# Optimization of Directivity and Signal-to-Noise Ratio of an Arbitrary Antenna Array

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**Abstract—In this paper, a unified formulation is made for the optimization of directivity and signal-to-noise ratio of an arbitrary array, with or without a constraint on the array $Q$-factor. When there is a constraint, the solution is reduced to that of a polynomial; when there is no constraint, the solution is given in a very simple form. First, it is shown that for a given array geometry there exists a finite permissible range of the $Q$-factor and this range reduces to zero for large spacings. Second, a detailed comparison between four well-known excitations (uniform, Hansen-Woodyard, optimum cophasal, and optimum) is made and the main results are as follows. 1) The Hansen-Woodyard excitation yields a directivity higher than that of the uniform only when the element spacing is somewhat smaller than a half wavelength ($\lambda/2$), but at the price of much higher $Q$. On the other hand, it is much lower than that of the optimum excitation. 2) For spacing less than $\lambda/2$, the optimum excitation is strongly tapered toward the ends of the array and approximately antiphasa (i.e., 0, 180° 0, 180°, $\cdots$); whereas for spacing greater than or equal to $\lambda/2$, it is nearly uniform and cophasal. 3) For large spacings, the directivity of uniform excitation is nearly optimum. For small spacings, the optimum directivity becomes much higher than all others, but is always associated with an enormously large $Q$-factor. Therefore in this case a constraint of the $Q$-factor is important. 4) Hansen-Woodyard and uniform arrays have the interesting property that their sensitivity factors are independent of spacing.**

**The optimization of signal-to-noise ratio is also demonstrated. In particular, the result shows that although an improvement in gain over the uniform excitation is very difficult to realize in practice, a substantial improvement in signal-to-noise ratio is entirely practical. Other numerical results and some extensions of the theory to aperture antennas are also included.**

## I. Introduction

IN RECENT years there has been a renewal of interest in the optimization of directivity, or directive gain (or simply the gain, for short) of antenna arrays. Twenty years ago Uzkov [1] formulated this problem in a very elegant manner by employing linear transformations. Ten years later Gilbert and Morgan [2], and Uzsoky and Solymar [3] investigated the same problem and introduced the sensitivity, or the tolerance factor which is considered to be of great importance in dealing with arrays with closely spaced elements. More recently Tai [4] extended the investigation to arrays of dipoles and made a very complete computation for broadside and end-fire linear arrays. However, he confined his computation to cophasal excitations and did not investigate the $Q$-factor (or the super-gain ratio) of the array.

A parallel problem which is of great current interest is the minimization of antenna noise. With the advent of many low noise devices, the performance of a receiving system is often limited by the noise in the antenna. The noise which originated from the antenna itself and the transmission lines, being independent of the scan, can be suppressed in some manner. The noise which is originated externally from the environment of the antenna[1] can be minimized if the excitation is properly chosen. Therefore, in this case it is not the gain, but rather the signal-to-noise ratio (SNR) which becomes an important design criterion.

In this paper, a unified formulation is made for optimizing the gain or the signal-to-noise ratio of any arbitrary array of either isotropic or nonisotropic elements, with or without a constraint on the $Q$-factor. A detailed study of the $Q$-factor is also made, and it shows that there exists a permissible range of the $Q$-factor which is dependent only on the array geometry and, most of all, on the element spacing. In case the gain of a linear array is optimized under no constraint on the $Q$-factor, the solution is reduced to a very simple form, which is in agreement with that obtained by others. Moreover, in this case, a quantitative comparison is made for four different excitations: uniform, Hansen-Woodyard, optimum cophasal, and unrestricted optimum. When there is a constraint on the $Q$-factor, the problem becomes much more difficult. Uzsoky and Solymar [3], and Solymar [5] used the same formulation for the optimum gain problem, but they actually gave neither the solution nor the numerical method by which the solution could be obtained. In this paper it is shown that the solution can be reduced to that of a polynomial for which many standard techniques are available. To illustrate this, some numerical examples are considered. Last, it will be shown that the solution given here is also applicable to arrays of continuous excitations. With a little modification it can also be generalized to the case where the desired signal source is extended over a region in space, instead of a point.

## II. Formulation of the Problem

An arbitrary array whose excitation is characterized by $N$ degrees of freedom is considered here. It is well

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[1] It may be of interest to note that the available noise output of any antenna in an environment of uniform temperature distribution is constant regardless of how the elements are connected or weighted.

known that for a given number of degrees of freedom in a sufficiently small aperture, or for a given aperture with a sufficiently large number of degrees of freedom, in theory, very high gain can be obtained at the expense of an astronomically large super-gain ratio. It will be seen later that in this case the matrix which must be inverted in order to determine the optimum excitation becomes ill-conditioned. Therefore, it is not only difficult to achieve such an excitation in practice, but also difficult to evaluate it in theory. To make the optimization more meaningful one must impose a constraint on the super-gain ratio or some similar quantity.

Taylor [6] introduced the super-gain ratio for a line source, but his definition is applicable to sectionally continuous excitation functions. For any practical discrete array the pattern function becomes periodic in the reduced angular variable $u$, or $\sin\theta$; thus his definition ceases to be meaningful [8]. Uzsoky and Solymar [2], on the other hand, introduced the $Q$-factor of a linear discrete array through a circuit approach. Now, by properly defining the scalar product of the excitation function, these two factors not only can be unified but also can be applied to arrays of high dimensions.

Let us introduce the *bra-ket* notations:[2] $\langle J$ denotes a row vector $[J_1, J_2, \cdots, J_N]$ and $J\rangle$ a column vector, namely the transpose of $\langle J$. Then $\langle JJ^*\rangle$ denotes the scalar product, or $\sum|J_K|^2$, between $\langle J$ and $J^*\rangle$, which is the complex conjugate of $J\rangle$. The $Q$-factor of an array is defined by

$$Q = \frac{\langle JJ^*\rangle}{\int_{4\pi} |P(\theta, \phi)|^2 d\Omega} \tag{1}$$

where $P(\theta, \phi)$ is the pattern function of the discrete array with element excitations equal to $J_1, J_2, \cdots, J_N$, and $d\Omega$, the increment of solid angle, is equal to $\sin\theta d\theta d\phi$. For the case of continuous or sectionally continuous aperture excitation function $J(\bar{r})$, the scalar product $\langle JJ^*\rangle$, as defined in the functional space, equals $\int|J(\bar{r})|^2 d\bar{r}$. Then the $Q$-factor becomes identical to Taylor's super-gain ratio in the case of linear array. As shown by Uzsoky and Solymar [3] the sensitivity factor $S$ which is a measure of the mean square variation of the maximum field with respect to the mean square deviation of the excitation, is related to the $Q$-factor and gain $G$ by

$$S = Q/G.$$

As the aperture decreases, the $Q$-factor increases at a much faster rate than the maximum gain $G$; as a result, $S$ increases very rapidly. Thus, the constraint on the $Q$-factor to a reasonably low value is equivalent to restricting the optimum design within a practical tolerance.

[2] This notation was introduced by P. A. M. Dirac in his immortal book *The Principles of Quantum Mechanics*. London: Oxford University Press. It becomes more widely used today; see, e.g., B. Friedman, *Principles and Techniques of Applied Mathematics*. New York: Wiley, 1961.

First we consider an array of $N$ isotropic elements placed at some prescribed positions in space. Let the current in the $n$th element be $J_n \exp -j\psi_n{}^0$, where

$$\psi_n{}^0 = 2\pi(x_n \sin\theta_0 \cos\phi_0 + y_n \sin\theta_0 \sin\phi_0 + z_n \cos\theta_0);$$

$(x_n, y_n, z_n)$ = Cartesian coordinates of the $n$th element position measured in wavelength; $(\theta_0, \phi_0)$ = angular coordinates of the beam maximum. In this definition $J_n$ becomes real for cophasal excitation and complex otherwise. Again let $\langle J = [J_1, \cdots, J_N]$; then the pattern is given by

$$P(\theta, \phi) = \sum_{n=1}^{N} J_n \exp j(\psi_n - \psi_n{}^0) = \langle JV^*\rangle \tag{2}$$

where $\langle V = [\exp j(\psi_1{}^0 - \psi_1), \cdots, \exp j(\psi_N{}^0 - \psi_N)]$. Now let us assume that the desired signal is in the direction of the beam maximum $(\theta_0, \phi_0)$. Then the received signal power is proportional to $\langle JCJ^*\rangle$ where $C$ is a dyadic, defined by $V_1^*\rangle\langle V_1$ and $\langle V_1 = \langle V|_{\theta=\theta_0, \phi=\phi_0} = [1, \cdots, 1]$. Let us assume that the noise be incoherent and distributed in space according to $T(\theta, \phi)$ in thermal temperature. Then the antenna noise power is proportional to $\langle JAJ^*\rangle$ where the matrix $A$ is hermitian and defined by

$$A = \frac{1}{4\pi}\int_{4\pi} V^*\rangle\langle V\, T(\theta, \phi)\, d\Omega. \tag{3}$$

The most general problem then is to maximize the signal-to-noise ratio:

$$\text{SNR} = \frac{\langle JCJ^*\rangle}{\langle JAJ^*\rangle} \tag{4}$$

under the constraint

$$Q = \frac{\langle JJ^*\rangle}{\langle JBJ^*\rangle} = \text{a given value} \tag{5}$$

where the dyadic $B = A|_{T=1}$. Since $B$ is hermitian and positive definite, it is well known that $Q^{-1}$ is bounded between the smallest and the largest eigenvalue of $B$. As the spacing between adjacent elements becomes very large the quadratic form $\langle JBJ^*\rangle$ becomes nearly a sphere in the $N$-dimensional space spanned by $J_1, J_2, \cdots, J_N$ and the value of $Q$ is nearly unity. Thus the assignment of the value of $Q$-factor *cannot be completely arbitrary*; otherwise, it will lead to no solution. Moreover, if the assigned value of $Q^{-1}$ happens to be equal to one of the eigenvalues of $B$, it can be shown that $J$ must be equal to the corresponding eigenvector of $B$. Consequently, SNR is completely determined. More details will be given later.

In the above formulation, the noise temperature distribution in the Fraunhofer region $T(\theta, \phi)$ is assumed known. This is not impractical, since the noise temperature distribution of the sky is available in literature [7] which indicates that a ribbon region in the celestial sphere containing the milky way has an average temperature much higher than the rest of the sky, depending on the frequency. On the ground, the temperature is almost uniform and equal to 290° K, except for some man-made noise sources and images of the sources in the sky.

## III. Solutions and Discussions

To determine the excitation for maximum signal-to-noise ratio under a constraint on the $Q$-factor, the method of Lagrange may be used. Introducing a scalar multiplier $\Lambda$, the solution is then obtained by making

$$L = \frac{\langle JCJ^*\rangle}{\langle JAJ^*\rangle} + \Lambda \frac{\langle JJ^*\rangle}{\langle JBJ^*\rangle} \tag{6}$$

stationary with respect to the vector $J$ and scalar $\Lambda$. By putting the first variation of $L$ with respect to that of $J$ equal to zero, the solution is (Appendix I),

$$J^*\rangle = qK^{-1}V_1^*\rangle \tag{7}$$

where

$$q = \langle JCJ^*\rangle^{-1}\langle JAJ^*\rangle\langle V_1J^*\rangle, \; K = A + pQB - pI$$

$$p = \Lambda Q\langle JAJ^*\rangle^2\langle JCJ^*\rangle^{-1}, \qquad I = \text{identity matrix.}$$

It may be noted that since only the relative magnitudes of $J_1, \cdots, J_N$, or the directions of the vector $J$ are of interest, the scalar $q$ which multiplies all components may be disregarded. Thus the only unknown in the solution (7) is the scalar $p$, which is proportional to the Lagrange multiplier $\Lambda$. Inserting (7) into (5) one obtains a characteristic equation for $p$,

$$\langle V_1K^{-1}(QB - I)K^{-1}V_1^*\rangle = 0. \tag{8}$$

Since the unknown $p$ is contained in $K$, a direct numerical solution for the above equation is rather difficult. It appears that this difficulty might have hindered some workers from obtaining numerical results in similar problems. To circumvent this difficulty one may see that (8) states merely that the vector $V_1$ is orthogonal to the vector $K^{-1}(QB-I)K^{-1}V_1^*$. It follows that the latter must be in the complementary space of the vector $V_1$. A complete set $\{V_n\}$ with $V_1$ as one of its elements can be easily constructed.

For simplicity, $\{V_n\}$ has been chosen such that

$$\begin{aligned} \langle V_1 &= [1, 1, 1, \cdots, 1] \\ \langle V_2 &= [-1, 1, 0, \cdots, 0] \\ \langle V_3 &= [-1, 0, 1, 0, \cdots, 0] \\ &\cdots\cdots \\ \langle V_N &= [-1, 0, \; \cdots, 0, 1]. \end{aligned} \tag{9}$$

It is clear that these vectors are independent and $V_1$ is orthogonal to all other vectors. Then the vector $K^{-1}(QB-I)K^{-1}V_1^*$ must be a linear combination of the vectors $V_2, V_3, \cdots, V_N$. Let it be

$$K^{-1}(QB - I)K^{-1}V_1^*\rangle = \sum_{n=2}^{N} h_nV_n^*\rangle$$

where $\{h_n\}$ is a set of constant scalars. Using the definition of $K$ and rearranging the above equation, one has

$$WH\rangle = 0 \tag{10}$$

where $W$ is a matrix with vectors $V_1\rangle, W_2\rangle, \cdots, W_N\rangle$ as columns, namely

$$W = [V_1, W_2, \cdots, W_N],$$

$$\begin{aligned} W_n\rangle = {} & p^2(QB - I)V_n^*\rangle + 2pAV_n^*\rangle \\ & + A(QB - I)^{-1}AV_n^*\rangle, \\ & \text{for } n = 2, 3, \cdots, N. \end{aligned}$$

$$\langle H = [-1, h_2, h_3, \cdots, h_N].$$

Since $H$ is not a null vector, the determinant of $W$ in (10) must vanish, i.e.,

$$\det [V_1, W_2, \cdots, W_N] = 0. \tag{11}$$

This results in a polynomial of $2(N-1)$ degrees in the only unknown $p$ and thus the solutions can be numerically determined. One of them will give the absolute maximum of the signal-to-noise ratio. Once $p$ is known, the current is determined from (7) and the problem is therefore solved. It may be noted that (11) is more accessible to numerical solution than (8) since it does not involve the inversion of a matrix which contains the unknown $p$. It should also be pointed out that all the above derivations are made for arrays with isotropic elements. However, they can be easily generalized for arrays with directive elements. To this end, it is only necessary to redefine dyadics $A$ and $B$ such that $V^*\rangle\langle V$ will be weighted by the normalized power pattern of an individual element.

For numerical evaluation, one needs first the explicit forms of $A$ and $B$. Fortunately, elements of $B$, denoted by $b_{nm}$, can be integrated out in closed form for planar arrays with either isotropic elements or vertical dipoles (see Appendix II). Assuming that the antenna elements are in the $xy$-plane, the elements of $B$ are given by

$$\begin{aligned} b_{nm} = b_{mn}^* = e^{-j\psi_{mn}^0} & \left[\frac{\sin 2\pi\rho_{nm}}{2\pi\rho_{nm}} + k\frac{\cos 2\pi\rho_{nm}}{(2\pi\rho_{nm})^2} \right. \\ & \left. - k\frac{\sin 2\pi\rho_{nm}}{(2\pi\rho_{nm})^3}\right], \\ & \text{for } n<m \end{aligned} \tag{12}$$

$$b_{nn} = \frac{2}{3}$$

where

$\psi_{nm}^0 = 2\pi\rho_{nm} \sin\theta_0 \cos(\phi_0 - \alpha_{nm})$

$\rho_{nm} = \sqrt{(x_n-x_m)^2 + (y_n-y_m)^2}$

$\alpha_{nm} = \tan^{-1}[(y_n-y_m)/(x_n-x_m)], \; 0 \le \alpha_{nm} < \pi.$

$k=0$ for isotropic elements, and $k=1$ for vertical dipoles.

TABLE I

FORMULAS FOR OPTIMUM GAIN AND SNR OF AN ARBITRARY ARRAY

| | Current | Gain | SNR | $Q$-factor |
|---|---|---|---|---|
| Definition | $J\rangle$ | $\dfrac{\lvert\langle JV_1^*\rangle\rvert^2}{\langle JBJ^*\rangle}$ | $\dfrac{\lvert\langle JV_1^*\rangle\rvert^2}{\langle JAJ^*\rangle}$ | $\dfrac{\langle JJ^*\rangle}{\langle JBJ^*\rangle}$ |
| Uniform Current Excitation | $V_1\rangle$ | $\dfrac{N^2}{\langle V_1BV_1\rangle}$ | $\dfrac{N^2}{\langle V_1AV_1\rangle}$ | $\dfrac{N}{\langle V_1BV_1\rangle}$ |
| Optimum Gain without Constraint on $Q$ | $B^{*-1}V_1\rangle$ | $\langle V_1B^{-1}V_1\rangle$ | $\dfrac{\langle V_1B^{-1}V_1\rangle^2}{\langle V_1B^{-1}AB^{-1}V_1\rangle}$ | $\dfrac{\langle V_1B^{-2}V_1\rangle}{\langle V_1B^{-1}V_1\rangle}$ |
| Optimum Gain with a Prescribed $Q$ | $F^{*-1}V_1\rangle$ | $\dfrac{\langle V_1F^{-1}V_1\rangle^2}{\langle V_1F^{-1}BF^{-1}V_1\rangle}$ | $\dfrac{\langle V_1F^{-1}V_1\rangle^2}{\langle V_1F^{-1}AF^{-1}V_1\rangle}$ | a given constant |
| Optimum SNR without Constraint on $Q$ | $A^{*-1}V_1\rangle$ | $\dfrac{\langle V_1A^{-1}V_1\rangle^2}{\langle V_1A^{-1}BA^{-1}V_1\rangle}$ | $\langle V_1A^{-1}V_1\rangle$ | $\dfrac{\langle V_1A^{-2}V_1\rangle}{\langle V_1A^{-1}BA^{-1}V_1\rangle}$ |
| Optimum SNR with a Prescribed $Q$ | $K^{*-1}V_1\rangle$ | $\dfrac{\langle V_1K^{-1}V_1\rangle^2}{\langle V_1K^{-1}BK^{-1}V_1\rangle}$ | $\dfrac{\langle V_1K^{-1}V_1\rangle^2}{\langle V_1K^{-1}AK^{-1}V_1\rangle}$ | a given constant |

Actual current in the $n$th element $=J_ne^{-j\psi_n{}^0}$

$\langle J=[J_1, J_2, \cdots, J_N]$, $\langle V_1=[1, 1, \cdots, 1]$

$\psi_n=2\pi(x_n \sin\theta\cos\phi+y_n\sin\theta\sin\phi+z_n\cos\theta)$

$\psi_n{}^0=\psi_n|_{\theta=\theta_0,\ \phi=\phi_0}$

$K=A+p(QB-I)$, $\quad F=K|_{A=B}$

$A=\frac{1}{4}\pi\int_{4\pi}V^*\rangle\langle VT(\theta, \phi)d\Omega$, $\quad B=A|_{T=1}$

$\langle V=[\exp j(\psi_1{}^0-\psi_1), \cdots, \exp j(\psi_N{}^0-\psi_N)]$

As for the dyadic $A$, the explicit form of its elements depends on the temperature distribution. In case of simple distribution functions, the integration can also be carried out in closed form. In general, the quadrature method may be used.

Now it is of interest to discuss the following special cases.

*a) Optimum Gain without Constraint:* To optimize the gain of an array is the same as to optimize the signal-to-noise ratio with $T(\theta, \phi)=1$ and $\Lambda=0$. Therefore, it is not necessary to solve (11) and the solution to the current vector $J$ is simply given by (7) with $A=B$ and $p=0$, namely, $J^*\rangle= B^{-1}V_1^*\rangle$. The optimum gain is equal to $\langle V_1^*B^{-1}V_1\rangle$, or the sum of all the elements of $B^{-1}$. These results have been previously obtained by others [3], [4], [8], [9]. For convenience, they are summarized in row 3 of Table I along with the expressions for the $Q$-factor and the signal-to-noise ratio.

*b) Optimum Gain under a Constraint on the Q-factor:* For the solution in this case, it is only necessary to set $T(\theta, \phi)=1$ or $A=B$. Then $J^*\rangle=F^{-1}V_1^*\rangle$ where $F=B+p(QB-I)$ and $p$ is determined from the determinantal equation (11). Expressions for $G$ and the signal-to-noise ratio are given in row 4 of Table I. This problem was considered by Gilbert and Morgan [2], and Uzsoky and Solymar [3] but the solution of $J$ was not explicitly obtained. It may be remarked that when the spacings between adjacent elements are sufficiently large, as noted before, the $Q$-factor becomes nearly constant. Thus the solution reduces to nearly the same as that in case a).

*c) Optimum Signal-to-Noise Ratio without Constraint:* This case is the same as case a) except that $A$ is not necessarily equal to $B$, depending on the temperature distribution. Likewise, it is not necessary to solve the determinantal equation (11) and the optimum excitation $J\rangle$ is given by $A^{*-1}V_1\rangle$. The results are listed in row 5 of Table I.

*d) Optimum Signal-to-Noise Ratio with Constraint:* This is the most general case considered in this paper. The optimum excitation and the corresponding signal-to-noise ratio are listed in row 6 of Table I along with the formula for the gain $G$.

*e) Uniform Excitation:* The uniform excitation has received a great attention, not only because of its simplicity, but also because of its excellent performance, nearly optimum when the spacing is large. For the purpose of comparison, formulas of relevant quantities are also listed in Table I.

It may be emphasized that the above results are applicable to any arbitrary array, linear or curved, planar or three-dimensional, uniform or nonuniformly spaced, with isotropic or directive elements. As a result, the possible cases which can be analyzed numerically are almost unlimited. For maximum gains of linear broadside and cophasal endfire arrays with no constraint on the $Q$-factor, the reader should refer to Tai [4], whereas for complete and incomplete circular and

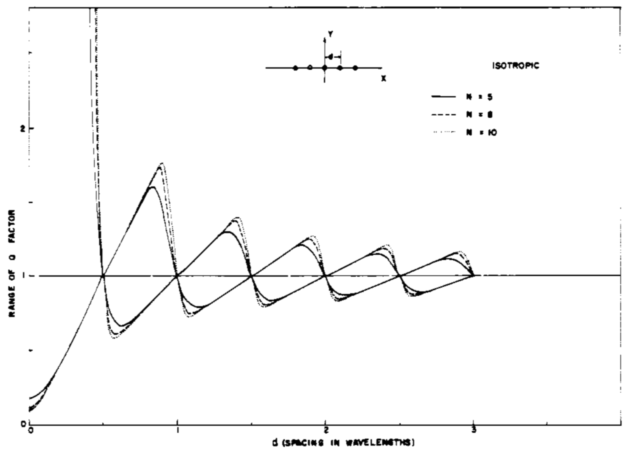


Fig. 1. Permissible range of the $Q$-factor for linear array of $N$ isotropic elements.

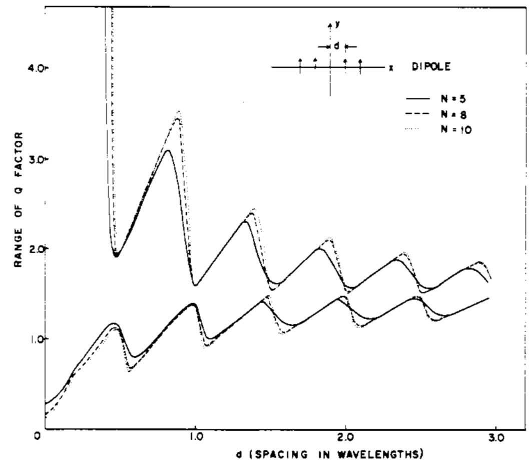


Fig. 2. Permissible range of the $Q$-factor for linear array of $N$ vertical dipoles.

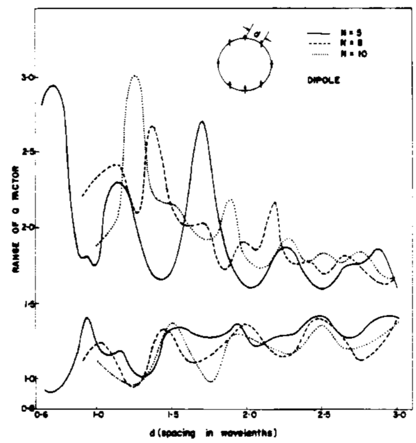


Fig. 3. Permissible range of the $Q$-factor for circular array of $N$ dipoles which are perpendicular to the plane of the array.

elliptical arrays, linear and planar arrays under various conditions, the reader should refer to a technical report by the present authors [11]. In the following, however, we shall first discuss the $Q$-factor of a few typical arrays; second, make a comparison study of a few well-known linear arrays; and third, investigate a semicircular array which is optimized under various conditions. The reason for choosing a circular geometry is simply to demonstrate the generality of the theory and also to consider a more realistic example for the maximization of signal-to-noise ratio.

## IV. Permissible Range of the $Q$-Factor

From the definition (5), $Q$ is independent of the magnitude of the vector $J$. Let the norm of $J$ be unity; then $Q^{-1} = \langle JBJ^* \rangle$, the locus of which is the surface of an ellipsoid in the $N$-dimensional space, since $B$ is positive definite. Thus the value of $Q$-factor is bounded between the smallest and the largest semi-axes of the ellipsoid, or the corresponding eigenvalues of $B$. Since from (12), elements of the matrix $B$ are proportional to $(\sin 2\pi\rho_{nm}/2\pi\rho_{nm})$, $\rho_{nm}$ being the relative spacing between elements, they become negligibly small for widely spaced arrays except for diagonal elements which equal unity. Thus, in this case the ellipsoid becomes nearly a sphere and the $Q$ approaches a constant, regardless of the geometry of the array.

For a linear array of isotropic elements uniformly spaced at multiples of a half wavelength, (12) shows that $B$ is an identical matrix; therefore, $Q=1$, independent of the excitation $J$. From Table I, row 3, it is seen that for these spacings, the maximum gain becomes equal to the number of elements. Therefore, it follows that the optimum gain curve vs. spacing must oscillate about these points and finally approach a constant value which is equal to the number of elements; otherwise, the gain would be completely independent of spacing. In the case of dipoles, the general characteristics remain the same, except that $Q \neq 1$ even for spacings which are multiples of a half wavelength. As spacing increases, $Q$ approaches the value 1.5.

Figure 1 shows the upper and lower bounds of the $Q$-factor as a function of spacing for a linear array with 5, 8, and 10 isotropic elements, respectively. It is seen that for spacing smaller than a certain value where the super-gain effect is possible, the upper bound increases very sharply. This critical spacing is approximately $0.41\lambda$ for 5 elements, and 0.45 for 10 elements. Fig. 2 is similar to Fig. 1 except that it is for vertical dipoles. It is seen that the general characteristics remain the same. To show how the range of $Q$-factor behaves for other geometry, the corresponding graphs for a circular array of vertical dipoles are plotted in Fig. 3. Despite the lack of regularity as compared with linear arrays, the gross feature seems to remain the same.

## V. Uniform, Hansen-Woodyard, and Optimum Arrays

The optimum gain of a linear array with no constraint on the $Q$-factor has been the subject of intensive study in the past [2], [4], [9]. In particular, Tai concluded that as the element spacing is greater than a half wavelength, the optimum gain is only slightly higher than that of a uniform excitation. For spacing smaller than that, super-gain effect prevails and the gain may reach an extremely high value. But he did not investigate the $Q$-factor, nor did he compute the maximum gain for a

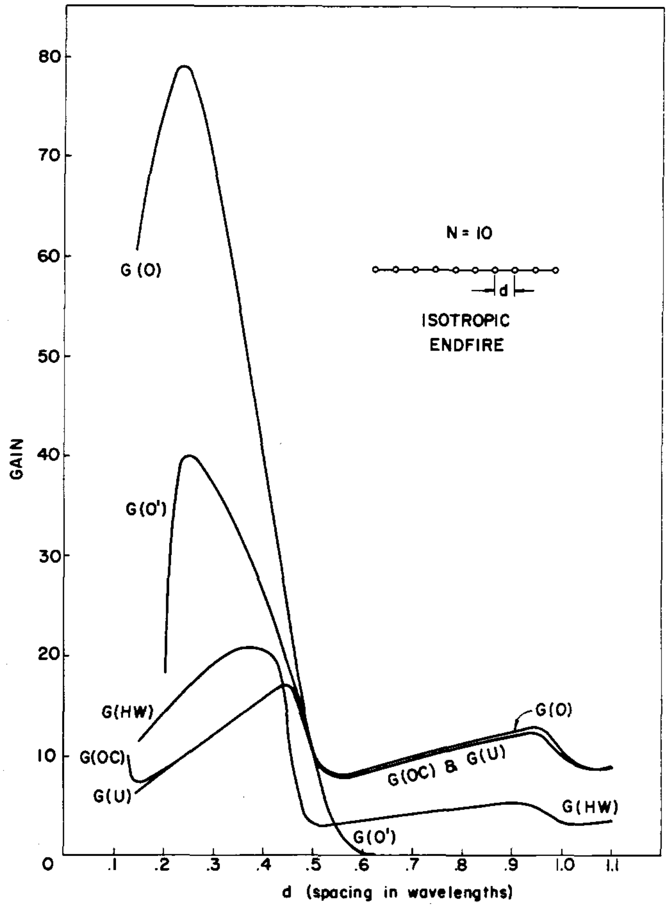


Fig. 4. Directive gain of a linear uniformly spaced end-fire array of 10 isotropic elements with various excitations: $U$: uniform, $OC$: optimum cophasal, $HW$: Hansen-Woodyard, $O$: optimum, $O'$: optimum with phase replaced by antiphase.

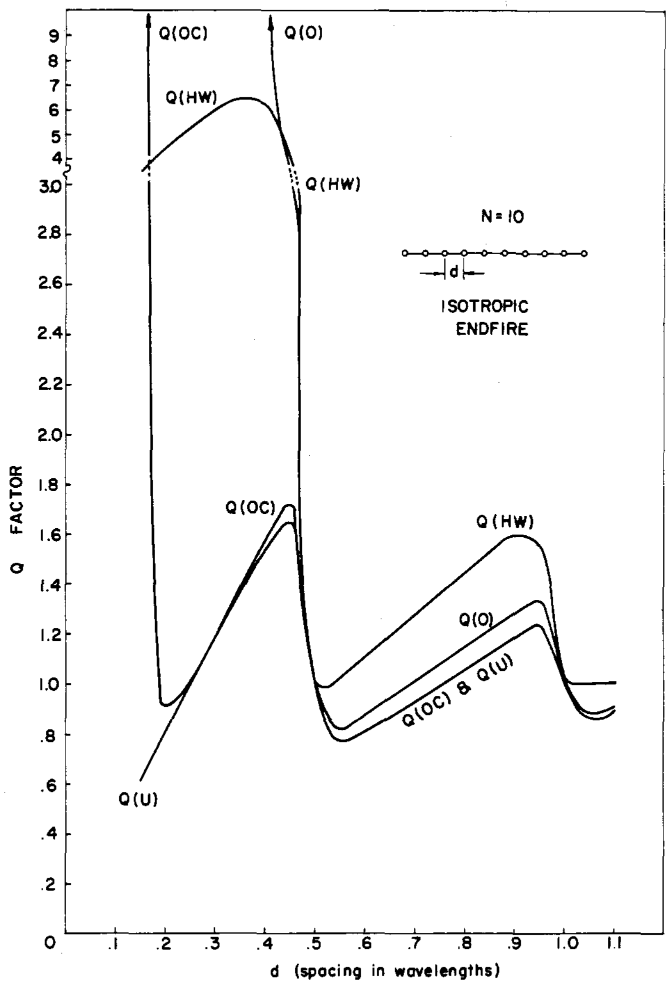


Fig. 5. $Q$-factor of the same array as in Fig. 4.

more general type of array where the excitation is not necessarily cophasal. In the following, we shall make use of the general solution given in Section II and compute the gain and the $Q$-factor of a typical end-fire array of 10 isotropic elements with four different excitations: a) uniform, b) Hansen-Woodyard, c) optimum cophasal, and d) optimum, i.e., one with no restriction on the phase. For simplicity, they will be identified by the symbols $U$, $HW$, $OC$, and $O$, respectively. These gain curves are plotted in Fig. 4. It is interesting to see that the $HW$ excitation yields a gain generally much lower than that of a uniform excitation unless the spacing is less than $0.44\lambda$. (This value may vary with the number of elements; for example, $0.3\lambda$ for a four-element array.) This is expected, since $HW$'s theory [10] is based on a continuous excitation function of constant magnitude and does not apply to discrete arrays. They surmise that it is approximately true for closely spaced elements. However, from the present theory, an exact solution can be obtained, and the truly optimum gain (as denoted by the curve $O$) is many times higher than theirs. This result, however impressive as it may be, shows only part of the story because, as Fig. 5 indicates, the $Q$-factors for both of these excitations reach extremely high values very rapidly at small spacings. On the other hand, there is no great difference among the gain curves denoted by $U$, $OC$, and $O$ for spacing greater than $\lambda/2$, and all equal 10, the number of elements, at $d=$a multiple of $\lambda/2$. The difference between $G(U)$ and $G(OC)$ is even smaller unless $d$ decreases below $0.15\lambda$. After that $G(OC)$ may become very large but again this event is accompanied by a rapid growth in the $Q$-factor. However, the converse is not necessarily true. For example for $d>\lambda/2$, $Q(HW)$ is higher than $Q(U)$ but $G(HW)$ is lower than $G(U)$.

In view of the results in Figs. 4 and 5, it may appear that for $d<0.44\lambda$ the $HW$ excitation offers a good compromise between the uniform and the optimum, since its gain and $Q$-factor fall between those of these two excitations. However, one may actually find an infinite number of such solutions by merely using the formula in row 5 of Table I for any prescribed value of the $Q$-factor in the permissible range.

For academic interest, we have plotted for various spacings the phase and amplitude of the optimum excitations whose gains and $Q$-factors have been designated by $G(O)$ and $Q(O)$ in Figs. 4 and 5. These are shown in Figs. 6 and 7. It is interesting to find that the phase variation is almost perfectly linear (except for very small spacings). This implies that the optimum phase shift per element is a constant which depends on

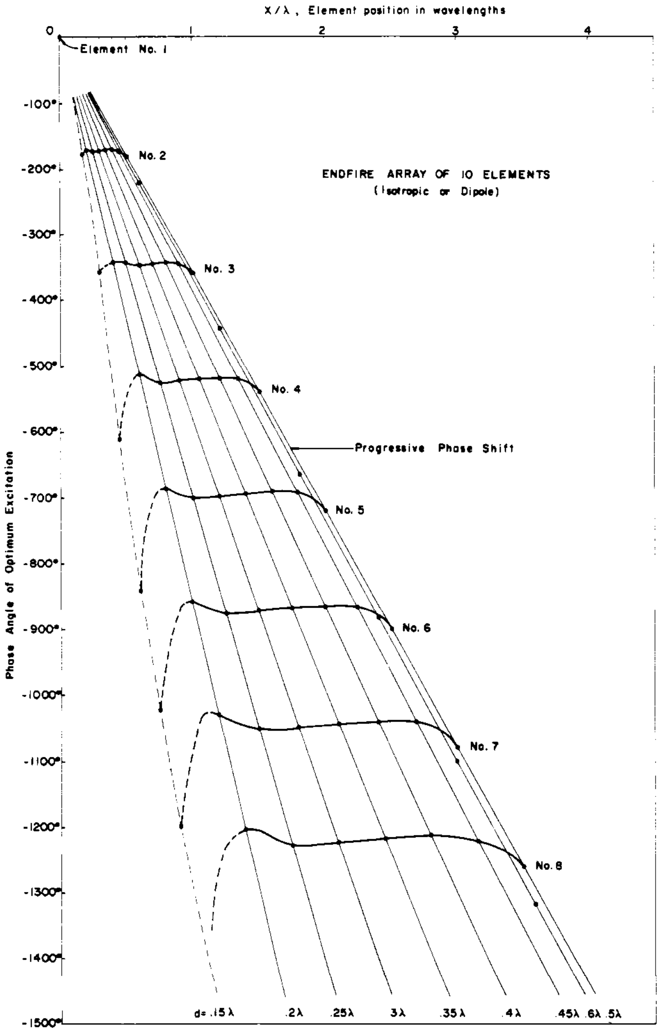


Fig. 6. Phase of the optimum excitation for the array in Fig. 4. Only the phase angles of the first eight elements are shown vs. their positions for various spacings. Those of the remaining elements can be linearly extrapolated. For spacing $d$ less than $\lambda/2$, the optimum excitation is nearly of antiphase; for spacing larger than $\lambda/2$, nearly cophasal. For $d=0.15\lambda$ the data are less accurate.

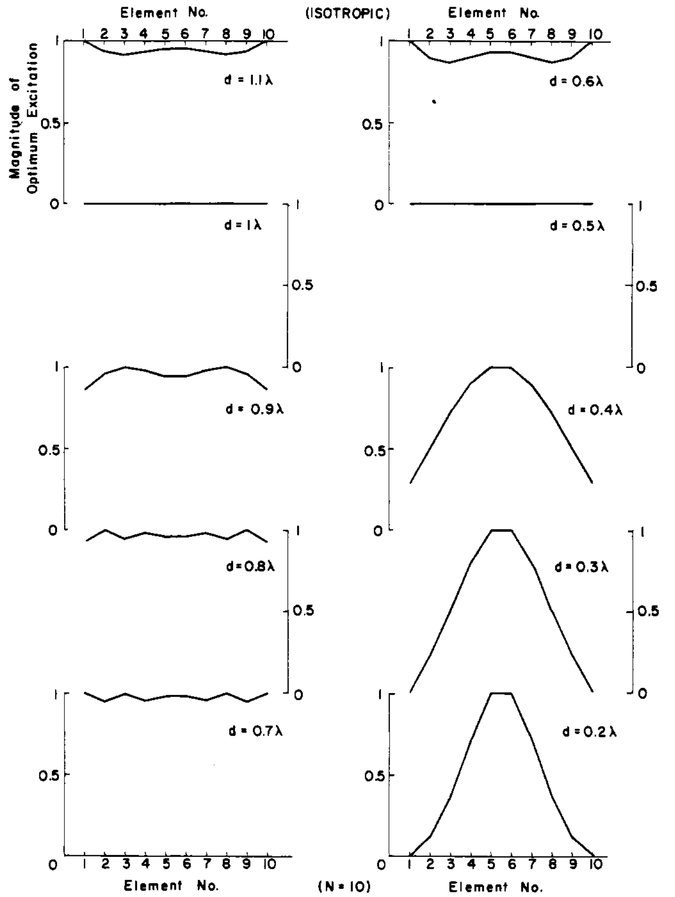


Fig. 7. Magnitude of the optimum excitation for the array in Fig. 4. For spacing $d$ larger than $\lambda/2$, it is nearly uniform.

the spacing and the total number of elements. For spacing $d<\lambda/2$, the optimum phase is larger than the cophasal value (or the progressive phase shift). For $d>\lambda/2$ it is nearly cophasal (exactly cophasal if $d$ is a multiple of $\lambda/2$). If the optimum phase angles of each element, say No. 2, for various spacings from $0.2\lambda$ to $0.5\lambda$, are joined by a curve as shown in Fig. 6, one finds that this curve is nearly a horizontal line with a constant phase angle, approximately equal to a multiple of $\pi$. In other words, the optimum excitation has the interesting property that it is nearly antiphasal for $d\leq\lambda/2$ and nearly cophasal for $d\geq\lambda/2$.

At this point it appears to be of interest to investigate the deterioration which may result if the exact antiphase is used in place of the optimum phase. The resulting gain curve is shown as $G(O')$ in Fig. 4. This curve also passes through the point $G=10$ at $d=\lambda/2$ as it should. However, for $d>\lambda/2$, the gain (in the end-fire direction) drops to near zero, whereas for $d<\lambda/2$ it is still substantially larger than those of uniform, optimum cophasal, and $HW$ excitations.

Figure 7 shows the magnitude of optimum excitation. It is clearly seen that for $d\geq\lambda/2$, it is nearly equal to a constant. From this and the results in Fig. 6, it is again clear that the optimum gain can not be much different from that of the uniform excitation for large spacings.

So far we have considered an end-fire array with 10 isotropic elements. In case of dipoles, the optimum excitation has an almost identical phase characteristics as that shown in Fig. 6 but a slightly different magnitude variation.

Figure 8 shows the sensitivity factor vs. the spacing for the previously mentioned four different excitations. It is interesting to see that both uniform and $HW$ excitations have a constant sensitivity factor, i.e., independent of spacing. It can be easily shown that it equals $N^{-1}$ for the uniform excitation, and $N\tan^2[\pi/2(N-1)]$ or $N\sin^2[\pi/2(N-1)]$, depending on whether $N$ is even or odd, respectively, for the $HW$ excitation. As $N$ becomes very large, the sensitivity factor for the $HW$ excitation approaches $\pi^2/4N$, which is 2.46 times that for the uniform excitation.

For optimum and optimum cophasal excitations, the sensitivity factors are nearly equal to that of the uniform case until the spacing becomes less than a certain number, about $0.45\lambda$ for the former and $0.175\lambda$ for the latter. After that spacing the sensitivity factor increases

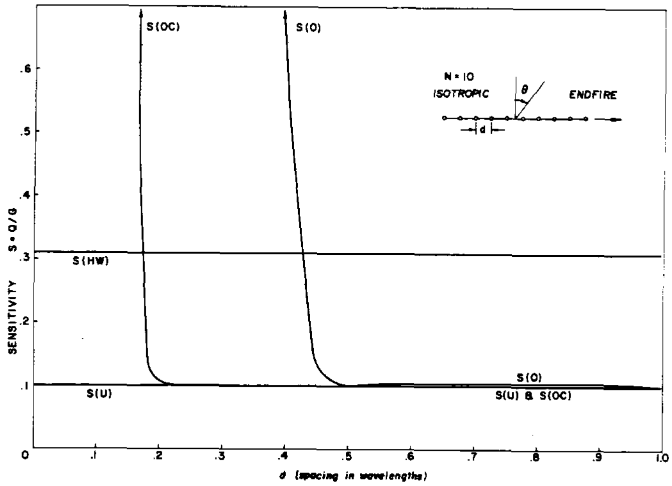


Fig. 8. Sensitivity factor of the same array (under various excitations) as in Fig. 4. It is independent of spacing for uniform and Hansen-Woodyard excitations.

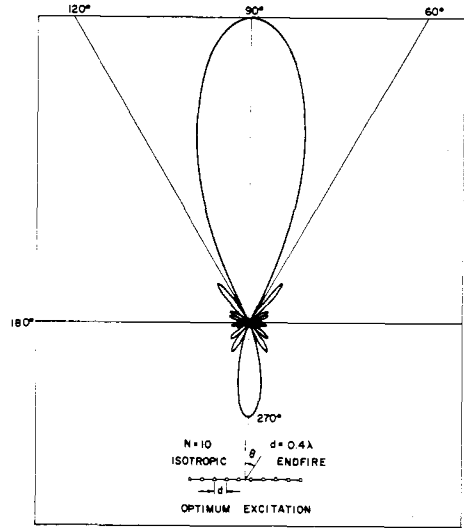


Fig. 10. Radiation pattern of a linear end-fire array of 10 isotropic elements spaced at 0.4λ, with optimum excitation (see Figs. 4–7).

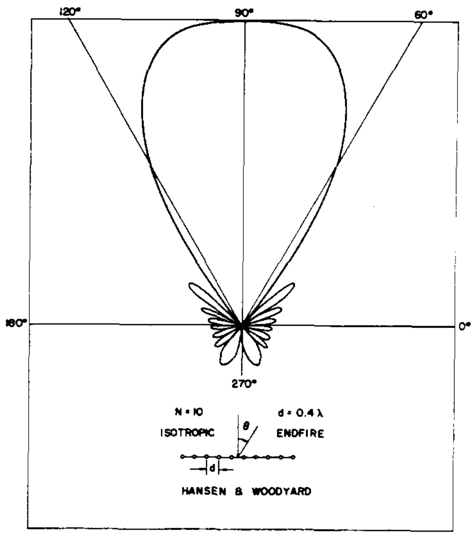


Fig. 9. Radiation pattern of a Hansen-Woodyard array of 10 isotropic elements spaced at 0.4λ (see Figs. 4 and 5).

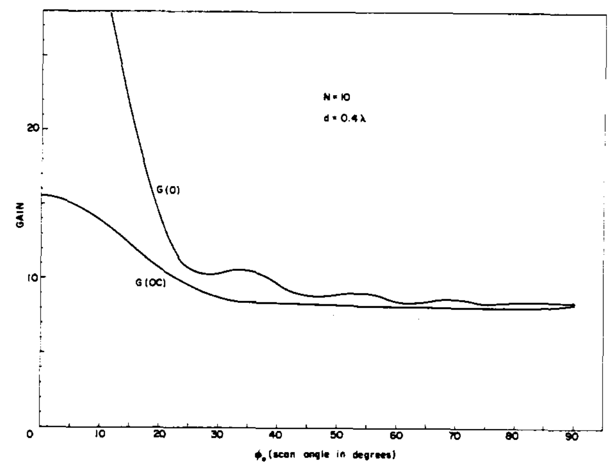


Fig. 11. Directive gain of a linear array of 10 isotropic elements with optimum ($O$) and optimum cophasal ($OC$) excitations vs. the scan angle of the main beam (measured from the array axis).

sharply. This once again indicates the super-gain effect.

To satisfy our curiosity, the pattern functions for the end-fire arrays with 10 isotropic elements spaced at 0.4λ as discussed above are also evaluated. First, Fig. 9 shows the pattern for $HW$ excitation. Since it is almost identical to that of either uniform or optimum cophasal excitation except for some minor details in sidelobes, the latter is omitted here. Figure 10 shows the pattern of the same array except for optimum excitation given in Figs. 6 and 8. Clearly the beamwidth is much narrower.

So far we have considered only the end-fire array. In case of broadside, $\theta_0 = 0$. Then from (12) $\psi_{nm}{}^0 = 0$ and $b_{nm}$ becomes real. Therefore from row 3 of Table I, the optimum currents also become real. This implies that for broadside array the gain of optimum excitation $G(O)$ is identically equal to that of optimum cophasal excitation $G(OC)$. Since $G(O)$ of an end-fire array can be much greater than $G(OC)$ for small spacings as discussed above, one expects that $G(O)$ approaches $G(OC)$ as the beam scans from end fire to broadside. This is indeed the case; in fact for the example considered above, i.e., an array of 10 isotropic elements spaced at 0.4λ, $G(O)$ drops to a value only slightly higher than $G(OC)$ as the beam scans to 25° from the end-fire as shown in Fig. 11. After that angle $G(O)$ oscillates slightly until the scan angle reaches broadside. For more details the reader may refer to the report of these authors [11]. In that report one may also find similar results for circular arc arrays under various excitations as discussed in this section. However, for small spacings the optimum excitation is not cophasal even if the beam is in the "broadside" direction, i.e., perpendicular to the largest dimension of the array.

## VI. Optimum Circular and Elliptical Arrays

Optimum excitations for circular and elliptical arrays under various conditions have been determined [11]. Obviously it is impossible to show all the results here. As an example, we plot the maximum gain and the $Q$-factor of a semicircular array of cophasally excited dipoles vs. the radius as shown in Figs. 12 and 13, re-

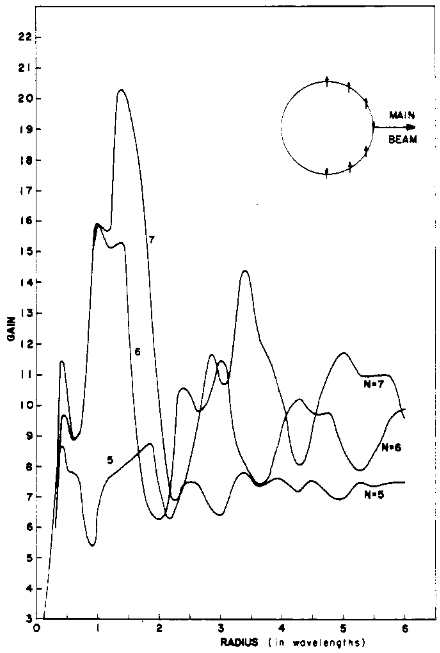


Fig. 12. Directive gain of a semicircular cophasal array with $N$ dipoles which are perpendicular to the plane of the array. For radius near zero, the data are less accurate.

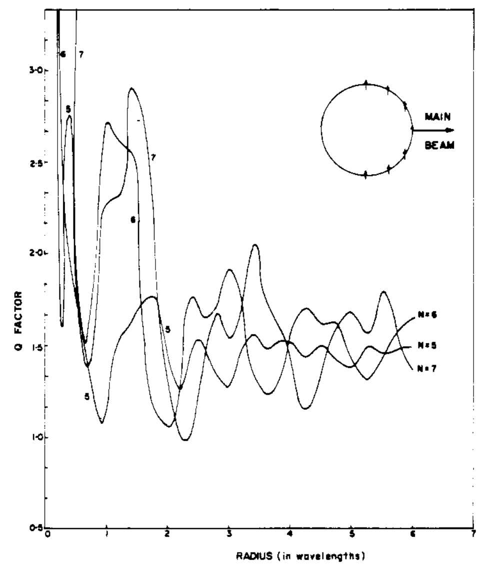


Fig. 13. $Q$-factor of the same array as in Fig. 12.

spectively. The dipoles are assumed to be perpendicular to the plane of the array and the main beam along the line of symmetry as shown. In the case of seven elements, for example, the highest maximum gain is about 20.3 at the radius of 1.4λ, whereas the lowest is about 7 at the radius of 2.2λ. Such a large difference shows that from the gain point of view the radius should be properly chosen. It is also interesting to see that the $Q$-factors for these two radii are about 2.9 and 1, respectively, which are at almost the same ratio as the gains. In other words, both designs will have the same sensitivity factor.

As a second example, a semicircular array of 9 isotropic elements is considered. For the sake of comparing the performances under various optimum conditions, we assume that the array lies in the upper half $xz$-plane as

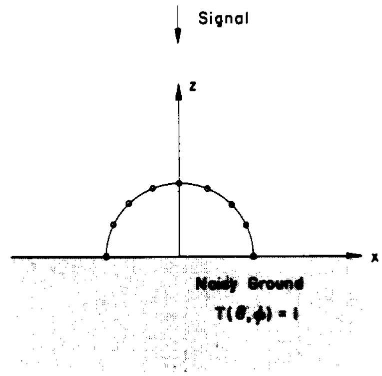


Fig. 14. A semicircular array of 9 isotropic elements above a ground of uniform thermal noise temperature distribution.

shown in Fig. 14. We also assume that the signal is in the $z$-direction (i.e., $\theta_0=0$) and a thermal noise of constant value is distributed in the lower semi-infinite space defined by $\pi/2<\theta\leq\pi$. This simulates the condition for an arch-shaped array above a hot ground which may be used perhaps for radio astronomy or space communication. (Clearly there is no difficulty in extending this technique to a much more complicated situation.) The optimum cophasal excitation, the gain, the signal-to-noise ratio and the $Q$-factor under four different optimization conditions as indicated in Table I are computed as shown in Table II for the radius of one wavelength and in Table III for the radius of 0.25 wavelength. The performances of the corresponding uniform excitation are also evaluated and given in both tables. For the first case, since the radius is equal to one wavelength, very large values of the $Q$-factor are not possible. It is significant to see that although both the gain and the $Q$-factor under these four optimizations differ little from those of the uniform excitation, there is a substantial difference in signal-to-noise ratio; namely, 35.5 for the uniform excitation against 81.6 for the optimum signal-to-noise ratio with no constraint. Antenna experimentalists have found great difficulty in achieving a gain significantly higher than that of uniform excitation; however, these results show that a *substantial improvement* in the signal-to-noise ratio over the uniform excitation is entirely possible *without* paying a high price on the $Q$-factor. It is also important to see that the optimum signal-to-noise ratio with no constraint is also larger than that under optimum gain; i.e., 81.6 against 55. Thus, in general, the optimization in the signal-to-noise ratio should not be substituted by the optimization in gain.

For the case when the radius is equal to 0.25 (Table III) the improvement in the signal-to-noise ratio is even greater, i.e., from 6.63 for the uniform excitation against 47.1 for the optimum signal-to-signal ratio under no constraint, or 21.8 with a prescribed $Q$-factor of 20. Since in this case the elements are closely spaced, this improvement is achieved at the expense of an extremely high $Q$ as seen in rows 2 and 4. This example also serves

TABLE II
OPTIMUM SEMICIRCULAR ARRAY OF NINE COPHASALLY EXCITED ISOTROPIC ELEMENTS (RADIUS = 1).

| | Current | Gain | SNR | $Q$-factor |
|---|---|---|---|---|
| Uniform excitation | $J_1=J_2=J_3=J_4=J_5=J_6=J_7=J_8=J_9=1$ | 8.24 | 35.5 | 0.916 |
| Optimum gain without constraint on $Q$-factor | $J_1=J_9=1.123$<br>$J_2=J_8=1.29$<br>$J_3=J_7=.881$<br>$J_4=J_6=.757$<br>$J_5=.600$ | 8.71 | 55.0 | 1.03 |
| Optimum gain with a prescribed $Q$-factor | $J_1=J_9=1.082$<br>$J_2=J_8=1.218$<br>$J_3=J_7=.898$<br>$J_4=J_6=.816$<br>$J_5=.659$ | 8.67 | 50.5 | 1.0<br>(prescribed) |
| Optimum SNR without constraint on $Q$-factor | $J_1=J_9=11.436$<br>$J_2=J_8=15.396$<br>$J_3=J_7=10.446$<br>$J_4=J_6=3.746$<br>$J_5=-.421$ | 7.76 | 81.6 | 1.14 |
| Optimum SNR with a prescribed $Q$-factor | $J_1=J_9=5.835$<br>$J_2=J_8=7.719$<br>$J_3=J_7=7.451$<br>$J_4=J_6=5.223$<br>$J_5=2.664$ | 8.44 | 55.1 | 1.0<br>(prescribed) |

Element positions: (in wavelengths)
$x_1=-x_9=1.000$ $z_1=z_9=0.0$
$x_2=-x_8=0.924$ $z_2=z_8=0.383$
$x_3=-x_7=0.707$ $z_3=z_7=0.707$
$x_4=-x_6=0.383$ $z_4=z_6=0.924$
$x_5=0.0$ $z_5=1.0$

Main beam: $\theta_0=0$.

Thermal noise distribution: $T(\theta,\phi)=\begin{cases}1, & \text{for } \pi/2<\theta\leq\pi\\ 0, & \text{otherwise.}\end{cases}$

TABLE III
OPTIMUM SEMICIRCULAR ARRAY OF NINE COPHASALLY EXCITED ISOTROPIC ELEMENTS (RADIUS $r=0.25$).

| | Current | Gain | SNR | $Q$-factor |
|---|---|---|---|---|
| Uniform excitation | $J_1=J_2=J_3=J_4=J_5=J_6=J_7=J_8=J_9=1$ | 2.19 | 6.63 | 0.244 |
| Optimum gain without constraint on $Q$-factor | $J_1=J_9=5.23$<br>$J_2=J_8=-15.74$<br>$J_3=J_7=34.81$<br>$J_4=J_6=-55.83$<br>$J_5=66.69$ | 3.63 | 37.8 | $3.76\times10^3$ |
| Optimum gain with a prescribed $Q$-factor | $J_1=J_9=2.24$<br>$J_2=J_8=-2.92$<br>$J_3=J_7=3.35$<br>$J_4=J_6=-2.23$<br>$J_5=2.37$ | 3.25 | 20.2 | 20.0<br>(prescribed) |
| Optimum SNR without constraint on $Q$-factor | $J_1=J_9=58.86$<br>$J_2=J_8=-179.6$<br>$J_3=J_7=412.72$<br>$J_4=J_6=-686.83$<br>$J_5=836.80$ | 3.52 | 47.1 | $3.26\times10^3$ |
| Optimum SNR with a prescribed $Q$-factor | $J_1=J_9=12.80$<br>$J_2=J_8=-15.58$<br>$J_3=J_7=19.70$<br>$J_4=J_6=-18.96$<br>$J_5=25.87$ | 3.19 | 21.8 | 20.0<br>(prescribed) |

Element positions: (in wavelengths)
$x_1=-x_9=r$ $z_1=z_9=0$
$x_2=-x_8=0.9239r$ $z_2=z_8=0.3827r$
$x_3=-x_7=0.7071r$ $z_3=z_7=0.7071r$
$x_4=-x_6=0.3827r$ $z_4=z_6=0.9239r$
$x_5=0$ $z_5=r$.

Main beam: $\theta_0=0$.

Thermal noise distribution: $T(\theta,\phi)=\begin{cases}1, & \text{for } \pi/2<\theta\leq\pi\\ 0, & \text{otherwise.}\end{cases}$

to show the importance of constrained optimization, should the results be made practically meaningful.

## VII. Generalization to Aperture Antennas

Often one is interested in optimizing an aperture antenna. In the following we shall briefly indicate how the general theory can be applied to this case. Since the numerical procedure remains the same as before, the details will be omitted.

As a first example let us consider a linear aperture which has been investigated extensively by many workers. Usually the distribution is expressed in terms of trigonometrical functions, although it can be also expressed in terms of other functions such as polynomials, spheroidal functions, etc. For simplicity, let us assume the aperture distribution to be

$$f(z) = \begin{cases} l^{-1} \sum_{n=-N}^{N} I_n \exp(-j2n\pi z/l), & |z| \leq l \\ 0 & , \quad |z| > l \end{cases} \tag{13}$$

where $l$ is the length of the aperture in wavelength. Then the pattern function is given by

$$P(u) = \int_{-l/2}^{l/2} f(z) e^{-juz} dz = \sum_{n=-N}^{N} I_n \frac{\sin[(ul/2) + n\pi]}{[(ul/2) + n\pi]}$$

where $u = 2\pi \sin\theta$, $\theta$ being the observation angle measured from the normal of the aperture. For further simplification one may assume that $f(z)$ be even; then, $I_n = I_{-n}$ and

$$P(u) = \langle JV^* \rangle = \sum_{n=0}^{N} (-1)^n \epsilon_n I_n \frac{U \sin U}{U^2 - n^2\pi^2} \tag{14}$$

where $\epsilon_0 = 1$ and $\epsilon_n = 2$ for $n \neq 0$, and $U = ul/2$. The objective is to determine $\{I_n\}$ under four different optimization conditions discussed earlier. To this end, one simply defines

$$\langle J = I_1, I_2, \cdots, I_N]$$

$$\langle V = \left[ \frac{\sin U}{U}, \frac{-2U \sin U}{U^2 - \pi^2}, \cdots, \frac{(-1)^N 2U \sin U}{U^2 - N^2\pi^2} \right]$$

and then applies the formulas in Table I for the solutions.

Similarly we may consider as a second example a circular aperture. For this case let the aperture distribution be circularly symmetric and expressed in terms of the circular polynomials $R_{2n}(\rho)$ [12]; i.e.,

$$f(\rho) = \sum_{n=0}^{N} I_{2n} R_{2n}(\rho) \tag{15}$$

where

$$R_{2n}(\rho) = \sum_{k=0}^{n} (-1)^k \frac{(2n-k)!}{k![(n-k)!]^2} \rho^{2(n-k)}$$

and $\{R_{2n}(\rho)\}$ is an orthogonal set with a weighting

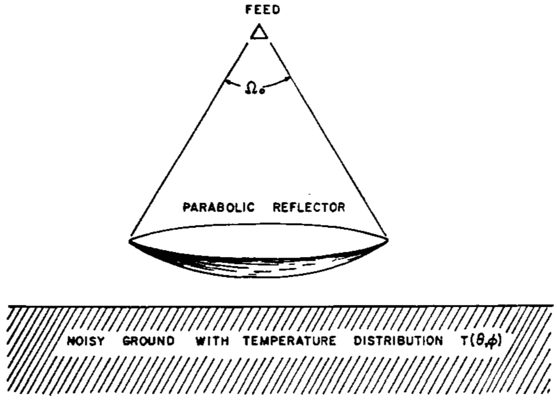


Fig. 15. A parabolic reflector over a noisy ground.

function $\rho$ for $0 \leq \rho \leq 1$. By using the properties of $R_{2n}(\rho)$, it can be shown [8] that the pattern function for the excitation of (15) is[3]

$$P(u) = 2\pi \sum_{n=0}^{N} (-1)^n I_{2n} \frac{J_{2n+1}(u)}{u} \tag{16}$$

where $J_{2n+1}(u)$ is the Bessel function of the first kind and order $2n+1$. Now let us define

$$\langle J = [I_0, I_2, \cdots, I_{2N}]$$

$$\langle V = \left[ \frac{J_1(u)}{u}, \frac{-J_3(u)}{u}, \cdots, \frac{(-1)^N J_{2N+1}(u)}{u} \right];$$

then the solutions to $\langle J$ under various optimization conditions are again given in Table I.

A problem of current interest is to design a feed for a paraboloidal reflector in a noisy environment as shown in Fig. 15, such that the signal-to-noise ratio is maximized. The feed is assumed to have a circular aperture with a distribution given by (15) and the reflector is pointed to the signal source. This problem differs from the previous ones in that the desired signal source as seen from the feed is extended over a solid angle $\Omega_0$, instead of a point as considered previously. Therefore, the signal power $\langle JCJ^* \rangle$ becomes proportional to

$$\left| \int_{\Omega_0} \langle JV^* \rangle d\Omega \right|^2 = \langle J \int_{\Omega_0} V^* d\Omega \rangle \langle \int_{\Omega_0} V d\Omega J^* \rangle.$$

Thus the dyadic $C$ is given by

$$C = \int_{\Omega_0} V^* d\Omega \rangle \langle \int_{\Omega_0} V d\Omega = V_1^* \rangle \langle V_1.$$

[3] Bessel function $J_{2n+1}(u)$ should not be confused with the current vector $J$. The former is always expressed explicitly with its argument.

With this definition of $C$, the solution of $J$ is once again given by the formulas in Table I. A few other problems can also be solved by this theory. Obviously it is not the intent of this paper to discuss all of them in detail. It is only hoped that the few examples given here have already illustrated the generality of the theory.

## VIII. Conclusion

In this paper the solution to the optimum current distribution of an arbitrary array for maximum directive gain or maximum signal-to-noise ratio, with or without a constraint on the array $Q$-factor, is obtained. The results in its full generality are applicable to any array, linear or curved, uniformly or nonuniformly spaced, planar or three-dimensional, discrete or continuous, with isotropic or directive elements. Under the special conditions that the noise temperature distribution becomes a constant and there is no constraint on the $Q$-factor, the solution reduces to that for maximum directivity obtained by others.

Optimum current distributions for maximum directivity of arrays of various geometries such as circles, ellipses, circular and elliptical arcs, with or without constraints, have been numerically determined [11]. Optimum distributions for maximum signal-to-noise ratio under some assumed temperature distributions have also been obtained. But, only a few examples are given in this paper.

It is believed that the present analysis is sufficiently general to cover almost all cases of interest except those arrays whose element spacings are also subject to optimization. The reason that such a general theory can be established is simply due to the fact that the optimization of current distribution is a linear problem whereas the optimization of element spacings is a highly nonlinear problem. However, it should be noted that for large element spacings, say greater than two or three wavelengths, the directivity becomes nearly a constant; therefore, optimization is no longer necessary whether the array be uniformly or nonuniformly spaced.

Some points of interest are summarized in the following. 1) As already concluded by Tai [4] for the case of cophasal broadside and end-fire arrays, the uniform excitation is nearly optimum for maximum directivity if the spacing is about $\lambda/2$ or greater. In this paper it is found that this is true for any array, cophasal or noncophasal (but optimum), linear or curved, whether the main beam is in the broadside, the end fire, or any other direction.

2) For spacing less than a certain critical number, generally somewhat smaller than $\lambda/2$, the directivity for the optimum end-fire array increases sharply; whereas the directivity for the optimum cophasal excitation remains nearly the same as that of the uniform excitation until the spacing reaches an even smaller value; after that it may increase sharply also. In the case of broadside array, the optimum excitation is cophasal.

3) High gain (i.e., higher than that of uniform excitation) is always associated with a high $Q$-factor, whereas the converse is not necessarily true. For small spacings, super-gain effect prevails and both the $Q$-factor and the sensitivity factor increase sharply. Therefore, a constraint on the $Q$-factor is necessary in order to make the solution practically meaningful.

4) For a given number of elements, the maximum directivity may increase and may also decrease with the spacing. In other words, the optimum gain curve oscillates with the spacing; thus there exist many spacings where the optimum gain reaches a maximum. For largest maximum gain, the proper spacing should be used.

5) Hansen-Woodyard excitation yields a directivity higher than that of uniform excitation only if the spacing is small. For large spacing, it is actually much lower. In fact, even for small spacings, the Hansen-Woodyard excitation is but one of many possible designs which have both the gain and the $Q$-factor between those of the optimum and the uniform excitations. However, the Hansen-Woodyard excitation, like the uniform, has the interesting property that its sensitivity factor is a constant, independent of the spacing, and dependent only on the number of elements, and it is about three times higher than that of the uniform excitation. Thus the field is three times more sensitive to variations in the Hansen-Woodyard excitation than those in the uniform excitation.

6) In the case of end-fire array the phase of optimum excitation differs only slightly from the antiphase excitation, i.e., 0, 180°, 0, 180°, $\cdots$, if the spacing is not greater than $\lambda/2$. Moreover, the optimum phase shift between adjacent elements is almost a constant.

7) For any array there exists a finite permissible range of the $Q$-factor. Bounds of this range are equal to the inverse of the largest and the smallest eigenvalues of a positive definite hermitian matrix which depends solely on the relative positions of the elements. For small spacing, somewhat less than $\lambda/2$, this range is extremely large; whereas for spacing greater than that, the range reduces to near zero. Therefore, for large spacings, regardless of how the array is excited, the $Q$-factor is always nearly equal to a constant which equals 1 for isotropic elements and 1.5 for dipoles. Furthermore, for isotropic elements spaced at multiples of $\lambda/2$, the $Q$-factor is identically equal to 1. This is also generally true for arrays of other geometry.

8) It is well known that the super-gain, (or a gain substantially higher than that of the uniform excitation) has not been realized in practice. Nevertheless, the present theory provides us with, among other things, complete information on the upper bound of the gain and the values of its associated $Q$-factor and sensitivity factor. Most important of all, this investigation (see Section VI) shows that a substantial improvement in signal-to-noise ratio over that of the uniform excitation is totally possible in practice.

## Appendix I

The purpose of this appendix is to furnish the intermediate steps from (6) to (7). By putting the first variation of $L$ with respect to $\delta J$ equal to zero, one has

$$\begin{aligned}\delta L = \langle JAJ^*\rangle^{-2}\{&\langle \delta JCJ^*\rangle + \langle JC\delta J^*\rangle\langle JAJ^*\rangle \\ &- [\langle \delta JAJ^*\rangle + \langle JA\delta J^*\rangle]\langle JCJ^*\rangle\} \\ &+ \Lambda\langle JBJ\rangle^{-2}\{[\langle \delta JJ^*\rangle + \langle J\delta J^*\rangle]\langle JBJ^*\rangle \\ &- [\langle \delta JBJ^*\rangle + \langle JB\delta J^*\rangle]\langle JJ^*\rangle\} = 0. \qquad (17)\end{aligned}$$

It follows after rearrangements,

$$\langle \delta J\left\{\frac{CJ^*\rangle\langle JAJ^*\rangle - AJ^*\rangle\langle JCJ^*\rangle}{\langle JAJ^*\rangle^2} + \Lambda\frac{J^*\rangle\langle JBJ^*\rangle - BJ^*\rangle\langle JJ^*\rangle}{\langle JBJ^*\rangle^2}\right\} + \left\{\frac{\langle JAJ^*\rangle\langle JC - \langle JCJ^*\rangle\langle JA}{\langle JAJ\rangle^{*2}} + \Lambda\frac{\langle JBJ^*\rangle\langle J - \langle JJ^*\rangle\langle JB}{\langle JBJ^*\rangle^2}\right\}\delta J^*\rangle = 0. \qquad (18)$$

Since matrixes $A$, $B$, and $C$ are hermitian, one has

$$\begin{aligned}\langle JA\delta J^*\rangle &= \langle \delta JAJ^*\rangle^* \\ \langle JB\delta J^*\rangle &= \langle \delta JBJ^*\rangle^* \qquad (19)\\ \langle JC\delta J^*\rangle &= \langle \delta JCJ^*\rangle^*.\end{aligned}$$

Making use of (19), (18) can be reduced to

$$\langle \delta J(\quad)\rangle + \langle \delta J(\quad)\rangle^* = 0 \qquad (20)$$

where

$$(\quad) = \frac{CJ^*\rangle\langle JAJ^*\rangle - AJ^*\rangle\langle JCJ^*\rangle}{\langle JAJ^*\rangle^2} + \Lambda\frac{J^*\rangle\langle JBJ^*\rangle - BJ^*\rangle\langle JJ^*\rangle}{\langle JBJ^*\rangle^2}.$$

Therefore,

$$\mathrm{Re}\langle \delta J(\quad)\rangle = 0.$$

Since $\delta J$ is arbitrary, one has

$$\frac{CJ^*\rangle\langle JAJ^*\rangle - AJ^*\rangle\langle JCJ^*\rangle}{\langle JAJ^*\rangle^2} + \Lambda\frac{J^*\rangle\langle JBJ^*\rangle - BJ^*\rangle\langle JJ^*\rangle}{\langle JBJ^*\rangle^2} = 0. \qquad (21)$$

By using the fact that $C = V_1^*\rangle\langle V_1$, and normalizing $\langle JJ^*\rangle = 1$ (since both the signal-to-noise ratio and $Q$ are independent of the norm $\|J\|$), (21) is reduced to

$$V_1^*\rangle\langle V_1J^*\rangle - AJ^*\rangle\langle JCJ^*\rangle\langle JAJ^*\rangle^{-1} + \Lambda QJ^*\rangle\langle JAJ^*\rangle - \Lambda Q^2BJ^*\rangle\langle JAJ^*\rangle = 0. \qquad (22)$$

Since $\langle V_1J^*\rangle \neq 0$, one can divide throughout the equation by this quantity and obtain (7).

## Appendix II

In this appendix, (12) will be derived. By definition, the elements of matrix $B$ for a vertical dipole with a pattern function $\sin\theta$, are given by, for $n \leq m$,

$$\begin{aligned}b_{nm} = b_{nm}^* &= \frac{1}{4\pi}\int_{4\pi} \sin^2\theta \exp j(\psi_{nm} - \psi_{nm}^0)\, d\Omega \\ &= \frac{e^{-j\psi_{nm}^0}}{4\pi}\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin^2\theta \\ &\qquad \exp j[2\pi\rho_{nm}\sin\theta\cos(\phi - \alpha_{nm})] \\ &= \frac{e^{-j\psi_{nm}^0}}{2}\int_0^\pi J_0(2\pi\rho_{nm}\sin\theta)\sin^3\theta d\theta \\ &= e^{-j\psi_{nm}^0}\left[\frac{\sin 2\pi\rho_{nm}}{2\pi\rho_{nm}} + \frac{\cos 2\pi\rho_{nm}}{(2\pi\rho_{nm})^2} - \frac{\sin 2\pi\rho_{nm}}{(2\pi\rho_{nm})^3}\right].\end{aligned}$$

In carrying out the integrations, the following formula has been used,

$$J_{\nu+\mu+1}(z) = \frac{z^{\nu+1}}{2^\nu\Gamma(\nu+1)}\int_0^{\pi/2} J_\mu(z\sin\theta)\sin^{\mu+1}\theta\cos^{2\nu+1}\theta d\theta,$$

$$(\mathrm{Re}\,\mu > -1, \quad \mathrm{Re}\,\nu > -1)$$

which, for example, is listed in Magnus and Oberhettinger, *Functions of Mathematical Physics*. New York: Chelsea, p. 30.

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