A8B37SAS: Homework from lecture 7 - 31.3.2020

First of all, I would like to *apologize* for any notational anomalies I'm using that might cause any kind of confusion. Some of the conventions I'm using in the following text are that $\theta(t)$ is the Heaviside step function or $\int_{\mathbb{R}} = \int_{-\infty}^{\infty}$.

1 Laplace and Z transforms of a convolution of functions

Theorem 1.1. *Let $f, g \in L_0$ (meaming that the generally complex functions are at least partially continuous and are of the exponential of lesser order of growth) have Laplace transforms $F(p), G(p)$. Then a convolution of those functions $h(t) \equiv f(t) * g(t)$ has Laplace transform*

$$H(p) \equiv \mathcal{L}[h(t)](p) = F(p)G(p). \tag{1}$$

Proof. Since (for Laplace transform to exist and for Fubini's theorem to work) we consider only measurable functions with value zero in the region of $\mathrm{Re}[f(t)] < 0$, we can write for the Laplace transform of the convolution of two such functions

$$\begin{aligned} H(p) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \theta(\tau) f(\tau) \theta(t-\tau) g(t-\tau) \, \mathrm{d}\tau \right) e^{-pt} \, \mathrm{d}t = \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \theta(\tau) f(\tau) e^{-p\tau} \theta(t-\tau) g(t-\tau) e^{-p(t-\tau)} \, \mathrm{d}t \right) \mathrm{d}\tau. \end{aligned}$$

Furthermore, we can substitute $u \coloneqq t - \tau$, which yields

$$\begin{aligned} H(p) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \theta(\tau) f(\tau) e^{-p\tau} \theta(u) g(u) e^{-pu} \, \mathrm{d}u \right) \mathrm{d}\tau = \\ &= \int_{\mathbb{R}} \theta(\tau) f(\tau) e^{-p\tau} \, \mathrm{d}u \int_{\mathbb{R}} \theta(u) g(u) e^{-pu} \, \mathrm{d}\tau = F(p)G(p). \end{aligned}$$

□

Theorem 1.2. *Let's assume that sequences $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \in Z_0$ (meaning that the generally complex sequences are of the exponential or lesser order of growth) and $(c_n)_{n\in\mathbb{N}_0} = (a_n)_{n\in\mathbb{N}_0} * (b_n)_{n\in\mathbb{N}_0}$. Then is true that*

$$\mathcal{Z}[c_n](z) = \mathcal{Z}[a_n](z)\mathcal{Z}[b_n](z). \tag{2}$$

Proof. We will prove this one directly. Starting from the right, we get

$$\mathcal{Z}[a_n](z)\mathcal{Z}[b_n](z) = \sum_{k\in\mathbb{N}_0} \frac{a_k}{z^k} \sum_{\ell\in\mathbb{N}_0} \frac{b_\ell}{z^\ell} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \frac{1}{z_n} = \mathcal{Z}[c_n](z).$$

□