Fast Estimation of Main Intermodulation Products Using Volterra Series

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***Abstract*— The most accurate procedure for computing the intermodulation products is to find a steady-state period of the signal, and then to determine the spectrum by means of fast Fourier transform. However, this method needs a time-consuming numerical integration over many periods of the faster signal even for accelerated extrapolation methods for determining the steady state. In the paper, an efficient method for a fast estimation of the main intermodulation products is presented. The method uses higher-order Volterra series in a simple multistep algorithm, which is compatible with a typical structure of the frequency-domain part of software tools. The method is demonstrated by an illustrative example first, which clearly shows possible incorrect interpretation of the Volterra series. Afterwards, the efficiency of the algorithm is demonstrated by a fast estimation of the main intermodulation products of a low-voltage low-power CMOS RF four-quadrant multiplier.**

***Index Terms*— Steady-state algorithm, fast Fourier transform, numerical integration, Volterra series, four-quadrant multiplier.**

I. Introduction

A natural and accurate method for determining the intermodulation products is to find a steady-state period, and then to compute the spectrum of that signal by means of fast Fourier transform. This algorithm has been implemented into author's experimental software tool C.I.A. (Circuit Interactive Analyzer [1]) with an automatic determination of unknown period for autonomous circuits. Basic theory of the steady-state analysis is described in [2], some improvements of the classical methods (especially automating the procedure for the autonomous circuits) are defined in [3]. As the implemented method for a numerical integration (which is necessary for the steady-state algorithm) is accurate and very flexible (it is based on an efficient recurrent form of Newton interpolation polynomial [4]), computed intermodulation products are available even for higher orders.

However, the numerical integration must be performed over many periods of the faster signal and therefore the analysis is time-consuming. For this reason, another method for a fast estimation of the main interpolation products has also been implemented based on Volterra series. A brief introduction to using the Volterra series is shown in [5], a more comprehensive analysis can be found in [6].

A disadvantage of many of the implementations of the Volterra series consists in a creation of a new (relatively large and isolated) block of the program. In this paper, a simple modification of the method is defined, which is compatible with the frequency-domain part of the program—the algorithm is built into and uses a relatively large part of the AC analysis.

II. Description of the Algorithm

The system of nonlinear algebraic-differential equations of a circuit is generally defined in the implicit form

$$\boldsymbol{f}\left[\,\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t\,\right] = \boldsymbol{0}. \tag{1}$$

As the resulting formulae arisen from a usage of the Volterra series are very complicated, consider for a simplicity of the explanation that a circuit system comprises only two equations, i.e. (1) can be written in the simpler form

$$f_1\left(x_1, x_2, \dot{x}_1, \dot{x}_2, t\right) = 0, \quad f_2\left(x_1, x_2, \dot{x}_1, \dot{x}_2, t\right) = 0. \tag{2}$$

The Taylor expansion of the functions f_1 and f_2 with the inclusion of second-order terms in a linearization center $^{(0)}$ is the following (of course, higher-order terms are necessary for the higher-order intermodulation products):

$$\begin{aligned}
&f_{1,2}^{(0)} + \frac{\partial f_{1,2}}{\partial x_1}^{(0)} \Delta x_1 + \frac{\partial f_{1,2}}{\partial x_2}^{(0)} \Delta x_2 + \\
&\frac{\partial f_{1,2}}{\partial \dot{x}_1}^{(0)} \Delta \dot{x}_1 + \frac{\partial f_{1,2}}{\partial \dot{x}_2}^{(0)} \Delta \dot{x}_2 + \\
&\frac{\partial^2 f_{1,2}}{\partial x_1 \partial x_2}^{(0)} \Delta x_1 \Delta x_2 + \frac{\partial^2 f_{1,2}}{\partial x_1 \partial \dot{x}_1}^{(0)} \Delta x_1 \Delta \dot{x}_1 + \\
&\frac{\partial^2 f_{1,2}}{\partial x_1 \partial \dot{x}_2}^{(0)} \Delta x_1 \Delta \dot{x}_2 + \frac{\partial^2 f_{1,2}}{\partial x_2 \partial \dot{x}_1}^{(0)} \Delta x_2 \Delta \dot{x}_1 + \\
&\frac{\partial^2 f_{1,2}}{\partial x_2 \partial \dot{x}_2}^{(0)} \Delta x_2 \Delta \dot{x}_2 + \frac{\partial^2 f_{1,2}}{\partial \dot{x}_1 \partial \dot{x}_2}^{(0)} \Delta \dot{x}_1 \Delta \dot{x}_2 + \\
&\frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial x_1^2}^{(0)} \Delta x_1^2 + \frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial x_2^2}^{(0)} \Delta x_2^2 + \\
&\frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial \dot{x}_1^2}^{(0)} \Delta \dot{x}_1^2 + \frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial \dot{x}_2^2}^{(0)} \Delta \dot{x}_2^2.
\end{aligned} \tag{3}$$

A natural linearization center for this type of analysis is the operating point, i.e. the static version of (2)

$$f_1\left(x_{1_0}, x_{2_0}, 0, 0, 0\right) = 0, \quad f_2\left(x_{1_0}, x_{2_0}, 0, 0, 0\right) = 0$$

must be solved in advance.

The next step is the standard frequency analysis, i.e. solving the system of the two equations

$$\begin{aligned}
&F_1(\omega) + \frac{\partial f_1}{\partial x_1}^{(0)} \Delta X_1 + \frac{\partial f_1}{\partial x_2}^{(0)} \Delta X_2 + \\
&j\omega \frac{\partial f_1}{\partial \dot{x}_1}^{(0)} \Delta X_1 + j\omega \frac{\partial f_1}{\partial \dot{x}_2}^{(0)} \Delta X_2 = 0,
\end{aligned}$$

TABLE I

COMPARISON OF THE RESULTS OBTAINED USING FAST FOURIER TRANSFORM WITH THE APPROXIMATION BY VOLTERRA SERIES

m	$\lvert\Delta V_2'\rvert$ (FFT)	$\arg(\Delta V_2')$ (FFT)	$\lvert\Delta V_2'\rvert$ (Volterra)	$\arg(\Delta V_2')$ (Volterra)
0	0.63 mV	81°	0.59 mV	82°
$0.\overline{3}$	0.75 mV	75°	0.85 mV	73°
0.5	0.81 mV	73°	1 mV	70°

$$F_2(\omega) + \frac{\partial f_2}{\partial x_1}^{(0)} \Delta X_1 + \frac{\partial f_2}{\partial x_2}^{(0)} \Delta X_2 + j\omega \frac{\partial f_2}{\partial \dot{x}_1}^{(0)} \Delta X_1 + j\omega \frac{\partial f_2}{\partial \dot{x}_2}^{(0)} \Delta X_2 = 0,$$

which must be solved for the two frequencies ω_1 and ω_2. In this way, we obtain the first-order products $\Delta X_1(\omega_1)$, $\Delta X_1(\omega_2)$, $\Delta X_2(\omega_1)$, and $\Delta X_2(\omega_2)$. The terms $F_1(\omega)$ and $F_2(\omega)$ represent independent signal sources of the circuit.

The second-order intermodulation products can be estimated using the second-order terms in (3) as the signal sources of the circuit (instead of the independent ones), i.e. the system

$$\begin{aligned}
&\frac{\partial f_{1,2}}{\partial x_1}^{(0)} \Delta X_1' + \frac{\partial f_{1,2}}{\partial x_2}^{(0)} \Delta X_2' + \\
&j\omega \frac{\partial f_{1,2}}{\partial \dot{x}_1}^{(0)} \Delta X_1' + j\omega \frac{\partial f_{1,2}}{\partial \dot{x}_2}^{(0)} \Delta X_2' + \\
&\frac{\partial^2 f_{1,2}}{\partial x_1 \partial x_2}^{(0)} \Delta X_1 \Delta X_2 + j\omega \frac{\partial^2 f_{1,2}}{\partial x_1 \partial \dot{x}_1}^{(0)} \Delta X_1^2 + \\
&j\omega \frac{\partial^2 f_{1,2}}{\partial x_1 \partial \dot{x}_2}^{(0)} \Delta X_1 \Delta X_2 + j\omega \frac{\partial^2 f_{1,2}}{\partial x_2 \partial \dot{x}_1}^{(0)} \Delta X_2 \Delta X_1 + \\
&j\omega \frac{\partial^2 f_{1,2}}{\partial x_2 \partial \dot{x}_2}^{(0)} \Delta X_2^2 - \omega^2 \frac{\partial^2 f_{1,2}}{\partial \dot{x}_1 \partial \dot{x}_2}^{(0)} \Delta X_1 \Delta X_2 + \\
&\frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial x_1^2}^{(0)} \Delta X_1^2 + \frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial x_2^2}^{(0)} \Delta X_2^2 - \\
&\omega^2 \frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial \dot{x}_1^2}^{(0)} \Delta X_1^2 - \omega^2 \frac{1}{2} \frac{\partial^2 f_{1,2}}{\partial \dot{x}_2^2}^{(0)} \Delta X_2^2 = 0
\end{aligned}$$

must be solved for the four frequencies $\omega_1 + \omega_1$, $\omega_2 + \omega_2$, $\omega_1 + \omega_2$, and $\omega_1 - \omega_2$, which gives the (second-order) harmonic products $\Delta X_1'(\omega_1 + \omega_1)$, $\Delta X_1'(\omega_2 + \omega_2)$, $\Delta X_2'(\omega_1 + \omega_1)$, $\Delta X_2'(\omega_2 + \omega_2)$, and the intermodulation products $\Delta X_1'(\omega_1 + \omega_2)$, $\Delta X_1'(\omega_1 - \omega_2)$, $\Delta X_2'(\omega_1 + \omega_2)$, and $\Delta X_2'(\omega_1 - \omega_2)$. The higher-order products can be determined in the analogical way using the higher-order terms [6].

III. ILLUSTRATIVE EXAMPLE

All the steps of the analysis by means of the Volterra series can be clearly demonstrated by a simple circuit in Fig. 1.

The two signal sources have the magnitude $0.1\,\mathrm{V}$. The first source has the frequency $1\,\mathrm{GHz}$, and the second one $0.25\,\mathrm{GHz}$; the conductance G is $0.1\,\mathrm{S}$, and the capacitance C is $10\,\mathrm{pF}$. The standard diode is not (statically) opened in any part of the period due to small magnitude of the signal sources. Therefore, the current i_1 is only determined by the junction capacitance

$$i_1 = C_{J0}(1 - m v_1)\, \dot{v}_1,$$

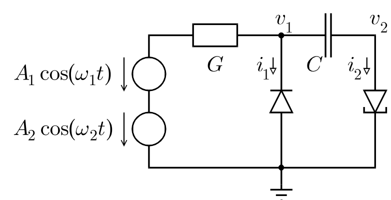

Fig. 1. Simple illustrative circuit analyzed by FFT and Volterra series.

where the term $C_{J0}(1 - m v_1)$ can be considered a simple linear approximation of the classical relation for the junction capacitance at $\phi_0 = 1\,\mathrm{V}$ by Maclaurin expansion. The zero-bias junction capacitance C_{J0} is $10\,\mathrm{pF}$, and the grading coefficient m changes from zero (i.e. the diode is replaced by a linear capacitor) through $0.\overline{3}$ (the diode with a linear junction) to 0.5 (the diode with an abrupt junction).

The tunnel diode can be approximated by a quadratic polynomial

$$i_2 = P_1 v_2 + P_2 v_2^2$$

in this analysis with respect to the small magnitude of the signal sources. The coefficients of the polynomial are $P_1 = 0.2\,\mathrm{S}$ and $P_2 = -1\,\mathrm{S/V}$. A current of the capacitance part of the tunnel-diode model can be neglected in this task.

The **first** step of the algorithm consists in determining the operating point, which is very easy:

$$v_{1_0} = A_1 + A_2, \quad v_{2_0} = 0.$$

The **second** step of the algorithm is the standard frequency analysis, i.e. solving the system (note that $A(\omega)$ is equal to A_1 for $\omega = \omega_1$, and correspondingly equal to A_2 for $\omega = \omega_2$)

$$G[\Delta V_1 - A(\omega)] + j\omega C_{J0}[1 - m(A_1 + A_2)]\Delta V_1 + j\omega C(\Delta V_1 - \Delta V_2) = 0,$$

$$j\omega C(\Delta V_2 - \Delta V_1) + P_1 \Delta V_2 = 0.$$

The system of two equations for the two variables $\Delta V_1(\omega)$ and $\Delta V_2(\omega)$ can be solved using Cramer rule, i.e.

$$\Delta V_1(\omega) = \frac{(P_1 + j\omega C)\, G A(\omega)}{\{G + j\omega[C_{J0}(1 - m(A_1 + A_2)) + C]\}(P_1 + j\omega C) + \omega^2 C^2},$$

$$\Delta V_2(\omega) = \frac{j\omega C G A(\omega)}{\{G + j\omega[C_{J0}(1 - m(A_1 + A_2)) + C]\}(P_1 + j\omega C) + \omega^2 C^2}.$$

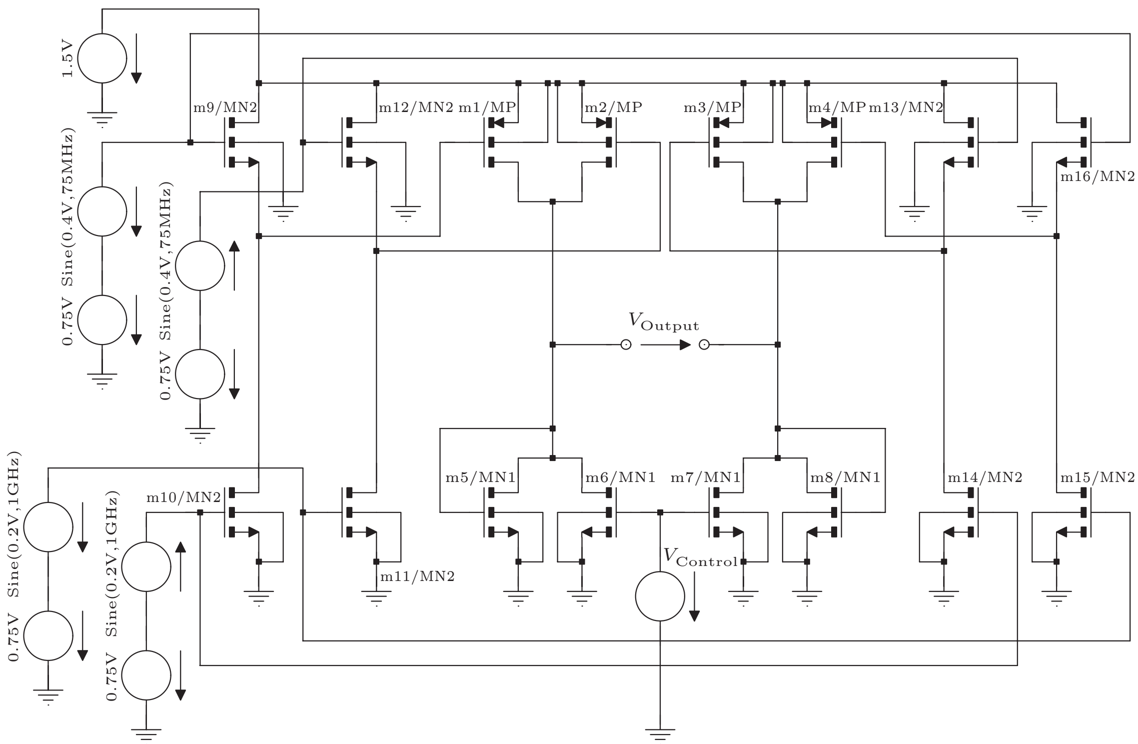

Fig. 2. Low-voltage low-power CMOS RF four-quadrant multiplier with the symmetrical low- (input signal) and high-frequency (local oscillator) sources.

TABLE II

MAIN INTERMODULATION PRODUCTS DETERMINED ACCURATELY BY THE STEADY-STATE ANALYSIS AND FAST FOURIER TRANSFORM AFTERWARDS (f_1+f_2, f_1-f_2, f_1+3f_2, f_1-3f_2, $3f_1+f_2$, AND $3f_1-f_2$; 3[RD]-ORDER PRODUCTS f_1+2f_2, f_1-2f_2, $2f_1+f_2$, AND $2f_1-f_2$ ARE NEGLIGIBLE)

V_{Control}	$V_{\text{Output},1.075\,\text{GHz}}$	$V_{\text{Output},0.925\,\text{GHz}}$	$V_{\text{Output},1.225\,\text{GHz}}$	$V_{\text{Output},0.775\,\text{GHz}}$	$V_{\text{Output},3.075\,\text{GHz}}$	$V_{\text{Output},2.925\,\text{GHz}}$
1 V	8.97 mV	10.1 mV	0.333 mV	0.377 mV	0.217 mV	0.228 mV
1.1 V	10.2 mV	11.5 mV	0.371 mV	0.485 mV	0.218 mV	0.229 mV
1.2 V	11.9 mV	13.6 mV	0.501 mV	0.712 mV	0.22 mV	0.232 mV
1.3 V	14.5 mV	16.8 mV	0.817 mV	1.18 mV	0.227 mV	0.239 mV
1.4 V	18.3 mV	21.2 mV	1.28 mV	1.85 mV	0.238 mV	0.252 mV
1.5 V	22 mV	25.7 mV	1.22 mV	1.67 mV	0.273 mV	0.29 mV

The **third** step of the algorithm is solving the system with the second derivatives for some harmonic or intermodulation product. Let us chose the product $\omega_1+\omega_2$, e.g. From the set of the second derivatives of the function f_1, only the derivative

$$\frac{\partial^2 f_1}{\partial v_1 \partial \dot{v}_1} = -mC_{J0}$$

is nonzero. In the first equation, that derivative is multiplied by the factor

$$j\left(\omega_1+\omega_2\right)\Delta V_1^2,$$

and ΔV_1 is a superposition of the frequency components ω_1 and ω_2. Therefore, it can be written in the form

$$j\left(\omega_1+\omega_2\right)\left(\Delta V_1\left(\omega_1\right)+\Delta V_1\left(\omega_2\right)\right)^2.$$

As a source of the product $\omega_1+\omega_2$, only the term

$$j\left(\omega_1+\omega_2\right)2\,\Delta V_1\left(\omega_1\right)\Delta V_1\left(\omega_2\right)$$

has meaning, but *not all* (that is just the source of frequent errors): corresponding analogy of the term $2\,\Delta V_1\left(\omega_1\right)\Delta V_1\left(\omega_2\right)$ in the time domain contains a factor of the type

$$\cos\left(\omega_1 t+\varphi_1\right)\cos\left(\omega_2 t+\varphi_2\right),$$

which can be expressed in the form

$$\tfrac{1}{2}\left[\cos\left(\left(\omega_1+\omega_2\right)t+\varphi_1+\varphi_2\right)+\cos\left(\left(\omega_1-\omega_2\right)t+\varphi_1-\varphi_2\right)\right]$$

—therefore, the term $2\,\Delta V_1\left(\omega_1\right)\Delta V_1\left(\omega_2\right)$ generates the intermodulation products both $\omega_1+\omega_2$ and $\omega_1-\omega_2$; the term

$$\Delta V_1\left(\omega_1\right)\Delta V_1\left(\omega_2\right)$$

is the source of the intermodulation product $\omega_1+\omega_2$, and the term

$$\Delta V_1\left(\omega_1\right)\Delta V_1^*\left(\omega_2\right)$$

is the source of the intermodulation product $\omega_1-\omega_2$ (the conjugate value is induced by the phase difference $\varphi_1-\varphi_2$).

The following step is analogical—from the set of the second derivatives of the function f_2, only the derivative

$$\frac{\partial^2 f_2}{\partial v_2^2} = 2P_2$$

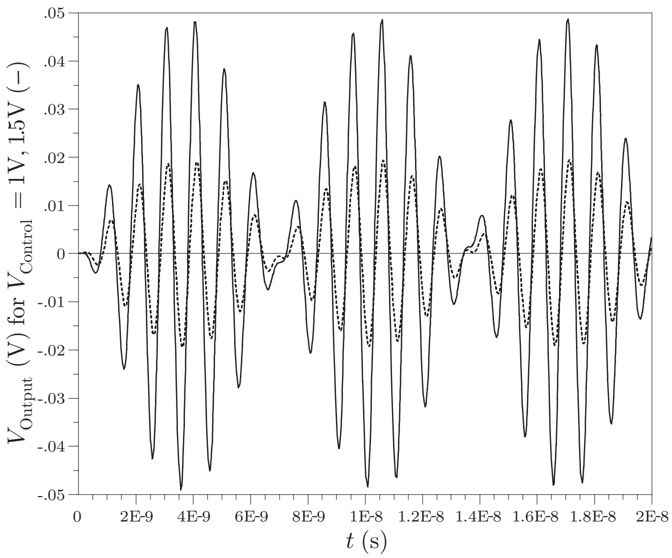

Fig. 3. The dependence of the output voltage on the controlling voltage.

is nonzero. Therefore, the system of the linear complex equations for the intermodulation product $\omega_1 + \omega_2$ can be written in the form

$$G\Delta V_1' + j(\omega_1+\omega_2)\,C_{J0}\,(1-m\,(A_1+A_2))\,\Delta V_1' + j(\omega_1+\omega_2)\,C\,(\Delta V_1' - \Delta V_2') - j(\omega_1+\omega_2)\,mC_{J0}\Delta V_1(\omega_1)\,\Delta V_1(\omega_2) = 0,$$

$$j(\omega_1+\omega_2)\,C\,(\Delta V_2' - \Delta V_1') + P_1\Delta V_2' + P_2\Delta V_2(\omega_1)\,\Delta V_2(\omega_2) = 0.$$

The system can be solved by the Cramer rule again with the following result for the second variable:

$$\Delta V_2' = \frac{1}{D} \times \left\{-P_2\left[G + j(\omega_1+\omega_2)\left[C_{J0}\,(1-m\,(A_1+A_2)) + C\right]\right] \times \Delta V_2(\omega_1)\,\Delta V_2(\omega_2) - (\omega_1+\omega_2)^2 CmC_{J0}\Delta V_1(\omega_1)\,\Delta V_1(\omega_2)\right\},$$

$$D = \{G + j(\omega_1+\omega_2)\left[C_{J0}\,(1-m\,(A_1+A_2)) + C\right]\} \times \left[P_1 + j(\omega_1+\omega_2)\,C\right] + (\omega_1+\omega_2)^2\,C^2.$$

In this case, the magnitude and argument of the intermodulation product $\omega_1 + \omega_2$ has been checked by the fast Fourier transform of the output (the resulting signal has the period 4 ns). The comparison is shown in Table I. It is clear that the estimation by the Volterra series is relatively precise for $m = 0$, and the inaccuracy is about 25 % for $m = 0.5$ (naturally, more nonlinear circuit corresponds to more inaccurate results).

IV. Practical Example of CMOS RF IC Design

Let us consider a four-quadrant CMOS RF multiplier in Fig. 2 [7]. All the parameters of the CMOS model have been kindly granted by Prof. Salama. However, they have been transformed to the new ones required by author's "smoothed" model with *suppressed* discontinuities [8]. The output voltage is strongly dependent on the controlling one connected to the gates of m6 and m7 transistors. A comparison of two output signals corresponding to different controlling voltages is shown in Fig. 3. For the controlling voltages 1 and 1.5 V, the magnitudes of the output signal are about 20 and 50 mV.

TABLE III
Main Intermodulation Products Estimated by Volterra Series ($f_1 + f_2$ and $f_1 - f_2$)

$\lvert V_{\text{Output}}\rvert_{\max}$	V_{Control}	$V_{\text{Output},1.075\,\text{GHz}}$	$V_{\text{Output},0.925\,\text{GHz}}$
19.3 mV	1 V	7.02 mV	7.57 mV
22.6 mV	1.1 V	7.59 mV	8.29 mV
26.4 mV	1.2 V	8.38 mV	9.27 mV
33.1 mV	1.3 V	9.47 mV	10.6 mV
41.9 mV	1.4 V	11 mV	12.5 mV
49 mV	1.5 V	13 mV	15 mV

However, the intermodulation products $f_1 + f_2$ and $f_1 - f_2$ can be estimated faster using the Volterra series—the results are shown in Table III. Similarly to the illustrative example, the results are compared with an outcome of the steady-state algorithm followed by FFT—significant intermodulation products are shown in Table II. The error of the estimation depends on the quality of CMOS models (smoothing is necessary), and is acceptable for lesser magnitudes of the signal, of course.

V. Conclusion

An algorithm for a fast estimation of the main interpolation products has been presented in a simple form, which could be an add-on of the standard frequency analysis, and which can efficiently reuse procedures of that analysis. The precision of the algorithm has been illustrated by an analysis of an RF four-quadrant multiplier. For a medium level of nonlinearities, the estimation can be used as a fast approximative analysis.

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