B2M17CADA - Project I

15th May 2023   Martin Šimák

Task *Analysis of a stripline using finite differences*
Determine the properties of a shielded coplanar stripline depicted in figure 1 designed on a suspended dielectric substrate ($h = 1$ mm, $\epsilon_r = 3.05$) using finite differences and the method of successive over-relaxation. Find the strip's width w and gap g between the strips so that the characteristic impedance is $Z_0 = 50\ \Omega$. The distance of the shielding walls from the strips should be larger than $20h$ to ensure that the line's impedance is not affected by finite dimensions.
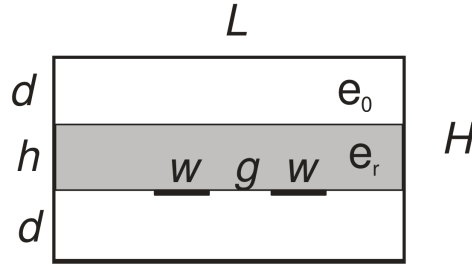

Figure 1: Task geometry

(a) Illustrate equipotencial countours and the distribution of electric field.

(b) Evaluate the capacity per unit length and the characteristic impedance Z_0.

(c) Plot the value of Z_0 as a function of number of iterations.

(d) Find the charge density on a strip.

(e) Determine numerically optimal value of the relaxation coefficient α. Compare with the analytical solution

$$\alpha = \frac{8 - \sqrt{64 - 16r^2}}{r^2}, \tag{1}$$

where $r = \cos(\pi/N_x) + \cos(\pi/N_y)$ and $N_x \times N_y$ are the dimensions of the discrete mesh.

1 Theoretical introduction

Generally speaking, finite-difference methods are a class of numerical analysis techniques used for solving differential equations. This is done via breaking down the computational domain (spatial or temporal in most cases) into a finite number of steps which allows for the replacement of differential operators, such as derivatives, with differences.

Methods based on finite differences are very useful for converting nonlinear differential equations into a system of linear equations which can be efficiently solved by matrix algebra techniques, e.g. Gauss-Seidel method and its variants. Since these linear algebra computations are efficiently implemented in modern computers, finite-difference methods are one of the most common approaches to the numerical solution of differential equations.

In our case, we use finite differences to approximate the differential operator from Gauss's law of electrostatics which can be expressed in the integral form as

$$\oint_S \epsilon \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = Q, \tag{2}$$

where S is a closed surface encompassing the concerned structure. However, in a plane, equation 2 simplifies to

$$\oint_C \epsilon \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{c} = q_l, \tag{3}$$

where C is a closed curve with linear charge density $q_l = Q/l$. Furthermore, assuming the problem can be considered electrostatic, $\boldsymbol{E}$ is an irrotational vector field, hence we can put $\boldsymbol{E} = -\operatorname{grad}\varphi$, simplifying equation 3 even further into the form of

$$-\oint_C \epsilon \operatorname{grad}\varphi \cdot \mathrm{d}\boldsymbol{c} = q_l,$$

$$\oint_C \epsilon \frac{\partial\varphi}{\partial n} \, \mathrm{d}c = -q_l. \tag{4}$$

The integrand from equation 4 represents the flux of electric field

$$\mathrm{d}\Psi = \epsilon \frac{\partial\varphi}{\partial n} \, \mathrm{d}c$$

through C enclosing a node in the mesh. This situation is depicted in figure 2 and can be expressed as

$$\Psi_1 = \epsilon_1 \frac{\varphi_1 - \varphi_0}{h} \frac{h}{2} + \epsilon_2 \frac{\varphi_1 - \varphi_0}{h} \frac{h}{2},$$

$$\Psi_2 = \epsilon_1 \frac{\varphi_2 - \varphi_0}{h} h,$$

$$\Psi_3 = \epsilon_1 \frac{\varphi_3 - \varphi_0}{h} \frac{h}{2} + \epsilon_2 \frac{\varphi_3 - \varphi_0}{h} \frac{h}{2},$$

$$\Psi_4 = \epsilon_2 \frac{\varphi_4 - \varphi_0}{h} h.$$

Figure 2: Modified Gauss's law around a node of potential φ_0

However, from Gauss's law, we know that $\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 = 0$ because there is no charge present. This yields

$$\frac{\epsilon_1 + \epsilon_2}{2}\varphi_0 = \frac{1}{4}\left(\frac{\epsilon_1 + \epsilon_2}{2}\varphi_1 + \epsilon_1\varphi_2 + \frac{\epsilon_1 + \epsilon_2}{2}\varphi_3 + \epsilon_2\varphi_4\right). \tag{5}$$
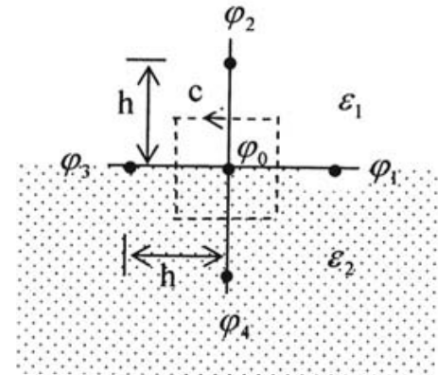
This final equation serves as the finite-difference approach to solving the electrostatic potential distribution in the *free nodes* of the configuration, i.e., nodes not in the electrodes or planar dielectric interfaces. In our environment, the permittivities are $\epsilon_1 = \epsilon_0$, $\epsilon_2 = \epsilon_r \epsilon_0$ and a constant voltage is connected to the strips. The rest of the procedure is successive over-relaxation.

Second part of the calculation yields charge density from which we can extract information about the capacity per unit length and the characteristic impedance.

Again, we utilize the equation 4 which now has a non-zero right-hand side because the curve encompasses the charged strips. The situation is similar to a one-electrode problem in figure 3.
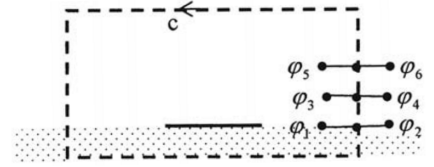

Figure 3: Modified Gauss's law around a charged electrode

Adapting this knowledge, we can write

$$\Psi = \epsilon_1 \frac{\varphi_2 - \varphi_1}{2h} \frac{h}{2} + \epsilon_2 \frac{\varphi_2 - \varphi_1}{2h} \frac{h}{2} + \epsilon_2 \frac{\varphi_4 - \varphi_3}{2h} h + \epsilon_2 \frac{\varphi_6 - \varphi_5}{2h} h + \cdots = q_l \tag{6}$$

which serves as the main computational formula for further results.

2   Implementation

The code below represents the core computational loop. The first step is to update the values of electric potential over the whole mesh using equation 5. Then, for a given iteration, the script computes charge density using equation 6. Results are saved for every iteration in order to track the evolution of impedance values. The loop breaks once the previous relaxation step hasn't improved the solution more than the threshold value 0.001.

```matlab
while Rmax > 1e-3
    iter = iter + 1;
    % Step 1: SOR method
    for j = 2:Nx-1
    for i = 2:Ny-1
        if i == IF1
            R(i,j) = (eps_r*V(i+1,j)+1*V(i-1,j)+er_avg*V(i,j-1)+ ...
                er_avg*V(i,j+1))/4/er_avg-V(i,j);
        elseif i == IF2
            R(i,j) = (1*V(i+1,j)+eps_r*V(i-1,j)+er_avg*V(i,j-1)+ ...
                er_avg*V(i,j+1))/4/er_avg-V(i,j);
        else
            R(i,j) = (V(i+1,j)+V(i-1,j)+V(i,j-1)+V(i,j+1))/4-V(i,j);
        end
            V(i,j) = V(i,j)+alpha*R(i,j);
        V(IF1-1,Nx/2-g/2-w/2:Nx/2-g/2+w/2) = V0;
        V(IF1-1,Nx/2+g/2-w/2:Nx/2+g/2+w/2) = V0;
    end
    end
    Rmax = max(max(R));
    % Step 2: numerical evaluation of the intergral
    qA = 0; qB = 0; qC = 0; qD = 0;
    for i=top:1:bottom
        if i < IF1
            qA = qA + eps_r*(V(i,left+1) - V(i,left-1))/2;
            qC = qC + eps_r*(V(i,right-1) - V(i,right+1))/2;
        elseif i == IF1
            qA = qA + er_avg*(V(i,left+1) - V(i,left-1))/2;
            qC = qC + er_avg*(V(i,right-1) - V(i,right+1))/2;
        else
            qA = qA + (V(i,left+1) - V(i,left-1))/2;
            qC = qC + (V(i,right-1) - V(i,right+1))/2;
        end
    end
    for j=left:1:right
            qB = qB + (V(bottom+1,j) - V(bottom-1,j))/2;
            qD = qD + eps_r*(V(top-1,j) - V(top+1,j))/2;
    end
    Q(iter) = eps_0*(qA + qB + qC + qD);
end
C = Q/(V0-Vgnd);
Z = 1./(c*C);
```

3   Results

Figure 4 shows the resultant electric field distribution, whereas in figure 5, we can inspect the evolution of characteristic impedance throughout the run.
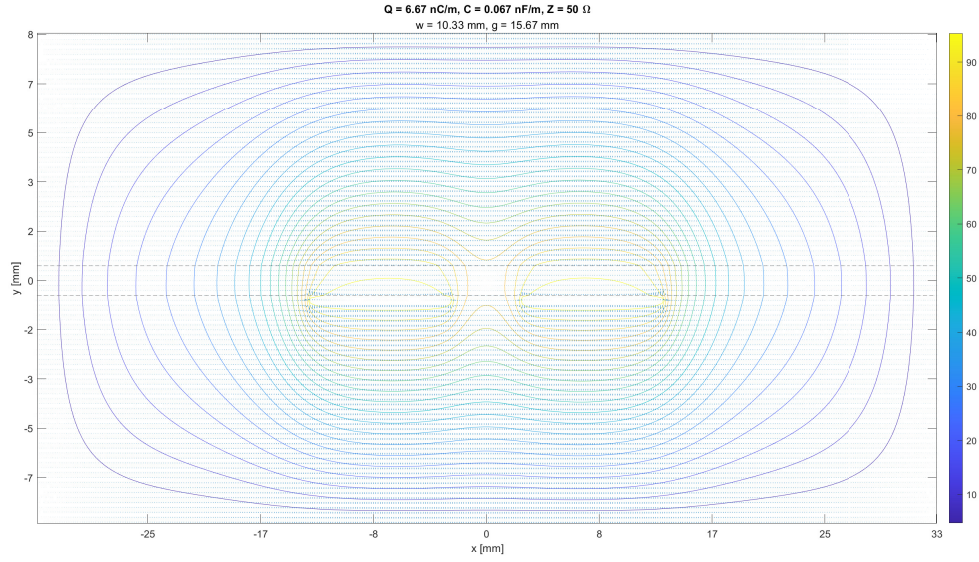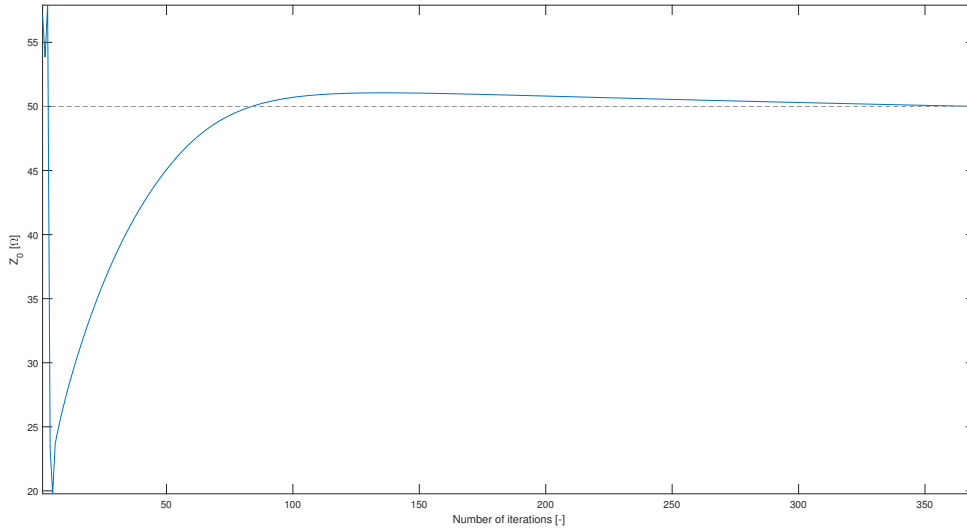

Figure 4: Output field

Figure 5: Evolution of the characteristic impedance over iterations

Numerical determination of the optimal value of α yielded the number 1.91 which is compliant with the analytical solution of $\alpha_0 = 1.96$.