CS5242 : Neural Networks and Deep Learning

Lecture 2: Linear Algebra

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Outline

- Review of linear algebra :
  - Vectors
  - Matrices
  - Matrix-vector multiplication
  - Matrix-matrix multiplication
  - Inner and outer products
  - Tensors

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Vectors

Vector Addition:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$$

Scalar Multiplication:

$$2 . \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

Inner Product

- a.k.a. scalar product :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Example : $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 2 + 6 = 9$

Inner Product

- Vector representation :

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \vec{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

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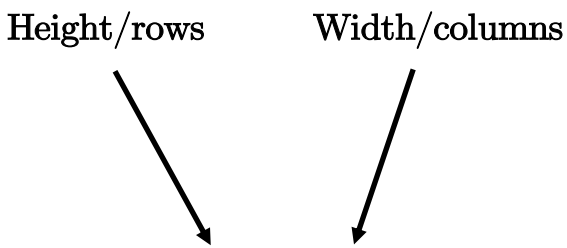
Matrices

$$W = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$ is a 3 x 2 matrix

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} \\ w_{21} & w_{22} & w_{23} & w_{24} & w_{25} \\ w_{31} & w_{32} & w_{33} & w_{34} & w_{35} \end{bmatrix}$$



is a 3 x 5 matrix

Transpose of a matrix

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 5 & 0 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

is a 3 x 5 matrix

$$W^T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 5 & 2 \\ 1 & 0 & 2 \\ 1 & 5 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

is a 5 x 3 matrix

Matrix operations

Matrix Addition:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 10 & 10 \\ 20 & 20 \\ 30 & 30 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ 23 & 24 \\ 35 & 36 \end{bmatrix}$$

Scalar multiplication:

$$2. \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix}$$

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Matrix-Vector Multiplication

$$\begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} \\ w_{21} & w_{22} & w_{23} & w_{24} & w_{25} \\ w_{31} & w_{32} & w_{33} & w_{34} & w_{35} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (\text{row}_1) \cdot \vec{\mathbf{x}} \\ (\text{row}_2) \cdot \vec{\mathbf{x}} \\ (\text{row}_3) \cdot \vec{\mathbf{x}} \end{bmatrix}$$

3 x 5 matrix

input vector has 5 entries

output vector has 3 entries

Example 1

$$W = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \vec{\mathbf{x}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$W\vec{\mathbf{x}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-4+6 \\ 4-10+12 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Example 2

$$W = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \vec{\mathbf{x}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \qquad \vec{\mathbf{b}} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$W\vec{\mathbf{x}} + \vec{\mathbf{b}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

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Example 1

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 3 & 3 \\ 8 & 4 & 5 & 5 \end{bmatrix}$$

$$(2 \text{ x } 3) \qquad (3 \text{ x } 4) \qquad = \qquad (2 \text{ x } 4)$$

(Row 2) . (col 3) = (entry 2,3)

Matrix-Matrix multiplication = batch of Matrix-Vector multiplication

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \\ 3 & 4 \end{bmatrix}$$

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Inner and outer products

$$\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{\mathbf{y}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = \underset{(1 \times 3)}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{(3 \times 1)}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} = \underset{(1 \times 1)}{\begin{bmatrix} 8 \end{bmatrix}}$$

Inner product

$$\vec{\mathbf{x}} \otimes \vec{\mathbf{y}} = \underset{(3 \times 1)}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \underset{(1 \times 3)}{\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}} = \underset{(3 \times 3)}{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix}}$$

Outer product

Matrix-Matrix multiplication = sum of outer products

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 5 & 3 \\ 0 & 2 \end{bmatrix}$$

This outer product

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 5 & 3 \\ 0 & 2 \end{bmatrix}$$

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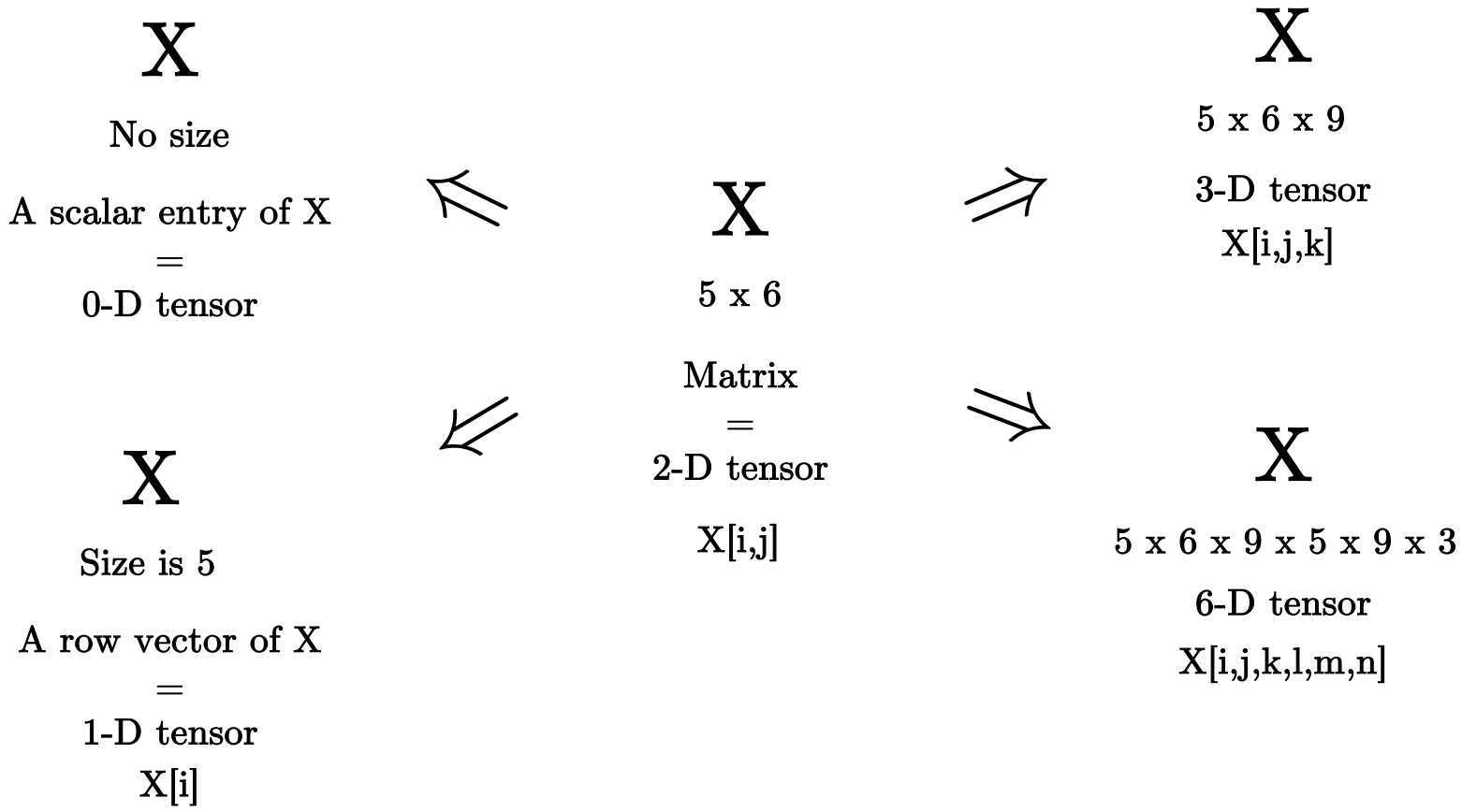
Tensors

- Generalization of matrices to higher dimensions:



Questions?