CHAPTER I

Differentiable Manifolds

1. *Differentiable manifolds*

A *pseudogroup of transformations* on a topological space S is a set Γ of transformations satisfying the following axioms:

(1) Each $f \in \Gamma$ is a homeomorphism of an open set (called the domain of f) of S onto another open set (called the range of f) of S;

(2) If $f \in \Gamma$, then the restriction of f to an arbitrary open subset of the domain of f is in Γ;

(3) Let $U = \bigcup_i U_i$ where each U_i is an open set of S. A homeomorphism f of U onto an open set of S belongs to Γ if the restriction of f to U_i is in Γ for every i;

(4) For every open set U of S, the identity transformation of U is in Γ;

(5) If $f \in \Gamma$, then $f^{-1} \in \Gamma$;

(6) If $f \in \Gamma$ is a homeomorphism of U onto V and $f' \in \Gamma$ is a homeomorphism of U' onto V' and if $V \cap U'$ is non-empty, then the homeomorphism $f' \circ f$ of $f^{-1}(V \cap U')$ onto $f'(V \cap U')$ is in Γ.

We give a few examples of pseudogroups which are used in this book. Let $\mathbf{R}^n$ be the space of n-tuples of real numbers $(x^1, x^2, \ldots, x^n)$ with the usual topology. A mapping f of an open set of $\mathbf{R}^n$ into $\mathbf{R}^m$ is said to be of *class* C^r, $r = 1, 2, \ldots, \infty$, if f is continuously r times differentiable. By class C^0 we mean that f is continuous. By class C^ω we mean that f is real analytic. The *pseudogroup* $\Gamma^r(\mathbf{R}^n)$ *of transformations of class* C^r of $\mathbf{R}^n$ is the set of homeomorphisms f of an open set of $\mathbf{R}^n$ onto an open set of $\mathbf{R}^n$ such that both f and f^{-1} are of class C^r. Obviously $\Gamma^r(\mathbf{R}^n)$ is a pseudogroup of transformations of $\mathbf{R}^n$. If $r < s$, then $\Gamma^s(\mathbf{R}^n)$ is a subpseudogroup of $\Gamma^r(\mathbf{R}^n)$. If we consider only those $f \in \Gamma^r(\mathbf{R}^n)$ whose Jacobians are positive everywhere, we obtain a subpseudogroup of $\Gamma^r(\mathbf{R}^n)$. This subpseudogroup, denoted by $\Gamma^r_o(\mathbf{R}^n)$, is called the *pseudogroup of orientation-preserving transformations of class* C^r *of* $\mathbf{R}^n$. Let $\mathbf{C}^n$ be the space of n-tuples of complex numbers with the usual topology. The *pseudogroup of holomorphic* (i.e., complex analytic) *transformations of* $\mathbf{C}^n$ can be similarly defined and will be denoted by $\Gamma(\mathbf{C}^n)$. We shall identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, when necessary, by mapping $(z^1, \ldots, z^n) \in \mathbf{C}^n$ into $(x^1, \ldots, x^n, y^1, \ldots, y^n) \in \mathbf{R}^{2n}$, where $z^j = x^j + iy^j$. Under this identification, $\Gamma(\mathbf{C}^n)$ is a subpseudogroup of $\Gamma^r_o(\mathbf{R}^{2n})$ for any r.

An *atlas* of a topological space M compatible with a pseudogroup Γ is a family of pairs (U_i, φ_i), called *charts*, such that

(a) Each U_i is an open set of M and $\bigcup_i U_i = M$;

(b) Each φ_i is a homeomorphism of U_i onto an open set of S;

(c) Whenever $U_i \cap U_j$ is non-empty, the mapping $\varphi_j \circ \varphi_i^{-1}$ of $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$ is an element of Γ.

A *complete* atlas of M compatible with Γ is an atlas of M compatible with Γ which is not contained in any other atlas of M compatible with Γ. Every atlas of M compatible with Γ is contained in a unique complete atlas of M compatible with Γ. In fact, given an atlas $A = \{(U_i, \varphi_i)\}$ of M compatible with Γ, let $\tilde{A}$ be the family of all pairs (U, φ) such that φ is a homeomorphism of an open set U of M onto an open set of S and that

$$\varphi_i \circ \varphi^{-1}\colon \varphi(U \cap U_i) \to \varphi_i(U \cap U_i)$$

is an element of Γ whenever $U \cap U_i$ is non-empty. Then $\tilde{A}$ is the complete atlas containing A.

If Γ' is a subpseudogroup of Γ, then an atlas of M compatible with Γ' is compatible with Γ.

A *differentiable manifold* of class C^r is a Hausdorff space with a fixed complete atlas compatible with $\Gamma^r(\mathbf{R}^n)$. The integer n is called the dimension of the manifold. Any atlas of a Hausdorff space compatible with $\Gamma^r(\mathbf{R}^n)$, enlarged to a complete atlas, defines a differentiable structure of class C^r. Since $\Gamma^r(\mathbf{R}^n) \supset \Gamma^s(\mathbf{R}^n)$ for $r < s$, a differentiable structure of class C^s defines uniquely a differentiable structure of class C^r. A differentiable manifold of class C^ω is also called a *real analytic manifold*. (Throughout the book we shall mostly consider differentiable manifolds of class C^∞. By a *differentiable manifold* or, simply, *manifold*, we shall mean a differentiable manifold of class C^∞.) A *complex* (*analytic*) *manifold* of complex dimension n is a Hausdorff space with a fixed complete atlas compatible with $\Gamma(\mathbf{C}^n)$. An *oriented* differentiable manifold of class C^r is a Hausdorff space with a fixed complete atlas compatible with $\Gamma_o^r(\mathbf{R}^n)$. An oriented differentiable structure of class C^r gives rise to a differentiable structure of class C^r uniquely. Not every differentiable structure of class C^r is thus obtained; if it is obtained from an oriented one, it is called *orientable*. An orientable manifold of class C^r admits exactly two orientations if it is connected. Leaving the proof of this fact to the reader, we shall only indicate how to *reverse the orientation* of an oriented manifold. If a family of charts (U_i, φ_i) defines an oriented manifold, then the family of charts (U_i, ψ_i) defines the manifold with the reversed orientation where ψ_i is the composition of φ_i with the transformation $(x^1, x^2, \ldots, x^n) \to (-x^1, x^2, \ldots, x^n)$ of $\mathbf{R}^n$. Since $\Gamma(\mathbf{C}^n) \subset \Gamma_o^r(\mathbf{R}^{2n})$, every complex manifold is oriented as a manifold of class C^r.

For any structure under consideration (e.g., differentiable structure of class C^r), an *allowable* chart is a chart which belongs to the fixed complete atlas defining the structure. From now on, by a chart we shall mean an allowable chart. Given an allowable chart (U_i, φ_i) of an n-dimensional manifold M of class C^r, the system of functions $x^1 \circ \varphi_i, \ldots, x^n \circ \varphi_i$ defined on U_i is called a *local coordinate system* in U_i. We say then that U_i is a *coordinate neighborhood*. For every point p of M, it is possible to find a chart (U_i, φ_i) such that $\varphi_i(p)$ is the origin of $\mathbf{R}^n$ and φ_i is a homeomorphism of U_i onto an open set of $\mathbf{R}^n$ defined by $|x^1| < a, \ldots, |x^n| < a$ for some positive number a. U_i is then called a *cubic neighborhood* of p.

In a natural manner $\mathbf{R}^n$ is an oriented manifold of class C^r for any r; a chart consists of an element f of $\Gamma_o^r(\mathbf{R}^n)$ and the domain of f. Similarly, $\mathbf{C}^n$ is a complex manifold. Any open subset N of a manifold M of class C^r is a manifold of class C^r in a natural manner; a chart of N is given by $(U_i \cap N, \psi_i)$ where (U_i, φ_i) is a chart of M and ψ_i is the restriction of φ_i to $U_i \cap N$. Similarly, for complex manifolds.

Given two manifolds M and M' of class C^r, a mapping $f\colon M \to M'$ is said to be differentiable of class C^k, $k \leqq r$, if, for every chart (U_i, φ_i) of M and every chart (V_j, ψ_j) of M' such that

$f(U_i) \subset V_j$, the mapping $\psi_j \circ f \circ \varphi_i^{-1}$ of $\varphi_i(U_i)$ into $\psi_j(V_j)$ is differentiable of class C^k. If $u^1, \ldots, u^n$ is a local coordinate system in U_i and $v^1, \ldots, v^m$ is a local coordinate system in V_j, then f may be expressed by a set of differentiable functions of class C^k:

$$v^1 = f^1(u^1, \ldots, u^n), \ldots, v^m = f^m(u^1, \ldots, u^n).$$

By a *differentiable mapping* or simply, a *mapping*, we shall mean a mapping of class C^∞. A differentiable function of class C^k on M is a mapping of class C^k of M into $\mathbf{R}$. The definition of a *holomorphic* (or *complex analytic*) mapping or function is similar.

By a *differentiable curve* of class C^k in M, we shall mean a differentiable mapping of class C^k of a closed interval $[a, b]$ of $\mathbf{R}$ into M, namely, the restriction of a differentiable mapping of class C^k of an open interval containing $[a, b]$ into M. We shall now define a *tangent vector* (or simply a *vector*) at a point p of M. Let $\mathfrak{F}(p)$ be the algebra of differentiable functions of class C^1 defined in a neighborhood of p. Let $x(t)$ be a curve of class C^1, $a \leqq t \leqq b$, such that $x(t_0) = p$. The vector tangent to the curve $x(t)$ at p is a mapping X: $\mathfrak{F}(p) \rightarrow \mathbf{R}$ defined by

$$Xf = (df(x(t))/dt)_{t_0}.$$

In other words, Xf is the derivative of f in the direction of the curve $x(t)$ at $t = t_0$. The vector X satisfies the following conditions:

(1) X is a linear mapping of $\mathfrak{F}(p)$ into $\mathbf{R}$;

(2) $X(fg) = (Xf)g(p) + f(p)(Xg) \qquad \text{for } f, g \in \mathfrak{F}(p).$

The set of mappings X of $\mathfrak{F}(p)$ into $\mathbf{R}$ satisfying the preceding two conditions forms a real vector space. We shall show that the set of vectors at p is a vector subspace of dimension n, where n is the dimension of M. Let $u^1, \ldots, u^n$ be a local coordinate system in a coordinate neighborhood U of p. For each j, $(\partial/\partial u^j)_p$ is a mapping of $\mathfrak{F}(p)$ into $\mathbf{R}$ which satisfies conditions (1) and (2) above. We shall show that the set of vectors at p is the vector space with basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$. Given any curve $x(t)$ with $p = x(t_0)$, let $u^j = x^j(t)$, $j = 1, \ldots, n$, be its equations in terms of the local coordinate system $u^1, \ldots, u^n$. Then

$$(df(x(t))/dt)_{t_0} = \Sigma_j \, (\partial f/\partial u^j)_p \cdot (dx^j(t)/dt)_{t_0}\, *,$$

* For the summation notation, see Summary of Basic Notations.

which proves that every vector at p is a linear combination of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$. Conversely, given a linear combination $\Sigma\, \xi^j(\partial/\partial u^j)_p$, consider the curve defined by

$$u^j = u^j(p) + \xi^j t, \qquad j = 1, \ldots, n.$$

Then the vector tangent to this curve at $t = 0$ is $\Sigma\, \xi^j(\partial/\partial u^j)_p$. To prove the linear independence of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$, assume $\Sigma\, \xi^j(\partial/\partial u^j)_p = 0$. Then

$$0 = \Sigma\, \xi^j(\partial u^k/\partial u^j)_p = \xi^k \qquad \text{for } k = 1, \ldots, n.$$

This completes the proof of our assertion. The set of tangent vectors at p, denoted by $T_p(M)$ or T_p, is called the *tangent space* of M at p. The n-tuple of numbers $\xi^1, \ldots, \xi^n$ will be called the *components* of the vector $\Sigma\, \xi^j(\partial/\partial u^j)_p$ with respect to the local coordinate system $u^1, \ldots, u^n$.

Remark. It is known that if a manifold M is of class C^∞, then $T_p(M)$ coincides with the space of X: $\mathfrak{F}(p) \to \mathbf{R}$ satisfying conditions (1) and (2) above, where $\mathfrak{F}(p)$ now denotes the algebra of all C^∞ functions around p. From now on we shall consider mainly manifolds of class C^∞ and mappings of class C^∞.

A *vector field* X on a manifold M is an assignment of a vector X_p to each point p of M. If f is a differentiable function on M, then Xf is a function on M defined by $(Xf)(p) = X_p f$. A vector field X is called *differentiable* if Xf is differentiable for every differentiable function f. In terms of a local coordinate system $u^1, \ldots, u^n$, a vector field X may be expressed by $X = \Sigma\, \xi^j(\partial/\partial u^j)$, where ξ^j are functions defined in the coordinate neighborhood, called the *components* of X with respect to $u^1, \ldots, u^n$. X is differentiable if and only if its components ξ^j are differentiable.

Let $\mathfrak{X}(M)$ be the set of all differentiable vector fields on M. It is a real vector space under the natural addition and scalar multiplication. If X and Y are in $\mathfrak{X}(M)$, define the bracket $[X, Y]$ as a mapping from the ring of functions on M into itself by

$$[X, Y]f = X(Yf) - Y(Xf).$$

We shall show that $[X, Y]$ is a vector field. In terms of a local coordinate system $u^1, \ldots, u^n$, we write

$$X = \Sigma\, \xi^j(\partial/\partial u^j), \qquad Y = \Sigma\, \eta^j(\partial/\partial u^j).$$

Then

$$[X, Y]f = \Sigma_{j,k}(\xi^k(\partial\eta^j/\partial u^k) - \eta^k(\partial\xi^j/\partial u^k))(\partial f/\partial u^j).$$

This means that $[X, Y]$ is a vector field whose components with respect to $u^1, \ldots, u^n$ are given by $\Sigma_k(\xi^k(\partial\eta^j/\partial u^k) - \eta^k(\partial\xi^j/\partial u^k))$, $j = 1, \ldots, n$. With respect to this bracket operation, $\mathfrak{X}(M)$ is a Lie algebra over the real number field (of infinite dimensions). In particular, we have Jacobi's identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$
$$\text{for } X, Y, Z \in \mathfrak{X}(M).$$

We may also regard $\mathfrak{X}(M)$ as a module over the algebra $\mathfrak{F}(M)$ of differentiable functions on M as follows. If f is a function and X is a vector field on M, then fX is a vector field on M defined by $(fX)_p = f(p)X_p$ for $p \in M$. Then

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$
$$f, g \in \mathfrak{F}(M), \qquad X, Y \in \mathfrak{X}(M).$$

For a point p of M, the dual vector space $T_p^*(M)$ of the tangent space $T_p(M)$ is called the space of *covectors* at p. An assignment of a covector at each point p is called a *1-form* (*differential form of degree* 1). For each function f on M, the *total differential* $(df)_p$ of f at p is defined by

$$\langle (df)_p, X\rangle = Xf \qquad \text{for } X \in T_p(M),$$

where $\langle\ ,\ \rangle$ denotes the value of the first entry on the second entry as a linear functional on $T_p(M)$. If $u^1, \ldots, u^n$ is a local coordinate system in a neighborhood of p, then the total differentials $(du^1)_p, \ldots, (du^n)_p$ form a basis for $T_p^*(M)$. In fact, they form the dual basis of the basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$ for $T_p(M)$. In a neighborhood of p, every 1-form ω can be uniquely written as

$$\omega = \Sigma_j f_j\, du^j,$$

where f_j are functions defined in the neighborhood of p and are called the *components* of ω with respect to $u^1, \ldots, u^n$. The 1-form ω is called *differentiable* if f_j are differentiable (this condition is independent of the choice of a local coordinate system). We shall only consider differentiable 1-forms.