**Antenna directivity**

$$D(\theta,\phi) = \frac{U(\theta,\phi)}{U_0} \tag{1}$$

Where $U_0 = \frac{P_r}{4\pi}$ is the radiated power distributed over the sphere (full solid angle in steradian). It is the radiation intensity of isotropic source (reference).

The radiation intensity is related to radial power flow expressed by electric field and plane wave relation (we are far from antenna, theoretically at infinite distance)

$$U(\theta,\phi) = r^2 S_r = r^2 \frac{1}{2Z_0} |\boldsymbol{E}(\theta,\phi)|^2 \tag{2}$$

The radiated power can be expressed by integration of the radial power flow through the enclosing sphere $\mathrm{d}A = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ or equivalently by integrating the radiation intensity over full solid angle $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$. Note that in far field $\boldsymbol{E} \propto 1/r$, $|\boldsymbol{E}|^2 \propto 1/r^2$ so the radial distance cancels with the $\mathrm{d}A$ element.

$$P_r = \oiint S_r \mathrm{d}A = \frac{1}{2Z_0} \oiint |\boldsymbol{E}(\theta,\phi)|^2 \mathrm{d}A = \frac{1}{2Z_0} \oiint r^2 |\boldsymbol{E}(\theta,\phi)|^2 \mathrm{d}\Omega \tag{3}$$

Inserting into (1) we obtain

$$D(\theta,\phi) = \frac{U(\theta,\phi)}{U_0} = \frac{4\pi U(\theta,\phi)}{P_r} = \frac{4\pi |\boldsymbol{E}(\theta,\phi)|^2}{\oiint |\boldsymbol{E}(\theta,\phi)|^2 \mathrm{d}\Omega} = \frac{|\boldsymbol{E}(\theta,\phi)|^2}{(1/4\pi) \oiint |\boldsymbol{E}(\theta,\phi)|^2 \mathrm{d}\Omega} \tag{4}$$

In the denominator we can see radiated power averaged over the full solid angle $4\pi$.

*Maximum directivity*

In most cases we are interested in the maximum directivity. Therefore, it is required to find the direction $(\theta,\phi)$ with maximum intensity $\boldsymbol{E}_{max}$ so directivity become a number

$$D_{max} = \frac{4\pi |\boldsymbol{E}_{max}|^2}{\oiint |\boldsymbol{E}(\theta,\phi)|^2 \mathrm{d}\Omega} = \frac{4\pi}{\oiint |\boldsymbol{f}_n(\theta,\phi)|^2 \mathrm{d}\Omega} = \frac{4\pi}{\Delta\Omega} \tag{5}$$

Where

$$\boldsymbol{f}_n(\theta,\phi) = \left| \frac{\boldsymbol{E}(\theta,\phi)}{\boldsymbol{E}_{max}} \right|^2 \tag{6}$$

Is normalized power pattern and $\Delta\Omega$ is called the beam area.

In (5) we used that

$$D_{max} = \frac{U_{max}}{U_0} = \frac{\frac{P_r}{\Delta\Omega}}{\frac{P_r}{4\pi}} = \frac{4\pi}{\Delta\Omega} \tag{7}$$