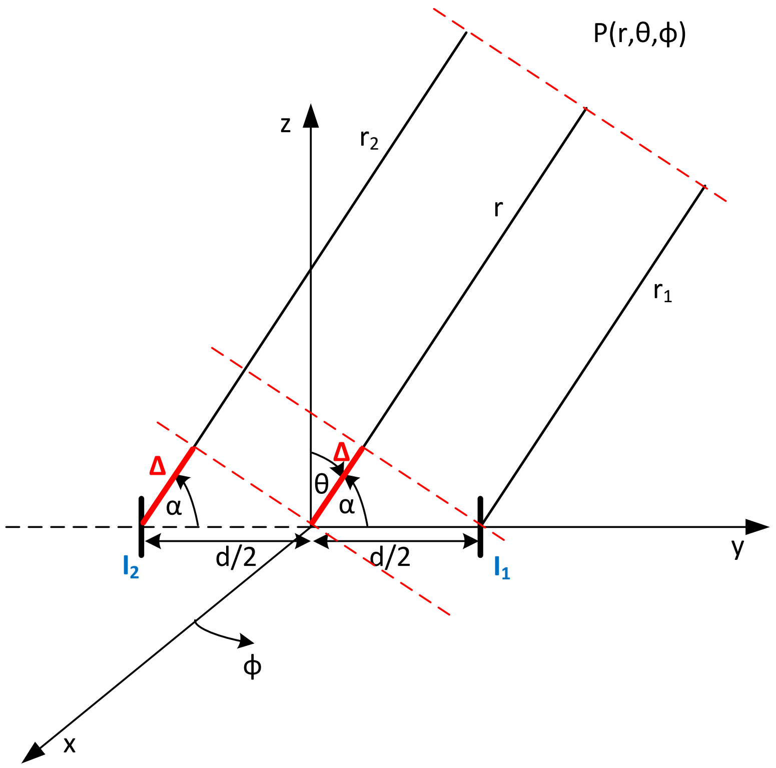
Consider two infinitesimal current elements (Hertz elementary dipoles) separated by distance d and excited in antiphase currents ($I_1 = -I, I_2 = I$) as shown in figure below.



a) Find the far zone electric and magnetic fields
b) For the case $d \ll \lambda$, find the ratio of the radiated power of the pair of current elements to that of a single current element IL

Solution

a) For a point $P(r, \theta, \phi)$ in the far zone

$$r_1 \cong r - \Delta = r - \frac{1}{2} d \cos \alpha$$

$$r_2 \cong r + \Delta = \mathrm{r} + \frac{1}{2} d \cos \alpha$$

$$\cos \alpha = \boldsymbol{r_0} \cdot \boldsymbol{y_0} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \cdot (0,1,0) = \sin \theta \sin \phi$$

In the above, the unit position vector $\boldsymbol{r_0}$ has been expressed in spherical coordinates.

$$r_1 \cong r - \frac{1}{2} d \sin \theta \sin \phi$$

$$r_2 \cong r + \frac{1}{2} d \sin \theta \sin \phi$$

An elementary dipole of length L aligned to z-axis produces the following electric field:

$$E_\theta(r, \theta) = 30 jkIL \frac{e^{-jkr}}{r} \sin \theta$$

The far-zone electric field of the discussed arrangement is hence given by the superposition (sum) of the two dipoles with currents $+I$ and $-I$

$$E_\theta(r,\theta) = 30jkIL\frac{e^{-jkr_2}}{r_2}\sin\theta + 30jk(-I)L\frac{e^{-jkr_1}}{r_1}\sin\theta$$

Since we are in far-field, we can approximate r_2 and r_1 by r in the denominators:

$$\frac{e^{-jkr_2}}{r_2} \cong \frac{e^{-jk\left(r+\frac{1}{2}d\sin\theta\sin\phi\right)}}{r} = \frac{e^{-jkr}}{r}e^{-jk\frac{1}{2}d\sin\theta\sin\phi}$$

$$\frac{e^{-jkr_1}}{r_1} \cong \frac{e^{-jk\left(r-\frac{1}{2}d\sin\theta\sin\phi\right)}}{r} = \frac{e^{-jkr}}{r}e^{+jk\frac{1}{2}d\sin\theta\sin\phi}$$

$$E_\theta(r,\theta,\phi) = 30jkIL\frac{e^{-jkr}}{r}\sin\theta\left[e^{-j\frac{1}{2}kd\sin\theta\sin\phi} - e^{+j\frac{1}{2}kd\sin\theta\sin\phi}\right]$$

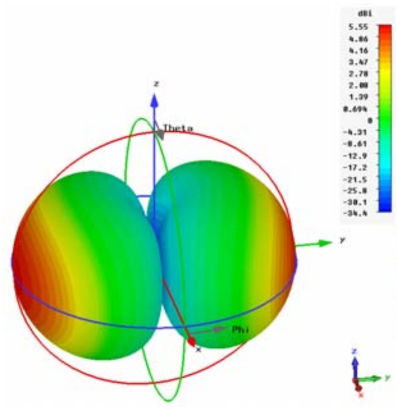
Noting that $\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) \rightarrow \left(e^{jx} - e^{-jx}\right) = 2j\sin x$

If we set $x = -\frac{1}{2}kd\sin\theta\sin\phi$, and further use $\sin(x) = -\sin(-x)$ the result is

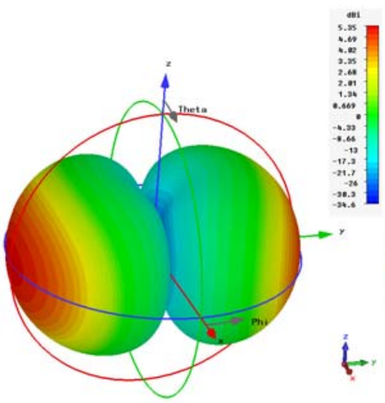
$$E_\theta(r,\theta,\phi) = 60kIL\frac{e^{-jkr}}{r}\sin\theta\sin\left(\frac{1}{2}kd\sin\theta\sin\phi\right) = C\frac{e^{-jkr}}{r}\mathrm{ElementPattern}\cdot\mathrm{ArrayFactor}$$

Full 3D radiation patterns for separations $d = 0.1\lambda, 0.25\lambda, 0.5\lambda, 0.75\lambda, 1\lambda$ are shown below. Note that the composite pattern is now function of ϕ also, due to the transformation $\cos\alpha = \sin\theta\sin\phi$.

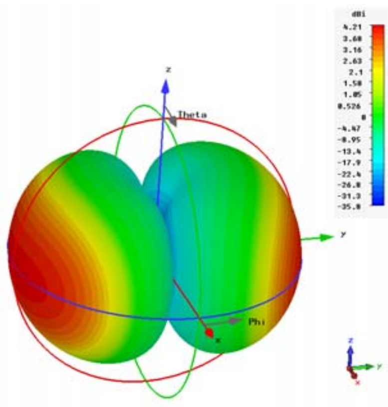
Magnetic field in the far-zone is simply given as $H_\phi(r,\theta,\phi) = \frac{E_\theta(r,\theta,\phi)}{Z_0}$



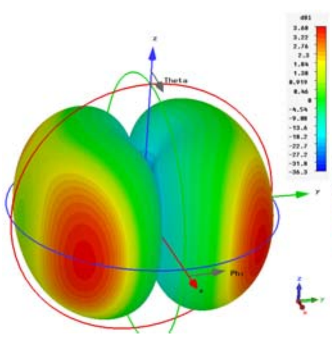
$d = 0.1\lambda$



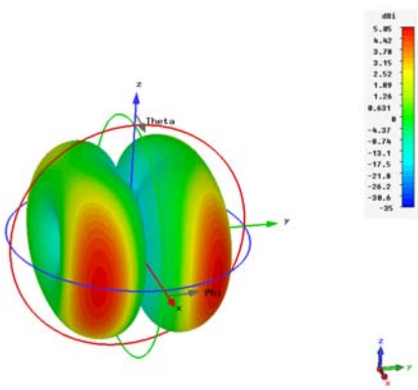
$d = 0.25\lambda$



$d = 0.5\lambda$



$d = 0.75\lambda$



$d = 1\lambda$

b) radiated power

For $d \ll \lambda$, $kd \to 0$ and the factor $\frac{1}{2}kd\sin\theta\sin\phi$ is small. Using Taylor approximation

$$\sin\left(\frac{1}{2}kd\sin\theta\sin\phi\right) \approx \frac{1}{2}kd\sin\theta\sin\phi$$

$$E_\theta(r,\theta,\phi) = 60kIL\frac{e^{-jkr}}{r}\sin\theta\frac{1}{2}kd\sin\theta\sin\phi = 30k^2dIL\frac{e^{-jkr}}{r}\sin^2\theta\sin\phi$$

The radiated power is obtained by integration of the radial power flow $S_r = \frac{|E_\theta(r,\theta,\phi)|^2}{2Z_0}$ over sphere:

$$\begin{aligned}P_{rad} &= \oiint_S \boldsymbol{S}\cdot d\boldsymbol{S} = \frac{1}{2Z_0}\oiint_S |E_\theta(r,\theta,\phi)|^2 dS \\ &= \frac{(30k^2dIL)^2}{2Z_0}\oiint_S \frac{(\sin^2\theta\sin\phi)^2}{r^2} r^2\sin\theta\, d\theta d\phi \\ &= \frac{(30k^2dIL)^2}{2Z_0}\int_0^\pi \sin^5\theta d\theta \int_0^{2\pi} \sin^2\phi d\phi = \frac{(30k^2dIL)^2}{2Z_0}\frac{16}{15}\pi = 4k^4d^2(IL)^2\end{aligned}$$

The radiated power of one dipole was already evaluated to be $10(kIL)^2$ so the requested ratio is

$$\frac{P_{rad}(two\ dipoles)}{P_{rad}(one\ dipole)} = \frac{4k^4d^2(IL)^2}{10(kIL)^2} = \frac{2}{5}k^2d^2 = 15.78\left(\frac{d}{\lambda}\right)^2$$

This arrangement is equivalent to one horizontal dipole located $d/2$ above PEC ground plane.

If the currents will be in-phase, $I_1 = I_2 = I$, we have

$$E_\theta(r,\theta,\phi) = 30jkIL\frac{e^{-jkr}}{r}\sin\theta\left[e^{-j\frac{1}{2}kd\sin\theta\sin\phi} + e^{+j\frac{1}{2}kd\sin\theta\sin\phi}\right]$$

$=60jkIL\frac{e^{-jkr}}{r}\sin\theta\cos\left(\frac{1}{2}kd\sin\theta\sin\phi\right)$

For $d \ll \lambda$, $kd \to 0$ and the factor $\frac{1}{2}kd\sin\theta\sin\phi$ is small. Using Taylor approximation

$$\cos\left(\frac{1}{2}kd\sin\theta\sin\phi\right) \approx 1$$

and the radiation pattern will be very similar to the dipole alone.