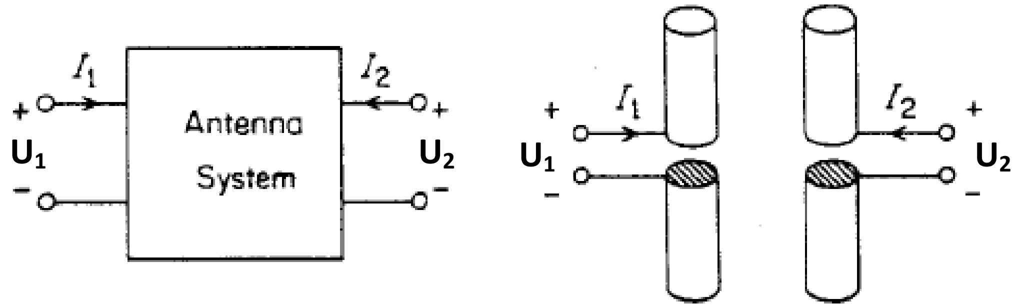
Mutual impedance between two linear antennas



Self impedance

$$Z_{11} = \left.\frac{U_1}{I_1}\right|_{I_2=0}$$

$$Z_{22} = \left.\frac{U_2}{I_2}\right|_{I_1=0}$$

The self impedance is measured with open terminals at the second antenna

Mutual impedance Z_{12}, Z_{21} accounts for electromagnetic coupling between distant antennas and is defined as:

$$Z_{21} = \left.\frac{U_2}{I_1}\right|_{I_2=0}$$

$$Z_{12} = \left.\frac{U_1}{I_2}\right|_{I_1=0}$$

The system of antennas can now be described by matrix equation

$$[Z][I] = [U]$$

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$U_1 = I_1 Z_{11} + I_2 Z_{12}$$

$$U_2 = I_2 Z_{22} + I_1 Z_{21}$$

It there is no mutual coupling ($Z_{12} = Z_{12} = 0$), the

input (driving) impedance at the terminals is:

$$Z_{in1} = \frac{U_1}{I_1} = Z_{1,} \quad Z_{in2} = \frac{U_1}{I_1} = Z_1$$

Evaluation of self/mutual impedances:

Input self impedance (referred to current I_0 in maximum or at the antenna terminals)

Derivation starts with the complex power

$$P_{in} = P_r + \mathrm{j}P_x = -\frac{1}{2}\int_V \boldsymbol{E}\cdot\boldsymbol{J}^* \,\mathrm{d}V,$$

where $\boldsymbol{E}$ is electric field generated by current density $\boldsymbol{J}$ flowing in antenna volume V. Since we are interested both in real and imaginary parts of the power as seen at antenna terminals, we insert

$$\boldsymbol{E} = -\mathrm{j}\omega\boldsymbol{A} - \nabla\varphi = -j\omega\boldsymbol{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{A})$$

After some manipulation and finally using $Z_{in} = 2P_{in}/|I_0|^2$, we obtain self-impedance for general 3D surface current distribution as:

$$Z_{11} = R_r + jX_A = \frac{j30}{k|I_0|^2}\iint\limits_{VV\prime}[\Psi - \Upsilon]\frac{e^{-jkR_{11}}}{R_{11}}dVdV'$$

Where $\Psi = k^2\boldsymbol{J}(\boldsymbol{r})\cdot\boldsymbol{J}^*(\boldsymbol{r}')$ originated from vector potential and $\Upsilon = \nabla\cdot\boldsymbol{J}(\boldsymbol{r})\nabla'\cdot\boldsymbol{J}^*(\boldsymbol{r}')$ originated from scalar potential

If the antenna is linear with assumed current $I(z) = \mathrm{I}_0\sin\left[k\left(\frac{l}{2} - |z|\right)\right]$:

$$Z_{11} = R_r + jX_A = \frac{j30}{k|I_0|^2}\int_{-l/2}^{l/2}\int_{-l/2}^{l/2}\left[k^2 I(z)I^*(z') - \frac{dI(z)}{dz}\frac{dI^*(z')}{dz'}\right]\frac{e^{-jkR_{11}}}{R_{11}}dzdz'$$

Kernel of the integral is the Green's function $\frac{e^{-jkR}}{R}$, the distance between interacting currents is

$R_{11} = \sqrt{(z - z') + a^2}$, where a is radius.

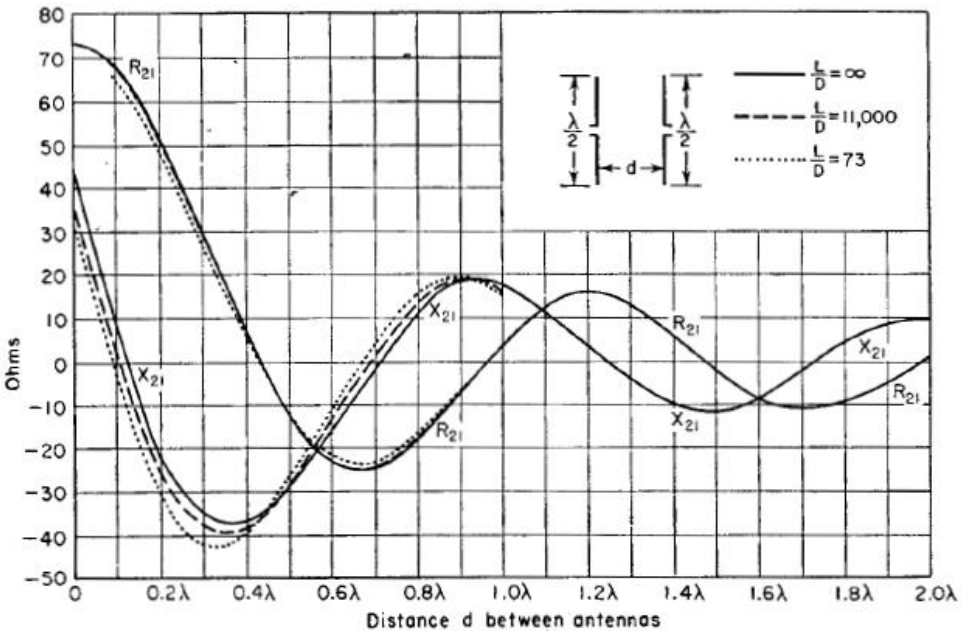
Mutual impedance is calculated in the same way, but the distance R_{12} now points from one antenna to another

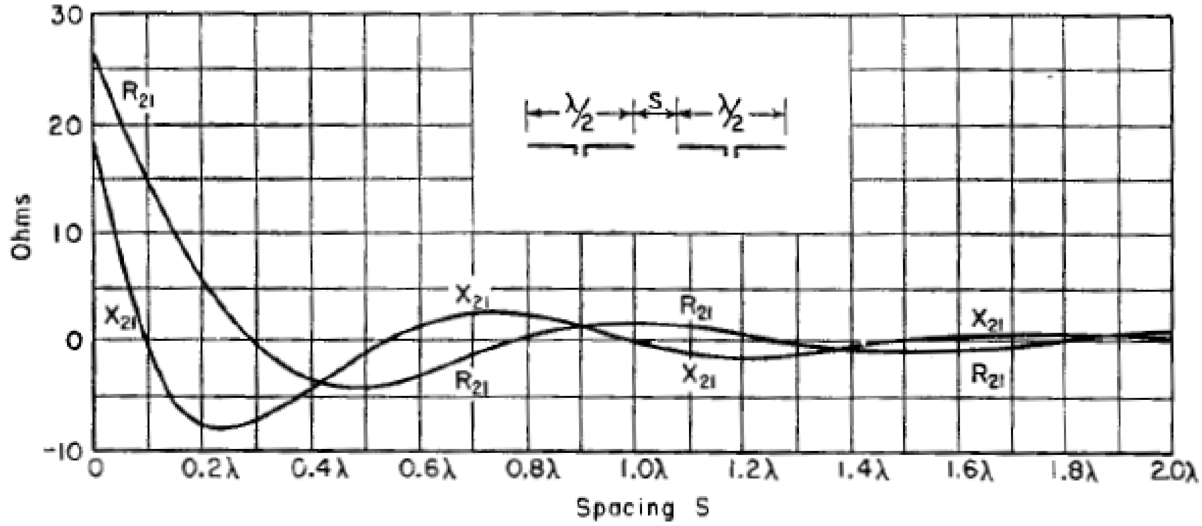
$$Z_{12} = R_{12} + jX_{12} = \frac{j30}{k|I_{10}|^2|I_{20}|^2}\int_{-l/2}^{l/2}\int_{-l/2}^{l/2}\left[k^2 I_1(z)I_2^*(z') - \frac{dI_1(z)}{dz}\frac{dI_2^*(z')}{dz'}\right]\frac{e^{-jkR_{12}}}{R_{12}}dzdz'$$

For case of side-by-side dipoles separated by d: $R_{12} = \sqrt{(z - z')^2 + a^2 + d^2}$

Actual evaluation of self and mutual impedance is implemented in function *imped*, see *help imped*.

Figures below show mutual impedance of $\lambda/2$ dipoles with sinusoidal current assumed

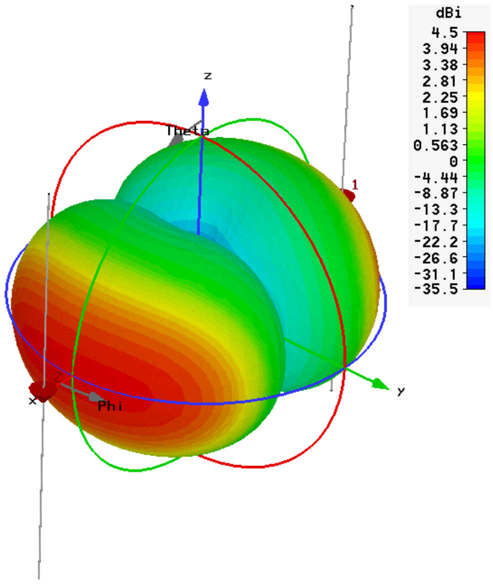

Ohms
Distance d between antennas
R_{21}
X_{21}
$\frac{\lambda}{2}$
d
$\frac{L}{D}=\infty$
$\frac{L}{D}=11,000$
$\frac{L}{D}=73$

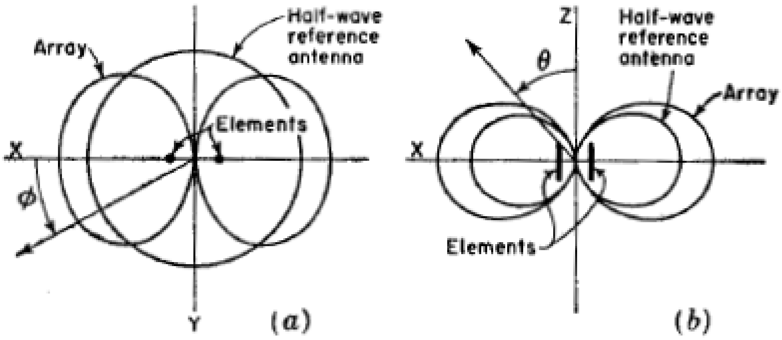

Ohms
Spacing S
R_{21}
X_{21}
$\lambda/2$
S

Example:

Evaluate input impedance Z_1, Z_2 for the array of infinitely thin $L = \lambda/2$ dipoles separated by $d = \lambda/2$. The dipole are fed out-of-phase, thus $I_1 = -I_2$. This array is called end-fire.

See script *Two_dipole_array.m*





The problem is symmetric, so $Z_{11} = Z_{22}$ and $Z_{12} = Z_{12}$

Solving the matrix equation $[Z][I] = [U]$ to get the input impedance $Z_1 = \frac{U_1}{I_1}, Z_2 = \frac{U_2}{I_2}$:

$$U_1 = I_1 Z_{11} + I_2 Z_{12}$$

$$U_2 = I_2 Z_{22} + I_1 Z_{21}$$

$$U_1 = I_1 Z_{11} - I_1 Z_{12} = I_1(Z_{11} - Z_{12})$$

$$U_2 = I_2 Z_{22} - I_2 Z_{21} = I_2(Z_{22} - Z_{21})$$

$$Z_1 = \frac{U_1}{I_1} = Z_2 = \frac{U_2}{I_2} = (Z_{11} - Z_{12})$$

using *imped* we get:

Z_{11}= imped(0.5,0) = 73.0790 +42.5151i Ω

Z_{12}= imped(0.5,0.5,0.5) = -12.5234 -29.9079i Ω

The input impedance is

$$Z_{in} = (Z_{11} - Z_{12}) = 73.0790 + 42.5151\text{i} - (-12.5234 - 29.9079\text{i}) = 85.6024 + 72.4231i$$

Currents at elements for constant input power

Let the total input (real) power to the array be *P*. Assuming no heat losses, the power P_1 in element 1 and power P_2 in element 2 are

$$P_1 = \frac{1}{2} I_1^2 (R_{11} - R_{12})$$

$$P_2 = \frac{1}{2} I_2^2 (R_{22} - R_{12})$$

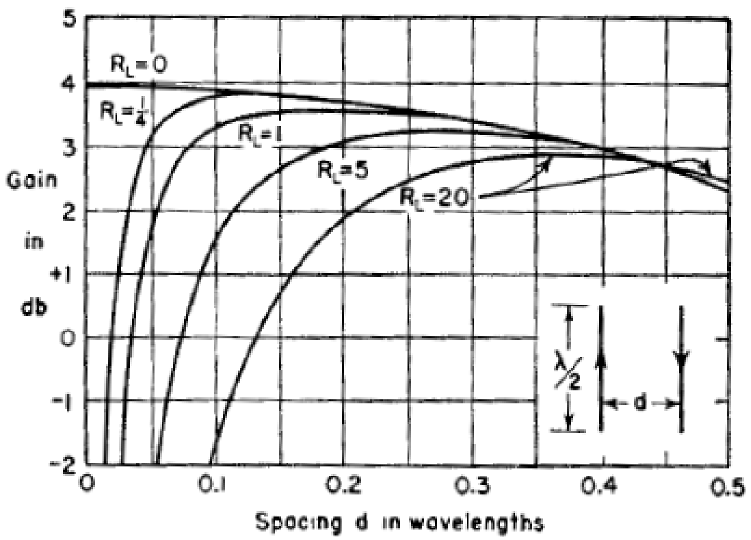
However $R_{11} = R_{22}$ and $I_1 = -I_2$ so the total input power is

$$P = P_1 + P_2 = I_1^2 (R_{11} - R_{12})$$

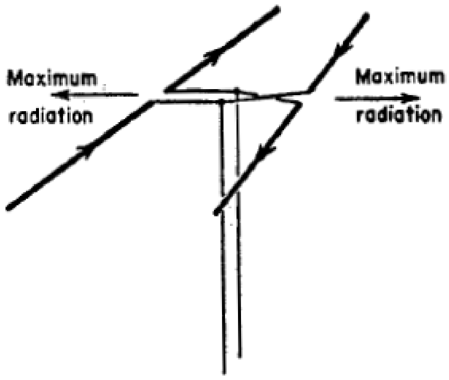
The amplitude of current at each element for input power *P* is function of distance because

$$I_1(d) = \sqrt{\frac{P}{R_{11} - R_{12}(d)}}$$

For closely spaced arrays ($d < 0.5\lambda$) the efficiency is issue due to high amplitude of currents. Below is shown gain in dBd for array with and without looses, significant improvement compared to one dipole is obtained.



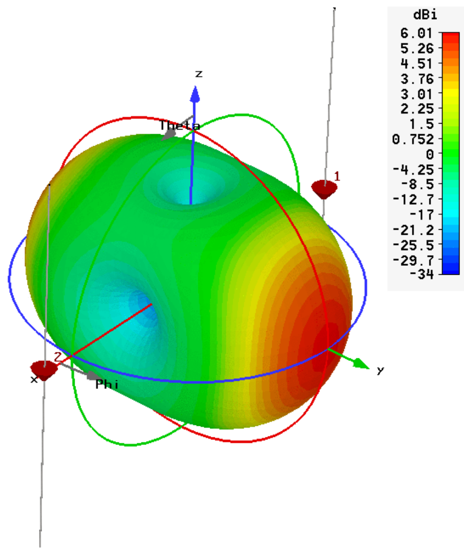
This simple array with out-of-phase closely spaced currents (*d* is usually between $0.25\lambda - 0.5\lambda$) has been invited by Kraus and is called the 8WJK



Similarly, considering now in-phase currents $I_1 = I_2$, the input impedance is

$$Z_1 = \frac{U_1}{I_1} = Z_2 = \frac{U_2}{I_2} = (Z_{11} + Z_{12}) = 60.5556 + 12.6072i$$

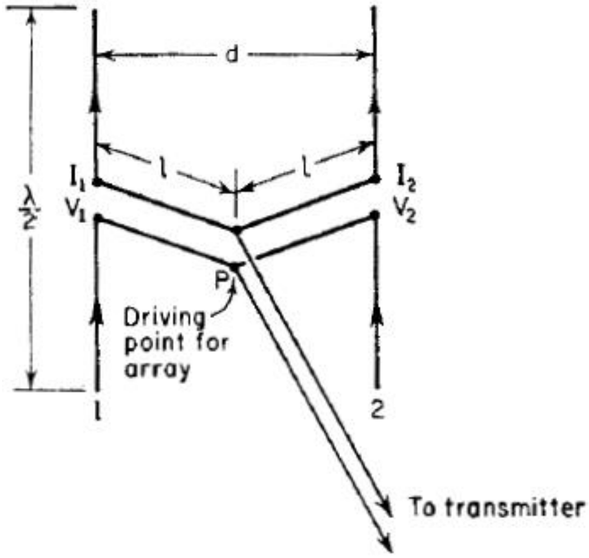
Radiation pattern of array of two in-phase $\lambda/2$ dipoles separated by $d = \lambda/2$. This array is called broadside.



Example of exciting such array by a 600Ω line

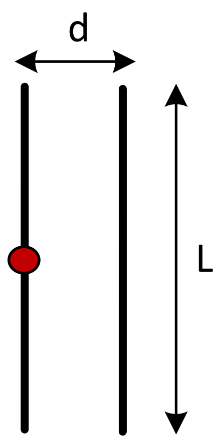
The input driving impedance at the terminals of both dipoles is $60.5 + 12.6i\ \Omega$ (imaginary part can eliminated by slight reducing length of dipoles), so let's take $Z_1 = Z_2 = 60\Omega$.

At point P (see fig below) 600Ω is needed, thus we have to connect two 1200Ω lines in parallel. This requires $\lambda/4$ transformer, e.g. line with characteristic impedance $\mathrm{Z}_0 = \sqrt{1200 \times 60} = 269\Omega$



Example

Consider two $L = \lambda/2$ dipoles (one active, one parasitic) with radius $a = \lambda/100$. The separation distance is $d = \lambda/10$, see figure below. The (active) dipole at left is excited by 100V, the right one is short-circuited (parasitic). Determine input impedance and current at the middle of dipoles. Sinusoidal current distribution is assumed.



Matrix equation is set-up:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

and is solved for unknown current amplitudes

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

where $U_1 = 100$, $U_2 = 0$.

Impedance matrix entries are evaluated using imped function

$$Z_{11} = Z_{22} = imped(0.5,1/100) = 73.0198 \ + 38.7675\mathrm{i}$$

$$Z_{12} = Z_{21} = imped(0.5,0.5,0.1) = 67.2870 \ + \ 7.5326i$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 73.0198 \ + 38.7675\mathrm{i} & 67.2870 \ + \ 7.5326i \\ 67.2870 \ + \ 7.5326i & 73.0198 \ + 38.7675\mathrm{i} \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

→

$$I_1 = 0.6056 \ - \ 1.6547i$$

$$I_2 = 0.0371 \ + \ 1.4426i$$

Input driving impedance is

$$Z_{in} = \frac{U_1}{I_1} = \frac{100}{0.6056 \ - \ 1.6547i} = 19.5062 \ + 53.2964i$$