Proofs and Summations

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1 Notion of proof in Mathematics

This first chapter is two-fold. First, we will present new and simple ways for proving results in discrete Mathematics and second, we will establish results of the most important classical summations.

1.1 All means are good

We are interested in this course in learning useful techniques to prove mathematical results. But what is really a "mathematical proof" (especially in the area of Computer Science)? René Thom, a famous french mathematician, defined it as "rigorous process that creates a state of evidence for educated readers who leads their adherence". Traditional proofs are based on logics and use inductive arguments. Formal proofs can be automatically derived using computer.s that can achieve today very high performances. But there is clearly a limit of formal proofs given at the high-school level, may be this is the reason why Maths are not popular (or at least, seen as hard)... It usually leads to teach mathematical recipes that the pupils do not always understand.

It is not always easy to find how to start a proof. In the real life, it is common to have no idea about what we are looking for. We should first be convinced that the target result is true and then, find a way to prove it. A lot of students have already some problems at this early step. Thus, we should develop intuition and learn through well-chosen examples using a lot of various techniques.

1.2 Simple proofs for counting

A lot of situations in Computer Science involve positive natural numbers (integers). A nice way for proving related results is to represent them by sets of items (bullets, squares) and to use one of the following principles:

- Fubini's principle is to count such items in two different ways. Obviously, the answer is the same, and thus, it provides the value we are looking for.
- Cantor's principle is to establish a one-to-one correspondence between two sets of items, one being straightforward to compute.

Another way is to restrict particularly to geometric figures where the counting is done by computing basic surfaces.

1.3 From a problem to a proof

Let us remark that solving in not only proving. Solving a problem requires much more, including a model the studied phenomenon, determine adequate notations and explicit the rules that govern the phenomenon.

2 Summations

Some specific summations are very useful in Computer Science and algorithms design. Let us present the most remarquable ones.

In particular, we define the *polygonal* numbers as sums of n terms:

- $n = 1 + 1 + ... + 1$ n times.
- $1 + 2 + 3 + ... + n$
- $1 + 3 + 5 + ... + (2n - 1)$
- ...

2.1 Triangular numbers

Triangular numbers are defined as the sum of the n first integers:

$\Delta_n = \sum_{i=1}^{n} i$.

There exist many proofs for this result, the simplest one is obtained in writing this sum forward and backward and gathering the terms two by two as follows:

$2.\Delta_n = 1 + 2 + 3 + ... + (n-1) + n + n + (n-1) + ... + 2 + 1 =$
$[1 + n] + [2 + (n-1)] + [3 + (n-2)] + ... + [n+1] =$
n times $n + 1$.

Thus, $\Delta_n = \frac{(n+1).n}{2}$

Figure 1 shows this result by using a geometric argument. The sum is represented by a series of boxes of size 1. The result is the sum of the surface of the big triangle – which is obviously half a square ($\frac{n^2}{2}$) – by n times half the surface of a basic box $\frac{1}{2}$. Thus, $\frac{n^2}{2} + n.\frac{1}{2} = \frac{(n+1).n}{2}$

There are several interesting properties on Δ_n:

What is the value of $\Delta_n + \Delta_{n-1}$?

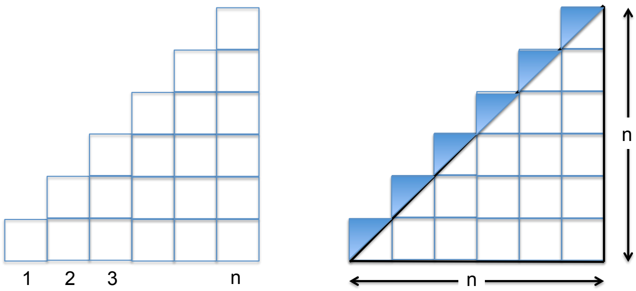

Figure 1: Geometric proof for computing triangular numbers

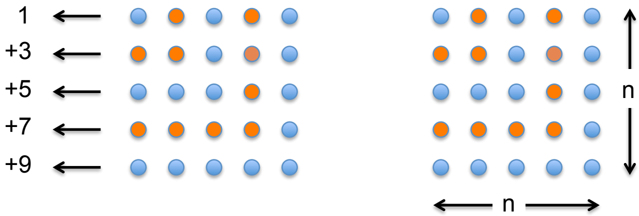

Figure 2: Fubini's principle for the sum of odd numbers

2.2 Sum of odd numbers

The question here is to determine $\sum_{q=0}^{n-1} 2q + 1$.

This result may also be stablished by using Fubini's principle and Cantor's principle as shown in the two following figures. Figure 2 gives two ways for computing the number of points (by the sum of odd numbers in the left and by the surface of the square in the right). Figure 3 evidences a one-by-one correspondence of the red points.

2.3 Tetrahedral numbers

The sum of the Δ_n is denoted by Θ_n and it is called a tetrahedral numbers:

$\Theta_n = \sum_{i=1}^{n} \Delta_i$.

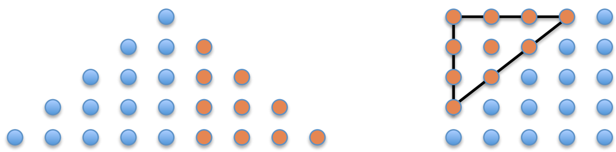

Figure 3: Cantor's principle for the sum of odd numbers

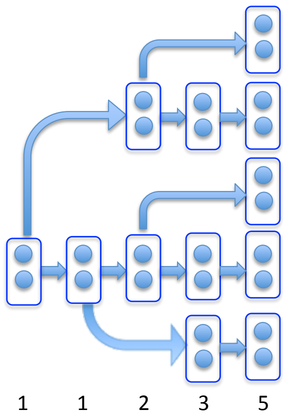

Figure 4: Construction of the Fibonacci progression

We can show that: $\Theta_n = \frac{n.(n+1).(n+2)}{6}$

A way to prove this expression is to write it as $\frac{1}{3}.\frac{n.(n+1)}{2}.(n+2)$. This way, we can consider three copies of Θ_n. Remember, the way we established the closed formula for triangular numbers was based on 2 copies arranged in the right form.

2.4 Other useful sums

- Sum of n first squares $\square_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- Alternate sum of squares $\sum_{k=1}^{n}(-1)^{(k+1)}k^2$
- Sum of n first cubes $C_n = \sum_{k=1}^{n} k^3 = \Delta_n^2$

3 Fibonacci numbers

In the original problem introduced by Leonardo of Pisa (Fibonacci) in the middle age, this number is the number of pairs of rabbits that can be produced at the n-th generation. Starting by a single pair of rabbits and assuming that each pair produces a new pair of rabbits at each generation during only two generations.

Definition Given the two numbers F(0) = 1 and F(1) = 1, the Fibonacci numbers are obtained by the following expression: F(n+1) = F(n)+F(n−1).

Notice that it is a special case of $u_{n+1} = a.u_n + b.u_{n-1}$ for $a = b = 1$.

4 Appendix: Classical results

Discrete Mathematics are based on integers. A *closed form* of an expression involving integers is obtained when the general term is not expressed by a quantity involving itself, but by a direct expression of n.

4.1 Basic identities

Let us start by a rather simple question "what is the value of the following expression $1 + x + x^2 + ... + x^n$?"

We can easily evaluate $(1 - x).(1 + x + x^2 + ... + x^n)$
$= 1 - x + x - x^2 + x^2... - x^n + x^n - x^{n+1} = 1 - x^{n+1}$
thus, $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$

We deduce the more general form:
$(1 - x).(x^p + x^{p+1} + ... + x^q) = x^p - x^{q+1}$

Similarly, it is easy to compute the following finite sum:
$1 - x + x^2 + ... + (-1)^n x^n$.
The expression is obtained by multiplying it by $(1 + x)$:
$(1 + x)(1 - x + x^2 + ... + (-1)^n x^n) = 1 + (-1)^n x^{n+1}$

4.2 Numerical progressions on integer sequences

Arithmetic progression

Definition. Given u_1, the current terms of an arithmetic progression are given by $u_{n+1} = u_n + r$ for $n \geq 1$.

Then, it is possible to determine the current term and the sum of the sequence by the following closed form.
$u_n = u_1 + (n - 1).r$
$S_n = n\frac{u_1+u_n}{r}$

Geometric progression

Definition. Given u_1, the current terms of a geometric progression are given by the successive values $u_{n+1} = q.u_n$ for $n \geq 1$.

$u_n = u_1.q^{n-1}$
The sum of the sequence is given by a closed form : $S_n = u_1\frac{1-q^n}{1-q}$ $(q \neq 1)$.

Numerical progression involving two consecutive terms

Definition. Given u_0 and u_1, the current term of such a progression is $u_{n+1} = a.u_n + b.u_{n-1}$

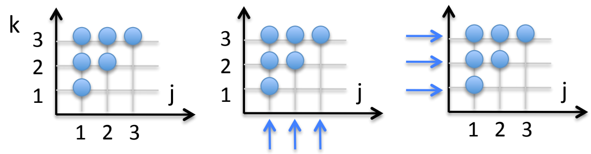

Figure 5: Handling double sums

Let consider the equation $x^2 - ax - b = 0$ whose discriminant is $\Delta = a^2 + 4b$. if this polynomial of degree 2 has two distinct roots x_0 and x_1, then, the general solution has the form : $u_n = \lambda.x_0^n + \mu.x_1^n$

4.3 Double sums and more

We usually use the symbol $\sum$ for a compact representation of a summation. For instance, $1 + 2 + 3 + ... + (n-1) + n$ is written shortly as: $\sum_{i=1}^{n} i$.

Such a representation may concern several indices: $\sum_{1 \leq j,k \leq n} a_{j,k}$.

For instance, the summation $\sum_{1 \leq j \leq k \leq 3} a_{j,k}$ corresponds to:

$a_{1,1} + a_{1,2} + a_{2,2} + a_{1,3} + a_{2,3} + a_{3,3}$.

An easy way of handling such abstract forms is to draw explicitly the domains of the indices.

The previous sum can also be written by two consecutive sums:
$\sum_{j=1}^{3} \sum_{k=j}^{3} a_{j,k} = (a_{1,1} + a_{1,2} + a_{1,3}) + (a_{2,2} + a_{2,3}) + a_{3,3}$.
This sum can also be written as:
$\sum_{k=1}^{3} \sum_{j=1}^{k} a_{j,k} = a_{1,1} + (a_{1,2} + a_{2,2}) + (a_{1,3} + a_{2,3} + a_{3,3})$.

4.4 Permutations and binomial coefficients

Let us consider n « objects » (indexed by integers).
Definition. A permutation is an ordered arrangement of the objects.

The number of different permutations is denoted by $n!$

The proof is straightforward : the first object in any permutations can be chosen in n different ways. Once it is chosen, the second one may be chosen in $n-1$ ways, etc..

There are used in some well-known formula like the following Newton formula:

Let a and b be two integers, $(a+b)^n = \sum_{k=0}^{k=n} \binom{n-k}{k} a^k.b^{n-k}$

which is the generalized expression of the classical form: $(a+b)^2 = a^2 + 2ab + b^2$

5 Infinite sums – Series

Given a sequence (a_n), a series is the sum of the consecutive terms. It may be finite or not. What are the main issues here? First, some of the previous techniques used while calculating the finite sums are no more valid for infinite sums.

A natural definition is $\sum_{k\geq 1} = lim_{n\to\infty} \sum_{k=1}^{n}$. But we have to be careful with the definition. For instance, let us consider the following paradoxal situation: we want to determine the value of $S = \sum_{k\geq 0} \frac{1}{2^k}$. Defining the infinite sum as the limit leads to the value 2. Indeed, $2S = S + 2$... But this way applied to the sum: $\sum_{k\geq 0} 2^k$, we obtain the value -1! This is obviously not correct since the sum of increasing positive numbers should be positive. The reason is that the terms of the series grows to $+\infty$.

$S_n = \sum_{1\leq k\leq n} a_k$ is called the partial sum of the series. Σ denotes the infinite sum. The series converges if Σ is bounded.

Proposition. If the series $\sum_k a_k$ converges, then, $a_k \to 0$.

The condition is not sufficient.

The series x^k diverges if $x \geq 1$.

The proof is easy since the finite sum is equal to $\frac{1-x^k}{1-x}$

A particular case is the geometric series: The convergence issues of a geometric series depend on q. If $q < 1$, the infinite geometric series converges and $S = \frac{u_1}{1-q}$

6 Harmonic series

$H_n = \sum_{i=1}^{n} \frac{1}{i}$

Analysis. It is easy to prove that H_n tends toward ∞ by grouping the terms according to powers of 2. Each group is between $\frac{1}{2}$ and 1. Moreover, this way of bounding the sum tells us about its value (actually, we know the value at a factor of 2):

$\frac{log(n)+1}{2} < H_n < log(n) + 1$. Thus, $H_n = \mathcal{O}(log(n))$

This bound can be improved by using the continuous analog of the series $\frac{1}{x}$ involving the Euler constant.