Picture Calculus for QM

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This talk has been very much influenced by Bob Coecke's views on QM.

Thank you, Bob.

I am not a physicist. That's bad...

I am a category theorist. So what?

Hear, hear:

> *We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.*
>
> Gregory Moore and Nathan Seiberg: Classical and quantum conformal field theory, *Comm. Math. Physics* 123 (1989), 177–254.

Wow! That's a bit depressing...

No, it isn't! CT means doing physics all the time!

What does Category Theory Bring to Quantum Physics and Quantum Computing?

In both physics and computing (and everyday life for that matter):

1. We manipulate data. These data have various types.
2. We can concatenate these manipulations: both sequentially and in parallel.

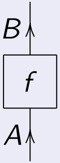
The above is essentially what category theory is about!

Pictorial Notation

Category Theory

$$\begin{tikzcd} B \\ A \arrow[u, "f"'] \end{tikzcd}$$

Picture Calculus



Intuition: A, B are the state spaces, f is the transformation.
So it's wires and boxes (plus axioms — later) instead of vector spaces, matrices, linear transformations, etc.

Picture Calculi in Physics and Category Theory

1. Roger Penrose: picture calculus for spinors (∼1971).
2. André Joyal, Ross Street: picture calculus for tensor categories (∼1980).
3. Samson Abramsky, Bob Coecke, Duško Pavlović, Peter Selinger, and others. . . : picture calculi for QM (∼2000).

Quantum Teleportation

Introduced in

> Charles H. Bennett, Giles Brassard, Claude Crépeau, Richard Josza, Asher Peres and Williams K. Wooters: Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Physical Review Letters* 70 (1993), 1895–1899.

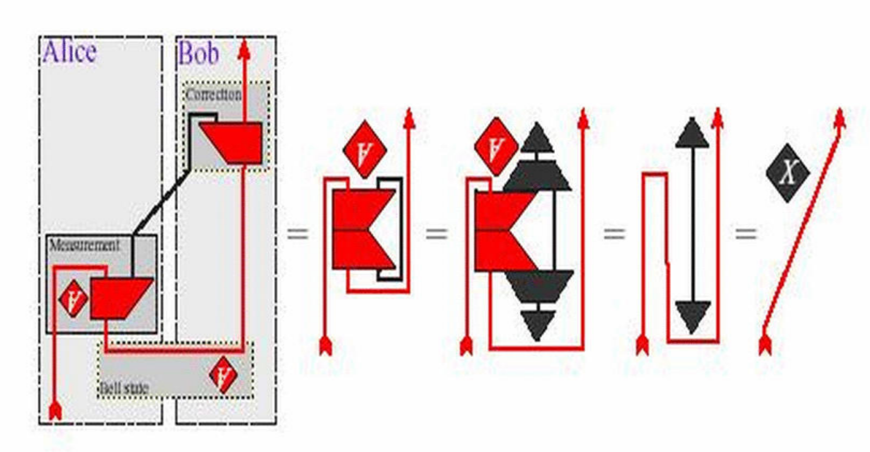
Physically implemented

> D. Boschi, S. Branca, F. De Martini, L. Hardy and S. Popescu in 1998, see also ArXiv: `quant-ph/9710013`

How Teleportation Works (Roughly)

1. Two parties: Alice and Bob, sharing an EPR pair.
2. Alice teleports a particle to Bob in that she measures a certain state and informs Bob about the result via a classical channel.

The categorical expression of Quantum Teleportation:



(B. Coecke, D. Pavlović, 2006)

Superdense Coding

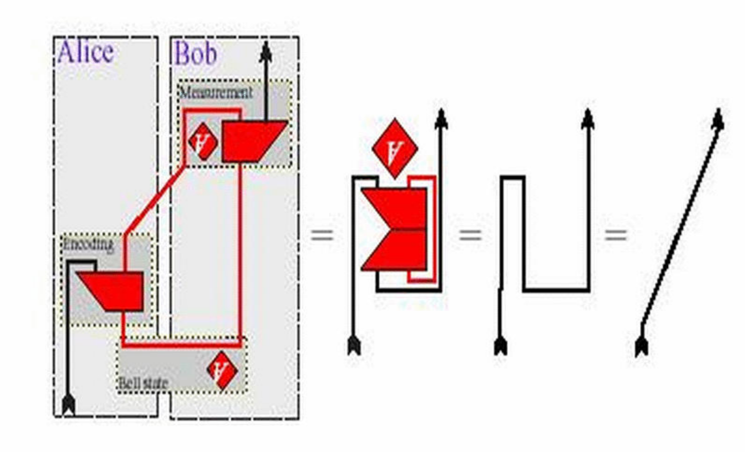
Introduced in

> Charles H. Bennett and Stephen J. Wiesner: Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, *Physical Review Letters* 69 (1992), 2881–2884.

How Superdense Coding Works (Roughly)

1. Two parties: Alice and Bob, sharing an EPR pair.
2. Alice encodes two classical bits into one q-bit and sends it to Bob via a quantum channel. Bob retrieves the message in that he measures a certain state.

Superdense Coding in Picture Calculus:



(B. Coecke, D. Pavlović, 2006)

Support from the Fathers of QM

. . . I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space anymore.

John von Neumann in a letter to George David Birkhoff, 13 November 1935

And Remember

In mathematics you don't understand things. You just get used to them.

John von Neumann (1903–1957)

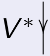
Primitive Notions

A (Labelled) Box: 

A (Labelled) Wire: 

Basic Axioms

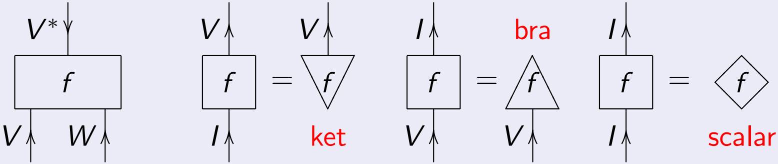
The Void Wire: 

The Involution: 

The void wire can be omitted from any picture.

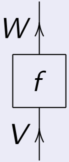
Operators

These are made from wires and boxes, e.g.,

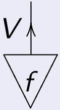


In Classical Model

V = finitely dimensional complex Hilbert space
V^* = space conjugate to V
I = the 1-dimensional space (complex numbers)

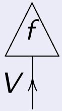


linear map from V to W



linear map from $\mathbb{C}$ to V, i.e., $|f\rangle$

In Classical Model

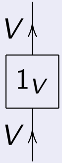


linear map from V to $\mathbb{C}$, i.e., $\langle f|$

linear map from $\mathbb{C}$ to $\mathbb{C}$, i.e., a scalar

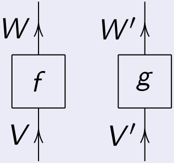
Sequential Composition

Glue together the corresponding wires: this composition is associative and has units:

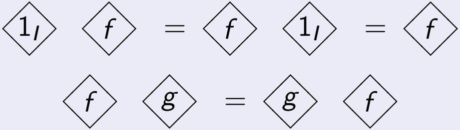


Thus, 1_V can be replaced by wire V in any picture.

Parallel Composition



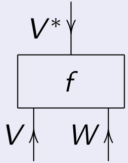
Lemma



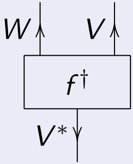
Adjoint Operators

For every operator f there is a unique adjoint $f^\dagger$ obtained just by symmetry along the centre of the box.

For example:



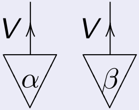
its adjoint is

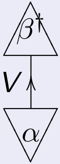


In Classical Model

adjoint = transpose of the conjugate

Observe:

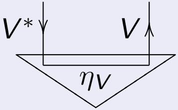
for kets 

the scalar 

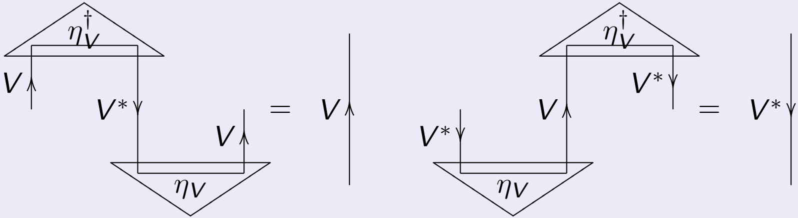
is the inner product $\langle\beta|\alpha\rangle$.

Bell States and Yanking

For every V there is a Bell state



such that Yanking Axioms hold:



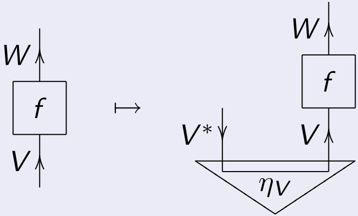
This is Already Quite a Powerful Beast

1. One can prove basic facts about transposes, adjoints, unitary, self-adjoint and positive operators.
2. Traces can be defined.
3. One can prove the Hilbert-Schmidt Correspondence.
4. Spectral Decomposition Theorem and Born's Rule can be derived. (Requires biproducts.)
5. ... and more.

But there is a serious drawback: linear algebra sneaks in!

The Calculus at Work, No 1: The Hilbert-Schmidt Correspondence

The map



is a bijection.

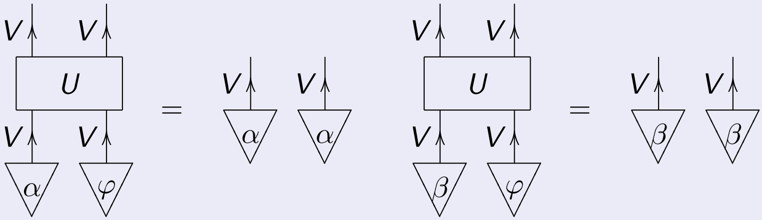
In Classical Model

There is a bijection

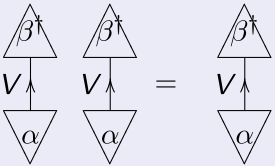
$$\mathrm{Lin}(V, W) \cong V^* \otimes W$$

The Calculus at Work, No 2: The No-Cloning Theorem

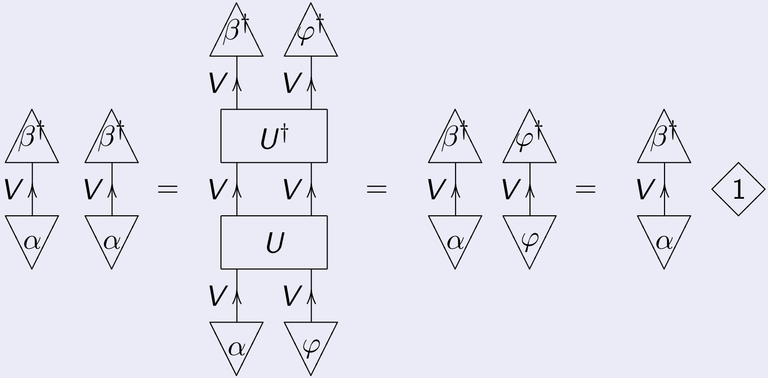
Suppose



holds for states $|\alpha\rangle$, $|\beta\rangle$ and $|\varphi\rangle$. Then we have the equality



Proof of The No-Cloning Theorem

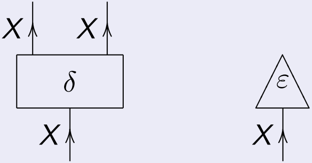


The Goal

To get rid of linear algebra once for all. This is done by distinguishing classical from quantum.

What does Distinguish Classical from Quantum?

Classical data can be copied and deleted:



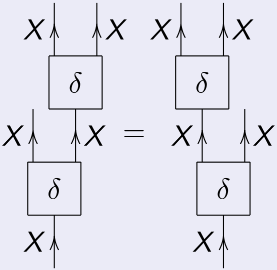
In quantum world, Bell States prohibit copying (entanglement).

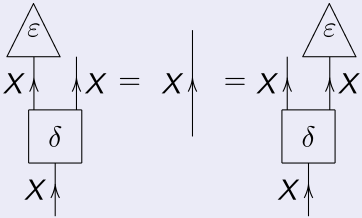
Entanglement Prohibits Copying

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "f"] \arrow[d, "\delta_{\mathbb{C}}"'] & \mathbb{C} \oplus \mathbb{C} \arrow[d, "\delta_{\mathbb{C}\oplus\mathbb{C}}"] \\
\mathbb{C} \otimes \mathbb{C} \arrow[r, "f \otimes f"'] & (\mathbb{C} \oplus \mathbb{C}) \otimes (\mathbb{C} \oplus C)
\end{tikzcd}
$$

$$
\begin{tikzcd}
|1\rangle \arrow[r, mapsto] \arrow[d, mapsto] & |0\rangle + |1\rangle \arrow[d, mapsto] \\
|1\rangle \otimes |1\rangle \arrow[r, mapsto] & (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \neq (|0\rangle \otimes |0\rangle) + (|0\rangle \otimes |0\rangle)
\end{tikzcd}
$$

Axioms For Copying and Deleting





This Allows Us to Avoid Linear Algebra Altogether

1. Bob Coecke and Duško Pavlović (2006): define quantum measurements and spectral decomposition without sums — hence no linear algebra, new models.
2. Bob Coecke (2006): Generalized Hadamard gates — axiomatic reasons why teleportation and superdense coding is possible.

References

1. B. Coecke: *Kindergarten Quantum Mechanics*, `http://fr.arxiv.org/abs/quant-ph/0510032`
2. S. Abramsky and B. Coecke: *Categorical Semantics of Quantum Protocols*, `http://fr.arxiv.org/abs/quant-ph/0402130`
3. B. Coecke and D. Pavlović: Quantum measurements without sums, to appear in *Mathematics of Quantum Computing and Technology* (2006)
4. P. Selinger: Dagger Compact Closed Categories and Completely Positive Maps, *Proceedings of the 3rd International Workshop on Quantum Programming Languages, Chicago, June 30 - July 1, 2005*