Czech Technical University in Prague,
Faculty of Electrical Engineering

Master's thesis

Dual Circularly Polarized Waveguide Antenna

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MASTER'S THESIS ASSIGNMENT

I. Personal and study details

Student's name:	**Šimák Martin**
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Study program:	**Electronics and Communications**
Specialisation:	**Radio Communications and Systems**

II. Master's thesis details

Master's thesis title in English:

Dual Circularly Polarized Waveguide Antenna

Master's thesis title in Czech:

Duálně kruhově polarizovaná vlnovodová anténa

Guidelines:

Research different polarizers in a metallic waveguide, consider round or square transversal shape of the guide. Inspired by circularly polarized patch antenna with chamfered corners, adapt this technique to the waveguide technology (the frequency should be used to be appropriate for easy fabrication, say around 5 GHz). Perform 2D eigenmode analysis of round and square waveguides with inserted metal triangles, and compare the results with the theory of patch antennas of such shapes. Choose one of the waveguide with the polarizer and optimize it to provide the best bandwidth and radiation properties, also design the transition from coaxial cable, preferably with the ability to excite both RHCP and LHCP patterns. Add a small horn (say 15 dBi) to the waveguide and finally optimise the whole structure. Build and measure the whole structure, compare to simulation results.

Bibliography / sources:

1/ Polarizers on sections of square waveguides with inner corner ridges | IEEE Conference Publication | IEEE Xplore
2/ Compact reconfigurable waveguide circular polarizer | IEEE Conference Publication | IEEE Xplore
3/ Design of Wideband Quad-Ridge Waveguide Polarizer | IEEE Conference Publication | IEEE Xplore
4/ Optimum-Iris-Set Concept for Waveguide Polarizers | IEEE Journals & Magazine | IEEE Xplore
5/ Novel square/rectangle waveguide septum polarizer | IEEE Conference Publication | IEEE Xplore
6/ Broadband Septum Polarizer With Triangular Common Port | IEEE Journals & Magazine | IEEE Xplore
7/ New Tunable Iris-Post Square Waveguide Polarizers for Satellite Information Systems | IEEE Conference Publication | IEEE Xplore
8/ Hexagonal waveguides: New class of waveguides for mm-wave circulaly polarized horns | IEEE Conference Publication | IEEE Xplore
9/ Hexagonal Waveguide Based Circularly Polarized Horn Antennas for Sub-mm-Wave/Terahertz Band | IEEE Journals & Magazine | IEEE Xplore
Bow-Tie-Shaped Radiating Element for Single and Dual Circular 10/ Polarization | IEEE Journals & Magazine | IEEE Xplore
11/ A Wideband Circularly Polarized Horn Antenna With a Tapered Elliptical Waveguide Polarizer | IEEE Journals & Magazine | IEEE Xplore

Name and workplace of master's thesis supervisor:

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Date of master's thesis assignment: **04.09.2024** Deadline for master's thesis submission: ______

Assignment valid until: **15.02.2026**

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Supervisor's signature

Head of department's signature

prof. Mgr. Petr Páta, Ph.D.
Dean's signature

III. Assignment receipt

The student acknowledges that the master's thesis is an individual work. The student must produce his thesis without the assistance of others, with the exception of provided consultations. Within the master's thesis, the author must state the names of consultants and include a list of references.

Date of assignment receipt

Student's signature

Contents

Introduction

Do not write a couple of words on literature survey. Present the difference construction and design choices, compare them both theoretically and by the results presented in gathered papers, but in their respective sections of the design process. Use them a nice foreword for each of the parts, going through existing approaches.

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Throughout the theoretical chapter, I omit using the common *del*, or *nabla*, notation for the vector differential operator ∇ which gives rise to the formally proper differential operators of gradient (∇), divergence ($\nabla\cdot$), curl ($\nabla\times$), and sometimes even the Laplace operator ($\nabla \cdot \nabla$ or ∇^2). Instead, I will use the standard notations of grad, div, curl, and Δ, respectively. While I am aware of the mnemonic merits it brings when working in coordinates, this does not come to fruition as I do not carry out any computations in this text. On the other hand, there are various reasons to avoid it, such as that it promotes a notational ambiguity with the covariant derivative used in differential geometry, or to distinguish the individual operators at first sight better.

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Chapter 1

Electrodynamics of guided waves

This chapter establishes the theoretical foundation for the analysis and design of a waveguide-based antenna system. Beginning with Maxwell's equations and general description of electromagnetic fields in various settings relevant to this work, the wave equations governing guided modes are derived, and their solutions are analysed to elucidate the behaviour of electromagnetic fields within waveguides. This analysis provides a framework for understanding the operation of structures designed in the following chapters. While focusing on the essential elements of waveguide theory, this chapter provides a comprehensive treatment of the subject and establishes the notation used throughout this work.

The exposition endeavours to build upon the foundations laid in [1], [2], incorporating personal insights and notational preferences to present a cohesive foundation for further chapters.

1.1 Fundamentals of electrodynamics

In this section, the fundamental principles of electromagnetism are presented, starting with Maxwell's equations, which encapsulate the relationships between electric and magnetic fields and their sources. The response of different materials to these fields is then explored through the introduction of *constitutive relations*. Finally, the focus is shifted to the behaviour of electromagnetic fields at material boundaries, providing essential *boundary conditions* for solving electromagnetic problems.

1.1.1 Maxwell's equations

The differential form of Maxwell's equations, as presented below, constitutes the cornerstone of classical electromagnetism, providing a complete[1] framework for analysing electromagnetic phenomena at any point in space and time. These equations summarize the relations between *electric field intensity* $\boldsymbol{E}$ and *magnetic flux density* $\boldsymbol{B}$ and their sources due to charge densities ρ, current densities $\boldsymbol{J}$, or the changing of the fields themselves. To ensure the validity of these expressions, let us assume that the field vectors are well-behaved functions, exhibiting continuity and possessing continuous derivatives. This assumption holds for most electromagnetic systems, with exceptions

[1] For actual completeness (save for some special properties stemming from interactions in matter), the equations must also be supplemented by the Lorentz's force law $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.

arising at interfaces between distinct media where abrupt changes in charge and current densities may occur.

$$\operatorname{div}\boldsymbol{E} = \frac{1}{\epsilon_0}\rho_{\mathrm{e}}, \tag{1.1a}$$
$$\operatorname{curl}\boldsymbol{E} = -\mu_0\boldsymbol{J}_{\mathrm{m}} - \partial_t\boldsymbol{B}, \tag{1.1b}$$
$$\operatorname{div}\boldsymbol{B} = \mu_0\rho_{\mathrm{m}}, \tag{1.1c}$$
$$\operatorname{curl}\boldsymbol{B} = \mu_0\boldsymbol{J}_{\mathrm{e}} + \mu_0\epsilon_0\partial_t\boldsymbol{E}, \tag{1.1d}$$

There is one oddity about equations (1.1) and that is the inclusion of the magnetic charge density ρ_{m} and magnetic current density $\boldsymbol{J}_{\mathrm{m}}$ which are part of the 'generalized concept'. Although these quantities, in spite of diligent search, were never physically observed, their introduction establishes a pleasing balance in Maxwell's equations while being theoretically sound as well. This concept is further utilized when solving advanced physical problems in applied physics and engineering. This is facilitated by the introduction of equivalent magnetic charge and current which can be used to conveniently express fields as if generated by these fictitious sources, especially in problems where the exact form of the electromagnetic field would otherwise be complicated to elucidate.

Mathematically, equations (1.1), like any differential equations, form a complete problem only when supplemented with suitable boundary conditions in a more traditional sense, such as behaviour of the vector fields 'in infinity'. These are typically 'obvious' from the problem-solving context, e.g., fields vanishing at large distance from localized charge distribution, etc.

1.1.2 Electromagnetic properties of matter

Although Maxwell's equations in their fundamental form (1.1) provide a complete description of electromagnetic phenomena, an alternative formulation offers a more convenient approach for analysing materials susceptible to electric and magnetic polarization. Within such media, the total electric charge density ρ_{e} can be expressed as a sum of the *free charge* density ρ_{f}, which constitutes the *actual source* charge,[2] and the *bound charge* $\rho_{\mathrm{b}} = -\operatorname{div}\boldsymbol{P}$, produced by an electric polarization $\boldsymbol{P}$ of the material. Moreover, changing electric fields also induce changing polarization, producing *polarization current* $\boldsymbol{J}_{\mathrm{p}} = \partial_t\boldsymbol{P}$ which add to the *free current* $\boldsymbol{J}_{\mathrm{f}}$. Similarly to electric polarization, a magnetic polarization $\boldsymbol{M}$ results in a bound current $\boldsymbol{J}_{\mathrm{b}} = \operatorname{curl}\boldsymbol{M}$. These effects, inherently connected to the susceptibility of materials to be polarized, hence influence the total electromagnetic field in their vicinity. This led to the introduction of convenient field quantities that account for the presence of such media.

Within the framework of Maxwell's equations, *electric flux density* $\boldsymbol{D}$ (also called the *electric displacement field*) and the *magnetic field intensity* $\boldsymbol{H}$ offer a more convenient representation, explicitly separating the free and bound sources. This approach allows for expressing Maxwell's equations in a form that directly relates the fields to the free charge and free current, which are sources that can be controlled directly. Using these field quantities, equations (1.1) read

$$\operatorname{div}\boldsymbol{D} = \rho_{\mathrm{f}}, \tag{1.2a}$$
$$\operatorname{curl}\boldsymbol{E} = -\boldsymbol{J}_{\mathrm{m}} - \partial_t\boldsymbol{B}, \tag{1.2b}$$
$$\operatorname{div}\boldsymbol{B} = \mu_0\rho_{\mathrm{m}}, \tag{1.2c}$$
$$\operatorname{curl}\boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}} + \partial_t\boldsymbol{D}, \tag{1.2d}$$

While equations (1.2) effectively express electromagnetic laws within media, their

[2] It is important to reinforce the idea that the magnetic charge and current are fictitious 'source' quantities. Therefore, they are already, by definition, purely *free* quantities.

hybrid notation, involving both $\boldsymbol{E}$ and $\boldsymbol{D}$, and both $\boldsymbol{B}$ and $\boldsymbol{H}$, necessitates the use of *constitutive relations*. These relations, which establish correspondence between the respective electric and magnetic field quantities, are material-dependent and reflect the specific response of a medium to electric and magnetic fields. In general, these relationships can be expressed as

$$\boldsymbol{D} = \hat{\epsilon} * \boldsymbol{E}, \tag{1.3a}$$
$$\boldsymbol{B} = \hat{\mu} * \boldsymbol{H}, \tag{1.3b}$$

where $\hat{\epsilon}$ and $\hat{\mu}$ are the material's *permittivity* and *permeability*, respectively, and the asterisk denotes *convolution*.

Remark 1.1.1. In formulations of akin to equations (1.2), with emphasis on the separation of free and bound sources, some authors prefer to further dissect the free current, too. This current is generally conceptualized as the current 'not directly tied to the bound charges' within a material. To name a few commonly recognized, *convection current*, *beam current*, or *conduction current*. It is the conduction current which is, in electrical engineering, especially worth mentioning because it arises from the movement of charges, typically electrons, that can move freely throughout the material. This kinetic energy of charges in conductors is the main cause of losses in waveguides and can be expressed by

$$\boldsymbol{J}_{\mathrm{c}} = \hat{\sigma} * \boldsymbol{E}, \tag{1.4}$$

where $\hat{\sigma}$ is the material's *conductivity*. Equation (1.4), together with equations (1.3), completes the required set of constitutive relations.

The *constitutive parameters* $\hat{\epsilon}$, $\hat{\mu}$, and $\hat{\sigma}$, generally represented as complex second-rank tensors, establish the relationship between the applied electromagnetic fields and the material's response. The functional dependencies of these tensors provide a classification scheme for material properties:

- *Linearity:* A material is classified as linear if its constitutive parameters are independent of the applied field strength; otherwise, it is considered nonlinear.
- *Homogeneity:* If the constitutive parameters are invariant with respect to position within the material, it is deemed homogeneous; conversely, spatial dependence indicates an inhomogeneous medium.
- *Isotropy:* Materials exhibiting constitutive parameters independent of the applied field's direction are classified as isotropic. Conversely, direction-dependent parameters signify an anisotropic material, with crystals being a prime example.
- *Dispersion:* Materials whose constitutive parameters exhibit frequency dependence are categorized as dispersive. While some materials demonstrate negligible frequency dependence and can be effectively considered nondispersive, all materials encountered in practice exhibit some degree of dispersion.

Example 1.1.2 [Constitutive relations in free space]**.** In the simplest case of free space, equations (1.3a), (1.3b), and (1.4) become

$$\hat{\epsilon} = \epsilon_0 \approx 8.854 \times 10^{-12}\ \mathrm{F\,m^{-1}}, \tag{1.5a}$$
$$\hat{\mu} = \mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}, \tag{1.5b}$$
$$\hat{\sigma} = \sigma_0 = 0\ \mathrm{S\,m^{-1}}. \tag{1.5c}$$

1.1.3 Boundary conditions

While the differential forms of Maxwell's equations are powerful tools for analysing electromagnetic fields within continuous media, material boundaries introduce discontinuities that require special treatment. These discontinuities in the fields E, B, D, and H arise at interfaces between media with different electrical properties or at surfaces carrying charge or current densities. To accurately describe the behaviour of the fields across such boundaries, Maxwell's equations in their integral form, which naturally incorporate these discontinuities, are more convenient. This form is obtained by applying integral theorems from vector calculus to equations (1.2) which then take on the form of

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} = Q_\mathrm{e}, \tag{1.6a}$$

$$\oint_{\partial S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_S \boldsymbol{J}_\mathrm{m} \cdot \mathrm{d}\boldsymbol{a} - \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}, \tag{1.6b}$$

$$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = Q_\mathrm{m}, \tag{1.6c}$$

$$\oint_{\partial S} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{J}_\mathrm{e} \cdot \mathrm{d}\boldsymbol{a} + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a}. \tag{1.6d}$$

where S is any closed surface.

Consider a boundary between two different media. The first medium is characterized by permittivity ϵ_1 and permeability μ_1, while the second medium is characterized by permittivity ϵ_2 and permeability μ_2. At this interface, electric and magnetic surface charge densities, denoted by q_f and q_m respectively, may be present. Additionally, electric and magnetic surface current densities, denoted by j_f and j_m respectively, may also exist. The general *boundary conditions* for electrodynamics are then obtained by applying equations equations (1.6) to arbitrary surfaces encompassing a portion of the interface, yielding

$$\boldsymbol{e}_n \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = q_\mathrm{f}, \tag{1.7a}$$

$$-\boldsymbol{e}_n \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = \boldsymbol{j}_\mathrm{m}, \tag{1.7b}$$

$$\boldsymbol{e}_n \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = q_\mathrm{m}, \tag{1.7c}$$

$$\boldsymbol{e}_n \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{j}_\mathrm{f}. \tag{1.7d}$$

1.2 Electromagnetic waves

Having established the foundations of electromagnetism, the focus is now shifted to one of its most significant consequences: the existence of electromagnetic waves. In this section, the manner in which Maxwell's equations predict the propagation of these waves is explored. The wave equations for the electric and magnetic fields are derived, revealing their interconnected nature and their ability to sustain each other even in the absence of sources. Subsequently, the simplest and most fundamental solutions to these equations, *monochromatic plane waves*, are investigated. Finally, the confinement and guidance of these plane waves within conducting cavities is examined, laying the groundwork for understanding waveguides and resonant structures.

1.2.1 The wave equations

Maxwell's equations provide a comprehensive description of electromagnetic phenomena, but their coupled nature can make them challenging to solve directly. To facilitate analysis, particularly in source-free regions, it's often advantageous to decouple the equations and express them in terms of the electric and magnetic fields individually. Inside regions with no *free* charge or *free* current, Maxwell's equations (1.2) take on the form of

$$\operatorname{div} \boldsymbol{D} = 0, \tag{1.8a}$$

$$\operatorname{curl} \boldsymbol{E} = -\partial_t \boldsymbol{B}, \tag{1.8b}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{1.8c}$$

$$\operatorname{curl} \boldsymbol{H} = \sigma \boldsymbol{E} + \partial_t \boldsymbol{D}. \tag{1.8d}$$

Furthermore, if the medium is *linear* and *homogeneous*, equation (1.8d) can be fully expressed in terms of $\boldsymbol{E}$. With this simplification, applying the curl to equations (1.8b) and (1.8d) yields

$$\Delta \boldsymbol{E} = \mu\sigma\partial_t \boldsymbol{E} + \mu\epsilon\partial_t^2 \boldsymbol{E}, \tag{1.9a}$$

$$\Delta \boldsymbol{B} = \mu\sigma\partial_t \boldsymbol{B} + \mu\epsilon\partial_t^2 \boldsymbol{B}. \tag{1.9b}$$

Therefore, electric and magnetic fields in linear homogeneous media both clearly satisfy the wave equation with a linear damping term $\mu\sigma\partial_t$, introduced by conductive losses. Moreover, in regions with $\sigma = 0$, such as free space or ideal insulators, equations (1.9) simplify even more to

$$\Delta \boldsymbol{E} = \mu\epsilon\partial_t^2 \boldsymbol{E}, \tag{1.10a}$$

$$\Delta \boldsymbol{B} = \mu\epsilon\partial_t^2 \boldsymbol{B}, \tag{1.10b}$$

taking on the form of classical wave equations which are ubiquitous in physics. This also immediately gives rise to the formula

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{\epsilon_r \mu_r}} \tag{1.11}$$

for the speed of electromagnetic waves in linear homogeneous media.

Remark 1.2.1. Compared with the original Maxwell's equations (1.2), these equations form two systems of partial differential equations of second order but are now decoupled and provide us with an additional solving method for given boundary-value problems. However, it is important to note that the wave equations (1.9a) and (1.9b) were derived from Maxwell's equations (1.8) by differentiation. This impedes their mathematical equivalence. More specifically (as stated in [2]), whereas every solution to Maxwell's equations is also a solution for the wave equations, the converse is not true.

1.2.2 Monochromatic plane waves

The electromagnetic theory presented thus far describes general vector fields that vary in space and time. However, as shown in section 1.2, electromagnetic fields in source-free regions exhibit wave behaviour. Nonetheless, these time-varying vector fields remain complex and challenging to analyse in practical systems. Consider the elementary solution to the wave equation

$$\hat{\boldsymbol{\psi}}(\boldsymbol{r}, t) = \hat{\boldsymbol{\Psi}}_0 \exp\left[\mathrm{i}\left(\hat{\boldsymbol{k}} \cdot \boldsymbol{r} - \omega t\right)\right]. \tag{1.12}$$

Here, $\hat{\boldsymbol{k}}$ is the complex *wave vector* indicating the direction of wave propagation, and ω is the angular frequency of the wave. Equation (1.12) is expressed in terms of a *complex* *wave function* with a *complex amplitude* $\hat{\mathbf{\Psi}}_0 \equiv \mathbf{\Psi}_0 e^{i\varphi}$. This quantity encapsulates both the *real amplitude* $\mathbf{\Phi}_0$ and the *phase shift* φ, of the physical wave. A sinusoidal wave representing this solution in physical reality can be extracted from equation (1.12) using the *Euler's formula*, yielding

$$\boldsymbol{\psi}(\boldsymbol{r}, t) = \mathrm{Re}\left[\hat{\mathbf{\Psi}}_0 \exp\left[\mathrm{i}\left(\hat{\boldsymbol{k}} \cdot \boldsymbol{r} - \omega t + \varphi\right)\right]\right] = \mathrm{Re}\left[\hat{\boldsymbol{\psi}}(\boldsymbol{r}, t)\right]. \tag{1.13}$$

Remark 1.2.2. Clearly, if equation (1.13) satisfies equations (1.10) and Maxwell's equations, the same holds true for equation (1.12), as the imaginary part differs from the real part only by the replacement of sine with cosine.

Equation (1.12) serves as as an established elementary solution to the general wave equation, and hence also to equations (1.10). Substituting this solution into equations (1.9), it becomes evident that these 'lossy wave equations' also admit plane-wave solutions. Furthermore, this substitution allows for the derivation of a general formula for the complex *wave number*

$$\hat{k}^2 = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = \mu\epsilon\omega^2 + \mathrm{i}\mu\sigma\omega. \tag{1.14}$$

In the context of equation (1.12), it is evident that the real part of the complex wave number $\hat{k}$ is the *actual* wave number as it determines the change of phase with spatial propagation. For this reason, the real part is simply denoted k and is called the *phase constant*. In contrast, the imaginary part of $\hat{k}$ is responsible for the exponential damping, or attenuation, in conductive media, and hence is called the *attenuation constant*.

Waves described by equation (1.12) are called *monochromatic*, or *time-harmonic*, *plane* waves. Monochromaticity refers to the fact that the wave oscillates at a single frequency ω through time, while planarity indicates that the fields are uniform over every plane perpendicular to the direction of propagation. Although less common, plane waves could alternatively be called *space-harmonic*,[3] as both of these terms signify a sinusoidal dependence on a given variable. In the case of monochromaticity, the variable is time, oscillating with an angular frequency ω. Similarly, planarity reflects the waveform repetition in the spatial coordinates, projected into the propagation direction, with a well-defined spatial frequency k.

The significance of this particular solution stems from the fact that, in practice, any wave we will be dealing with can be expressed as a linear combination of these monochromatic plane waves, i.e.,

$$\hat{\boldsymbol{\psi}}(\boldsymbol{r}, t) = \int_{\mathbb{R}^3} \hat{\mathbf{\Psi}}_0(\boldsymbol{k}) \exp\left[\mathrm{i}\left(\hat{\boldsymbol{k}} \cdot \boldsymbol{r} - \omega t\right)\right] \,\mathrm{d}\boldsymbol{k}. \tag{1.15}$$

This superposition principle mathematically reflects the Fourier transform over every plane wave corresponding to a given frequency ω. With this formally sound mathematical description, the existence of a unique linear combination for 'any wave we will be dealing with', as vaguely stated above, can be rigorously established through the following theorem.

Theorem 1.2.3 [Dirichlet-Jordan test]. *Let f be a function in $L^1(-\infty, \infty)$ and of bounded variation in a neighbourhood of the point x. Then*

$$\frac{1}{\pi} \lim_{M\to\infty} \int_0^M \mathrm{d}u \int_{\mathbb{R}} f(t) \cos(u(x-t))\,\mathrm{d}t = \lim_{\epsilon\to 0} \frac{f(x+\epsilon) + f(x-\epsilon)}{2}. \tag{1.16}$$

If f is continuous in an open interval, then the integral on the left-hand side converges uniformly in the interval, and the limit on the right-hand side is $f(x)$.

[3] Therefore, monochromatic plane waves are something one could call *spacetime-harmonic* or simply *harmonic*.

More details on this mathematical theory can be found in [3]. A version of theorem 1.2.3, retaining the original form due to Dirichlet, is often used in signal processing. More details on that formulation can be found, e.g., in [4].

Since any physically realizable signal is square-integrable and has compact support,[4] this text's attention is confined to monochromatic plane waves. Therefore, the fields take on the form of

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = \hat{\boldsymbol{E}}_0 \exp\left[\mathrm{i}\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{r} - \omega t\right)\right], \tag{1.17a}$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \hat{\boldsymbol{B}}_0 \exp\left[\mathrm{i}\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{r} - \omega t\right)\right], \tag{1.17b}$$

where $\hat{\boldsymbol{E}}_0$ and $\hat{\boldsymbol{B}}_0$ are complex amplitudes.

As discussed in remark 1.2.1, satisfying the wave equations does not guarantee solutions to Maxwell's equations. Substituting the solutions of the wave equations into Maxwell's equations is necessary, as it might refine the solutions or yield more information. As an example, consider the plane waves in vacuum.

Example 1.2.4 [Monochromatic plane waves in free space]**.** In free space, from equations (1.8a) and (1.8c) it follows that

$$\hat{\boldsymbol{E}}_0 \cdot \boldsymbol{e}_z = \hat{\boldsymbol{B}}_0 \cdot \boldsymbol{e}_z = 0, \tag{1.18}$$

i.e., the electromagnetic fields are *transverse*. Furthermore, either of equations (1.8b) and (1.8d) yields

$$\hat{\boldsymbol{B}}_0 = \frac{k}{\omega}\left(\boldsymbol{e}_z \times \hat{\boldsymbol{E}}_0\right) = \frac{1}{c}\left(\boldsymbol{e}_z \times \hat{\boldsymbol{E}}_0\right). \tag{1.19}$$

Drawing a conclusion, in free space, $\boldsymbol{E}$ and $\boldsymbol{B}$ are *in phase* and *mutually perpendicular*. Moreover, defining the *polarization vector* as a unit vector in the direction of electric field oscillations, i.e.,

$$\boldsymbol{e}_n \cdot \boldsymbol{E} = \boldsymbol{E} \qquad\qquad \|\boldsymbol{e}_n\| = 1, \tag{1.20}$$

the actual solution to Maxwell's equations in free space takes the form of

$$\boldsymbol{E}(\boldsymbol{r},t) = E_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \varphi)\boldsymbol{e}_n, \tag{1.21}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c} E_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \varphi)(\boldsymbol{k} \times \boldsymbol{e}_n). \tag{1.22}$$

As will soon become evident, electromagnetic waves confined in waveguides are generally not transverse. The result we have just obtained is unique to lossless regions, i.e., regions with $\sigma = 0$, where the solution is not subject to boundary conditions.

1.2.3 Guided waves

Assumptions:

- Materials: PEC with no surface sources, i.e., according to equations (1.7)

$$\boldsymbol{e}_n \times \boldsymbol{E} = 0, \tag{1.23a}$$

$$\boldsymbol{e}_n \cdot \boldsymbol{B} = 0. \tag{1.23b}$$

  Free charges and currents will be induced on the surface in such a way as to enforce these constraints.

[4]In the field of signal processing, these signal properties are often described as having *finite energy* and *duration*, respectively.

- Waves: Monochromatic waves that propagate down the tube, so $\boldsymbol{E}$ and $\boldsymbol{B}$ have the generic form (WLOS: $\hat{\boldsymbol{k}} \cdot \boldsymbol{e}_z = \hat{\boldsymbol{k}}$)

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = \hat{\boldsymbol{E}}_0(x, y) \exp\left[\mathrm{i}\left(\hat{k}z - \omega t\right)\right], \tag{1.24a}$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r}, t) = \hat{\boldsymbol{B}}_0(x, y) \exp\left[\mathrm{i}\left(\hat{k}z - \omega t\right)\right]. \tag{1.24b}$$

$$\hat{\boldsymbol{E}}_0(x, y) = E_x(x, y)\boldsymbol{e}_x + E_y(x, y)\boldsymbol{e}_y + E_z(x, y)\boldsymbol{e}_z, \tag{1.25a}$$

$$\hat{\boldsymbol{B}}_0(x, y) = B_x(x, y)\boldsymbol{e}_x + B_y(x, y)\boldsymbol{e}_y + B_z(x, y)\boldsymbol{e}_z. \tag{1.25b}$$

- Maxwell's equations: sourceless, i.e., equations (1.8) which we will express in terms $\boldsymbol{E}$ and $\boldsymbol{B}$ in this sections due to the assumed linearity.

From equations (1.8b) and (1.8d) for a linear medium

$$\partial_x E_y - \partial_y E_x = \mathrm{i}\omega B_z \tag{1.26a}$$

$$\partial_y E_z - \mathrm{i}kE_y = \mathrm{i}\omega B_x \tag{1.26b}$$

$$\mathrm{i}\hat{k}E_x - \partial_x E_z = \mathrm{i}\omega B_y \tag{1.26c}$$

$$\partial_x B_y - \partial_y B_x = -\frac{\mathrm{i}\omega}{c^2}E_z \tag{1.26d}$$

$$\partial_y B_z - \mathrm{i}\hat{k}B_y = -\frac{\mathrm{i}\omega}{c^2}E_x \tag{1.26e}$$

$$\mathrm{i}\hat{k}B_x - \partial_x B_z = -\frac{\mathrm{i}\omega}{c^2}E_y \tag{1.26f}$$

Equations b, c, e, and f can be solved for E_x, E_y, B_x, and B_y:

$$E_x = \frac{\mathrm{i}}{(\omega/c)^2 - \hat{k}^2}\left(\hat{k}\partial_x E_z + \omega\partial_y B_z\right), \tag{1.27a}$$

$$E_y = \frac{\mathrm{i}}{(\omega/c)^2 - \hat{k}^2}\left(\hat{k}\partial_y E_z - \omega\partial_x B_z\right), \tag{1.27b}$$

$$B_x = \frac{\mathrm{i}}{(\omega/c)^2 - \hat{k}^2}\left(\hat{k}\partial_x B_z - \frac{\omega}{c^2}\partial_y E_z\right), \tag{1.27c}$$

$$B_y = \frac{\mathrm{i}}{(\omega/c)^2 - \hat{k}^2}\left(\hat{k}\partial_y B_z + \frac{\omega}{c^2}\partial_x E_z\right). \tag{1.27d}$$

Substituting all equations a, b, c, and d into equations (1.8a) and (1.8c) yields uncoupled equations for E_z and B_z:

$$\left(\partial_x^2 + \partial_y^2 + \left(\frac{\omega}{c}\right)^2 - \hat{k}^2\right)E_z = 0, \tag{1.28a}$$

$$\left(\partial_x^2 + \partial_y^2 + \left(\frac{\omega}{c}\right)^2 - \hat{k}^2\right)B_z = 0. \tag{1.28b}$$

Remark 1.2.5. TE, TM and TEM solutions stemming from equation above. Reason why a hollow waveguide cannot have TEM modes.

- $E_z = 0$ implies $\partial_x E_x + \partial_y E_x = 0$, given by equation (1.8a).
- $B_z = 0$ implies $\partial_x E_y - \partial_y E_x = 0$, given by equation (1.8d) in the z-coordinate.

Since $\hat{\boldsymbol{E}}_0$ is a function of x and y only, these equations translate to $\operatorname{div}\hat{\boldsymbol{E}}_0 = 0$ and $\operatorname{curl}\hat{\boldsymbol{E}}_0 = 0$. Therefore, it can be expressed as the gradient of a scalar potential which satisfies the Laplace's equation $\Delta\hat{\boldsymbol{E}}_0 = 0$. However, the boundary condition on electric field, i.e., equation (1.23a), enforces an equipotential at the conductor surface. Since Laplace's equation admits no local extrema, the potential must be constant throughout, and hence the electric field zero.

Let us go for TE, i.e., $B_z = 0$, in a rectangular shape of height a oriented in the x-direction and width b in the y-direction. By *separation of variables*, i.e., putting $B_z(x, y) = X(x)Y(y)$, equation (1.28b) reads

$$YX'' + XY'' + \left[\left(\frac{\omega}{c}\right)^2 - \hat{k}^2\right] XY = 0. \tag{1.29}$$

Divide by B_z and note that the x- and y-dependent terms must be constant:

$$\frac{1}{X}X'' = -k_x^2, \tag{1.30a}$$

$$\frac{1}{Y}Y'' = -k_y^2, \tag{1.30b}$$

while

$$-k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - \hat{k}^2 = 0. \tag{1.31}$$

The equations for X and Y are simple second order ODRs with the general solution

$$A\sin(\lambda x) + B\cos(\lambda x). \tag{1.32}$$

Boundary condition (1.23b) enforces vanishing B_x, which can be expressed, according to equation (1.27c), as $\partial_x B_z$, at $x = 0$ and $x = a$. This implies $A = 0$ and

$$k_x = \frac{m\pi}{a}, \quad m \in \mathbb{N}_0. \tag{1.33}$$

Similarly, for Y,

$$k_y = \frac{n\pi}{b}, \quad n \in \mathbb{N}_0. \tag{1.34}$$

Together, the particular solution takes the form of

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right). \tag{1.35}$$

That is the TE_{mn} mode and the wave number is

$$\hat{k} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi m}{a}\right)^2 - \left(\frac{\pi n}{b}\right)^2}. \tag{1.36}$$

Clearly, if

$$f < \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \equiv f_{mn}, \tag{1.37}$$

where f is frequency from $\omega = 2\pi f$, the wave number is imaginary. On such frequencies, the travelling wave becomes *evanescent*, i.e., exponentially attenuated. For this reason, we talk about f_{mn} as the *cutoff frequency* for the TE_{mn} mode in question.

TODO: Derive the *guide wavelength* and try to think if there is anything else worth talking about.

Conclusion

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Glossary of Symbols

curl	curl
Δ	Laplace operator
div	divergence
ϵ	permittivity
grad	gradient
μ	permeability
$\partial\Omega$	boundary of set Ω
∂_ξ	partial derivative w.r.t. variable ξ
ρ_e	electric charge density
ρ_m	magnetic charge density
σ	conductivity
$\boldsymbol{B}$	magnetic flux density
$\boldsymbol{D}$	electric flux density
$\boldsymbol{E}$	electric field intensity
$\boldsymbol{e}_\xi$	unit vector in the ξ-coordinate axis
$\boldsymbol{H}$	magnetic field intensity
$\boldsymbol{J}_\mathrm{e}$	source electric current density
$\boldsymbol{J}_\mathrm{m}$	magnetic current density
$\boldsymbol{k}$	wave vector

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