Connections on Differentiable Manifolds

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Why is differential geometry important?

- It provides an elegant approach to both differential and integral calculus on general geometries.

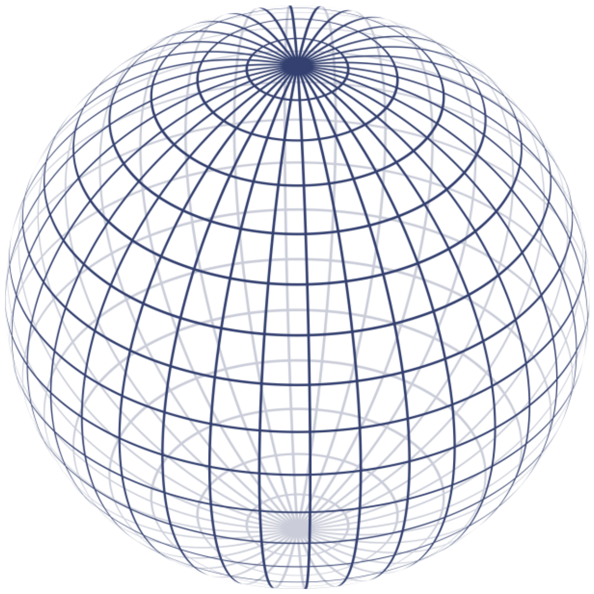
Aims of the thesis are to:

(a) explain the basic concepts of differential geometry;

(b) show equivalence of connections and covariant derivatives.

What is a smooth manifold?

Smooth manifold is a generalised geometrical object which retains the "nice properties" of Euclidean spaces, which we utilise in the definitions of differential calculus, locally.



Picture courtesy of user Geek3 at Wikipedia Commons

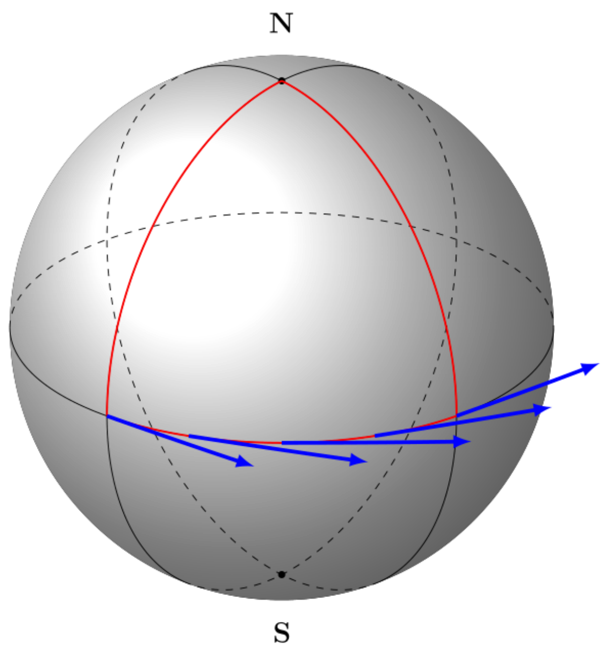
How to find out that a tangent vector stays the same?

1. The vector does not change its direction and magnitude.

This approach leads to some form of a derivative—Covariant derivative.

2. The vector stays the same in nearby places.

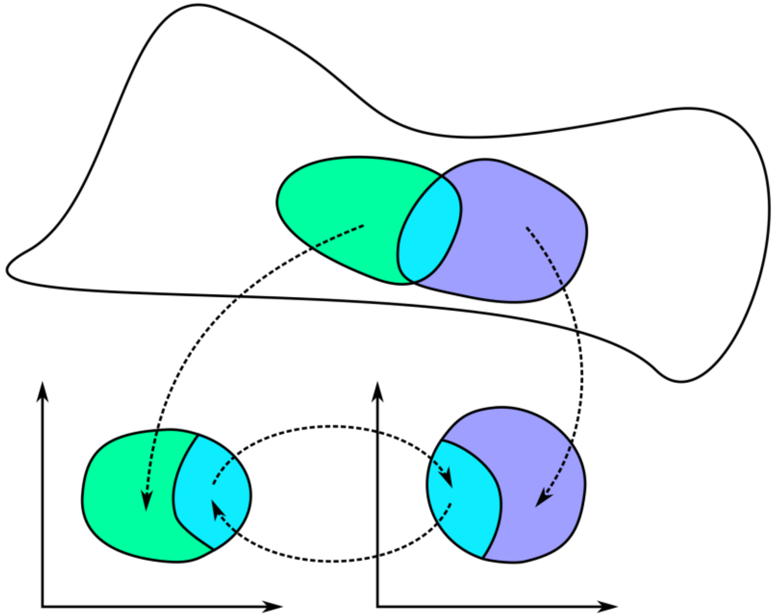
This approach leads to a connection of tangent spaces—Connection on tangent bundle.



The image has been created using TikZ macros by Till Tantau.

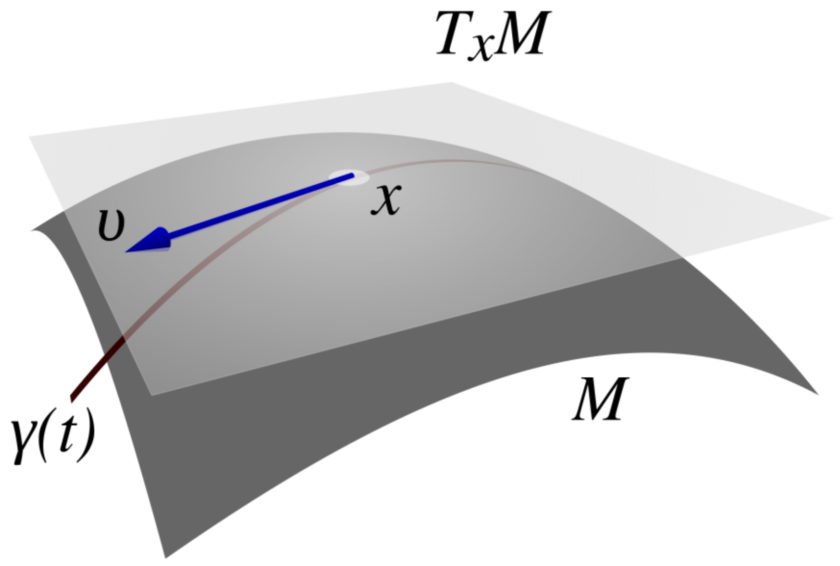
How to describe a smooth manifold?

Smooth manifolds are “isomorphic” to an n-dimensional Euclidean space on separate “coordinate patches”.



What is a tangent vector to a manifold?

- At each point of the smooth manifold, we define a linear space of tangent vectors.
- Rough idea: tangent vector is a derivative of a curve.

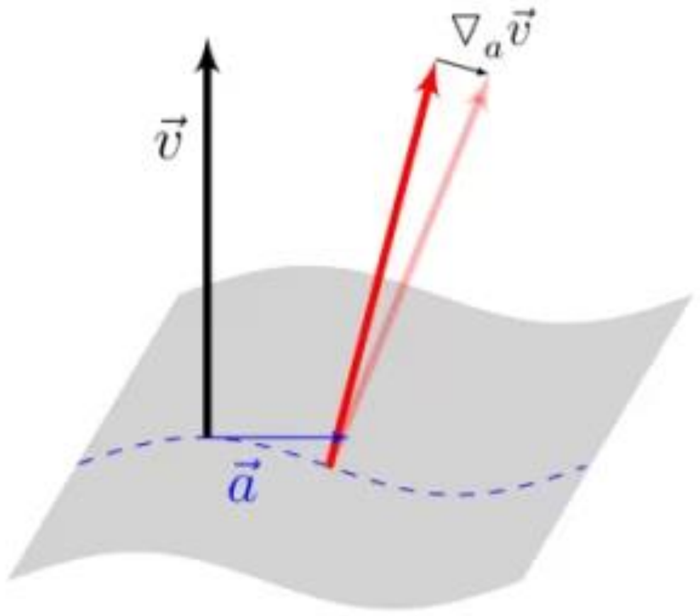


The image has been released into the public domain.

What is a covariant derivative?

$\nabla_{\vec{a}}\vec{v}$

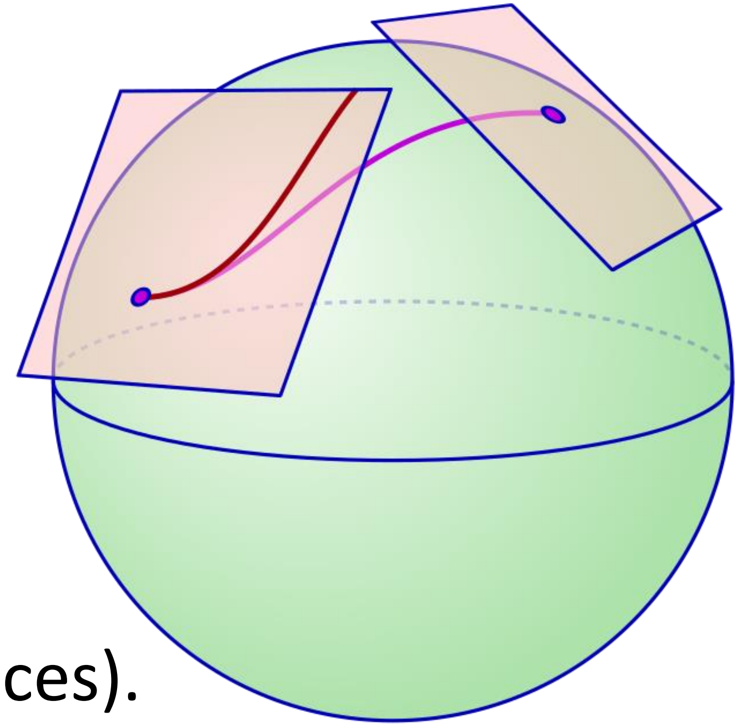
"Derivative of $\vec{v}$ in direction of $\vec{a}$",
satisfying certain properties.



Credit: Johann at https://www.naturelovesmath.com/en/mathematical-physics/

What is a connection?

Provides us with a linear isomorphism of “nearby” tangent spaces.
Formally, it is a map $K: TTM \to TM$, satisfying certain properties.



TM = tangent bundle (collection of tangent spaces).

Main result of the thesis

“Giving a covariant derivative on a manifold is equivalent to giving the means of parallel transport—connection form—on said manifold.”

Questions raised in opponent’s report

(1) Yes, definition of a smooth manifold on page 17 is wrong. I have overlooked the fact that the map φ is required to be an embedding into $\mathbb{R}^N$, i.e., it needs to be smooth, injective and its derivative needs to be injective too.

This way, we can conduct the usual parametrisation of a line.

(2) Indeed not; we do not need it. This confusion has arisen from the possibility of defining two distinct connection forms on manifolds. The connection form C is different from the connection form K and I should have emphasised that. In literature, the connection form C is often called the “horizontal connection form”.

Thank you for your attention