Mathematics for computer science

Homework 1

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Abstract

This report sumarise, explains and refers to the answers we have designed for the first M̈athematics for computer scienceḧomework.
In this report, we will use the following notations:

- N and K are two finite sets of size n and k.
- N will be represented as

$$N = \{n_i | \text{i bellongs to [0, n-1]}\} \tag{1}$$

- The set of expected f function will include the partial functions: a function f bellonging to this set may be undefined on a specific point of its input set N.
  If a function f is undefined on a point x bellonging to N, we will always write

$$f(x) = \epsilon \tag{2}$$

1 Question 1

A function
$$f : N \to K \tag{3}$$
is an application which associates to each element n, bellonging to N, at most one element k bellonging to K.
Thus, building such a function f is equivalent to build a word

$$w = k_0...k_{n-1} \text{ where } \forall i \in [0, n-1] k_i \in K \cup \{\epsilon\} \tag{4}$$

As we have no restriction on f, we have k+1 different choices to choose any one of the ki. This choice is independant from the choice of any kj where j is different from i. Thus, the number of different word

$$w = k_0...k_{n-1} \tag{5}$$

is
$$\prod_{i=0}^{n-1} (k+1) = (k+1)^n \tag{6}$$
Thus, the number of different functions

$$f : K \to N \tag{7}$$

with no restriction on f is
$$(k+1)^n \tag{8}$$
This result would be
$$k^n \tag{9}$$
if we only consider the non partial functions.

2 Question 2

Using the same arguments as previously, we can say that our problem is equivalent to find all the different words

$$w = k_0...k_{n-1} \text{ where } \forall i \in [0, n-1] k_i \in K \cup \{\epsilon\} \tag{10}$$

Let w0 such a word, and fw0 the corresponding injective function. We have:

$$\forall i, j \in [0, n-1] \text{ with } i \neq j, f_{\omega 0}(n_i) = f_{\omega 0}(n_j) \Longrightarrow n_i = n_j \tag{11}$$

which is absurde by definition of ni and nj. So

$$\forall i, j \in [0, n-1], i \neq j \Longrightarrow f_{\omega 0}(n_i) \neq f_{\omega 0}(n_j) \tag{12}$$

Using this condition we can conclude that

- To chose the character k0 of w0 among K union epsilon, we have (k+1) choices.
- To chose the character k1 of w0 among K union epsilon excluding K0, we have (k+1 -1) choices for each k0.

- To chose the character k2 of w0 among K union epsilon excluding K0, K1, we have (k+1 -2) choices for each k0, k1.
- To chose the character ki of w0 among K union epsilon excluding K0, K1, ...ki-1, we have (k+1 - i) choices for each k0, k1, ...ki-1.

Thus, the number of different words w = k0....kn-1 which respect the condition (12), and the number of injective function is:

$$\prod_{i=0}^{n-1} k+1-i = \frac{(k+1)!}{(k-n+1)!} \text{ (as k } \geq \text{ n by definition of an injective function)} \tag{13}$$

This result would be

$$\frac{k!}{(k-n)!} \tag{14}$$

if we only consider the non partial functions.

3 Question 3

Let's f a surjection between N and K. So:

- Each element form K is mapped to at least 1 element from N
- Each element from N is mapped to at most 1 element from K

Thus, according to pigeonhole principle, n and k must to respect the rule:

$$n \geq k \tag{15}$$

Let f such a surjective function, and w its corresponding world:
w = w0, ..., wn-1 where wi = f(ni).
According to the definition of a surjective function, the word w should contain, at least, once every ki bellonging to K. Thus, a word w must be a combination of

$$k_0, ..., k_{k-1} \quad c_0, ..., c_{n-k-1} \text{ where } c_i \in K \cup \{\epsilon\} \tag{16}$$

But we know that the number of different

$$k_0, ..., k_{k-1} \quad \text{is} k! \tag{17}$$

and the number of different

$$c_0, ..., c_{n-k-1} \text{ where } c_i \in K \cup \{\epsilon\} \quad \text{is } (k+1)^{n-k} \tag{18}$$

Thus, the number of different word w, and the number of surjective functions is

$$(k+1)^{(n-k)} k! \text{ with } n \geq k \tag{19}$$

4 Question 4

By definition of an bijection, the input and output sets must have the same cardinals. Thus, in the following answer, we will consider the condition k = n respected.
Building a bijection between N and K is equivalent to build a word

$$\omega = k_0, k_1, ..., k_{n-1} \text{ where } \forall i, j \in [0, n-1] k_i \neq k_j \tag{20}$$

Thus,

- To chose the character k0 among K, we have k choices.
- To chose the character k1 among K excluding K0, we have (k - 1) choices for each k0.
- To chose the character k2 among K excluding K0, K1, we have (k - 2) choices for each k0, k1.
- To chose the character ki among K excluding K0, K1, ...ki-1, we have (k - i) choices for each k0, k1, ..., ki-1.

Thus, the number of different bijective function from N to K is

$$\prod_{i=0}^{n-1} k - i = \prod_{i=0}^{n-1} n - i = n! \tag{21}$$