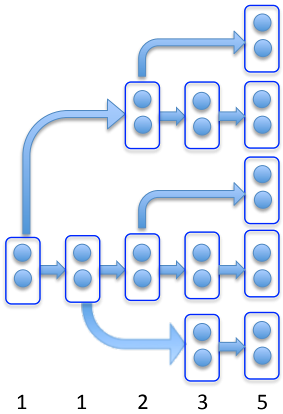

Figure 1: Construction of the Fibonacci progression

Fibonacci

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Lecture notes **Maths for Computer Science** – *MOSIG 1 – 2015*

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1 Fibonacci numbers

In the original problem introduced by Leonardo of Pisa (Fibonacci) in the middle age, this number is the number of pairs of rabbits that can be produced at the n-th generation. Starting by a single pair of rabbits and assuming that each pair produces a new pair of rabbits at each generation during only two generations.

Definition Given the two numbers F(0) = 1 and F(1) = 1, the Fibonacci numbers are obtained by the following expression: F(n+1) = F(n)+F(n−1).

Notice that it is a special case of $u_{n+1} = a.u_n + b.u_{n-1}$ for $a = b = 1$.

2 Some recurrence on Fibonacci numbers

Fibonacci numbers are defined by the following numerical progression: $n \geq 2$ $F(n) = F(n-1) + F(n-2)$, $F(0) = F(1) = 1$. These numbers have nice properties, like the following ones.

- The relation $F(n-1).F(n) = \sum_{k=0}^{n-1} F^2(k)$ (for $n \geq 1$) can be proved by using a geometric argument (see figure **??**). Show it by induction.

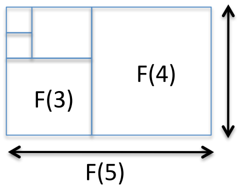
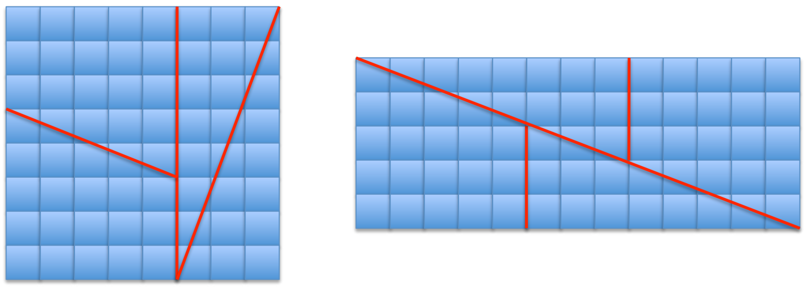


Figure 2: Geometric interpretation of the relation $F(4).F(5) = F^2(0) + F^2(1) + F^2(2) + F^2(3) + F^2(4)$.



- $F(n+1)^2 = 4.F(n).F(n-1) + F(n-2)^2$ for $n \geq 2$
- Show the following Cassini's identity: $F(n-1).F(n+1) = F^2(n) + (-1)^{n+1}$ for $n \geq 1$.

  Here is the proof:

  $F(n).F(n+2) = F(n)(F(n+1)+F(n))$ by definition of the Fibonacci progression

  $= F(n).F(n+1) + F(n)^2$

  from the recurrence hypothesis, we have

  $F^2(n) = F(n-1).F(n+1) - (-1)^{n+1} = F(n-1).F(n+1) + (-1)^{n+2}$
  Thus,

  $F(n).F(n+2) = F(n).F(n+1) + F(n-1).F(n+1) + (-1)^{n+2}$

  $= F(n+1)(F(n)+F(n-1)) + (-1)^{n+2}$ and again, since $F(n)+F(n-1) = F(n+1)$

  $= F^2(n+1) + (-1)^{n+2}$

- The previous result (Cassini's identity) is the basis of a geometrical paradox (one of the favorite puzzle of Lewis Carroll). Consider a chess board and cut it into 4 pieces as shown in figure **??**, then reassemble them into a rectangle.

  The surface of the square is F_n^2 while the rectangle is $F_{n+1}.F_{n-1}$. The Cassini identity is applied for $n = 8$. The paradox comes from the

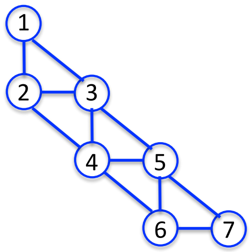


Figure 3: Counting paths from node 1 to node n ($n = 7$)

wrong representation of the diagonal of the rectangle which does not coincide with the hypothenus of the rectangle triangles of sides F_{n-1} and F_{n-2}. In other words, it always remains (for any n) an empty space (corresponding to the unit size of the basic case of the chess board). The greater n, the better the paradox because the surface of this basic case becomes more tiny.

- $F(n)$ is the number of paths from node 1 to n in the following family of graphs of figure **??**. Show how this number is related to Fibonacci's numbers.