Chapter 3

Signal Spectra – the Relationship between the Time Domain and the Frequency Domain

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Outline

- Preview
- Periodic Signals
- Spectra of Digital Waveforms
- Spectrum Analyzers
- Representation of Nonperiodic Waveforms
- Representation of Random (Data) Signals

Preview

- Spectrum of Signals
  - The frequency spectrum of the signals is the most important aspect of the ability to satisfy the EMC requirements.
- What Dominates the Bandwidth of Digital Signals
  - Through the following discussion, we will find out that the rise/fall time of the digital signals dominates the bandwidth of the digital signal.
  - Fundamental frequency is not as important as we expect.

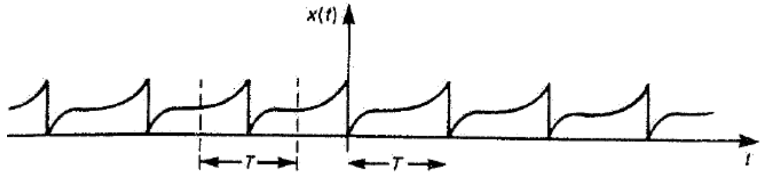
Periodic Signals

- Category of Infinite Signals
  - Periodic Signal - Deterministic
    - Time-domain signal or waveforms that occur repetitively in time are referred to as periodic signals.
    - Clock and data signals are two examples of them.
  - Random Signals -Nondeterministic
    - Signals whose time behavior is not known but can only be described statistically.
    - Data streams in digital products are examples of them.
    - Data streams are nondeterministic otherwise no information would be conveyed.

Periodic Signals

- Representation of Periodic Signals
  - General Form
    - A periodic signal has the property

$$x(t \pm kT) = x(t), \; k = 1, 2, 3, \ldots$$



    - where T is the period
    - and the fundamental frequency of the signal f_0 is

$$f_0 = \frac{1}{T} \longrightarrow \omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

    - The average power is defined as

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{t_1}^{t_1+T} x^2(t)dt$$

Nonperiodic signals are referred as energy signals since their energy is finite.

    - And the energy in a signal is defined as

$$E = \int_{-\infty}^{\infty} x^2(t)dt$$

Periodic signals are referred as power signals since their power is finite.

Periodic Signals

- Representation of Periodic Signals
  - General Form
    - Periodic signals can be represented as linear combinations of basis functions

$$x(t) = \sum_{n=0}^{\infty} c_n \phi_n(t)$$

$$= c_0 \phi_0(t) + c_1 \phi_1(t) + c_2 \phi_2(t) + \dots$$

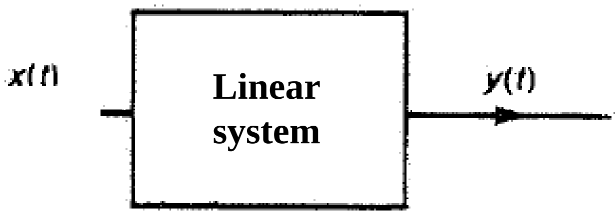
    - For this signal passing through a linear system

$$\phi_n(t) \rightarrow y_n(t)$$

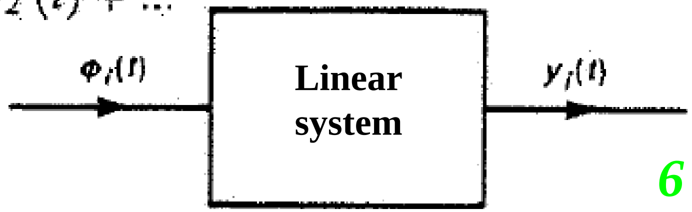
$$y(t) = \sum_{n=0}^{\infty} c_n y_n(t)$$

$$= c_0 y_0(t) + c_1 y_1(t) + c_2 y_2(t) + \dots$$

Superpositions of response of basis functions



(a)



Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - Trigonometric Form
    - The trigonometric form has basis functions, $\Phi_0=1$ and $\Phi_n=\cos(n\omega_0 t)$, $\sin(n\omega_0 t)$ for $n=1, 2, 3, \ldots, \infty$.
    - The expansion coefficients are real for this case.
  - Complex-Exponential Form
    - The periodic signal could be expressed as

$$x(t) = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$= \cdots + c_{-2}e^{-j2\omega_0 t} + c_{-1}e^{-j\omega_0 t} + c_0 + c_1 e^{j\omega_0 t} + c_2 e^{j2\omega_0 t} + \cdots$$

$$\phi_n = e^{jn\omega_0 t}$$

$$= \cos(n\omega_0 t) + j\sin(n\omega_0 t) \qquad \text{for} \quad -\infty, \ldots, -1, 0, 1, \ldots, \infty$$

Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - Complex-Exponential Form
    - The complex coefficients c_n are calculated from

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}\,dt$$

    - and for n=0

$$c_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\,dt$$

$$= \frac{\text{area under curve over one period}}{T}$$

$$= \text{average value of } x(t)$$

    - which is a real number.

Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - Complex-Exponential Form
    - Since $c_{-n} = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{jn\omega_0 t}\,dt = c_n^*$ $\quad c_n = |c_n|\angle c_n = |c_n|e^{j\angle c_n} \rightarrow c_n^* = |c_n|e^{-j\angle c_n}$
    - The complex-exponential form may be rewritten as

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_n^* e^{-jn\omega_0 t}$$

    - After some manipulations, we have

$$x(t) = c_0 + \sum_{n=1}^{\infty} |c_n| e^{j(n\omega_0 t + \angle c_n)} + \sum_{n=1}^{\infty} |c_n| e^{-j(n\omega_0 t + \angle c_n)}$$

$$= c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n)$$

Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - Complex-Exponential Form
    - The expansion coefficients for the one-sided spectrum (positive frequencies only) are obtained from doubling the magnitudes for the double-sided spectrum $c_n^+=2|c_n|$ and the dc component c_0 remains unchanged.
    - Also, this could be rewritten as a function of sines

$$x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \sin(n\omega_0 t + \angle c_n + 90^\circ)$$

Expanding this will transforms this equation to the trigonometric form

Periodic Signals

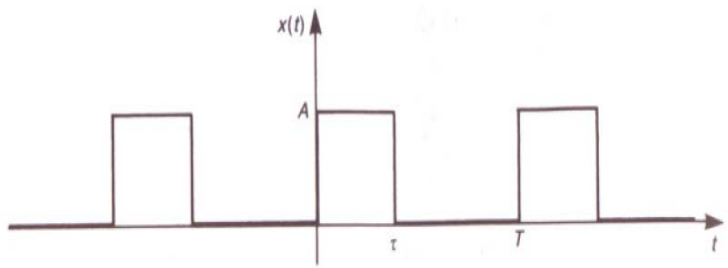
- The Fourier Series Representation of Periodic Signals
  - A Periodic “Square Wave” Pulse Train
    - The expansion coefficients are obtained as

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} e^{-jn\omega_0 t} x(t)\,dt$$

$$= \frac{1}{T}\int_0^{\tau} e^{-jn\omega_0 t} A\,dt + \cancel{\frac{1}{T}\int_{\tau}^{T} e^{-jn\omega_0 t} \times 0\,dt}$$

$$= \frac{A}{jn\omega_0 T}\left(1 - e^{-jn\omega_0 \tau}\right)$$

$$= \frac{A\tau}{T} e^{-jn\omega_0 \tau/2} \frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}$$



Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - A Periodic "Square Wave" Pulse Train
    - which could be decomposed as

$$|c_n| = \frac{A\tau}{T}\left|\frac{\sin(\frac{1}{2}n\omega_0\tau)}{\frac{1}{2}n\omega_0\tau}\right| \quad \xrightarrow{\omega_0=2\pi/T} \quad |c_n| = \frac{A\tau}{T}\left|\frac{\sin(n\pi\tau/T)}{n\pi\tau/T}\right|$$

$$\angle c_n = \pm\tfrac{1}{2}n\omega_0\tau \qquad \angle c_n = \pm\frac{n\pi\tau}{T}$$

$$\xrightarrow{n/T=f} \quad \frac{A\tau}{T}\left|\frac{\sin(\pi f\tau)}{\pi f\tau}\right|$$

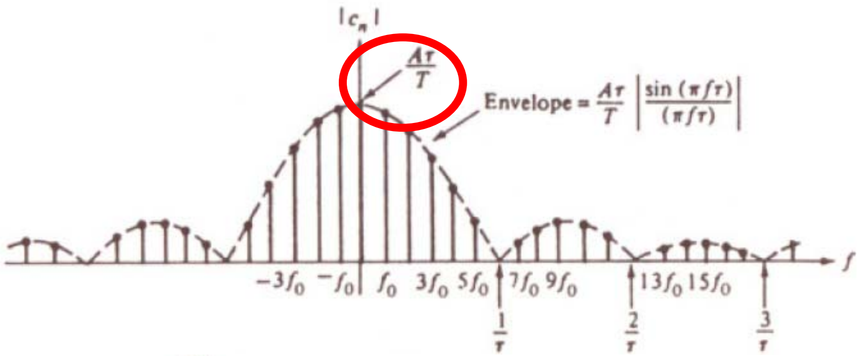
which is a sinc function of the form $\sin x/x$

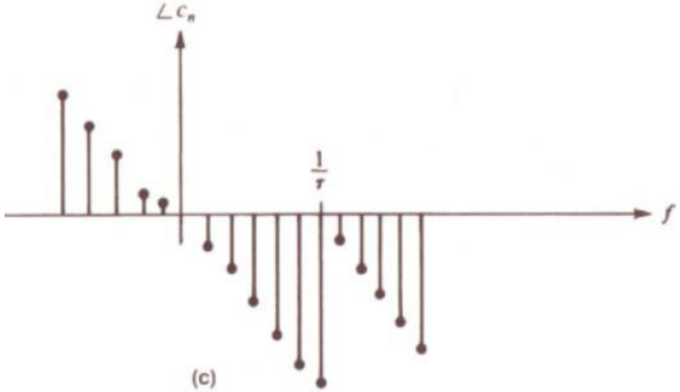
- The duty cycle is defined as

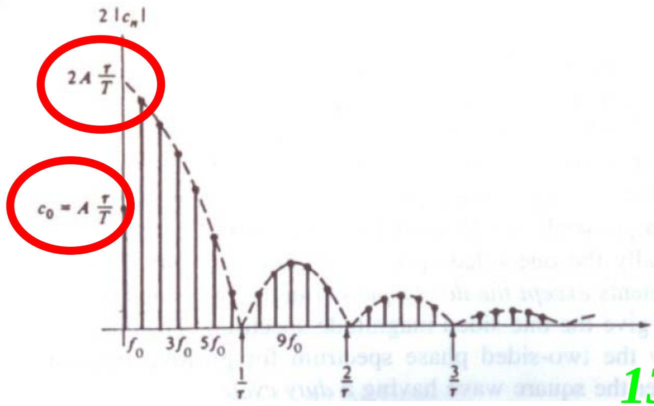
$$D = \frac{\tau}{T}$$

Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - A Periodic “Square Wave” Pulse Train
    - The magnitudes and phases for c_n and $2c_n$ are







Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - A Periodic "Square Wave" Pulse Train
    - An interesting result occurs when D=1/2, the even harmonics becomes zeros.

$$\frac{\tau}{T} = \frac{1}{2}$$

$$|c_n| = \frac{A}{2}\left|\frac{\sin(n\pi/2)}{n\pi/2}\right|$$

$$= \frac{A}{n\pi} \qquad n = 1, 3, 5, \ldots$$

$$= 0 \qquad n = 2, 4, 6, \ldots$$

$$\angle c_n = \angle\frac{-n\omega_0\tau}{2} + \angle\sin\left(\tfrac{1}{2}n\omega_0\tau\right)$$

$$= \angle\frac{-n\pi}{2} + \angle\sin\left(\frac{n\pi}{2}\right)$$

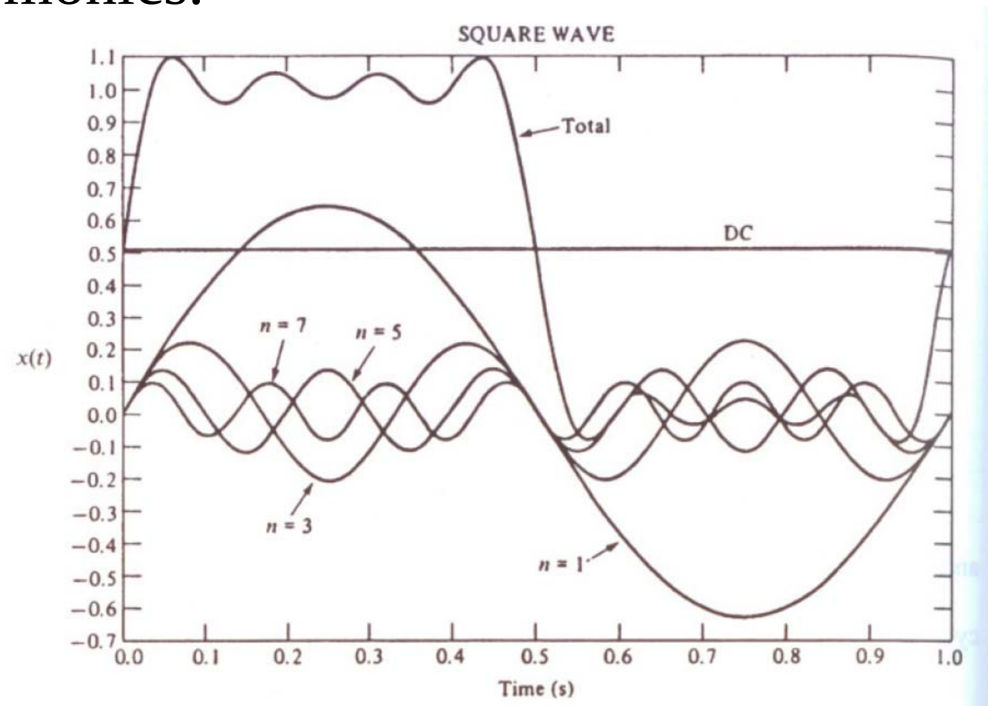
$$= -90^\circ \qquad n = 1, 3, 5, \ldots$$

Substitute the value of n in thest terms one by one, you could prove this.

$$\boxed{c_0 = A/2}$$

Periodic Signals

- The Fourier Series Representation of Periodic Signals
  - Example 3.1
    - A periodic square wave pulse train composed of 7 harmonics.



How many harmonics are needed for reconstructing the original signal? This is determined by the BW of the signal.

Periodic Signals

- Response of Linear Systems to Periodic Input Signals
  - Linear System
    - For a linear system, if the input signal is of the form

      $$x(t) = X\cos(\omega t + \phi_x)$$

    - The output signal is of the same form

      $$y(t) = Y\cos(\omega t + \theta_y)$$

    - If the impulse response for the linear system is

      $$H(j\omega) = |H(j\omega)|\underline{/H(j\omega)}$$

    - The output response could be written as

      $$Y\,\underline{/\theta_y} = H(j\omega)X\,\underline{/\phi_x} \longrightarrow \begin{cases} Y = |H(j\omega)|X \\ \theta_y = \underline{/H(j\omega)} + \phi_x \end{cases}$$

Periodic Signals

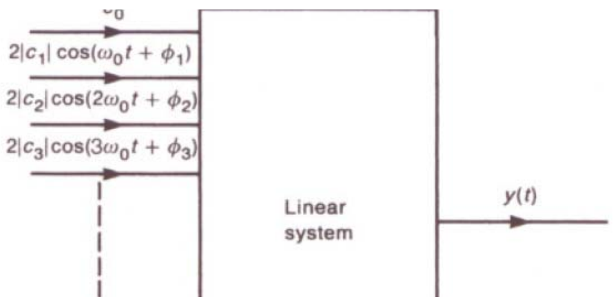
- Response of Linear Systems to Periodic Input Signals
  - Linear System
    - Thus, suppose that $x(t)$ is periodic and written in the form of Fourier series

$$x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n)$$

    - The output response could be obtained via

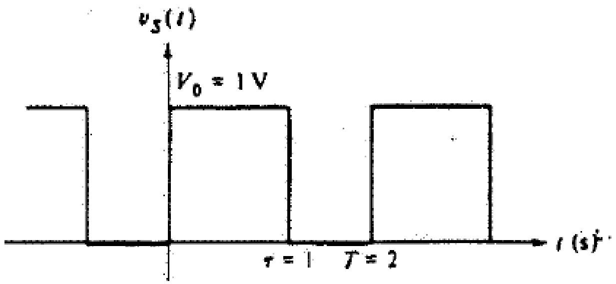
$$y(t) = c_0 H(0) + \sum_{n=1}^{\infty} 2|c_n| \, |H(jn\omega_0)| \cos[n\omega_0 t + \angle c_n + \angle H(jn\omega_0)]$$

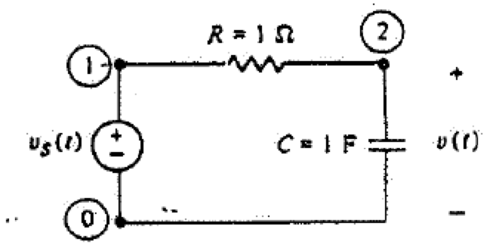
This result is obtained through the theorem of superposition also.

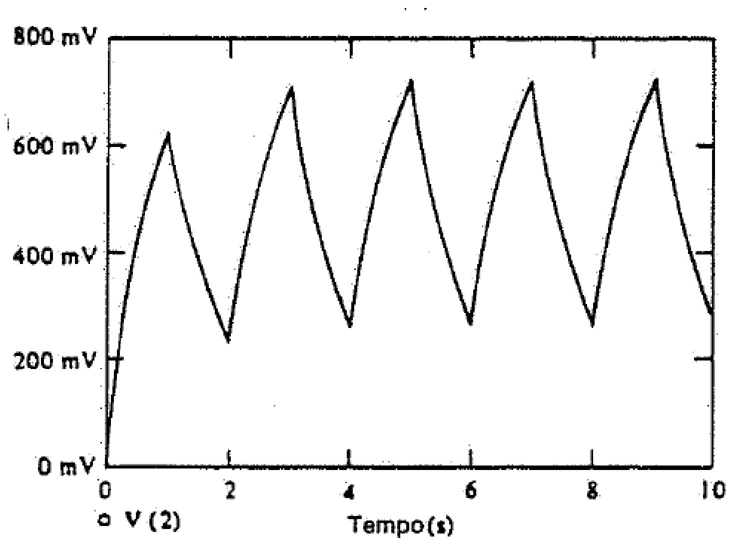


Periodic Signals

- Response of Linear Systems to Periodic Input Signals
  - Example 3.2
    - We could see that the pulse width is too small compared to the RC time constant, thus, the capacitor could not reach it maximum value of 1V.



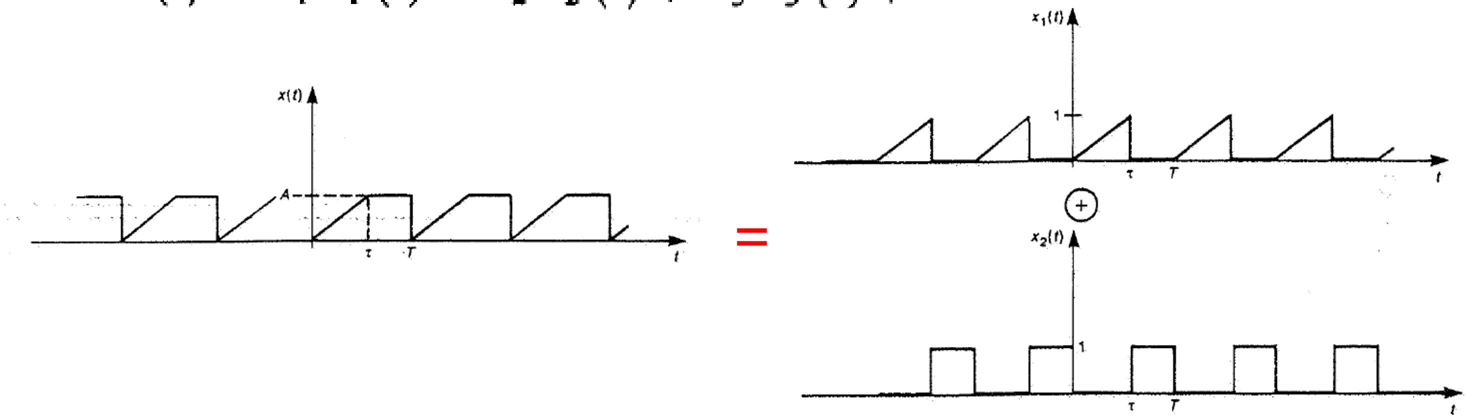




Periodic Signals

- Important Computational Techniques
  - Linearity
    - Any waveform or function can be written as a linear combination of two or more functions as

$$x(t) = A_1\, x_1(t) + A_2\, x_2(t) + A_3\, x_3(t) + \ldots$$



    - If for example, $x_n(t)$ are expressed in the form of Fourier series

$$x_1(t) = \sum_{n=-\infty}^{\infty} c_{1n}\, e^{jn\omega_0 t} \qquad x_2(t) = \sum_{n=-\infty}^{\infty} c_{2n}\, e^{jn\omega_0 t}$$

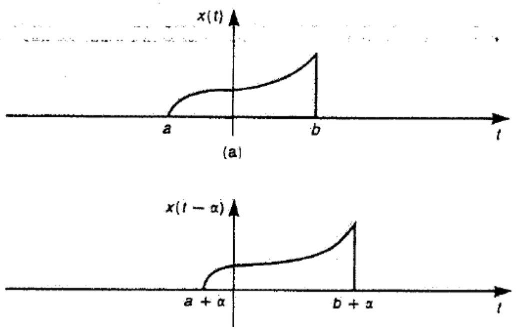
Periodic Signals

- Important Computational Techniques
  - Linearity
    - Then by linearity, we obtain

$$x(t) = x_1(t) + x_2(t) = \sum_{n=-\infty}^{\infty} (c_{1n} + c_{2n}) e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

  - Time-Shifting
    - If $x(t)$ is shifted ahead in t by an amount α (delayed in time by α), then the Fourier series of $x(t-\alpha)$ is

$$\begin{aligned} x(t-\alpha) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0(t-\alpha)} \\ &= \sum_{n=-\infty}^{\infty} \underbrace{c_n e^{-jn\omega_0 \alpha}}_{c_n'} e^{jn\omega_0 t} \end{aligned}$$



Therefore we multiply the expansion coefficients of $x(t)$ by $\exp(-jn\omega_0\alpha)$ to obtain the expansion coefficients of $x(t-\alpha)$.

Periodic Signals

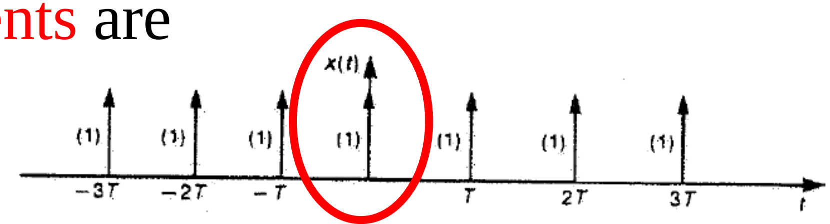
- Important Computational Techniques
  - Unit Impulse Function $\delta(t)$
    - The definition of the impulse function is

$$\delta(t) = \begin{cases} 0 & \text{per } t < 0 \\ 0 & \text{per } t > 0 \\ \int_{0^-}^{0^+} \delta(t)\, dt = 1 \end{cases}$$

    - Consider a periodic train of unit impulse functions

$$x(t) = \delta(t \pm kT), \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

    - The expansion coefficients are

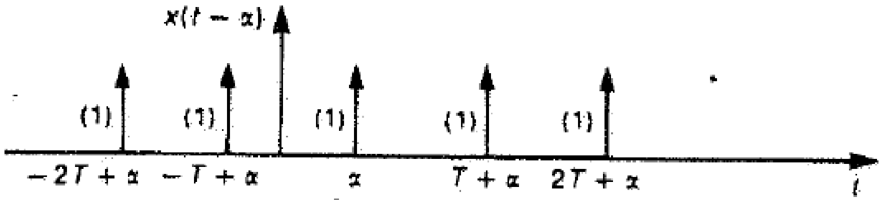


$$\begin{aligned} c_n &= \frac{1}{T} \int_0^T \delta(t) e^{-jn\omega_0 t}\, dt \\ &= \frac{1}{T} \int_{0^-}^{0^+} \delta(t) e^{jn\omega_0 t}\, dt = \frac{1}{T} \int_{0^-}^{0^+} \delta(t)\, dt \\ &= \frac{1}{T} \end{aligned}$$

Periodic Signals

- Important Computational Techniques
  - Unit Impulse Function $\delta(t)$
    - If the pulse train is shifted ahead in t by α, the expansion coefficients become

$$c_n = \frac{1}{T}\, e^{-jn\omega_0\alpha}$$



  - Influence of Derivative on Expansion Coefficients
    - If $x(t)$ is represented with the complex-exponential Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

Periodic Signals

- Important Computational Techniques
  - Influence of Derivative on Expansion Coefficients
    - The *k*th derivative is represented as

$$\frac{d^k x(t)}{dt^k} = \sum_{n=-\infty}^{\infty} c_n^{(k)} e^{jn\omega_0 t}$$

    - Also, by differentiating the original $x(t)$, the *k*th derivative could be written as

$$\frac{d^k x(t)}{dt^k} = \sum_{n=-\infty}^{\infty} \underbrace{(jn\omega_0)^k c_n}_{c_n^{(k)}} e^{jn\omega_0 t}$$

    - Thus, the expansion coefficients are related by

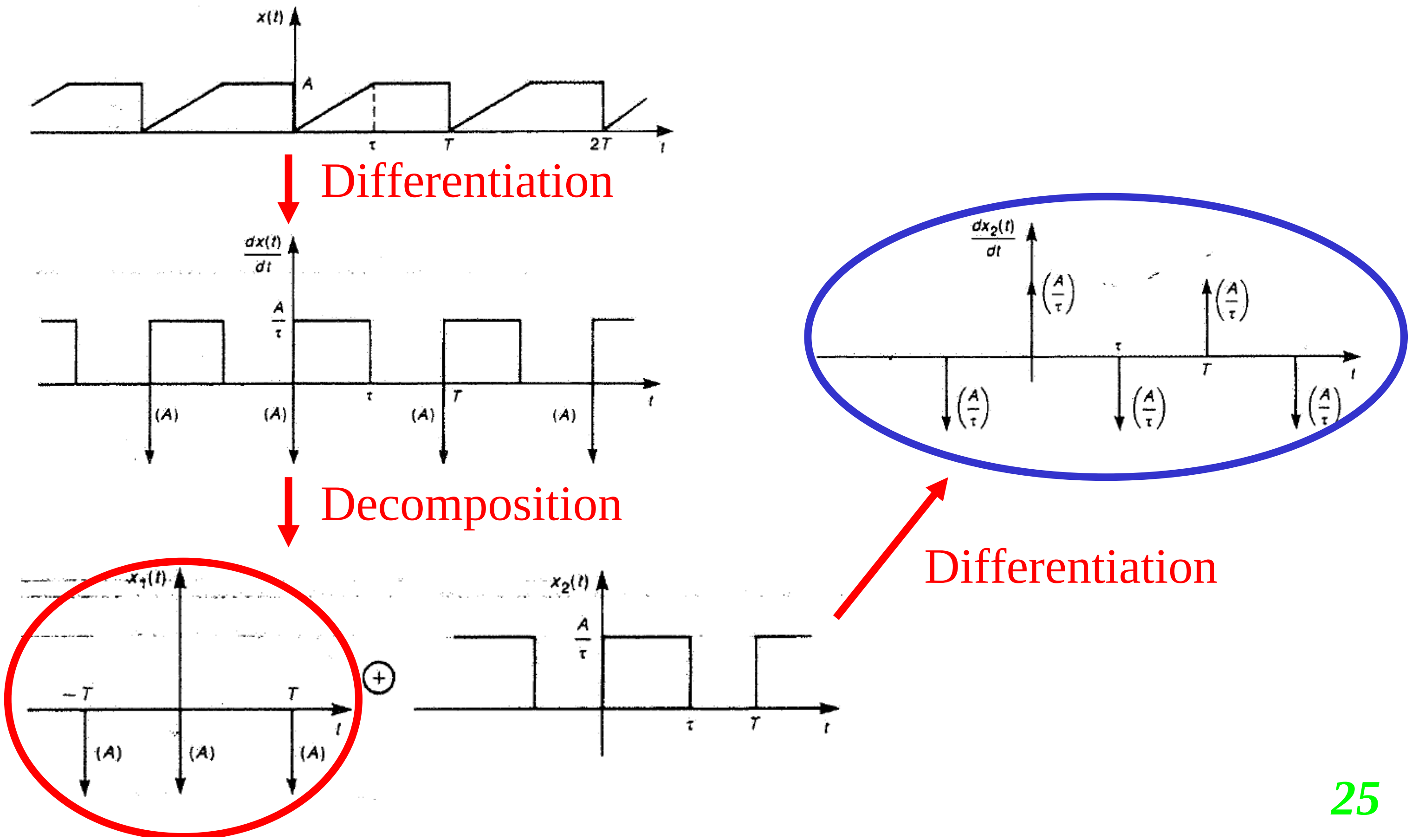
$$c_n = \frac{1}{(jn\omega_0)^k} c_n^{(k)} \qquad n \neq 0$$

Periodic Signals

- Important Computational Techniques
  - Procedures for Obtaining the Expansion Coefficients
    - The technique is to *repeatedly differentiate* the function until the first occurrence of an impulse function.
    - If some part are not impulse functions, *continue differentiate* that part until the occurrence of an impulse function.
    - Through the properties of linearity, time-shifting, unit impulse response, and derivative, we could reconstruct the original signal *in a simple way*.

Periodic Signals

- Important Computational Techniques
  - Example 3.4
    - Consider the wave shown below



Periodic Signals

- Important Computational Techniques
  - Example 3.4
    - The expansion coefficients for those enclosed by red and blue circles are

$$c_{1n}^{(1)} = -\frac{A}{T} \qquad\qquad c_{2n}^{(2)} = \frac{A}{\tau}\frac{1}{T} - \frac{A}{\tau}\frac{1}{T}e^{-jn\omega_0\tau}$$

    - Thus, the expansion coefficients for the original signal are

$$c_n = \frac{1}{jn\omega_0}c_{1n}^{(1)} + \frac{1}{(jn\omega_0)^2}c_{2n}^{(2)} \qquad n \neq 0$$

$$= -\frac{1}{jn\omega_0}\frac{A}{T} + \frac{1}{(jn\omega_0)^2}\left(\frac{A}{\tau}\frac{1}{T} - \frac{A}{\tau}\frac{1}{T}e^{-jn\omega_0\tau}\right)$$

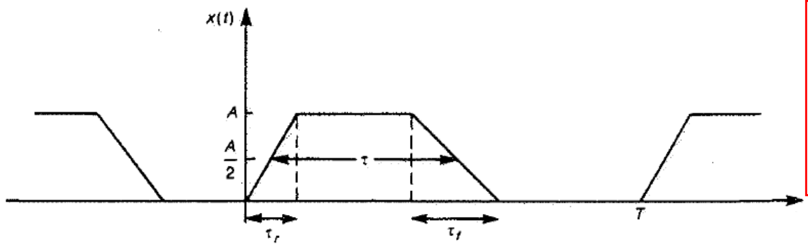
$$= j\frac{A}{n\omega_0 T}\left[1 + j\frac{1}{n\omega_0\tau}\left(1 - e^{-jn\omega_0\tau}\right)\right]$$

$$= j\frac{A}{n\omega_0 T} - j\frac{A}{n\omega_0 T}\frac{\sin(\frac{1}{2}n\omega_0\tau)}{\frac{1}{2}n\omega_0\tau}e^{-jn\omega_0\tau/2}$$

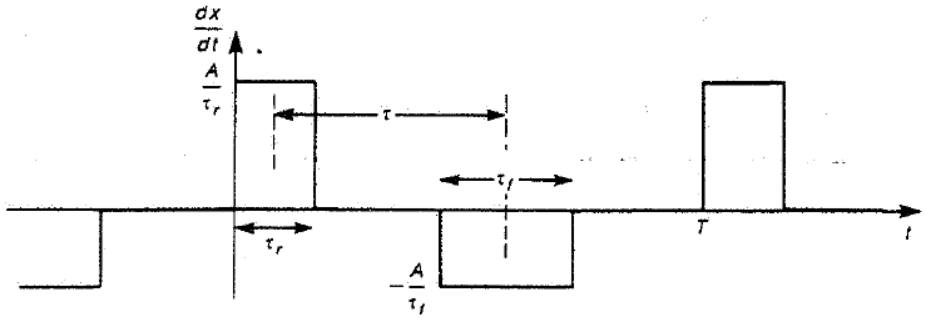
Spectra of Digital Waveforms

- The Spectrum of Trapezoidal (Clock) Waveforms
  - Periodic, Trapezoidal Pulse Train

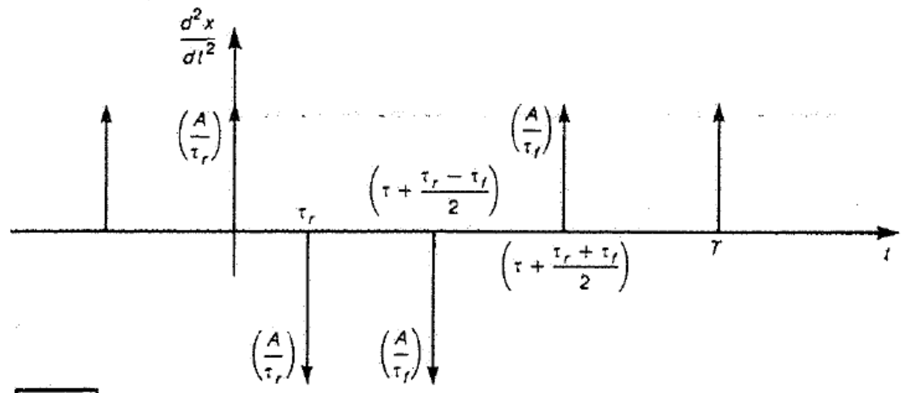
τ_r: rise time, generally 10~90%
τ_f: fall time, generally 10~90%
τ : pulse width, generally midpoints



Differentiation



Differentiation



Spectra of Digital Waveforms

- The Spectrum of Trapezoidal (Clock) Waveforms
  - Periodic, Trapezoidal Pulse Train
    - Since the coefficients for the 2nd derivative are

$$
\begin{aligned}
c_n^{(2)} &= \frac{1}{T}\frac{A}{\tau_r} - \frac{1}{T}\frac{A}{\tau_r} e^{-jn\omega_0\tau_r} - \frac{1}{T}\frac{A}{\tau_f} e^{-jn\omega_0[\tau+(\tau_r-\tau_f)/2]} \qquad (7.61)\\
&\quad + \frac{1}{T}\frac{A}{\tau_f} e^{-jn\omega_0[\tau+(\tau_r+\tau_f)/2]}\\
&= \frac{A}{T}\left[\frac{1}{\tau_r} e^{-jn\omega_0\tau_r/2}\left(e^{jn\omega_0\tau_r/2} - e^{-jn\omega_0\tau_r/2}\right)\right.\\
&\quad \left. - \frac{1}{\tau_f} e^{-jn\omega_0\tau_r/2} e^{-jn\omega_0\tau}\left(e^{-jn\omega_0\tau_f/2} - e^{-jn\omega_0\tau_f/2}\right)\right]\\
&= j\frac{A}{2\pi n}(n\omega_0)^2 e^{-jn\omega_0(\tau+\tau_r)/2}\left[\frac{\sin(\frac{1}{2}n\omega_0\tau_r)}{\frac{1}{2}n\omega_0\tau_r} e^{jn\omega_0\tau/2} - \frac{\sin(\frac{1}{2}n\omega_0\tau_f)}{\frac{1}{2}n\omega_0\tau_f} e^{-jn\omega_0\tau/2}\right]
\end{aligned}
$$

Spectra of Digital Waveforms

- The Spectrum of Trapezoidal (Clock) Waveforms
  - Periodic, Trapezoidal Pulse Train
    - Thus, the coefficients for the original signal are

$$c_n = \frac{1}{(jn\omega_0)^2} c_n^{(2)} \quad n \neq 0$$

$$= -\frac{c_n^{(2)}}{(n\omega_0)^2}$$

$$= -j\frac{A}{2\pi n} e^{-jn\omega_0(\tau+\tau_r)/2}\left[\frac{\sin(\frac{1}{2}n\omega_0\tau_r)}{\frac{1}{2}n\omega_0\tau_r} e^{jn\omega_0\tau/2} - \frac{\sin(\frac{1}{2}n\omega_0\tau_f)}{\frac{1}{2}n\omega_0\tau_f} e^{-jn\omega_0\tau/2}\right]$$

- If $\tau_r = \tau_f$, we obtain

$$c_n = A\frac{\tau}{T}\frac{\sin(\frac{1}{2}n\omega_0\tau)}{\frac{1}{2}n\omega_0\tau}\frac{\sin(\frac{1}{2}n\omega_0\tau_r)}{\frac{1}{2}n\omega_0\tau_r} e^{-jn\omega_0(\tau+\tau_r)/2}$$

Notice that the result can be placed in the form of the product of two sin$x$/x functions.

Spectra of Digital Waveforms

- The Spectrum of Trapezoidal (Clock) Waveforms
  - Periodic, Trapezoidal Pulse Train
    - For the one-sided spectrum, the signal could be expressed as

$$x(t) = c_0 + \sum_{n=1}^{\infty} |c_n^+| \cos(n\omega_0 t + \angle c_n)$$

    - where

$$|c_n^+| = 2\,|c_n| = 2A\,\frac{\tau}{T}\left|\frac{\sin(n\pi\tau/T)}{n\pi\tau/T}\right|\left|\frac{\sin(n\pi\tau_r/T)}{n\pi\tau_r/T}\right| \quad \text{per } n \neq 0$$

$$c_0 = A\,\frac{\tau}{T}$$

    - and the angle is

$$\tau_r = \tau_f$$

$$\angle c_n = \pm\, n\pi\frac{\tau + \tau_r}{T}$$

$$\omega_0 = 2\pi/T$$

Spectra of Digital Waveforms

- The Spectrum of Trapezoidal (Clock) Waveforms
  - Periodic, Trapezoidal Pulse Train
    - Suppose $\tau = T/2$, which means the duty cycle is 50%, the first sine term becomes

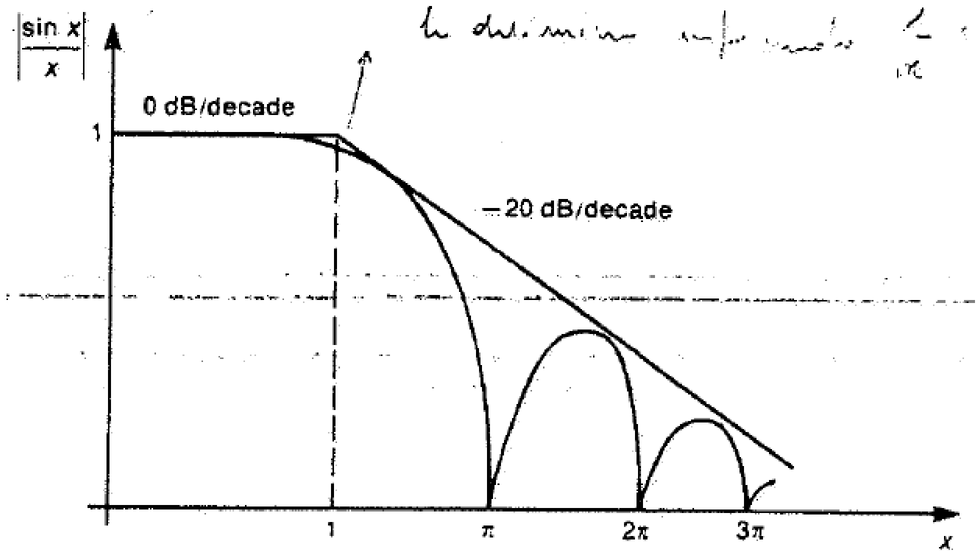
$$|\sin(n\pi\tau/T)|/|n\pi\tau/T| = |\sin \tfrac{1}{2} n\pi|/|\tfrac{1}{2} n\pi|$$

    - which is zero for even *n*.
    - Therefore there are (theoretically) no even harmonics for a 50% duty cycle.
    - The odd-harmonic level are quite stable for slight variations in duty cycle but the even-harmonic level are unstable.
    - Hence, we should prevent from using even $n/T=nf$.

Spectra of Digital Waveforms

- Spectral Bounds for Trapezoidal Waveforms
  - Bounds on the sin*x*/*x* Function
    - For a sinc function, the response is shown below

$$\left|\frac{\sin x}{x}\right| \le \begin{cases} 1 & \text{for small } x \\ \dfrac{1}{|x|} & \text{for large } x \end{cases}$$



Spectra of Digital Waveforms

- Spectral Bounds for Trapezoidal Waveforms
  - Effect of Rise/Falltime on Spectral Content
    - The discrete spectrum could be replaced with a continuous envelope

$$|c_n^+| = 2\,|c_n| = 2A\,\frac{\tau}{T}\left|\frac{\sin(n\pi\tau/T)}{n\pi\tau/T}\right|\left|\frac{\sin(n\pi\tau_r/T)}{n\pi\tau_r/T}\right| \quad \text{per } n \neq 0$$

$$\xrightarrow{\boxed{f = n/T}} = 2A\,\frac{\tau}{T}\left|\frac{\sin(\pi\tau f)}{\pi\tau f}\right|\left|\frac{\sin(\pi\tau_r f)}{\pi\tau_r f}\right|$$

    - Transforming in to dB, we obtain

$$= 20\log_{10}\left(2A\,\frac{\tau}{T}\right) + 20\log_{10}\left|\frac{\sin(\pi\tau f)}{\pi\tau f}\right|$$

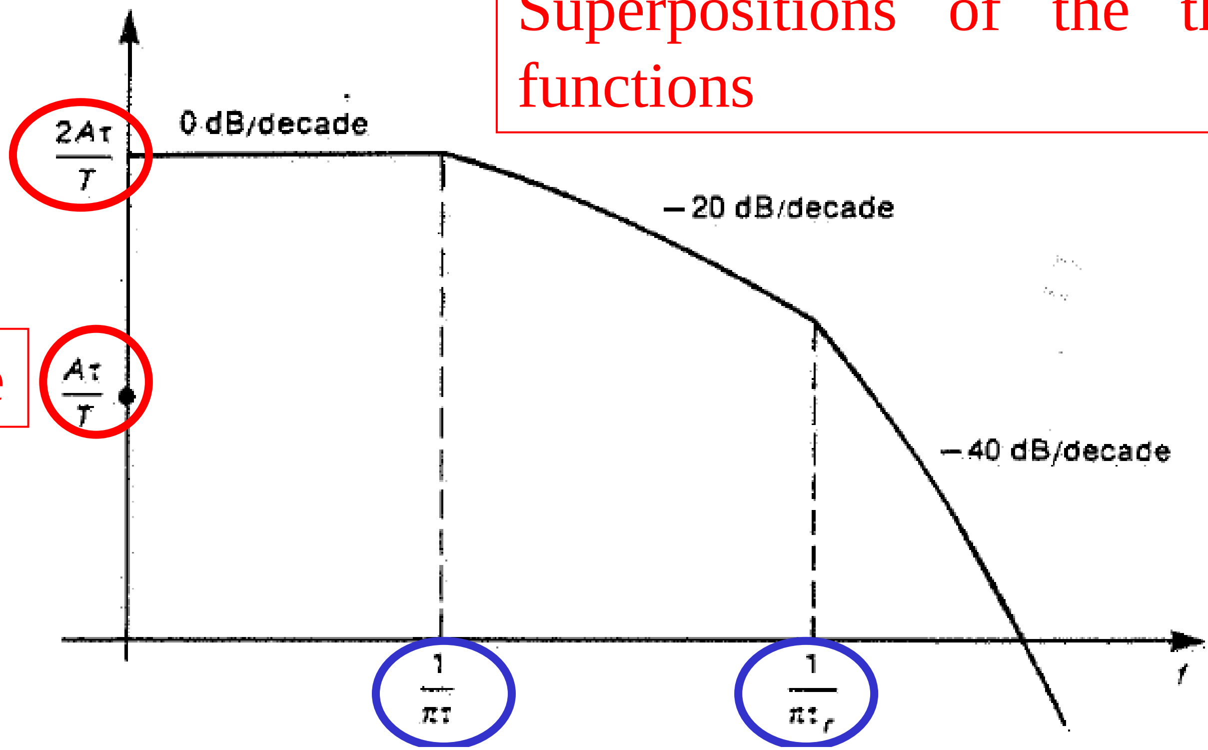
$$+ 20\log_{10}\left|\frac{\sin(\pi\tau_r f)}{\pi\tau_r f}\right|$$

Spectra of Digital Waveforms

- Spectral Bounds for Trapezoidal Waveforms
  - Effect of Rise/Falltime on Spectral Content
    - The Bode plot is shown below

Superpositions of the three functions

DC value



Spectra of Digital Waveforms

- Spectral Bounds for Trapezoidal Waveforms
  - Effect of Rise/Falltime on Spectral Content
    - It is clear that the high-frequency content of a trapezoidal pulse train is due primarily to the rise/falltime of the pulse.
    - Thus, in order to reduce the high-frequency spectrum, which could in turn reduce the emissions of a product, we should increase the rise/falltimes of the clock and/or data pulses

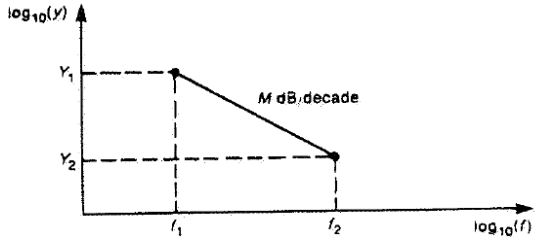
Spectra of Digital Waveforms

- Spectral Bounds for Trapezoidal Waveforms
  - Interpolation on the dB Scale
    - Since on the dB scale, we have the relations

Y and f are natural values.

$$\log_{10} Y_2 - \log_{10} Y_1 = M(\log_{10} f_2 - \log_{10} f_1)$$

$$\longrightarrow \log_{10} Y_2 = \log Y_1 + M \log_{10}\left(\frac{f_2}{f_1}\right)$$



- This relation is applied to result in

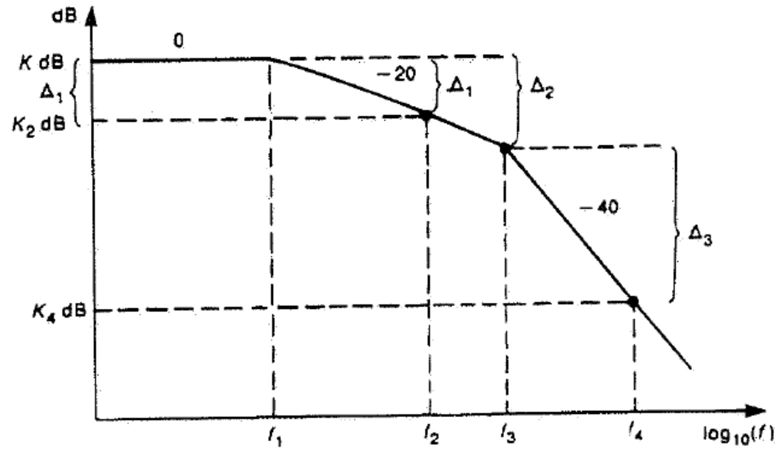
$$K_2 \text{ dB} = K \text{ dB} + \Delta_1$$

$$K_4 \text{ dB} = K \text{ dB} + \Delta_2 + \Delta_3$$

$$\Delta_1 = -20 \log_{10}\left(\frac{f_2}{f_1}\right)$$

$$\Delta_2 = -20 \log_{10}\left(\frac{f_3}{f_1}\right)$$

$$\Delta_3 = -40 \log_{10}\left(\frac{f_4}{f_3}\right)$$



Spectra of Digital Waveforms

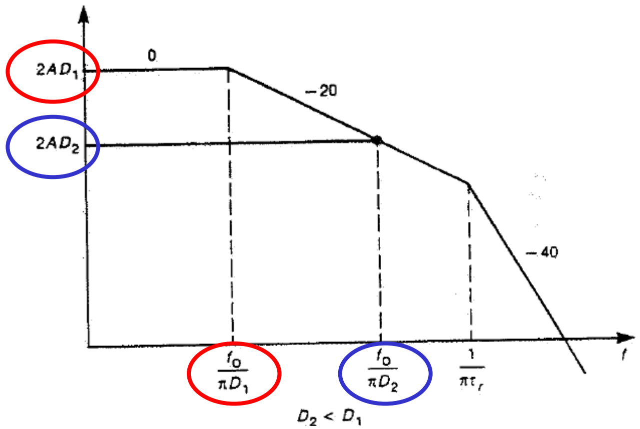
- Bandwidth of Digital Waveforms
  - Definition of Bandwidth
    - From the Bode plot, the bandwidth is defined as 3 times the second breakpoint or $3/\pi\ \tau_r$. This is approximately $1/\tau_r$. Hence we might choose the bandwidth of a digital clock signal as $BW=1/\tau_r$.
    - It is interesting that the first null in the true spectrum occurs at $f=1/\tau_r$.
    - Typically the lower frequencies of the spectrum affect the level of the pulse, while the higher frequencies affect the sharp edges.
    - Judging the BW from the power is *unfair* since 96% of the total average power is contained in the dc term and the first harmonic for this square wave.

Spectra of Digital Waveforms

- Effect of Repetition Rate and Duty Cycle
  - Effect of Duty Cycle on the Spectral Bounds
    - Since $D = \frac{\tau}{T}$, the one-sided spectrum for the trapezoidal waveform could be rewritten as

$$|c_n^+| = 2AD \left|\frac{\sin(n\pi D)}{n\pi D}\right| \left|\frac{\sin(n\pi\tau_r f_0)}{n\pi\tau_r f_0}\right| \quad \text{per } n \neq 0$$

$$c_0 = AD$$



Therefore, if we reduce the pulsewidth (i.e. reduce the duty cycle), we will lower the starting level and will also move the first breakpoint out in frequency.

Spectra of Digital Waveforms

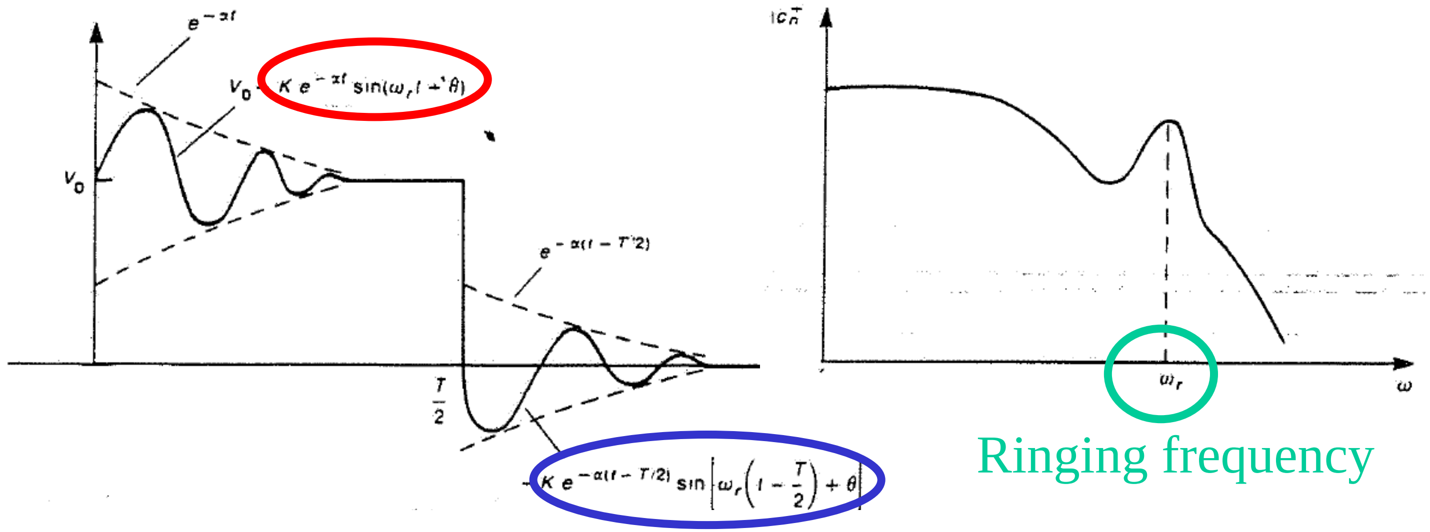
- Effect of Repetition Rate and Duty Cycle
  - Effect of Duty Cycle on the Spectral Bounds
    - It is a simple matter to show that the first breakpoint for the smaller duty cycle D_2 will lie on the -20dB/decade segment for the larger duty cycle D_1.
    - Therefore reducing the duty cycle (the pulsewidth) reduces the low-frequency spectral content of the waveform, but does not affect the high-frequency content.

Spectra of Digital Waveforms

- Effect of Ringing (Undershoot/Overshoot)
  - Ringing
    - Inductance and capacitance of PCB lands and wires in a digital system can cause a phenomenon referred to as ringing.
    - Quite often a discrete resistor is placed in series with the output land of the driving gate to damp this and provide a smooth transition.
    - Also, ferrite beads or matching the transmission lines could be used to solve this problem.

Spectra of Digital Waveforms

- Effect of Ringing (Undershoot/Overshoot)
  - Ringing
    - The responses in time and frequency domain are shown below



Consequently, undershoot/overshoot will have the effect of increasing the emissions about the ringing frequency.

Spectra of Digital Waveforms

- Effect of Ringing (Undershoot/Overshoot)
  - Ringing
    - For a typical ringing problem, the expansion coefficients are expressed as the superposition of

$$
\begin{aligned}
c_n &= c_{n\,\text{onda quadra}} + \frac{1}{T}\int_0^{T/2} Ke^{-\alpha t}\sin(\omega_r t+\theta)\,e^{-jn\omega_0 t}\,dt \\
&\quad - e^{-jn\omega_0 T/2}\left[\frac{1}{T}\int_0^{T/2} Ke^{-\alpha t}\sin(\omega_r t+\theta)\,e^{-jn\omega_0 t}\,dt\right] \\
&= c_{n\,\text{onda quadra}} + \left(1-e^{-jn\omega_0 T/2}\right)\frac{1}{T}\int_0^{T/2} Ke^{-\alpha t}\sin(\omega_r t+\theta)\,e^{-jn\omega_0 t}\,dt \\
&= \boxed{\frac{V_0}{2}\,\frac{\sin(\frac{1}{4}n\omega_0 T)}{\frac{1}{4}n\omega_0 T}\,e^{-jn\omega_0 T/4}} + \boxed{\frac{K}{2}\,\frac{\sin(\frac{1}{4}n\omega_0 T)}{\frac{1}{4}n\omega_0 T}\,e^{-jn\omega_0 T/4}\,\frac{p\omega_r}{p^2+2\alpha p+\alpha^2+\omega_r^2}}
\end{aligned}
\qquad (7.73)
$$

Square wave

Ringing wave

Spectra of Digital Waveforms

- Use of Spectral Bounds
  - In Computing Bounds on the Output Spectrum of a Linear System
    - A linear system having input $x(t)$, output $y(t)$, and impulse response $h(t)$ has an output spectrum

$$Y(jn\omega_0) = H(jn\omega_0)X(jn\omega_0)$$

    - Thus the magnitude and phase spectrum of the output are

$$|Y(jn\omega_0)| = |H(jn\omega_0)| \times |X(jn\omega_0)|$$

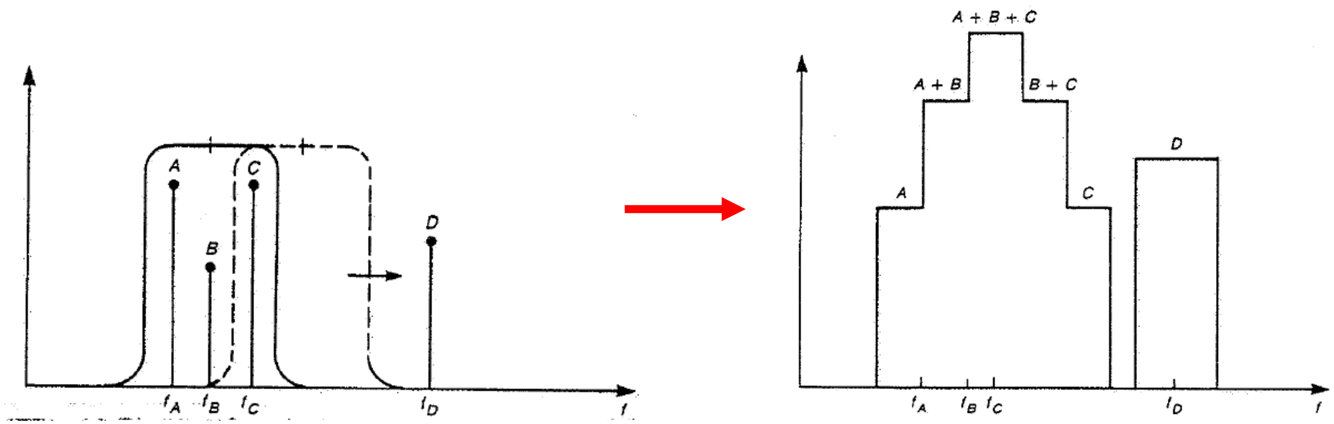
$$\angle Y(jn\omega_0) = \angle H(jn\omega_0) + \angle X(jn\omega_0)$$

    - Transforming into dB format

$$20\log_{10}|Y(jn\omega_0)| = 20\log_{10}|H(jn\omega_0)| + 20\log_{10}|X(jn\omega_0)|$$

Spectrum Analyzers

- Basic Principles
  - Definition
    - Spectrum analyzers are devices that display the magnitude spectrum for periodic signals.
    - The devices are radio receivers having a bandpass filter that is swept in time.
    - If the resolution bandwidth is too large, incorrect output spectrum is obtained.

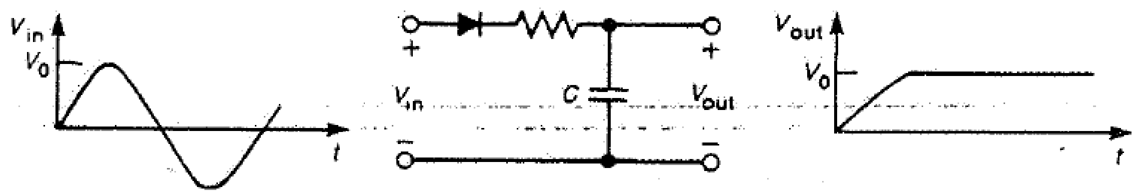


Spectrum Analyzers

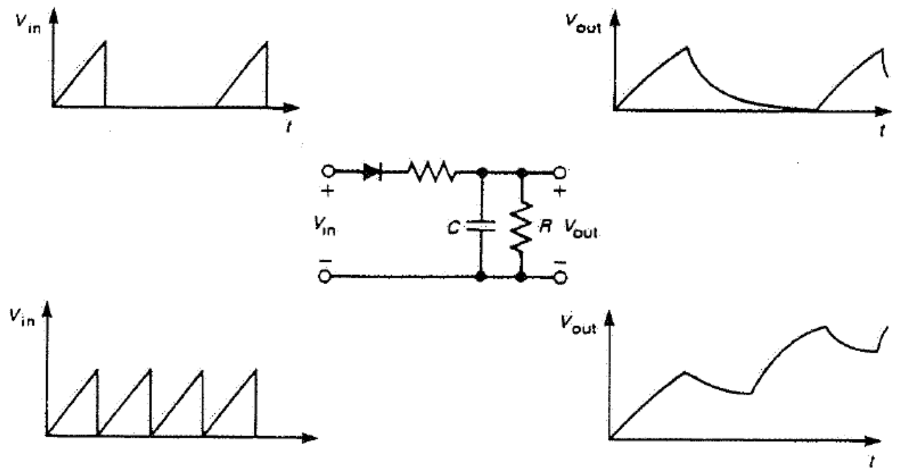
- Basic Principles
  - Resolution Bandwidth
    - The bandwidth is 6 dB bandwidth, where the response is reduced by 6 dB from its maximum level at the center frequency.
    - In order to obtain the lowest possible level on the SA display, we should choose as small as a bandwidth as possible.
    - We should attempt to choose clock and data repetition rates such that none of the harmonics of any signal in the system will be closer than the measurement bandwidth of the SA.

Spectrum Analyzers

- Peak versus Quasi-Peak
  - Peak Detector
    - Actually the peak value displayed on SA is RMS.



  - Quasi-Peak Detector
    - The regulatory requirements are measured with such kind of detector.



Representation of Nonperiodic Waveforms

- The Fourier Transform
  - Envelope of Spectral Components
    - From a square pulse train to a single square pulse ($T \rightarrow \infty$, $f_o = 1/T \rightarrow 0$)

$$|c_n| = \frac{A\tau}{T}\left|\frac{\sin(n\pi\tau/T)}{n\pi\tau/T}\right| \xrightarrow{T\rightarrow\infty,\ f_o=1/T\rightarrow 0} = \frac{A\tau}{T}\,\frac{\sin(\pi f\tau)}{\pi f\tau}$$

The spectrum of a single pulse is a continuum of frequency components.

  - General Form
    - Fourier Transform

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}\,dt \xrightarrow{T\rightarrow\infty,\ n\omega_o=\omega} \mathscr{F}\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

    - Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$x(t)$ consists of a continuum of complex sinusoids.

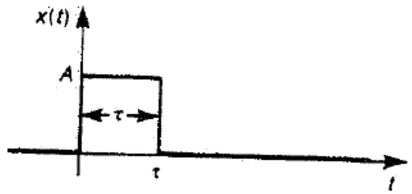
Representation of Nonperiodic Waveforms

- The Fourier Transform
  - Example for a Single Pulse
    - The Fourier transform of a single pulse is

$$\begin{aligned} X(j\omega) &= \int_0^{\tau} A e^{-j\omega t}\, dt \\ &= -\frac{A}{j\omega}\left(e^{-j\omega\tau} - 1\right) \\ &= -\frac{A}{j\omega} e^{-j\omega\tau/2}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) \\ &= A\tau \frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau} e^{-j\omega\tau/2} \end{aligned}$$

$$|X(j\omega)| = A\tau \left| \frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau} \right|$$

$$\angle X(j\omega) = \pm \frac{1}{2}\omega\tau$$



    - From this, we observe that the coefficients of the complex-exponential Fourier series of a periodic train of such pulse could be obtained from the single pulse

$$\boxed{c_n = \frac{1}{T} X(jn\omega_0)}$$

Representation of Nonperiodic Waveforms

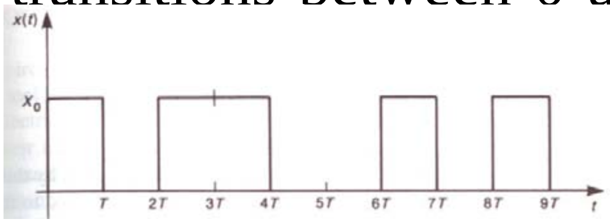
- Response of Linear Systems to Nonperiodic Inputs
  - Same as Periodic Signals
    - All the properties derived for periodic functions and the Fourier series—linearity, superposition, differentiation, time shifting, impulse functions—apply to the Fourier transform.
    - The Fourier transform of the output of a linear system is the product of the Fourier transforms of the input to that system and the impulse response of that system.

$$Y(j\omega) = H(j\omega)X(j\omega)$$

Representation of Random (Data) Signals

- Autocorrelation Function and Power Spectral Density
  - Autocorrelation Function
    - A random waveform that transitions between 0 and X_0 can be described as

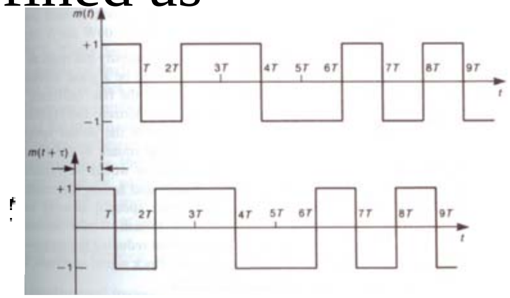
$$x(t) = \frac{1}{2}X_0[1 + m(t)]$$



- Where $m(t)$ is a random variable that assumes values of ±1 with equal probability in the bit interval $nT < t < (n+1)T$
- The autocorrelation function is defined as

$$R_x(\tau) = \overline{x(t)x(t+\tau)}$$

$$= \lim_{t \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)\,dt$$

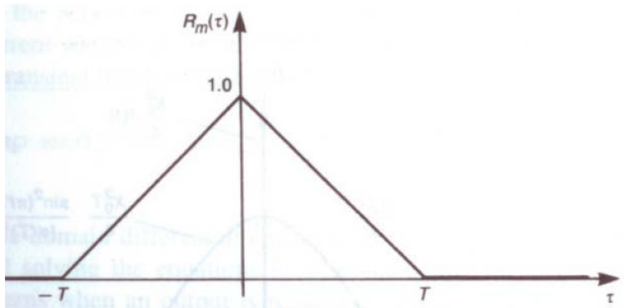


Representation of Random (Data) Signals

- Autocorrelation Function and Power Spectral Density
  - Autocorrelation Function
    - which could be simplified as

$$
\begin{aligned}
R_x(\tau) &= \tfrac{1}{4}X_0^2\overline{[1+m(t)][1+m(t+\tau)]} \\
&= \tfrac{1}{4}X_0^2[1+\overline{m(t)}+\overline{m(t+\tau)}+\overline{m(t)m(t+\tau)}] \\
&= \tfrac{1}{4}X_0^2[1+\overline{m(t)m(t+\tau)}] \\
&= \tfrac{1}{4}X_0^2[1+\boxed{R_m(\tau)}]
\end{aligned}
$$

$$
\begin{aligned}
R_m(\tau) &= 1-\frac{|\tau|}{T} \qquad \text{for } |\tau| < T \\
&= 0 \qquad \text{for } |\tau| > T
\end{aligned}
$$



Representation of Random (Data) Signals

- Autocorrelation Function and Power Spectral Density
  - Power Spectral Density
    - The power spectral density (according to Wiener-Kinchine theorem) is defined as

$$G_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} \, d\tau \quad \text{W/Hz}$$

    - The average power associated with this signal is

$$P_{av} = \int_{-\infty}^{\infty} G_x(f) \, df \quad \text{W}$$

Pulse Code Modulation - None Return to Zero

    - Thus, for the PCM-NRZ waveform, the power spectral density is

$$G_x(f) = \frac{X_0^2}{4} \delta(f) + \frac{X_0^2 T}{4} \frac{\sin^2(\pi f T)}{(\pi f T)^2} \quad \text{W/Hz}$$

Representation of Random (Data) Signals

- Autocorrelation Function and Power Spectral Density
  - Power Spectral Density
    - This could also be obtained from the Fourier series of square pulse train by replacing A, τ in the square wave with X_0, T, respectively, and squaring the result to give power.

This makes sense since $m(t)$ is of equal probability in each interval. Half of the intervals are 1s and half of the intervals are 0s (like a square pulse train).