Various ways for solving a problem case study on the sum of squares

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Methodological aspects.

Objective of this lecture:

to study various methods for finding/proving a mathematical expression involving numbers.

Case stydu: computing the sum of squares

1. Guess an answer, prove it by induction

n	0	1	2	3	4	5	6	7	8	9	10
n^2	0	1	4	9	16	25	36	49	64	81	100
Sn	0	1	5	14	30	55	91	140	204	285	385

1. Compute the first ranks and derive a formula
2. Open a book [Sloane, Handbook of integer sequences - Academic Press 1973] or go to the Internet (this problem has been extensively studied before).

$S_n = n(n+1)(2n+1)/6$

Guess:

$S_n = n(n+1/2)(n+1)/3$. Check the first ranks…

Proof.

Basis: trivially $S_0 = 0$ and the expression holds

Induction step: Assume the expression is true for all integers lower than n-1.

$S_n = S_{n-1} + n^2$

$3S_n = (n-1)(n-1/2)n + 3n^2$

$= n^3 + 3/2n^2 + 1/2n$

$= n(n+1/2)(n+1)$

2. Asymptotic behaviour
2.1. A rough upper bound

A first attempt is to guess the order of magnitude of the summation.

$$S_n = \Sigma_{1 \le k \le n} k^2$$

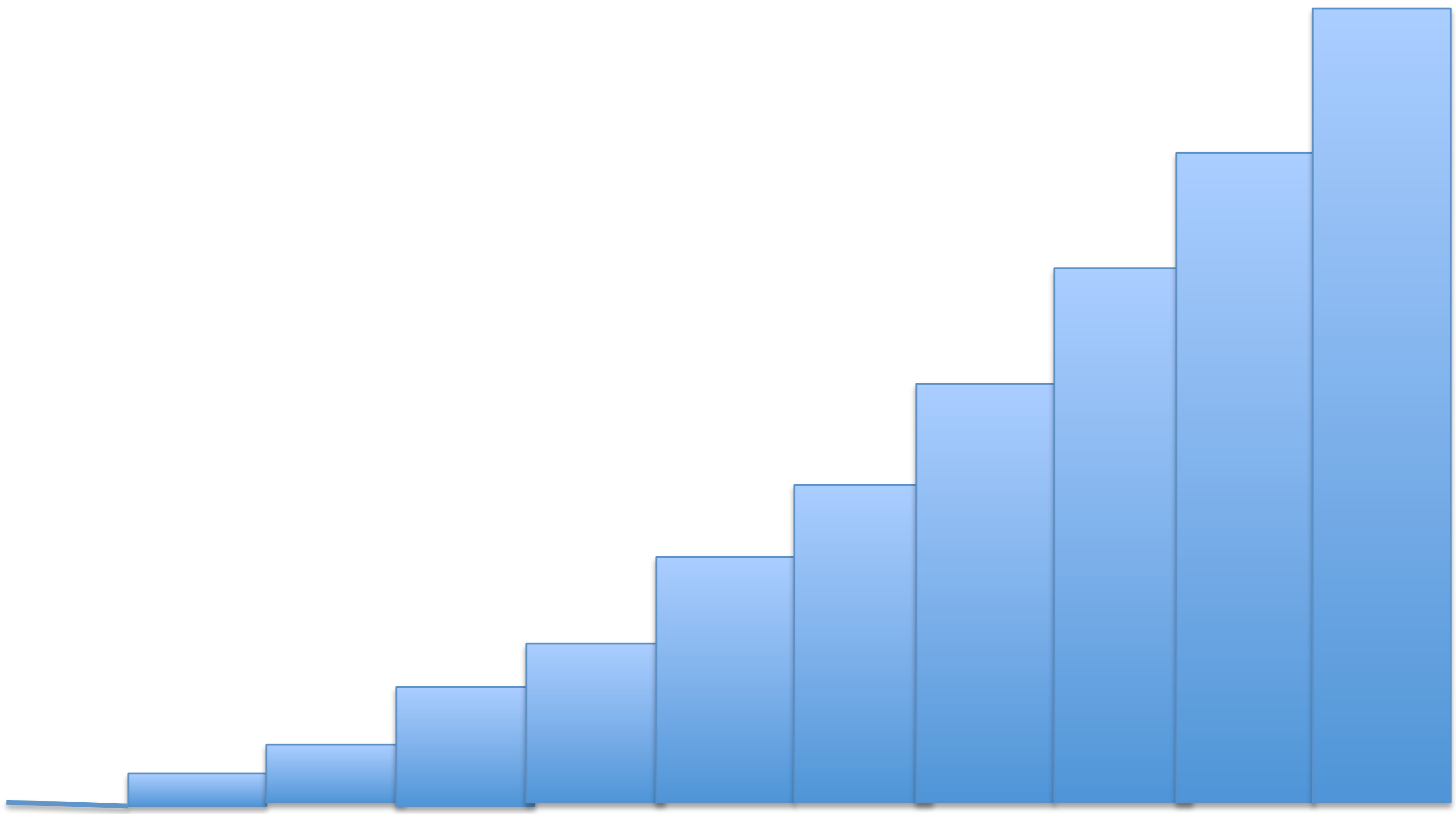
An easy way is for instance to bound every k by n (because $k \le n$)

Thus, $S_n \le \Sigma_{1 \le k \le n} n^2 = n^3$

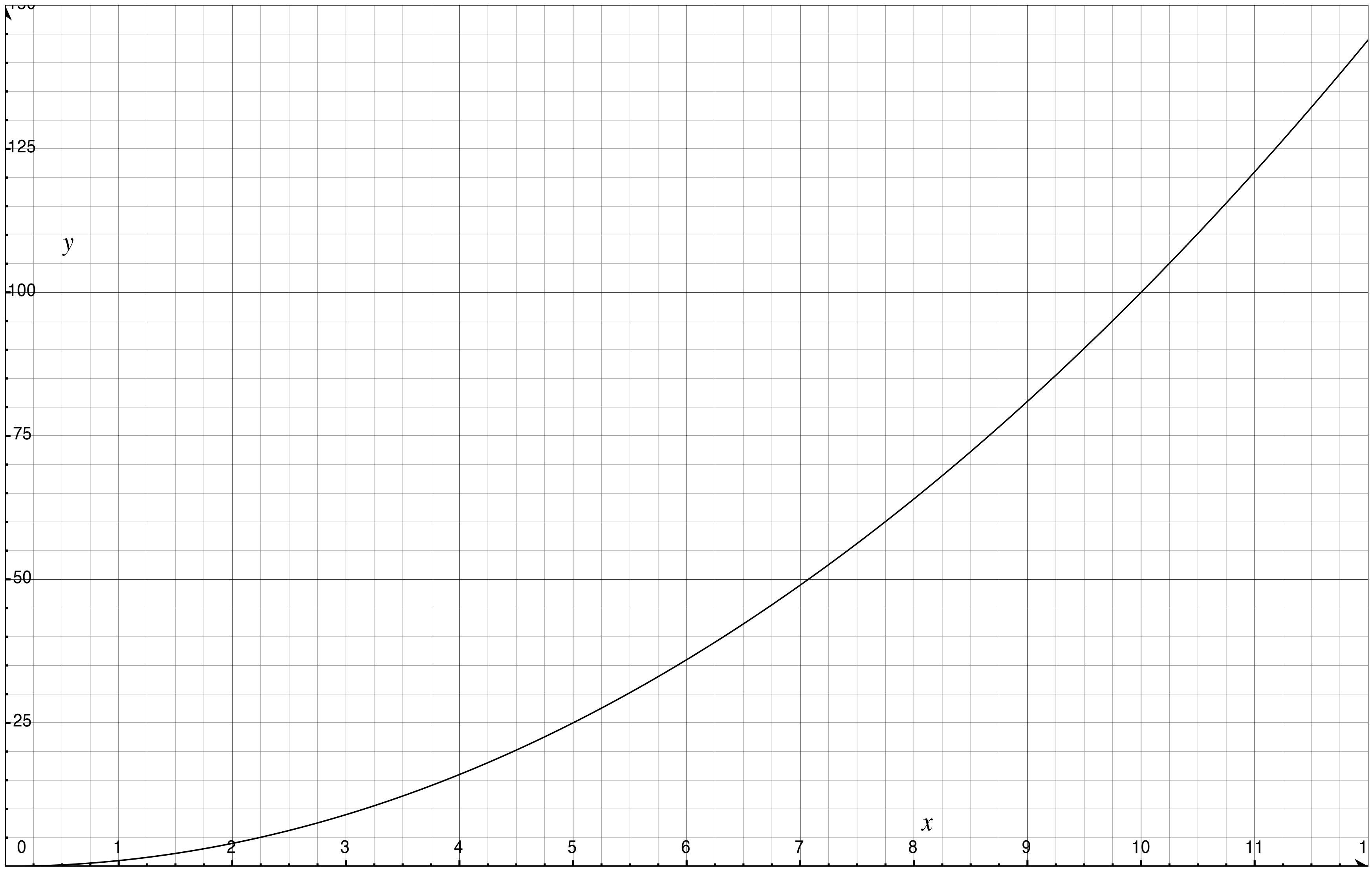
2. Asymptotic behaviour

2.2. Replace sums by integrals

Alternatively, let us consider the continuous problem for determining an asymptotic value (more accurate than the previous value).



$f(x) = x^2$



The sum is approached by the integral of:

x^2dx between 0 and $n = 1/3\ n^3$

We can even refine by lower bound and upper bound and determine exactly the preponderant term…

3. Polynomial

We guess a progression in degree 3:

$S(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

We have 4 equations for determining the 4 unknowns a_i:

$S(0) = 0$, thus, $a_0 = 0$

$S(1) = 1$, $a_1 + a_2 + a_3 = 1$

$S(2) = 5$, $2a_1 + 4a_2 + 8a_3 = 5$

$S(3) = 14$, $3a_1 + 9a_2 + 27a_3 = 14$

We obtain:

$a_0 = 0$

$a_1 = 1/6$

$a_2 = 1/2$

$a_3 = 1/3$

$$
\begin{aligned}
S(x) &= 1/6\ x + 1/2\ x^2 + 1/3\ x^3 \\
&= x(1 + 3x + 2x^2)/6 \\
&= x(x+1)(2x+1)/6
\end{aligned}
$$

4. Perburb the sum

The perturbation method is to create an equation by extracting the first or the last term of the sum (then, write the sum in a different manner) :

$S_{n+1} = S_n + (n+1)^2$

$S_{n+1} = \Sigma_{1 \le k \le n+1} k^2 = \Sigma_{0 \le k \le n} (k+1)^2$

$= \Sigma_{0 \le k \le n} (k^2 + 2k + 1) = S_n + 2\Sigma_{0 \le k \le n} k + n+1$

Thus, $S_n + (n+1)^2 = S_n + 2\Sigma_{0 \le k \le n} k + n+1$

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Thus, $S_n + (n+1)^2 = S_n + 2\sum_{0 \le k \le n} k + n+1$

We lose!

This is a way to compute the sum of first integers…

$(n+1)^2 = 2\Sigma_{0 \le k \le n} k + n+1$

Let us try with sum of the cubes in order to get S_n

$C_{n+1} = C_n + (n+1)^3$

$C_{n+1} = \Sigma_{1 \le k \le n+1} k^3 = \Sigma_{0 \le k \le n} (k+1)^3$

$= \Sigma_{0 \le k \le n} (k^3 + 3k^2 + 3k + 1)$

$= C_n + 3S_n + 3n(n+1)/2 + n+1$

Thus, $(n+1)^3 = 3S_n + 3n(n+1)/2 + n+1$

$3S_n = (n+1)^3 - 3n(n+1)/2 – (n+1)$

$= (n+1)((n+1)^2 + 2n + 1 – 3n/2 – 1)$

5. Expand and contract

The idea here is to replace the original sum by a more complicated double sum that can be simplified afterwards.

$S_n = \sum_{1 \le k \le n} k^2 = \sum_{1 \le j \le k \le n} k$

because

$S_n = 1 + 2+2 + 3+3+3 + \ldots + (n+n+n\ldots)$ n times

$$\begin{aligned} = 1 + 2 + 3 + \ldots + n \\ + 2 + 3 + \ldots + n \\ + 3 + \ldots + n \\ + \ldots \end{aligned}$$

$S_n = \Sigma_{1\leq j\leq k\leq n}\, k + \Sigma_{1\leq j\leq n}\, \Sigma_{j\leq k\leq n}\, k$

Let us write $\Sigma_{j\leq k\leq n}\, k$ as $\Sigma_{1\leq k\leq n}\, k - \Sigma_{1\leq k\leq j-1}\, k$

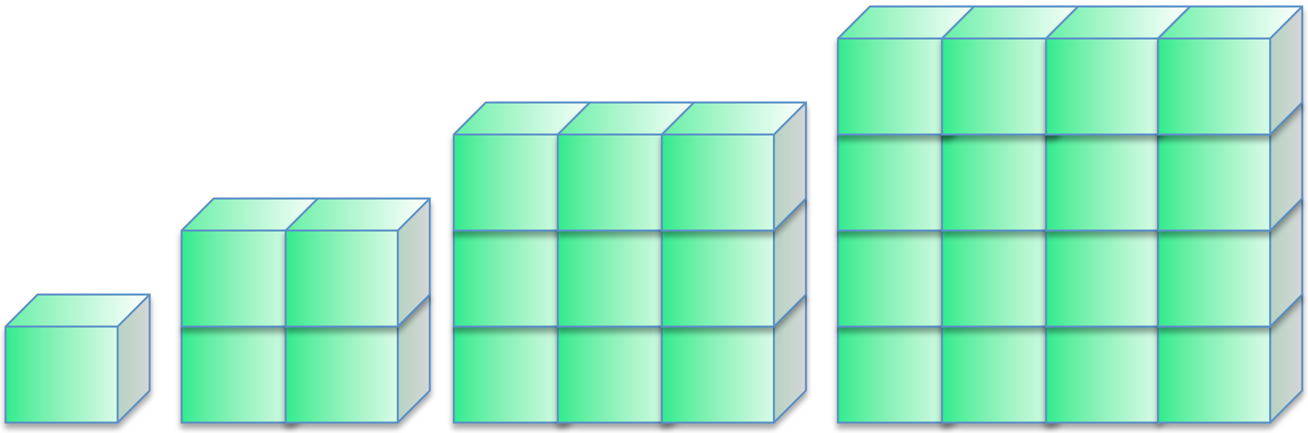
$= \Sigma_{1\leq j\leq n}\, (n-j+1)(n+j)/2$

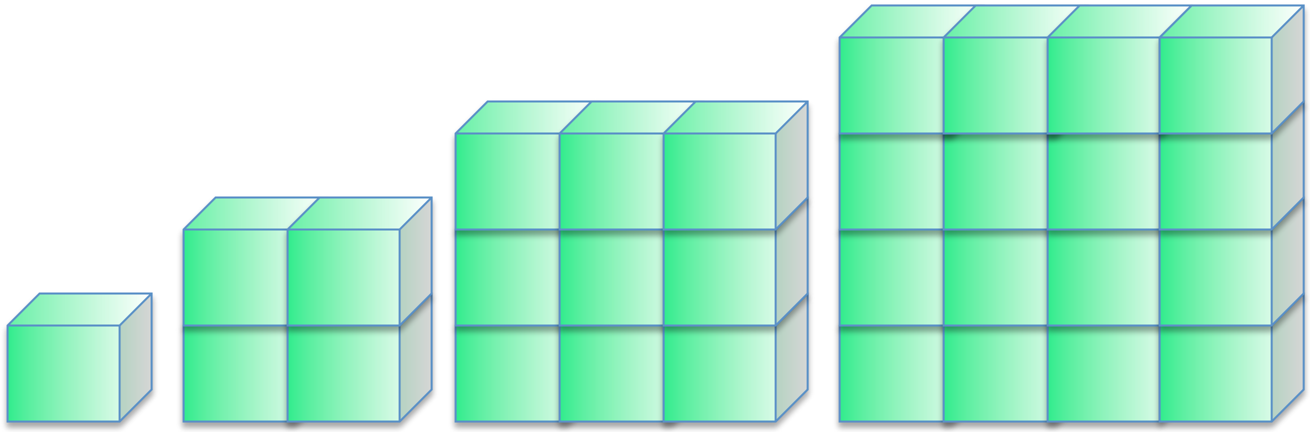
$= 1/2\, \Sigma_{1\leq j\leq n}\, n(n+1)+j-j^2$

$= n^2(n+1)/2 + n(n+1)/4 - S_n/2$

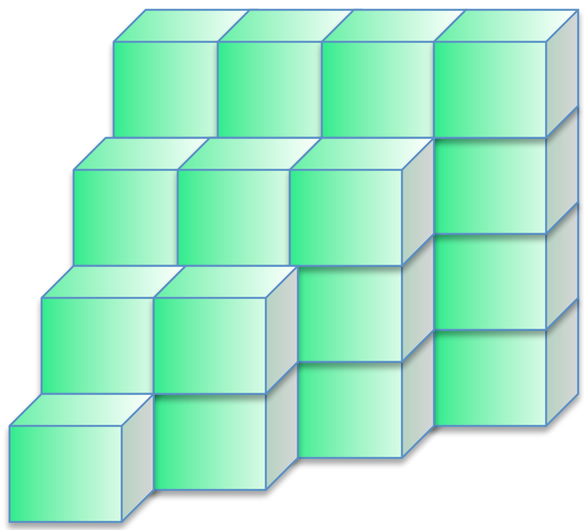
Thus, $3S_n/2 = (n+1)(n^2+n/2)/2$

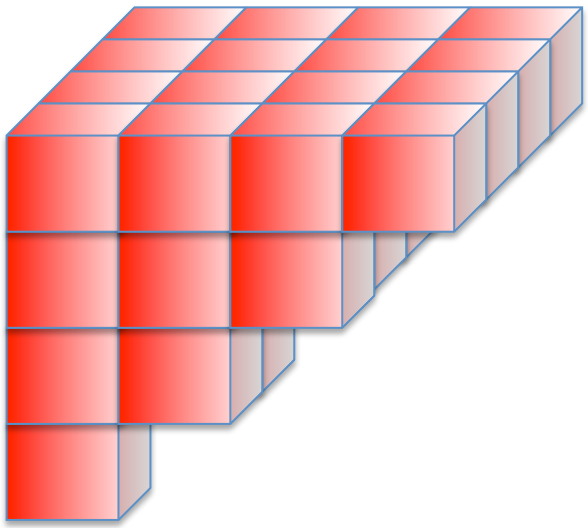
6. Geometrical proof



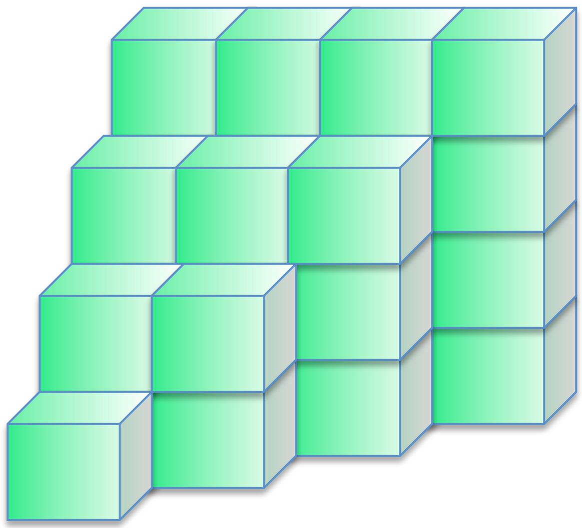


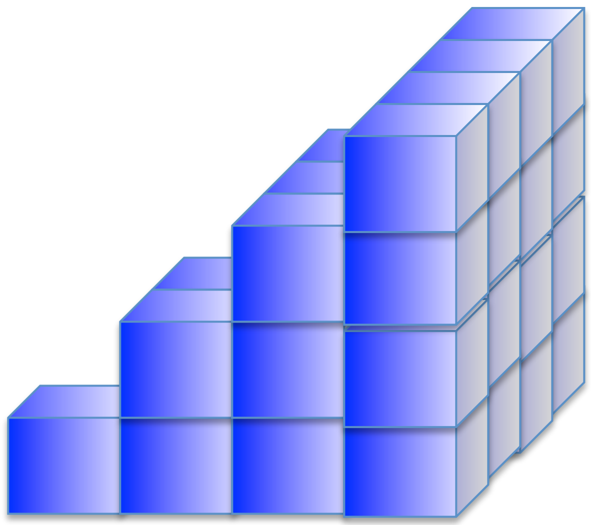
$$1 + 4 + 9 + 16 = S_n$$

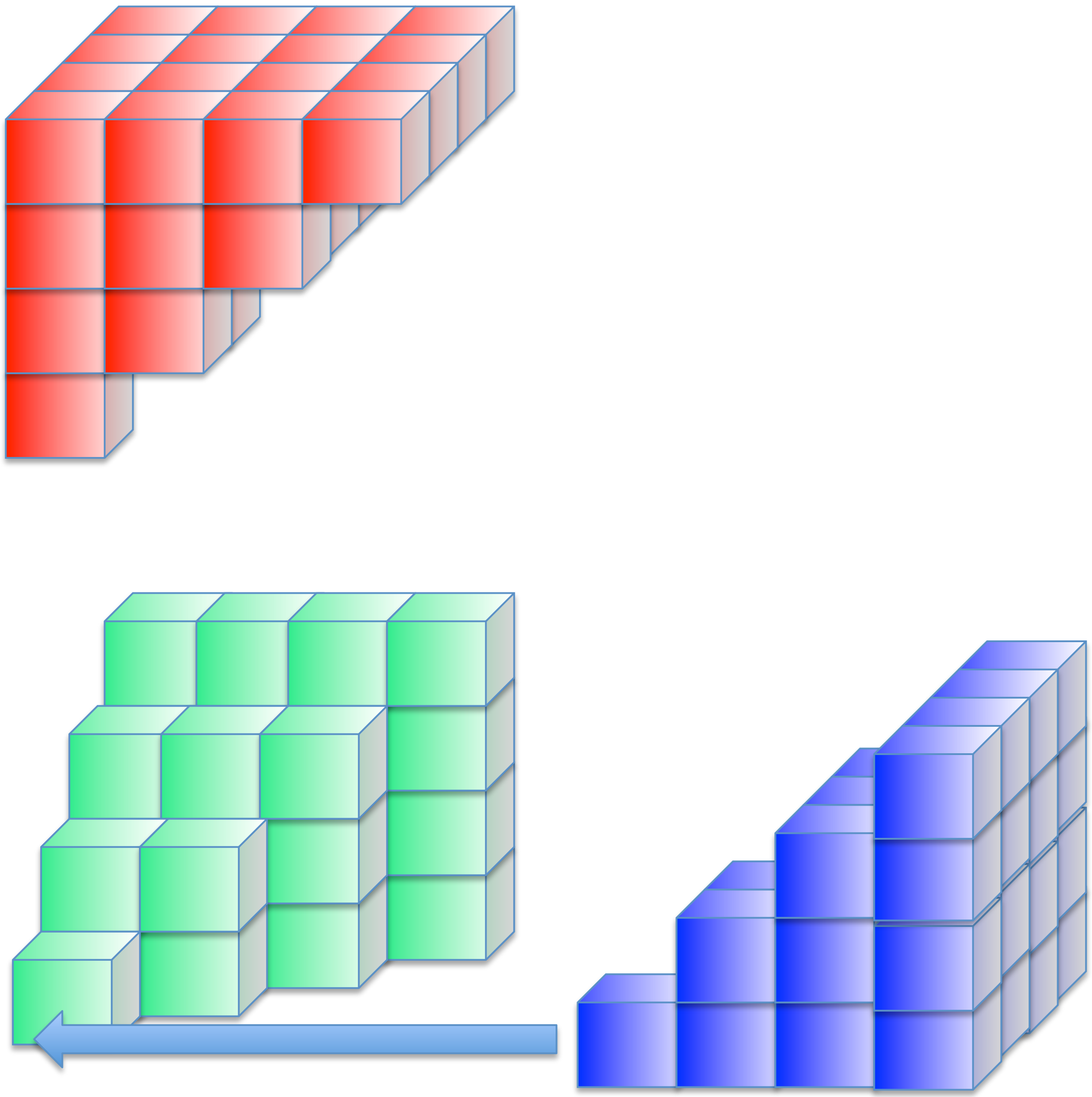


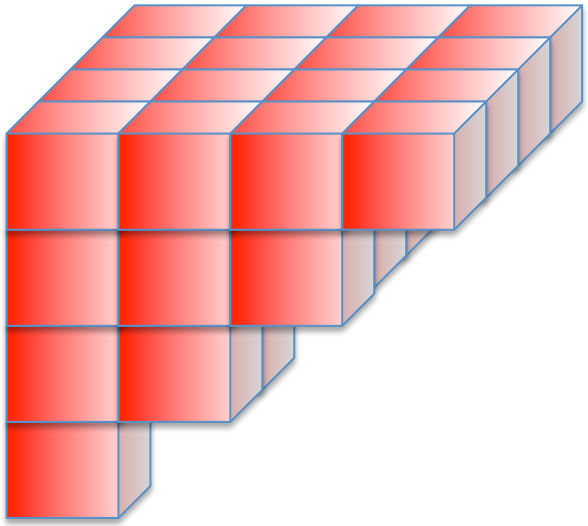


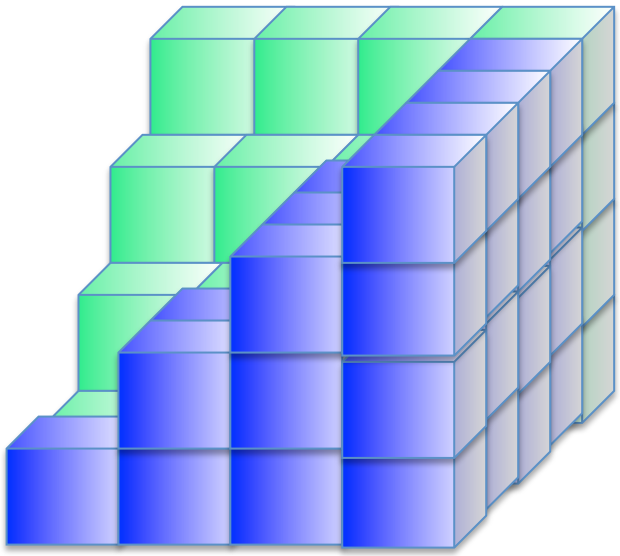
3 times S_n

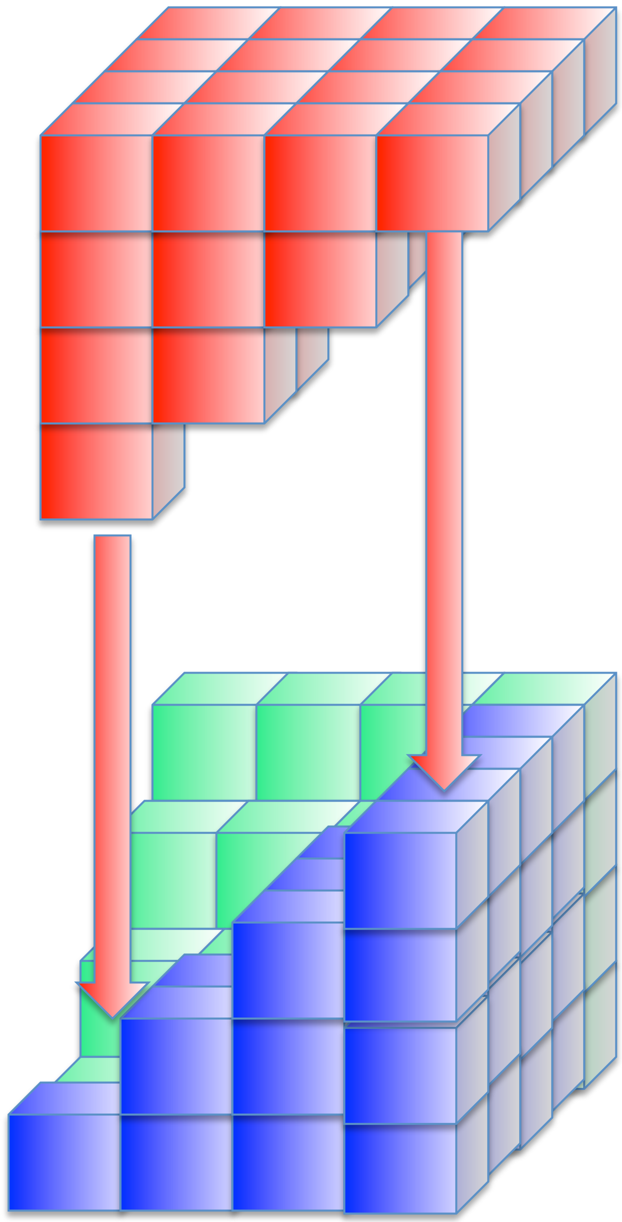


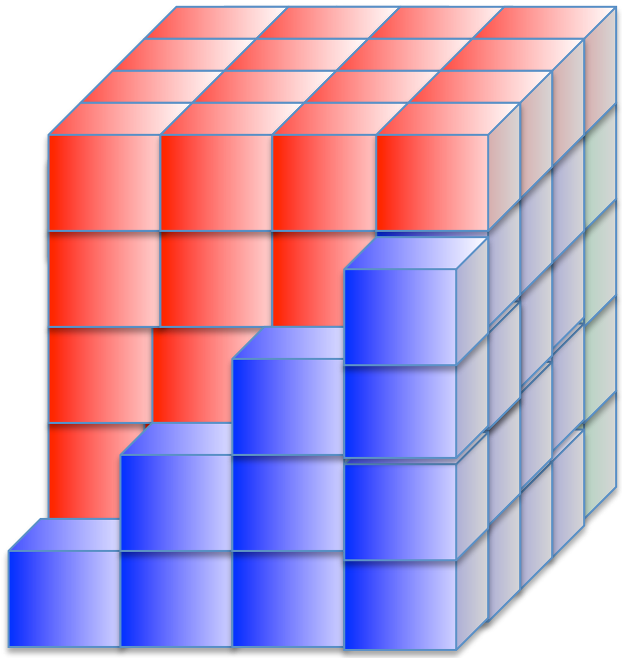












$$3\,S_n = n.n.(n+1) + \tfrac{1}{2}\,n.(n+1)$$
$$= n.(n+1).(n+\tfrac{1}{2})$$