UE Mathematics for Computer Science

Homework 1, October 2015

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Task 1.

According to the statement of the problem:

f: N -> K, where *N* is n-set, i.e. *N* = $\{a_1, a_2, \ldots, a_n\}$, *K* = $\{x_1, x_2, \ldots, x_k\}$.

It means that, if there are no restrictions for *f*, *f(a_i)* takes any value from $\{x_1, x_2, \ldots, x_k\}$, i.e. k possibilities. As to input (parameter) of *f* can be given any of the variables $\{a_1, a_2, \ldots, a_n\}$ => number of functions equals k^n.

Task 2.

If n > k, this is not possible to construct injective function on such sets, as there will be a, b $\in$ $\{a_1, a_2, \ldots, a_n\}$ such that $a \neq b$ and $f(a) = f(b)$, as the number of function values is less than the number of parameters.

If n <= k, than by definition of injection, if f(x) = f(y) => x = y. This implies that if $x \neq y$ => $f(x) \neq f(y)$. So different values from $\{a_1, a_2, \ldots, a_n\}$ should take different function values.

f(a_1) can take any of k possible values $\{x_1, x_2, \ldots, x_k\}$,

f(a_2) can take any of k - 1 possible values $\{x_1, x_2, \ldots, x_k\}$ / *f(a_1)*,

f(a_3) can take any of k - 2 possible values $\{x_1, x_2, \ldots, x_k\}$ / {*f(a_1), f(a_2)*},

…

f(a_n) can take any of k - n + 1 possible values $\{x_1, x_2, \ldots, x_k\}$ / {*f(a_1), f(a_2), …, f(a_{n-1})*}.

Thus, the number of such function equals k * (k-1) * (k - n + 1) = k! / (k - n)!.

Task 3.

We denote the number of surjective functions *f: N -> K* as $S_{n,k}$. Us we proved before, the number of functions with no restrictions is equal to k^N. Let's suppose that surjection take place only for one x_i from K = $\{x_1, x_2, \ldots, x_k\}$, so $x_i = f(a_j)$. The number of such functions equals the number of ways to pick x_i (C_K^1) multiply by the number of functions without restrictions on K' ={ x_1 , x_2 , …, x_k} / {x_i} equal to $(k-1)^N$ for the rest values, and equals C_K^1* $(k-1)^N$. Now let's suppose that surjection takes place only for x_i, x_j from K = {x_1 , x_2 , …, x_k}, so $x_i = f(a_k)$, $x_j = f(a_m)$. The number of such functions equals the number of ways to pick x_i , x_j (C_K^2) multiply by the number of functions $(k-2)^n$ for the rest values, and equals C_K^2* $(k-2)^N$. Continuing in the same and applying Inclusion-exclusion principle, we will get that the number of surjective functions equals:

$S_{n,k}$ = k^n - C_k^1* $(k-1)^n + C_k^2$* $(k-2)^n$ -...± C_k^{k-1} 1^n = $\sum_{i=0}^{k-1} (-1)^i C_k^i$* $(k-i)^n$

Task 4.

By definition of a bijection *f: N -> K* is bijection if:

1) $\forall a \in N,\ b \in N (f(a) = f(b) \Rightarrow a = b)$
2) $\forall y \in K,\ \exists x \in N$ *such that* $f(x) = y$

It can be obtained that $\forall y \in K,\ \exists$ *only one* $x \in N$ *such that* $f(x) = y$, as if $\exists$ *only one* $x1 \in N$ *such that* $f(x1) = y$ => x1 = x. So it means that to every value from K corresponse only one value from N and vice versa. That is why for a bijection the number of elements in set N equals number of elements in set K, i.e. k = n. We could notice that:

f(a_1) can take any of k possible values {x_1 , x_2 , …, x_k},

f(a_2) can take any of k - 1 possible values {x_1 , x_2 , …, x_k} / *f(a_1)*,

f(a_3) can take any of k - 2 possible values {x_1 , x_2 , …, x_k} / {*f(a_1), f(a_2)*},

…

f(a_n) can take any of k - n + 1 = k - k + 1 = 1 possible values {x_1 , x_2 , …, x_k} / {*f(a_1), f(a_2), …, f(a_{n-1})*}.

Thus, the number of bijective function equals k * (k-1) * 1 = k! = n!