

## 4.2 The Finsler Bundle

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Before we are able to define connection forms abstractly, we have to introduce the so-called

Finsler Bundle  $T^*M$  of a manifold  $M$

Formally, the Finsler Bundle is a ~~smooth~~ manifold  $T^*M$  equipped with two smooth projections

$$\begin{array}{ccc} T^*M & \xrightarrow{p_2} & TM \\ p_1 \downarrow & & \\ TM & & \end{array}$$

that make the square

$$\begin{array}{ccc} T^*M & \xrightarrow{p_2} & TM \\ p_1 \downarrow & & \downarrow \pi_M \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

commutative and that has a certain universal property making the above square a pullback. Before we state the property, let us show how a pullback of two maps between sets is formed.

4.2.1 Example let  $\begin{array}{ccc} & Y & \\ & \downarrow g & \\ X & \xrightarrow{f} & Z \end{array}$  be a diagram of

sets and mappings and let us consider the set  $P = \{(x, y) \mid f(x) = g(y)\}$ , equipped with the projections  $p_1: P \rightarrow X$  and  $p_2: P \rightarrow Y$ ,  $(x, y) \mapsto x$  and  $(x, y) \mapsto y$ .

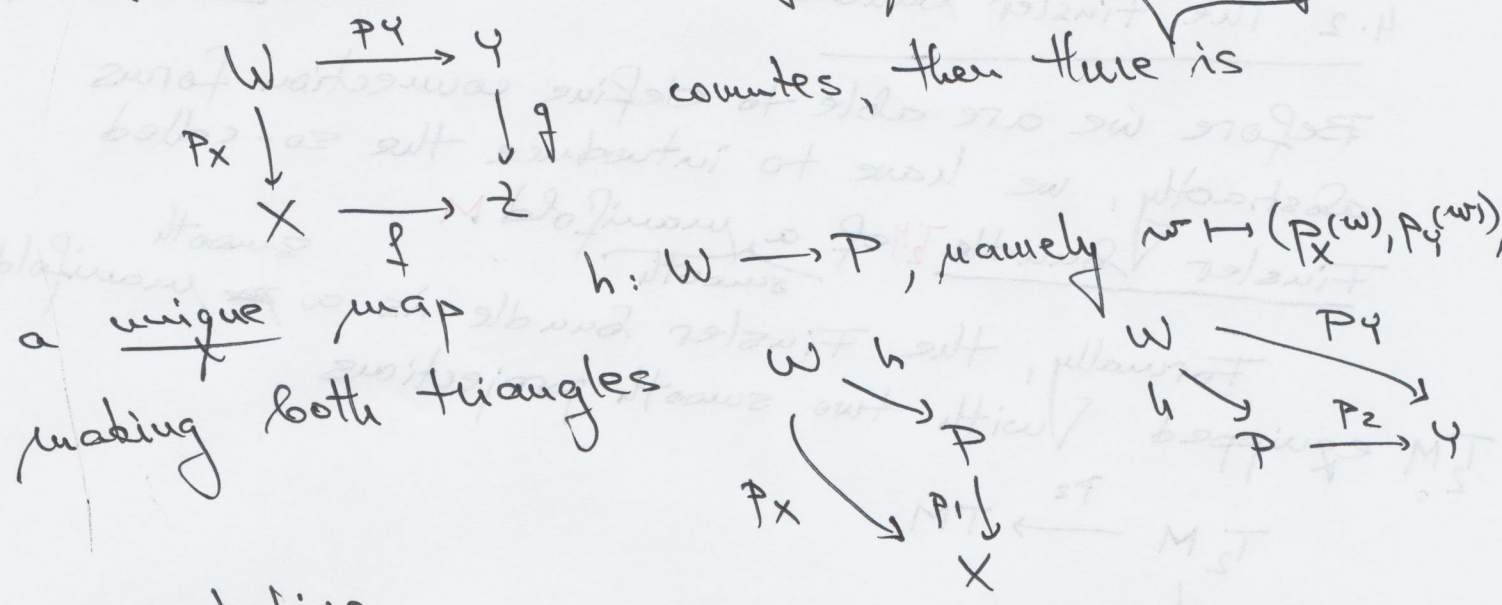
Then it is clear that the square

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ p_1 \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

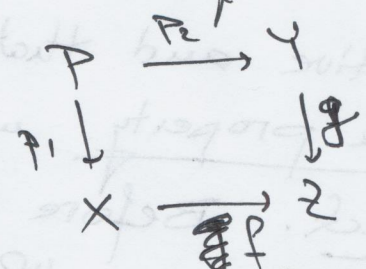
commutes.



Moreover: whenever any square clearly commutes, then there is



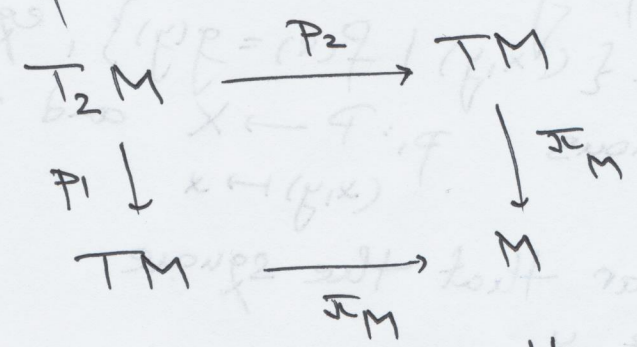
In fact, it is easy to see that the above universal property (i.e., the existence of a unique map  $h$ ) determines  $P$  in the square the set



up to a unique isomorphism (i.e., a bijection).  
 <end of example>

### 4.2.2 Definition Define the Finsler bundle

of a manifold  $M$  to be the pullback



in the realm of ~~smooth~~ smooth manifolds and smooth maps.

is in the spirit of Category Theory. It can be unravelled into more elementary terms as follows:

- the points of  $T_2M$  are tuples  $(p, v_p, w_p)$  with  $p \in M$ ,  $v_p \in T_p M$ ,  $w_p \in T_p M$
- the topology of  $T_2M$  is that of a subspace of  $TM \times TM$

See, e.g.) Szilasi & al. for more details.

the maps  $\pi_1$  and  $\pi_2$  are the obvious (smooth) projections  $(p, v_p, w_p) \mapsto (p, v_p)$  and  $(p, v_p, w_p) \mapsto (p, w_p)$ , respectively.

~~It is also trivial to observe that the~~  
< end of Remark >

We use the ~~definition~~ universal property of the Tinsler bundle to define a smooth certain

map  $TTM \xrightarrow{\tau_M} T_2M$

that we will use later.



# 4.2.4 Proposition

For every manifold  $M$  counts.

the square

$$\begin{array}{ccc} TTM & \xrightarrow{T\pi_M} & TM \\ \pi_{TM} \downarrow & & \downarrow \pi_M \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

Thus, there is a unique smooth map

$$TTM \xrightarrow{\tau_M} T_2M$$

making the triangles

$$\begin{array}{ccc} TTM & \xrightarrow{\tau_M} & T_2M \\ \pi_{TM} \searrow & & \downarrow \pi_1 \\ & & TM \end{array}$$

$$\begin{array}{ccc} TTM & \xrightarrow{T\pi_M} & TM \\ \tau_M \searrow & & \downarrow \pi_2 \\ T_2M & \xrightarrow{\pi_2} & TM \end{array}$$

commutative.

Proof The proposition will follow from the universal property of a pullback, once we show that the square

$$\begin{array}{ccc} TTM & \xrightarrow{T\pi_M} & TM \\ \pi_{TM} \downarrow & & \downarrow \pi_M \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

counts. In fact, we prove a more general result: for every smooth map  $N \xrightarrow{f} M$ , the square

$$\begin{array}{ccc} TN & \xrightarrow{Tf} & TM \\ \pi_N \downarrow & & \downarrow \pi_M \\ N & \xrightarrow{f} & M \end{array}$$

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We will use the "trivialisation technique", mentioned in Rem 2.3.4 and Ex 2.3.5.

Given  $(U, \ast)$ , we are to prove that

$$\begin{array}{ccc} (p, v) & \xrightarrow{\quad} & (f(p), f'(p) \cdot v) \\ \downarrow & & \downarrow \\ p & \xrightarrow{\quad} & f(p) \end{array}$$

holds, which is trivial. □

We are now ready to state the abstract definition of the connection form on a manifold.

4.2.5 Definition A smooth map  $C: T_2M \rightarrow TTM$  is called a connection form, if the following three properties hold:

(1) The diagram

$$\begin{array}{ccc} T_2M & \xrightarrow{C} & TTM \\ & \searrow \text{id} & \downarrow \tau_M \\ & & T_2M \end{array}$$

commutes, where  $\tau_M$  is the mapping of Proposition 4.2.4

(2) The diagram

$$\begin{array}{ccc} T_2M & \xrightarrow{C} & TTM \\ & \searrow \pi_1 & \swarrow \pi_{TM} \\ & & TM \end{array}$$

commutes, and  $C$  is linear in every fibre.



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(3) The diagram

$$\begin{array}{ccc}
 T_2 M & \xrightarrow{C} & TTM \\
 & \searrow p_2 & \swarrow T\pi_M \\
 & & TM
 \end{array}$$

commutes,  
and  $C$  is  
linear in every  
fibre.

4.2.6 Remark Definition 4.2.5 is best understood

by the trivialisation technique. In  $(U, \pi)$

we have  $C: (p, v, w) \mapsto (p, v, w, \Gamma_p(v, w))$   
due to condition 1. Conditions 2 and 3  
then state that  $\Gamma_p(v, w)$  is linear in  $v$  and  $w$ ,  
respectively. After renaming, we have obtained  
the connection form

$$(p, k, u) \mapsto (p, k, u, -\Gamma_p(k, u))$$

of Example ??  $\leftarrow$  refer to the appropriate.

4.2.7 Remark There exists an equivalent  
description of the connection form that is  
more pleasant to work with.

Namely, define a smooth map

$$TTM \xrightarrow{K} TM$$

in a local trivialisation by putting

$$((p, v), (k, w)) \mapsto (p, w + \Gamma_p(k, v))$$



and observe that it has the following | Sect 4.2/7  
two properties

(1) The square

$$\begin{array}{ccc} \text{TTM} & \xrightarrow{K} & \text{TM} \\ \pi_M \downarrow & & \downarrow \pi_M \\ \text{TM} & \xrightarrow{\pi_M} & M \end{array}$$

counts,  
and it is  
linear  
in every fibre.

Indeed

$$\begin{array}{ccc} ((p, v), (k, w)) & \xrightarrow{\quad} & (p, w + \Gamma_p(k, v)) \\ \downarrow & & \downarrow \\ (p, \underbrace{\pi'_M((k, w))}_{(p, v) = k}) & \xrightarrow{\quad} & p \end{array}$$

~~Moreover,~~

(2) The square

$$\begin{array}{ccc} \text{TTM} & \xrightarrow{K} & \text{TM} \\ \pi_{\text{TM}} \downarrow & & \downarrow \pi_M \\ \text{TM} & \xrightarrow{\pi_M} & M \end{array}$$

counts,  
and  
it is  
linear  
in every  
fibre.

Indeed,

$$\begin{array}{ccc} ((p, v), (k, w)) & \xrightarrow{\quad} & (p, w + \Gamma_p(k, v)) \\ \downarrow & & \downarrow \\ (p, v) & \xrightarrow{\quad} & p \end{array}$$