Q1 (25 points)

Theorem 10.4 of Lecture 10 states that the size-biased permutation of the two-parameter Poisson-Dirichlet distribution is given by the two-parameter GEM distribution. Using the notation of the theorem, show that for any integer $m \geq 1$

$$\mathbb{E}\left[P_{\tau_3}^m(\alpha,\theta]=\mathbb{E}\left[V_3^m(\alpha,\theta)\right].$$

Using this result to derive the case for one-parameter model. (Justification of all steps is required to get the full credit.)

Proof:

For brevity, denote $P_k(\alpha,\theta)$ as P_k, $\mathbf{P}(\alpha,\theta)$ as $\mathbf{P}$, $V_k(\alpha,\theta)$ as V_k, $\mathbf{V}(\alpha,\theta)$ as $\mathbf{V}$. Fix $m\in\mathbb{N}$. We begin by showing that

$$E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right)=E\left(U_3^m\right):$$

To begin, since $\mathbf{P}\sim PD(\alpha,\theta)$, by definition 10.2 (lecture 10), the order statistics of $\mathbf{V}$ are precisely $\mathbf{P}$. We have already shown in class that

$$E\left({P_{\tau_1}}^m\right)=E\left(V_1^m\right).$$

Thus, $P_{\tau_1}\overset{d}{=}V_1$. Since the order statistics of $\mathbf{V}$ are equal to $\mathbf{P}$, we claim that $P_{\tau_1}=V_1$. Indeed, if this was not the case, then there exists $k\in\mathbb{N}-\{1\}$ $P_{\tau_1}=V_k$, which implies that $V_1=V_k$. Then

$$E\left(V_1\right)=E\left(V_k\right)\implies E\left(U_1\right)=E\left(\left(1-U_1\right)\ldots\left(1-U_{k-1}\right)U_k\right)=E\left(1-U_1\right)\quad\text{By definition 10.1.}$$

Following by defintion 10.1, we have that

$$E\left(U_1\right)=\frac{1-\alpha}{1+\theta}.$$

And,

$$E\left(V_k\right)=E\left(\left(1-U_1\right)\left(1-U_2\right)\ldots\left(1-U_{k-1}\right)U_k\right)=E\left(1-U_1\right)E\left(1-U_2\right)\ldots E\left(1-U_{k-1}\right)E\left(U_k\right)\quad\text{(By independence).}$$

$$=\frac{\theta+\alpha}{(1-\alpha)+(\theta+\alpha)}\frac{\theta+2\alpha}{(1-\alpha)+(\theta+2\alpha)}\cdots\frac{\theta+(k-1)\alpha}{(1-\alpha)+(\theta+(k-1)\alpha)}\frac{1-\alpha}{(1-\alpha)+(\theta+k\alpha)}<\frac{1-\alpha}{(1-\alpha)+(\theta+k\alpha)}\ \text{Since } \alpha\in[0,1).$$

And so

$$E\left(V_k\right)<\frac{1-\alpha}{\theta+(k-1)\alpha}\leq\frac{1-\alpha}{1+\theta}=E\left(V_1\right).$$

Thus, for $k\in\mathbb{N}-\{1\}$, $E\left(V_k\right)<E\left(V_1\right)$, which cannot be the case if $V_1\overset{d}{=}V_k$. As a result, $P_{\tau_1}=V_1$. It can be shown in a very similar way that since $P_{\tau_2}\overset{d}{=}V_2$ (shown in class), $P_{\tau_2}=V_2$:

Suppose, for the sake of a contradiction, that $P_{\tau_2}\overset{d}{=}V_j$ for some $j\in\mathbb{N}-\{2\}$ which implies that $V_2\overset{d}{=}V_j$. We have already shown that V_1 is not equal in distribution to V_k for $k\in\mathbb{N}-\{1\}$, thus V_2 is not equal in distribution to V_1. So, if $P_{\tau_2}\overset{d}{=}V_j$ for some $j\in\mathbb{N}-\{2\}$, it follows that $j\neq 1$, so $j\in\mathbb{N}-\{1,2\}$. Note that:

$$E\left(V_2\right)=E\left(\left(1-U_1\right)U_2\right)=E\left(1-U_1\right)E\left(U_2\right)=\frac{\theta+\alpha}{(1-\alpha)+(\theta+\alpha)}\frac{1-\alpha}{(1-\alpha)+(\theta+2\alpha)}.$$

And,

$$E\left(V_j\right)=E\left(\left(1-U_1\right)\ldots\left(1-U_{j-1}\right)U_j\right)=E\left(1-U_1\right)\ldots E\left(1-U_{j-1}\right)E\left(U_j\right)$$

$$=\frac{\theta+\alpha}{(1-\alpha)+(\theta+\alpha)}\cdots\frac{\theta+(j-1)\alpha}{(1-\alpha)+(\theta+(j-1)\alpha)}\frac{1-\alpha}{(1-\alpha)+(\theta+j\alpha)}<$$

$$\frac{\theta+\alpha}{(1-\alpha)+(\theta+\alpha)}\frac{1-\alpha}{(1-\alpha)+(\theta+j\alpha)} \quad \text{Since } \alpha\in[0,1)\text{, and } j\in\mathbb{N}-\{1,2\}.$$

And so,

$$E\left(V_j\right) < \frac{\theta+\alpha}{(1-\alpha)+(\theta+\alpha)}\frac{1-\alpha}{(1-\alpha)+(\theta+j\alpha)} \le \frac{\theta+\alpha}{(1-\alpha)+(\theta+\alpha)}\frac{1-\alpha}{(1-\alpha)+(\theta+\alpha)} = E\left(V_2\right).$$

Thus, for $j\in\mathbb{N}-\{1,2\}$, we conclude that $E\left(V_j\right)\neq E\left(V_2\right)$ which cannot be the case if $V_2 \stackrel{d}{=} V_j$. As a result, $P_{\tau_2}=V_2$

So, we have:

$$\begin{aligned}
E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right) &= E_{\mathbf{P},\tau_1,\tau_2}\left(E_{\tau_3}\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m \middle| \mathbf{P},\tau_1,\tau_2\right)\right) \quad \text{By double expectation}\\
&= E\left(\sum_{n=1}^{\infty}\left(\frac{P_n}{1-P_{\tau_1}-P_{\tau_2}}\right)^m \frac{P_n}{1-P_{\tau_1}-P_{\tau_2}} I_{n\neq\tau_1,n\neq\tau_2}\right)\\
&= E\left(\sum_{n=1}^{\infty}\left(\frac{P_n}{1-P_{\tau_1}-P_{\tau_2}}\right)^{m+1} I_{P_n\neq P_{\tau_1},P_n\neq P_{\tau_2}}\right)\\
&= E\left(\sum_{n=1}^{\infty}\left(\frac{P_n}{1-V_1-V_2}\right)^{m+1} I_{P_n\neq V_1,P_n\neq V_2}\right) \quad \text{As we have shown, } V_1=P_{\tau_1},\ V_2=P_{\tau_2}\\
&= E\left(\sum_{n=3}^{\infty}\left(\frac{V_{(n)}}{1-V_1-V_2}\right)^{m+1}\right) \quad \text{Since } \mathbf{P} \text{ are the order statistics of } \mathbf{V} \quad \dagger.
\end{aligned}$$

Note that since $V_k=\prod_{n=1}^{k-1}\left(1-U_k\right)U_k$, and $U_i\sim Beta\left(1-\alpha,\theta+i\alpha\right)$, we have that $V_k\in[0,1]$, which means that the sequence $\left(\sum_{n=1}^{b}V_b^{m+1}, b\in\mathbb{N}\right)$ in a monotone sequence. Moreover, since:

$$\sum_{n=1}^{\infty}V_n=1 \quad a.s.$$

We can conclude that

$$0\le\sum_{n=1}^{\infty}V_n^{m+1}\le\sum_{n=1}^{\infty}V_n=1 \quad a.s$$

Thus, the sequence $\left(\sum_{n=1}^{b}V_b^{m+1}, b\in\mathbb{N}\right)$ is monotone and bounded, thus by the monotone convergence theorem, the series $\sum_{n=1}^{\infty}V_n^{m+1}$ converges pointwise. This convergence is absolute since $V_n\in[0,1]$ for each $n\in\mathbb{N}$, so $\sum_{n=1}^{\infty}V_n=\sum_{n=1}^{\infty}V_{(n)}$ (since the convergence is absolute, it can be reordered). Thus, plugging this result into †:

$$\begin{aligned}
E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right) &= E\left(\sum_{n=3}^{\infty}\left(\frac{V_{(n)}}{1-V_1-V_2}\right)^{m+1}\right)\\
&= E\left(\sum_{n=1}^{\infty}\left(\frac{V_n}{1-V_1-V_2}\right)^{m+1}\right)
\end{aligned}$$

.

Since $\left(\sum_{n=1}^{b}V_b^{m+1}, b\in\mathbb{N}\right)$ is a bounded monotone sequence that converges almost surely, the sequence $\left(\sum_{n=1}^{b}\left(\frac{V_n}{1-V_1-V_2}\right)^{m+1}, b\in\mathbb{N}\right)$ is a bounded monotone sequence that converges almost surely, thus by the monotone convergence theorem:

$$
\begin{aligned}
E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right) &= E\left(\sum_{n=3}^{\infty}\left(\frac{V_n}{1-V_1-V_2}\right)^{m+1}\right) \\
&= \sum_{n=3}^{\infty} E\left(\left(\frac{V_n}{1-V_1-V_2}\right)^{m+1}\right) \\
&= \sum_{n=3}^{\infty} E\left(\left(\frac{\prod_{i=1}^{n-1}\left(1-U_i\right)U_n}{1-U_1-\left(1-U_1\right)U_2}\right)^{m+1}\right) \\
&= \sum_{n=3}^{\infty} E\left(\left(\frac{\prod_{i=1}^{n-1}\left(1-U_i\right)U_n}{\left(1-U_1\right)\left(1-U_2\right)}\right)^{m+1}\right) \\
&= \sum_{n=3}^{\infty} E\left(\left(\prod_{i=3}^{n-1}\left(1-U_i\right)U_n\right)^{m+1}\right) \\
&= \sum_{n=3}^{\infty} E\left(U_n^{m+1}\right)\prod_{i=3}^{n-1} E\left(\left(1-U_i\right)^{m+1}\right) \quad \text{By independence} \\
&= E\left(U_3^{m+1}\right) + \sum_{n=4}^{\infty} E\left(U_n^{m+1}\right)\prod_{i=3}^{n-1} E\left(\left(1-U_i\right)^{m+1}\right) \ddagger .
\end{aligned}
$$

Since $U_n \sim Beta\left(1-\alpha, \theta+n\alpha\right)$,

$$
\begin{aligned}
E\left(U_n^{m+1}\right) &= \int_0^1 \frac{\Gamma\left(1-\alpha+\theta+n\alpha\right)}{\Gamma\left(1-\alpha\right)\Gamma\left(\theta+n\alpha\right)} u^{m+1}u^{1-\alpha+1}\left(1-u\right)^{\theta+n\alpha-1}\,du \\
&= \frac{\Gamma\left(1-\alpha+\theta+n\alpha\right)}{\Gamma\left(1-\alpha\right)\Gamma\left(\theta+n\alpha\right)} \int_0^1 u^{m+2+\alpha-1}\left(1-u\right)^{\theta+n\alpha-1} \\
&= \frac{\Gamma\left(1-\alpha+\theta+n\alpha\right)}{\Gamma\left(1-\alpha\right)\Gamma\left(\theta+n\alpha\right)} \frac{\Gamma\left(m+2-\alpha\right)\Gamma\left(\theta+n\alpha\right)}{\Gamma\left(m+2-\alpha+\theta+n\alpha\right)} \\
&= \frac{\Gamma\left(1-\alpha+\theta+n\alpha\right)}{\Gamma\left(1-\alpha\right)} \frac{\Gamma\left(m+1+1-\alpha\right)}{\Gamma\left(m+1+1-\alpha+\theta+n\alpha\right)} \\
&= \frac{\Gamma\left(m+1+1-\alpha\right)}{\Gamma\left(1-\alpha\right)} \frac{\Gamma\left(1+\theta+\left(n-1\right)\alpha\right)}{\Gamma\left(m+1+1+\theta+\left(n-1\right)\alpha\right)} \\
&= \frac{\left(1-\alpha\right)_{(m+1)}}{\left(1+\theta+\left(n-1\right)\alpha\right)_{(m+1)}} \quad \clubsuit
\end{aligned}
$$

Since $U_i \sim Beta\left(1-\alpha, \theta+i\alpha\right)$,

$$
\begin{aligned}
E\left((1-U_i)^{m+1}\right) &= \int_0^1 \frac{\Gamma(1-\alpha+\theta+i\alpha)}{\Gamma(1-\alpha)\Gamma(\theta+i\alpha)}(1-u)^{m+1}u^{1-\alpha+1}(1-u)^{\theta+i\alpha-1}du \\
&= \frac{\Gamma(1-\alpha+\theta+i\alpha)}{\Gamma(1-\alpha)\Gamma(\theta+i\alpha)}\int_0^1 (1-u)^{m+1+\theta+i\alpha}u^{1-\alpha+1}du \\
&= \frac{\Gamma(1-\alpha+\theta+i\alpha)}{\Gamma(1-\alpha)\Gamma(\theta+i\alpha)}\left(\frac{\Gamma(m+1+1-\alpha+\theta+i\alpha)}{\Gamma(1-\alpha)\Gamma(m+1+\theta+i\alpha)}\right)^{-1} \\
&= \frac{\Gamma(1+\theta+(i-1)\alpha)}{\Gamma(\theta+i\alpha)}\left(\frac{\Gamma(m+1+1+\theta+(i-1)\alpha)}{\Gamma(m+1+\theta+i\alpha)}\right)^{-1} \\
&= \frac{\Gamma(m+1+\theta+i\alpha)}{\Gamma(\theta+i\alpha)}\frac{\Gamma(1+\theta+(i-1)\alpha)}{\Gamma(m+1+1+\theta+(i-1)\alpha)} \\
&= \frac{(\theta+i\alpha)_{(m+1)}}{(1+\theta+(i-1)\alpha)_{(m+1)}}
\end{aligned}
$$

.

Note:

$$
\begin{aligned}
\frac{1}{(1+\theta+(i-1)\alpha)_{(m+1)}} &= \frac{1}{(1+\theta+(i-1)\alpha)\cdots(m-1+1+\theta+(i-1)\alpha)(m+1+\theta+(i-1)\alpha)} \\
&= \frac{\theta+(i-1)\alpha}{(\theta+(i-1)\alpha)(1+\theta+(i-1)\alpha)\cdots(m+\theta+(i-1)\alpha)(m+1+\theta+(i-1)\alpha)} \\
&= \frac{\theta+(i-1)\alpha}{(\theta+(i-1)\alpha)_{(m+1)}(m+1+\theta+(i-1)\alpha)}
\end{aligned}
$$

So,

$$
\begin{aligned}
E\left((1-U_i)^{m+1}\right) &= \frac{(\theta+i\alpha)_{(m+1)}}{(1+\theta+(i-1)\alpha)_{(m+1)}} \\
&= (\theta+i\alpha)_{(m+1)}\frac{\theta+(i-1)\alpha}{(\theta+(i-1)\alpha)_{(m+1)}(m+1+\theta+(i-1)\alpha)} \quad \diamondsuit
\end{aligned}
$$

Applying results $\clubsuit, \diamondsuit$ to $\ddagger$:

$$
E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right) =
$$

$$
\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} + \sum_{n=4}^{\infty}\frac{(1-\alpha)_{(m+1)}}{(1+\theta+(n-1)\alpha)_{(m+1)}}\prod_{i=3}^{n-1}(\theta+i\alpha)_{(m+1)}\frac{\theta+(i-1)\alpha}{(\theta+(i-1)\alpha)_{(m+1)}(m+1+\theta+(i-1)\alpha)}
$$

$$
= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} + \sum_{n=4}^{\infty}\frac{(1-\alpha)_{(m+1)}}{(1+\theta+(n-1)\alpha)_{(m+1)}}\left(\prod_{i=3}^{n-1}\frac{(\theta+i\alpha)_{(m+1)}}{(\theta+(i-1)\alpha)_{(m+1)}}\right)\left(\prod_{i=3}^{n-1}\frac{\theta+(i-1)\alpha}{m+1+\theta+(i-1)\alpha}\right)
$$

We simplify $\prod_{i=3}^{n-1}\frac{(\theta+i\alpha)_{(m+1)}}{(\theta+(i-1)\alpha)_{(m+1)}}$:

$$
\begin{aligned}
\prod_{i=3}^{n-1}\frac{(\theta+i\alpha)_{(m+1)}}{(\theta+(i-1)\alpha)_{(m+1)}} &= \frac{\cancel{(\theta+3\alpha)_{(m+1)}}}{(\theta+2\alpha)_{(m+1)}}\frac{\cancel{(\theta+4\alpha)_{(m+1)}}}{\cancel{(\theta+3\alpha)_{(m+1)}}}\cdots\frac{\cancel{(\theta+(n-2)\alpha)_{(m+1)}}}{\cancel{(\theta+(n-3)\alpha)_{(m+1)}}}\frac{(\theta+(n-1)\alpha)_{(m+1)}}{\cancel{(\theta+(n-2)\alpha)_{(m+1)}}} \\
&= \frac{(\theta+(n-1)\alpha)_{(m+1)}}{(\theta+2\alpha)_{(m+1)}}
\end{aligned}
$$

As a result:

$$E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right)=$$

$$\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+\sum_{n=4}^{\infty}\frac{(1-\alpha)_{(m+1)}}{(1+\theta+(n-1)\,\alpha)_{(m+1)}}\frac{(\theta+(n-1)\alpha)_{(m+1)}}{(\theta+2\alpha)_{(m+1)}}\left(\prod_{i=3}^{n-1}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)$$

$$=\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+\frac{(1-\alpha)_{(m+1)}}{(\theta+2\alpha)_{(m+1)}}\sum_{n=4}^{\infty}\frac{(\theta+(n-1)\alpha)_{(m+1)}}{(1+\theta+(n-1)\,\alpha)_{(m+1)}}\left(\prod_{i=3}^{n-1}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)$$

Note that:

$$\frac{(\theta+(n-1)\alpha)_{(m+1)}}{(1+\theta+(n-1)\,\alpha)_{(m+1)}}=\frac{\theta+(n-1)\alpha}{\cancel{1+\theta+(n-1)\alpha}}\frac{\cancel{1+\theta+(n-1)\alpha}}{\cancel{2+\theta+(n-1)\alpha}}\cdots\frac{\cancel{m+\theta+(n-1)\alpha}}{m+1+\theta+(n-1)\alpha}$$

$$=\frac{\theta+(n-1)\alpha}{m+1+\theta(n-1)\alpha}$$

Thus,

$$E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right)=\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+\frac{(1-\alpha)_{(m+1)}}{(\theta+2\alpha)_{(m+1)}}\sum_{n=4}^{\infty}\frac{\theta+(n-1)\alpha}{m+1+\theta(n-1)\alpha}\left(\prod_{i=3}^{n-1}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)$$

$$=\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+\frac{(1-\alpha)_{(m+1)}}{(\theta+2\alpha)_{(m+1)}}\sum_{n=4}^{\infty}\left(\prod_{\mathbf{i=3}}^{\mathbf{n}}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)$$

$$=\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+\frac{(1-\alpha)_{(m+1)}}{(\theta+2\alpha)_{(m+1)}}\sum_{n=4}^{\infty}\frac{\theta+2\alpha}{m+1+\theta+2\alpha}\left(\prod_{\mathbf{i=4}}^{n}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)$$

$$=\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+\frac{(1-\alpha)_{(m+1)}}{(\theta+2\alpha)_{(m+1)}}\frac{\theta+2\alpha}{m+1+\theta+2\alpha}\sum_{n=4}^{\infty}\left(\prod_{i=4}^{n}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)$$

$$=\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+(1-\alpha)_{(m+1)}\frac{\theta+2\alpha}{(\theta+2\alpha)_{(m+1)}\,(m+1+\theta+2\alpha)}\sum_{n=4}^{\infty}\left(\prod_{i=4}^{n}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)\heartsuit$$

Following from direct calculation:

$$\frac{\theta+2\alpha}{(\theta+2\alpha)_{(m+1)}\,(m+1+\theta+2\alpha)}=\frac{\cancel{\theta+2\alpha}}{\cancel{(\theta+2\alpha)}(1+\theta+2\alpha)(2+\theta+2\alpha)\cdots(m+\theta+2\alpha)(m+1+\theta+2\alpha)}$$

$$=\frac{1}{(1+\theta+2\alpha)_{(m+1)}}$$

Plugging this result into $\heartsuit$:

$$E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right)=$$

$$\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}+\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}}\sum_{n=4}^{\infty}\left(\prod_{i=4}^{n}\frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(1 + \sum_{n=4}^{\infty} \left(\prod_{i=4}^{n} \frac{\theta+(i-1)\,\alpha}{m+1+\theta+(i-1)\,\alpha}\right)\right)$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(1 + \frac{\theta+3\alpha}{m+1+\theta+3\alpha} + \frac{\theta+3\alpha}{m+1+\theta+3\alpha}\frac{\theta+4\alpha}{m+1+\theta+4\alpha} + \dots\right)$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(1 + \frac{(\theta+3\alpha)\,/\alpha}{(m+1+\theta+3\alpha)/\alpha} + \frac{(\theta+3\alpha)/\alpha}{(m+1+\theta+3\alpha)/\alpha}\frac{(\theta+3\alpha+\alpha)/\alpha}{(m+1+\theta+3\alpha+\alpha)/\alpha} + \dots\right)$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(1 + \frac{(\theta+3\alpha)\,/\alpha}{(m+1+\theta+3\alpha)/\alpha} + \frac{(\theta+3\alpha)/\alpha}{(m+1+\theta+3\alpha)/\alpha}\frac{(\theta+3\alpha)/\alpha+1}{(m+1+\theta+3\alpha)/\alpha+1} + \dots\right)$$

Let $a = \dfrac{\theta+3\alpha}{\alpha}$, $c = \frac{m+1+\theta+3\alpha}{\alpha}$, then

$$\frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(1 + \frac{(\theta+3\alpha)\,/\alpha}{(m+1+\theta+3\alpha)/\alpha} + \frac{(\theta+3\alpha)/\alpha}{(m+1+\theta+3\alpha)/\alpha}\frac{(\theta+3\alpha)/\alpha+1}{(m+1+\theta+3\alpha)/\alpha+1} + \dots\right)$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(1 + \frac{a}{c} + \frac{a}{c}\frac{a+1}{c+1} + \dots\right)$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(\frac{c-1}{c-a-1}\right) \quad \text{By Guass's Hypergeometric theorem}$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \left(\frac{c-1}{c-a-1}\right)$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \frac{\frac{m+1+\theta+3\alpha}{\alpha}-1}{\frac{m+1+\theta+3\alpha-\theta-3\alpha}{\alpha}-1}$$

$$= \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \frac{\frac{m+1+\theta+2\alpha}{\alpha}}{\frac{m+1-\alpha}{\alpha}} = \frac{(1-\alpha)_{(m+1)}}{(1+\theta+2\alpha)_{(m+1)}} \frac{m+1+\theta+2\alpha}{m+1-\alpha}$$

$$= \frac{(1-\alpha)(1+1-\alpha)\dots(m-1+1-\alpha)\cancel{(m+1-\alpha)}}{(1+\theta+2\alpha)(1+1+\theta+2\alpha)(2+1+\theta+2\alpha)\dots(m-1+1+\theta+2\alpha)\cancel{(m+1+\theta+2\alpha)}} \frac{\cancel{m+1+\theta+2\alpha}}{\cancel{m+1-\alpha}}$$

$$= \frac{(1-\alpha)_{(m)}}{(1+\theta+2\alpha)_{(m)}}$$

Note that:

$$E(U_3^m) = \int_0^1 u^m \frac{\Gamma(1-\alpha+\theta+3\alpha)}{\Gamma(1-\alpha)\Gamma(\theta+3\alpha)} u^{1-\alpha-1}(1-u)^{\theta+3\alpha-1}du$$

$$= \int_0^1 \frac{\Gamma(1-\alpha+\theta+3\alpha)}{\Gamma(1-\alpha)\Gamma(\theta+3\alpha)} u^{m+1-\alpha-1}(1-u)^{\theta+3\alpha-1}du$$

$$= \frac{\Gamma(1-\alpha+\theta+3\alpha)}{\Gamma(1-\alpha)\Gamma(\theta+3\alpha)} \int_0^1 u^{m+1-\alpha-1}(1-u)^{\theta+3\alpha-1}du$$

$$= \frac{\Gamma(1-\alpha+\theta+3\alpha)}{\Gamma(1-\alpha)\cancel{\Gamma(\theta+3\alpha)}} \frac{\Gamma(m+1-\alpha)\cancel{\Gamma(\theta+3\alpha)}}{\Gamma(m+1-\alpha+\theta+3\alpha)}$$

$$= \frac{\Gamma(1+\theta+2\alpha)}{\Gamma(1-\alpha)} \frac{\Gamma(m+1-\alpha)}{\Gamma(m+1+\theta+2\alpha)}$$

$$= \frac{\Gamma(m+1-\alpha)}{\Gamma(1-\alpha)} \frac{\Gamma(1+\theta+2\alpha)}{\Gamma(m+1+\theta+2\alpha)}$$

$$= \frac{(1-\alpha)_{(m)}}{(1+\theta+2\alpha)_{(m)}} = E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right)$$

Thus, we can conclude that

$$E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right)=E(U_3^m)\quad\triangle$$

To conclude the proof, we now consider:

$$\begin{aligned}
E\left(P_{\tau_3}^m\right) &= E\left((1-P_{\tau_1}-\tau_2)^m\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\right)\\
&= E_{\mathbf{P},\tau_1,\tau_2}\left\{E\left[(1-P_{\tau_1}-P_{\tau_2})^m\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\middle|\,\mathbf{P},\tau_1,\tau_2\right]\right\}\\
&= E_{\mathbf{P},\tau_1,\tau_2}\left\{(1-P_{\tau_1}-P_{\tau_2})^m\,E\left[\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\middle|\,\mathbf{P},\tau_1,\tau_2\right]\right\}\\
&= E\left((1-P_{\tau_1}-P_{\tau_2})^m\sum_{n=1}^{\infty}\left(\frac{P_n}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\frac{P_n}{1-P_{\tau_1}-P_{\tau_2}}I_{n\neq\tau_1,n\neq\tau_2}\right)\\
&= E\left((1-P_{\tau_1}-P_{\tau_2})^m\sum_{n=1}^{\infty}\left(\frac{P_n}{1-P_{\tau_1}-P_{\tau_2}}\right)^m\frac{P_n}{1-P_{\tau_1}-P_{\tau_2}}I_{P_n\neq P_{\tau_1},P_n\neq P_{\tau_2}}\right)\\
&= E\left((1-P_{\tau_1}-P_{\tau_2})^m\sum_{n=1}^{\infty}\left(\frac{P_n}{1-P_{\tau_1}-P_{\tau_2}}\right)^{m+1}I_{P_n\neq P_{\tau_1},P_n\neq P_{\tau_2}}\right)\\
&= E\left((1-V_1-V_2)^m\sum_{n=1}^{\infty}\left(\frac{P_n}{1-V_1-V_2}\right)^{m+1}I_{P_n\neq V_1,P_n\neq V_1}\right)\quad\text{By the argument on page 2}\\
&= E\left((1-V_1-V_2)^m\sum_{n=3}^{\infty}\left(\frac{V_n}{1-V_1-V_2}\right)^{m+1}\right)\quad\text{By the argument on page 2}\\
&= E\left((1-U_1-(1-U_1)U_2)^m\sum_{n=3}^{\infty}\left(\frac{\prod_{i=1}^{n-1}(1-U_i)\,U_n}{1-U_1-(1-U_1)\,U_2}\right)^{m+1}\right)\\
&= E\left((1-U_1)^m(1-U_2)^m\sum_{n=3}^{\infty}\left(\frac{\prod_{i=1}^{n-1}(1-U_i)\,U_n}{(1-U_1)\,(1-U_2)}\right)^{m+1}\right)\\
&= E\left((1-U_1)^m(1-U_2)^m\sum_{n=3}^{\infty}\left(\prod_{i=3}^{n-1}(1-U_i)\,U_n\right)^{m+1}\right)\\
&= E\left((1-U_1)^m(1-U_2)^m\right)E\left(\sum_{n=3}^{\infty}\left(\prod_{i=3}^{n-1}(1-U_i)\,U_n\right)^{m+1}\right)\quad\text{As the } U_i's \text{ are independent}\\
&= E\left((1-U_1)^m(1-U_2)^m\right)E\left(\sum_{n=3}^{\infty}\left(\frac{\prod_{i=1}^{n-1}(1-U_i)\,U_n}{(1-U_1)(1-U_2)}\right)^{m+1}\right)\\
&= E\left((1-U_1)^m(1-U_2)^m\right)E\left(\sum_{n=3}^{\infty}\left(\frac{\prod_{i=1}^{n-1}(1-U_i)\,U_n}{1-U_1-(1-U_1)U_2}\right)^{m+1}\right)\\
&= E\left((1-U_1)^m(1-U_2)^m\right)E\left(\sum_{n=3}^{\infty}\left(\frac{V_n}{1-V_1-V_2}\right)^{m+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= E\left((1-U_1)^m(1-U_2)^m\right) E\left(\sum_{n=3}^{\infty}\left(\frac{V_{(n)}}{1-V_1-V_2}\right)^{m+1}\right)\\
&= E\left((1-U_1)^m(1-U_2)^m\right) E\left(\left(\frac{P_{\tau_3}}{1-P_{\tau_1}-P_{\tau_2}}\right)^{m}\right) \quad \text{See page 2}\\
&= E\left((1-U_1)^m(1-U_2)^m\right) E\left(U_3^m\right) \text{ by } \triangle\\
&= E\left((1-U_1)^m(1-U_2)^m U_3^m\right)\\
&= E\left(((1-U_1)(1-U_2)U_3)^m\right)\\
&= E\left(V_3^m\right) \quad \text{By the definition of } V_3.
\end{aligned}
$$

Thus, since m was arbitrary, for all $m \in \mathbb{N}$, $E\left(P_{\tau_3}^m\right) = E\left(V_3^m\right)$. The one parameter case follows by setting $\alpha = 0$, concluding the proof.