<the ideas for chapter 5>

5.1 From connection forms to covariant derivatives

Given $K : TTM \longrightarrow TM$, i.e., on a trivializing $(U,*)$, given

$$((p,u)\,(k,v)) \longmapsto (p, v+\Gamma_p(u,k))$$

and

$$U \xrightarrow{X} TM \qquad\qquad U \xrightarrow{Y} TM$$
$$p \longmapsto (p, x(p)) \qquad\qquad p \longmapsto (p, y(p))$$

This needs a bit of an explanation

form the composite

$$U \xrightarrow{X} TM \cong TU \xrightarrow{TY} TTM \xrightarrow{K} TM$$

and you obtain a covariant derivative.

More in detail

$$U \xrightarrow{X} TM \cong TU \xrightarrow{TY} TTM$$
$$p \longmapsto (p, x(p)) \longmapsto ((p, y(p)), (x(p), y'(p)\cdot x(p)))$$

$$TTM \xrightarrow{K} TM$$

$$((p, y(p)), (x(p), y'(p)\cdot x(p))) \longmapsto (p, \underbrace{y'(p)\cdot x(p) + \Gamma_p(y(p), x(p))}_{\nabla_X Y|_p})$$

5.2 From covariant derivatives to conn forms

$\nabla \rightsquigarrow \Gamma$ (on a trivialising (U, x))

Define $K : ((p,u),(k,v)) \longmapsto (p, v + \Gamma_p(k,u))$