For fun, I've made a process that is analogous to the Chinese Restaurant process/Hoppe's Urn which I created by imagining what goes on through my dog's head at dinner. I present:

The Hunrgy Dog Process

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- A dinner table has infinite people, each with infinite amount of food (treats)
- call the person that the dog chooses to get a treat from *a victim*
- The dog chooses its victims for each treat independently
- The dog first picks its first victim
- Subsequently, the dog will either get a treat from a known victim with a probability proportional to the number of treats received from that victim (i.e. the mass assigned to that victim increases by 1 for each time they give a treat) or they will choose someone new with a probability proportional to $\theta > 0$ (i.e. the mass assigned to choosing someone new to get a treat from is θ).
- Label the dogs victims in the order in which the dog received the treats from (so the first victim will be labeled 1 and so on).
- For $i \in \mathbb{N}$, let X_i denote the victim that the dog chose to receive the $i'th$ treat from.
- The sequence $\{X_1, X_2, \ldots\}$ is called the *Hungry Dog Process*

Theorem 6.1

Theorem 6.1: Given $n \geq 1$, let $N_{i(n)}$ denote the total number of treats among the first n treats recieved that came from the ith victim. Then we have:

$$\lim_{n \to \infty} \frac{N_{i(n)}}{n} = P_i \text{ a.s} \tag{1}$$

and

$$\sum_{n=1}^{\infty} P_i = 1, \text{ a. s..} \tag{2}$$

In addition,

$$(P_1, P_2, P_3 \ldots) \stackrel{d}{=} (V_1, V_2, V_3, \ldots) \tag{3}$$

Proof: For $i \in \mathbb{N}$ let η_i be the indicator for if the $i'th$ treat comes from a new victim. Then, note that whenever the dog receives a treat it's either from a new victim and the mass assigned to that victim is one, or the treat is from an old victim and their mass increases by 1. Thus, each time the dog receives a treat, the total mass of all the options the dog has to get treats from (whether it be from an old victim or someone new) increases by 1. Moreover, since everybody is new at the initial step, we start with mass θ. Thus

$$\mathbf{P}\left(\text{Dog chooses a new victim to recieve treat } i\right) = \mathbf{P}\left(\eta_i = 1\right) = \frac{\theta}{\theta + i - 1}.$$

and,

$$\mathbf{P}\left(\eta = 0\right) = 1 - \mathbf{P}\left(\eta_i = 1\right) = \frac{i-1}{\theta + i - 1}.$$

Let τ_i denote the index for the first time that the dog got the treat from victim i. That is to say,

$$\tau_i = \inf\{k : K \geq 1, X_k = i\}.$$

With the above definition, τ_i will also be the index for the first time that the dog chose a new victim after having first received treats from $i-1$ victims. Thus,

$$\tau_i = \min\{j > \tau_{i-1} : \eta_j = 1\}.$$

Note that the person who gave the person who gave the dog the first treat is the first victim, so $\tau_1 = 1$.

Now, assume that the dog has received m treats. Then the victims that the dog has received treats $1, 2, \ldots, m$ are $X_1, X_2, X_3, \ldots, X_m$. For the next draws let everybody else at the table who has not given a treat to the dog be indistinguishable in the sense that receiving a treat from anybody else would be like receiving a treat from the same victim; call this victim, victim 0.

For the next n draws, we wish to track whether the dog revisits a previous victim, or whether the dog has received a treat from someone new (victim 0). Let $k \in \{1, 2, \ldots, n\}$. If the $(m+k)'th$ treat was received by a previous victim, let X_k^m to be the index of the first treat that the dog received from victim X_{m+k}. If the $(m+k)'th$ treat was received from someone new (victim 0), let $X_k^m = 0$. Since η_j is the indicator for whether the $j'th$ treat is received from someone new, η_j is equivalent to an indicator for whether or not treat j was the first treat victim X_j gave to the dog (since in this case X_j would be a new victim), we have that

$$X_k^m = \begin{cases} j, \text{If } \eta_j = 1 \text{ and } X_{m+k} = X_j, \\ 0, \text{otherwise} \end{cases}.$$

Then $X_1^m, X_2^m, \ldots$ is a $0, 1, 2, \ldots, m$ valued sequence. To see that $X_1^m, X_1^m, \ldots$ is indeed a Polya urn sequence, relabel the first m treats as being received from their own separate victims in the sense that:

- If victim X_k was a new victim at step k, relabel that victim to just k but assign the initial mass of victim k to be the number of treats that victim has given to the dog out of the first m treats (this number is then just $N_{X_k}(m)$)
- If X_k was not a new victim at step k, still we relabel that victim to just k but assign 0 mass to that victim (as the mass of that victim has already been accounted for)

Thus, the sequence $X_1^m, X_2^m, X_3^m, \ldots$ are labels for the victim the dog revisited in the next iterations. Moreover, since η_k is the indicator for whether or not the $k'th$ treat was from a new victim, the mass associated with the new labels $1, 2, \ldots, m$ is:

$$\alpha_k^m = \eta_k N_{X_k}(m)$$

Letting victim 0 have mass θ,

$$\alpha_0^m = \theta$$

Now, $X_1^m, X_2^m, \ldots$ is a $(0, 1, 2, \ldots, m)$ valued sequence, where each label has initial mass $\alpha_0^m, \ldots, \alpha_m^m$. Moreover, if the next treat is from victim 0, then since the probability of getting a treat from someone is proportional to the number of times a treat has been received from that person, the mass of Victim 0 increases by one. Similarly, if the next treat is from victim k, their mass again increases by one. Letting the victim labels $(0, 1, 2, \ldots, m)$ be colours of balls in an urn, and letting the event of receiving a treat from victim k be analogous to drawing a ball of color k we have that given $X_1, X_2, X_3, \ldots, X_m$, we see that the sequence $X_1^m, X_2^m, \ldots$ is a polya urn sequence. Define,

$$N_i^m(n) = \#\text{Treats received from victim } i \text{ in the next } n \text{ treats} = \#\{X_l^m = i, 1 \leq l \leq n\}, i = 0, 1, 2, \ldots, m$$

Note that the initial total mass of the number of balls in the urn is the sum of the masses of each colour:

$$\sum_{n=0}^{m} \alpha_n^m = \sum_{n=1}^{m} \alpha_n^m + \alpha_0^m = \sum_{n=1}^{m} \alpha_n^m + \theta$$

Another way to compute the total initial mass of the number of balls in the urn is just to add the mass of victim 0 plus the number of treats given which is $\theta + m$. Thus we conclude that:

$$\sum_{n=0}^{m} \alpha_n^m = \sum_{n=1}^{m} \alpha_n^m + \theta = m + \theta$$

So the proportion of treats received by victim i in $n+m$ draws

$$\frac{\text{Mass of victim i after } n+m}{\text{total mass of victims}} = \frac{\text{Number of treats from victim } i \text{ in the next n treats} + \text{Initial mass of victim } i}{n+m+\theta}.$$

$$= \frac{N_i^{(m)}(n) + \alpha_i^m}{n+m+\theta}.$$

Define the limiting proportion of treats the dog receives from victim i:

$$\lim_{n\to\infty} \frac{N_i^{(m)}(n) + \alpha_i^m}{n+m+\theta}.$$

By the polya urn construction of the Dirichlet, given $X_1, X_2, \ldots, X_m$ we have, with probability 1 (by theorem 4.4 in the lecture slides):

$$\lim_{n\to\infty} \left(\frac{N_0^{(m)} + \alpha_0^m}{n+m+\theta}, \frac{N_1^{(m)} + \alpha_1^m}{n+m+\theta}, \ldots, \frac{N_n^{(m)} + \alpha_n^m}{n+m+\theta} \right) = (Q_0^m, Q_1^m, \ldots, Q_m^m) \sim Dirichlet\left(\alpha_0^m, \alpha_1^m, \ldots, \alpha_m^m\right).$$

For whatever the values of $X_1, \ldots, X_m$, the convergence is always guaranteed (it's just that the parameters of the distribution are determined by $X_1, \ldots, X_m$). Accordingly, we conclude by integrating out the condition that,

$$\lim_{n\to\infty} \frac{N_i^{(m)}(n) + \alpha_i^m}{n+m+\theta} = Q_i^m \quad a.s$$

We now want to show that Q_i^m does not depend on m. Note that the mass of victim i after $n+m$ treats is simply the sum of the mass after the first m treats plus the number of treats that victim i gave out of the next n treats so:

$$\alpha_i^{(n+m)} = N_i^m(n) + \alpha_i^m.$$

As a consequence (almost surely):

$$\begin{aligned}
Q_i^m &= \lim_{n\to\infty} \frac{N_i^m(n) + \alpha_i^m}{n+m+\theta} \\
&= \lim_{n\to\infty} \frac{\alpha_i^{(n+m)}}{n+m+\theta} \\
&= \lim_{n\to\infty} \frac{\alpha_i^{n+m}}{n} \text{ Since } \theta \text{ and } m \text{ do not grow w.r.t } n \\
&= \lim_{n\to\infty} \frac{\alpha_i^n}{n} \text{ Since } n+m \text{ is approximately } n \text{ if } n \text{ is large .}
\end{aligned}$$

The last term does depend on m, so Q_i^m can be denoted as Q_i. We now show that $\sum_{n=1}^{\infty} Q_n = 1, a.s.$. First, since, conditionally on $X_1, X_2, \ldots, X_m$

$$(Q_0^m, Q_1^m, \ldots, Q_m^m) \sim \textit{Dirichlet}\left(\alpha_0^m, \alpha_1^m, \ldots, \alpha_m^m\right).$$

Which implies $Q_0^m \mid X_1, X_2, \ldots, X_m \sim \textit{Beta}\left(\alpha_0^m = \theta, \sum_{n=1}^m \alpha_n^m - \alpha_0^m = m - \theta\right)$. By the law of total expectation,

$$E(Q_0) = E\left(E\left(Q_0^m \mid X_1, X_2, \ldots, X_m\right)\right) = E\left(\frac{\theta}{m+\theta}\right) = \frac{\theta}{m+\theta}.$$

Note that for each m, the sum of the limiting proportions treats given by victims $0, 1, \ldots, m$ must sum to 1, thus

$$\sum_{n=1}^m Q_n = 1 \implies \sum_{n=1}^m Q_n = 1 - Q_0^m$$

Hence,

$$E\left(1 - \sum_{n=1}^m Q_n\right) = E\left(E\left(1 - \sum_{n=1}^m Q_n \mid X_1, \ldots, X_m\right)\right) = E(E\left(Q_0^m \mid X_1, X_2, \ldots, X_m\right)) = \frac{\theta}{m+\theta} \to 0 \text{ as } m \to \infty.$$

Thus $\lim_{m\to\infty} E\left(1 - \sum_{n=1}^\infty Q_n\right) = 0 \implies \lim_{m\to\infty} E\left(\sum_{n=1}^m Q_m\right) = 1$.

Moreover, since for each i

$$Q_i = \lim_{n\to\infty} \frac{\alpha_i^n}{n} \geq 0$$

So, for each $m_1 < m_2 \leq 1$, $0 \leq \sum_{n=1}^{m_1} Q_n \leq \sum_{n=1}^{m_2} Q_n \leq 1$ implying the sequence $\sum_{n=1}^1 Q_n, \sum_{n=1}^2 Q_n, \sum_{n=1}^3 Q_n, \ldots$ is increasing and bounded. As a result, by the monotone convergence theorem,

$$\lim_{m\to\infty} E\left(\sum_{n=1}^m Q_n\right) = E\left(\lim_{m\to\infty} \sum_{n=1}^m Q_m\right) = 1$$

Now to show that $\lim_{m\to\infty} \sum_{n=1}^m Q_n = 1$ a.s.

Suppose, for the sake of a contradiction, that $\mathbf{P}\left(\lim_{m\to\infty} \sum_{n=1}^m Q_n \neq 1\right) > 0$. We have shown that $\sum_{n=1}^m Q_m \leq 1$ for each m, so $\mathbf{P}\left(\lim_{m\to\infty} \sum_{n=1}^m Q_n \neq 1\right) > 0 \iff \mathbf{P}\left(\lim_{m\to\infty} \sum_{n=1}^m Q_n < 1\right) > 0$. For the sake of notation let $W = \lim_{m\to\infty} \sum_{n=1}^m Q_n$. Then since $\mathbf{P}(W < 1) > 0$, there exists $\epsilon > 0$ such that $\mathbf{P}(W < 1 - \epsilon) > 0$, thus:

$$\begin{aligned}
E(W) &= E\left(WI(W < 1-\epsilon) + WI(W \geq 1-\epsilon)\right) \\
&= E\left(WI(W < 1-\epsilon)\right) + E\left(WI(W \geq 1-\epsilon)\right) \\
&= \int WI(W < 1-\epsilon)\, d\mathbf{P} + \int WI(W \geq 1-\epsilon)\, d\mathbf{P} \\
&\leq (1-\epsilon)\,\mathbf{P}(W < 1-\epsilon) + 1 \cdot \mathbf{P}(W \geq 1-\epsilon) \\
&= \mathbf{P}(W < 1-\epsilon) + \mathbf{P}(W \geq 1-\epsilon) - \epsilon\mathbf{P}(W < 1-\epsilon) \\
&= 1 - \epsilon \cdot \mathbf{P}(W < 1-\epsilon) < 1
\end{aligned}$$

.

Which is a contradiction since $E(W) = 1$; we may conclude that:

$$W = \sum_{n=1}^\infty Q_n = 1 \text{ a.s.} \quad \triangle.$$

Now, since τ_i denotes the index of the first treat that victim i gives, we have that $X_{\tau_i} = i$ and $\eta_{\tau_i} = 1$. Thus $\alpha_{\tau_i^n} = \eta_{\tau_i} N_{X_{\tau_i}}(n) = 1 \cdot N_i(n)$. Accordingly, we may write, the limiting proportion of treats received by victim i (denoted P_i) as:

$$P_i = Q_{\tau_i} = \lim_{n\to\infty} \frac{\alpha_{\tau_i}^n}{n} = \lim_{n\to\infty} \frac{N_i(n)}{n} \quad a.s \text{ which concludes (1) in theorem 6.1.}$$

Moreover, if $i \notin \{\tau_1, \tau_2, \ldots\}$ then it's not an index for the first time that a victim gives a dog a treat by definition of τ_i. Consequently, victim X_i is not a new victim, so $\eta_i = 0$, thus $\alpha_i^n = \eta_i N_{X_i}(n) = 0$. So, in the case that $i \notin \{\tau_1, \tau_2, \ldots\}$,

$$Q_i = \lim_{n\to\infty} \frac{\alpha_i^n}{n} = \lim_{n\to\infty} \frac{0}{n} = 0$$

Hence,

$$\sum_{n=1}^{\infty} Q_n = \sum_{i\in\{\tau_1,\tau_2,\ldots\}}^{\infty} Q_i = \sum_{n=1}^{\infty} P_i.$$

And so we conclude that (noting the result $\triangle$)

$$\sum_{n=1}^{\infty} P_i = 1 \text{ a.s., concluding equation (2) in theorem 6.1.}$$

It remains to show that:

$$(P_1, P_2, \ldots) \stackrel{d}{=} (V_1, V_2, \ldots).$$

Let

$$U_1 = P_1.$$

and, for $n \in \mathbb{N}$:

$$U_n = \frac{P_n}{\sum_{i=n}^{\infty} P_i}.$$

We show that, for $n \geq 2$ that $P_n = (1 - U_1,) \ldots (1 - U_{n-1}) U_n$ using induction.

Base Case (n=2) Note that

$$\begin{aligned}
(1 - U_1) U_2 &= (1 - P_1) \left(\frac{P_2}{\sum_{i=2}^{\infty} P_i} \right) \\
&= \left(\sum_{i=2}^{\infty} P_i - P_1 \right) \left(\frac{P_2}{\sum_{i=2}^{\infty} P_i} \right) \\
&= \left(\sum_{i=2}^{\infty} P_i \right) \left(\frac{P_2}{\sum_{i=2}^{\infty} P_i} \right) \\
&= P_2
\end{aligned}$$

.

Concluding the base case.

Induction Hypothesis Fix $n \in N$ and assume that for $k \in \{1, 2, \ldots, n\}$

$$P_k = (1 - U_1) \ldots (1 - U_{k-1}) U_k.$$

Then

$$U_{n+1} = \frac{P_{n+1}}{\sum_{i=n+1}^{\infty} P_{n+1}} = \frac{P_{n+1}}{\sum_{i=1}^{\infty} P_i - \sum_{i=1}^{n} P_i}.$$
$$= \frac{P_{n+1}}{1 - \sum_{i=1}^{n} P_i} \quad a.s.$$

Thus, (almost surely)

$$\begin{aligned}
P_{n+1} &= \left(1 - \sum_{i=1}^{n} P_i\right)(U_{n+1}) \\
&= (1 - U_1 - (1 - U_1) U_2 - (1 - U_1)(1 - U_2) U_2 - \ldots - (1 - U_1) \cdots (1 - U_{n-1})) U_n) U_{n+1} \quad \text{(induction hypothesis)} \\
&= (1 - U_1) [1 - U_2 - (1 - U_2) U_3 - \ldots - (1 - U_2) \cdots (1 - U_{n-1})) U_n] U_{n+1} \\
&= (1 - U_1)(1 - U_2)(1 - U_3 - \ldots - (1 - U_3) \cdots (1 - U_{n-1}) U_n)) U_{n+1} \\
&= . \\
&= . \qquad \text{(Proceeding iteratively)} \\
&= . \\
&= (1 - U_1) \ldots (1 - U_n) U_{n+1}
\end{aligned}$$
.

Thus, by induction, we conclude that $\forall n \in \mathbb{N}$ where $n \geq 2$,

$$P_n = (1 - U_1) \ldots (1 - U_{n-1}) U_n.$$

Thus, $P_1 = U_1$, $P_2 = (1 - U_1) U_2$, and for $n \geq 3$, $P_n = (1 - U_1) \ldots (1 - U_{n-1}) U_n$.
We now prove the following claim.

Claim: $U_1, U_2, \ldots$ are *iid Beta* $(1, \theta)$ random variables.

Proof:
Given $n \in \mathbb{N}$. Conditioning on the treats that have been received up to when the $n'th$ victim gave their first treat to the dog. Consider the sequence $\{X_1^{\tau_n}, X_2^{\tau_n}, \ldots\}$. Then, from before we know that the sequence is a polya urn sequence and

$$(Q_1, Q_2, \ldots, Q_{\tau_n}, Q_0) \sim Dirichlet\left(\alpha_1^{\tau_1}, \alpha_2^{\tau_n} \ldots, \alpha_{\tau_n}^{\tau_n}, \alpha_0^{\tau_n} = \theta\right).$$

Since $Q_i = 0$ for $i \notin \{\tau_1, \tau_2, \ldots\}$ we have:

$$(Q_{\tau_1}, Q_{\tau_2}, \ldots, Q_{\tau_n}, Q_0) \sim Dirichlet\left(\alpha_{\tau_1}^{\tau_n}, \ldots, \alpha_{\tau_n}^{\tau_n}, \alpha_0^{\tau_n} = \theta\right).$$

Note that Q_0 is the limiting proportion of treats that came from victim 0, which is a label for the victims that were not in the original n victims that already gave the dog a treat. Thus this proportion is equal to the sum of the proportions of treats that came from victims $n+1, n+2, \ldots$. That is to say,

$$Q_0 = \sum_{i=n+1}^{\infty} P_i.$$

Moreover, we know that $Q_{\tau_i} = P_i$ thus $(Q_{\tau_1}, Q_{\tau_2}, \ldots, Q_{\tau_n}, Q_0) \sim Dirichlet\left(\alpha_{\tau_1}^{\tau_n}, \ldots, \alpha_{\tau_n}^{\tau_n}, \alpha_0^{\tau_n} = \theta\right)$ implies that:

$$\left(P_1, P_2, \ldots, P_n, \sum_{i=n+1}^{\infty} P_i\right) \sim Dirichlet\left(\alpha_{\tau_1}^{\tau_n}, \ldots, \alpha_{\tau_n}^{\tau_n}, \alpha_0^{\tau_n} = \theta\right).$$

For $i \in \{1, 2, \ldots, n\}$ let $\alpha_i = \alpha_{\tau_i}^{\tau_n}$, and let $\alpha_{n+1} = \alpha_0^{\tau_n} = \theta$. So

$$\left(P_1, P_2, \ldots, P_n, \sum_{i=n+1}^{\infty} P_i\right) \sim Dirichlet\left(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}\right).$$

Now to show that $U_1, U_2, \ldots \sim Beta\left(1, \theta\right)$, for $i \in \{1, \ldots, n+1\}$ let

$$Y_i \sim Gamma\left(\alpha_i, 1\right).$$

Let $V = \sum_{i=1}^{n+1} Y_i$. Then, by the Dirichlet Gamma algebra:

$$\left(P_1, P_2, \ldots, P_n, \sum_{i=n+1}^{\infty} P_i\right) \stackrel{d}{=} \left(\frac{Y_1}{V}, \ldots, \frac{Y_n}{V}, \frac{Y_{n+1}}{V}\right) \sim Dirichlet\left(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} = \theta\right).$$

Thus, for a measurable function f,

$$f\left(P_1, P_2, \ldots, P_n, \sum_{i=n+1}^{\infty} P_i\right) \stackrel{d}{=} f\left(\frac{Y_1}{V}, \ldots, \frac{Y_n}{V}, \frac{Y_{n+1}}{V}\right).$$

Consider the function $f\left(x_1, x_2, \ldots, x_{n+1}\right) = \dfrac{x_n}{x_n + x_{n+1}}$. Since the function is continuous, it is measurable. So,

$$\frac{P_n}{P_n + \sum_{i=n+1}^{\infty} P_i} \stackrel{d}{=} \frac{\frac{Y_n}{V}}{\frac{Y_n}{V} + \frac{Y_{n+1}}{V}}.$$

Note that

$$\frac{\frac{Y_n}{V}}{\frac{Y_n}{V} + \frac{Y_{n_1}}{V}} = \frac{Y_n}{Y_n + Y_{n+1}}.$$

Which has a $Beta\left(\alpha_n, \alpha_{n+1} = \theta\right)$ distribution by the Beta-Gamma algebra. Consequently,

$$\frac{P_n}{P_n + \sum_{i=n+1}^{\infty} P_i} = \frac{P_n}{\sum_{i=n}^{\infty} P_i} = U_n \sim Beta\left(\alpha_n, \theta\right).$$

Note that $\alpha_n = \alpha_{\tau_n}^{\tau_n} = \#\{1 \leq k \leq \tau_n : X_k = \tau_n\} = 1$. This means,

$$U_n \sim Beta\left(1, \theta\right).$$

Now, since $Beta\left(1, \theta\right)$ clearly does not depend on $X_1, \ldots, X_{\tau_n}$ (since 1 and θ are constants), integrating out the condition we have:

$$U_n \sim Beta\left(1, \theta\right).$$

Since n is arbitrary, it follows that

$$U_1, U_2, \ldots \sim Beta\left(1, \theta\right).$$

We now show that $U_1, U_2, \ldots$ are independent.

From Connor and Mosimann (1969), since $\left(P_1, P_2, \ldots P_n, \sum_{i=n+1}^{\infty} P_i\right)$ has a Dirichlet distribution, P_1 is neutral (the definition of neutrality is discussed in the paper). Moreover, since

$$U_n = \frac{P_n}{\sum_{i=n}^{\infty} P_i} = \frac{P_n}{\sum_{i=1}^{\infty} P_i - \sum_{i=1}^{n-1} P_i} = \frac{P_n}{1 - P_1 - P_2 - \ldots - P_{n-1}}.$$

We conclude, by theorem 1 found in Connor and Mosimann (1969), that

$$U_1, U_2, \ldots, U_n$$

are independent (conditionally on $X_1, X_2, \ldots, X_n$. Moreover, since $U_n \sim Beta(1, \theta)$ for each $n \in \mathbb{N}$ (and the parameters are constants so they don't depend on $X_1, X_3, \ldots, X_{\tau_n}$, integrating out the the condition we have that

$$U_1, U_2, \ldots \overset{iid}{\sim} Beta(1, \theta) \quad \text{Satisfying (3) in theorem 6.1}$$

Concluding the proof of theorem 6.1.