Solutions to exercises and problems in Lee's *Introduction to Smooth Manifolds*

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1 Topological Manifolds

Exercise 1.1. *Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to* any *open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.*

Every open ball is homeomorphic to the the open unit ball centred at the origin (we can map any two balls onto each other by linear scaling and a translation, both of which are continuous with continuous inverses). Note that open unit ball is homeomorphic to $\mathbb{R}^n$ itself via the homeomorphism

$$f : \text{int}(\mathbb{B}^n) \to \mathbb{R}^n, \ x \mapsto \frac{x}{1 - \|x\|}.$$

Hence, it is enough to show that we obtain an equivalent definition of a topological manifold if we require that U be homeomorphic to an open ball. First, suppose that at every $p \in M$, there is an open neighbourhood U such that $\widehat{U} \in \mathbb{R}^n$ is open and $\phi : U \to \widehat{U}$ is open. Then $\phi(p) \in B \subseteq \widehat{U}$, where B is some open ball. Then $\phi^{-1}(B)$ is an open subset of M containing p that is homeomorphic to an open ball in $\mathbb{R}^n$. Conversely, if every point $p \in M$ is contained in a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$, then M already satisfies the given definition.

Exercise 1.6. *Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.*

Let $[x]$ and $[y]$ be distinct points of $\mathbb{RP}^n$, that is, distinct 1-dimensional subspaces of $\mathbb{R}^{n+1}$, and let them be spanned by the unit vectors x and y respectively. Since $\mathbb{S}^n$ is Hausdorff, it is not hard to see that we can find pairwise disjoint open sets $U, \overline{U}, V, \overline{V} \subseteq \mathbb{S}^n$ such that $x \in U, -x \in \widetilde{U}, y \in V, -y \in \widetilde{V}$, where $\overline{A} = -a : a \in A$. Let $\widehat{U} = U \cup \widetilde{U}$ and $\widehat{V} = V \cup \widetilde{V}$. Define

$$\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^n, \ p \mapsto \frac{p}{\|p\|}.$$

We claim that $U = \pi(\phi^{-1}(\widehat{U}))$ and $V = \pi(\phi^{-1}(\widehat{V}))$ are disjoint and contain $[x]$ and $[y]$ respectively, and that U and V are open. It is clear that $[x] \in U$ and $[y] \in V$. Let $[p] \in U \cap V$. Then $[p] = \pi(u) = \pi(v)$ for some $u \in \phi^{-1}(\widehat{U})$, $v \in \phi^{-1}(\widehat{V})$. Then $u = \lambda v$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Hence, $\phi(u) = \pm\phi(v)$. But this implies that $\phi(u) \in \widehat{U} \cap \widehat{V}$, a contradiction. We conclude that $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B}$ be a countable basis for $\mathbb{R}^n$. We claim that $\pi(\mathcal{B}) = \{\pi(B) : B \in \mathcal{B}\}$ is a basis for $\mathbb{RP}^n$. Define $f_t : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, p \mapsto tp$ for every $t \in \mathbb{R} \setminus \{0\}$. Note that f_t is continuous and has a continuous inverse $f_{1/t}$. Hence, for if U is open, then $f_t(U)$ is also open. We claim that $\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R} \setminus \{0\}} f_t(U)$ for every open set $U \in \mathbb{R}^n \setminus \{0\}$. Let $p \in \pi^{-1}(\pi(U))$. Then $\pi(p) \in \pi(U)$, implying that there is a $u \in U$ such that u spans the same vector space as p. Hence, $p = \lambda u$ for some non-zero λ, and therefore $p \in \bigcup_{t \in \mathbb{R} \setminus \{0\}} f_t(U)$. Conversely, suppose that $p \in \bigcup_{t \in \mathbb{R} \setminus \{0\}} f_t(U)$. Then $p = f_\lambda(u) = \lambda u$ for some $u \in U$ and some non-zero λ. Hence, $\pi(p) = \pi(\lambda u) = \pi(u)$, so $p \in \pi^{-1}(\pi(U))$, proving the claim. We are now ready to prove second-countability, equivalently to prove that $\pi(\mathcal{B})$ is a basis. Let $[p] \in \pi(B_1) \cap \pi(B_2)$ for two basis sets $B_1, B_2 \in \mathcal{B}$. Then $p \in \pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2))$, which is open by our previous work. Since this set is nonempty, there is a basis set B_3 contained in $\pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2))$. Then $\pi(B_3) \subseteq \pi(B_1) \cap \pi(B_2)$, showing that $\pi(\mathcal{B})$ is a basis, and therefore showing that $\mathbb{RP}^n$ is a manifold.

Exercise 1.7. *Show that $\mathbb{RP}^n$ is compact. [Hint: show that the restriction of π to $\mathbb{S}^n$ is surjective.]*

Let $[p] = \pi(p) \in \mathbb{RP}^n$. Then $\pi(p/\|p\|) = \pi(p)$, and $p/\|p\| \in \mathbb{S}^n$. Then the restriction of π to $\mathbb{S}^n$ is surjective. Since $\mathbb{S}^n$ is compact, and the image of a compact set under a continuous function is compact, $\mathbb{RP}^n$ is compact.

Exercise 1.14. *Suppose $\mathfrak{X}$ is a locally finite collection of subsets of a topological space M.*

(a) *The collection $\{\overline{X} : X \in \mathfrak{X}\}$ is also locally finite.*
(b) $\overline{\bigcup_{X \in \mathfrak{X}} X} = \bigcup_{X \in \mathfrak{X}} \overline{X}.$

Let's prove (b) first. Let $x \in \overline{\bigcup_{X \in \mathfrak{X}} X}$. Then every open set U containing x intersects $\bigcup_{X \in \mathfrak{X}} X$. By local finiteness, choose an open set U containing x intersecting $X_1, \ldots, X_n \in \mathfrak{X}$. Suppose $x \notin \bigcup_{X \in \mathfrak{X}} \overline{X}$. Then, for each $i \in \{1, \ldots, n\}$, there is an open set U_i that contains x but does not intersect X_i. Then $U \cap U_1 \cap \cdots \cap U_n$ contains x, is open, but does not intersect $\bigcup_{X \in \mathfrak{X}} X$, which is a contradiction. Conversely, suppose that $x \in \bigcup_{X \in \mathfrak{X}} \overline{X}$. Then $x \in \overline{X}$ for some $X \in \mathfrak{X}$; thus, every open set containing x intersects X, and therefore intersects $\bigcup_{X \in \mathfrak{X}} X$ as well. Hence, $x \in \overline{\bigcup_{X \in \mathfrak{X}} X}$.

For (a), let $x \in M$, and let U be an open set containing x and intersecting $X_1, \ldots, X_n \in \mathfrak{X}$. Let $\widetilde{\mathfrak{X}} = \mathfrak{X} \setminus \{X_1, \ldots, X_n\}$. Since U contains x but does not intersect any of the sets in $\widetilde{\mathfrak{X}}$, we have $x \notin \overline{\bigcup_{X \in \widetilde{\mathfrak{X}}} X}$. Hence, $U \setminus \overline{\bigcup_{X \in \widetilde{\mathfrak{X}}} X} = U \setminus \bigcup_{X \in \widetilde{\mathfrak{X}}} \overline{X}$ is an open set containing x and intersecting only sets in $\{\overline{X_1}, \ldots, \overline{X_n}\}$. Therefore, $\{\overline{X} : X \in \mathfrak{X}\}$ is locally finite.

Exercise 1.18. *Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

($\Rightarrow$) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two atlases for M determining the same smooth structure. By Proposition 1.17 (a), these two atlases are contained in a unique maximal smooth atlas $\mathcal{A}$. Since any two charts in $\mathcal{A}$ are smoothly compatible, then any two charts in $\mathcal{A}_1 \cup \mathcal{A}_2$ are smoothly compatible. Since each $\mathcal{A}_i$ covers M, $\mathcal{A}_1 \cup \mathcal{A}_2$ covers M and is therefore a smooth atlas.

($\Leftarrow$) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be smooth atlases and let $\mathcal{A}_1 \cup \mathcal{A}_2$ be a smooth atlas. By part (a), $\mathcal{A}_1$ and $\mathcal{A}_2$ are each contained in a maximal smooth atlas. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are each contained in $\mathcal{A}_1 \cup \mathcal{A}_2$, they are contained in the same maximal atlas, and therefore they determine the same smooth structure on M.

Exercise 1.20. *Every smooth manifold has a countable basis of regular coordinate balls.*

We're not going to worry about our coordinate balls being centred at 0, since a ball in $\mathbb{R}^n$ can always be mapped to a ball centred at 0 in $\mathbb{R}^n$ via a translation. Let M be a topological n-manifold, and let $\{U_i, \phi_i\}$ be an atlas of charts for M. We may assume that the atlas is countable, since M is second-countable. Note that $\phi_i(U_i) \subseteq \mathbb{R}^n$ has a countable basis of regular coordinate balls, namely those of the for $B_r(x)$, such that $x \in \mathbb{Q}^n$, $r \in \mathbb{Q}$, and such that $B_r(x) \subseteq B_{r'}(x) \subseteq \phi_i(U_i)$ for some $r' > r$. We claim that $\mathcal{B} = \{\phi_i^{-1}(B_r(x))\}$ forms a basis of regular coordinate balls for the topology on M. Let $B_1, B_2 \in \mathcal{B}$ intersect non-trivially. Then $B_1 \cap B_2 \subseteq U_i$ for some i, and $\phi_i(B_1 \cap B_2) \subseteq \phi_i(U_i) \subseteq \mathbb{R}^n$. There is a rational coordinate ball B whose closure is contained in another coordinate ball with the same centre. Hence, $\phi_i^{-1}(B) = B_1 \cap B_2$. Hence, $\mathcal{B}$ is a basis. It remains to show that the elements of $\mathcal{B}$ are regular. Let $\phi_i^{-1}(B_r(x)) \in \mathcal{B}$. Then $\phi_i(\overline{\phi_i^{-1}(B_r(x))}) = \overline{B_r(x)} \subseteq B_{r'}(x) \subseteq \phi_i(U_i)$. Hence, $\overline{\phi_i^{-1}(B_r(x))} \subseteq \phi_i^{-1}(B_{r'}(x))$, so $\phi_i^{-1}(B_r(x))$ is a regular coordinate ball.

Exercise 1.39. *Let M be a topological n-manifold with boundary.*

(a) Int$M$ *is an open subset of M and a topological n-manifold without boundary.*
(b) ∂M *is a closed subset of M and a topological $(n-1)$-manifold without boundary.*
(c) *M is a topological manifold if and only if $\partial M = \varnothing$.*
(d) *If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.*

For (a), let $x \in \operatorname{Int}M$. Then $x \in U$, where (U, ϕ) is an interior chart. We claim that $U \subseteq \operatorname{Int}M$. Let $y \in U$. Then $\phi(y) \in \phi(U) \subseteq \mathbb{R}^n$. Since $\phi(U)$ is open, there is an open set $\widehat{V}$ such that $\phi(y) \in \widehat{V} \subseteq \phi(U)$. Then $y \in \phi^{-1}(\widehat{V}) \subseteq U$. Therefore, $(\phi^{-1}(\widehat{V}), \phi|_{\phi^{-1}(\widehat{V})})$ is an interior chart containing y, so $y \in \operatorname{Int}M$. Hence, $U \subseteq \operatorname{Int}M$, which proves that $\operatorname{Int}M$ is open. Since $\operatorname{Int}M$ is a subspace of M, which is Hausdorff and second-countable, we have that $\operatorname{Int}M$ is Hausdorff and second-countable. By definition, every point of $\operatorname{Int}M$ is contained in an interior chart, so $\operatorname{Int}M$ is a topological n-manifold.

For (b), by the theorem on topological invariance of the boundary, $\partial M = M \setminus \operatorname{Int}M$. By part (a), ∂M is closed. ∂M is Hausdorff and second-countable since it is a subspace of M, which is Hausdorff and second-countable. It remains to show that every $x \in \partial M$ is in some chart (U, ϕ) with $\phi(U) \subseteq \mathbb{R}^{n-1}$. Let $x \in \partial M$, and let (V, ψ) be a boundary chart containing x. Then $\psi(x) \in \partial \mathbb{H}^n$. Let $\widehat{U} = \psi(V) \cap \partial \mathbb{H}^n$. Note that $\widehat{U}$ contains x, and is open in $\partial \mathbb{H}^n \cong \mathbb{R}^{n-1}$. Let $U = \psi^{-1}(\widehat{U}) = V \cap \partial M$. Then U is open in ∂M, and $(U, \psi|_U)$ is a chart for $x \in \partial M$. This proves that ∂M is an $(n-1)$-manifold.

For (c), let M be a topological manifold. By definition, every point is in an interior chart. By the theorem on topological invariance of the boundary, $\partial M = \varnothing$. Conversely, let $\partial M = \varnothing$. Again, by the theorem on the topological invariance of the boundary, every point is then an interior point. M then satisfies the definition of a topological manifold.

For (d), let M be a 0-manifold. Then every point $p \in M$ is in a chart $(\{p\}, \phi)$, where $\phi : \{p\} \to \mathbb{R}^0$ is a homeomorphism. Hence, every point of M is an interior point, so $\partial M = \varnothing$. By part (c), M is a 0-manifold.

Exercise 1.41. *Let M be a topological manifold with boundary.*

(a) *M has a countable basis of precompact coordinate balls and half-balls.*

(b) *M is locally compact.*

(c) *M is paracompact.*

(d) *M is locally path-connected.*

(e) *M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.*

(f) *The fundamental group of M is countable.*

For (a), let $\{(U_i, \phi_i)\}$ be a collection of charts covering M. We may assume that this collection is countable, since M has a countable basis by definition. Say (U, ϕ) is an interior chart. Then $\phi(U) \subseteq \mathbb{R}^n$ has a countable basis of balls whose closures lie in $\phi(U)$. Similarly, if (U, ϕ) is a boundary chart, then $\phi(U)$ has a countable basis of balls and half-balls whose closures all lie in $\phi(U)$. Let $\mathcal{B}$ be the collection of preimages of all the coordinate balls and half-balls discussed above, under the respective coordinate maps. Then $\mathcal{B}$ is a countable basis for M. It remains to show that the sets in $\mathcal{B}$ are precompact. Let $B \in \mathcal{B}$, and let ϕ be its coordinate map. Then $B \subseteq U$, where (U, ϕ) is a coordinate chart (the ϕ in this chart is an extension of the ϕ in (B, ϕ)). Then $\phi(\overline{B}) = \overline{\phi(B)} \subseteq \phi(U)$ is compact, so $\overline{B}$ is compact, where $\overline{B}$ is the closure of B in U. Since M is Hausdorff, $\overline{B}$ is closed in M. It follows that $\overline{B}$ is also the closure of B in M. This proves (a).

For (b), part (a) implies that every $x \in M$ is contained in a precompact coordinate ball. Hence, M is locally compact.

For (c), let $\mathcal{X}$ be an open cover of M, and let $\mathcal{B}$ be an arbitrary basis for the topology on M. We will prove the stronger result that $\mathcal{X}$ has a countable, open refinement consisting of elements of $\mathcal{B}$. Let $\{K_j\}_{j=1}^\infty$ be an exhaustion of M by compact sets; that is $X = \bigcup_{j=1}^\infty K_j$ and $K_j \subseteq \operatorname{Int}K_{j+1}$. Let $A_j = K_{j+1} \setminus \operatorname{Int}K_j$ and let $B_j = \operatorname{Int}K_{j+2} \setminus K_{j-1}$. Then A_j is compact and B_j is open, and $A_j \subseteq B_j$. Every $x \in A_j$ is contained in some basis set $B \subseteq X \cap B_j$. Since these basis sets cover the compact set A_j, there is a finite cover of A_j. Taking the union of these basis sets over $j \in \mathbb{N}$, we obtain a countable refinement of $\mathcal{X}$ consisting of sets in $\mathcal{B}$. The cover is locally finite, since the basis sets are all contained in some B_j, B_j and $B_{j'}$ don't intersect if $|j - j'| > 2$.

For (d), by part (a) M has a basis of coordinate balls and half-balls, which are all path-connected. Hence, M is locally path-connected.

For (e), let $\{C_i\}$ be the collection of connected components of M. Then $\{C_i\}$ is an open cover of M. Since M is second-countable, $\{C_i\}$ has a countable subcover. But since the C_i are disjoint, we must have that $\{C_i\}$ is countable to begin with. Hence, M has countably many components. By definition, each component is an

open subset. As subspaces of M, the C_i are Hausdorff and second-countable. Let $x \in C_i$ and let (U, ϕ) be a chart containing x. Then $(U \cap C_i, \phi|_{U \cap C_i})$ is a chart in C_i containing x. Thus, C_i is a connected topological manifold with (possibly empty) boundary.

For (f), let $\mathcal{B}$ be a countable basis of coordinate balls and half-balls. For any $B_1, B_2 \in \mathcal{B}$, the intersection $B_1 \cap B_2$ has countably many components. To see this, note that $B_1 \cap B_2$ is open in M, and therefore its components are open in M. Moreover, every component is itself a connected manifold with boundary, and therefore is locally path-connected. Since connected and locally path-connected spaces are path-connected, every component is path-connected. Since each component contains a distinct basis set, $B_1 \cap B_2$ has countably many components. Let $\mathcal{X}$ be a set containing a single point from each component of $B_1 \cap B_2$ for every pair $B_1, B_2 \in \mathcal{B}$. For every pair $x, x' \in \mathcal{X}$ contained in some $B \in \mathcal{B}$, let $h^B_{x,x'}$ be a fixed path from x to x' in B. Select a point $p \in \mathcal{X}$ as the basepoint of the fundamental group $\pi_1(M, p)$. Define a special loop to be a loop starting at p that is the concatenation of paths of the form $h^B_{x,x'}$. Since the set of all special loops is countable, it suffices to show that every loop based at p is homotopic to some special loop. Let $f : [0, 1] \to M$ be an arbitrary loop based at p. Since $f([0, 1])$ is compact, there are finitely many basis sets $B_1, \ldots, B_k \in \mathcal{B}$ that cover it. Hence, there exist $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $f([a_{i-1}, a_i]) \subseteq B_i$ for each i. Each $f(a_i)$ is contained in the same component of $B_i \cap B_{i+1}$ as some $x_i \in \mathcal{X}$. Let g_i be a path from x_i to $f(a_i)$, where g_0 and g_k are both equal to the constant path at p. Let f_i be the path obtained by restricting f to $[a_{i-1}, a_i]$,

$$f \simeq f_1 \cdot f_2 \cdot \ \cdots \ \cdot f_k \simeq g_0 \cdot f_1 \cdot \overline{g_1} \cdot g_1 \cdot f_2 \cdot \overline{g_2} \cdot \ \cdots \ \cdot g_{k-1} \cdot f_k \cdot \overline{g_k}.$$

Since each B_i is simply connected, and the endpoints of $g_{i-1} \cdot f_i \cdot g_i$ are x_{i-1} and x_i, $g_{i-1} \cdot f_i \cdot g_i$ is homotopic to $h^{B_i}_{x_{i-1}, x_i}$. Hence, f is homotopic to a special loop, concluding the proof.

Exercise 1.42. *Show that every smooth manifold with boundary has a countable basis consisting of regular coordinate balls and half-balls*

This is a straightforward adaptation of Exercise 1.20.

Exercise 1.43. *Show that the smooth manifold chart lemma (Lemma 1.35) holds with "$\mathbb{R}^n$" replaced with "$\mathbb{R}^n$ or $\mathbb{H}^n$" and "smooth manifold" replaced by "smooth manifold with boundary."*

Just make the same replacements in the proof of Lemma 1.35. The only external result that is used is Exercise A.22, which is a result for general topological spaces.

Exercise 1.44. *Suppose M is a smooth n-manifold with boundary, and U is an open subset of M. Prove the following statements:*

(a) *U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, ϕ) for M such that $V \subseteq U$ defines a smooth structure on U. With this topology and smooth structure, U is called an* **open submanifold with boundary**.

(b) *If $U \subseteq \text{Int} M$, then U is actually a smooth manifold (without boundary); in this case, we call it an* **open submanifold of M**.

(c) *$\text{Int} M$ is an open submanifold of M (without boundary).*

For (a), every point $p \in U$ is contained in the domain of some chart (V, ϕ). Then $(U \cap V, \phi|_{U \cap V})$ is a chart whose domain is contains p and is contained in U. Thus, U is a topological n-manifold with boundary. These charts clearly cover U, so the form an atlas. They are smoothly compatible, since this atlas is a subset of the maximal smooth atlas for M. Hence, this collection of charts defines a smooth structure on U.

For (b), we need to show that all of the charts are in fact interior charts. Let (V, ϕ) be a smooth chart with $V \subseteq U$. If $\phi(V) \subseteq \mathbb{H}^n$ with some $p \in V$ such that $\phi(p) \in \partial \mathbb{H}^n$. Then p is a boundary point and an interior point, which is impossible by the theorem on the topological invariance of the boundary.

For (c), by exercise 1.39 $\text{Int} M$ is open in M. Take $U = \text{Int} M$ in part (b).

Problem 1.1. *Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the* **line with two origins**.)

Let $\pi : X \to M, x \mapsto [x]$. Let $[(x,1)] \in M$ for some x (we are including the possibility that $x = 0$, the $[(0,-1)]$ case is similar). Then there is an open set $(a,b) \times \{1\}$ in X such that $x \in (a,b)$. We claim that the map $\phi : \pi((a,b) \times \{1\}) \to (a,b), [(y,\pm 1)] \mapsto y$ is a homeomorphism. If $[(y,\pm 1)] = [(y',\pm 1)]$, then $y = y'$, so ϕ is well-defined. Let $\phi([(y,\pm 1)]) = \phi([(y',\pm 1)]) \Rightarrow y = y'$. If $y = y' = 0$, then $[(y,\pm 1)] = [(y',\pm 1)] = [(0,1)]$, since the points $[(y,\pm 1)]$ and $[(y',\pm 1)]$ must be in $\pi((a,b) \times \{1\})$. Otherwise, we can ignore the ± 1 since it does not affect the equivalence class, so $[(y,\pm 1)] = [(y',\pm 1)]$. Thus, ϕ is injective. It is also clearly surjective, and thus bijective. Let $U \subseteq (a,b)$ be open. Then $\phi^{-1}(U) = \pi(U \times \{1\})$. Since $\pi^{-1}(\pi(U \times \{1\})) = U \times \{\pm 1\}$ is open in X, we have that $\pi(U \times \{1\})$ is open in M. Let $U \subseteq \pi((a,b) \times \{1\})$ be open in M. Then $U \times \{\pm 1\}$ is open in X, and therefore U is open in $(a,b) \subseteq \mathbb{R}$. Thus, π is a homeomorphism and M is locally Euclidean.

Let $U \in X$ be open. Then $\pi^{-1}(\pi(U)) = U \cup r(U)$, where r is the reflection across the x-axis. Since $U \cup r(U)$ is open, $\pi^{-1}(\pi(U))$ is open. Hence, π is an open map. By the same argument given in Exercise 1.6, $\pi(\mathcal{B})$ is a basis for M if $\mathcal{B}$ is a basis for X. Since X is a subspace of a second-countable space, M is second-countable.

Let U and V be open sets in M containing $[(0,1)]$ and $[(0,-1)]$ respectively. Hence, $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open sets in X containing $(0,1)$ and $(0,-1)$ respectively. Hence, U and V contain subsets $(-\epsilon, \epsilon) \times \{1\}$ and $(-\epsilon, \epsilon) \times \{-1\}$, respectively, for some $\epsilon > 0$. Then $\pi((-\epsilon, \epsilon) \times \{1\}) \subseteq U$ and $\pi((-\epsilon, \epsilon) \times \{-1\}) \subseteq V$. However, these two sets have non-trivial intersection. We conclude that $[(0,1)]$ and $[(0,-1)]$ cannot be separated by open sets, and therefore M is not Hausdorff.

Problem 1.2. *Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.*

Let M be a disjoint union of uncountably many copies of $\mathbb{R}$. Let $x \in M$. Let R be the component containing x. Then the homeomorphism $\phi : R \to \mathbb{R}$ defines a chart whose domain contains x. Therefore, M is locally Euclidean. The disjoint union of Hausdorff spaces is Hausdorff, so M is Hasudorff. Since manifolds have countably many components, M cannot be a manifold. We conclude that M is not second-countable since it satisfies the other conditions of the definition.

Problem 1.3. *A topological space is said to be σ-**compact** if it can be expressed as a union of countably many compact spaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.*

($\Rightarrow$) Let M be a topological manifold. Then M is second countable. By Lemma 1.10, M has a countable basis of precompact coordinate balls. Then M is the union of the closures of these basis sets, which are compact.

($\Leftarrow$) Let M be locally Euclidean, Hausdorff, and σ-compact. Let $\{K_i\}_{i \in \mathbb{N}}$ be a countable cover of M by compact sets. Since M is locally Euclidean, M has an open cover of coordinate balls. Each K_i is then covered by finitely many such coordinate balls, so we can assume that M is covered by countably many coordinate balls. Let (B, ϕ_B) be such a coordinate ball. Then $\phi(B) \subseteq \mathbb{R}^n$ has a countable basis $\mathcal{B}_B$. Let $\phi_B^{-1}(\mathcal{B}_B)$ be collection of preimages of sets in $\mathcal{B}_B$ under ϕ_B. Then $\bigcup_B \phi_B^{-1}(\mathcal{B}_B)$ is a basis for M. Therefore, M is second-countable.

Problem 1.4. *Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.*

(a) *Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.*

(b) *Give an example to show that the converse to (a) may be false.*

(c) *Now that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.*

For (a), let $p \in M$ be arbitrary. Then $p \in U$ for some $U \in \mathcal{U}$, and U is a neighbourhood of p that intersects at most finitely many of the other sets in $\mathcal{U}$.

For (b), let $M = \mathbb{R}$, and let $\mathcal{U}$ contain $\mathbb{R}$, and the intervals $(i, i+1)$ for each $i \in \mathbb{Z}$. Then $\mathcal{U}$ is locally finite since every $x \in \mathbb{R}$ is contained in some open interval of unit length, and every such open interval intersects at most three of the sets in $\mathcal{U}$.

For (c), let $U \in \mathcal{U}$. For every $x \in \overline{U}$, let U_x be an open set containing x such that U_x intersects finitely many sets in $\mathcal{U}$. Then $\{U_x\}_{x \in X}$ is an open cover for $\overline{U}$, which is compact. Hence, it has a finite subcover

$\{U_x^{(1)}, \ldots, U_x^{(k)}\}$. Since each $U_x^{(i)}$ intersects finitely many sets in $\mathcal{U}$ and they cover U, we conclude that U intersects finitely many sets in $\mathcal{U}$.

Problem 1.5. *Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components.*

($\Rightarrow$) If M is second-countable, it is a topological manifold. Hence, it is countably many components and is paracompact by the results from the section.

($\Leftarrow$) Let M be paracompact and have countably many components. We can assume that M is connected: if each connected component has a countable basis, and there are countably many components, then M has a countable basis.

Let $\mathcal{U}$ be an open cover of M by coordinate charts and let (U, ϕ) be such a chart. Note that $\phi(U)$ has a basis of precompact balls, where the closures of said balls lie in $\phi(U)$. Pulling back these balls to M yields a covering of M by balls whose closure in U is compact, and thus closed because M is Hausdorff. Thus, the closure of a coordinate ball in U is the same as its closure in M, and therefore we have an open cover of precompact coordinate charts for M. Since M is paracompact, the open cover constructed has a locally finite open refinement. Since M is Hausdorff, the this locally finite refinement also consists of precompact coordinate charts. Call this open cover $\mathcal{U}$.

Let $U \in \mathcal{U}$ be arbitrary, and define $\mathcal{V}_0 = \{U\}$. By part (c) of the preceding problem, U intersects finitely many sets in $\mathcal{U}$. Denote the collection of these sets $\mathcal{W}_1$, and define $\mathcal{V}_1 = \mathcal{W}_1 \setminus \mathcal{V}_0$. Suppose we have define $\mathcal{V}_n \subseteq \mathcal{U}$. Let $\mathcal{W}_{n+1}$ be the collection of all sets in $\mathcal{U}$ that intersect sets in $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_n$, and define $\mathcal{V}_{n+1} = \mathcal{W}_{n+1} \setminus (\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_n)$. By construction, each $\mathcal{V}_i$ contains finitely many sets. We will show that $\bigcup_{i=0}^{\infty} \mathcal{V}_i$ covers M. Since M is locally Euclidean, it is locally path-connected, and because we are assuming that M is connected, it is path-connected. Let $y \in M$ and $x \in U$ be arbitrary, and let $f : [0, 1] \to M$ be a path from x to y. Since $f([0, 1])$ is compact, it is covered by finitely many of the sets in $\mathcal{U}$, and therefore there are numbers $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $f([a_{i-1}, a_i]) \subseteq U_i$ for some $U_i \in \mathcal{U}$. Then $f([a_0, a_1]) = f([0, a_1]) \subseteq U_1 = U$. Suppose that we have shown that $f([a_{j-1}, a_j])$ is contained in the union of the sets in $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_{j-1}$. Then $f(a_j) \in V$ for some $V \in \mathcal{V}_0 \cup \cdots \cup \mathcal{V}_{j-1}$. Hence, $f([a_j, a_{j+1}])$ must be contained in the union of sets in $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_j$, since it intersects the sets in $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_{j-1}$. Therefore, y is contained in a set in some $\mathcal{V}_k$, which proves the claim. Hence, we have a countable cover of M by precompact coordinate charts. Each of these charts has a countable basis, and the union of all these bases is a basis for the topology on M. Hence, M is second countable.

Problem 1.6. *Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones.*

Following the hint, for every $s > 0$ we show that $F_s : \mathbb{B}^n \to \mathbb{B}^n, x \mapsto |x|^{s-1} x$ is a homeomorphism, and a diffeomorphism if and only if $s = 1$. Note that F_s is not defined at $x = 0$ for $s \leq 1$. However, $|F_s(x)| = |x|^s$, so defining $F_s(0) = 0$ makes F_s a continuous function. Moreover, $|F_s(x)| = |x|^s \leq 1$, so the image of F_s does indeed lie in $\mathbb{B}^n$. The inverse of F_s is given by $F_s^{-1}(x) = |x|^{\frac{1}{s}-1} x$, where we define $F_s^{-1}(0) = 0$. Clearly, F_s^{-1} is continuous, so F_s is a homeomorphism for each $s > 0$.

Away from $x = 0$, F_s is a diffeomorphism since F_s and F_s^{-1} are products or quotients of smooth non-vanishing functions. Moreover, $F_1 = \text{id}$ is a diffeomorphism. Now suppose that $s \neq 1$. It suffices to show that F_s is not smooth for $s < 1$, since the same proof will show that F_s^{-1} is not smooth for $s > 1$. If F_s is smooth, then it should have continuous derivatives of all orders. However, the partial derivative of the first component of F_s with respect to the first variable is

$$\frac{\partial}{\partial x_1} |x|^{s-1} x_1 = (s-1)|x|^{s-3} x_1^2 + |x|^{s-1}.$$

To see that this quantity is not continuous at 0, we can take its limit as $x \to 0$ along the x_1-axis. Setting $x_2 = \cdots = x_n = 0$, we find

$$\frac{\partial}{\partial x_1} |x|^{s-1} x_1 = (s-1)|x_1|^{s-3} x_1^2 + |x_1|^{s-1} = s|x_1|^{s-1} \to \infty,$$

since $s < 1$. This concludes our proof of the statement in the hint.

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Given that M has a smooth structure, let $\mathcal{A}$ be a locally finite atlas for M. Let $x \in M$ and let x be contained in the domains of the charts $(U_1, \phi_1), \ldots, (U_k, \phi_k)$. Define Let $\mathcal{A}_0 = \mathcal{A}$, and inductively define

$$\mathcal{A}_i = \begin{cases} \mathcal{A}_{i-1} \setminus \{(U_i, \phi_i)\} & \text{if } U_{i-1} \text{ is covered by the domains of other charts in } \mathcal{A}_{i-1} \\ \mathcal{A}_{i-1} & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, k$. By induction, $\mathcal{A}_k$ is an atlas for M. Hence, the charts in $\mathcal{A}_k$ cover M. Since $x \in M$, there must be at least one $i \in \{1, \ldots, k\}$ such that $(U_i, \phi_i) \in \mathcal{A}_k$. Hence, U_i is not covered by the domains in of charts in $\mathcal{A}_{i-1}$, and therefore is not covered by the domains of charts in $\mathcal{A}_k \subseteq \mathcal{A}_{i-1}$. Thus, there is a $p \in U_i$ that is covered only by U_i. Moreover, $\phi_i(U_i) \setminus \{\phi_i(p)\}$ can be covered by open balls $B_\alpha = B_{r_\alpha}(x_\alpha)$ entirely contained in $\phi_i(U_i) \setminus \{\phi_i(p)\}$. There is also a ball $B = B_r(\phi_i(p)) \subseteq \phi_i(U_i)$. Pulling back these balls, we get that $\mathcal{A}' = \mathcal{A}_k \setminus \{(U_i, \phi_i)\}] \cup \{(\phi_i^{-1}(B_i), \phi_i)\} \cup \{(\phi_i^{-1}(B), \phi_i)\}$ is a smooth atlas for M such that p is only contained the domain of $(\phi_i^{-1}(B), \phi_i)$. The coordinate maps are smoothly compatible, since the new maps are obtained by restricting ϕ, which is smoothly compatible with all of the other coordinate maps. There is a diffeomorphism $f : B \to \mathbb{B}^n$ obtained by translating and then scaling. Replacing $(\phi_i^{-1}(B), \phi_i)$ with $(\phi_i^{-1}(B), f \circ \phi_i)$ in $\mathcal{A}'$, we obtain a new smooth atlas $\mathcal{Z}$.

For every $s > 0$, define $\mathcal{Z}_s$ to be the set of charts in $\mathcal{Z}$, except $(\phi_i^{-1}(B), f \circ \phi_i)$ is replaced with $(\phi_i^{-1}(B), F_s \circ f \circ \phi_i)$. We claim that each $\mathcal{Z}_s$ is a smooth atlas, and that $\mathcal{Z}_s$ is not smoothly compatible with $\mathcal{Z}_{s'}$ for $s \neq s'$. Clearly, every pair of charts not including $(\phi_i^{-1}(B), F_s \circ f \circ \phi_i)$ is smoothly compatible. Now consider the pair of charts $(\phi_i^{-1}(B), F_s \circ f \circ \phi_i)$ and (U, ϕ) and suppose that $\phi_i^{-1}(B) \cap U \neq \varnothing$. The transition function is then $F_s \circ f \circ \phi_i \circ \phi^{-1}$. Since $\phi_i^{-1}(B) \cap U$ does not contain p and F_s is a diffeomorphism away from 0, we have that the transition function is a diffeomorphism. This shows that $\mathcal{Z}_s$ is a smooth atlas, and therefore determines a smooth structure for each s. For $s \neq s'$, the charts $(\phi_i^{-1}(B), F_s \circ f \circ \phi_i)$ and $(\phi_i^{-1}(B), F_{s'} \circ f \circ \phi_i)$ are not smoothly compatible, since

$$F_s \circ f \circ \phi_i \circ (F_{s'} \circ f \circ \phi_i)^{-1} = F_s \circ f \circ \phi_i \circ \phi_i^{-1} \circ f^{-1} \circ F_{s'}^{-1} = F_s \circ F_{s'}^{-1} = F_{s/s'}$$

(the last equality is a standard computation) is not a diffeomorphism. We conclude that $\mathcal{Z}_s$ and $\mathcal{Z}_{s'}$ are not smoothly compatible and therefore there are uncountably many different smooth structures on M.

Problem 1.7. *Let N denote the **north pole** $(0, \ldots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the **south pole** $(0, \ldots, 0, -1)$. Define the **stereographic projection** $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by*

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) *For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $(\tilde{\sigma}(x), 0)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called **stereographic projection from the south pole**.)*

(b) *Show that σ is bijective, and*

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

(c) *Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ and $\tilde{\sigma}$ are called **stereographic coordinates**.)*

(d) *Show that this smooth structure is the same as the one defined in Example 1.31.*

For (a), let $x = (x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$. The line through N and x is parametrized by $u^1 = x^1 t, \ldots, u^n = x^n t, u^{n+1} = (x^{n+1} - 1)t + 1$. The intersection of this line with the $u^{n+1} = 0$ occurs when

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$t = 1/(1 - x^{n+1})$. Hence, the intersection point is $(\sigma(x), 0)$, as desired. Similarly, if $x = (x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{S\}$, then the line through S and x is parametrized by $u^1 = x^1 t, \ldots, u^n = x^n t, u^{n+1} = (x^{n+1} + 1)t - 1$. Setting $u^{n+1} = 0$, we find that the intersection point is $(\tilde{\sigma}(x), 0)$.

For (b), it suffices to check that the given definition for σ^{-1} is indeed the inverse of σ. It is straightforward to check that $|\sigma^{-1}(u)| = 1$ for all $u \in \mathbb{R}^n$, and the last coordinate of $\sigma^{-1}(u)$ is $\frac{|u|^2 - 1}{|u|^2 + 1} < 1$. Hence, $\sigma^{-1}(\mathbb{R}^n) \subseteq \mathbb{S}^n \setminus \{N\}$. Showing that $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$ are the identity maps is a straightforward calculation.

For (c), the transition map is

$$\tilde{\sigma} \circ \sigma^{-1}(u^1, \ldots, u^n) = \frac{(u^1, \ldots, u^n)}{|u|}.$$

Note that the domain of $\tilde{\sigma} \circ \sigma^{-1}$ is $\sigma(\mathbb{S}^n \setminus \{N, S\})$. Since $\sigma(S) = 0$, this shows that $u = 0$ is not in the domain of $\tilde{\sigma} \circ \sigma^{-1}$. Hence, $\tilde{\sigma} \circ \sigma^{-1}$ is smooth. A similar computation shows that $\sigma \circ \tilde{\sigma}^{-1}$ is smooth. Since $\mathbb{S}^n \setminus \{N\}$ and $\mathbb{S}^n \setminus \{S\}$ form an open cover of $\mathbb{S}^n$, we conclude that $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \tilde{\sigma})$ define a smooth structure on $\mathbb{S}^n$.

For (d), there are three "types" of charts from Example 1.31 to consider. The first is $(U_{n+1}^+, \phi_{n+1}^+)$, which contains N. The second is $(U_{n+1}^-, \phi_{n+1}^-)$, which contains S. Finally, the third type is $(U_i^\pm, \phi_i^\pm)$ for $i = 1, \ldots, n$, which contain neither N nor S. We will show that each type of chart is smoothly compatible with σ; the computation showing that they are smoothly compatible with $\tilde{\sigma}$ is similar. The first functions for the first and second type of chart are

$$\phi_{n+1}^\pm \circ \sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1},$$

which are both smooth. The inverses of these transition functions are

$$\sigma \circ (\phi_{n+1}^\pm)^{-1}(u^1, \ldots, u^n) = \frac{(u^1, \ldots, u^n)}{1 \mp \sqrt{1 - |u|^2}}$$

Hence, $\sigma \circ (\phi_{n+1}^-)^{-1}$ is smooth. To see that $\sigma \circ (\phi_{n+1}^+)^{-1}$ is smooth, note that its domain is $(\phi_{n+1}^+)(U_{n+1}^+ \setminus \{N\})$, which does not include 0. For the third type of chart, we have

$$\phi_i^\pm \circ \sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, \widehat{2u^i}, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1},$$

which is smooth, and

$$\sigma \circ (\phi_i^\pm)^{-1}(u^1, \ldots, u^n) = \frac{(u^1, \ldots, u^{i-1}, \sqrt{1 - |u|^2}, u^i, \ldots, u^{n-1})}{1 - u^n},$$

which is also smooth, since $u^n \neq 1$ in $U_i^\pm$. Hence, the smooth atlas from Example 1.31 is smoothly compatible with the smooth atlas described in this problem. Hence, they determine the same smooth structures on $\mathbb{S}^n$.

Problem 1.8. *By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An **angle function** on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.*

($\Rightarrow$) Let θ be an angle function on a subset $U \subseteq \mathbb{S}^1$. For a contradiction, suppose that $U = \mathbb{S}^1$. Since the image of a connected and compact space under a continuous function is connected and compact, we have that $\theta(\mathbb{S}^1) = [a, b] \subseteq \mathbb{R}$ for some $a \leq b$. We now show that $\tilde{\theta} : \mathbb{S}^1 \to [a, b], z \mapsto \theta(p)$ is a homeomorphism; this will give us a contradiction, since $[a, b]$ is simply connected, but $\mathbb{S}^1$ is not. It suffices to show that $\tilde{\theta}$ is bijective, since a continuous bijection with compact domain has a continuous inverse. By definition, $\tilde{\theta}$ is surjective. If $\tilde{\theta}(z) = \tilde{\theta}(z')$, then $z = e^{i\theta(z)} = e^{i\theta(z')} = z'$, so $\tilde{\theta}$ is injective, and therefore bijective.

($\Leftarrow$) Let $U \subset \mathbb{S}^1$ be a proper open subset. Suppose that $p \in \mathbb{S}^1 \setminus U$. We will construct an angle function θ on $\mathbb{S}^1 \setminus \{p\}$. The restriction of θ to U is therefore also an angle function. Note that $\theta(z) = -i \ln(z)$ is an angle function, where we must take a branch of the complex logarithm along the ray through p and the origin.

Let $\theta : U \to \mathbb{R}$ be an angle function. We proved in that θ is injective in the $\Rightarrow$ direction above. Hence, by the theorem on invariance of domain, θ is an open map and therefore a homeomorphism onto its image. This shows that (U, θ) is a chart for S^1. Suppose that $U = \mathbb{S}^1 \setminus \{N\}$. Since $\theta^{-1}(x) = e^{i\theta(\theta^{-1}(x))} = e^{ix}$ corresponds to $(\cos x, \sin x) \subseteq \mathbb{R}^n$. Let σ be the stereographic projection from the north discussed in the previous problem. Then

$$\sigma \circ \theta^{-1}(x) \;=\; \frac{\cos x}{1 - \sin x} \;=\; \tan\left(\frac{x}{2} + \frac{\pi}{4}\right),$$

which is a diffeomorphism from any interval of the form $(\frac{\pi}{2} + 2\pi n, \frac{5\pi}{2} + 2\pi n)$ to $\mathbb{R}$, where $n \in \mathbb{Z}$ (proving the above identity is a fun trigonometry exercise). Now, let $U \subset \mathbb{S}^1$ be an arbitrary proper open subset. By rotating $\mathbb{S}^1$, we may assume that $N \notin U$. Then $\sigma \circ \theta^{-1}$ is the restriction of the above formula on each connected component of U. Thus, (U, σ) is smoothly compatible with the stereographic coordinate charts, and is thus a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

Problem 1.9. Complex projective n-space, *denoted by* $\mathbb{CP}^n$, *is the set of all 1-dimensional complex-linear subspaces of* $\mathbb{C}^{n+1}$, *with the quotient topology inhereted from the natural projection* $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. *Show that* $\mathbb{CP}^n$ *is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for* $\mathbb{RP}^n$.

Let $\widetilde{U}_i \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ be the open set of points where where $z^i \neq 0$, and let $U_i = \pi(\widetilde{U}_i)$. It is not too hard to show that $\pi^{-1}(\pi(\widetilde{U}_i)) = \widetilde{U}_i$. Hence, U_i is open. Moreover, since the $\widetilde{U}_i$ cover $\mathbb{C}^{n+1} \setminus \{0\}$ and π is surjective, the U_i form an open cover of $\mathbb{CP}^n$. Define $\phi_i : U_i \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ by

$$\phi_i[z^1, \ldots, z^{n+1}] \;=\; \left(\frac{z^1}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^{n+1}}{z^i}\right).$$

Since $\phi_i[z^1, \ldots, z^{n+1}]$ is unchanged by multiplying $(z^1, \ldots, z^{n+1})$ by a non-zero constant, ϕ_i is well-defined. Moreover, ϕ_i is bijective, which is not too hard to see. We claim that $\pi|_{\widetilde{U}_i} : \widetilde{U}_i \to U_i$ is a quotient map; to show this, we must show that π is an open map, which was done in Exercise 1.6. Hence, by the characteristic property of quotient maps, ϕ_i is continuous if and only if $\phi_i \circ \pi|_{\widetilde{U}_i}$ is continuous. Since

$$\phi_i \circ \pi|_{\widetilde{U}_i}(z^1, \ldots, z^{n+1}) \;=\; \left(\frac{z^1}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^{n+1}}{z^i}\right),$$

ϕ_i is continuous. The inverse of ϕ_i is given by

$$\phi_i^{-1}(z^1, \ldots, z^n) \;=\; [z^1, \ldots, z^{i-1}, 1, z^i, \ldots, z^n].$$

Let $f_i : \mathbb{C}^n \to \mathbb{C}^{n+1}, (z^1, \ldots, z^n) \mapsto (z^1, \ldots, z^{i-1}, 1, z^i, \ldots, z^n)$; note that f_i is continuous. Then $\phi_i^{-1} = \pi \circ f_i$ is continuous. Thus ϕ_i is a homeomorphism, and we have shown that $\mathbb{CP}^n$ is locally Euclidean.

To prove that $\mathbb{CP}^n$ is Hausdorff, we will need the following fact from topology: if X is a compact Hausdorff space and G is a compact topological group acting continuously on X, then the orbit space X/G is Hausdorff. Let $X = \mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ and $G = \mathbb{S}^1 \subseteq \mathbb{C}$, and let $\mathbb{S}^1 \times \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1} : (\lambda, z) \mapsto \lambda z$ be the group action. Then the quotient space $\mathbb{S}^{2n+1}/\mathbb{S}^1$ is Hausdorff. We will now prove that $\mathbb{S}^{2n+1}/\mathbb{S}^1$ is homeomorphic to $\mathbb{CP}^n$. We will denote the equivalence classes in $\mathbb{S}^{2n+1}/\mathbb{S}^1$ and $\mathbb{CP}^n$ by $[\cdot]_{\mathbb{S}^{2n+1}/\mathbb{S}^1}$ and $[\cdot]_{\mathbb{CP}^n}$, respectively. We claim that the map $\bar{i} : \mathbb{S}^{2n+1}/\mathbb{S}^1 \to \mathbb{CP}^n, [z^1, \ldots, z^{n+1}]_{\mathbb{S}^{2n+1}/\mathbb{S}^1} \mapsto [z^1, \ldots, z^{n+1}]_{\mathbb{CP}^n}$ is a homeomorphism. The fact that $\bar{i}$ is well-defined and bijective is easily verified. For continuity, observe the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{S}^{2n+1} & \xrightarrow{\ i\ } & \mathbb{C}^{2n+1} \setminus \{0\} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathbb{S}^{2n+1}/\mathbb{S}^1 & \xrightarrow{\ \bar{i}\ } & \mathbb{CP}^n
\end{array}
$$

where i is the inclusion map, and the maps denoted π are the corresponding quotient maps. Commutativity and the continuity of $\pi \circ i$ imply that $\bar{i} \circ \pi$ is continuous. The universal property of quotient maps implies that $\bar{i}$ is continuous. Since the inverse of a continuous function with compact domain is continuous, we have that $\bar{i}$ is a homeomorphism. It follows that $\mathbb{CP}^n \cong \mathbb{S}^{2n+1}/\mathbb{S}^1$, and is therefore Hausdorff.

The proof that $\mathbb{CP}^n$ is second-countable is a straightforward adaptation of the proof that $\mathbb{RP}^n$ is second-countable given in Exercise 1.6.

We conclude the problem by showing that charts constructed above are smoothly compatible, and therefore define a smooth structure on $\mathbb{CP}^n$. Let (U_i, ϕ_i) and (U_j, ϕ_j) be two charts, with $i > j$. The transition function is

$$\phi_j \circ \phi_i^{-1}(z^1, \ldots, z^n) = \left(\frac{z^1}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^{i-1}}{z^j}, \frac{1}{z^j}, \frac{z^i}{z^j}, \ldots, \frac{z^n}{z^j} \right).$$

Since $z^j \neq 0$ for $(z_1, \ldots, z^n) \in \phi_i(U_i \cap U_j)$, the above map is well defined, and therefore a diffeomorphism from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$.

Problem 1.10. *Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\phi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\binom{I_k}{B}$, where I_k denotes the $k \times k$ identity matrix.*

As described in Example 1.36, if $\pi_P : \mathbb{R}^n \to P$ and $\pi_Q : \mathbb{R}^n \to Q$ are the projections onto P and Q, then the map $\pi_P|_S : S \to P$ is an isomorphism. Hence, there are vectors $b_1, \ldots, b_k \in \mathbb{Q}$ such that $e_i + q_i \in S$ for each $i \in \{1, \ldots, k\}$. Since $\{e_1 + b_1, \ldots, e_k + b_k\}$ gets mapped to the basis $\{e_1, \ldots, e_k\}$ for P, it must be a basis for S. Arranging the b_i into columns, we have a $(n-k) \times k$ matrix B such that S is spanned by the columns of $\binom{I_k}{B}$. Now let's show that this B is unique. Suppose B' is a $(n-k) \times k$ matrix such that the columns of $\binom{I_k}{B'}$ form a basis for S. Since $\pi_P|_S(e_i + b_i') = e_i$ and $\pi_P|_S$ is an isomorphism, we must have $b_i' = b_i$ for each i. Hence, $B = B'$, so B is unique.

Since $(\pi_Q|_S) \circ (\pi_P|_S)^{-1}(e_i) = (\pi_Q|_S)(e_i + b_i) = b_i$, the matrix representation of $(\pi_Q|_S) \circ (\pi_P|_S)^{-1}$ is B. Hence, $\phi(S) = B$.

Problem 1.11. *Let $M = \overline{\mathbb{B}^n}$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$.*

$\mathbb{B}^n$ is Hausdorff and second-countable, since it is a subspace of $\mathbb{R}^n$. Let $U_i^+ = \{(x^1, \ldots, x^n) \in \overline{\mathbb{B}^n} : x^i > 0\}$ and $U_i^- = \{(x^1, \ldots, x^n) \in \overline{\mathbb{B}^n} : x^i < 0\}$. Then the $U_i^\pm$ form an open cover of $\overline{\mathbb{B}^n}$. Define

$$\phi_i^\pm : U_i^\pm \to \mathbb{B}_i^{n,\pm}, \ (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^{i-1}, x^i \mp \sqrt{1 - (x^1)^2 - \cdots - \widehat{(x^i)^2} - \cdots - (x^n)^2}, x^{i+1}, \ldots, x^n),$$

where $\mathbb{B}_i^{n,+} = \{(x^1, \ldots, x^n) \in \mathbb{B}^n : x^i \geq 0\}$ and $\mathbb{B}_i^{n,-} = \{(x^1, \ldots, x^n) \in \mathbb{B}^n : x^i \leq 0\}$. The inverse of $\phi_i^\pm$ is then

$$(\phi_i^\pm)^{-1} : \mathbb{B}_i^{n,\pm} \to U_i^\pm, \ (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^{i-1}, x^i \pm \sqrt{1 - (x^1)^2 - \cdots - \widehat{(x^i)^2} - \cdots - (x^n)^2}, x^{i+1}, \ldots, x^n).$$

Since both $\phi_i^\pm$ and $(\phi_i^\pm)^{-1}$ are continuous, we conclude that $(U_i^\pm, \phi_i^\pm)$ are boundary charts covering M, so M is a topological manifold.

Let $(x^1, \ldots, x^n) \in \mathbb{S}^{n-1}$. Say $(x^1, \ldots, x^n) \in U_i^\pm$. Then $x^i = \pm\sqrt{1 - (x^1)^2 - \cdots - \widehat{(x^i)^2} - \cdots - (x^n)^2}$. Then $\phi_i^\pm(x^1, \ldots, x^n) = (x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n)$, so every point in $\mathbb{S}^{n-1}$ is a boundary point. If $(x^1, \ldots, x^n)$ lies in $\mathbb{B}^n$, then it lies in the interior chart $(\mathbb{B}^n, \mathrm{id})$, and is thus an interior point.

Note that $\phi_i^\pm$ and $(\phi_i^\pm)^{-1}$ can be extended to maps from $\mathbb{R} \times \cdots \times \mathbb{R} \times (-1, 1) \times \mathbb{R} \times \cdots \times \mathbb{R}$ to itself, where $(-1, 1)$ is the ith factor, and . Since both extensions of $\phi_i^\pm$ and $(\phi_i^\pm)^{-1}$ are both diffeomorphisms, the transition functions are all smooth, so the charts given above define a smooth structure on M. Moreover, since every chart is smoothly compatible with $(\mathbb{B}^n, \mathrm{id})$, every interior chart is smoothly compatible with the standard smooth structure on $\mathbb{B}^n$.

Problem 1.12. *Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).*

We will show that if M is a smooth m-manifold and N is a smooth n-manifold with boundary, then $M \times N$ is a smooth manifold with boundary, and $\partial(M \times N) = M \times \partial N$. The general case follows by induction. First, $M \times N$ is Hausdorff and second-countable, since both M and N are. Given charts (U, ϕ) and (V, ψ) for M and N, respectively, we let $(U \times V, \phi \times \psi)$ be a chart for $M \times N$. The collection of all such charts then gives $M \times N$ the structure of a smooth manifold with boundary, as we now show. Let $(U_1 \times V_1, \phi_1 \times \psi_1)$ and $(U_2 \times V_2, \phi_2 \times \psi_2)$ be two intersecting charts. Note that $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \times \psi_1^{-1})$ has domain $(\phi_1 \circ \psi_1)([U_1 \times V_1] \cap [U_2 \times V_2]) = \phi_1(U_1 \cap U_2) \times \psi_1(V_1 \cap V_2)$. Since $\psi_1(V_1 \cap V_2)$ can be extended to an open set such that $\psi_2 \circ \psi_1^{-1}$ is smooth, we have that the product charts are smoothly compatible. This, $M \times N$ is a smooth manifold with boundary.

Let $(x, y) \in \partial(M \times N)$. Then (x, y) is in the domain of some boundary chart $(U \times V, \phi \times \psi)$. Since $\phi(U)$ is open in $\mathbb{R}^m$, it follows that (V, ψ) must be a boundary chart. Moreover, $\psi(y)$ should lie on the boundary of $\mathbb{H}^n$, since otherwise there is an interior chart (V', ψ') whose domain contains y', and thus $(U \times V', \phi \times \psi')$ is an interior chart containing (x, y'). Thus, $y \in \partial N$, so $(x, y) \in M \times \partial N$. Conversely, let $(x, y) \in M \times \partial N$. Then there is a boundary chart (V, ψ) such that $y \in V$ and $\psi(y)$ is in $\partial \mathbb{H}^n$. Then, if u is in the domain of a chart (U, ϕ), then $(\phi \times \psi)(x, y) = (\phi(x), \psi(y)) \in \partial \mathbb{H}^{m+n}$, so $(x, y) \in \partial(M \times N)$. Hence, $\partial(M \times N) = M \times \partial N$.

2 Smooth Maps

Exercise 2.1. *Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.*

Let $f, g \in C^\infty(M)$. We want to show that $f + g$ and fg are also smooth functions. By Exercise 2.3, $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are smooth for every coordinate chart (U, ϕ). Then $(f + g) \circ \phi^{-1} = (f \circ \phi^{-1}) + (g \circ \phi^{-1})$ and $(fg) \circ \phi^{-1} = (f \circ \phi^{-1})(g \circ \phi^{-1})$ are smooth. Thus $C^\infty(M)$ is a commutative ring, since multiplication in $\mathbb{R}$ is commutative. Since smooth functions can be multiplied by real constants without affecting smoothness, $C^\infty(M)$ is also a commutative and associative algebra over $\mathbb{R}$, since multiplication of functions is commutative and associative.

Exercise 2.2. *Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.*

The chart (U, id) defines a smooth structure on U. Let f be smooth in the sense defined in the section. Then, for each chart $p \in U$, there is a chart (V, ψ) whose domain contains p and such that $f \circ \psi^{-1}$ is smooth. Since $\psi \circ \text{id}^{-1} = \psi$ is smooth, we have that $(f \circ \psi^{-1}) \circ \psi = f$ is smooth on V. Since this holds for every chart, we conclude that f is smooth in the ordinary calculus sense. Now suppose that f is smooth in the sense of ordinary calculus. Then, since every $p \in U$ and $f \circ \text{id}^{-1}$ is smooth, f is smooth in the sense of the section.

By Exercise 1.44, if $U \subseteq \mathbb{H}^n$ is open, then (U, id) gives U a smooth manifold structure, since $\text{id} : U \to U$ can be extended to all of $\mathbb{R}$. Let $f : U \to \mathbb{R}$ be smooth in the sense of the section. Then for every $p \in M$, there is an open set $V \ni p$ in U and a chart (V, ψ) such that $f \circ \psi^{-1}$ is smooth on some open extension of V. Since $\psi \circ \text{id}^{-1} = \psi$ is also smooth on some open extension of V, we conclude that f is smooth on some open extension of V, in the sense of ordinary calculus. Conversely, let $f : U \to \mathbb{R}$ be smooth in the sense of ordinary calculus. Then $f = f \circ \text{id}^{-1}$ is smooth on some open extension of U. Then f is smooth in the sense defined in the section.

Exercise 2.3. *Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for* every *smooth chart (U, ϕ).*

Let (U, ϕ) be an arbitrary chart, and let $p \in U$. Since f is smooth, there is a chart (V, ψ) such that $p \in V$ and $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth. Since the transition function $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is smooth, we have that $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1}) = f \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}^k$ is smooth. Since $p \in U$ was chosen arbitrarily, we conclude that $f \circ \phi^{-1}$ is smooth on all of $\phi(U)$.

Exercise 2.7. *Prove propositions 2.5 and 2.6.*

We begin with Proposition 2.5.

(Smoothness $\Rightarrow$ (b)) By smoothness of F, there exists smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for every α there exists a β such that $F(U_\alpha) \subseteq V_\beta$ and $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is smooth. Now, let (U_α, ϕ_α) and (V_β, ψ_β) be arbitrary charts in the atlases described above. Let $(V_{\beta'}, \psi_{\beta'})$ be a chart such that $F(U_\alpha) \subseteq V_{\beta'}$. We have that $\psi_{\beta'} \circ F \circ \phi_\alpha^{-1}$ is smooth from $\phi_\alpha(U_\alpha)$ to $\psi_{\beta'}(V_{\beta'})$, and $\psi_\beta \circ \psi_{\beta'}^{-1}$ is smooth from $\psi_{\beta'}(V_\beta \cap V_{\beta'})$ to $\psi_\beta(V_\beta \cap V_{\beta'})$. Hence, $(\psi_\beta \circ \psi_{\beta'}^{-1}) \circ (\psi_{\beta'} \circ F \circ \phi_\alpha^{-1}) = \psi_\beta \circ F \circ \phi_\alpha^{-1}$ is smooth from $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$ (there is no issue with extending the codomain here).

((b) $\Rightarrow$ (a)) Since $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$, for every $p \in M$, $p \in U_\alpha$ for some α and $F(p) \in V_\beta$ for some β. Let $U = U_\alpha$ and $V = V_\alpha$. By continuity of F, $U \cap F^{-1}(V)$ is open. By assumption, $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$ is smooth.

((a) $\Rightarrow$ Smoothness) Let p be in M, and let $(\widehat{U}, \phi)$ and (V, ψ) be charts containing p and $F(p)$, respectively. Taking $U = \widehat{U} \cap F^{-1}(V)$, we find that F is smooth.

We turn to Proposition 2.6.

For (a), suppose that every $p \in M$ has a neighbourhood U such that $F|_U$ is smooth. Then there is a chart $(\widetilde{U}, \tilde{\phi})$ of U containing p and a chart (V, ψ) of N containing $F(p)$ such that $F(\widetilde{U}) \subseteq V$ and $\psi \circ F \circ \phi^{-1}$ is smooth. Since $\widetilde{U}$ is open in U and U is open in M, we have that $(\widetilde{U}, \tilde{\phi})$ is a chart of M containing p. Thus F is smooth.

For (b), let $U \subseteq M$ be an open submanifold with its standard smooth structure. Let $p \in U$ and let (U', ϕ) and (V, ψ) be charts containing p and $F(p)$ such that $\psi \circ F \circ \phi^{-1}$ is smooth. Then the restriction $(U \cap U', \phi)$ is a chart in U having all the required properties for smoothness. Thus, $F|_U$ is smooth.

Exercise 2.9. *Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to* every *pair of smooth charts for M and N is smooth.*

Let (U, ϕ) and (V, ψ) be an arbitrary pair of charts for M and N, respectively. If $U \cap F^{-1}(V) = \varnothing$, then there is nothing to check, so suppose that $p \in U \cap F^{-1}(V)$, so that $F(p) \in V$. Since F is smooth, there is a chart $(\widetilde{U}, \tilde{\phi})$ containing p and a chart $(\widetilde{V}, \tilde{\psi})$ such that $F(\widetilde{U}) \subseteq \widetilde{V}$ and $\tilde{\psi} \circ F \circ \tilde{\phi}^{-1} : \tilde{\phi}(\widetilde{U}) \to \tilde{\psi}(\widetilde{V})$ is smooth. Since $\tilde{\phi} \circ \phi^{-1} : \phi(U \cap \widetilde{U}) \to \tilde{\phi}(U \cap \widetilde{U})$ and $\psi \circ \tilde{\psi}^{-1} : \tilde{\psi}(V \cap \widetilde{V}) \to \psi(V \cap \widetilde{V})$ are smooth, it follows that

$$(\psi \circ \tilde{\psi}^{-1}) \circ (\tilde{\psi} \circ F \circ \tilde{\phi}^{-1}) \circ (\tilde{\phi} \circ \phi^{-1}) \;=\; \psi \circ F \circ \phi^{-1} : \phi(U \cap \widetilde{U} \cap F^{-1}(V)) \to \psi(V).$$

Since $p \in U \cap F^{-1}(V)$ was arbitrary, we conclude that $\psi \circ F \circ \phi^{-1}$ is smooth on all of $\phi(U \cap F^{-1}(V))$.

Exercise 2.11. *Prove parts* (a)-(c) *of Proposition 2.10.*

For (a), let $c : M \to N, p \mapsto c$ be a constant map, where c is some fixed point in N. Let $p \in M$ be arbitrary, and let (U, ϕ) be a chart containing p and (V, ψ) a chart containing $F(p) = c$. Since $U \cap F^{-1}(V) = U \cap M = U$ is open and $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V), x \mapsto \psi(c)$ is smooth (as a constant map between Euclidean spaces), we have that $c : M \to N$ is smooth.

For (b), let $\mathrm{id} : M \to M$ be the identity map. Let $p \in M$ and let (U, ϕ) be a chart containing p. Then (U, ϕ) is a chart containing $\mathrm{id}(p) = p$, and $\phi \circ \mathrm{id} \circ \phi^{-1} : \phi(U) \to \phi(U)$ is smooth since it is the identity map, and the identity map on a subset of Euclidean space is smooth.

For (c), let $U \subseteq M$ be an open submanifold and let $i : U \hookrightarrow M$ be the inclusion map. Let $p \in U$ and let (V, ϕ) be a chart in U containing p. Then (V, ϕ) is also a chart in M containing $i(p) = p$. Then, $\phi \circ i \circ \phi^{-1} : \phi(U) \to \phi(U)$ is the identity map, which is smooth. Hence $i : U \hookrightarrow M$ is smooth.

Exercise 2.16. *Prove Proposition 2.15.*

For (a), let $F : M \to N$ and $G : N \to P$ be diffeomorphisms. We that $G \circ F : M \to P$ is a bijection and is smooth by Proposition 2.10. Since F^{-1} and G^{-1} are also smooth, the composition $F^{-1} \circ G^{-1} = (G \circ F)^{-1}$ is also smooth by Proposition 2.10. We conclude that $G \circ F$ is a diffeomorphism.

For (b), let $F_i : M_i \to N_i$ be diffeomorphisms between smooth manifolds for $i = 1, \ldots, n$. We want to show that

$$F_1 \times \cdots \times F_n : M_1 \times \cdots \times M_n \to N_1 \times \cdots \times N_n, \; (p_1, \ldots, p_n) \mapsto (F_1(p_1), \ldots, F_n(p_n))$$

is a diffeomorphism. The inverse of $F_1 \times \cdots \times F_n$ is $F_1^{-1} \times \cdots \times F_n^{-1}$. By Proposition 2.12, $F = F_1 \times \cdots \times F_n$ is smooth if and only if each $\pi_i \circ F$ is smooth for each i, where π_i is the projection from $N_1 \times \cdots N_n \mapsto N_i$. Since $(\pi_i \circ F)(p_1, \ldots, p_n) = F_i(p_i)$, we have that $\pi_i \circ F = F_i \circ \pi_i$, where the second π_i is the projection from $M_1 \times \cdots \times M_n$ to M_i. Since F_i and π_i are smooth, $\pi_i \circ F$ is smooth, and therefore F is smooth. A similar proof shows that F^{-1} is smooth.

For (c), let $F : M \to N$ be a diffeomorphism. Then F and F^{-1} are continuous by Proposition 2.4, so F is a homeomorphism. Then F is and open map since homeomorphisms are open maps.

For (d), let $F : M \to N$ be a diffeomorphism, let $U \subseteq M$ be an open submanifold, and let $F|_U : U \to F(U)$ be the restriction. Note that $F|_U = F \circ i$, where $i : U \hookrightarrow M$ is the inclusion. By Exercise 2.11, $F|_U$ is smooth. The same argument applies to $(F|_U)^{-1} : F(U) \to U$, since $F(U)$ is an open submanifold by part (c).

For (e), M is always diffeomorphic to M, since $\mathrm{id} : M \to M$ is a diffeomorphism; being diffeomorphic is reflexive. If F is a diffeomorphism, then so is F^{-1}; being diffeomorphic is symmetric. Finally, since the composition of diffeomorphisms is a diffeomorphism, being diffeomorphic is transitive. Hence, being diffeomorphic is an equivalence relation.

Exercise 2.19. *Use Theorem 1.46 to prove Theorem 2.18.*

Let $F(p) \in F(\partial M)$. Then $p \in \partial M$, so there is a boundary chart (U, ϕ) containing p. Then $(F(U), \phi \circ F^{-1})$ is a boundary chart, since F and ϕ are diffeomorphisms and thus homeomorphisms. Hence, $F(p) \in \partial N$. Hence, $F(\partial M) \subseteq \partial N$; applying F^{-1} to both sides, we have $\partial M \subseteq F^{-1}(\partial N)$. But F^{-1} is a diffeomorphism, so the same argument given above shows that $F^{-1}(\partial N) \subseteq \partial M$. Thus $F(\partial M) = \partial N$.

Since $\mathrm{Int}M$ is open by Exercise 1.39, F restricts to a diffeomorphism from $\mathrm{Int}M$ to $F(\mathrm{Int}M)$ by Proposition 2.15. Since $F(\partial M) = \partial N$, and every point is either an interior point or a boundary point, we have that $F(\mathrm{Int}M) = \mathrm{Int}N$.

Exercise 2.24. *Show how the proof of Theorem 2.23 needs to be modified for the case in which M has nonempty boundary.*

We need our bases to be composed of regular balls and half balls. We also need to extend our definition of bump functions to include coordinate half balls. Otherwise the proof is essentially the same.

Exercise 2.27. *Give a counterexample to show that the conclusion of the extension lemma can be false if A is not closed.*

Let $M = \mathbb{R}$ (with its standard smooth structure) and let $A = (-\infty, 0) \cup (0, \infty)$. Let $f : A \to \mathbb{R}$ be defined by $f(x) = 1$ if $x > 0$ and $f(x) = -1$ if $x < 0$. Note that f is smooth since it is smooth on each component of A. Any extension of f to $\mathbb{R}$ is discontinuous, and therefore not smooth, so the extension lemma does not hold here.

Problem 2.1. *Define $f : \mathbb{R} \to \mathbb{R}$ by*
$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, ϕ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

Since f is not continuous, it is not smooth in the sense we have defined in this chapter. Moreover, f is smooth away from $x = 0$ since it is constant there. Let $\epsilon > 0$, let $U = (-\epsilon, \epsilon)$, and let $V = (\frac{1}{2}, \frac{3}{2})$. Then U contains $x = 0$ and V contains $f(x) = 1$. Moreover, we let (U, id) and (V, id) be charts for $\mathbb{R}$. Then $\mathrm{id} \circ f \circ \mathrm{id}^{-1} = f$ is the constant map on $f(U \cap f^{-1}(V)) = [0, \epsilon)$, and is therefore smooth.

Problem 2.2. *Prove Proposition 2.12.*

Let $F : N \to M_1 \times \cdots \times M_k$ be smooth. By Proposition 2.10, in order to show that $\pi_i \circ F$ is smooth, it suffices to show that the projection $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth. Let $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, and let $(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k)$ be a product chart containing $(p_1, \ldots, p_k)$. Then (U_i, ϕ_i) is a chart

containing $p_i \in M_i$. The coordinate representation of $\phi_i \circ \pi_i \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is just the projection from $\phi_1(U_1) \times \cdots \times \phi_k(U_k)$ to $\phi_i(U_i)$, which is smooth. Hence, π_i is smooth.

Conversely, let $\pi_i \circ F$ be smooth for each $i = 1, \ldots, k$. Let $p \in N$, let (V, ψ) be a chart containing p and let (U_i, ϕ_i) be charts containing $\pi_i \circ F(p)$ such that $\pi_i \circ F(V) \subseteq U_i$ and $\phi_i \circ \pi_i \circ F \circ \psi^{-1} : \psi(V) \to \phi_i(U_i)$ is smooth for each $i = 1, \ldots, k$. Now, then $(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k)$ is a chart for $M_1 \times \cdots \times M_k$ such that $F(V) \subseteq U_1 \times \cdots \times U_k$. Then the coordinate representation $(\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1} : \psi(V) \to \phi_1(U_1) \times \cdots \times \phi_k(U_k)$ is smooth, since the projection onto each $\phi_i(U_i)$ is equal to $\phi_i \circ \pi_i \circ F \circ \psi^{-1}$, which is smooth.

Problem 2.3. *For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.*

(a) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ *is the* **nth power map** *for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.*

(b) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ *is the* **antipodal map** *$\alpha(x) = -x$.*

(c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ *is given by $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of $\mathbb{S}^3$ as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.*

For (a), let $z \in \mathbb{S}^1$, let (U, θ) be an angle coordinate chart containing z, and let (V, ϕ) be an angle coordinate chart containing z^n. Then $\phi \circ p_n \circ \theta^{-1}(x) = \phi \circ p_n(e^{ix}) = \phi(e^{inx}) = nx + 2k\pi$ for some k, which is constant on each component of $\theta(U \cap p_n^{-1}(V))$. Note that $U \cap p_n^{-1}(V)$ is open, since p_n is continuous. Hence, p_n is smooth.

For (b), let $p \in \mathbb{S}^n$, and assume that $(\mathbb{S}^n \setminus \{N\}, \sigma)$ is the stereographic chart from the north and it contains p. Then $(\mathbb{S}^n \setminus S, \tilde{\sigma})$ contains $\alpha(p)$, where $\tilde{\sigma}$ is the stereographic projection from the south. A computation shows that $\tilde{\sigma} \circ \alpha \circ \sigma^{-1}(u) = -u$, which is smooth. Hence, α is smooth.

For (c), writing F in real coordinates, we have

$$F(x^1, x^2, x^3, x^4) = (2x^1 x^3 + 2x^2 x^4, 2x^1 x^4 - 2x^2 x^3, (x^3)^2 + (x^4)^2 - (x^1)^2 - (x^2)^2),$$

so F is continuous, since it is the restriction of a continuous function. A computation shows that

$$\sigma_{\mathbb{S}^2} \circ F \circ \sigma_{\mathbb{S}^3}^{-1}(u_1, u_2, u_3) = \frac{(8u^1 u^3 + 4u^2(|u|^2 - 1), 4u^1(|u|^2 - 1) - 8u^2 u^3)}{1 + (2u^1)^2 + (2u^2)^2 - (2u^3)^2 - (|u|^2 - 1)^2},$$

which is smooth on $\sigma_{\mathbb{S}^3}(\mathbb{S}^3 \setminus \{N\} \cap F^{-1}(\mathbb{S}^2 \setminus \{N\}))$. Here $\sigma_{\mathbb{S}^n}$ is the stereographic projection from the north of $\mathbb{S}^n$. Similar computations for different pairs of charts show that F is indeed a smooth function.

Problem 2.4. *Show that the inclusion map $\overline{\mathbb{B}^n} \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}^n}$ is regarded as a smooth manifold with boundary.*

We give $\overline{\mathbb{B}^n}$ the smooth structure described in 1.11. Let $x \in \mathbb{B}^n$ be contained in some chart $(U_i^\pm, \phi_i^\pm)$. Then $i(x) = x$ is contained in the chart $(\mathbb{R}^n, \mathrm{id})$, where $i : \overline{\mathbb{B}^n} \hookrightarrow \mathbb{R}^n$ is the inclusion. Then $\mathrm{id} \circ i \circ (\phi_i^\pm)^{-1} = (\phi_i^\pm)^{-1}$, which is smooth, as discussed in Problem 1.11.

Problem 2.5. *Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.*

(a) *Show that f is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.*

(b) *Show that f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.*

For (a), the coordinate representation $\psi \circ f \circ \mathrm{id}^{-1} = \psi \circ f$ is smooth, since $\psi(x) = x^3$ and f are both smooth in the sense of regular calculus.

For (b), the coordinate representation is $\mathrm{id} \circ f \circ \psi^{-1} = f \circ \psi^{-1}$, and $f \circ \psi^{-1}(x) = f(x^{1/3})$. Let $f \circ \psi^{-1}$. We appeal to *Faà di Bruno's formula*, which states that for smooth functions F and G

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} F(G(x)) = \sum \frac{n!}{m_1! \cdots m_n!} F^{(m_1 + \cdots + m_n)}(G(x)) \prod_{j=1}^{n} \left(\frac{G^{(j)}(x)}{j!} \right)^{m_j},$$

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where the sum ranges over all n-tuples of nonnegative integers $(m_1, \ldots, m_n)$ such that $m_1 + 2m_2 + \cdots + nm_n = n$. Since we are assuming $f \circ \psi^{-1}$ is smooth, we can apply the above formula to $F = f \circ \psi^{-1}$ and $G = \psi$. Note that $\psi^{(j)}(0) = 0$ for all $j \neq 3$. Hence, if $f^{(n)}(0) \neq 0$, then there must be an n-tuple $(m_1, \ldots, m_n)$ such that $m_j = 0$ for all $j \neq 3$. Then $n = 3m_3$. This proves the first direction.

For the other direction, we first prove the following fact by induction on k: if $F \in C^k(\mathbb{R})$, then so is $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})$. For $k = 0$, the result is clear, since $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})$ is continuous. For the inductive step, let $F \in C^k(\mathbb{R})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{k+\frac{1}{3}} F(x^{\frac{1}{3}}) \right] \;=\; x^{(k-1)+\frac{1}{3}} F(x^{\frac{1}{3}}) + \frac{1}{3} x^{(k-1)+\frac{1}{3}} x^{\frac{1}{3}} F'(x^{\frac{1}{3}}).$$

By induction, $x^{(k-1)+\frac{1}{3}} F(x^{\frac{1}{3}}) \in C^{k-1}(\mathbb{R})$. Since $xF'(x) \in C^{k-1}(\mathbb{R})$, we also have $\frac{1}{3} x^{(k-1)+\frac{1}{3}} F'(x^{\frac{1}{3}}) \in C^{k-1}(\mathbb{R})$ by induction. Since it derivative is $C^{k-1}(\mathbb{R})$, we have that $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}}) \in C^k(\mathbb{R})$, concluding the proof of the claim.

Suppose that $f^{(n)}(0) = 0$ for all non-multiples of 3. Using the Taylor remainder theorem to write f up to the $3m$th term:

$$f(x) \;=\; f(0) + \frac{f^{(3)}(0)}{3!} x^3 + \cdots + \frac{f^{(3m)}(0)}{(3m)!} x^{3m} + x^{3m+1} F_{3m+1}(x)$$

for some smooth function F_{3m+1}. Then

$$f(x^{\frac{1}{3}}) \;=\; f(0) + \frac{f^{(3)}(0)}{3!} x + \cdots + \frac{f^{(3m)}(0)}{(3m)!} x^m + x^{m+\frac{1}{3}} F_{3m+1}(x^{\frac{1}{3}}).$$

By the claim, $x^{m+\frac{1}{3}} F_{3m+1}(x^{\frac{1}{3}}) \in C^m(\mathbb{R})$. Hence, $f \circ \psi^{-1} \in C^m(\mathbb{R})$ for all $m \geq 0$, and is therefore smooth.

Problem 2.6. *Let $P : \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. (Such a function is said to be **homogenous of degree** d.) Show that the map $\widetilde{P} : \mathbb{RP}^k \to \mathbb{RP}^k$ defined by $\widetilde{P}([x]) = [P(x)]$ is well defined and smooth.*

If $[x] = [y]$, then $x = \lambda y$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then $\widetilde{P}([x]) = [P(x)] = [P(\lambda y)] = [\lambda^d P(y)] = [P(y)] = \widetilde{P}([y])$. Therefore, $\widetilde{P}$ is well defined.

Let $[x] \in U_i$ and suppose that $\widetilde{P}([x]) \in U_j$. The coordinate representation $\phi_j \circ \widetilde{P} \circ \phi_i^{-1}$ takes a point $(u^1, \ldots, u^k) \in \phi_i(U_i)$ to

$$\frac{(P_1(\alpha), \ldots, P_{j-1}(\alpha), P_{j+1}(\alpha), \ldots, P_{n+1}(\alpha))}{P_j(\alpha)},$$

where $\alpha = \phi_i(u^1, \ldots, u^k) = (u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^k)$ and P_m is the mth component of P. Since P is smooth, each P_m is smooth. Therefore, the coordinate representation is smooth, and thus so is $\widetilde{P}$.

Problem 2.7. *Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.*

We begin by proving the suggested claim from the hint, namely that if $f_1, \ldots, f_k \in C^\infty(M)$ have disjoint support and are nonzero, then they are linearly independent. Consider a linear combination of these functions set to 0:

$$\alpha_1 f_1 + \cdots + \alpha_k f_k \;=\; 0.$$

For each $i = 1, \ldots, k$, choose some $x_i \in M$ such that $f_i(x_i) \neq 0$. Since $f_j(x_i) = 0$ for each $j \neq i$, we have $\alpha_i f(x_i) = 0 \Rightarrow \alpha_i = 0$. Since i was arbitrary, the f_i are linearly independent.

Now, let (U, ϕ) be some chart and let $k \in \mathbb{N}$ be arbitrary. It is not hard to see that there are k disjoint open balls $B_{r_i}(\phi(x_i))$ in $\phi(U)$. By Proposition 2.25, there are smooth bump functions that are positive on $\phi^{-1}(\overline{B_{r_i/2}(\phi(x_i))})$ and supported on $\phi(B_{r_i}(\phi(x_i)))$. Since the support of all these balls are disjoint, the corresponding bump functions are linearly independent. Since $k \in \mathbb{N}$ was arbitrarily chosen, we have that $C^\infty(M)$ is infinite dimensional.

Problem 2.8. *Define $F : \mathbb{R}^n \to \mathbb{RP}^n$ by $F(x^1, \ldots, x^n) = [x^1, \ldots, x^n, 1]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G : \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G(z^1, \ldots, z^n) = [z^1, \ldots, z^n, 1]$.*

We claim that $F : \mathbb{R}^n \to U_{n+1}$ is a diffeomorphism, where U_{n+1} is an open submanifold of $\mathbb{RP}^n$. We begin by showing that F is injective; suppose that $[x^1, \ldots, x^n, 1] = [y^1, \ldots, y^n, 1]$. Then $(x^1, \ldots, x^n, 1) = \lambda(y^1, \ldots, y^n, 1)$, so $1 = \lambda \cdot 1 \Rightarrow \lambda = 1$. To see that F is surjective, let $[x^1, \ldots, x^{n+1}] \in U_{n+1}$. Then $F(\frac{x^1}{x^{n+1}}, \ldots, \frac{x^n}{x^{n+1}}) = [x^1, \ldots, x^{n+1}]$. The coordinate representation of F is

$$\phi_{n+1} \circ F : \mathbb{R}^n \to \phi_{n+1}(U_{n+1}), \ (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n),$$

which is of course smooth. The inverse of F is given by

$$F^{-1} : U_{n+1} \to \mathbb{R}^n, \ [x^1, \ldots, x^{n+1}] \mapsto \left(\frac{x^1}{x^{n+1}}, \ldots, \frac{x^n}{x^{n+1}} \right).$$

The coordinate representation is $F^{-1} \circ \phi_{n+1}^{-1} = (\phi_{n+1} \circ F)^{-1} = \text{id}$ which is smooth. Hence, F is a diffeomorphism. It remains to show that U_{n+1} is open and dense in $\mathbb{RP}^n$. We know that U_{n+1} is open since it is the domain of a chart of $\mathbb{RP}^n$.

For density, let $[x^1, \ldots, x^{n+1}] \in \mathbb{RP}^n \setminus U_{n+1}$ and let $U \subseteq \mathbb{RP}^n$ be an open set containing $[x^1, \ldots, x^{n+1}]$. Since π is surjective, we have that $U = \pi(\pi^{-1}(U))$. Since π is continuous, we have that $\pi^{-1}(U)$ is open, and therefore it contains a point $(y^1, \ldots, y^{n+1})$ such that $y^{n+1} \neq 0$. Hence, $[y^1, \ldots, y^{n+1}] \in U \cap U_{n+1}$. Thus, every open set containing $[x^1, \ldots, x^{n+1}]$ intersects U_{n+1}, and therefore U_{n+1} is dense in $\mathbb{RP}^n$.

Doing the same for $G : \mathbb{C} \to \mathbb{CP}^n$ is completely analogous.

Problem 2.9. *Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\widetilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G : \mathbb{C} \to \mathbb{CP}^1$ is the map from Problem 2.8:*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\ \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \end{array}$$

Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n \neq 0$. Define $P : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$ by

$$P(z^1, z^2) = (a_n(z^1)^n + a_{n-1}(z^1)^{n-1}z^2 + \cdots + a_1 z^1(z^2)^{n-1} + a_0(z^2)^n, (z^2)^n).$$

To see that the image of this map does not include 0, not that if $z^2 = 0$, then $z_1 \neq 0$ and $P(z^1, 0) = (a_n(z^1)^n, 0) \neq 0$. Since P is homogeneous of degree n, $\tilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1, [z^1, z^2] \mapsto [P(z^1, z^2)]$ is well defined. It follows directly from the definition that $\tilde{p} \circ G = G \circ p$. Since P is smooth, we have that $\tilde{p}$ is smooth by Problem 2.6. If there is another smooth $\tilde{p}'$ making the diagram commute, then $\tilde{p}'$ must agree with $\tilde{p}$ on the image of G, which is dense in $\mathbb{CP}^1$. Since $\tilde{p}$ and $\tilde{p}'$ are smooth, they are equal.

Problem 2.10. *For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.*

(a) *Show that F^* is a linear map.*

(b) *Suppose M and N are smooth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.*

(c) *Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.*

For (a), let $f, g \in C^\infty(M)$ and let $\alpha, \beta \in \mathbb{R}$. Then $F^*(\alpha f + \beta g) = (\alpha f + \beta g) \circ F = \alpha f \circ F + \beta g \circ F$. Hence, F^* is linear.

For (b), let $F : M \to N$ be smooth, and let $f \in C^\infty(N)$. Since the composition of smooth functions is smooth, $F^*(f) = f \circ F \in C^\infty(M)$. Hence, $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Conversely, suppose that $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $\mathcal{B} = \{(V, \psi)\}$ be an atlas of regular coordinate balls in N and let $\mathcal{A} = \{(U, \phi)\}$ any atlas for M. By the extension lemma for smooth functions (Lemma 2.26), each coordinate map $\psi : V \to \mathbb{R}^n$ can be extended to a smooth map $\psi : N \to \mathbb{R}^n$, note that each extended component function $\psi^i : N \to \mathbb{R}$ is in

$C^\infty(N)$. Let $p \in M$ be arbitrary and be contained in a smooth chart (U, ϕ), and let $F(p)$ be in the chart (V, ψ). We want to show that $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$ is smooth. Note that $U \cap F^{-1}(V)$ is open since F is assumed to be continuous. We have that $\psi \circ F : M \to \mathbb{R}^n$ is smooth, since each of the component functions are $F^*(\psi^i) \in C^\infty(M)$. The restriction to $U \cap F^{-1}(V)$ is also smooth, and since ϕ^{-1} is smooth, we have that $\psi \circ F \circ \phi^{-1}$ is smooth. Hence, F is smooth.

For (c), let F be a diffeomorphism. Then $F^*(C^\infty(N)) \subseteq C^\infty(M)$ and $(F^{-1})^*(C^\infty(M)) \subseteq C^\infty(N)$. Since F is invertible, F^* must be injective. We conclude that $F^* : C^\infty(N) \to C^\infty(M)$ is a bijection. It is also linear by part (a). Finally, if $f, g \in C^\infty(N)$, then $F^*(fg) = (fg) \circ F = (f \circ F)(g \circ G)$. Thus, F^* is an algebra isomorphism when restricted to $C^\infty(N)$. Conversely, suppose that F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$. Then $F^*(C^\infty(N)) \subseteq C^\infty(M)$ and $(F^{-1})^*(C^\infty(M)) \subseteq C^\infty(N)$, so part (b) implies that F and F^{-1} are smooth, and therefore that F is a diffeomorphism.

Problem 2.11. *Suppose V is a real vector space of dimension $n \geq 1$. Define the **projectivization of V**, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^n$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \ldots, E_n)$ for V, the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E[v^1, \ldots, v^n] = [v^i E_i]$ (where the brackets denote equivalence classes) is a diffeomorphism.*

Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear homeomorphism (V and W have topologies inherited from any of their norms and the topologies are independent of the chosen norms). Define $\widetilde{T} : \mathbb{P}(V) \to \mathbb{P}(W), [v] \mapsto [T(v)]$. Linearity of T guarantees that $\widetilde{T}$ is well defined. If $[T(v_1)] = [T(v_2)]$, then $T(v_1) = \lambda T(v_2) = T(\lambda v_2) \Rightarrow v_1 = \lambda v_2 \Rightarrow [v_1] = [v_2]$, so $\widetilde{T}$ is injective; $\widetilde{T}$ is clearly surjective, so it is bijective.

We want to show that $\widetilde{T}$ is a homeomorphism. Let $\pi_V : V \to \mathbb{P}(V)$ and $\pi_W : W \to \mathbb{P}(W)$ be the natural projections, and let $U \subseteq \mathbb{P}(W)$ be open. $\widetilde{T}^{-1}(U)$ is open if and only $\pi_V^{-1}(\widetilde{T}^{-1}(U)) = (\widetilde{T} \circ \pi_V)^{-1}(U)$ is. But $\widetilde{T} \circ \pi_V = \pi_W \circ T$, which is continuous, hence $\pi_V^{-1}(\widetilde{T}^{-1}(U))$ is open, so $\widetilde{T}^{-1}(U)$ is open. This show that $\widetilde{T}$ is continuous. The same idea shows that $\widetilde{T}^{-1}$ (which is given by $\widetilde{T}^{-1}(w) = [T^{-1}(w)]$) is continuous, since T^{-1} is also a linear homeomorphism. We conclude that $\mathbb{P}(V)$ and $\mathbb{P}(W)$ are homeomorphic. Since V is homeomorphic to $\mathbb{R}^n$ via a linear homeomorphism taking any basis of V to the standard basis of $\mathbb{R}^n$, we have that $\mathbb{P}(V) \cong \mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^{n-1}$. Since $\mathbb{RP}^{n-1}$ is a topological $(n-1)$-manifold, $\mathbb{P}(V)$ is too.

Fix a basis $(B_1, \ldots, B_n)$ for V, and let $T : \mathbb{R}^n \to V$ be the linear homeomorphism that sends the standard basis vector e_i to B_i for each $i = 1, \ldots, n$. Let $\widetilde{T} : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ be the induced homeomorphism. Consider the standard smooth atlas $\{(U_i, \phi_i)\}_{i=1}^n$ for $\mathbb{RP}^{n-1}$. I claim that $\{(\widetilde{T}(U_i), \phi_i \circ \widetilde{T}^{-1}\}$ is a smooth atlas for $\mathbb{P}(V)$. Indeed, the transition functions are $(\phi_i \circ \widetilde{T}^{-1}) \circ (\phi_j \circ \widetilde{T}^{-1})^{-1} = \phi_i \circ \phi_j^{-1}$ which are smooth. Now let $(E_1, \ldots, E_n)$ be a basis for V, and let $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ be defined as above. Then the coordinate representation of E is

$$(\phi_i \circ \widetilde{T}^{-1}) \circ E \circ \phi_j^{-1}(x^1, \ldots, x^{n-1}) = \phi_i[T^{-1}F(x^1, \ldots, x^{j-1}, 1, x^j, \ldots, x^{n-1})],$$

where $F : \mathbb{R}^n \to V$ is defined by $F(v^1, \ldots, v^n) = v^i E_i$. Since $T^{-1}F$ is an invertible linear map from $\mathbb{R}^n$ to itself, it is a diffeomorphism. Since ϕ_i and $\pi : \mathbb{R}^n \to \mathbb{RP}^{n-1}$ are smooth, the coordinate representation is smooth. Therefore, E is smooth. The coordinate representation of E^{-1} is

$$\phi_i \circ E^{-1} \circ (\phi_j \circ \widetilde{T}^{-1})^{-1}(x^1, \ldots, x^{n-1}) = \phi_i[FT^{-1}(x^1, \ldots, x^{j-1}, 1, x^j, \ldots, x^{n-1})],$$

so E^{-1} is smooth as well. Therefore, E is a diffeomorphism.

Let $\mathcal{A}$ be a smooth atlas for $\mathbb{P}(V)$ such that $\widetilde{T}$ is a diffeomorphism, and let $(U, \phi) \in \mathcal{A}$ be a smooth chart. Then $\phi_i \circ \widetilde{T}^{-1} \circ \phi^{-1}$ is smooth for each i. But this just means that $\mathcal{A}$ is smoothly compatible with the smooth structure described in this section. This proves uniqueness of the smooth structure.

Problem 2.12. *State and prove the analogue of Problem 2.11 for complex vector spaces.*

If V is a complex vector space of dimension ≥ 1, define the projectivization of V, denoted by $\mathbb{P}(V)$ to be the set of all 1-dimensional complex linear subspaces of V, with the quotient topology induced by the map

$\pi : V \setminus \{0\} \to \mathbb{P}(V)$ that sends x to its span. Then $\mathbb{P}(V)$ is a topological $(2n-2)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \ldots, E_n)$ fpr V, the map $E : \mathbb{CP}^{n-1} \to \mathbb{P}(V)$ defined by $E[z^1, \ldots, z^n] = [z^i E_i]$ is a diffeomorphism.

The proof of the previous problem only used the fact that all n-dimensional real vector spaces are isomorphic (and homeomorphic) via a linear map that sends bases to bases, as well as the structure of the coordinate charts on $\mathbb{RP}^{n-1}$. Since the same fact is true of complex vector spaces, and the coordinate charts on $\mathbb{CP}^{n-1}$ are defined completely analogously to the coordinate charts on $\mathbb{RP}^{n-1}$, the proof of the above statement is given in the previous problem, with the exception that we must replace $\mathbb{R}$ with $\mathbb{C}$ whenever it appears.

Problem 2.13. *Suppose that M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.*

Let $\mathcal{X} = \{X_\alpha\}_\alpha$ be an open cover, and let $(\psi_\alpha)_\alpha$ be a partition of unity subordinate to it. Let $U_\alpha = \{p \in M : \psi_\alpha(p) > 0\} = \psi_\alpha^{-1}((0, \infty))$. Then U_α is open, since ψ_α is continuous. Since $\sum_\alpha \psi_\alpha(p) = 1$ for all $p \in M$, we have that the U_α cover M. Moreover, since $U_\alpha \subseteq \text{supp}(\psi_\alpha)$ and $\{\sup(\psi_\alpha)\}$ is locally finite, $\{U_\alpha\}_\alpha$ is locally finite. Moreover, $U_\alpha \subseteq X_\alpha$, so $\{U_\alpha\}_\alpha$ is a refinement of $\mathcal{X}$. Hence, M is paracompact.

Problem 2.14. *Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.*

By Theorem 2.29, there are functions $g, h : M \to [0, \infty)$ such that $g^{-1}(0) = A$ and $h^{-1}(0) = B$. It is immediate to check that $\frac{g}{g+h}$ has the desired property.

3 Tangent Vectors

Exercise 3.5. *Prove Lemma 3.4.*

For (a), let $f_1 : M \to \mathbb{R}$ be the constant function at 1. Then

$$v f_1 = v(f_1^2) = f_1(p) v f_1 + f_1(p) v f_1 = 2 v f_1,$$

so $v f_1 = 0$. If f is the constant function at $c \in \mathbb{R}$, then $vf = v(cf_1) = c \cdot v f_1 = 0$.

For (b), we have

$$v(fg) = f(p)vg + g(p)vf = 0.$$

Exercise 3.7. *Prove Proposition 3.6.*

For (a), let $u, v \in T_p M$ be derivations, let $a, b \in \mathbb{R}$, and let $f \in C^\infty(N)$. Then

$$dF_p(au + bv)(f) = (au + bv)(f \circ F) = au(f \circ F) + bv(f \circ F) = a dF_p(u)(f) + b dF_p(v)(f).$$

Since f was arbitrary, we have that $dF_p(au + bv) = a dF_p(u) + b dF_p(v)$.

For (b), let $v \in T_p M$ be a derivation and let $f \in C^\infty(P)$. Then

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = dF_p(v)(f \circ G) = dG_{F(p)} \circ dF_p(v)(f).$$

For (c), we have

$$d(\text{Id}_M)_p(v)(f) = v(f \circ \text{Id}_M) = vf.$$

For (d), let $F : M \to N$ be a diffeomorphism. Parts (b) and (c) imply $\text{Id}_{T_p M} = d(\text{Id}_M)_p = d(F^{-1} \circ F)_p = d(F^{-1})_{F(p)} \circ dF_p$ and $\text{Id}_{T_p N} = d(\text{Id}_N)_{F(p)} = d(F \circ F^{-1})_{F(p)} = dF_p \circ d(F^{-1})_{F(p)}$. This shows that dF_p is an isomorphism and that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Exercise 3.17. *Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where*

$$\tilde{x} = x, \quad \tilde{y} = y + x^3.$$

Let p be the point $(1,0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p,$$

even though the coordinate functions x and $\tilde{x}$ are identically equal.

The map $\phi : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x, y + x^3)$ is smooth with smooth inverse $\phi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x, y - x^3)$. Therefore, $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$. Now,

$$\left.\frac{\partial}{\partial x}\right|_p = \left.\frac{\partial \tilde{x}}{\partial x}\right|_p \frac{\partial}{\partial \tilde{x}} + \left.\frac{\partial \tilde{y}}{\partial x}\right|_p \frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial \tilde{x}} + 3\frac{\partial}{\partial \tilde{y}} \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p.$$

Exercise 3.19. *Suppose M is a smooth manifold with boundary. Show that TM has a natural topology and smooth structure making it into a smooth manifold with boundary, such that if $(U, (x^i))$ is any smooth boundary chart for M, then rearranging the coordinates in the natural chart $(\pi^{-1}(U), (x^i, v^i))$ for TM yields a boundary chart $(\pi^{-1}(U), (v^i, x^i))$.*

We begin by defining the topology on TM, which I believe was not done explicitly in Lee. Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be a smooth atlas for M. For every $(U, \phi) \in \mathcal{A}$, we have that

$$\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n, (p, v^i \, \partial/\partial x^i|_p) \mapsto (\phi(p), v^i e_i)$$

is a bijection. Hence, we obtain a topology on $\pi^{-1}(U)$ by pulling back the topology on $\phi(U) \times \mathbb{R}^n$. We then define a subset of TM to be open if its intersection with each $\pi^{-1}(U)$ is open; it is not hard to check that this defines a topology on TM. We define the coordinate charts as in the proof of Proposition 3.18, and the proof that they define a smooth structure on TM is the same.

It remains to show that if (U, ϕ) is a boundary chart, then $(\pi^{-1}(U), \tilde{\phi})$ is a boundary chart (after some rearranging of coordinates). Simply note that $\phi(U) \subseteq \mathbb{H}^n$, so $\tilde{\phi}(\phi^{-1}(U)) = \phi(U) \times \mathbb{R}^n \subseteq \mathbb{H}^n \times \mathbb{R}^n \cong \mathbb{H}^{2n}$. Moreover, $\tilde{\phi}(p, v) \in \partial\mathbb{H}^{2n}$ for any $p \in U$ such that $\phi(p) \in \partial\mathbb{H}^n$. Hence, $(\pi^{-1}(U), \tilde{\phi})$ is a boundary chart. The statement of the exercise asks us to rearrange coordinates so that the last coordinate of the chart intersects the boundary, but this is not so important.

Exercise 3.27. *Show that any (covariant or contravariant) functor from C to D takes isomorphisms in C to isomorphisms in D.*

Let $\mathcal{F}$ be a covariant functor from C to D, and let $f \in \hom_\mathsf{C}(X, Y)$ be an isomorphism. Then $\mathrm{Id}_{\mathcal{F}(X)} = \mathcal{F}(f^{-1} \circ f) = \mathcal{F}(f^{-1}) \circ \mathcal{F}(f)$ and $\mathrm{Id}_{F(Y)} = \mathcal{F}(f \circ f^{-1}) = \mathcal{F}(f) \circ \mathcal{F}(f^{-1})$. Hence, $\mathcal{F}(f)$ is an isomorphism, with inverse $\mathcal{F}(f^{-1})$. The proof is similar in the case where $\mathcal{F}$ is a contravariant functor.

Problem 3.1. *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.*

($\Rightarrow$) Let dF_p be the zero map for each $p \in M$. Let $q \in N$ be in the image of F. We will show that $F^{-1}(q)$ is both open and closed. Since $\{q\} \subseteq N$ is closed and F is smooth, $F^{-1}(q)$ is closed. On the other hand, let $p_0 \in M$ be such that $F(p_0) = q$, and let (B, ϕ) be a coordinate ball centred at p_0. If p_1 is any other point in B, then there is a straight-line smooth path $\gamma : [0,1] \to \phi(B)$ such that $\gamma(0) = \phi(p_0)$ and $\gamma(1) = \phi(p_1)$. Then $\phi^{-1} \circ \gamma$ is a smooth path from p_0 to p_1 and $F \circ \phi^{-1} \circ \gamma$ is a smooth path from $F(p_0)$ to $F(p_1)$. By Proposition 3.24, $(F \circ \phi^{-1} \circ \gamma)'(t) = dF((\phi^{-1} \circ \gamma)'(t)) = 0$ for every $t \in [0,1]$.

We will now show that a smooth curve with velocity identically equal to 0 is a constant path. Let P be a smooth manifold, and let $\gamma : J \to N$ be a path with velocity identically equal to zero (where $J \subseteq \mathbb{R}$ is some interval). Let (V, ψ) be a chart containing $\gamma(t)$ for some t. Then $\psi \circ \gamma$ is a curve in $\psi(V) \subseteq \mathbb{R}^n$, and $(F \circ \gamma)'(t) = dF(\gamma'(t)) = dF(0) = 0$. A smooth curve of velocity 0 in $\mathbb{R}^n$ is constant, and therefore γ is constant.

Hence, the curve $F \circ \phi^{-1} \circ \gamma$ is constant, so $F(p_0) = F(p_1) = q$. We conclude that $B \subseteq F^{-1}(q)$, so $F^{-1}(q)$ is open. Therefore, $F^{-1}(q)$ is either a component or a union of components. We conclude that F is constant on each component of M.

($\Leftarrow$) If F is constant, then since dF_p only depends on F in a neighbourhood of p, and F is constant on the component of p, we have that dF_p is the zero map for each p.

Problem 3.2. *Prove Proposition 3.14.*

For each $i = 1, \ldots, k$, let $\beta_i : M_i \to M_1 \times \cdots \times M_k, x \mapsto (p_1, \ldots, p_{i-1}, x, p_{i+1}, \ldots, p_k)$, and let

$$\beta : T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k \to T_p(M_1 \times \cdots \times M_k), \ v_1 \oplus \cdots \oplus v_k \mapsto d(\beta_1)_{p_1}(v_1) + \cdots + d(\beta_k)_{p_k}(v_k).$$

It is straightforward to check that β is a linear map. Let α be the map defined in Proposition 3.14. Note that $\pi_i \circ \beta_i$ is the identity on M_i if $i = j$, and is the constant map at p_j otherwise. It follows from this observation and part (b) of Proposition 3.6 that $\alpha \circ \beta$ is the identity on $T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k$. Then α is surjective. Since $T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k$ and $T_p(M_1 \times \cdots \times M_k)$ have the same dimension (namely $\sum_i \dim M_i$), and α is a surjection, α is an isomorphism.

If one of the M_i is a smooth manifold with boundary, then $M_1 \times \cdots \times M_k$ is a smooth manifold with boundary. Hence it has a well defined tangent space at each point, and differentials of maps are defined. The same proof then applies in this case.

Problem 3.3. *Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.*

By the preceding problem, the tangent space $T_{(p,q)}(M \times N)$ can be identified with $T_p M \oplus T_q N$ for every pair $(p, q) \in M \times N$. Thus, we can define a map

$$F : T(M \times N) \to TM \times TN, \ ((p, q), u \oplus v) \mapsto ((p, u), (q, v)).$$

We claim that F is a diffeomorphism. The inverse of F is given by $F^{-1}((p, u), (q, v)) = ((p, q), u \oplus v)$. Let (U, ϕ) and (V, ψ) be charts for M and N, respectively. Let $\pi_M, \pi_N, \pi_{M \times N}$ be projections from $TM, TN, T(M \times N)$ to $M, N, M \times N$, respectively. We have a chart $(\pi_{M \times N}^{-1}(U \times V), \alpha)$ for $T(M \times N)$, where

$$\alpha : \pi_{M \times N}^{-1}(U \times V) \to \phi(U) \times \psi(V) \times \mathbb{R}^m \times \mathbb{R}^n, \ \left((p, q), u^i \frac{\partial}{\partial x^i} \Big|_{(p,q)} \oplus v^j \frac{\partial}{\partial y^j} \Big|_{(p,q)} \right) \mapsto (\phi(p), \psi(q), u, v).$$

We also have a corresponding chart $(\pi_M^{-1}(U) \times \pi_N^{-1}(V), \beta)$ for $TM \times TN$, where

$$\beta : \pi_M^{-1}(U) \times \pi_N^{-1}(V) \to \phi(U) \times \mathbb{R}^m \times \psi(V) \times \mathbb{R}^n, \ \left(\left(p, u^i \frac{\partial}{\partial x^i} \Big|_p \right), \left(q, v^j \frac{\partial}{\partial y^j} \Big|_q \right) \right) \mapsto (\phi(p), u, \psi(q), v).$$

In both maps, $u = (u^1, \ldots, u^m)$ and $v = (v^1, \ldots, v^n)$. The coordinate representation of F is $\beta \circ F \circ \alpha^{-1}(x, y, u, v) = (x, u, y, v)$ and the coordinate representation of F^{-1} is $\alpha \circ F \circ \beta^{-1}(x, u, y, v) = (x, y, u, v)$. Both of these maps are smooth, so F is a diffeomorphism.

Problem 3.4. *Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.*

View $\mathbb{S}^1 \subseteq \mathbb{C}$. We have two angle charts (U, θ) and (V, ϕ) covering $\mathbb{S}^1$, where $U = \mathbb{S}^1 \setminus \{1\}$, $\theta : U \to (0, 2\pi)$, and $V = \mathbb{S}^1 \setminus \{-1\}$, $\phi : V \to (-\pi, \pi)$. Define $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ as follows: if $z \in U$, let $F(z, v \frac{d}{d\theta}|_z) = (z, v)$ and if $z \in V$, then let $F(z, v \frac{d}{d\phi}|_z) = (z, v)$. It is not hard to show that $\frac{d}{d\theta}|_z = \frac{d}{d\phi}|_z$ for $z \in U \cap V$; hence, F is well defined. Note that F restricted to $\pi^{-1}(U)$ or $\pi^{-1}(V)$ is a diffeomorphism from $\pi^{-1}(U)$ to $U \times \mathbb{R}$ and from $\pi^{-1}(V)$ to $V \times \mathbb{R}$, and it agrees on the overlap $\pi^{-1}(U) \cap \pi^{-1}(V)$. By the gluing lemma for smooth maps (Corollary 2.8), F is a diffeomorphism.

Problem 3.5. *Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centred at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no diffeomorphism with the same property.*

Let $T_1 \subseteq \mathbb{R}^2$ be the set $\{(x, y) : x \geq 0, |x| \geq |y|\}$. Let $r : \mathbb{R}^2 \to \mathbb{R}^2$ be the $\pi/2$ angle clockwise rotation about the origin, and let $T_2 = r(T_1)$, $T_3 = r^2(T_1)$, and $T_4 = r^3(T_1)$. Note that $\mathbb{R}^2 = \bigcup_{i=1}^4 T_i$. We begin by defining a homeomorphism $F_1 : T_1 \to T_1$. For $(x, y) \in T_1$, let

$$F_1(x, y) = \begin{cases} \sqrt{1 + \frac{y^2}{x^2}} (x, y) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

For $(x, y) \neq (0, 0)$, we have $|F_1(x, y)| = \sqrt{1 + \frac{y^2}{x^2}}|(x, y)| \leq \sqrt{2}|(x, y)|$, so $F_1(x, y)$ tends to 0 as (x, y) tends to $(0, 0)$ in T_1. Thus, F_1 is continuous. The inverse of F_1 is given by

$$
F_1^{-1}(x, y) = \begin{cases} \frac{x}{\sqrt{x^2 + y^2}}(x, y) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
$$

as one can check. In T_1, we have $\left|F_1^{-1}(x, y)\right| \leq |(x, y)|$, so F_1^{-1} is continuous on T_1. Thus, F_1 is a homeomorphism. We similarly obtain a homeomorphism of T_i via the function $F_i = r^{i-1} \circ F_1 \circ r^{-(i-1)}$. Finally, define

$$
F : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto F_i(x, y) \quad \text{if } (x, y) \in T_i.
$$

This function is well defined, since the F_i agree on the overlaps of their domains. By the pasting lemma, $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism. We check that $F(\mathbb{S}^1) = K$. Assume that $(x, y) \in \mathbb{S}^1 \cap T_1$. Then

$$
F(x, y) = F_1(x, y) = \sqrt{1 + \frac{y^2}{x^2}}(x, y) = \frac{1}{x}\sqrt{x^2 + y^2}(x, y) = \frac{1}{x}(x, y) = \left(1, \frac{y}{x}\right) \in K \cap T_1.
$$

Then $F(\mathbb{S}^1) \subseteq K$, since $\mathbb{S}^1$ and K are both invariant under rotations by $\pi/2$. Now let $(x, y) \in K \cap T_1$. Then

$$
F_1^{-1}(x, y) = F_1^{-1}(1, y) = \frac{1}{\sqrt{1 + y^2}}(1, y),
$$

so $\left|F_1^{-1}(x, y)\right| = 1$, meaning $F_1^{-1}(x, y) \in \mathbb{S}^1 \cap T_1$. Thus, $F(\mathbb{S}^1) = K$.

Suppose that there is a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$. Let $\gamma(t) : \mathbb{R} \to \mathbb{R}^2, t \mapsto (\cos t, \sin t)$. Note that $\gamma(t) \in \mathbb{S}^1$ and $\gamma'(t) \neq 0$ for all $t \in \mathbb{R}$. Let $t_c \in \mathbb{R}$ such that $F \circ \gamma(t_c) = (1, 1)$. Then, for some $\epsilon > 0$, there are intervals $I_- = (t_c - \epsilon, t_c)$ and $I_+ = (t_c, t_c + \epsilon)$ such that $F \circ \gamma(I_-) \subseteq \{1\} \times (-1, 1)$ and $F \circ \gamma(I_+) \subseteq (-1, 1) \times \{1\}$ (assuming that F sends γ counter-clockwise around K). Then

$$
F \circ \gamma(t) = \begin{cases} (1, y \circ F \circ \gamma(t)) & \text{if } t \in I_- \\ (x \circ F \circ \gamma(t), 1) & \text{if } t \in I_+ \end{cases}
$$

and

$$
(F \circ \gamma)'(t_c) = \frac{d(x \circ F \circ \gamma)}{dt}(t_c) \left.\frac{\partial}{\partial x}\right|_{(1,1)} + \frac{d(y \circ F \circ \gamma)}{dt}(t_c) \left.\frac{\partial}{\partial y}\right|_{(1,1)}.
$$

By continuity of $\frac{d(x \circ F \circ \gamma)}{dt}$ and $\frac{d(y \circ F \circ \gamma)}{dt}$, they must both vanish at t_c. Hence, $(F \circ \gamma)'(t_c) = 0$. But Proposition 3.24 implies $(F \circ \gamma)'(t_c) = dF(\gamma'(t_c))$. Since F is a homeomorphism, dF_p is an isomorphism for every $p \in \mathbb{R}^2$. In particular, $dF(\gamma'(t_c)) = dF_{\gamma(t_c)}(\gamma'(t_c)) \neq 0$, since $\gamma'(t_c) \neq 0$. Thus, F cannot be a diffeomorphism.

Problem 3.6. *Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = (e^{it}z^1, e^{it}z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.*

Letting $z^k = x^k + iy^k$ for $k = 1, 2$, we have that

$$
\gamma_z(t) = (x^1 \cos t - y^1 \sin t, x^1 \sin t + y^1 \cos t, x^2 \cos t - y^2 \sin t, x^2 \sin t + y^2 \cos t).
$$

Let (U_1^+, ϕ_1^+) be the coordinate chart for $\mathbb{S}^3$ described in Example 1.4. The coordinate representation of γ_z with respect to this chart is

$$
\phi_1^+ \circ \gamma_z : \gamma_z^{-1}(U_1^+) \to \mathbb{R}^3, \ t \mapsto (x^1 \sin t + y^1 \cos t, x^2 \cos t - y^2 \sin t, x^2 \sin t + y^2 \cos t).
$$

Since γ_z is continuous, $\gamma_z^{-1}(U_1^+)$ is open, and therefore $\phi_1^+ \circ \gamma_z$ is smooth. A similar computation shows that the coordinate representation of γ_z is smooth with respect to every chart $(U_i^\pm, \phi_i^\pm)$. The velocity of γ_z at some t such that $\gamma_z(t) \in U_1^+$ is

$$
\gamma_z'(t) = (x^1 \cos t - y^1 \sin t, -x^2 \sin t - y^2 \cos t, x^2 \cos t - y^2 \sin t),
$$

in the coordinates of U_1^+. Since the first coordinate of $\gamma_z(t)$ is $x^1 \cos t - y^1 \sin t$ and is positive for $\gamma_z(t)$ in U_1^+, $\gamma'(t)$ has positive first in U_1^+ and is therefore nonzero. A similar calculation shows that $\gamma'(t)$ is nonzero in the other charts $(U_i^\pm, \phi_i^\pm)$.

Problem 3.7. *Let M be a smooth manifold with or without boundary and p be a point of M. let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi : \mathcal{D}_pM \to T_pM$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an isomorphism.*

Let's begin by checking that Φv is indeed a derivation at p. It is clearly linear. Let $f, g \in C^\infty(M)$. Then

$$(\Phi v)(fg) \;=\; v[fg]_p \;=\; f(p)v[g]_p + g(p)v[f]_g \;=\; f(p)(\Phi v)g + g(p)(\Phi v)f,$$

so Φv is a derivation at p.

To check that Φ is an isomorphism, we first note that it is clearly linear. For injectivity, let $\Phi v = 0$. Let $[f]_p$ be the germ of some pair (f, U), where U is open an contains p. Let B' be a coordinate ball contained in U and centred at p, and let B be a coordinate ball centred at p such that $\overline{B} \subseteq B$. Moreover, by the extension lemma for smooth functions, there is a smooth function ψ supported in U such that $\psi \equiv 1$ on $\overline{U}$. Then $\tilde{f} = \psi f$ is a smooth function such that $[\tilde{f}]_p = [f]_p$. Then $v[f]_p = v[\tilde{f}]_p = (\Phi v)(\tilde{f}) = 0$. Since $[f]_p$ was arbitrary, we conclude that $v = 0$, and therefore that Φ is injective.

Finally, let $w \in T_pM$ be an arbitrary derivation. Define $v \in \mathcal{D}_pM$ by $v[f]_p = wf$. Since $wf = wg$ if $f = g$ on some neighbourhood of p (Proposition 3.8), v is well defined. Then $(\Phi v)f = v[f]_p = wf$ for any $f \in C^\infty M$, so $\Phi v = w$. This proves surjectivity, and therefore proves that Ψ is an isomorphism.

Problem 3.8. *Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function defined on a neighbourhood of p. Show that the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective.*

Let $[\gamma_1] = [\gamma_2]$ for two smooth curves starting at p. Let (U, ϕ) be a chart containing p. In the coordinates of this chart, we have

$$\gamma_1'(0) \;=\; (x^i \circ \gamma_1)'(0) \left.\frac{\partial}{\partial x^i}\right|_p \;=\; (x^i \circ \gamma_2)'(0) \left.\frac{\partial}{\partial x^i}\right|_p \;=\; \gamma_2'(0),$$

since $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighbourhood of p. Hence, Ψ is well defined.

For injectivity, let $\gamma_1'(0) = \gamma_2'(0)$, and let f be any smooth real-valued function defined on a neighbourhood of p. Then

$$(f \circ \gamma_1)'(0) \;=\; d(f \circ \gamma)_0\left(\left.\frac{d}{dt}\right|_0\right) \;=\; df_{\gamma(0)} \circ d\gamma_0\left(\left.\frac{d}{dt}\right|_0\right) \;=\; df_{\gamma(0)} \circ \gamma_1'(0) \;=\; df_{\gamma(0)} \circ \gamma_2'(0) \;=\; (f \circ \gamma_2)'(0),$$

so $[\gamma_1] = [\gamma_2]$, proving injectivity.

For surjectivity, let v be an arbitrary derivation at p, and let (U, ϕ) be some coordinate chart containing p such that $\phi(p) = 0$. Then we can write

$$v \;=\; v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

in the coordinates of (U, ϕ). For some $\epsilon > 0$, we can define a curve

$$\tilde{\gamma} : [0, \epsilon) \to U, \ t \mapsto (tv^1, \dots, tv^n),$$

and then let $\gamma = \phi^{-1} \circ \tilde{\gamma}$. Then γ is a smooth curve starting at p, since it is the composition of smooth functions. Then

$$\gamma'(0) \;=\; \frac{d\gamma^i}{dt}(0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(0)} \;=\; \frac{d(\phi^i \circ \gamma)}{dt}(0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(0)} \;=\; \frac{d(tv^1)}{dt}(0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(0)} \;=\; v^i \left.\frac{\partial}{\partial x^i}\right|_p \;=\; v.$$

4   Submersions, Immersions, and Embeddings

Exercise 4.3. *Verify the claims made in the preceding example.*

For (a), let $(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k)$ be a smooth chart for $M_1 \times \cdots \times M_k$. Then π_i has the coordinate representation

$$\phi_i \circ \pi_i \circ (\phi_1 \times \cdots \times \phi_k)^{-1}(v_1, \ldots, v_k) \;=\; v_i.$$

Let $\dim M_i = n_i$ and let $v_i = (x_i^1, \ldots, x_i^{n_1})$. Thus, in coordinates, $d(\pi_i)_p$ is the $(n_1 + \cdots + n_k) \times n_i$ matrix $(\partial x_i^j / \partial x_m^l)$, where j ranges from 1 to n_i, m ranges from 1 to k, and for each m, l ranges from 1 to n_m. This matrix is of full rank, since the only nonzero terms occur when $j = l$ and $i = m$ simultaneously. Hence, $d(\pi_i)_p$ is surjective and is therefore a smooth submersion.

For (b), suppose that the smooth curve γ is a smooth immersion. Then $d\gamma_{t_0}$ is injective for every $t_0 \in J$. Then $\gamma'(t_0) = d\gamma_{t_0}(d/dt|_{t_0}) \neq 0$. Conversely, suppose $\gamma'(t_0) \neq 0$ for every $t_0 \in J$. Suppose that $d\gamma_{t_0}(v) = 0$ for some $v \in T_{t_0}J$. Since $T_{t_0}J$ is spanned by $d/dt|_{t_0}$, we have that $v = \alpha \, d/dt|_{t_0}$ for some $\alpha \in \mathbb{R}$. Then $0 = d\gamma_{t_0}(\alpha \, d/dt|_{t_0}) = \alpha d\gamma_{t_0}(d/dt|_{t_0}) = \alpha \gamma'(t_0)$, implying $\alpha = 0$. Hence, $d\gamma_{t_0}$ is injective, and therefore γ is a smooth immersion.

For (c), let (U, ϕ) be an open chart on M, and let $(\pi^{-1}(U), \tilde{\phi})$ be the corresponding chart on on TM. Then the coordinate representation of π with respect to these charts is $\phi \circ \pi \circ \tilde{\phi}^{-1}(x, v) = x$. A similar computation as the one done in (a) shows that $d\pi|_p$ is a surjection for every $p \in TM$, and therefore π is a submersion.

For (d), $dX_{(u,v)}$ has the matrix representation

$$dX_{(u,v)} \;=\; \begin{pmatrix} -2\pi \sin 2\pi u \cos 2\pi v & -2\pi (2 + \cos 2\pi u) \sin 2\pi v \\ -2\pi \sin 2\pi u \sin 2\pi v & 2\pi (2 + \cos 2\pi u) \cos 2\pi v \\ 2\pi \cos 2\pi u & 0 \end{pmatrix}.$$

Note that at least one of the $(1,2)$ or $(2,2)$ entries is nonzero. If $2\pi \cos 2\pi u = 0$, then the bottom row is a row of zeros, and the determinant of the 2×2 matrix formed by the top two rows is $\pm 8\pi$. Hence, the columns are linearly independent, and thus $dX_{(u,v)}$ has full rank. Therefore, X is a smooth immersion.

Exercise 4.4. *Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersion is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.*

Let $F : M \to N$ and $G : N \to P$ be smooth [submersions/immersions]. Then for every $p \in M$, dF_p and $dG_{F(p)}$ are [surjective/injective]. Hence, $dG_{F(p)} \circ dF_p = d(G \circ F)_p$ is [surjective/injective]. Therefore, $G \circ F$ is a smooth [submersion/immersion].

For the counterexample, let $f : \mathbb{R} \to \mathbb{R}^2, x \mapsto (x, x^2)$ and $g : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto y$. Note that g is a a submersion and therefore has constant rank 1. Moreover, $df_x = (1 \; 2x)^\top$, so f has constant rank 1. However, $g \circ f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ has derivative $d(g \circ f)_x = 2x$, so $g \circ f$ has rank 1 for each $x \neq 0$ and rank 0 at $x = 0$.

Exercise 4.7. *Prove Proposition 4.6.*

For (a), let $F : M \to N$ and $G : N \to P$ be local diffeomorphisms and let $p \in M$. Let $U \subseteq M$ be an open set containing p such that $F|_U : U \to F(U)$ is a diffeomorphism. Let $V \subseteq N$ be an open set containing $F(p)$ such that $G|_V : V \to G(V)$ is a a diffeomorphism. Then $G \circ F|_{U \cap F^{-1}(V)} : U \cap F^{-1}(V) \to (G \circ F)(U) \cap G(V)$ is a diffeomorphism, where $U \cap F^{-1}(V) \subseteq M$ is an open set containing p. Hence, $G \circ F$ is a local diffeomorphism.

For (b), let $F_i : M_i \to N_i$ be a local homeomorphism for $i = 1, \ldots, k$, and consider the product

$$F_1 \times \cdots \times F_k : M_1 \times \cdots \times M_k \to N_1 \times \cdots \times N_k.$$

Let $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$ and let U_i be open an open set containing p_i such that $F_i|_{U_i} : U_i \to F_i(U_i)$ is a diffeomorphism. By Exercise 2.16,

$$F_1|_{U_1} \times \cdots \times F_k|_{U_k} : U_1 \times \cdots \times U_k \to F_1(U_1) \times \cdots \times F_k(U_k)$$

is a diffeomorphism. Since this is the same map as

$$(F_1 \times \cdots \times F_k)|_{U_1 \times \cdots \times U_k} : U_1 \times \cdots \times U_k \to (F_1 \times \cdots \times F_k)(U_1 \times \cdots \times U_k),$$

we have that $F_1 \times \cdots \times F_k$ is a local diffeomorphism.

For (c), let $F : M \to N$ be a local diffeomorphism. Since every point $p \in M$ has an open neighbourhood U such that $F|_U : U \to F(U)$ is a diffeomorphism. Since diffeomorphisms are homeomorphisms, F is a local homeomorphism. Let $U \in M$ be an arbitrary open set. Let $p' \in F(U)$ and let $p \in U$ be such that $F(p) = p'$. Since U is open, we can choose an open set V such that $p \in V \subseteq U$ and $F|_V : V \to F(V)$ is a diffeomorphism. Then $F(V) \subseteq F(U)$ is open and contains p'. Thus, $F(U)$ is open and F is an open map.

For (d), let $F : M \to N$ be a local diffeomorphism and let $U \subseteq M$ be an open submanifold. Let $p \in U$ be arbitrary, and let $V \subseteq M$ be an open set such that $F|_V : V \to F(V)$ is a diffeomorphism. By Exercise 2.16, $(F|_V)|_{U \cap V} = F|_{(U \cap V)} : U \cap V \to F(U \cap V)$ is a diffeomorphism. Hence, $F|_U$ is a local diffeomorphism.

For (e), let $F : M \to N$ be a diffeomorphism and let $p \in M$. Note that M itself is an open set containing p and $F|_M = F$ is a diffeomorphism. Therefore, F is a local diffeomorphism.

For (f), let $F : M \to N$ be a bijective local diffeomorphism. To see that F is smooth, note that every point of M is contained in a chart (U, ϕ) such that $F|_U$ is a diffeomorphism. Then the coordinate representation of F in the chart (U, ϕ) is smooth, so F is smooth. To see that F^{-1} is also smooth, let $F(p) \in N$ and let $U \subseteq M$ be an open set containing p such that $F|_U$ is a diffeomorphism. Then $F^{-1}|_{F(U)} : F(U) \to U$ is also a diffeomorphism. Let (V, ψ) be a smooth chart containing $F(p)$ such that $V \subseteq F(U)$. Then $F^{-1}|_V$ is a diffeomorphism onto its image, and therefore F^{-1} has a smooth coordinate representation in (V, ψ). Therefore, F^{-1} is smooth and F is a diffeomorphism.

For (g), let $F : M \to N$ be a local diffeomorphism. Let $p \in M$ and let U be an open set containing p such that $F|_U : U \to F(U)$ is a diffeomorphism. We can then find a chart (V, ϕ) such that $V \subseteq U$. Then the coordinate representation of F in (V, ϕ) is a diffeomorphism, and therefore a local diffeomorphism by part (e). Conversely, suppose that every point $p \in M$ is contained in a chart (U, ϕ) such that $F(p)$ is in some chart (V, ψ) and $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U \cap F^{-1}(V)))$ is a local diffeomorphism. Since F is smooth (coordinate representations are only defined for smooth maps), $\phi(U \cap F^{-1}(V))$ is open in $\mathbb{R}^n$ (or $\mathbb{H}^n$). Since local diffeomorphisms are open maps, $\psi(F(U \cap F^{-1}(V)))$ is open in $\mathbb{R}^n$ (or $\mathbb{H}^n$). Since the restriction of a local diffeomorphism to an open submanifold is a local diffeomorphism, $(\psi \circ F \circ \phi^{-1}) \circ \phi|_{U \cap F^{-1}(V)} = \psi \circ F|_{U \cap F^{-1}(V)}$. Similarly, $\psi^{-1}|_{\psi(F(U \cap F^{-1}(V)))}$ is a local diffeomorphism, and therefore $F|_{U \cap F^{-1}(V)} : U \cap F^{-1}(V) \to F(U \cap F^{-1}(V))$ is a local diffeomorphism. Then there is an open set $W \subseteq U \cap F^{-1}(V)$ containing p such that $F|_W : W \to F(W)$ is a diffeomorphism. Hence, F is a local diffeomorphism.

Exercise 4.9. *Show that the conclusions of Proposition 4.8 still hold if N is allowed to be a smooth manifold with boundary, but not if M is.*

To show that the result holds when N has nonempty boundary, we just need to show that we can apply the inverse function theorem for manifolds when F is a smooth immersion and a smooth submersion. To do this, it suffices to show that $F(M) \subseteq \mathrm{Int} N$. This is the result of 4.2.

The inclusion map $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ is both a smooth immersion and a smooth submersion (it's differential is given by the identity). On the other hand, it is not a local diffeomorphism: if $U \subseteq \mathbb{H}^n$ is an open neighbourhood of 0, then $F(U)$ is not open in $\mathbb{R}^n$. In particular, $F(U)$ contains no open ball centred at 0.

Exercise 4.10. *Suppose M, N, P are smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Prove the following:*

(a) *If $G : P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.*

(b) *If in addition F is surjective and $G : N \to P$ is any map, then G is smooth if and only if $G \circ F$ is smooth.*

For (a), let G be smooth. Since local diffeomorphisms are smooth, $F \circ G$ is smooth. Conversely, suppose that $F \circ G$ is smooth. Let $p \in P$, and let $U \subseteq M$ be an open set containing $g(p)$ such that $F|_U : U \to F(U)$ is a diffeomorphism. Since G is continuous, $G^{-1}(U)$ is open, and therefore $(F \circ G)|_{G^{-1}(U)} : G^{-1}(U) \to F(U)$ is smooth. Then $(F|_U)^{-1} \circ (F \circ G)|_{G^{-1}(U)} = G|_{G^{-1}(U)} : G^{-1}(U) \to U$ is smooth. Thus, G is smooth.

For (b), let G be smooth. Since local diffeomorphisms are smooth, $G \circ F$ is smooth. Conversely, suppose that $G \circ F$ is smooth. Let $q \in N$ be arbitrary, and let $F(p) = q$ (using the fact that F is surjective). Let $U \subseteq M$ be an open neighbourhood op p such that $F|_U : U \to F(U)$ is a diffeomorphism. Then $G \circ F|_U$ is smooth, and so is $(G \circ F|_U) \circ (F|_U)^{-1} = G|_{F(U)} : F(U) \to N$. Thus, G is smooth.

Exercise 4.17. *Show that every composition of smooth embeddings is a smooth embedding.*

The composition of smooth immersions is a smooth immersion (Exercise 4.4). It remains to show that a composition of topological embeddings is a topological embedding. Let $F : M \to N$ and $G : N \to P$ be topological embeddings. Then $F : M \to F(M)$ and $G : N \to G(N)$ are homeomorphisms. Then $G|_{F(M)} : F(M) \to G \circ F(M)$ is a homeomorphism, so $G \circ F : M \to G \circ F(M)$ is a homeomorphism. Therefore $G \circ F$ is a topological embedding.

**Exercise 4.24.** *Give an example of a smooth embedding that is neither an open map nor a closed map.*

Let $F : \mathbb{R} \to \mathbb{R}^2$ be given by $F(t) = (e^t, 0)$. Then F is a continuous bijection onto $(0, \infty) \times \{0\}$ and $F^{-1}(x, 0) = \log x$ for $x > 0$. The inverse is continuous, so we have that F is a topological embedding. Next, $dF_t = (e^t \; 0)$ is of full rank for all t, so F is a smooth immersion and therefore is a smooth embedding. Note that $F(\mathbb{R}) = (0, \infty) \times \{0\}$ is neither open nor closed in $\mathbb{R}^2$ despite the fact that $\mathbb{R}$ is both open and closed in $\mathbb{R}$. Hence, F is neither an open nor a closed map.

Exercise 4.27. *Give an example of a smooth map that is a topological submersion but not a smooth submersion.*

Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(x) = x^3$. Then F is smooth and bijective with continuous inverse $F^{-1}(x) = x^{1/3}$. Therefore F is a topological submersion. However, $dF_0 = 0$, so F is not a smooth submersion.

Exercise 4.32. *Prove Theorem 4.31.*

By Theorem 4.30, there are smooth maps $F : N_1 \to N_2$ and $G : N_2 \to N_1$ such that $F \circ \pi_2 = \pi_1$ and $G \circ \pi_1 = \pi_2$. Then $G \circ F \circ \pi_2 = \pi_2$. Since π_2 is surjective, $G \circ F = \mathrm{Id}$. A similar argument shows that $F \circ G = \mathrm{Id}$. Hence, $G = F^{-1}$ and F is a diffeomorphism.
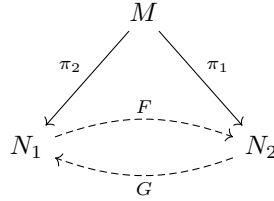
Exercise 4.34. *Prove Proposition 4.33.*

For (a), let $\pi : E \to M$ be a smooth covering map. Let $p \in M$ and let $q \in E$ such that $\pi(q) = p$. There is a neighbourhood U of p such that $V \subseteq E$ is the component of $\pi^{-1}(U)$ containing q. Then $\pi|_V : V \to U$ is a diffeomorphism, meaning π is a local diffeomorphism. By Proposition 4.8, π is also a smooth submersion (the fact that E and M can have nonempty boundary is not a problem for this direction of the proof of Proposition 4.8). Since local diffeomorphisms are open maps, π is an open map. Surjective open maps are quotient maps, so π is a quotient map.

For (b), if π is injective, then it is bijective. Since bijective local diffeomorphisms are diffeomorphisms, π is a diffeomorphism.

For (c), let $\pi : E \to M$ be a topological covering map that is also a smooth covering map. By part (a), π is also a local diffeomorphism. Conversely, let $\pi : E \to M$ be a topological covering map that is also a local diffeomorphism. Let $p \in M$ and let U be an evenly covered neighbourhood of p, and let $\pi^{-1}(U) = \bigcup_{\alpha \in A} U_\alpha$, where $\pi|_{U_\alpha}$ is a homeomorphism onto its image. Then $\pi|_{U_\alpha}$ is a bijective local diffeomorphism, and therefore a diffeomorphism.

Exercise 4.37. *Suppose $\pi : E \to M$ is a smooth covering map. Show that every local section of π is smooth.*

Let $U \subseteq M$ be open, and let $\sigma : U \to E$ be a local section. Note that every $p \in U$ is contained in an evenly covered open neighbourhood $U_p \subseteq U$. By Proposition 4.36, there is a unique smooth local section $\tau : U_p \to E$ of π such that $\tau(p) = \sigma(p)$. Then $\pi \circ \sigma|_{U_p} = \pi \circ \tau$, and since π is bijective on $\pi^{-1}(U_p)$, we have that $\sigma|_{U_p} = \tau$. Therefore, σ is smooth.

Exercise 4.38. *Suppose $E_1, \ldots, E_k$ and $M_1, \ldots, M_k$ are smooth manifolds (without boundary), and $\pi_i : E_i \to M_i$ is a smooth covering map for each $i = 1, \ldots, k$. Show that $\pi_1 \times \cdots \times \pi_k : E_1 \times \cdots \times E_k \to M_1 \times \cdots \times M_k$ is a smooth covering map.*

We already have that $\pi_1 \times \cdots \times \pi_k$ is smooth and surjective. Let $p_i \in M_i$ be an arbitrary point, and let U_i be an evenly covered neighbourhood of p_i for each $i = 1, \ldots, k$. Let U_{i,α_i} be a component of $\pi^{-1}(U_i)$ for each $i = 1, \ldots, k$. Since the product of diffeomorphisms is a diffeomorphism, we have that

$$(\pi_1)|_{U_{1,\alpha_1}} \times \cdots \times (\pi_k)|_{U_{k,\alpha_k}} : U_{1,\alpha_1} \times \cdots \times U_{k,\alpha_k} \to U_1 \times \cdots \times U_k$$

is a diffeomorphism. Since the connected components of $(\pi_1 \times \cdots \times \pi_k)^{-1}(U_1 \times \cdots \times U_k)$ are all of the form $U_{1,\alpha_1} \times \cdots \times U_{k,\alpha_k}$, this shows that $\pi_1 \times \cdots \times \pi_k$ is a smooth covering map.

Exercise 4.39. *Suppose that $\pi : E \to M$ is a smooth covering map. Since π is also a topological covering map, there is a potential ambiguity about what it means for a subset $U \subseteq M$ to be evenly covered: does π map the components of $\pi^{-1}(U)$ diffeomorphically onto U, or merely homeomorphically? Show that the two concepts are equivalent: if $U \subseteq M$ is evenly covered in the topological sense, then π maps each component of $\pi^{-1}(U)$ diffeomorphically onto U.*

Let V be a component of $\pi^{-1}(U)$. Note that every $p \in U$ has an open neighbourhood $U_p \subseteq U$ such that $\pi|_{V \cap \pi^{-1}(U_p)} : V \cap \pi^{-1}(U_p) \to U_p$ is a diffeomorphism. Hence, $\pi|_V$ is a bijective local diffeomorphism, and therefore a diffeomorphism. Hence, π maps the components of $\pi^{-1}(U)$ diffeomorphically onto U.

Exercise 4.42. *Prove Proposition 4.41.*

The facts that E is second-countable and Hausdorff are proven in exactly the same way as in the proof of Proposition 4.40. To show that E is a topological n-manifold with boundary, let $p \in E$ and first suppose that $\pi(p) \in \partial M$. Then there is an evenly covered open neighbourhood U of $\pi(p)$ such that (U, ϕ) is also a boundary chart for M. Letting V be the component of $\pi^{-1}(U)$ containing p, we find that $(V, \phi \circ \pi|_V)$ is a boundary chart containing p. We can similarly construct an interior boundary chart of any point $p \in E$ mapping to $\mathrm{Int}\, M$. Therefore, E is a topological n-manifold with boundary.

Let $p \in \partial E$ and let $U \subseteq M$ be an evenly covered open neighbourhood of $\pi(p)$, such that (U, ϕ) is a chart with domain U. As described above, we obtain a chart $(V, \phi \circ \pi|_V)$ containing p. Since p is a boundary point, $\phi(\pi(p)) \in \partial \mathbb{H}^n$, which implies that $\pi(p)$ is a boundary point of M. Then $p \in \pi^{-1}(\pi(p)) \subseteq \pi^{-1}(\partial M)$. Conversely, suppose that $p \in \pi^{-1}(\partial M)$. Let U be an evenly covered open neighbourhood of $\pi(p)$ that is also the domain of a chart (U, ϕ). Then $\phi(\pi(p)) \in \partial \mathbb{H}^n$, which implies that p is a boundary point of the chart $(V, \phi \circ \pi|_V)$, and therefore of E. Thus, we have proven $\partial E = \pi^{-1}(\partial M)$.

To prove the existence of a unique smooth structure on E making π into a smooth covering map, we need to show that the charts $(V, \phi \circ \pi|_V)$ constructed above are smoothly compatible. This is done in the same way as in the proof of Proposition 4.40. The proof of uniqueness will be given in Problem 4.9.

Exercise 4.44. *Prove Corollary 4.43.*

We know that any connected and locally simply connected space admits a unique universal cover. Since topological manifolds admit bases of coordinate balls, they are locally simply connected. Thus, if M is a connected (smooth) manifold, it admits a topological covering map $\pi : \widetilde{M} \to M$ where $\widetilde{M}$ is simply connected. By Proposition 4.40, $\widetilde{M}$ has a unique smooth structure making π into a smooth covering map. This proves the existence of a smooth covering manifold of M.

For uniqueness, suppose that $\widehat{\pi} : \widehat{M} \to M$ is another smooth covering of M by a simply connected space $\widehat{M}$. By uniqueness of topological universal coverings, there is a homeomorphism ϕ making the following diagram commute

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\phi} & \widehat{M} \\ {\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle \widehat{\pi}} \\ & M & \end{array} \ .$$

Every point of M has a neighbourhood that is evenly covered by π and $\widehat{\pi}$ (by intersecting two possibly different evenly covered sets). Let $V \subseteq M$ be a component of $\pi^{-1}(U)$. Then $\phi(V)$ is a component of $\widehat{\pi}^{-1}(U)$, and we have the following commutative diagram

$$\begin{array}{ccc} & U & \\ {}^{(\pi|_V)^{-1}}\swarrow & & \searrow {}^{(\widehat{\pi}|_{\phi(V)})^{-1}} \\ V & \xrightarrow{\ \phi|_V\ } & \phi(V) \end{array}\ .$$

By Theorem 4.29, since $(\pi|_V)^{-1}$ is a surjective smooth submersion (it is a diffeomorphism) $\phi|_V \circ (\pi|_V)^{-1}$ is smooth if and only if $\phi|_V$ is smooth. Since $\phi|_V \circ (\pi|_V)^{-1} = (\widehat{\pi}|_{\phi(V)})^{-1}$ is smooth, we have that $\phi|_V$ is smooth. Since M is covered by sets of the form V, ϕ is smooth. A similar argument shows that ϕ^{-1} is smooth, so ϕ is a diffeomorphism, which proves uniqueness of the universal covering manifold of M.

Exercise 4.45. *Generalize the preceding corollary to smooth manifolds with boundary.*

By Proposition 4.41, every smooth manifold with boundary M admits a universal covering $\pi : \widetilde{M} \to M$ where $\widetilde{M}$ is a smooth manifold with boundary.

We will prove uniqueness in a simpler way than we did in Exercise 4.44 that does not require the use of Theorem 4.29 (which is not stated for manifolds with boundary). Again, if there is another universal covering $\widehat{\pi} : \widehat{M} \to M$, then there is a homeomorphism ϕ making

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \phi\ } & \widehat{M} \\ {}_{\pi}\searrow & & \swarrow_{\widehat{\pi}} \\ & M & \end{array}\ .$$

commute. Let $U \subseteq M$ be an evenly covered neighbourhood, and let V be a component of $\pi^{-1}(U)$. Then $\phi(V)$ is a component of $\widehat{\pi}^{-1}(U)$, and we have that $\widehat{\pi}|_{\phi(V)} \circ \phi|_V = \pi|_V$. Since $\widehat{\pi}|_{\phi(V)}$ and $\pi|_V$ are diffeomorphisms, we have that $\phi|_V$ is a diffeomorphism. Then ϕ is a bijective local diffeomorphism, and therefore is a diffeomorphism. This concludes the proof of uniqueness.

Problem 4.1. *Use the inclusion map $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to show that Theorem 4.5 does not extend to the case in which M is a manifold with boundary.*

Denote the inclusion by $i : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$. Then i is smooth and $di_{(0,0)} = \mathrm{Id}$ is invertible. Suppose there are connected open neighbourhoods $U_0 \subseteq \mathbb{H}^n$ of 0 and $V_0 \subseteq \mathbb{R}^n$ of 0 such that $i|_{U_0} : U_0 \to V_0$ is a diffeomorphism. But U_0 is a manifold with nonempty boundary, while V_0 has empty boundary. This shows that Theorem 4.5 (the inverse function theorem for manifolds) does not hold when the domain is a manifold with boundary.

Problem 4.2. *Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F : M \to N$ is smooth. Show that if $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in \mathrm{Int}\,N$.*

Suppose there is a $p \in M$ such that $F(p) \in \partial N$. Let (U, ϕ) be a chart containing p and let (V, ψ) be a boundary chart containing $F(p)$. Suppose that the coordinates of U are x^i for $i = 1, \ldots, m$ and those of V are $y^i(x^1, \ldots, x^m)$ for $i = 1, \ldots, n$. Then $y^n(x^1, \ldots, x^n)$ has a local minimum at $(x^1, \ldots, x^m) = \phi(p)$. Therefore, $\partial y^n / \partial x^i|_{\phi(p)} = 0$ for each $i = 1, \ldots, m$, which implies that the bottom row of the matrix representation of dF_p is a row of zeros. Hence, dF_p is singular, a contradiction.

Problem 4.3. *Formulate and prove a version of the rank theorem for a map of constant rank whose domain is a smooth manifold with boundary.*

Formulation of the "domain manifold with boundary" version of the rank theorem: let M be a smooth manifold with boundary and let N be a smooth manifold of dimensions m and n, respectively, and let $F : M \to N$ be a smooth map with constant rank r. For each $p \in M$ there exist smooth charts (U, ϕ) for M centred at p and (V, ψ) for N centred at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form

$$\widehat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \ = \ (x^1, \ldots, x^r, 0, \ldots, 0).$$

We now begin the proof of this version of the rank theorem. After choosing smooth coordinates, we can replace M and N by open subsets $U \subseteq \mathbb{H}^m$ and $V \subseteq \mathbb{R}^n$. Moreover, we may assume that U intersects $\partial \mathbb{H}^m$,

otherwise the proof is the same as the one given in the text. By translating, we may assume that $p = 0$ and $F(p) = 0$. Since F is smooth, it has an extension $\widetilde{F} : \widetilde{U} \to V$, where $U \subseteq \widetilde{U}$ and $\widetilde{U}$ is open in $\mathbb{R}^m$. We will now show that there is a projection $\pi : \mathbb{R}^n \to \mathbb{R}^r$ such that $\pi \circ \widetilde{F}$ is a submersion. Since F has constant rank r, the matrix $(\partial \widetilde{F}^i / \partial x^j)$ has an $r \times r$ submatrix with nonzero determinant at $(x^1, \ldots, x^m) = 0$ (assume it is the upper-left submatrix by reordering coordinates). Therefore, by shrinking $\widetilde{U}$ if necessary, we may assume that this submatrix has non-vanishing determinant on all of $\widetilde{U}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^r$ be the projection onto the first r coordinates; then $\pi \circ \widetilde{F}$ is a submersion.

By the usual rank theorem, for each $q \in \widetilde{U}$, there are charts (U_0, ϕ_0) for $\widetilde{U}$ centred at q and (W, α) for $\pi(V)$ centred at $\pi \circ \widetilde{F}(q)$ such that $\pi \circ \widetilde{F}(U_0) \subseteq W$ and

$$\alpha \circ \pi \circ \widetilde{F} \circ \phi_0^{-1}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \; = \; (x^1, \ldots, x^r).$$

Note that $\alpha \circ \pi \circ \widetilde{F} \circ \phi_0^{-1} = \pi \circ (\alpha \times \mathrm{id}_{\mathbb{R}^{n-r}}) \circ \widetilde{F} \circ \phi_0^{-1}$, so

$$(\alpha \times \mathrm{id}_{\mathbb{R}^{n-r}}) \circ \widetilde{F} \circ \phi_0^{-1}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \; = \; (x^1, \ldots, x^r, R(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m))$$

for some smooth function R defined on $\phi_0(U_0)$. By shrinking U_0 if necessary, we may assume that $\phi_0(U_0)$ is an open cube. By the same argument given in the proof of the usual rank theorem, $R|_{\phi_0(U_0 \cap U)}$ is independent of $x^{r+1}, \ldots, x^m$ (this uses that F has constant rank r).

Define the open set $V_0 = \{(y^1, \ldots, y^r, y^{r+1}, \ldots, y^n) \subseteq (W \times \mathbb{R}^{n-r}) \cap V : (y^1, \ldots, y^r, 0, \ldots, 0) \subseteq \phi_0(U_0)\}$. Moreover, define $\psi_0 : V_0 \to \mathbb{R}^n$ by $\psi_0(v, w) = (v, w - R(v, 0))$, where $v = (y^1, \ldots, y^r)$ and $w = (y^{r+1}, \ldots, y^n)$. Note that ψ_0 is a diffeomorphism onto its image since it has inverse given by $\psi_0^{-1}(v, w) = (v, w + R(v, 0))$. Therefore, (V_0, ψ_0) is a chart for V centred at 0. Note that on $\phi_0(U_0 \cap U)$ we have

$$\psi_0 \circ (\alpha \times \mathrm{id}_{\mathbb{R}^{n-r}}) \circ F \circ \phi_0^{-1}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \; = \; (x^1, \ldots, x^r, 0, \ldots, 0)$$

since R is independent of $x^{r+1}, \ldots, x^m$ there.

What remains is to show how (U_0, ϕ_0) can be made into a boundary chart for U. Simple restriction will not work here and an idea similar to the one used in the proof of Theorem 4.15 will not work, since we do not have that F is an immersion. Unfortunately, I don't know how to do this step.

Problem 4.4. *Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.*

We claim that for every $t \in \mathbb{R}$, the set $\{e^{2\pi i \alpha(t+n)} : n \in \mathbb{Z}\}$ is dense in $\mathbb{S}^1$. Since multiplication by $e^{2\pi i \alpha t}$ defines a homeomorphism of $\mathbb{S}^1$, it is enough to show that $\{e^{2\pi i \alpha n} : n \in \mathbb{Z}\}$ is dense in $\mathbb{S}^1$. It is then enough to show that for every $N \in \mathbb{N}$ and every $k = 0, \ldots, N-1$, there is an $n \in \mathbb{Z}$ such that $\lfloor \alpha n \rfloor \in (\frac{k}{N}, \frac{k+1}{N})$, or equivalently, that there exists an $m \in \mathbb{Z}$ such that $\alpha n - m \in (\frac{k}{N}, \frac{k+1}{N})$. By Dirichlet's approximation theorem, there are $m', n' \in \mathbb{Z}$ such that $|\alpha n' - m'| < \frac{1}{N}$. If $\alpha n' - m' \in (-\frac{1}{N}, 0)$, replace m' and n' by $-m'$ and $-n'$, respectively, so that $\alpha n' - m' \in (0, \frac{1}{N})$. Note that we are using the fact that α is irrational here to justify our use of open intervals. Then, for some $l \in \mathbb{N}$, we have that $l(\alpha n' - m') = \alpha(ln') - (lm') \in (\frac{k}{N}, \frac{k+1}{N})$. Choose $m = lm'$ and $n = ln'$.

Now let $U \subseteq \mathbb{S}^1 \times S^1$ be open. We may then assume that U is of the form $U_1 \times U_2$, where $U_i \subseteq \mathbb{S}^1$ is open. Choose $t \in \mathbb{R}$ such that $e^{2\pi i t} \in U$. By our previous work, there is an $n \in \mathbb{Z}$ such that $e^{2\pi i \alpha(t+n)} \in U_2$. Hence,

$$\gamma(t+n) \; = \; (e^{2\pi i(t+n)}, e^{2\pi i \alpha(t+n)}) \; = \; (e^{2\pi i t}, e^{2\pi i \alpha(t+n)}) \in U_1 \times U_2.$$

Hence, $\gamma(\mathbb{R})$ intersects every open set in $\mathbb{T}^2$, implying that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

Problem 4.5. *Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space, as define in 1.9.*

(a) *Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.*

(b) *Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.*

For (a), let $\widetilde{U}_k = \{(z^1, \ldots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\} : z^k \neq 0\}$ and let $U_k = \pi(\widetilde{U}_k)$. Let $(z^1, \ldots, z^{n+1}) \in \widetilde{U}_k$. Then, with respect to the charts $(\widetilde{U}_k, \mathrm{id})$ and (U_k, ϕ_k) (see Problem 1.9), the coordinate representation of π is

$$\phi_k \circ \pi(z^1, \ldots, z^{n+1}) \; = \; \left(\frac{z^1}{z^k}, \ldots, \frac{z^{k-1}}{z^k}, \frac{z^{k+1}}{z^k}, \ldots, \frac{z^{n+1}}{z^k} \right),$$

which is smooth since $z^k \neq 0$. Hence, π is smooth and surjective. It remains to show that π is a submersion. Letting $z^j = x^j + iy^j$, we have that

$$\frac{z^j}{z^k} \;=\; \frac{x^j x^k + y^j y^k}{(x^k)^2 + (y^k)^2} + i\frac{x^k y^j - x^j y^k}{(x^k)^2 + (y^k)^2}.$$

In the coordinates of U_i, one can then check that $d\pi$ has a $2n \times 2n$ submatrix of the form

$$A \;=\; \begin{pmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \end{pmatrix},$$

where

$$D \;=\; \begin{pmatrix} \frac{x^k}{(x^k)^2+(y^k)^2} & \frac{-y^k}{(x^k)^2+(y^k)^2} \\ \frac{y^k}{(x^k)^2+(y^k)^2} & \frac{x^k}{(x^k)^2+(y^k)^2} \end{pmatrix}.$$

Then $\det A = 1$, so $d\pi$ is of full rank, and therefore π is a submersion.

For (b), we define a diffeomorphism $F : \mathbb{S}^2 \to \mathbb{CP}^1$ as follows:

$$F(x,y,z) \;=\; \begin{cases} [1, \frac{x}{1-z} + i\frac{y}{1-z}] & \text{if } (x,y,z) \in \mathbb{S}^2 \setminus \{N\} \\ [\frac{x}{1+z} - i\frac{y}{1+z}, 1] & \text{if } (x,y,z) \in \mathbb{S}^2 \setminus \{S\}. \end{cases}$$

We check that F is well defined: if $(x,y,z) \in \mathbb{S}^2 \setminus \{N, S\}$, then

$$\left[1, \frac{x}{1-z} + i\frac{y}{1-z}\right] \;=\; \left[\left(\frac{x}{1+z} - i\frac{y}{1+z}\right)\left(1, \frac{x}{1-z} + i\frac{y}{1-z}\right)\right] \;=\; \left[\frac{x}{1+z} - i\frac{y}{1+z}, 1\right].$$

Next, note that $F|_{\mathbb{S}^2 \setminus \{N\}} = \phi_2^{-1} \circ i \circ \sigma$, where ϕ_2 is one of the standard coordinate maps defined on $\mathbb{CP}^1$, i is the identification of $\mathbb{R}^2$ with $\mathbb{C}$, and σ is the stereographic projection from the north. Since each of these are diffeomorphisms (i is by definition essentially), we have that $F|_{\mathbb{S}^2 \setminus \{N\}}$ is a diffeomorphism onto its image. Similarly, $F|_{\mathbb{S}^2 \setminus \{S\}} = \phi_1^{-1} \circ \tau \circ i \circ \tilde{\sigma}$, where τ is complex conjugation and $\tilde{\sigma}$ is the stereographic projection from the south. Thus, $F|_{\mathbb{S}^2 \setminus \{S\}}$ is also a diffeomorphism onto its image. Since both restrictions are defined on open sets and they agree on the overlap, we have that F is a diffeomorphism onto its image. It remains to show that F is surjective. This is straightforward, since the image of $F|_{\mathbb{S}^2 \setminus \{N\}}$ is U_1, the image of $F|_{\mathbb{S}^2 \setminus \{S\}}$ is U_2, and $\mathbb{CP}^1 = U_1 \cup U_2$. Hence, $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^1$.

Problem 4.6. *Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.*

By Proposition 4.28, if F is a smooth submersion then it is an open map. Since M is open in itself, $F(M)$ is open. Since M is compact and F is continuous, $F(M)$ is compact. Since open sets in $\mathbb{R}^k$ are not compact, this is a contradiction. Hence, there is no smooth submersion from M to $\mathbb{R}^k$. Note that this result can easily be generalized: if M is a compact topological space, then there is no open map $F : M \to \mathbb{R}^k$ for $k > 0$.

Problem 4.7. *Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map $F : \widetilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\widetilde{N}$.*

Let $\tilde{\pi} : M \to \widetilde{N}$ be the same smooth surjective submersion as π, but viewed as a map into $\widetilde{N}$. Note that $\pi : M \to N$ is equal to $\mathrm{Id}_N^{-1} \circ \tilde{\pi}$ and is smooth. Therefore Id_N^{-1} is smooth. Note that $\mathrm{Id}_{\widetilde{N}} : \widetilde{N} \to \widetilde{N}$ is smooth, and therefore that $\mathrm{Id}_{\widetilde{N}} \circ \tilde{\pi} = \tilde{\pi}$ is smooth. Then Id_N is smooth, since $\mathrm{Id}_N \circ \pi = \tilde{\pi}$ is smooth. This shows that Id_N is a diffeomorphism.

Problem 4.8. *This problem shows that the converse of Theorem 4.29 is false. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.*

Since $\pi(x, 1) = x$ for all $x \in \mathbb{R}$, we have that π is surjective. Moreover, π is smooth since it is a polynomial. Suppose that $F : \mathbb{R} \to P$ is smooth; then $F \circ \pi$ is smooth by composition. Conversely, suppose that $F \circ \pi$ is smooth. Since $f : R \to \mathbb{R}^2, x \mapsto (x, 1)$ is smooth, we have that $F \circ \pi \circ f = F$ is smooth by composition. To see that π is not a submersion, note that

$$d\pi_{(x,y)} \;=\; \begin{pmatrix} y \\ x \end{pmatrix},$$

which has rank 0 at $(0, 0)$.

Problem 4.9. *Let M be a connected smooth manifold, and $\pi : E \to M$ be a topological covering map. Complete the proof of Proposition 4.40 by showing that there is only one smooth structure on E such that π is a smooth covering map.*

We will prove this for manifolds with boundary as well. We already know that E has a smooth structure making $\pi : E \to M$ a smooth covering map. Suppose that $\widetilde{E}$ is the same topological manifold as E, but with a possibly different smooth structure making $\tilde{\pi} : \widetilde{E} \to M$ into a smooth covering map, where $\tilde{\pi}$ is the same map as π, just viewed as a map whose domain is $\widetilde{E}$. We will prove that the identity map $\mathrm{Id} : E \to \widetilde{E}$ is a diffeomorphism; it suffices to show that Id is a local diffeomorphism since it is obviously bijective. Every point $p \in E$ is in the preimage of some evenly covered $V \subseteq M$. Let U be the component of $\pi^{-1}(U)$ containing p. Since there is no distinction between being smoothly evenly covered and topologically evenly covered (Exercise 4.39), $\mathrm{Id}(U)$ is also a component of $\tilde{\pi}^{-1}(V)$, and there is a smooth section $\sigma : V \to \mathrm{Id}(U)$ of $\tilde{\pi}$. Thus, $\mathrm{Id}|_U = \sigma \circ \pi|_U$ is a diffeomorphism, so Id is a local diffeomorphism, proving our claim.

Problem 4.10. *Show that the map $q : \mathbb{S}^n \to \mathbb{RP}^n$ defined in Example 2.13(f) is a smooth covering map.*

As discussed in Example 2.13(f), we have that q is smooth. By our solution to Exercise 1.7, we have that q is surjective as well. Consider the usual smooth charts (U_i, ϕ_i) for $\mathbb{RP}^n$ and the smooth charts $(V_i^{\pm}, \psi_i^{\pm})$ for $\mathbb{S}^n$ (see Example 1.4) for $i = 1, \ldots, n+1$. It is not too hard to see that $q^{-1}(U_i) = V_i^+ \sqcup V_i^-$. Hence, if we can show that $q|_{V_i^{\pm}} : V_i^{\pm} \to U_i$ is a diffeomorphism, we will be done. In coordinates, we have

$$\phi_i \circ q|_{V_i^{\pm}} \circ (\psi_i^{\pm})^{-1}(x^1, \ldots, x^n) \;=\; \pm \frac{(x^1, \ldots, x^n)}{\sqrt{1 - |x|^2}}.$$

Then $\phi_i \circ q|_{V_i^{\pm}} \circ (\psi_i^{\pm})^{-1} : \mathbb{B}^n \to \mathbb{R}^n$ is a smooth map with smooth inverse $f : \mathbb{R}^n \to \mathbb{B}^n$ given by

$$f(x^1, \ldots, x^n) \;=\; \frac{(x^1, \ldots, x^n)}{\sqrt{|x|^2 + 1}}.$$

Hence, $\phi_i \circ q|_{V_i^{\pm}} \circ (\psi_i^{\pm})^{-1}$ is a diffeomorphism, which means that $q|_{V_i^{\pm}}$ is a diffeomorphism.

Problem 4.11. *Show that a topological covering map is proper if and only if its fibres are finite, and therefore the converse of Proposition 4.46 is false.*

Let $\pi : E \to M$ be a topological covering map. Suppose that π is proper. Then $\{p\}$ is compact for every $p \in M$, which implies that $\pi^{-1}(p)$ is compact in E. Let U be an evenly covered neighbourhood of p. Then each component of $\pi^{-1}(U)$ contains exactly one point of $\pi^{-1}(p)$ since each component maps homeomorphically onto U. This collection of components then forms an open cover for $\pi^{-1}(p)$, and must therefore be finite by compactness. Thus, $\pi^{-1}(p)$ is finite.

Conversely, suppose that $\pi^{-1}(p)$ is finite for each $p \in M$. Let $K \subseteq M$ be compact and let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of $\pi^{-1}(K)$. Let $p \in K$ be arbitrary, and let $\pi^{-1}(p) = \{q_1^p, \ldots, q_{k_p}^p\}$. By the fact that π is an open map (topological covering maps are open) and $\pi^{-1}(p)$ is finite, we can choose open neighbourhoods V_i^p

of q_i^p such that $V_i^p \subseteq U_\alpha$ for each $i = 1, \ldots, k_p$ and for some $\alpha \in A$, and such that $\pi|_{V_i^p}$ is a homeomorphism onto its image $W^p = \pi(V_i^p)$. By compactness of K, let $\{W^{p_1}, \ldots, W^{p_n}\}$ be an open covering of K. We claim that collection of components of all the $\pi^{-1}(W^{p_i})$ forms a finite open covering of $\pi^{-1}(K)$. To see this, let $q \in \pi^{-1}(K)$, so that $\pi(q) \in K$. Then $\pi(q) \in W^{p_j}$ for some $j \in \{1, \ldots, n\}$. Then q is in one of the components of $\pi^{-1}(W^{p_j})$, as claimed. The fact that the covering is finite follows from the fact that the fibres are finite. For each component $V_i^{p_j}$ of W^{p_j}, choose a U_α that contains it. This finite subcollection of $\{U_\alpha\}_{\alpha \in A}$ is then a finite subcover of $\pi^{-1}(K)$. Hence, $\pi^{-1}(K)$ is compact, which proves that π is proper.

Since $\epsilon : \mathbb{R} \to \mathbb{S}^1$ defined in Example 2.13(b) is a smooth covering map with (countably) infinite fibres, it is not proper. Hence, the converse of Proposition 4.46 is false.

Problem 4.12. *Using the covering map $\epsilon^2 : \mathbb{R}^2 \to \mathbb{T}^2$ (see Example 4.35), show that the immersion $X : \mathbb{R}^2 \to \mathbb{R}^3$ defined in Example 4.2(d) descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\widetilde{X} : \mathbb{T}^2 \to \mathbb{R}^3$, and then show that $\widetilde{X}$ is a smooth embedding whose image is the given surface of revolution.*

It is straightforward to check that X is constant on the fibres of ϵ^2. Since ϵ^2 is a smooth submersion, Theorem 4.30 guarantees that there is a smooth map $\widetilde{X}$ making

$$
\begin{array}{ccc}
\mathbb{R}^2 & & \\
\epsilon^2 \downarrow & \searrow^{X} & \\
\mathbb{T}^2 & \xrightarrow[\widetilde{X}]{} & \mathbb{R}^3
\end{array}
$$

commute. Because ϵ^2 is actually a local diffeomorphism (Example 4.11) and X is a smooth immersion, it follows that $\widetilde{X}$ must be a smooth immersion. More specifically, at each $p \in \mathbb{R}^2$, we have $d\widetilde{X}_{\epsilon^2(p)} \circ d\epsilon_p^2 = dX_p$. Since $d\epsilon_p^2$ is invertible and dX_p is injective, it follows that $d\widetilde{X}_{\epsilon^2(p)}$ is injective for every p; surjectivity of ϵ^2 then implies that $\widetilde{X}$ is an immersion. We will now show that $\widetilde{X}$ is an injection. To this end, let $\widetilde{X}(e^{2\pi i s_1}, e^{2\pi i s_2}) = \widetilde{X}(e^{2\pi i t_1}, e^{2\pi i t_2})$. Then, by the definition of X,

$$
\begin{aligned}
(2 + \cos 2\pi s_1) \cos 2\pi s_2 &= (2 + \cos 2\pi t_1) \cos 2\pi t_2 \\
(2 + \cos 2\pi s_1) \sin 2\pi s_2 &= (2 + \cos 2\pi t_1) \sin 2\pi t_2 \\
\sin 2\pi s_1 &= \sin 2\pi t_1.
\end{aligned}
$$

Since $2 + \cos x$ is never zero, the first two equations imply that $\cos 2\pi s_2 = \cos 2\pi t_2$ and $\sin 2\pi s_2 = \sin 2\pi t_2$. Since $\cos x$ and $\sin x$ are never zero for the same x, at least one of the first two equations yields $2 + \cos 2\pi s_1 = 2 + \cos 2\pi t_1$, which gives $\cos 2\pi s_1 = \cos 2\pi t_1$. Putting this together with the third equation, we have that $(e^{2\pi i s_1}, e^{2\pi i s_2}) = (e^{2\pi i t_1}, e^{2\pi i t_2})$, which proves injectivity.

Since $\mathbb{T}^2$ is compact and $\widetilde{X}$ is an injective smooth immersion, Proposition 4.22 implies that $\widetilde{X} : \mathbb{T}^2 \to \mathbb{R}^3$ is an embedding.

Problem 4.13. *Define a map $F : \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Using the smooth covering map of Example 2.13(f) and Problem 4.10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.*

Since $F = G \circ \iota$, where $G : \mathbb{R}^3 \to \mathbb{R}^4$ is the map given by the same formula as F but extended to all of $\mathbb{R}^3$ and $\iota : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is the inclusion, we have that F is smooth. Note that $q : \mathbb{S}^2 \to \mathbb{RP}^2$ is a smooth submersion and that F is constant on the fibres $q^{-1}(x, y, x) = \{\pm(x, y, z)\}$. By Theorem 4.30, there is then a map $\widetilde{F}$ such that

$$
\begin{array}{ccc}
\mathbb{S}^2 & & \\
q \downarrow & \searrow^{F} & \\
\mathbb{RP}^2 & \xrightarrow[\widetilde{F}]{} & \mathbb{R}^4
\end{array}
$$

commutes. The commutativity of the diagram forces

$$
\widetilde{F}[x, y, z] = \frac{1}{x^2 + y^2 + z^2}(x^2 - y^2, xy, xz, yz).
$$

In the coordinates of the usual chart (U_1, ϕ_1), we have

$$\widetilde{F} \circ \phi_1^{-1}(u, v) = \widetilde{F}[1, u, v] = \frac{1}{1 + u^2 + v^2}(1 - u^2, u, v, uv),$$

and the matrix representation of $d\widetilde{F}_p$ for $p \in U_1$ is then

$$\frac{1}{(1 + u^2 + v^2)^2} \begin{pmatrix} -4u - 2uv^2 & 1 - u^2 + v^2 & -2uv & v - u^v + 3 \\ 2u^2v - 2v & -2uv & 1 + u^2 - v^2 & u + u^3 - uv^2 \end{pmatrix}.$$

The first minor is (up to a nonzero factor)

$$\begin{vmatrix} -4u - 2uv^2 & 1 - u^2 + v^2 \\ 2u^2v - 2v & -2uv \end{vmatrix} = 2v(1 + 2u^2 + v^2 + u^4 + u^2v^2),$$

which can only be zero if $v = 0$. But plugging $v = 0$ into the matrix representation of $d\widetilde{F}$ shows that the rows are linearly independent. Thus, $\widetilde{F}$ is an immersion on U_1. Similar computations show that $\widetilde{F}$ is an immersion on the other charts.

We now show that $\widetilde{F}$ is injective. To this end, let $\widetilde{F}[x, y, x] = \widetilde{F}[u, v, w]$. We may assume that (x, y, z) and (u, v, w) are normalized. Then

$$x^2 - y^2 = u^2 - v^2, \quad xy = uv, \quad xz = uw, \quad yz = vw.$$

Using these equations along with $x^2 + y^2 + z^2 = u^2 + v^2 + w^2 = 1$, it can be shown that $(x, y, z) = \pm(u, v, w)$, and therefore that $[x, y, z] = [u, v, w]$. Then $\widehat{F}$ is an injective smooth submersion with compact domain, and is therefore an embedding. Hence, $\widetilde{F} : \mathbb{RP}^2 \to \mathbb{R}^4$ is an embedding.