B. Bandwidth and Quality Factor of Lossless and Lossy Circular-Loop Antennas

The lossless and lossy circular wire-loop antennas considered here have a total wire length of approximately 2.18 m and a wire diameter of 1 mm such that the first natural resonant frequency of the circular-loop, that is, the first ω_0 where $X(\omega_0) = 0$ and $X'(\omega_0) > 0$, equals the first natural resonant frequency of the straight-wire antenna discussed above, namely, $f_0 = \omega_0/(2\pi) = 144$ MHz. The input impedance of the lossless circular-loop calculated with NEC is plotted in Fig. 8. One of the important differences to note in comparing the impedance of the circular-loop antenna to that of the straight-wire antenna is that the circular-loop antenna undergoes a natural antiresonance (at approximately 66 MHz) prior to the frequency where it undergoes its first natural resonance.

Fig. 9 compares the exact $Q, Q_B, Q_{\rm cv}$ and Q_Z for the lossless circular-loop antenna. Again the exact Q, the inverse of the exact matched VSWR bandwidth (Q_B), and the approximate

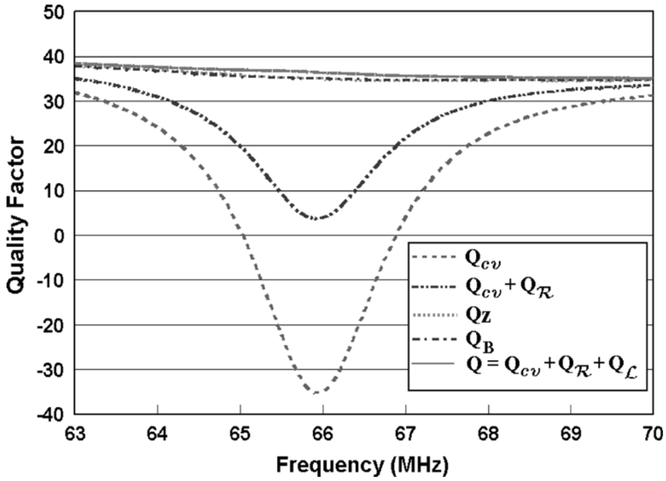

Fig. 10. Comparison of different methods for computing the quality factor in the first antiresonant region of the tuned, lossy circular-loop antenna with a radius of .348 m and a wire diameter of .5 mm.

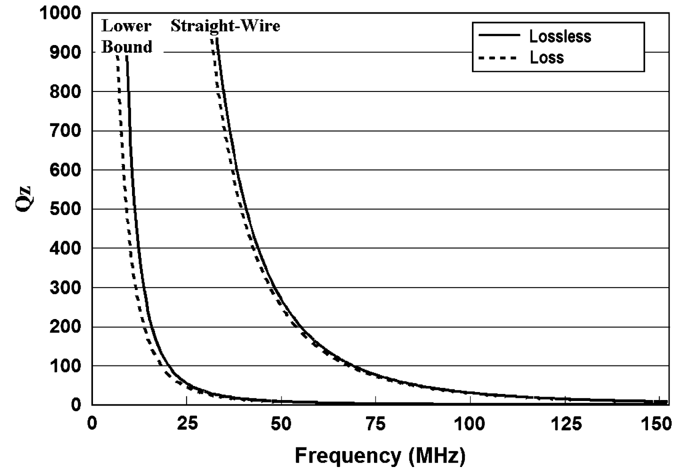

Fig. 11. Quality factors (as approximated by Q_Z) for the tuned, lossless and lossy straight-wire (wire diameter equal to .5 mm) antennas compared to the Collin-Rothschild lower bound for an electric-dipole antenna.

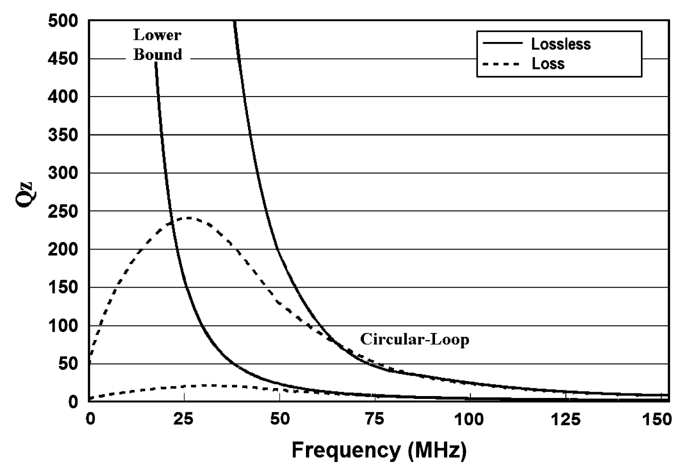

Fig. 12. Quality factors (as approximated by Q_Z) for the tuned, lossless and lossy circular-loop antennas (wire diameter equal to .5 mm) compared to the Collin-Rothschild lower bound for a magnetic-dipole antenna.

quality factor Q_Z computed from (96) are in excellent agreement over the whole frequency range. The conventional quality factor $Q_{\rm cv}$ computed from (81) gives accurate results at low frequencies and in resonant frequency ranges. However, it does not produce an accurate approximation to Q and inverse of bandwidth in antiresonant frequency ranges.

To demonstrate the significant contribution from the material-loss dispersion-energy term $W_{\mathcal{L}}$ to the exact Q of a lossy antenna, the quality factor and bandwidth of the lossy circular-loop antenna were computed. Loss was included in the NEC model of the circular wire-loop by specifying a finite copper-wire conductivity ($\sigma = 5.75 \times 10^7$ (Ohm-m)$^{-1}$). The wire diameter was reduced from 1 mm to 0.5 mm to increase the resistance of the wire. The exact Q, Q_B, and the approximate quality factors $Q_{\rm cv}$ and Q_Z computed for the lossy circular-loop antenna are presented in Fig. 10 in the frequency range around the first antiresonance. In Fig. 10, the terms in (80) comprising the expression for exact Q are shown separately to illustrate the significance of the $W_{\mathcal{L}}$ and $W_{\mathcal{R}}$ terms in calculating the exact Q. The conventional quality factor $Q_{\rm cv}$ equals the first term of the exact Q in (80). The curve labeled by $Q_{\rm cv} + Q_{\mathcal{R}}$ in Fig. 10 is a calculation of exact Q using only $Q_{\rm cv}$ and the far-field dispersion-energy term $Q_{\mathcal{R}} = -2\omega_0 W_{\mathcal{R}}/[|I_0|^2 R_0(\omega_0)]$. Note that the summation of these two terms does not give an accurate calculation of the exact Q. Once the material-loss dispersion-energy term $Q_{\mathcal{L}} = -2\omega_0 W_{\mathcal{L}}/[|I_0|^2 R_0(\omega_0)]$, as well as the far-field dispersion-energy term $Q_{\mathcal{R}}$, is included in the calculation of exact Q, close agreement is obtained with the inverse of the matched VSWR bandwidth and with the approximate quality factor Q_Z determined from (96).

In Figs. 11 and 12, the quality factors (as approximated by Q_Z) for the lossless and lossy straight-wire and circular wire-loop antennas, both having wire diameters of 0.5 mm, are compared to the Collin-Rothschild lower bounds on quality factor for the tuned electric or magnetic dipole antenna. This lower-bound quality factor $Q_{\rm lb}$ is given by [2], [4]

$$Q_{\rm lb} = \frac{1}{(ka)^3} + \frac{1}{ka} \tag{97}$$

where $k = 2\pi/\lambda$ is the free-space wave number and a is the radius of an imaginary sphere circumscribing the electrically small dipole antenna. For a lossy antenna, the lower bound in (97) is multiplied by the NEC-computed radiation efficiency η of each antenna [30]. It is obvious from Figs. 11 and 12 that the lower bounds on Q are dramatically lower than the actual Q for lossless and lossy straight-wire and circular wire-loop antennas at these frequencies below the first resonance or antiresonance. This discrepancy between the lower bound and actual quality factors implies that the contribution to the Q from the electric and magnetic fields inside the circumscribing sphere of radius a is the dominant contribution to the total Q even as, and especially as, the electrical size of the antennas becomes small.

C. Bandwidth and Quality Factor of Lossless Yagi Antenna

In Section IV-E, two techniques were described that allow us to reasonably remove the ambiguity in determining the exact Q that may arise from the value of $W_{\mathcal{R}}$ depending upon the chosen position of the origin of the coordinates with respect to the antenna. (This ambiguity does not exist for antennas having far-field magnitude patterns satisfying $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega = 0$.) To

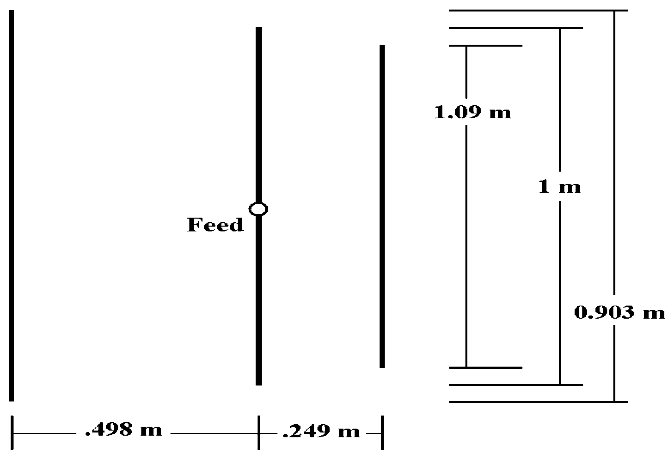

Fig. 13. Schematic of a 3-element, lossless Yagi antenna.

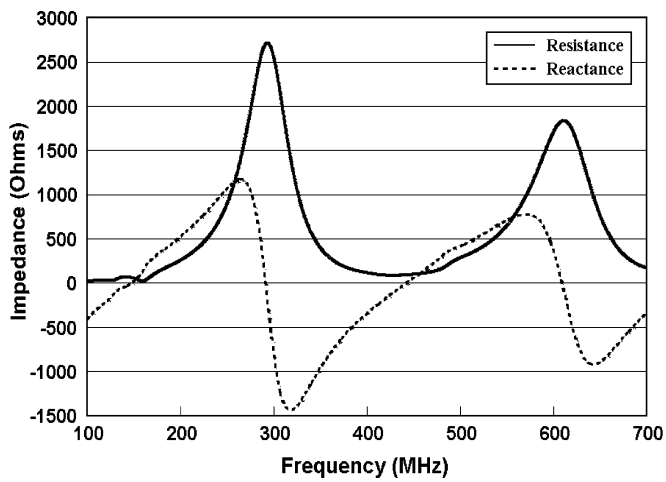

Fig. 14. Input impedance of the untuned, lossless, 3-element Yagi antenna.

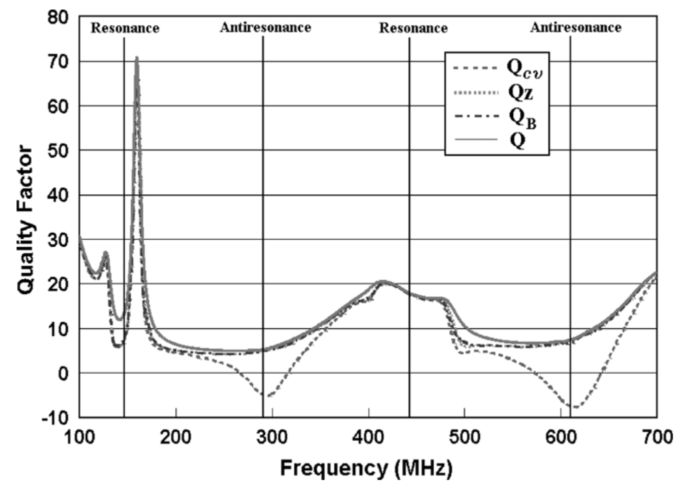

Fig. 15. Comparison of the $Q, Q_{\rm cv}, Q_Z$, and Q_B (1.5:1 matched VSWR bandwidth) for the tuned, lossless, 3-element Yagi antenna with the coordinate origin placed at the center of the driven element.

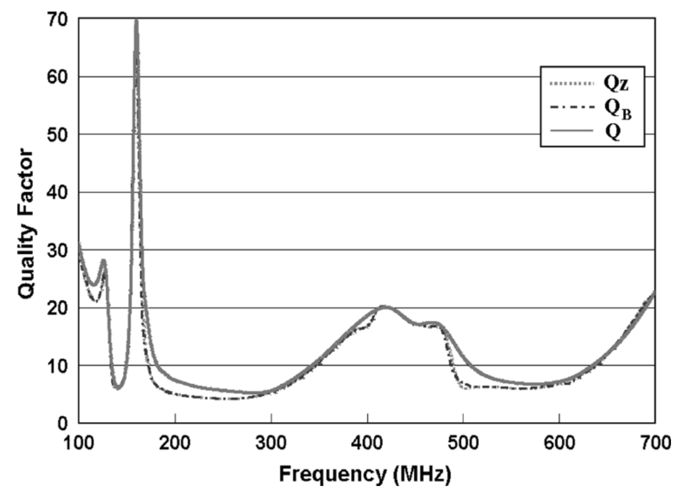

Fig. 16. Comparison of the Q, Q_Z, and Q_B (1.5:1 matched VSWR bandwidth) for the tuned, lossless, 3-element Yagi antenna with the coordinate origin shifted for each frequency by an amount determined from the shifts at the natural resonant and antiresonant frequencies.

illustrate the effectiveness of these techniques, the exact Q of a lossless Yagi antenna is determined and compared with the equivalent Q_B obtained from the inverse of the exact matched VSWR bandwidth. We emphasize, however, that one can approximate the exact bandwidth of a tuned antenna through all frequency ranges using (41), that is, using the inverse of Q_Z, if the impedance of the antenna is known, whether or not one computes an exact Q.

The Yagi antenna considered here consists of three perfectly conducting elements as shown in Fig. 13. Its untuned impedance is plotted in Fig. 14 over a frequency range that covers two natural resonant and two natural antiresonant frequencies. At frequencies near its first natural resonance, where the Yagi antenna is designed to operate, it has a directive radiation pattern and, as a result, $\int_{4\pi} \hat{\mathbf{r}}|\mathbf{F}|^2\, d\Omega$ is generally not equal to zero at these frequencies. For this reason, the exact Q determined from (80) might not accurately predict the inverse of the exact bandwidth of the antenna.

To illustrate these points the inverse of the exact bandwidth and the exact and approximate quality factors are plotted in Fig. 15. The center of the feed element is chosen as the coordinate origin for the calculation of the exact Q from (80). Fig. 15 shows that the approximate quality factor Q_Z determined from (96) and the Q_B determined in (95) from the inverse of the exact matched VSWR bandwidth are in excellent agreement at all frequencies, whereas $Q_{\rm cv}$ is inaccurate in the antiresonant regions. Comparing the exact Q with Q_B reveals that near the Yagi's first natural resonance, where it is designed to operate with a directive radiation pattern, the agreement is relatively poor.

As explained in Section IV-E, one can improve the agreement between the exact Q and the inverse of bandwidth Q_B by shifting the origin of the coordinate system with respect to the antenna to make $Q = |Q_{\rm cv}|$ at the natural resonant and antiresonant frequencies. Once the positions of the shifted origins are determined at the natural resonant and antiresonant frequencies, a linear interpolation between these shifted origins is performed to compute the appropriate shifted origins for frequencies between each natural resonance and antiresonance. The shifted origin for each natural resonant frequency is found by computing through trial and error the location of the coordinate origin that results in a calculated $W_{\mathcal{R}} = 0$. The shifted origin for each natural antiresonant frequency is found by computing the location of the coordinate origin that results in a calculated $W_{\mathcal{R}} = |I_0|^2 X'/2$. (Since the Yagi is lossless, $W_{\mathcal{L}} = 0$.) Fig. 16

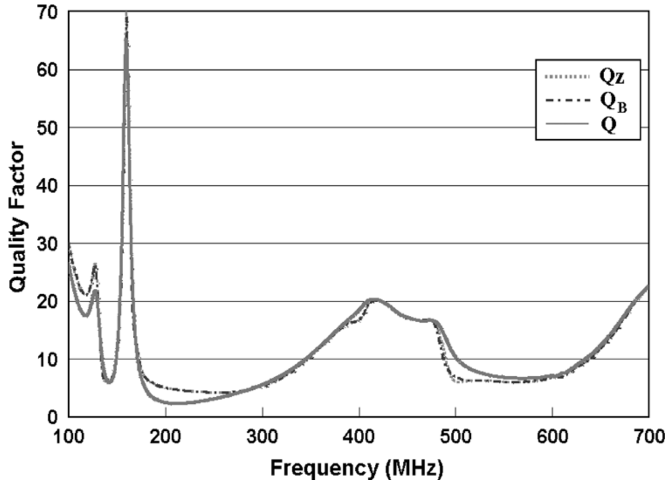

Fig. 17. Comparison of the Q, Q_Z, and Q_B (1.5:1 matched VSWR bandwidth) for the tuned, lossless, 3-element Yagi antenna with the coordinate origin placed at the center of the driven element, but with the exact Q at each frequency determined by interpolating between its values at the natural resonant and antiresonant frequencies.

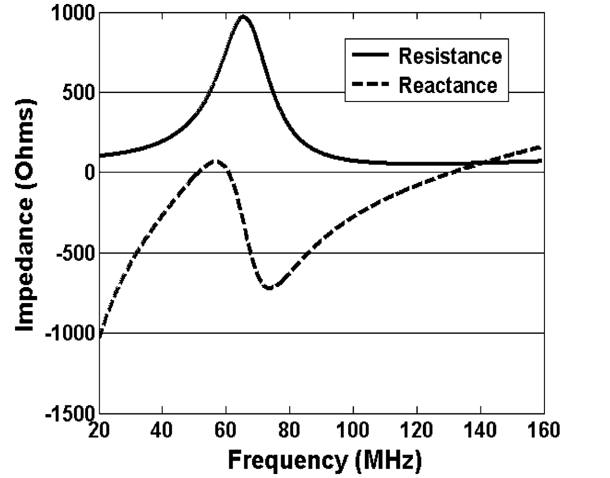

Fig. 18. Input impedance of the center-fed, untuned, lossless straight-wire antenna having a total length of 1 m and a wire diameter of 1 mm, and embedded in a lossy dispersive dielectric.

compares the exact Q computed with these shifted origins to Q_Z and Q_B. As one might expect, the major improvement in accuracy of the shifted-origin exact Q in Fig. 16 over the feed-element-origin exact Q in Fig. 15 occurs near the first natural resonance of the Yagi antenna.

The drawback of this shifted-origin technique is the large computer time required to calculate the frequency derivative of the far field of the antenna at each natural resonant and antiresonant frequency as the origin of the coordinates is shifted by trial and error to obtain the proper value of $W_{\mathcal{R}}$. As an alternative to this shifted-origin technique, we can first compute an initial exact Q using an origin near the center of the imaginary sphere that circumscribes the antenna. This exact Q will be calculated knowing that an ambiguity exists associated with the specified location of the coordinate origin. However, the ambiguity can be corrected at the natural resonant and antiresonant frequencies knowing $|Q_{\rm cv}|$ at each of these natural frequencies. If the differences between the exact Q calculated with the origin near the center of the circumscribing sphere and $|Q_{\rm cv}|$ are taken as corrections at the natural frequencies, we can interpolate these corrections between the natural resonant and antiresonant frequencies to arrive at a full set of corrections for all frequencies. This allows us to compute a corrected exact Q without having to determine the far fields and their frequency derivatives at each frequency for different coordinate origins. This interpolation technique was applied to the values of the exact Q initially calculated with the center of the feed element as the reference coordinate origin. Fig. 17 shows that the resulting interpolated exact Q compares favorably with Q_Z and Q_B as well as with the shifted-origin exact Q shown in Fig. 16.

D. Bandwidth and Quality Factor of a Straight-Wire Embedded in a Lossy Dispersive Dielectric

Our definitions of internal energies in (67a)–(67c) or (72), and thus Q include terms involving the frequency derivatives of the constitutive parameters. To confirm that these derivative terms should indeed be included as part of the energy used to define Q, we embed the lossless, center-fed, straight-wire antenna described in Section VI-A in a lossy dispersive dielectric material with Lorentz permittivity given by

$$\begin{aligned}\epsilon(\omega) &= \epsilon_r(\omega) - j\epsilon_i(\omega)\\ &= \epsilon_0\left[(\kappa_L+1)+\frac{\chi}{1-(\omega/\omega_L)^2+2j\xi(\omega/\omega_L)}\right]\end{aligned} \tag{98}$$

for ω through the first resonant frequency of the antenna, where the electric susceptibility constant $\chi = 1.2$, the offset relative permittivity constant $\kappa_L = .5$, the loss constant $\xi = .2$, and the Lorentz antiresonant frequency $\omega_L = (2\pi)50 \times 10^6$ Hz. For frequencies $f \lesssim 160$ MHz, we can model this embedded antenna by a constant inductance ($L = .45 \times 10^{-6}$ henrys) in series with a frequency dependent "radiation" resistance ($R_\epsilon = Z_f\omega^2\sqrt{(\epsilon_r/\epsilon_0}/(24\pi c^2)$ Ohms) and a lossy capacitance ($C = 2.5\times 10^{-12}\epsilon_r/\epsilon_0$ farads, $R_C = .4\times 10^{12}\epsilon_0/(\omega\epsilon_i)$ Ohms). The impedance $R(\omega)$ and $X(\omega)$ of the untuned embedded antenna is shown in Fig. 18, which agrees closely with the impedance (not shown) computed with the NEC code. The efficiency ($\eta = R_\epsilon/R$) of this antenna is less than 5% for frequencies less than 80 MHz and thus it is not a practical antenna throughout about the first half of the frequency range shown in Fig. 18.

Fig. 19 demonstrates the close agreement between the inverse of the exact bandwidth Q_B and the approximation Q_Z for the inverse of the bandwidth of the tuned antenna, as well as the failure in the antiresonant region of the conventional expression $Q_{\rm cv}$ for the quality factor of the tuned antenna. In the frequency range from about 30 to 70 MHz, the exact Q does not agree well with the inverse of the exact bandwidth Q_B (or with the approximation Q_Z) because the antenna material is both highly lossy and dispersive—so dispersive, in fact, that the value of $W(\omega_0)$ can become negative to make $Q(\omega_0)$ equal to zero at frequencies near 40 MHz and 60 MHz. Thus, as pointed out in Section IV-D, one would not expect the exact Q to be a highly accurate approximation (in this frequency range) to the inverse of the exact bandwidth. Most noteworthy in Fig. 19 is the quality factor $Q - \Delta Q$ where

$$\Delta Q = \frac{\omega_0^2\epsilon_r'(\omega_0)}{2|I_0^2|R_0(\omega_0)}\int\limits_{\rm dielectric}|E|^2\,d\mathcal{V} \tag{99}$$

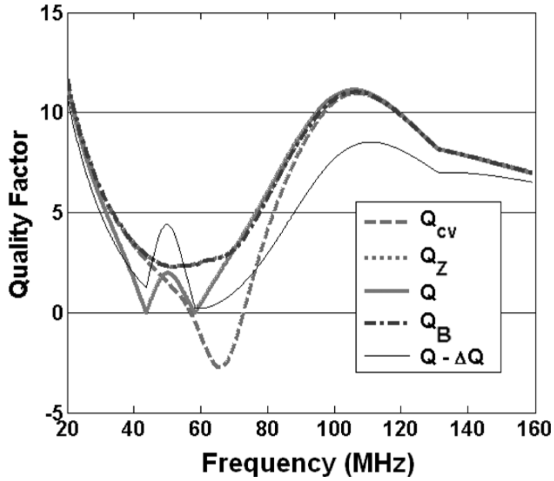

Fig. 19. Comparison of the exact $Q, Q_{\rm cv}, Q_Z, Q_B$ (1.5:1 matched VSWR bandwidth), and $Q - \Delta Q$ for the center-fed, tuned, lossless straight-wire antenna embedded in a lossy dispersive dielectric.

which is the amount that the Q is reduced by the omission of the ϵ_r' derivative term in the first integral of (72b). Fig. 19 shows that the omission of this ΔQ would produce a much less accurate value of the exact quality factor over the frequency range where the ϵ_r' term contributes significantly to $W(\omega_0)$.

VII. Conclusion

The input reactance of a general one-port linear antenna can vary over a large range of negative and positive values as the frequency of the antenna sweeps through successive natural resonances and antiresonances of the antenna. If, however, the antenna's input reactance is tuned to a value of zero at any frequency ω_0 by means of a series inductor or capacitor, a matched VSWR fractional bandwidth of the antenna can be defined that exists at every frequency ω_0. Moreover, this fractional bandwidth, $\mathrm{FBW}_V(\omega_0)$, is given approximately by the simple formula in (41) for any frequency ω_0 at which the antenna is tuned.

The internal energy $W(\omega_0)$ of the same general one-port linear antenna tuned at a frequency ω_0 is defined in (66)–(67) in terms of an integration of the electric and magnetic fields of the antenna. This internal energy excludes the radiation fields but includes terms involving the frequency derivative of the constitutive parameters. In defining $Q(\omega_0)$ as $\omega_0|W(\omega_0)|/P_A(\omega_0)$ in (78), the inclusion of these frequency-derivative terms allows $Q(\omega_0)$ to remain inversely proportional to the fractional bandwidth $\mathrm{FBW}_V(\omega_0)$, as expressed in (87), for antennas that contain materials with frequency dependent constitutive parameters; see Figs. 18 and 19. Although it is not impossible for $W(\omega_0)$ to have a negative value, and the values of the constitutive parameters may even be negative, it is proven that the internal energy density in lossless antenna material is always greater than or equal to zero; see (69).

For antennas with asymmetric far-field magnitude patterns, the value of the internal energy $W(\omega_0)$ and thus $Q(\omega_0)$ will depend upon the position of the origin of the coordinate system with respect to the antenna (because of the radiation fields that are subtracted to yield a finite value for the internal energy). A reasonable choice for the origin of the coordinate system is the center of the sphere that circumscribes the dominant sources of the antenna. Nonetheless, a simple procedure is given at the end of Section IV-E for eliminating this ambiguity and determining a well-defined "exact value" of $Q(\omega_0)$.

Although the value of $W(\omega_0)$ and thus $Q(\omega_0)$ can be determined, in principle, by integrating the electric and magnetic fields of the antenna throughout all space, it may be prohibitive in numerical practice to evaluate the exact $Q(\omega_0)$ by means of this direct volume integration. Therfore, an alternative expression for the exact $Q(\omega_0)$ is given in (80) in terms of the frequency derivative of the input reactance of the antenna and two "dispersion energies." One of the dispersion energies $(W_{\mathcal{R}}(\omega_0))$ requires an integration of the far-field and the other $(W_{\mathcal{L}}(\omega_0))$ requires an integration of the fields over the portions of the antenna material that exhibit loss. Each of these integrations is much less demanding than integrating the fields of the antenna over all space to compute the exact $Q(\omega_0)$.

In Section VI-D, this alternative expression in (80) for the exact $Q(\omega_0)$ is evaluated numerically for straight-wire and wire-loop lossy and lossless antennas, as well as for a Yagi antenna and a straight-wire antenna embedded in a lossy dispersive dielectric, over frequency ranges that cover several resonant and antiresonant frequency bands. In all cases, except for the last antenna in a frequency range where the efficiency was less than 5% (rendering the antenna impractical in this frequency range), the exact Q agreed closely with the inverse of the exact computed bandwidth and with the approximate formula for the quality factor and inverse bandwidth given in (87). In fact, all our comparisons to date for practical antennas indicate that the simple approximation, $\omega_0|Z_0'(\omega_0)|/[2R_0(\omega_0)]$, in (87) for $Q(\omega_0)$ and the inverse of the matched VSWR bandwidth is so accurate that it makes the evaluation of the exact Q and exact bandwidth practically unnecessary unless the frequency derivative of the input impedance of the antenna, specifically $|Z_0'(\omega_0)|$, is not readily computable.

Appendix A
Derivation of Expression in (64) for X_0'

To derive the expression (64) for $X_0'(\omega_0)$, begin by taking the frequency derivative of Maxwell's equations

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}, \quad \nabla \times \mathbf{H} = j\omega\boldsymbol{D} \tag{A.1}$$

to get

$$\nabla \times \mathbf{E}' = -j\mathbf{B} - j\omega\mathbf{B}', \quad \nabla \times \mathbf{H}' = j\boldsymbol{D} + j\omega\boldsymbol{D}'. \tag{A.2}$$

Scalar multiply the first equation in (A.2) by $\mathbf{H}^*$ and the complex conjugate of the second equation in (A.1) by $\mathbf{E}'$, then subtract the two resulting equations to get

$$\nabla \cdot (\mathbf{E}' \times \mathbf{H}^*) = -j\mathbf{B}\cdot\mathbf{H}^* - j\omega(\mathbf{B}'\cdot\mathbf{H}^* - \boldsymbol{D}^*\cdot\mathbf{E}'). \tag{A.3}$$

Similarly, scalar multiply the complex conjugate of the second equation in (A.2) by $\mathbf{E}$ and the first equation in (A.1) by $\mathbf{H}'^*$, then subtract the two resulting equations to get

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}'^*) = j\boldsymbol{D}^*\cdot\mathbf{E} - j\omega(\mathbf{B}\cdot\mathbf{H}'^* - \boldsymbol{D}'^*\cdot\mathbf{E}). \tag{A.4}$$

Subtracting (A.3) from (A.4) yields

$$\begin{aligned}\nabla \cdot (\mathbf{E} \times \mathbf{H}'^* - \mathbf{E}' \times \mathbf{H}^*) &= j(\mathbf{B}\cdot\mathbf{H}^* + \boldsymbol{D}^*\cdot\mathbf{E}) \\ &\quad + j\omega(\mathbf{B}'\cdot\mathbf{H}^* - \mathbf{B}\cdot\mathbf{H}'^* + \boldsymbol{D}'^*\cdot\mathbf{E} - \boldsymbol{D}^*\cdot\mathbf{E}'). \end{aligned} \tag{A.5}$$

Integrate (A.5) over the volume $\mathcal{V}_o(r)$ between the shielded power supply and a sphere of radius r that surrounds the antenna system, apply the divergence theorem, and take the limit as $r \to \infty$ to get

$$\begin{aligned} V_0' I_0^* - V_0 I_0'^* &= \lim_{r\to\infty}\left[\int_{\mathcal{V}_o(r)} j[(\mathbf{B}\cdot\mathbf{H}^* + \boldsymbol{D}^*\cdot\mathbf{E}) \right.\\ &\quad + \omega(\mathbf{B}'\cdot\mathbf{H}^* - \mathbf{B}\cdot\mathbf{H}'^* + \boldsymbol{D}'^*\cdot\mathbf{E} - \boldsymbol{D}^*\cdot\mathbf{E}')]\,d\mathcal{V}\\ &\quad \left. - r^2\int_{4\pi}(\mathbf{E}\times\mathbf{H}'^* - \mathbf{E}'\times\mathbf{H}^*)\cdot\hat{\mathbf{r}}\,d\Omega\right]. \end{aligned} \tag{A.6}$$

The left-hand side of (A.6) results from (13)–(14) and the fact mentioned in Section II that one of the basis fields in the feed line of the antenna can always be made independent of frequency. This implies from the frequency derivative of (14) that

$$\int_{S_0}[\mathbf{e}_0'(\boldsymbol{\rho})\times\mathbf{h}_0(\boldsymbol{\rho})]\cdot\hat{\mathbf{n}}_0 dS = \int_{S_0}[\mathbf{e}_0(\boldsymbol{\rho})\times\mathbf{h}_0'(\boldsymbol{\rho})]\cdot\hat{\mathbf{n}}_0\,dS = 0. \tag{A.7}$$

If the frequency derivatives in (A.6) are taken while holding I_0 constant with frequency, $I_0'^* = 0$ on the left-hand side of (A.6). Then substituting $[V_0' = (R_0' + jX_0')I_0]_{I_0=\mathrm{constant}}$ and taking the imaginary part of (A.6) produces

$$\begin{aligned} |I_0|^2 X_0'(\omega) &= \lim_{r\to\infty}\left[\int_{\mathcal{V}_o(r)} \mathrm{Re}\left[(\mathbf{B}\cdot\mathbf{H}^* + \boldsymbol{D}^*\cdot\mathbf{E})\right.\right.\\ &\quad \left. + \omega\left(\mathbf{B}_{I_0}'\cdot\mathbf{H}^* - \mathbf{B}\cdot\mathbf{H}_{I_0}'^* + \boldsymbol{D}_{I_0}'^*\cdot\mathbf{E} - \boldsymbol{D}^*\cdot\mathbf{E}_{I_0}'\right)\right]\,d\mathcal{V}\\ &\quad \left. - r^2\mathrm{Im}\int_{4\pi}\left(\mathbf{E}\times\mathbf{H}_{I_0}'^* - \mathbf{E}_{I_0}'\times\mathbf{H}^*\right)\cdot\hat{\mathbf{r}}\,d\Omega\right]. \end{aligned} \tag{A.8}$$

Taking the frequency derivative of $\lim_{r\to\infty} r^2 \int_{4\pi}(\mathbf{E}\times\mathbf{H}^*)\cdot\hat{\mathbf{r}}\,d\Omega$, which is real, shows that

$$\lim_{r\to\infty} r^2\mathrm{Im}\int_{4\pi}\left(\mathbf{E}\times\mathbf{H}_{I_0}'^* + \mathbf{E}_{I_0}'\times\mathbf{H}^*\right)\cdot\hat{\mathbf{r}}\,d\Omega = 0 \tag{A.9}$$

and thus (A.8) can be rewritten as

$$\begin{aligned} |I_0|^2 X_0'(\omega) &= \lim_{r\to\infty}\left[\mathrm{Re}\int_{\mathcal{V}_o(r)}(\mathbf{B}\cdot\mathbf{H}^* + \boldsymbol{D}^*\cdot\mathbf{E})\,d\mathcal{V}\right.\\ &\quad \left. + 2r^2\mathrm{Im}\int_{4\pi}(\mathbf{E}_{I_0}'\times\mathbf{H}^*)\cdot\hat{\mathbf{r}}\,d\Omega\right]\\ &\quad + \omega\mathrm{Re}\int_{\mathcal{V}_a}\left(\mathbf{B}_{I_0}'\cdot\mathbf{H}^* - \mathbf{B}\cdot\mathbf{H}_{I_0}'^*\right.\\ &\quad \left. + \boldsymbol{D}_{I_0}'^*\cdot\mathbf{E} - \boldsymbol{D}^*\cdot\mathbf{E}_{I_0}'\right)\,d\mathcal{V} \end{aligned} \tag{A.10}$$

where we have used the fact that the real part of the last integrand in (A.10) is zero for $\mathbf{r}$ in the part of $\mathcal{V}_o(r)$ that lies outside the antenna material (that is, outside $\mathcal{V}_a$).

Lastly, we evaluate the $d\Omega$ integral in (A.10) in terms of the far electric field pattern $\mathbf{F}(\theta,\phi)$ defined in (20) by expanding $\mathbf{E}$ and $\mathbf{H}$ in a Wilcox series [31]

$$\mathbf{E}(r,\theta,\phi) = \left[\mathbf{F}(\theta,\phi) + \frac{\mathbf{G}(\theta,\phi)}{r} + \mathbf{O}(1/r^2)\right]\frac{e^{-jkr}}{r} \tag{A.11}$$

$$\begin{aligned} \mathbf{H}(r,\theta,\phi) &= \frac{j}{\omega\mu_0}\nabla\times\mathbf{E}\\ &= \frac{j}{\omega\mu_0}\left[-jk\hat{\mathbf{r}}\times\mathbf{F} + \nabla\times\mathbf{F} - \frac{\hat{\mathbf{r}}\times\mathbf{F}}{r}\right.\\ &\quad \left. - jk\frac{\hat{\mathbf{r}}\times\mathbf{G}}{r} + \mathbf{O}(1/r^2)\right]\frac{e^{-jkr}}{r} \end{aligned} \tag{A.12}$$

and by taking the derivative with respect to frequency of $\mathbf{E}$ in (A.11) to get

$$\mathbf{E}'(r,\theta,\phi) = \left[-\frac{j}{c}r\mathbf{F} + \mathbf{F}' - \frac{j}{c}\mathbf{G} + \mathbf{O}(1/r)\right]\frac{e^{-jkr}}{r} \tag{A.13}$$

where c is the speed of light in free space. (Recall that $k = \omega/c$.) Crossing $\mathbf{E}'$ from (A.13) into $\mathbf{H}^*$ obtained from (A.12) and using a number of vector identities gives

$$\begin{aligned} &(\mathbf{E}'\times\mathbf{H}^*)\cdot\hat{\mathbf{r}}\\ &= -\frac{j\epsilon_0}{r^2}[r|\mathbf{F}|^2 + 2\mathrm{Re}(\mathbf{F}\cdot\mathbf{G}^*) + jc\mathbf{F}'\cdot\mathbf{F}^* + \mathbf{O}(1/r)] \end{aligned} \tag{A.14}$$

which when substituted into (A.10) yields

$$\begin{aligned} |I_0|^2 X_0'(\omega) &= \lim_{r\to\infty}\left[\mathrm{Re}\int_{\mathcal{V}_o(r)}(\mathbf{B}\cdot\mathbf{H}^* + \boldsymbol{D}^*\cdot\mathbf{E})\,d\mathcal{V} - 2\epsilon_0 r\int_{4\pi}|\mathbf{F}|^2\,d\Omega\right]\\ &\quad + \omega\mathrm{Re}\int_{\mathcal{V}_a}\left(\mathbf{B}_{I_0}'\cdot\mathbf{H}^* - \mathbf{B}\cdot\mathbf{H}_{I_0}'^* + \boldsymbol{D}_{I_0}'^*\cdot\mathbf{E} - \boldsymbol{D}^*\cdot\mathbf{E}_{I_0}'\right)d\mathcal{V}\\ &\quad + \frac{2}{Z_f}\mathrm{Im}\int_{4\pi}\mathbf{F}_{I_0}'\cdot\mathbf{F}^* d\Omega. \end{aligned} \tag{A.15}$$

an expression equal to (64) if $\omega = \omega_0$.

The term $\mathrm{Re}(\mathbf{F}\cdot\mathbf{G}^*)$ in (A.14) does not appear in (A.15) because it integrates to zero. This can be proven by expanding the fields of the antenna (outside an enclosing sphere) in a complete set of vector spherical wave functions [32, Secs. 7.11–7.14], [33, ch. 9] (see Appendix C) and noting that each of the spherical modes has a $\mathbf{G}(\theta,\phi)$ that is 90 degrees out of phase with $\mathbf{F}(\theta,\phi)$. Therefore, for each vector spherical mode, $\mathrm{Re}\int_{4\pi}\mathbf{F}\cdot\mathbf{G}^* d\Omega = 0$. Orthogonality of the vector spherical modes then demands that this result also holds for the complete sum of vector spherical modes, that is, for the total fields of the antenna. Specifically

$$r^2\int_{4\pi}\mathbf{E}\cdot\mathbf{E}^*\,d\Omega = \int_{4\pi}[|\mathbf{F}|^2 + 2\mathrm{Re}(\mathbf{F}\cdot\mathbf{G}^*)/r + \mathbf{O}(1/r^2)]\,d\Omega \tag{A.16}$$

and

$$r^2 \int_{4\pi} \mathbf{E}\cdot\mathbf{E}^*\, d\Omega = r^2 \int_{4\pi} \sum_{l=1}^{\infty}\sum_{m=-l}^{l} [|A_{lm}|^2 \mathbf{M}_{lm}\cdot\mathbf{M}_{lm}^* + |B_{lm}|^2 \mathbf{N}_{lm}\cdot\mathbf{N}_{lm}^*]\, d\Omega$$
$$= \int_{4\pi} \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \left[|A_{lm}|^2 \left(|\mathbf{F}_{lm}^M|^2 + \mathbf{O}(1/r^2)\right) + |B_{lm}|^2 \left(|\mathbf{F}_{lm}^N|^2 + \mathbf{O}(1/r^2)\right)\right] d\Omega \quad \text{(A.17)}$$

where in (A.17) we have used the asymptotic property, $\mathrm{Re}[\mathbf{F}_{lm}^{\{M,N\}}(\theta,\phi)\cdot\mathbf{G}_{lm}^{\{M,N\}*}(\theta,\phi)] = 0$, of the fields of vector spherical modes $[\mathbf{M}_{lm}(r,\theta,\phi), \mathbf{N}_{lm}(r,\theta,\phi)]$ for each degree l and order m. Equating the $1/r$ terms on the right-hand sides of (A.16) and (A.17) proves that $\mathrm{Re}\int_{4\pi}\mathbf{F}(\theta,\phi)\cdot\mathbf{G}^*(\theta,\phi)\, d\Omega = 0$ for the total fields of the antenna.

APPENDIX B
PROOF THAT LOSSLESS, SUSCEPTIBILITY AND INTERNAL ENERGIES ARE ≥ 0

The power per unit volume supplied to a macroscopic distribution of time dependent current $\boldsymbol{\mathcal{J}}(\mathbf{r},t)$ and polarization $[\boldsymbol{\mathcal{P}}(\mathbf{r},t), \boldsymbol{\mathcal{M}}(\mathbf{r},t)]$ densities by electromagnetic fields $[\boldsymbol{\mathcal{E}}(\mathbf{r},t), \boldsymbol{\mathcal{H}}(\mathbf{r},t)]$ is given by [12, eq. (2.174)]

$$P_{el}(\mathbf{r},t) = \left(\boldsymbol{\mathcal{J}} + \frac{\partial\boldsymbol{\mathcal{P}}}{\partial t}\right)\cdot\boldsymbol{\mathcal{E}} + \mu_0\frac{\partial\boldsymbol{\mathcal{M}}}{\partial t}\cdot\boldsymbol{\mathcal{H}} \quad \text{(B.1)}$$

or, since $\partial\boldsymbol{\mathcal{D}}/\partial t = \epsilon_0\partial\boldsymbol{\mathcal{E}}/\partial t + \partial\boldsymbol{\mathcal{P}}/\partial t + \boldsymbol{\mathcal{J}}$ and $\partial\boldsymbol{\mathcal{B}}/\partial t = \mu_0(\partial\boldsymbol{\mathcal{H}}/\partial t + \partial\boldsymbol{\mathcal{M}}/\partial t)$

$$P_{el}(\mathbf{r},t) = \frac{\partial\boldsymbol{\mathcal{D}}}{\partial t}\cdot\boldsymbol{\mathcal{E}} + \frac{\partial\boldsymbol{\mathcal{B}}}{\partial t}\cdot\boldsymbol{\mathcal{H}} - \frac{1}{2}\frac{\partial}{\partial t}(\epsilon_0|\boldsymbol{\mathcal{E}}|^2 + \mu_0|\boldsymbol{\mathcal{H}}|^2). \quad \text{(B.2)}$$

Expressing this electromagnetic power as a time derivative at each fixed $\mathbf{r}$, that is, $P_{el}(\mathbf{r},t) = \partial W_{el}(\mathbf{r},t)/\partial t$, integrating from the remote past $(-\infty)$ to the present time t (at each $\mathbf{r}$), and inserting P_{el} from (B.2) gives

$$W_{el}(\mathbf{r},t) - W_0 = \int_{-\infty}^{t}\left(\frac{\partial\boldsymbol{\mathcal{D}}}{\partial t'}\cdot\boldsymbol{\mathcal{E}} + \frac{\partial\boldsymbol{\mathcal{B}}}{\partial t'}\cdot\boldsymbol{\mathcal{H}}\right)dt' - \frac{1}{2}(\epsilon_0|\boldsymbol{\mathcal{E}}(\mathbf{r},t)|^2 + \mu_0|\boldsymbol{\mathcal{H}}(\mathbf{r},t)|^2) \quad \text{(B.3)}$$

where W_0 can be considered equal to the rest energy of the carriers (charged and polarized particles or molecules) of $\boldsymbol{\mathcal{J}}, \boldsymbol{\mathcal{P}}$, and $\boldsymbol{\mathcal{M}}$ in the remote past when these carriers are at rest with $(\boldsymbol{\mathcal{J}}, \boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{M}}) = 0$ and the fields are assumed equal to zero; that is, $W_{el}(\mathbf{r},-\infty) = W_0$ and $[\boldsymbol{\mathcal{E}}(\mathbf{r},-\infty), \boldsymbol{\mathcal{H}}(\mathbf{r},-\infty)] = 0$.

In general, even assuming a passive medium, $W_{el}(\mathbf{r},t) - W_0$ is not equal to the per unit volume reversible kinetic and potential energy of the carriers of $\boldsymbol{\mathcal{J}}, \boldsymbol{\mathcal{P}}$, and $\boldsymbol{\mathcal{M}}$ that reside at the position $\mathbf{r}$ and time t plus the irreversible energy lost by the carriers to frictional forces (ohmic losses) and radiation, for three reasons: 1) The carriers drift so that the ones that reside at $\mathbf{r}$ at some time t_1 are not the same carriers that reside at $\mathbf{r}$ at another time t_2; 2) the kinetic energy of the carriers can be transferred through collisions with each other and with the material lattice; and 3) the total electromagnetic force-power on classical moving charges and dipoles includes the self electromagnetic force-power on these charges and dipoles and thus the total electromagnetic force-power cannot, in general, be equated to the time rate of change of the mechanical (kinetic plus potential) momentum-energy of these particles [34, Sec. 5.1].

In a spatially nondispersive macroscopic distribution of $\boldsymbol{\mathcal{J}}, \boldsymbol{\mathcal{P}}$, and $\boldsymbol{\mathcal{M}}$, the charges and dipoles in each differential volume element $d\mathcal{V}$ are assumed unconnected to the charges and dipoles in all the other differential volume elements. Thus, the self electromagnetic force-power approaches zero faster than $d\mathcal{V}$ [12, p. 46] and reason 3) is not an issue. Moreover, the energy radiated by a volume element $d\mathcal{V}$ of macroscopic sources approaches zero faster than $d\mathcal{V}$. If, in addition, the conduction current $\boldsymbol{\mathcal{J}}$ is zero, we can assume a macroscopic model of the medium in which the carriers are "bound" by infinitesimal restoring "springs" (which can be lossy) to a rigid lattice such that the carrier drift is negligible. Furthermore, we can assume a model in which the bound carriers do not collide (or collide only with a rigid lattice) and, thus, reason 2) does not apply. Consequently, for a macroscopic distribution of bound $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{M}}$ in a passive, spatially nondispersive medium, $W_{el}(\mathbf{r},t) - W_0$ represents the reversible kinetic and potential energy plus the irreversible frictional energy loss (in the "springs") per unit volume at $(\mathbf{r},t)$ for a model with collisionless carriers bound by lossy "springs", under the assumption that this energy is zero in the remote past $(t = -\infty)$).

Since $W_{el}(\mathbf{r},t) - W_0$ begins at a value of zero and equals (for this model) the total reversible energy change plus frictional energy loss per unit volume of the carriers, $W_{el}(\mathbf{r},t) - W_0$ can never be negative, that is, $W_{el}(\mathbf{r},t) - W_0 \geq 0$ in this passive, spatially nondispersive medium of bound carriers and we can conclude from (B.3) that

$$W_{el}(\mathbf{r},t) - W_0 = \int_{-\infty}^{t}\left(\frac{\partial\boldsymbol{\mathcal{D}}}{\partial t'}\cdot\boldsymbol{\mathcal{E}} + \frac{\partial\boldsymbol{\mathcal{B}}}{\partial t'}\cdot\boldsymbol{\mathcal{H}}\right)dt' - \frac{1}{2}(\epsilon_0|\boldsymbol{\mathcal{E}}(\mathbf{r},t)|^2 + \mu_0|\boldsymbol{\mathcal{H}}(\mathbf{r},t)|^2) \geq 0 \quad \text{(B.4)}$$

for all $(\mathbf{r},t)$. We have not found this inequality (B.4) stated or proven elsewhere in the published literature, although Tonning [35] concludes that the integral alone in (B.4) is the electromagnetic energy absorbed per unit volume in a "reversible" lossless medium.

We shall now use the inequality in (B.4) to prove the inequalities in (69)–(70) of the main text *for a linear, passive, lossless, spatially nondispersive medium.* To do this first write the integrand of (B.4) as $[\partial\boldsymbol{\mathcal{F}}/\partial t']\cdot\boldsymbol{\mathcal{G}}$, where $\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{B}})$ and $\boldsymbol{\mathcal{G}} = (\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}})$ are six component vectors. Then the constitutive relations in (51) can be rewritten as simply $\mathbf{F} = \bar{\boldsymbol{\gamma}}\cdot\mathbf{G}$, where $\mathbf{F} = (\mathbf{D}, \mathbf{B})$ and $\mathbf{G} = (\mathbf{E}, \mathbf{H})$ are the corresponding frequency-domain fields and

$$\bar{\boldsymbol{\gamma}} = \begin{bmatrix} \bar{\boldsymbol{\epsilon}} & \bar{\boldsymbol{\tau}} \\ \bar{\boldsymbol{\nu}} & \bar{\boldsymbol{\mu}} \end{bmatrix}. \quad \text{(B.5)}$$

In a passive lossless medium (57) holds and can be rewritten as $\bar{\boldsymbol{\gamma}} = \bar{\boldsymbol{\gamma}}_t^*$. The energy inequality in (B.4)[3] can be rewritten in terms of $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}$ as (after integration by parts)

$$W_{el}(\mathbf{r},t) - W_0 = -\int_{-\infty}^{t} \boldsymbol{\mathcal{F}} \cdot \frac{\partial \boldsymbol{\mathcal{G}}}{\partial t'}\, dt' + \boldsymbol{\mathcal{F}}(t) \cdot \boldsymbol{\mathcal{G}}(t) - \frac{1}{2}[\boldsymbol{\mathcal{G}}(t) \cdot \bar{\mathbf{I}}_{\epsilon\mu} \cdot \boldsymbol{\mathcal{G}}(t)] \geq 0 \quad \text{(B.6a)}$$

with the 6×6 dyadic $\bar{\mathbf{I}}_{\epsilon\mu}$ given by

$$\bar{\mathbf{I}}_{\epsilon\mu} = \epsilon_0(\hat{\mathbf{x}}_1\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_3\hat{\mathbf{x}}_3) + \mu_0(\hat{\mathbf{x}}_4\hat{\mathbf{x}}_4 + \hat{\mathbf{x}}_5\hat{\mathbf{x}}_5 + \hat{\mathbf{x}}_6\hat{\mathbf{x}}_6). \quad \text{(B.6b)}$$

Next choose the time dependence of $\boldsymbol{\mathcal{G}}$ as

$$\begin{aligned} \boldsymbol{\mathcal{G}} &= \boldsymbol{\mathcal{G}}_1 \begin{Bmatrix} e^{\alpha t}\sin\omega t \\ \sin\omega t \end{Bmatrix} + \boldsymbol{\mathcal{G}}_2 \begin{Bmatrix} e^{\alpha t}\cos\omega t \\ \cos\omega t \end{Bmatrix}, \quad \begin{matrix} -\infty < t \leq 0 \\ 0 \leq t < +\infty \end{matrix} \\ &= [e^{\alpha t}u(-t) + u(t)](\boldsymbol{\mathcal{G}}_1 \sin\omega t + \boldsymbol{\mathcal{G}}_2 \cos\omega t) \end{aligned} \quad \text{(B.7)}$$

where $\boldsymbol{\mathcal{G}}_1$ and $\boldsymbol{\mathcal{G}}_2$ are arbitrary constant real vectors, $u(t)$ is the unit step function, and $\alpha > 0$ is a decay parameter that will later approach zero. Multiplying by $e^{-j\zeta t}$ and taking the Fourier transform of (B.7), applying the dyadic constitutive relations, expanding $\bar{\boldsymbol{\gamma}}(\zeta)$ for lossless media in the complex ζ-plane about $\zeta = \omega$, and then taking the inverse Fourier transform produces $\boldsymbol{\mathcal{F}}$ as

$$\begin{aligned} \boldsymbol{\mathcal{F}} = [e^{\alpha t}u(-t) + u(t)]&[\bar{\boldsymbol{\gamma}}_r \cdot (\boldsymbol{\mathcal{G}}_1 \sin\omega t + \boldsymbol{\mathcal{G}}_2 \cos\omega t) \\ &- \bar{\boldsymbol{\gamma}}_i \cdot (\boldsymbol{\mathcal{G}}_1 \cos\omega t - \boldsymbol{\mathcal{G}}_2 \sin\omega t) \\ &- \alpha e^{\alpha t}u(-t)[\bar{\boldsymbol{\gamma}}_r' \cdot (\boldsymbol{\mathcal{G}}_1 \cos\omega t - \boldsymbol{\mathcal{G}}_2 \sin\omega t) \\ &+ \bar{\boldsymbol{\gamma}}_i' \cdot (\boldsymbol{\mathcal{G}}_1 \sin\omega t + \boldsymbol{\mathcal{G}}_2 \cos\omega t)] + O(\alpha^2) \end{aligned} \quad \text{(B.8)}$$

wherein primes indicate differentiation with respect to ω and the subscripts "r" and "i" indicate the real and imaginary parts of a complex variable. Use has been made of $\bar{\boldsymbol{\gamma}} = \bar{\boldsymbol{\gamma}}_r - j\bar{\boldsymbol{\gamma}}_i$ with $\bar{\boldsymbol{\gamma}}_r$ being a symmetric dyadic ($\bar{\boldsymbol{\gamma}}_r = \bar{\boldsymbol{\gamma}}_{r,t}$) and $\bar{\boldsymbol{\gamma}}_i$ being an antisymmetric dyadic ($\bar{\boldsymbol{\gamma}}_i = -\bar{\boldsymbol{\gamma}}_{i,t}$) in lossless media. Inserting $\boldsymbol{\mathcal{G}}$, its time derivative, and $\boldsymbol{\mathcal{F}}$ from (B.7)–(B.8) into (B.6a); performing the integrations from $-\infty$ to t; and letting $\alpha \to 0$ produces

$$\begin{aligned} W_{el}(\mathbf{r},t) - W_0 = &\frac{1}{2}\boldsymbol{\mathcal{G}}_1 \cdot (\bar{\boldsymbol{\gamma}}_r - \bar{\mathbf{I}}_{\epsilon\mu}) \cdot \boldsymbol{\mathcal{G}}_1 \sin^2\omega t \\ &+ \frac{1}{2}\boldsymbol{\mathcal{G}}_2 \cdot (\bar{\boldsymbol{\gamma}}_r - \bar{\mathbf{I}}_{\epsilon\mu}) \cdot \boldsymbol{\mathcal{G}}_2 \cos^2\omega t \\ &+ \frac{1}{2}\boldsymbol{\mathcal{G}}_1 \cdot (\bar{\boldsymbol{\gamma}}_r - \bar{\mathbf{I}}_{\epsilon\mu}) \cdot \boldsymbol{\mathcal{G}}_2 \sin 2\omega t \\ &+ \frac{\omega}{4}\boldsymbol{\mathcal{G}}_1 \cdot \bar{\boldsymbol{\gamma}}_r' \cdot \boldsymbol{\mathcal{G}}_1 + \frac{\omega}{4}\boldsymbol{\mathcal{G}}_2 \cdot \bar{\boldsymbol{\gamma}}_r' \cdot \boldsymbol{\mathcal{G}}_2 \\ &- \frac{1}{2}\boldsymbol{\mathcal{G}}_1 \cdot (\omega\bar{\boldsymbol{\gamma}}_i)' \cdot \boldsymbol{\mathcal{G}}_2 \geq 0, \quad t \geq 0. \end{aligned} \quad \text{(B.9)}$$

[3]A passive lossless medium with constitutive relations $\mathbf{F} = \bar{\gamma} \cdot \mathbf{G}$ at every frequency is a spatially nondispersive linear medium that can be modeled by bound collisionless carriers. Although the actual medium may not conform to this idealized model, it is sufficient that the frequency dependent constitutive parameters can be obtained with the idealized model for (B.4) to apply.

Choosing t such that $\sin\omega t = 0$ reduces (B.9) to

$$\begin{aligned} \frac{1}{2}\boldsymbol{\mathcal{G}}_2 \cdot (\bar{\boldsymbol{\gamma}}_r - \bar{\mathbf{I}}_{\epsilon\mu}) \cdot \boldsymbol{\mathcal{G}}_2 + \frac{\omega}{4}\boldsymbol{\mathcal{G}}_1 \cdot \bar{\boldsymbol{\gamma}}_r' \cdot \boldsymbol{\mathcal{G}}_1 + \frac{\omega}{4}\boldsymbol{\mathcal{G}}_2 \cdot \bar{\boldsymbol{\gamma}}_r' \cdot \boldsymbol{\mathcal{G}}_2 \\ - \frac{1}{2}\boldsymbol{\mathcal{G}}_1 \cdot (\omega\bar{\boldsymbol{\gamma}}_i)' \cdot \boldsymbol{\mathcal{G}}_2 \geq 0, \quad t \geq 0 \end{aligned} \quad \text{(B.10)}$$

an inequality that holds for all values of the real six-vectors $\boldsymbol{\mathcal{G}}_1$ and $\boldsymbol{\mathcal{G}}_2$. Letting $\boldsymbol{\mathcal{G}}_2 = 0$ in (B.10) for all $\boldsymbol{\mathcal{G}}_1$, then letting $\boldsymbol{\mathcal{G}}_1 = 0$ for all $\boldsymbol{\mathcal{G}}_2$, shows that, because $\bar{\boldsymbol{\gamma}}_r$ is symmetric, each of the elements satisfies the inequalities

$$[(\omega\gamma_{kl,r})' - I_{kl,\epsilon\mu}] \geq \omega\gamma_{kl,r}'/2 \geq 0 \quad \text{(B.11)}$$

which expand to (70b) in the main text.

Replace $\boldsymbol{\mathcal{G}}_2$ with $-\boldsymbol{\mathcal{G}}_1$ and $\boldsymbol{\mathcal{G}}_1$ with $\boldsymbol{\mathcal{G}}_2$ in (B.9), add the resulting inequality to the original inequality in (B.9), and divide by two to obtain

$$\begin{aligned} \frac{1}{2}\boldsymbol{\mathcal{G}}_1 \cdot (\bar{\boldsymbol{\gamma}}_r - \bar{\mathbf{I}}_{\epsilon\mu}) \cdot \boldsymbol{\mathcal{G}}_1 \sin^2\omega t + \frac{1}{2}\boldsymbol{\mathcal{G}}_2 \cdot (\bar{\boldsymbol{\gamma}}_r - \bar{\mathbf{I}}_{\epsilon\mu}) \cdot \boldsymbol{\mathcal{G}}_2 \cos^2\omega t \\ \frac{\omega}{4}\boldsymbol{\mathcal{G}}_1 \cdot \bar{\boldsymbol{\gamma}}_r' \cdot \boldsymbol{\mathcal{G}}_1 + \frac{\omega}{4}\boldsymbol{\mathcal{G}}_2 \cdot \bar{\boldsymbol{\gamma}}_r' \cdot \boldsymbol{\mathcal{G}}_2 - \frac{1}{2}\boldsymbol{\mathcal{G}}_1 \cdot (\omega\bar{\boldsymbol{\gamma}}_i)' \cdot \boldsymbol{\mathcal{G}}_2 \geq 0, \\ t \geq 0 \end{aligned} \quad \text{(B.12)}$$

an inequality that holds for all values of $\boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{G}}_2$, and $t \geq 0$. Letting $\sin^2\omega t = \cos^2\omega t = 1/2$ shows that

$$\begin{aligned} \frac{1}{4}\boldsymbol{\mathcal{G}}_1 \cdot [(\omega\bar{\boldsymbol{\gamma}}_r)' - \bar{\mathbf{I}}_{\epsilon\mu}] \cdot \boldsymbol{\mathcal{G}}_1 + \frac{1}{4}\boldsymbol{\mathcal{G}}_2 \cdot [(\omega\bar{\boldsymbol{\gamma}}_r)' - \bar{\mathbf{I}}_{\epsilon\mu}] \cdot \boldsymbol{\mathcal{G}}_2 \\ - \frac{1}{2}\boldsymbol{\mathcal{G}}_1 \cdot (\omega\bar{\boldsymbol{\gamma}}_i)' \cdot \boldsymbol{\mathcal{G}}_2 \geq 0 \end{aligned} \quad \text{(B.13)}$$

or, by defining the complex six-vector $\mathbf{G} = \boldsymbol{\mathcal{G}}_2 + j\boldsymbol{\mathcal{G}}_1$, that the "susceptibility energy" is greater than zero

$$\frac{1}{4}\mathrm{Re}\{\mathbf{G}^* \cdot [(\omega\bar{\boldsymbol{\gamma}})' - \bar{\mathbf{I}}_{\epsilon\mu}] \cdot \mathbf{G}\} \geq 0 \quad \text{(B.14)}$$

which expands to (69) in the main text with $\mathbf{G} = (\mathbf{E}, \mathbf{H}), \bar{\boldsymbol{\gamma}}$ given in (B.5), and $\bar{\mathbf{I}}_{\epsilon\mu}$ in (B.6b). For $\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\tau}} = 0$ and $\bar{\boldsymbol{\mu}} = \mu\bar{\mathbf{I}}$, the inequality in (B.14) reduces to the inequality given in [36, p. 81].

Averaging (B.9) by integrating from $t = 0$ to $t = T$, dividing by T, and letting $T \to \infty$ also produces the result (B.13) and (B.14). With no dispersion, the left-hand side of (B.14) would reduce to $\mathrm{Re}\{\mathbf{G}^* \cdot [\bar{\boldsymbol{\gamma}} - \bar{\mathbf{I}}_{\epsilon\mu}] \cdot \mathbf{G}\}/4$, the per unit volume "average reactive energy" minus the "average energy stored in the fields", so that $\omega\mathrm{Re}\{\mathbf{G}^* \cdot \bar{\boldsymbol{\gamma}}' \cdot \mathbf{G}\}/4$ can be viewed as the per unit volume increase from $t = -\infty$ to $t = 0$ in the kinetic and potential energy of the carriers as the sinusoidal fields are built up in a lossless medium from an amplitude of zero at $t = -\infty$ to their final amplitude at $t = 0$.

Appendix C
Proof That Q Increases Rapidly With Decreasing Antenna Size

From (78) and (47) Q can be written as

$$Q(\omega_0) = \omega_0|W(\omega_0)|/P_A(\omega_0) = \omega_0\eta(\omega_0)|W(\omega_0)|/P_{\mathcal{R}}(\omega_0) \quad \text{(C.1)}$$

where $\eta(\omega_0) \leq 1$ is the efficiency of the antenna and $P_{\mathcal{R}}(\omega_0)$ is the power radiated by the antenna. With the help of $P_{\mathcal{R}}(\omega_0)$ from (19) and $W(\omega_0)$ from (66)–(67c), the Q in (C.1) can be expressed in terms of the electric and magnetic fields of the antenna by (C.2), shown at the bottom of the page, where

$$Q^{\text{in}}(\omega_0) = \frac{\eta\omega_0 Z_f}{2}\text{Re}\int_{\mathcal{V}_o(r_0)}\{\mathbf{E}^*\cdot(\omega_0\bar{\boldsymbol{\epsilon}})'\cdot\mathbf{E} + \mathbf{H}^*\cdot(\omega_0\bar{\boldsymbol{\mu}})'\cdot\mathbf{H} + \mathbf{E}\cdot[(\omega_0(\bar{\boldsymbol{\nu}}_t + \bar{\boldsymbol{\tau}}^*)]'\cdot\mathbf{H}^*\}\,d\mathcal{V} \Bigg/ \int_{4\pi}|\mathbf{F}|^2\,d\Omega \quad \text{(C.3)}$$

with $k_0 = \omega_0/c$. The volume $\mathcal{V}_o(r)$ has been divided into $\mathcal{V}_o(r_0)$ and $\mathcal{V}_o(r) - \mathcal{V}_o(r_0)$ where r_0 is the radius of the smallest spherical surface that is centered on the origin of the coordinate system and encloses the volume of the antenna and its power supply. (Recall that $\mathcal{V}_o(r)$ and thus $\mathcal{V}_o(r_0)$ excludes the volume $\mathcal{V}_p$ of the power supply.) We can also write

$$Q(\omega_0) = |Q_e(\omega_0) + Q_m(\omega_0) + Q^{\text{in}}(\omega_0) - \eta k_0 r_0| \quad \text{(C.4)}$$

where

$$\begin{aligned} Q_e(\omega_0) &= \frac{\eta k_0}{2}\frac{\lim_{r\to\infty}\left[\int_{r_0}^{r}\int_{4\pi}|\mathbf{E}|^2 r^2\,d\Omega\,dr - (r-r_0)\int_{4\pi}|\mathbf{F}|^2\,d\Omega\right]}{\int_{4\pi}|\mathbf{F}|^2\,d\Omega} \\ &= \frac{\eta k_0}{2}\frac{\int_{r_0}^{\infty}dr\int_{4\pi}(r^2|\mathbf{E}|^2 - |\mathbf{F}|^2)\,d\Omega}{\int_{4\pi}|\mathbf{F}|^2\,d\Omega} \end{aligned} \quad \text{(C.5)}$$

and similarly for $Q_m(\omega_0)$ by replacing $|\mathbf{E}|^2$ in (C.5) with $Z_f|\mathbf{H}|^2$. We can also write $Q^{\text{in}}(\omega_0) = Q_e^{\text{in}}(\omega_0) + Q_m^{\text{in}}(\omega_0) + Q_{\text{me}}^{\text{in}}(\omega_0)$, where the three terms on the right-hand side of this equation equal the corresponding three terms on the right-hand side of (C.3). Q_e and Q_m are the electric and magnetic reactive energies outside the sphere of radius r_0 defined by Fante [3].

It is a simple matter to prove that $Q_e > 0$ (and $Q_m > 0$) by expanding the electric field on the right-hand side of (C.5) (and the magnetic field on the right-hand side of the corresponding equation for Q_m) in a complete set of vector spherical waves [32, Sec. 7.11–7.14], [33, ch. 9]

$$\mathbf{E}(r,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}[A_{lm}\mathbf{M}_{lm}(r,\theta,\phi) + B_{lm}\mathbf{N}_{lm}(r,\theta,\phi)] \quad \text{(C.6)}$$

with

$$\begin{aligned} \mathbf{M}_{lm}(r,\theta,\phi) &= \nabla\times[\mathbf{r}h_l(k_0 r)P_l^m(\cos\theta)e^{-jm\phi}] \\ \mathbf{N}_{lm}(r,\theta,\phi) &= \nabla\times\mathbf{M}_{lm}(r,\theta,\phi)/k_0 \end{aligned} \quad \text{(C.7)}$$

to obtain [37]

$$Q_e(\omega_0) = \frac{\eta}{2}\int_{k_0 r_0}^{\infty}\left\{\sum_{l=1}^{\infty}[|\alpha_l|^2 H_l(x) + |\beta_l|^2 G_l(x)]\right\}dx \Bigg/ \sum_{l=1}^{\infty}[|\alpha_l|^2 + |\beta_l|^2] \quad \text{(C.8)}$$

wherein

$$\begin{aligned} H_l(x) &= x^2|h_l(x)|^2 - 1 \\ G_l(x) &= x^2[(l+1)|h_{l-1}(x)|^2 + l|h_{l+1}(x)|^2]/(2l+1) - 1. \end{aligned} \quad \text{(C.9)}$$

Spherical Hankel functions of the second kind (recall that $e^{j\omega_0 t}$ time dependence is assumed) are denoted by $h_l(x)$ and associated Legendre polynomials by $P_l^m(\cos\theta)$. Use has been made of the orthogonality relations for the vector spherical waves and the integration variable x has been inserted for $k_0 r$. The coefficients $|\alpha_l|^2$ and $|\beta_l|^2$ are defined as

$$\begin{aligned} |\alpha_l|^2 &= 4\pi\sum_{m=-l}^{l}\frac{(l+m)!}{(2l+1)(l-m)!}|A_{lm}|^2 \\ |\beta_l|^2 &= 4\pi\sum_{m=-l}^{l}\frac{(l+m)!}{(2l+1)(l-m)!}|B_{lm}|^2 \end{aligned} \quad \text{(C.10)}$$

and the integration over the far field squared is given by $\int_{4\pi}|\mathbf{F}|^2 d\Omega = k_0^{-2}\sum_{l=1}^{\infty}[|\alpha_l|^2 + |\beta_l|^2]$. It can be shown from [38, p. 255, (4)] that the integrals $\int_{k_0 r_0}^{\infty} H_l(x)\,dx$ and $\int_{k_0 r_0}^{\infty} G_l(x)\,dx$ are equal to Fante's Q_l' and Q_l, respectively. Similarly, $Q_m(\omega_0)$ is given by (C.8) with H_l and G_l interchanged.

Equation (10.1.27) in [39] implies that both $H_l(x)$ and $G_l(x)$ are monotonically decreasing functions of increasing x such that $H_l(x) > 0$ and $G_l(x) > 0$ for all x and both $H_l(x)$ and $G_l(x)$ decay as $O(1/x^2)$ as $x \to \infty$. Consequently, the radial integration in (C.8) converges (although this is proven implicitly by (64) as well) and

$$Q_e(\omega_0) > 0, \quad Q_m(\omega_0) > 0. \quad \text{(C.11)}$$

Moreover, as the radius r_0 of the antenna is decreased while maintaining its frequency, efficiency, and far-field pattern

$$Q(\omega_0) = \left|\frac{\eta k_0}{2}\frac{\lim_{r\to\infty}\left[\int_{r_0}^{r}\int_{4\pi}(|\mathbf{E}|^2 + Z_f^2|\mathbf{H}|^2)r^2\,d\Omega\,dr - 2r\int_{4\pi}|\mathbf{F}|^2\,d\Omega\right]}{\int_{4\pi}|\mathbf{F}|^2\,d\Omega} + Q^{\text{in}}(\omega_0)\right| \quad \text{(C.2)}$$

$\mathbf{F}(\theta,\phi)$, the quality factors $Q_e(\omega_0)$ and $Q_m(\omega_0)$ increase extremely fast. To prove this, truncate the infinite summations over l to a maximum value L that is large enough to make the far fields produced by the remaining spherical modes negligible, so that (C.8) can be written as[4]

$$Q_{\substack{e\\m}}(\omega_0) \approx \eta \int_{k_0 r_0}^{\infty} \left\{ \sum_{l=1}^{L} \left[|\alpha_l|^2 \left\{ \begin{matrix} H_l(x) \\ G_l(x) \end{matrix} \right\} + |\beta_l|^2 \left\{ \begin{matrix} G_l(x) \\ H_l(x) \end{matrix} \right\} \right] \right\} dx \Big/ \sum_{l=1}^{L} \left[|\alpha_l|^2 + |\beta_l|^2 \right]. \quad \text{(C.12)}$$

As $k_0 r_0$ becomes less than L, the functions $H_L(k_0 r_0)$ and $G_L(k_0 r_0)$ in (C.9) increase in value extremely rapidly. Therefore, as the radius r_0 of the antenna is reduced to a value smaller than L/k_0, the quality factors $Q_e(\omega_0)$ and $Q_m(\omega_0)$ become much greater than 1, provided the efficiency and the coefficients $|\alpha_l|^2$ and $|\beta_l|^2$ stay the same, as they (the coefficients) will, if the frequency and far-field pattern of the antenna is kept the same. Moreover, the rate of increase of $Q_e(\omega_0)$ and $Q_m(\omega_0)$ with decreasing $k_0 r_0$ grows rapidly with L. In view of (C.4), this implies that $Q(\omega_0)$, as well as $Q_e(\omega_0)+Q_m(\omega_0)$, becomes enormously large as the radius of the antenna is reduced below the value of L/k_0 (assuming $Q^{\rm in}(\omega_0)$ does not grow negatively at the same rate). The enormously high reactive fields with rapid spatial variation responsible for the enormously large $Q(\omega_0)$ prevent the practical realization of supergain above a few dB.

Most antennas do not have extremely high reactive fields and $k_0 r_0 \approx L$. In addition, an antenna is probably unnecessarily large in size if $k_0 r_0 \gg L$, because the same far-field pattern could be obtained by another antenna with a much smaller radius $r_0 \approx L/k_0$ without significantly increasing the reactive fields. Hypothetically, one could conceive of an extremely oversized tuned antenna with r_0 so much greater than L/k_0 that $Q_e(\omega_0)+Q_m(\omega_0)+Q^{\rm in}(\omega_0)$ in (C.4) would be less than $\eta k_0 r_0$. This, of course, implies that $W(\omega_0)$ is negative. It is difficult, however, to imagine that an antenna with $W(\omega_0) \le 0$ could be practical even if the antenna had a $Q^{\rm in}(\omega_0)$ that was highly negative because of the dominance of lossy materials with negative internal energy density; see Section VI-D.

Lastly, we consider antennas with nondispersive $\epsilon_r \ge 0$ and $\mu_r \ge 0$, so that $Q_e(\omega_0)+Q_e^{\rm in}(\omega_0) = Q_m(\omega_0)+Q_m^{\rm in}(\omega_0)$ with $Q_e^{\rm in}(\omega_0) \ge 0$ and $Q_m^{\rm in}(\omega_0) \ge 0$. Then we can write

$$\begin{aligned} Q(\omega_0) &= 2Q_e(\omega_0) + 2Q_e^{\rm in}(\omega_0) - \eta k_0 r_0 \\ &= 2Q_m(\omega_0) + 2Q_m^{\rm in}(\omega_0) - \eta k_0 r_0. \end{aligned} \quad \text{(C.13)}$$

For electrically small antennas $k_0 r_0 < 1, Q_e(\omega_0) \gg 1$, and $Q_m(\omega_0) \gg 1$, so that

$$Q(\omega_0) \approx 2Q_e(\omega_0) + 2Q_e^{\rm in}(\omega_0) = 2Q_m(\omega_0) + 2Q_m^{\rm in}(\omega_0) \quad \text{(C.14)}$$

and thus the larger of $[2Q_e(\omega_0), 2Q_m(\omega_0)]$ is the greatest lower bound for our defined $Q(\omega_0)$. In other words, the lower bound on quality factor given by Fante [3] is indeed the lower bound to our defined exact Q for electrically small antennas with nondispersive $\epsilon_r \ge 0$ and $\mu_r \ge 0$. Moreover, for electrically small antennas, the lower bounds on quality factor given by Chu [1] and Collin and Rothschild [2] are approximately equal to the lower bound of Fante [3]. For specified fields outside a closed surface S_a, the Q reaches its lower bound by generating the fields outside S_a with equivalent surface currents on S_a that produce zero fields inside S_a except for the magnetic or electric fields within a tuning inductor or capacitor depending on whether $Q_e(\omega_0)$ is greater than or less than $Q_m(\omega_0)$, respectively [40]. If $Q_e(\omega_0) = Q_m(\omega_0)$, as in the case of equal energy electric and magnetic dipole fields outside a sphere, no tuning inductor or capacitor would be required inside S_a.

Appendix D
Evaluation of Q From (80) for *RLC* Series- and Parallel-Circuit Antenna Models

To determine an approximate expression for the Q in (80) for a tuned antenna at a resonant or antiresonant frequency ω_0, we shall lump the radiation resistance of the antenna into a series *RLC* circuit for a resonance or into a parallel *RLC* circuit in series with a tuning inductor (L_s) or capacitor (C_s) for an antiresonance. (Adding the tuning capacitor or inductor explicitly to the series *RLC* circuit model for a resonance is unnecessary because the resulting circuit is still an *RLC* series circuit. Unlike the ordinary *RLC* series circuit, however, the resistance is allowed to be a function of frequency.) Although the inductance and capacitance in these *RLC* circuits are assumed independent of frequency for the antenna tuned at ω_0, this still allows a reactance that is frequency dependent.

Since the radiation resistance of the tuned antenna is lumped into the *RLC* circuit resistance, and the radiated fields of an *RLC* circuit are negligible, $W_{\mathcal{R}}$ in (80) is negligible for these circuit models. Therefore, Q in (80) is given approximately by

$$Q(\omega_0) \approx \frac{\omega_0}{2R_0(\omega_0)} \left[X_0'(\omega_0) - \frac{4}{|I_0|^2} W_{\mathcal{L}}(\omega_0) \right]. \quad \text{(D.1)}$$

Thus the task of this Appendix reduces to evaluating the right-hand side of (D.1) for tuned series and parallel *RLC* circuits.

Approximating a tuned antenna at a resonant frequency $(X_0'(\omega_0) > 0)$ by a series *RLC* circuit with elements $R_{\rm sc}(\omega), L_{\rm sc}$, and $C_{\rm sc}$, we have

$$R_0(\omega) = R_{\rm sc}(\omega). \quad \text{(D.2)}$$

The current I_0 is flowing through $R_{\rm sc}(\omega)$ and (76) becomes simply

$$W_{\mathcal{L}}(\omega_0) = \frac{R_{\rm sc}(\omega_0)}{2} {\rm Im}\left[(I_0)'_{I_0} I_0^* \right]. \quad \text{(D.3)}$$

Because the frequency derivative of I_0 with I_0 held constant with frequency is identically zero, that is, $(I_0)'_{I_0} = 0$, we have $W_{\mathcal{L}}(\omega_0) = 0$ and thus from (D.1)

$$Q(\omega_0) \approx \frac{\omega_0}{2R_0(\omega_0)} X_0'(\omega_0). \quad \text{(D.4)}$$

[4]In order for the spherical wave expansion in (C.6) to converge for all $r > r_0$, the values of the coefficients A_{lm} and B_{lm} and thus $|\alpha_l|^2$ and $|\beta_l|^2$ have to grow small extremely rapidly for l greater than a finite value because the values of $|h_l(x)|$ increase extremely rapidly as l grows larger than x.

Next approximate the tuned antenna at an antiresonant frequency ($X_0'(\omega_0) < 0$) with the tuning reactance $X_s(\omega)$ of an inductor or capacitor in series with a parallel *RLC* circuit with elements $R_{\rm pc}, L_{\rm pc}$, and $C_{\rm pc}$. For this tuned parallel *RLC* circuit, we find

$$R_0(\omega) = R(\omega) \approx \frac{R_{\rm pc}}{1+4R_{\rm pc}^2C_{\rm pc}^2(\omega-\omega_{\rm pc})^2} \tag{D.5}$$

$$X_0(\omega) = X(\omega) + X_s(\omega), \quad X_0(\omega_0) = X(\omega_0) + X_s(\omega_0) = 0 \tag{D.6}$$

$$X(\omega) \approx \frac{-2R_{\rm pc}^2C_{\rm pc}(\omega-\omega_{\rm pc})}{1+4R_{\rm pc}^2C_{\rm pc}^2(\omega-\omega_{\rm pc})^2} \tag{D.7}$$

where $\omega_{\rm pc} = 1/\sqrt{L_{\rm pc}C_{\rm pc}}$ is the antiresonant frequency of the untuned parallel *RLC* circuit. The "approximately equal" signs hold under the condition that $|\omega-\omega_{\rm pc}|/\omega_{\rm pc} \ll 1$ ($\lesssim 1/8$ will suffice). Taking the derivative of these expressions with respect to frequency and setting $\omega = \omega_0$ produces under the same condition

$$R_0'(\omega_0) = R'(\omega_0) \approx \frac{-8R_{\rm pc}^3C_{\rm pc}^2(\omega_0-\omega_{\rm pc})}{\left[1+4R_{\rm pc}^2C_{\rm pc}^2(\omega_0-\omega_{\rm pc})^2\right]^2} \tag{D.8}$$

$$X_0'(\omega_0) \approx X'(\omega_0) \approx -2R_{\rm pc}^2C_{\rm pc}\frac{\left[1-4R_{\rm pc}^2C_{\rm pc}^2(\omega_0-\omega_{\rm pc})^2\right]}{\left[1+4R_{\rm pc}^2C_{\rm pc}^2(\omega_0-\omega_{\rm pc})^2\right]^2}. \tag{D.9}$$

From (D.8) and (D.9) we find

$$\begin{aligned}|Z_0'(\omega_0)| &= \sqrt{[R_0'(\omega_0)]^2 + [X_0'(\omega_0)]^2} \\ &\approx \frac{2R_{\rm pc}^2C_{\rm pc}}{1+4R_{\rm pc}^2C_{\rm pc}^2(\omega_0-\omega_{\rm pc})^2}.\end{aligned} \tag{D.10}$$

With the current flowing through $R_{\rm pc}$ denoted by I_R, the formula (76) becomes

$$W_{\mathcal{L}}(\omega_0) = \frac{R_{\rm pc}}{2}{\rm Im}\left[(I_R)'_{I_0}I_R^*\right]. \tag{D.11}$$

Since

$$I_R = \frac{V_R}{R_{\rm pc}} = \frac{V_0Z(\omega_0)}{Z_0(\omega_0)R_{\rm pc}} = I_0\frac{Z(\omega_0)}{R_{\rm pc}} \tag{D.12}$$

where V_R is the voltage across the parallel *RLC* circuit, we have

$$(I_R)'_{I_0} = I_0\frac{Z'(\omega_0)}{R_{\rm pc}} \tag{D.13}$$

and from (D.11)

$$\begin{aligned}W_{\mathcal{L}}(\omega_0) &= \frac{|I_0|^2}{2R_{\rm pc}}{\rm Im}[Z'(\omega_0)Z^*(\omega_0)] \\ &= \frac{|I_0|^2}{2R_{\rm pc}}[X'(\omega_0)R(\omega_0) - R'(\omega_0)X(\omega_0)].\end{aligned} \tag{D.14}$$

With the aid of (D.5)–(D.9), (D.14) evaluates to

$$W_{\mathcal{L}}(\omega_0) \approx \frac{-|I_0|^2R_{\rm pc}^2C_{\rm pc}}{\left[1+4R_{\rm pc}^2C_{\rm pc}^2(\omega_0-\omega_{\rm pc})^2\right]^2} \tag{D.15}$$

which when substituted into (D.1) yields

$$\begin{aligned}Q(\omega_0) &\approx \frac{\omega_0}{2R_0(\omega_0)}\left[\frac{2R_{\rm pc}^2C_{\rm pc}}{1+4R_{\rm pc}^2C_{\rm pc}^2(\omega_0-\omega_{\rm pc})^2}\right] \\ &\approx \omega_0R_{\rm pc}C_{\rm pc}\end{aligned} \tag{D.16}$$

after inserting $X_0'(\omega_0)$ from (D.9). A comparison of (D.16) and (D.10) reveals the relationship

$$Q(\omega_0) \approx \frac{\omega_0}{2R_0(\omega_0)}|Z_0'(\omega_0)|. \tag{D.17}$$

Especially note from (D.9), (D.10), and (D.17) that

$$Q(\omega_0) \not\approx \frac{\omega_0}{2R_0(\omega_0)}|X_0'(\omega_0)| \tag{D.18}$$

except for $|\omega_0-\omega_{\rm pc}|/\omega_{\rm pc} \ll 1/(2Q_{\rm pc})$, where $Q_{\rm pc} = \omega_{\rm pc}R_{\rm pc}C_{\rm pc} = Q(\omega_{\rm pc})$ is the quality factor of the untuned parallel *RLC* circuit, that is, the quality factor at the antiresonant frequency of the untuned antenna. For most antennas $Q_{\rm pc} \gg 1$.

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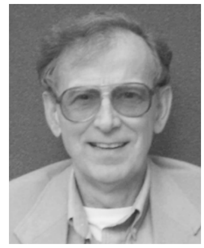

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