Mathematics for computer science: Homework 2

SID-LAKHDAR Riyane

15/11/2015

I understand what plagiarism entails and I declare that this report is my own, original work SID-LAKHDAR Riyane November the 15 th, 2015

Abstract

This report summarize, explains and refers to the answers we have designed for the second M̈athematics for computer scienceḧomework.
In this report, we will use the following notations:

- F_i the i th Fibonacci number, such as
  $F_0 = 0 \quad F_1 = 1 \quad F_2 = 1$
  $F_n = F_{n-1} + F_{n-2}$ if $n > 2$
- For a given integer n, we consider the set $\{b_i | b_i \in \{0, 1\}\}_{i \in [2,r]}$

For the clarity of our demonstrations, we have decided to start by showing the increment process of the system. Then we will answer to all the other questions in the order given by the topic.

1 Increment process of the Zeckendorf's system

Let an integer n $= \sum_{i=2}^{r} b_i F_i$, such as if $b_i = 1$ then $bi + 1 = 0$ and $bi - 1 = 0$ (assuming that this bits exist). To increment this integer, we can distinguish the two following cases:

1.1 Base case: r = 3

Let n an integer such as $n = b_3F_3 + b_2F_2$. To respect the Zeckendorf's system the only possible expressions for n are

- $n = 0 * F_3 + 0 * F_2 \leftrightarrow n + 1 = n + F_2 = 0 * F_4 + 0 * F_3 + 1 * F_2$
- $n = 1 * F_3 + 0 * F_2 \leftrightarrow n + 1 = n + F_2 = 0 * F_4 + 1 * F_3 + 1 * F_2 = 1 * F_4 + 0 * F_3 + 0 * F_2$
- $n = 0 * F_3 + 1 * F_2 \leftrightarrow n + 1 = n + F_1 = 0 * F_4 + 0 * F_3 + 1 * F_2 + 1 * F_1 = 0 * F_4 + 0 * F_3 + 1 * F_3 + 0 * F_2$

We can see that in all this cases, n+1 respects the Zeckendorf's system. We can also notice that adding F_2 (or F_1) to a Zeckendorf's system of length 2, creates a remainder of upper rank.

1.2 Generalizing the process

The base case showed us that adding 1 to the Zeckendorf's system creates a remainder which is a Fibonacci number F_i and which propagates. Adding such a remainder may have two results on the input Zeckendorf's system:

- Create two consecutive non null digits. If for $i \geq 2$, b_i and b_{i+1} equal 1, we can simply use the definition of a Fibonacci number to replace them by 0 and add 1 to b_{i+1}
- Create a digit equal to 2. To fight this behavior, we can notice that for $i \geq 2, 2F_i = F_i + F_i = F_{i-2} + F_{i-1} + F_i = F_{i-2} + F_{i+1}$. Thus if the digit $b_i = 2$ ($i \geq 2$) we can replace it by 0 and add 1 to the digit i-2 and i+1.

The problem with the solution to this two methods is that fixing each problem may generate the other problem. Thus, the order of applying this two algorithm is crucial.

2 Zeckendorf's theorem

For each integer n, let P(n) the property: "n can be written in the Zeckendorf's system previously described".

2.1 Existence of the decomposition for each integer

Let prove P by induction.

- We can notice that if $n = 1$ then $n = F_2$. Thus, P(1) is true.
- Let suppose n an integer bigger or equal 2, and let suppose that P(i) is true for each $i \leq n$. Let prove then that P(n+1) is true.
  If n+1 is a Fibonacci number then P(n+1) is true.
  Else, it exists $k \in \mathbb{N}$ such that $F_k < n+1 < F_{k+1}$ (as $n+1 > F_1 then F_k > F_1 and k > 1$). As $F_k \geq 1$ then $n+1-F_k < n+1$. Thus, according to the

recurrence hypothesis, $\delta = n+1-F_k = \sum_{i=2}^{r} b_i F_i$. Let find the value of the upper bound r of this sum. We know that

$$\begin{aligned} \delta + F_k = n+1 &< F_{k+1} \\ &< F_k + F_{k-1} \\ \delta &< F_{k-1} \end{aligned}$$

Thus $\delta = \sum_{i=2}^{k-2} b_i F_i$ and $\delta + F_k = n+1 = \sum_{i=2}^{k-2} b_i F_i + 0F_{k-1} + F_k$
As $\sum_{i=2}^{k-2} b_i F_i$ respects the Zeckendorf's system (according to the recurrence hypothesis), $n+1 = \sum_{i=2}^{k-2} b_i F_i + 0F_{k-1} + F_k$ also respects the system. Thus P(n+1) is true.
[1]

Thus, by induction we have proved the existence of the Zeckendorf's decomposition for each integer.

2.2 Uniqueness

Let suppose that the Zeckendorf's representation is not unique. Thus, for $n \in \mathbb{N}, n = \sum_{i=2}^{r} b_i F_i = \sum_{i=2}^{r'} b'_i F_i$. Let $E = i \in [2, r] | b_i = 1$, $E' = i \in [2, r'] | b'_i = 1$ and $c = i \in [2, min(r, r')] | b_i = b'_i = 1$.
We have

$$\begin{aligned} n &= \sum_{i \in C} F_i + \sum_{i \in E/C} F_i \\ &= \sum_{i \in C} F_i + \sum_{i \in E'/C} F_i \end{aligned}$$

Thus, $\sum_{i \in E/C} F_i = \sum_{i \in E'/C} F_i$.
As E/C and E'/C are the sets of non common digits in the two representations, the previous statement proves that the two representations are the same.

3 Examples and compare with binary representation

Let consider the integer n = 12345. We can notice that

$$\begin{aligned} n = 12345 &= 10946 + 987 + 377 + 34 + 1 \\ &= F_{21} + F_{16} + F_{14} + F_9 + F_2 \end{aligned}$$

As this representation respects the Zeckendorf's system, the Fibonacci representation of 12345 is $(10000101000010000001)_{zeckendorf}$.

- $(1)_{10} = (00001)_{zeckendorf}$. $(2)_{10} = (00010)_{zeckendorf}$

[1]This part of the demonstration is inspired from the demonstration of the compared growing theorem in the website: www.lesbonsprofs.com

- $(3)_{10} = (00100)_{zeckendorf}$. $(4)_{10} = (00101)_{zeckendorf}$
- $(5)_{10} = (01000)_{zeckendorf}$. $(6)_{10} = (01001)_{zeckendorf}$
- $(7)_{10} = (01010)_{zeckendorf}$. $(8)_{10} = (10000)_{zeckendorf}$