4.2 The Finsler bundle

Before we are able to define connection forms abstractly, we have to introduce the so-called **Finsler bundle** T_2M of a smooth manifold M.

Formally, the Finsler bundle is a smooth manifold T_2M equipped with two smooth projections

$$\begin{tikzcd} T_2M \arrow[r, "p_2"] \arrow[d, "p_1"'] & TM \\ TM & \end{tikzcd}$$

that make the square

$$\begin{tikzcd} T_2M \arrow[r, "p_2"] \arrow[d, "p_1"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

commutative and that has a certain **universal property** making the above square a **pullback**. Before we state the property, let us show how a pullback of two maps between sets is formed.

4.2.1 Example Let

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a diagram of sets and mappings and let us consider the set $P = \{(x,y) \mid f(x) = g(y)\}$, equipped with the projections $p_1 : P \to X$, $(x,y) \mapsto x$ and $p_2 : P \to Y$, $(x,y) \mapsto y$.

Then it is clear that the square

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes.

Moreover: whenever *any* square

$$\begin{tikzcd} W \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes, then there is clearly a *unique* map $h : W \to P$, namely $w \mapsto (p_X(w), p_Y(w))$, making both triangles

$$\begin{tikzcd} W \arrow[dr, "h"] \arrow[ddr, bend right, "p_X"'] & \\ & P \arrow[d, "p_1"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} W \arrow[dr, "h"'] \arrow[drr, bend left, "p_Y"] & & \\ & P \arrow[r, "p_2"] & Y \end{tikzcd}$$

commutative.

In fact, it is easy to see that the above *universal property* (i.e., the existence of a unique map h) determines the set P in the square

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

up to a unique isomorphism (i.e., a bijection).

<end of example>

4.2.2 Definition Define the *Finsler bundle* of a manifold M to be the pullback

$$\begin{tikzcd} T_2M \arrow[r, "p_2"] \arrow[d, "p_1"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

in the realm of ~~...~~ smooth manifolds and smooth maps.

4.2.3 Remark Definition 4.2.2 is in the spirit of Category Theory. It can be unravelled into more elementary terms as follows:

- the points of T_2M are triples (p, v_p, w_p) with $p \in M$, $v_p \in T_pM$, $w_p \in T_pM$
- the topology of T_2M is that of a subspace of $TM \times TM$
- See, e.g., Szilasi & al. for more details.

the maps p_1 and p_2 are the obvious (smooth) projections $(p, v_p, w_p) \longmapsto (p, v_p)$ and $(p, v_p, w_p) \longmapsto (p, w_p)$, respectively.

~~It is now trivial to observe that the~~

<end of Remark>

~~If~~

We use the ~~definition~~ universal property of the Finsler bundle to define a certain smooth map

$$TTM \xrightarrow{\tau_M} T_2M$$

that we will use later.

<u>4.2.4 Proposition</u> For every manifold M the square

$$\begin{tikzcd} TTM \arrow[r, "T\pi_M"] \arrow[d, "\pi_{TM}"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

commutes.

Thus, there is a unique smooth map

$$TTM \xrightarrow{\tau_M} T_2M$$

making the triangles

$$\begin{tikzcd} TTM \arrow[dr, "\tau_M"] \arrow[ddr, "\pi_{TM}"'] & \\ & T_2M \arrow[d, "p_1"] \\ & TM \end{tikzcd} \qquad \begin{tikzcd} TTM \arrow[d, "\tau_M"'] \arrow[dr, "T\pi_M"] & \\ T_2M \arrow[r, "p_2"'] & TM \end{tikzcd}$$

commutative.

<u>Proof</u> The proposition will follow from the universal property of a pullback, once we show that the square

$$\begin{tikzcd} TTM \arrow[r, "T\pi_M"] \arrow[d, "\pi_{TM}"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

commutes. In fact, we prove a more general result: for every smooth map $N \xrightarrow{f} M$, the square

$$\begin{tikzcd} TN \arrow[r, "Tf"] \arrow[d, "\pi_N"'] & TM \arrow[d, "\pi_M"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

We will use the "trivialisation technique", mentioned in Rem 2.3.4 and Ex 2.3.5. Given $(U,*)$, we are to prove that

$$\begin{tikzcd}
(p,v) \arrow[r, mapsto] \arrow[d, mapsto] & (f(p), f'(p).v) \arrow[d, mapsto] \\
p \arrow[r, mapsto] & f(p)
\end{tikzcd}$$

holds, which is trivial. ∎

We are now ready to state the abstract definition of the connection form on a manifold.

<u>4.2.5 Definition</u> A smooth map $C : T_2M \to TTM$ is called a <u>connection form</u>, if the following three properties hold:

(1) The diagram

$$\begin{tikzcd}
T_2M \arrow[r, "C"] \arrow[dr, "id"'] & TTM \arrow[d, "\tau_M"] \\
& T_2M
\end{tikzcd}$$

commutes, where τ_M is the mapping of Proposition 4.2.4

(2) The diagram

$$\begin{tikzcd}
T_2M \arrow[r, "C"] \arrow[dr, "p_1"'] & TTM \arrow[d, "\pi_{TM}"] \\
& TM
\end{tikzcd}$$

commutes, and C is linear in every fibre.

(3) The diagram

$$\begin{tikzcd} T_2 M \arrow[r, "C"] \arrow[dr, "p_2"'] & TTM \arrow[d, "T\pi_M"] \\ & TM \end{tikzcd}$$

commutes, and C is linear in every fibre.

4.2.6 <u>Remark</u> Definition 4.2.5 is best understood by the trivialisation technique. In $(U, *)$ we have

$$C : (p, v, w) \longmapsto (p, v, w, -\Gamma_p(v, w))$$

due to condition 1. Conditions 2 and 3 then state that $\Gamma_p(v,w)$ is linear in v and w, respectively. After renaming, we have obtained the connection form

$$(p, k, u) \longmapsto (p, k, u, -\Gamma_p(k, u))$$

of Example ?? ← refer to the appropriate.

4.2.7 <u>Remark</u> There exists and equivalent description of the connection form that is more pleasant to work with.

Namely, define a smooth map

$$TTM \xrightarrow{K} TM$$

in a local trivialisation by putting

$$((p, v)(k, w)) \longmapsto (p, w + \Gamma_p(k, v))$$

and observe that it has the following two properties

(1) The square
$$\begin{tikzcd} TTM \arrow[r, "K"] \arrow[d, "T\pi_M"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$
commutes, and it is linear in every fibre.

Indeed
$$\begin{tikzcd} ((p,v)\,(k,w)) \arrow[r, mapsto] \arrow[d, mapsto] & (p, w + \Gamma_p(k,v)) \arrow[d, mapsto] \\ (p, \underbrace{\pi'_M|_{(p,v)}(k,w)}_{=k}) \arrow[r, mapsto] & p \end{tikzcd}$$

~~Moreover~~,

(2) The square
$$\begin{tikzcd} TTM \arrow[r, "K"] \arrow[d, "\pi_{TM}"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$
commutes, and it is linear in every fibre.

Indeed,
$$\begin{tikzcd} ((p,v),(k,w)) \arrow[r, mapsto] \arrow[d, mapsto] & (p, w + \Gamma_p(k,v)) \arrow[d, mapsto] \\ (p,v) \arrow[r, mapsto] & p \end{tikzcd}$$