Handling numbers
Geometrical Proofs

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Content

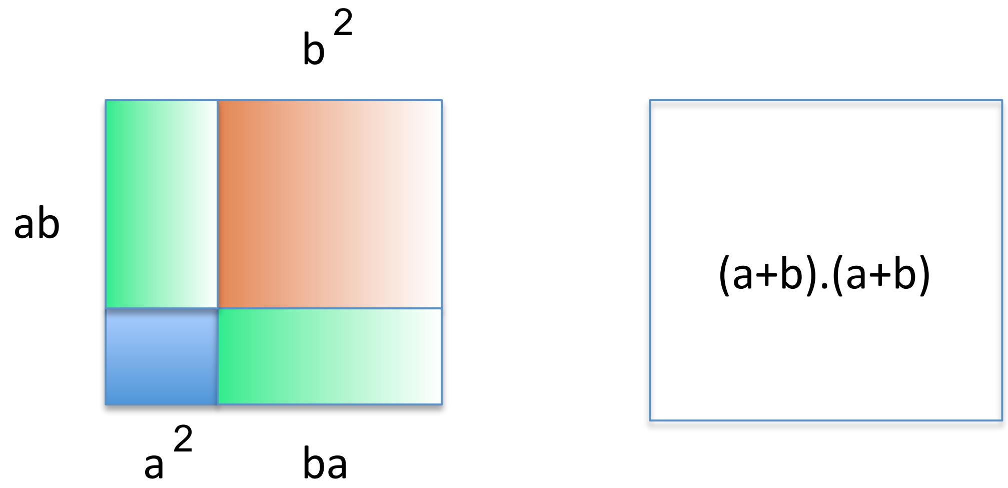
Preliminaries

- Basic identities
- Some properties

Summations

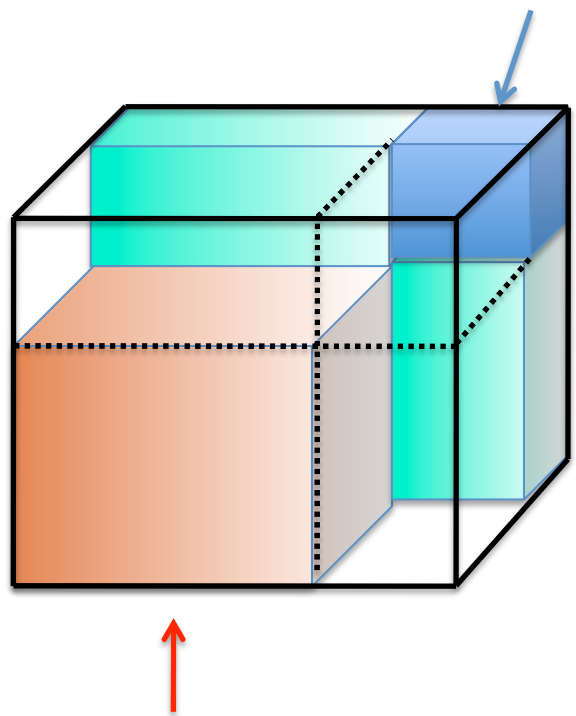
- Triangular numbers (sum of the n first integers)
- Sum of the first odd integers
- Sum of the first cubes
- Sum of the first squares

Starting with simple Identities

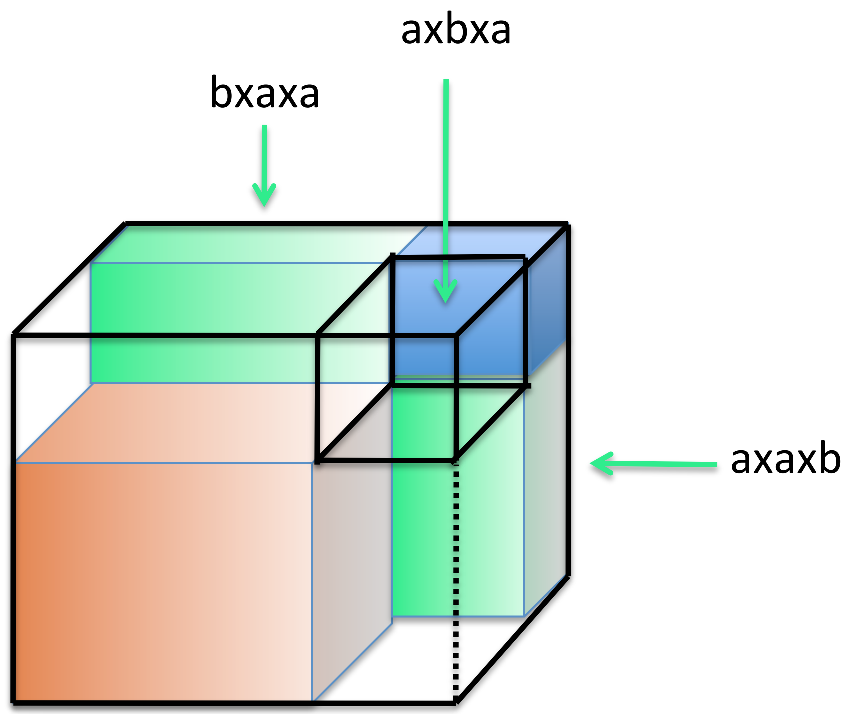


$$(a+b)^2 = a^2 + 2ab + b^2$$

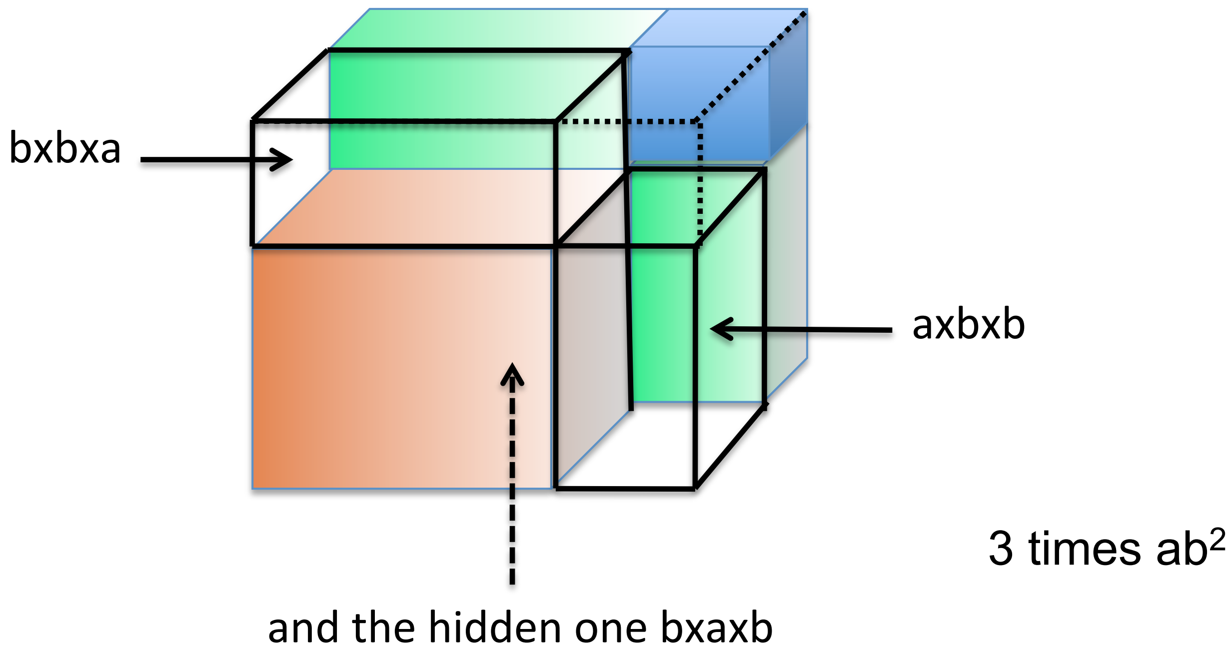
Computing $(a+b)^3$



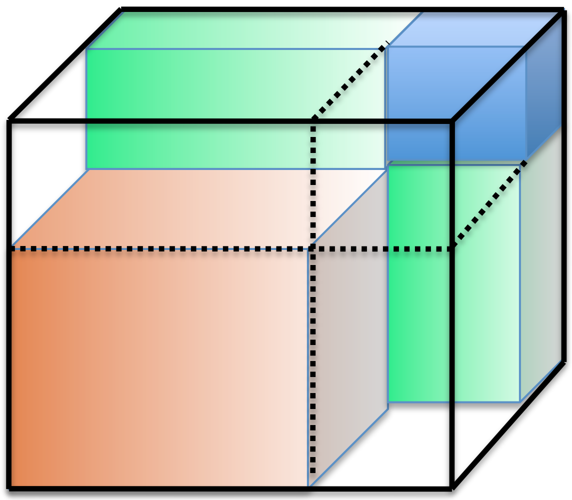
$a^3 + b^3$



3 times a^2b

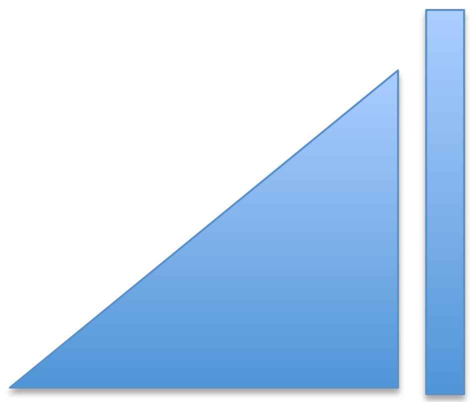

bxbxa
axbxb
and the hidden one bxaxb
3 times ab2

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$



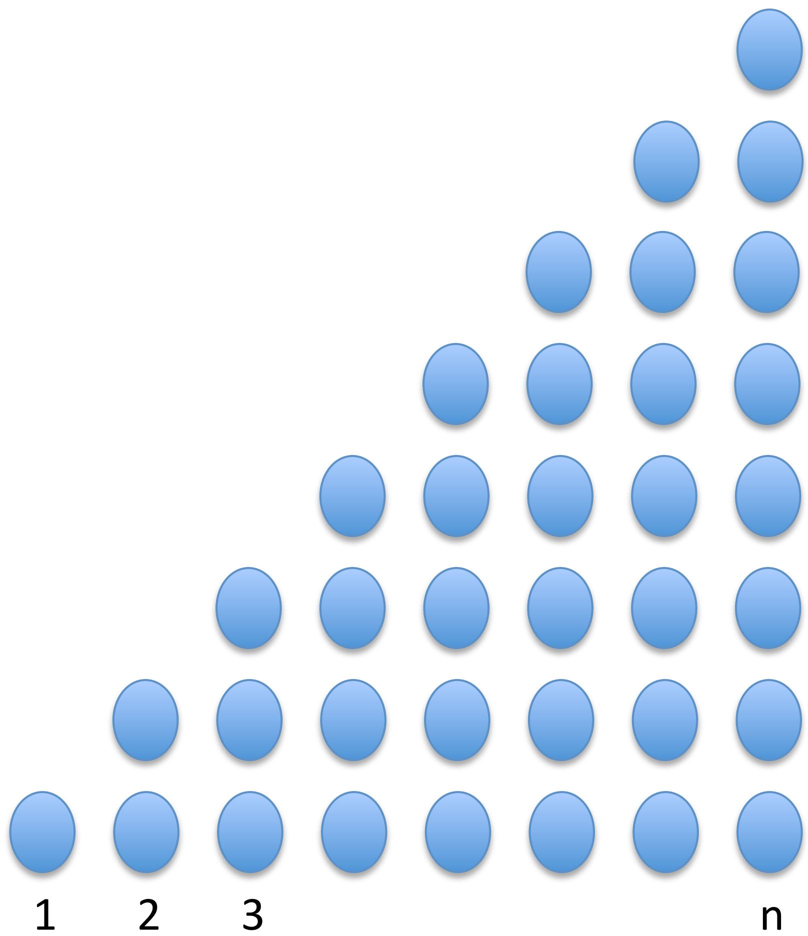
Triangular numbers
Sum of the n first integers

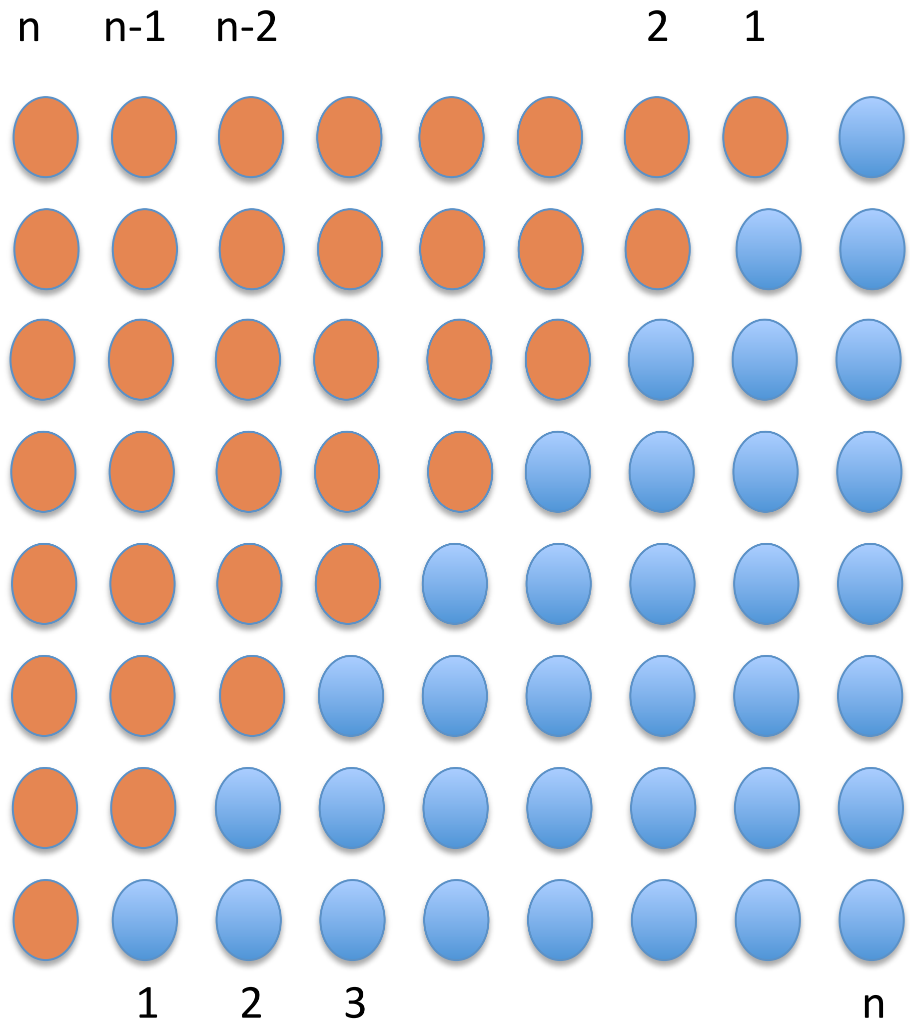
$\Delta_n = 1+2+3+ \ldots + (n-1) + n$

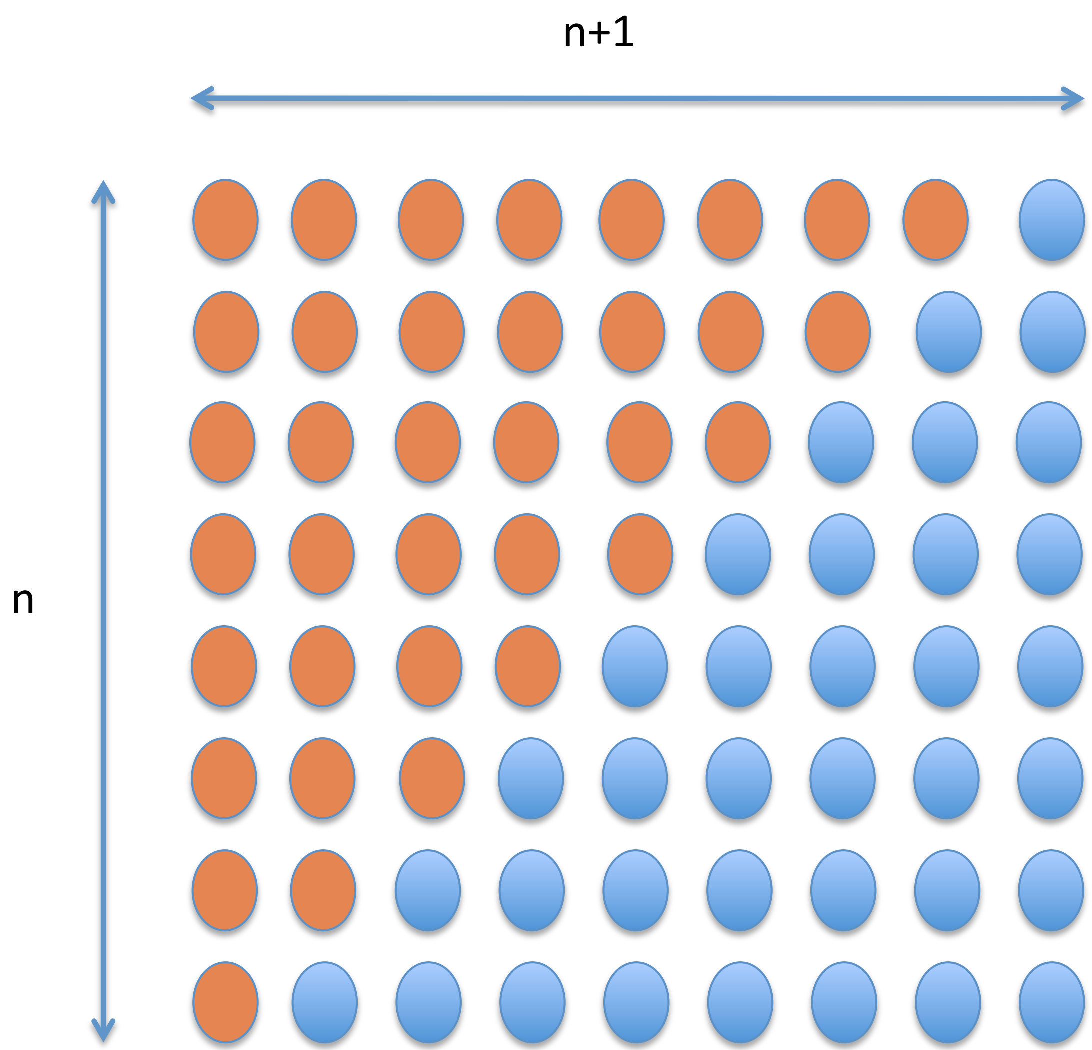


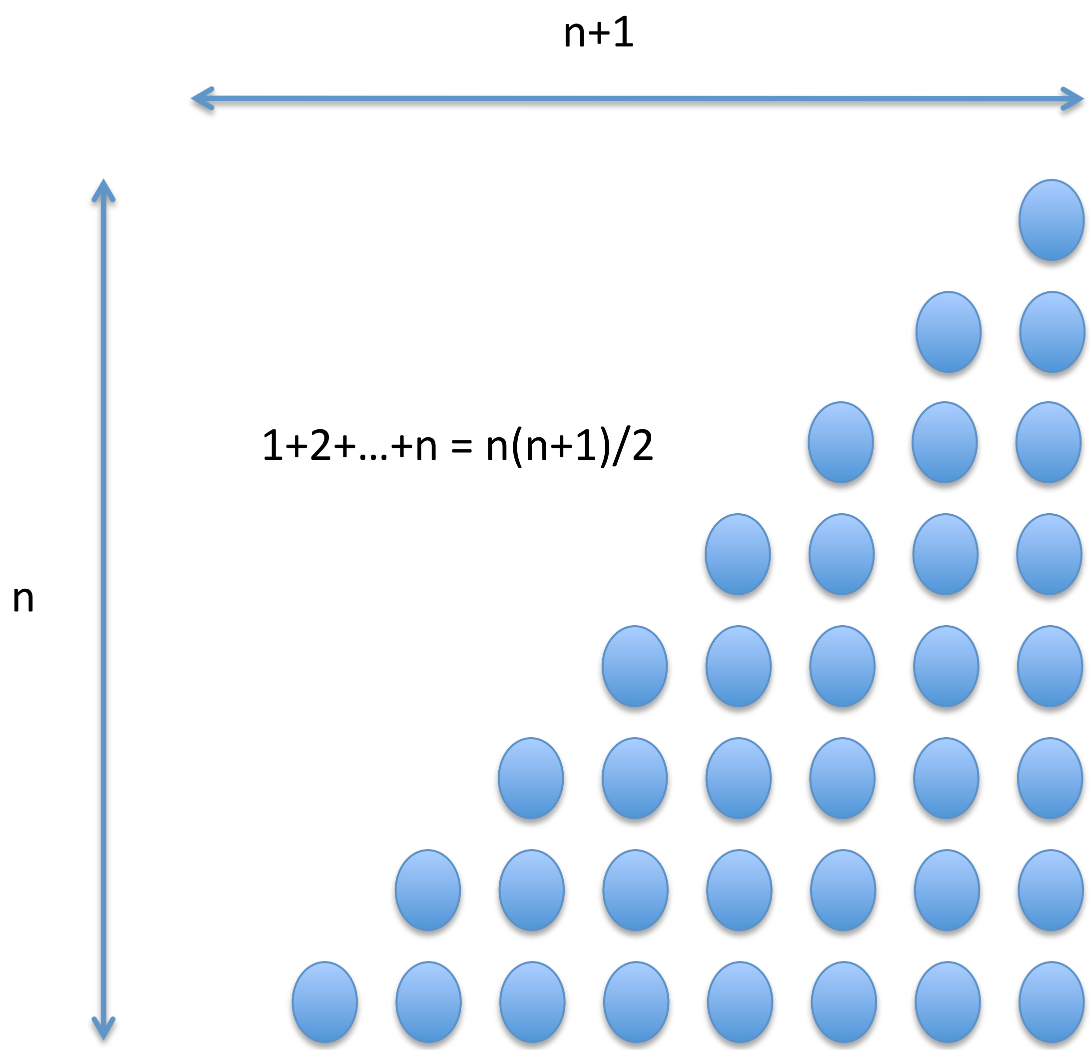
gnomon:
column

Δ_n

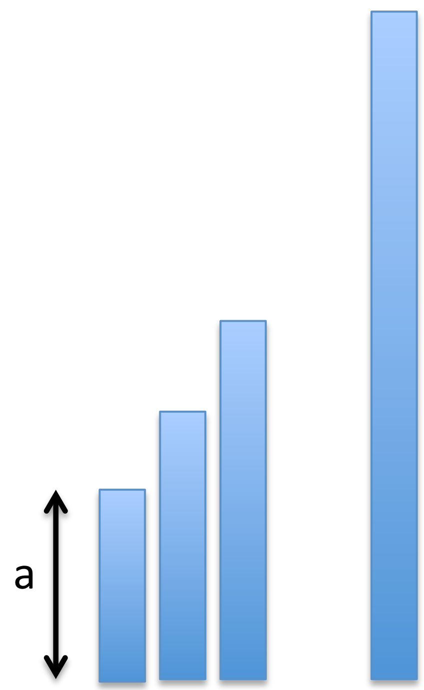



n
n-1
n-2
2
1
1
2
3
n


n+1
n


n+1
1+2+...+n = n(n+1)/2
n

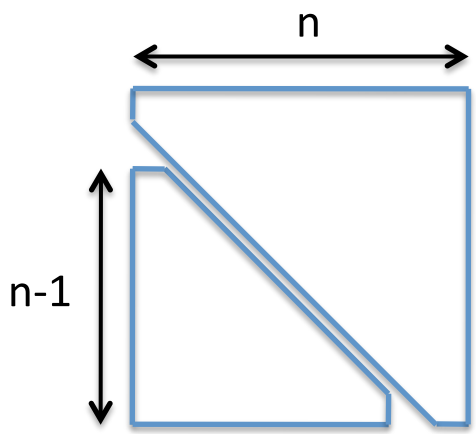
Extension
link with arithmetic progressions…



Average between the first and the last number.

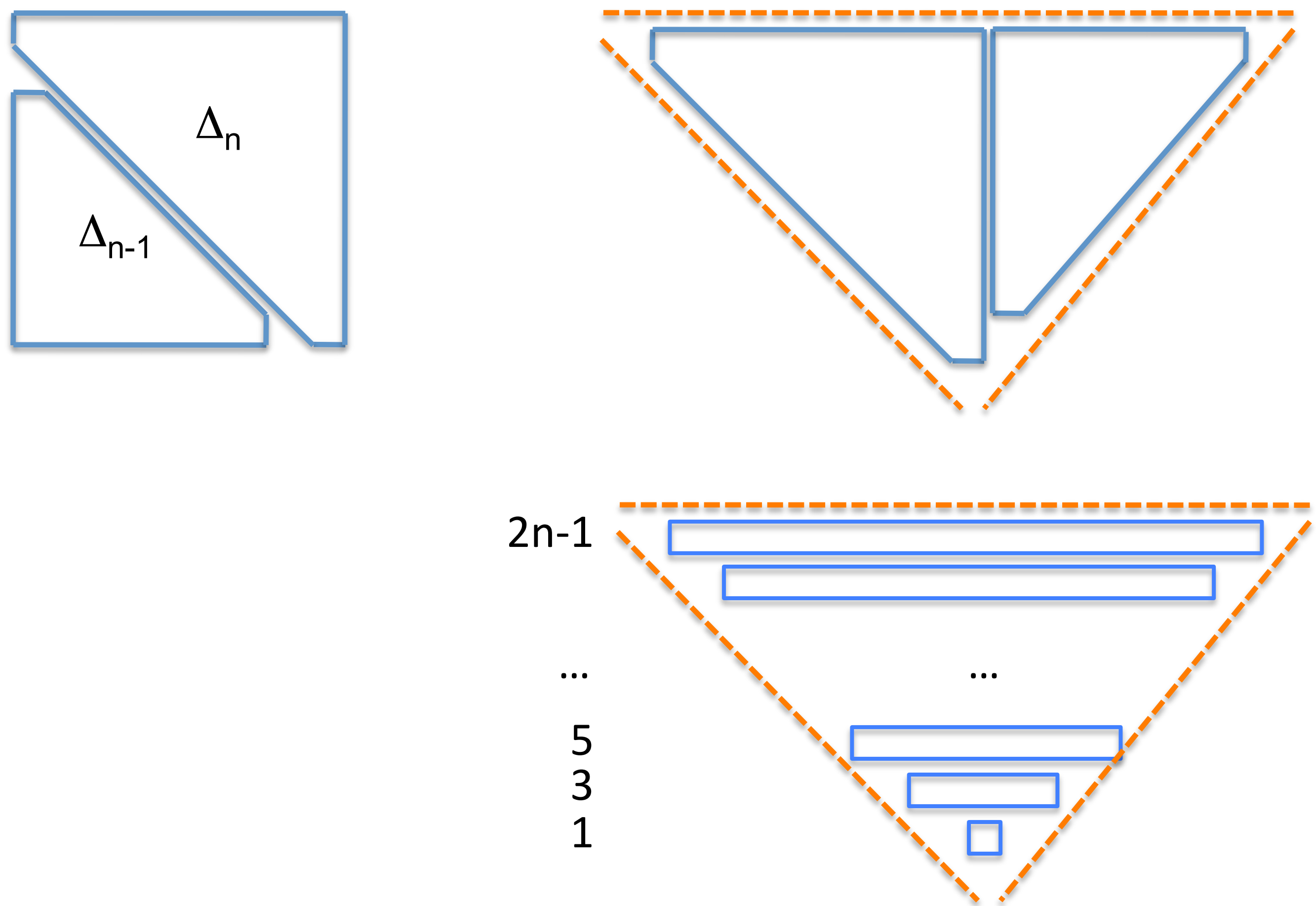
Some properties on triangular numbers (1)

$\Delta_n + \Delta_{n-1} = n^2$



Some properties on triangular numbers (2)

$$\begin{aligned}\Delta_n + \Delta_{n-1} &= 1 + \boxed{\begin{matrix}2\\1\end{matrix}} + \boxed{\begin{matrix}3\\2\end{matrix}} + \ldots + \boxed{\begin{matrix}n\\n-1\end{matrix}}\\ &= 1 + 3 + 5 + \ldots + (2n-1)\end{aligned}$$



Δ_n
Δ_{n-1}
2n-1
...
...
5
3
1

Some properties on triangular numbers (3)

Sum of triangular numbers over odd indices

$\Delta_{odd,n} = 1 + 3 + 5 + \ldots + n = \Sigma\, k^3$

			1			
		3		5		
	7		9		11	
13		15		17		19

$$1 \xrightarrow{+} 1^3$$

$$\boxed{3 \quad 5} \longrightarrow 2^3$$

$$\boxed{7 \quad 9 \quad 11} \longrightarrow 3^3$$

$$\boxed{13 \quad 15 \quad 17 \quad 19} \longrightarrow 4^3$$

$$\Delta_{odd,n} = \Sigma\, k^3$$

Some properties on triangular numbers (4)

The sum of cubes is a perfect square $\Delta_n^2 = \Sigma\, k^3$

Direct consequence of the three previous results.
This result may also be obtained simply by multiplicative tables

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25
	x2	x3	x4	x5

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25
	x2	x3	x4	x5

$\downarrow$ +

$$
\begin{aligned}
&1.(1+2+3+4+5)\\
+&2.(1+2+3+4+5)\\
+&3.(1+2+3+4+5)\\
+&4.(1+2+3+4+5)\\
+&5.(1+2+3+4+5)
\end{aligned}
$$

$$= (1+2+3+4+5)\ (1+2+3+4+5)$$

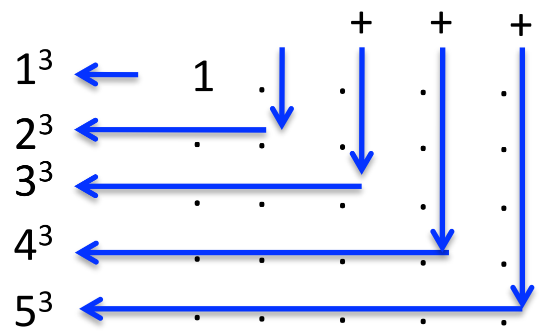
$2+4+2 = 2^3$

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

$$2+4+2 = 2^3$$

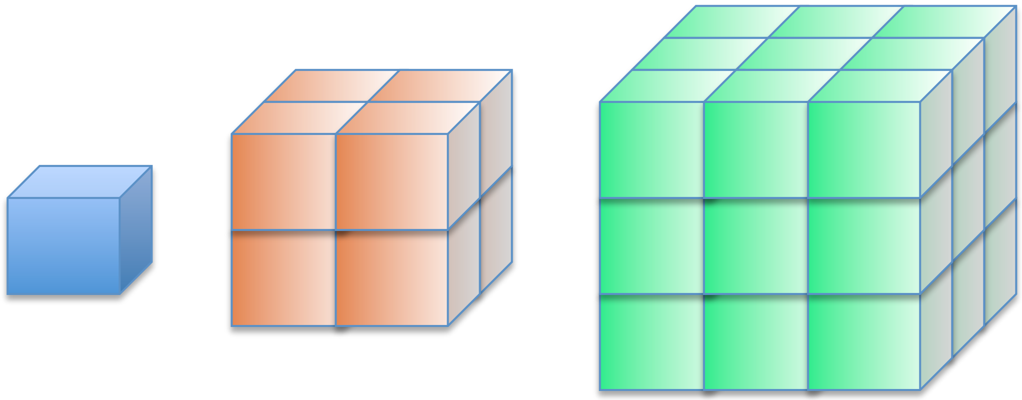
$$3+6+9+6+3 = 3^3$$

		+		
1	2	.	4	5
2	4	.	8	10
.	.	.	12	15
4	8	12	16	20
5	10	15	20	25

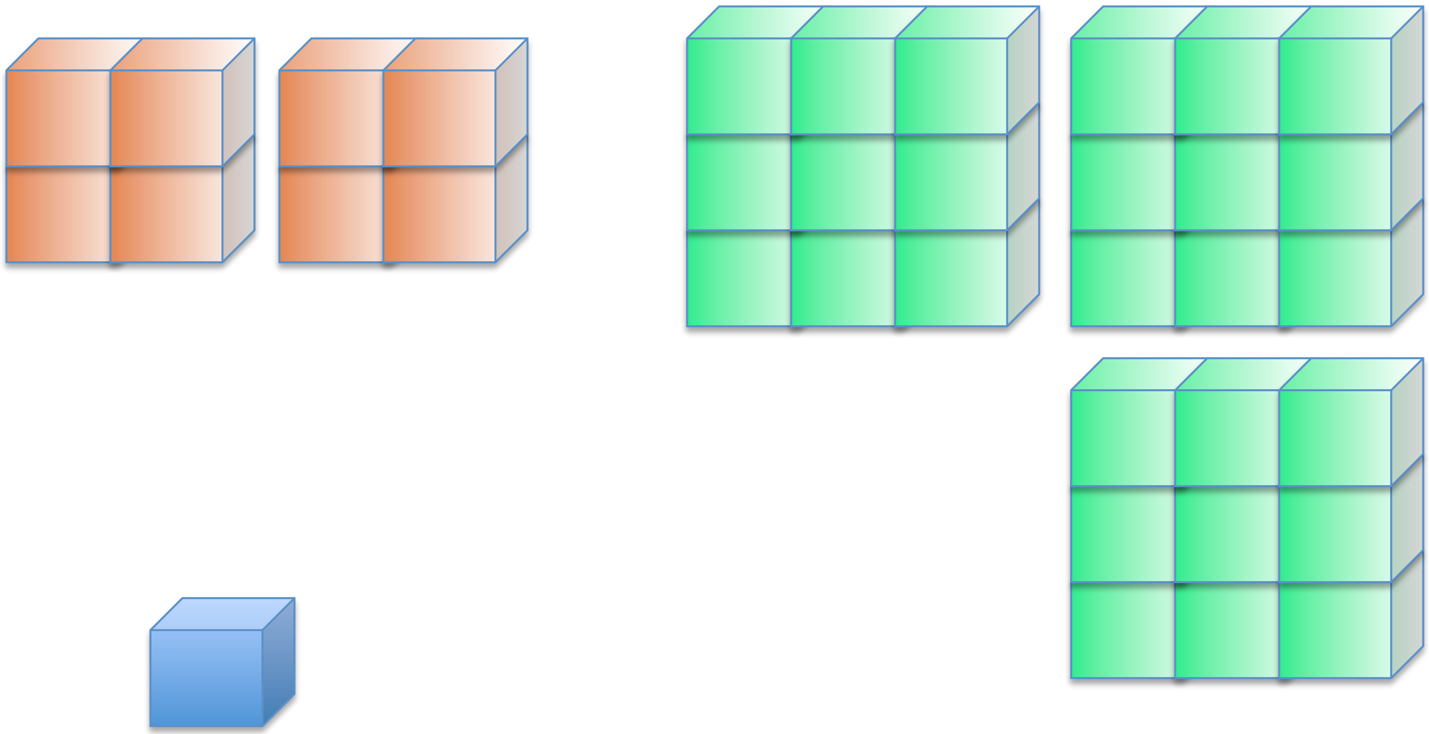


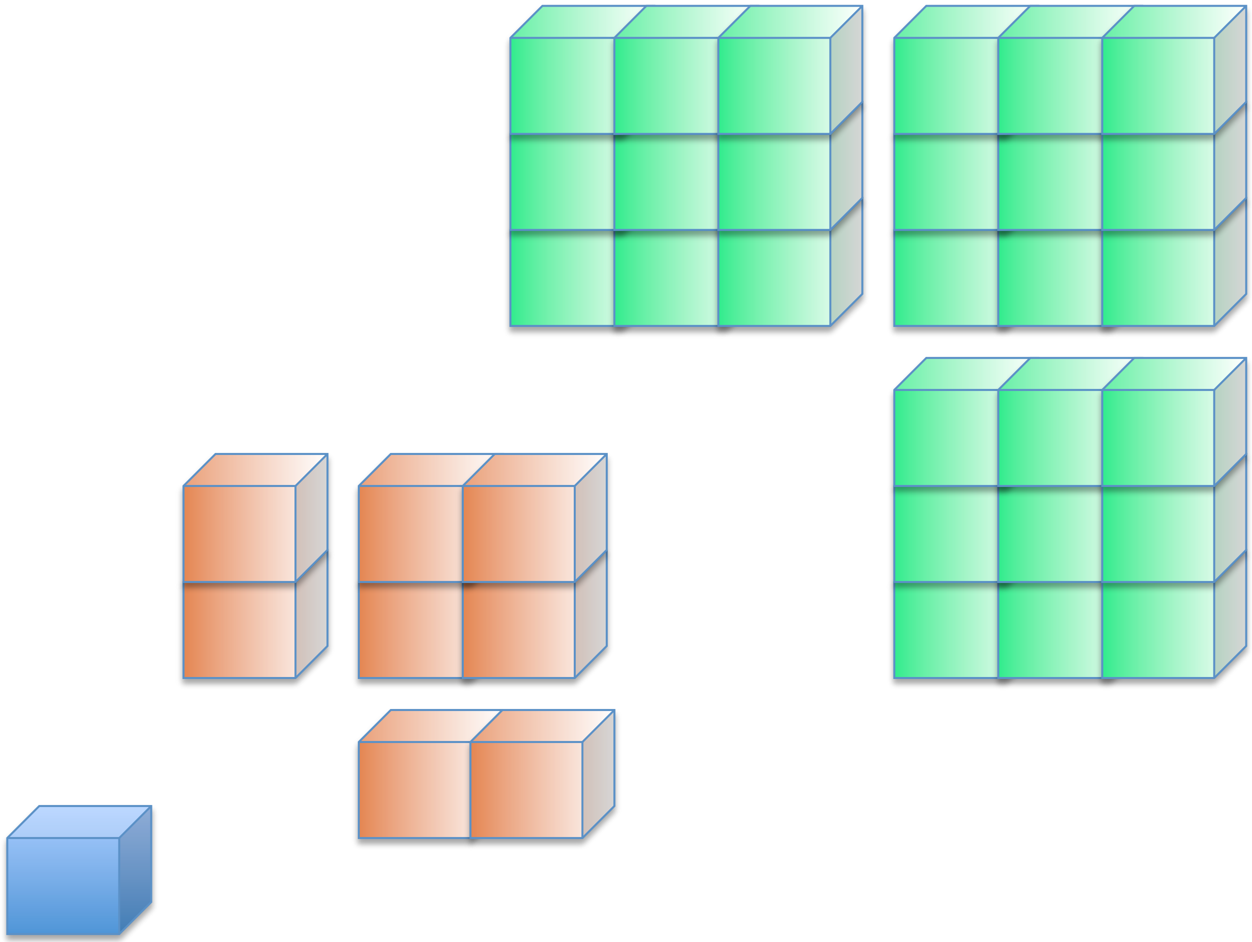
$= 1^3+2^3+3^3+4^3+5^3$

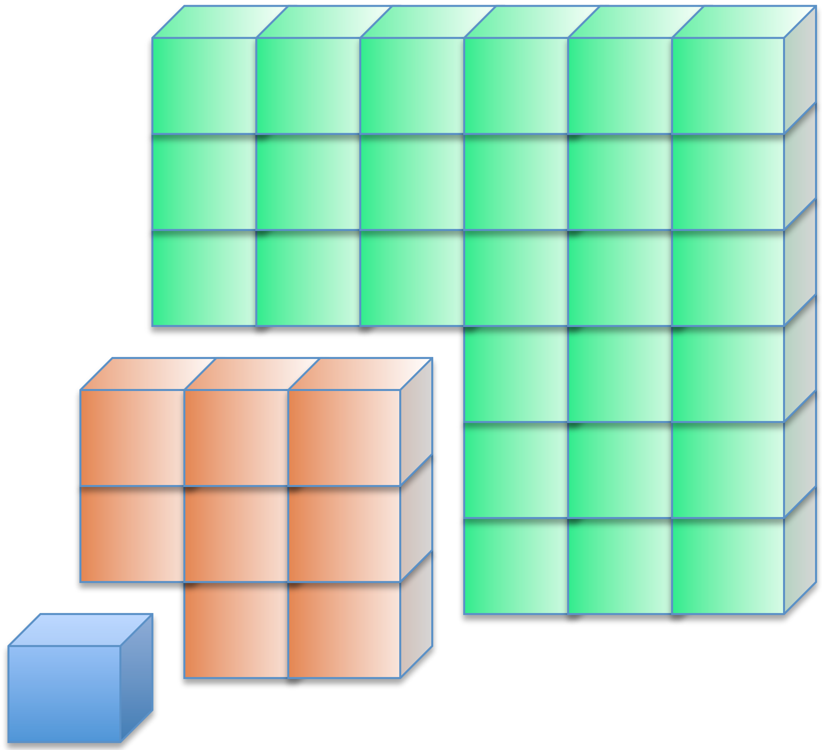
Sum of the n first cubes

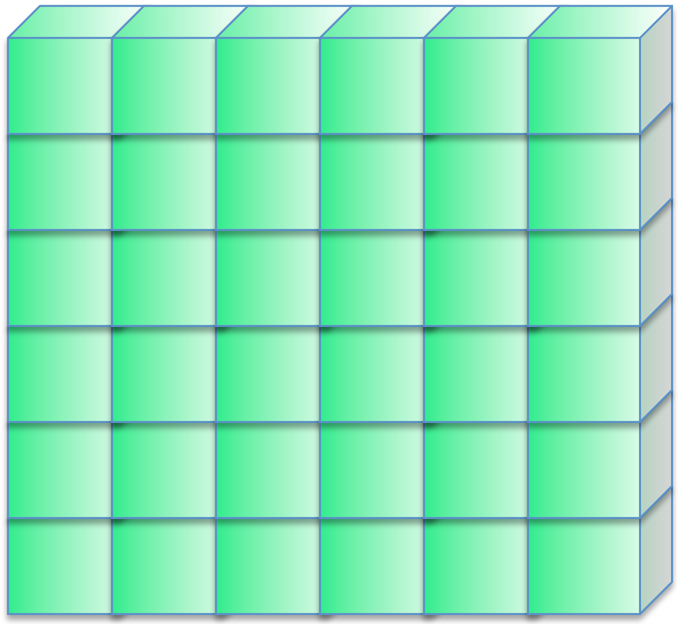


1^3 2^3 3^3





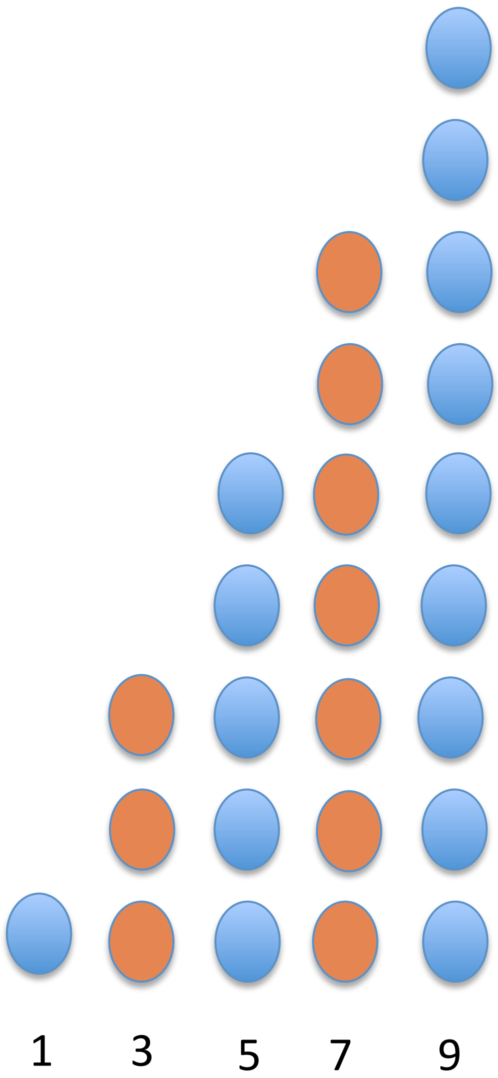




$$(1 + 2 + \ldots + n)^2$$

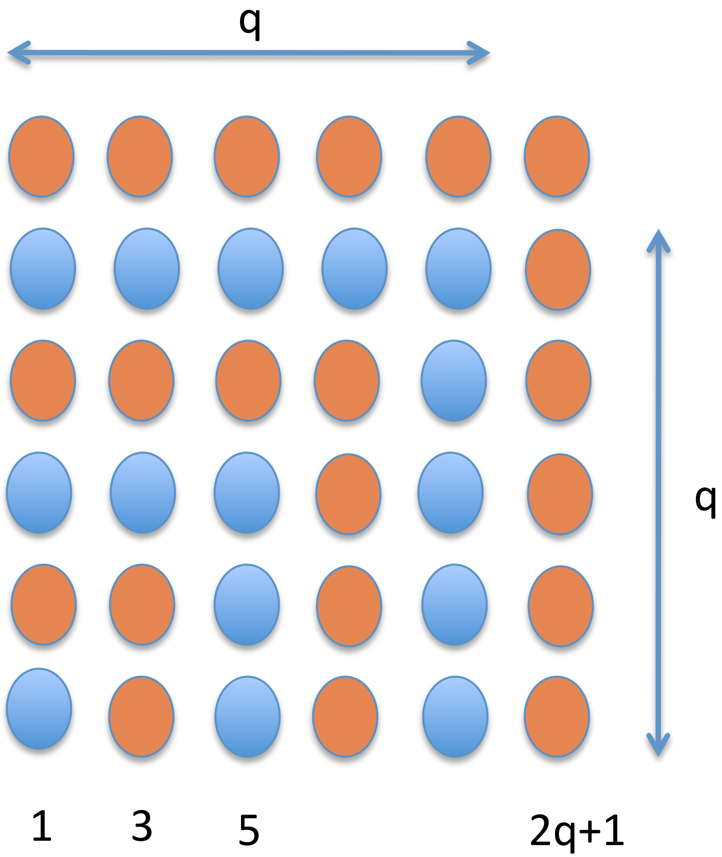
Polygonal numbers

1 + 1 + 1 + 1 + ...	Basic (counting)
1 + 2 + 3 + 4 + ...	Triangular numbers
1 + 3 + 5 + 7 + ...	Sum of odd (squares)



Sum of the first odd numbers

$$1+3+\ldots+(2q+1) = (q+1)^2$$



Tetrahedral numbers (tri-dimensional)

Determine the sum of consecutive triangular numbers

$T_n = \Delta_1 + \Delta_2 + \ldots + \Delta_n$

Example: T_4

1
+ 1+2 (3)
+ 1+2+3 (6)
+ 1+2+3+4 (10)

1

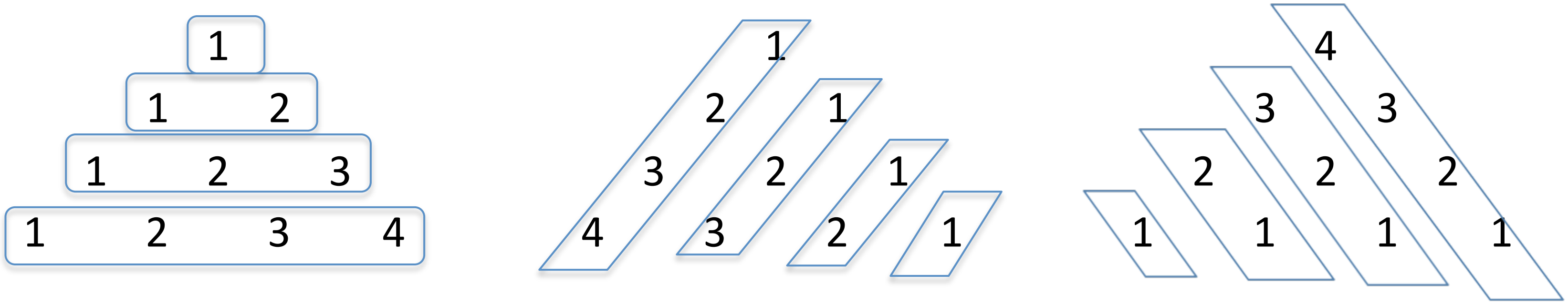
+ 1+2 (3)

+ 1+2+3 (6)

+ 1+2+3+4 (10)

			1			
		1		2		
	1		2		3	
1		2		3		4

$$
\begin{array}{l}
\quad 1 \\
+\ 1+2\ (3) \\
+\ 1+2+3\ (6) \\
+\ 1+2+3+4\ (10)
\end{array}
$$



$$3.(1 + 3 + 6 + 10)$$

$$
\begin{array}{ccccccc}
 & & & \boxed{1} & & & \\
 & & 1 & & 2 & & \\
 & 1 & & 2 & & 3 & \\
1 & & 2 & & 3 & & 4
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & \boxed{1} & & & \\
 & & 2 & & 1 & & \\
 & 3 & & 2 & & 1 & \\
4 & & 3 & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & \boxed{4} & & & \\
 & & 3 & & 3 & & \\
 & 2 & & 2 & & 2 & \\
1 & & 1 & & 1 & & 1
\end{array}
$$

$$3.(1 + 3 + 6 + 10)$$

$$= \boxed{1.6}$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \boxed{1} & & \boxed{2} & & \\
 & 1 & & 2 & & 3 & \\
1 & & 2 & & 3 & & 4
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \boxed{2} & & \boxed{1} & & \\
 & 3 & & 2 & & 1 & \\
4 & & 3 & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 4 & & & \\
 & & \boxed{3} & & \boxed{3} & & \\
 & 2 & & 2 & & 2 & \\
1 & & 1 & & 1 & & 1
\end{array}
$$

$$3.(1 + 3 + 6 + 10)$$

$$= 1.6 + \boxed{2.6}$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 2 & & \\
 & \boxed{1} & & \boxed{2} & & \boxed{3} & \\
1 & & 2 & & 3 & & 4
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 2 & & 1 & & \\
 & \boxed{3} & & \boxed{2} & & \boxed{1} & \\
4 & & 3 & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 4 & & & \\
 & & 3 & & 3 & & \\
 & \boxed{2} & & \boxed{2} & & \boxed{2} & \\
1 & & 1 & & 1 & & 1
\end{array}
$$

$$3.(1 + 3 + 6 + 10)$$

$$= 1.6 + 2.6 + \boxed{3.6}$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 2 & & \\
 & 1 & & 2 & & 3 & \\
\boxed{1} & & 2 & & 3 & & 4
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 2 & & 1 & & \\
 & 3 & & 2 & & 1 & \\
\boxed{4} & & 3 & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 4 & & & \\
 & & 3 & & 3 & & \\
 & 2 & & 2 & & 2 & \\
\boxed{1} & & 1 & & 1 & & 1
\end{array}
$$

$$3.(1 + 3 + 6 + 10)$$

$$= 1.6 + 2.6 + 3.6 + \boxed{4.6}$$

$$3.(1 + 3 + 6 + 10)$$

$$= 1.6 + 2.6 + 3.6 + 4.6 = \Delta_4 (4+2)$$

More generally, $T_n = \Delta_n (n+2)/3$

$$T_n = n(n+1)/2 \,.\, (n+2)/3 = n(n+1)(n+2)/6$$

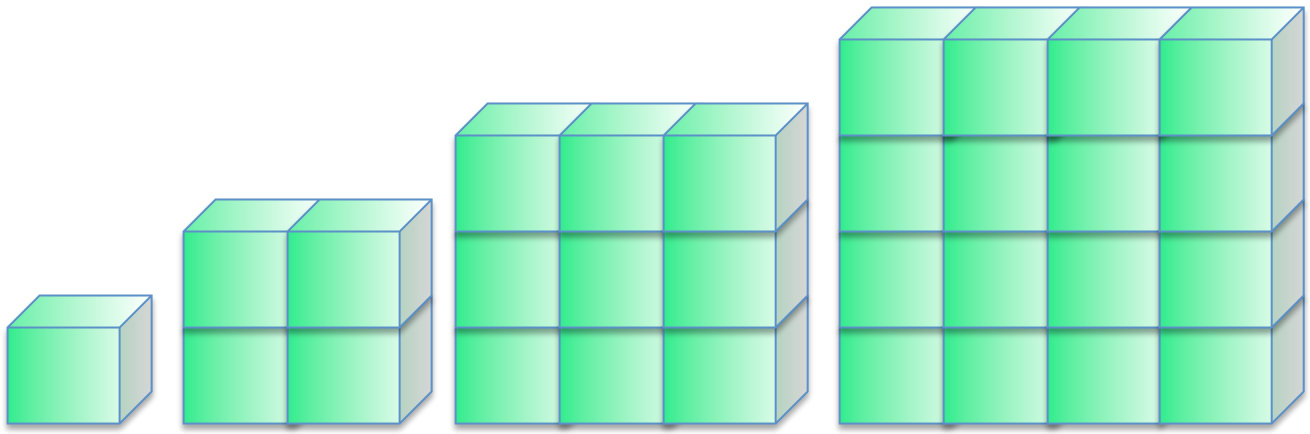
Synthesis

At this point, it is interesting to make two remarks:

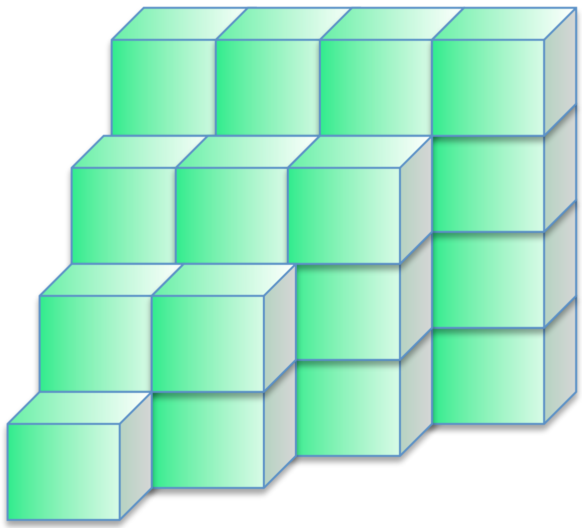
For the 1D case (sum of units), we get n
For the 2D case (triangular), we get $n(n+1)/2$
For the 3D case (tetrahedral), we get $n(n+1)(n+2)/6$

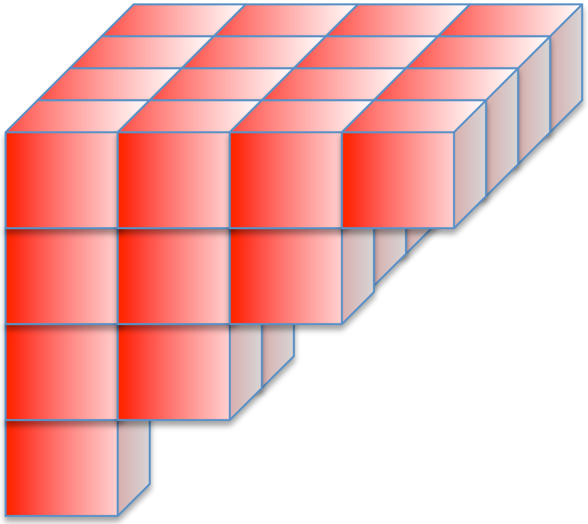
The proof of 2D case was obtained by two copies of 1D structure (lines)
The proof of the 3D case was obtained by three copies of 2D structure (triangles)

Sum of n first squares

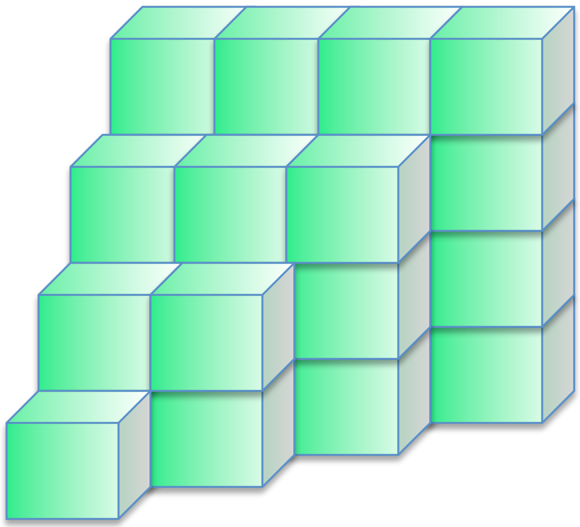


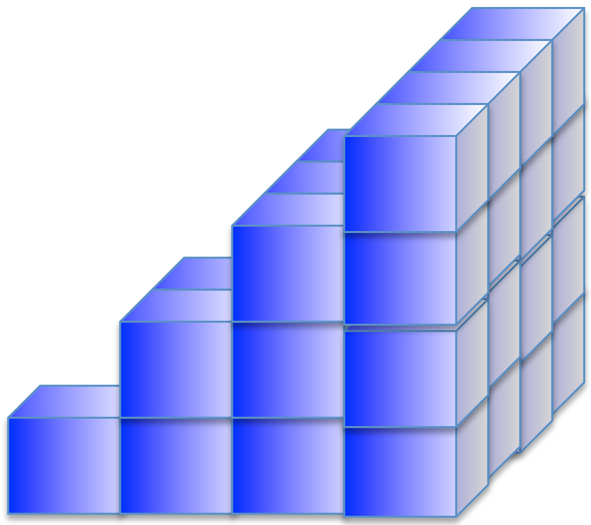
$$1 \quad + \quad 4 \quad + \quad 9 \quad + \quad 16 \quad = \quad S_n$$

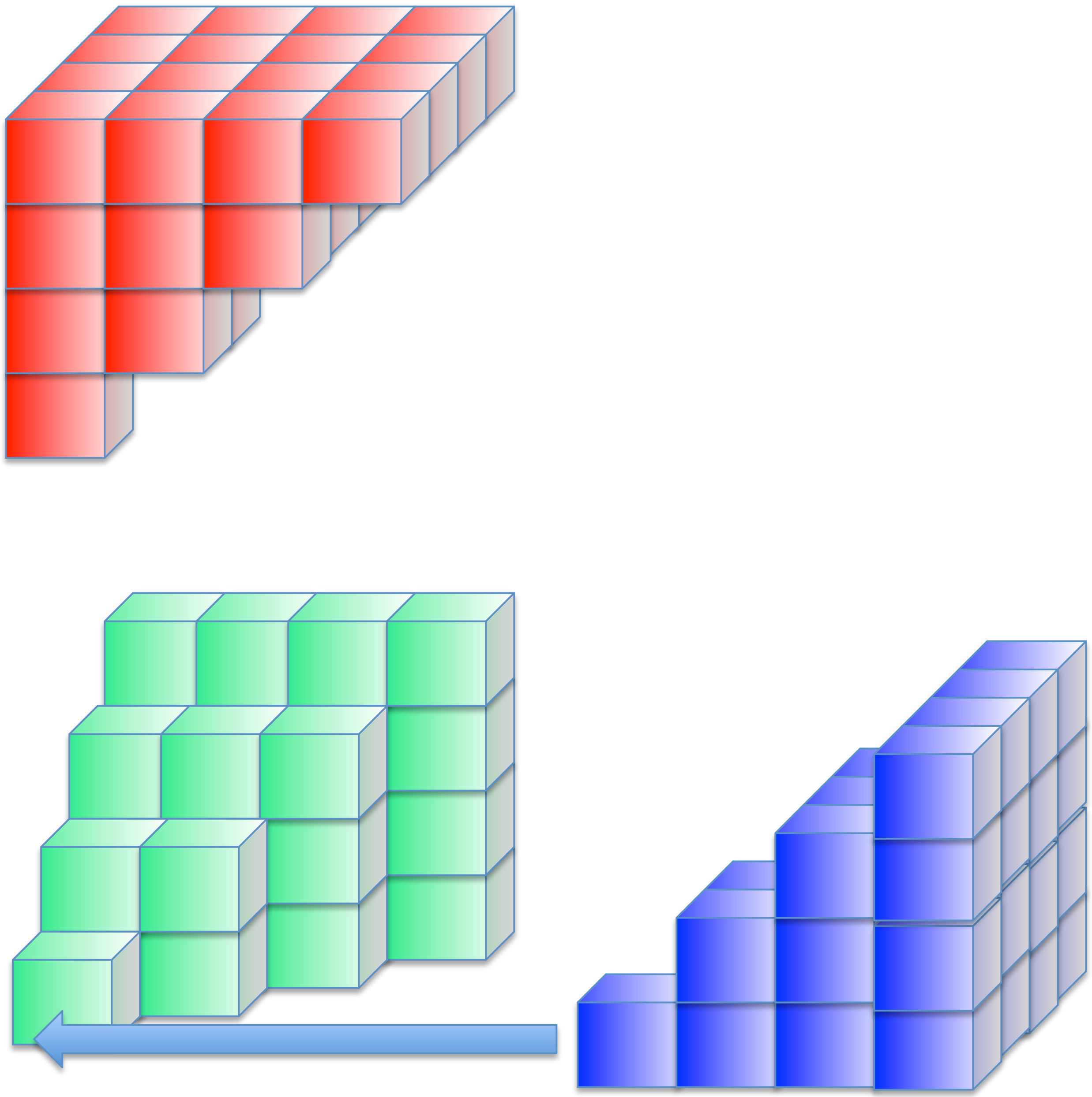


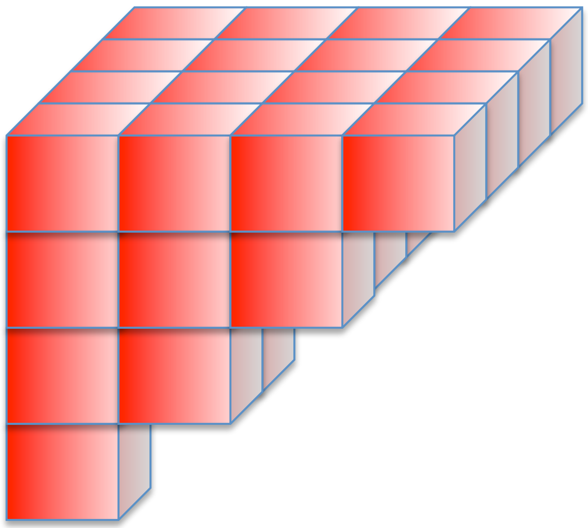


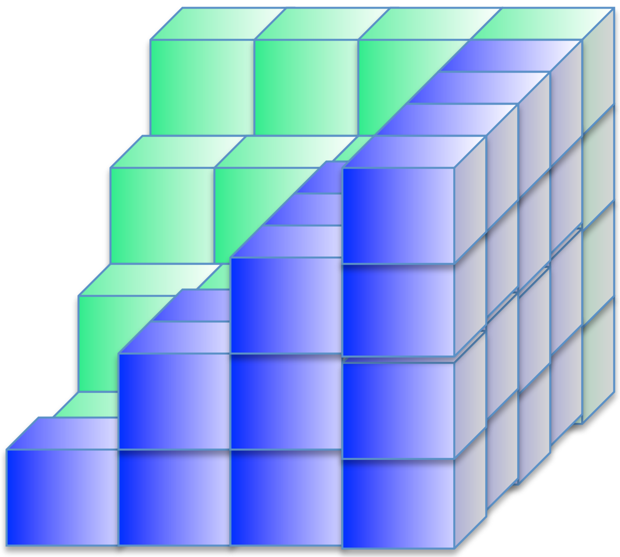
3 times S_n

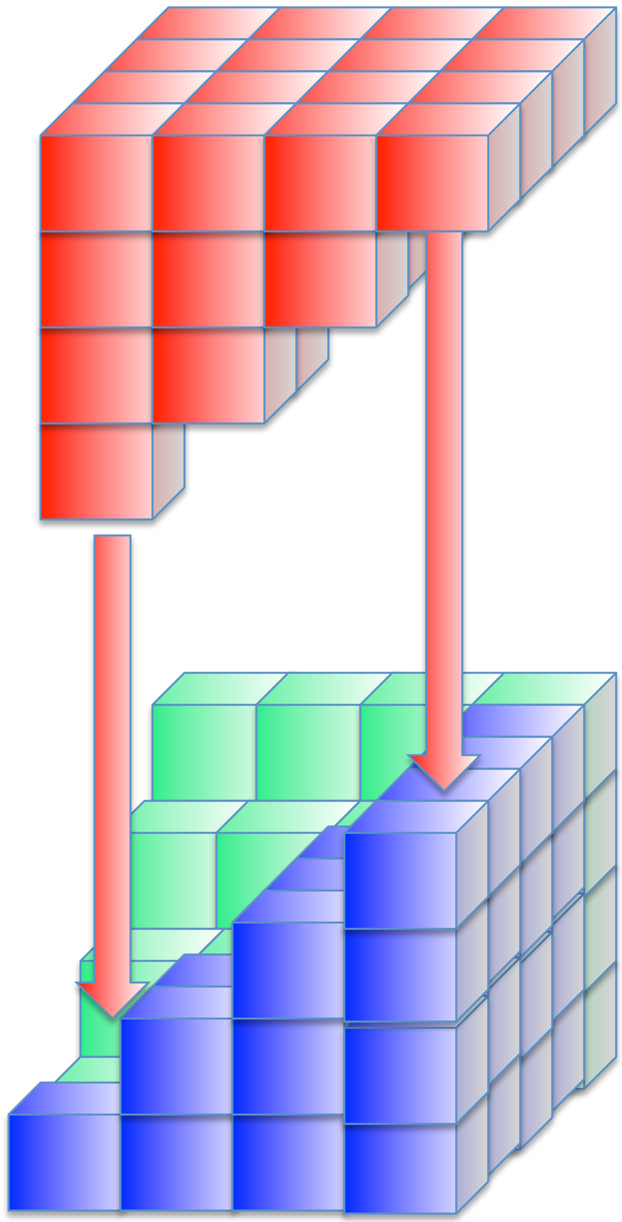


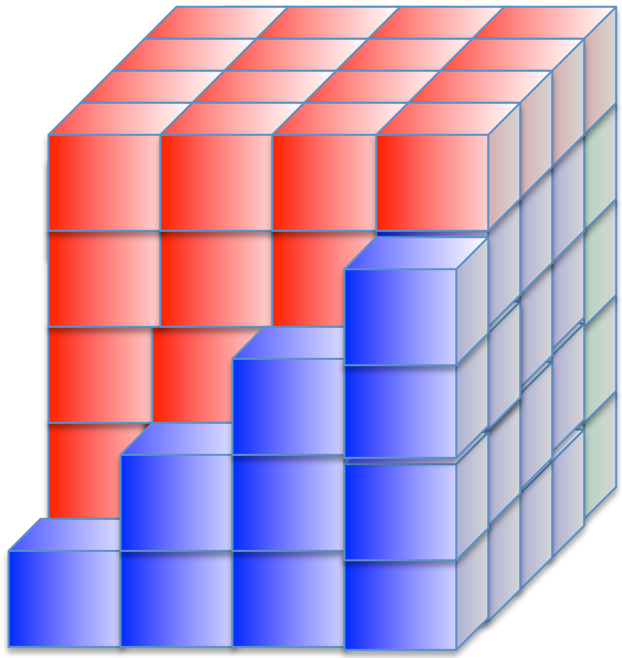


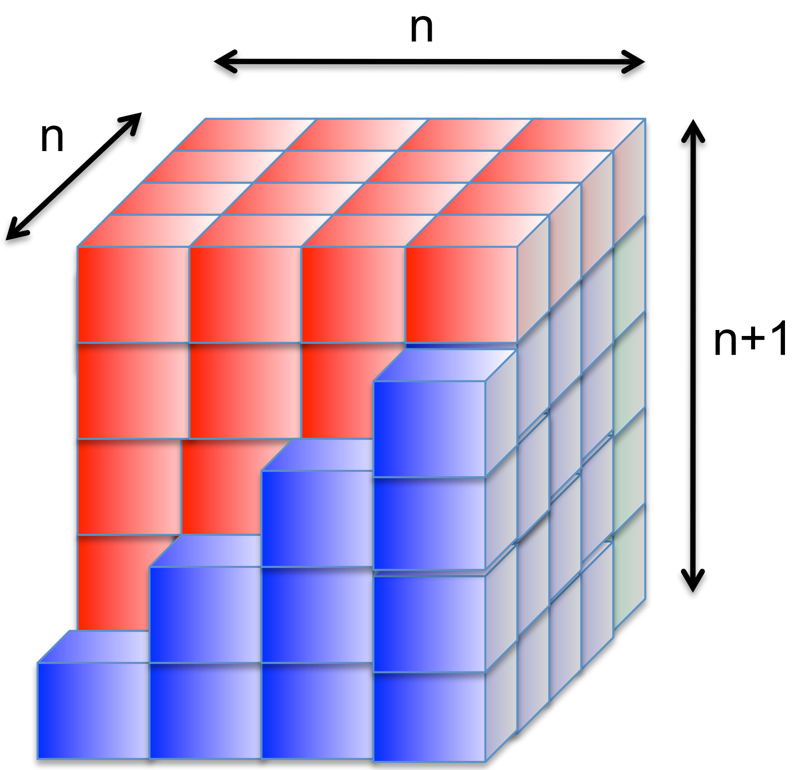




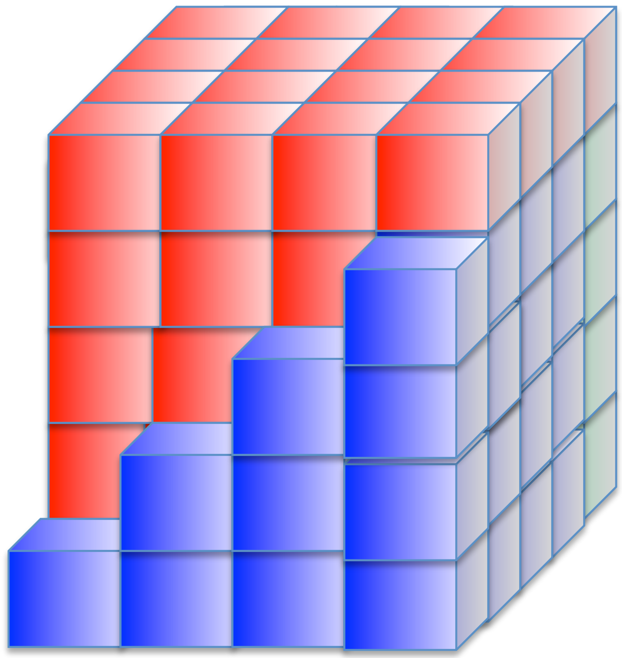







n
n
n+1

$$3\,S_n = n.n.(n+1) + \tfrac{1}{2}\,n.(n+1)$$
$$= n.(n+1).(n+\tfrac{1}{2})$$



References

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Conway and Guy, the book of numbers