$$e^j: \qquad e^j(e_i) = \delta_i^j.$$

$$\{e_1, \dots, e_n\} \to \{e'_1, \dots, e'_n\}, \qquad e'_i = A^j{}_i e_j.$$

$$v = v^m e_m = v'^m e'_m = v'^m A^n{}_m e_n = v'^b A^m{}_b e_m$$

$$\left(v^m - v'^b A^m{}_b\right) e_m = 0, \qquad v^m = v'^b A^m{}_b$$

$$v^m (A^{-1})^p{}_m = v'^b A^m{}_b (A^{-1})^p{}_m = v'^b \delta_b^p = v'^p$$

$$v'^p = (A^{-1})^p{}_m v^m = A_m{}^p v^m.$$

$$\alpha: v \to \alpha(v) \in \mathbb{R}$$

$$\{e_1, \dots, e_n\}, \{e'_1, \dots, e'_n\} \to \{e^1, \dots, e^n\}, \{e'^1, \dots, e'^n\}.$$

$$e^m(e_n) = \delta_n^m, \qquad e'^m(e'_n) = \delta_n^m$$

$$e'^m\left(A^d{}_n e_d\right) = A^d{}_n e'^m(e_d) = A^d{}_n C^m{}_f e^f(e_d) = C^m{}_d A^d{}_n = \delta_n^m, \qquad C = A^{-1}.$$

$$e'^m = (A^{-1})^m{}_n e^n.$$

$$\alpha(v) = \alpha_i e^i(v) = \alpha'_m e'^m(v) = \alpha'_m (A^{-1})^m{}_i e^i(v)$$

$$(\alpha_i - \alpha'_m (A^{-1})^m{}_i)\, e^i(v) = 0$$

$$\alpha_i A^i{}_n = \alpha'_m (A^{-1})^m{}_i A^i{}_n$$

$$\alpha'_n = A^i{}_n \alpha_i$$