# Image and Signal Processing

*Lecture and Personal Notes*

MoSIG

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This document belongs to: ______________________

# Contents

# Introduction

To understand what is **Digital Signal Processing** (*DSP*), let's examine what does each of its words mean.

**Signal** is any physical quantity that carries information.

**Processing** is a series of steps or operations to achieve a particular end.

> *it is easy to see that Signal Processing is used everywhere to extract information from signals or to convert information-carrying signals from one form to another. For example, our brain and ears take input speech signals, and then process and convert them into meaningful words.*

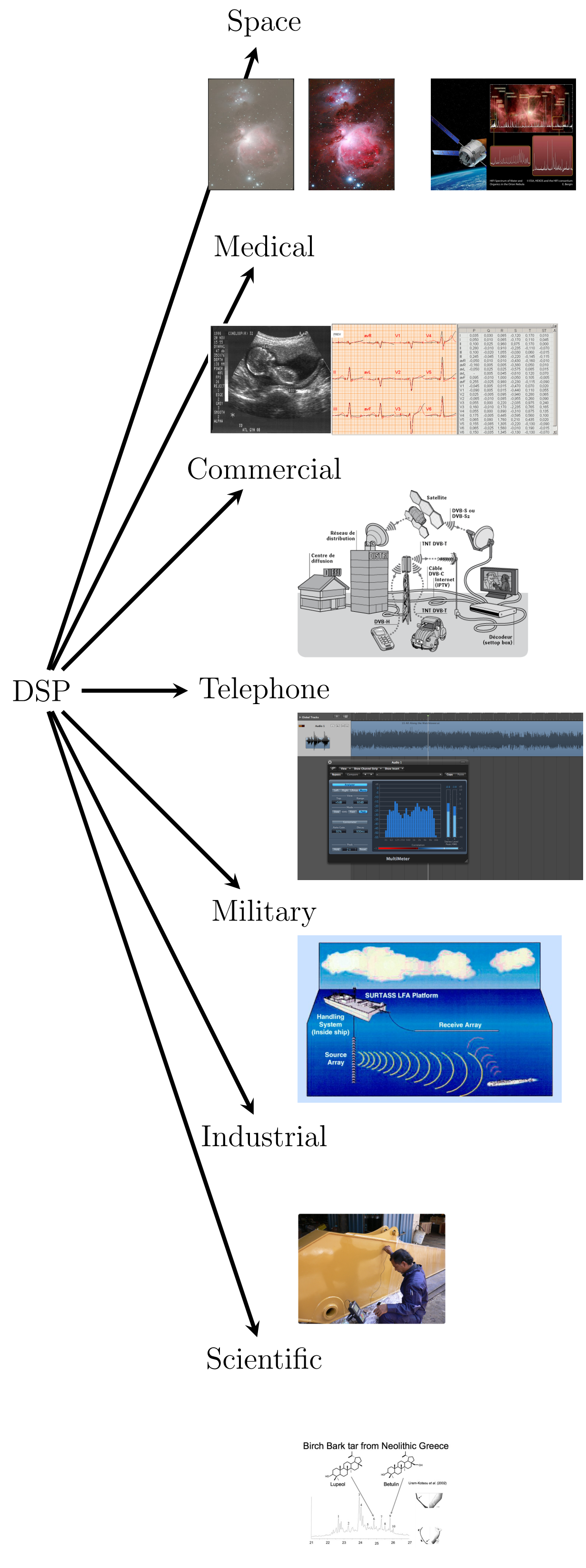
**Digital** means that the process is done by computers, microporcessors or logic circuits.

Digital Signal Processing is one of the most powerful technologies that will shape science and engineering in the twenty-first century. Revolutionary changes have already been made in a broad range of fields: communications, medical imaging, radar and sonar, high fidelity music reproduction, and oil prospecting, to name just a fiew. Each of these areas has developed a *deep* DSP technology, with its own algorithms, mathematics, and specialized techniques. This combination of brath and depth makes it impossible for any individual to master all of the DSP technology that has been developed. DSP education involves two tasks: learning general concepts that apply to the field as a whole, and learning specialized techniques for your particular area of interest. This course aims at the first task to ease the second one.

The roots of DSP are in the 1960s and 1970s when digital computers first became available. Computers where expensive during this era, and DSP was limited to only a few critical applications. Pioneering efforts were made in four key areas: *radar and sonar*, where national security was at risk; *oil exploration*, where large amounts of money could be made; *space exploration*, where the data are irreplaceable; and *medical imaging*, where lives could be saved. The personal computer revolution of the 1980s and 1990s caused DSP to explode with new applications. Rather than being motivated by military and government needs, DSP was suddenly driven by the commercial marketplace. Anyone who thought they could make money in the rapidly expanding field was suddenly a DSP vendor. DSP reached the public in such products as: mobile telephones, compact disk players, and electronic voice mail. Figure 1 illustrates a few of these varied applications.

This technological revolution occurred from top-down. In the early 1980s, DSP was taught as a *graduate* level course in electrical engineering. A decade later, DSP had become a standard part of the *undergraduate* curriculum. Today DSP is a *basic skill* needed by scientists and engineers in many fields. As an analogy, DSP can be compared to a previous technological revolution: *electronics*. While still the realm of electrical engineering, nearly every scientist and engineer has some background in basic circuit design. Without it, they would be lost in the technological world. DSP has the same future.

*Note: A large part of this introduction is extracted from the following Book:* The Scientist and Engineer's Guide to Digital Signal Processing By Steven W. Smith *which also inspired some part of the course and computer exercises. You can find this book online at this address:* ***http://www.dspguide.com/***




- Space photograph enhancement
- Data compression
- Intelligent sensory analysis by remote space probes
- Diagnostic imaging (CT, MRI, US, etc.)
- Electrocardiogram analysis
- Medical image storage/retrieval
- Image and sound compression
- Movie special effects
- Video conference calling
- Voice and data compression
- Echo reduction
- Signal multiplexing
- Radar / Sonar
- Ordnance guidance
- Secure communication
- Oil and mineral prospecting
- Process monitoring and control
- Nondestructive testing
- Earthquake recording and analysis
- Data acquisition
- Spectral analysis

Figure 1: Example of application fields of DSP.

# Notes

*Chapter I*

# Signal

## Contents



# 1 Definitions / Terminology

## 1.a Signal

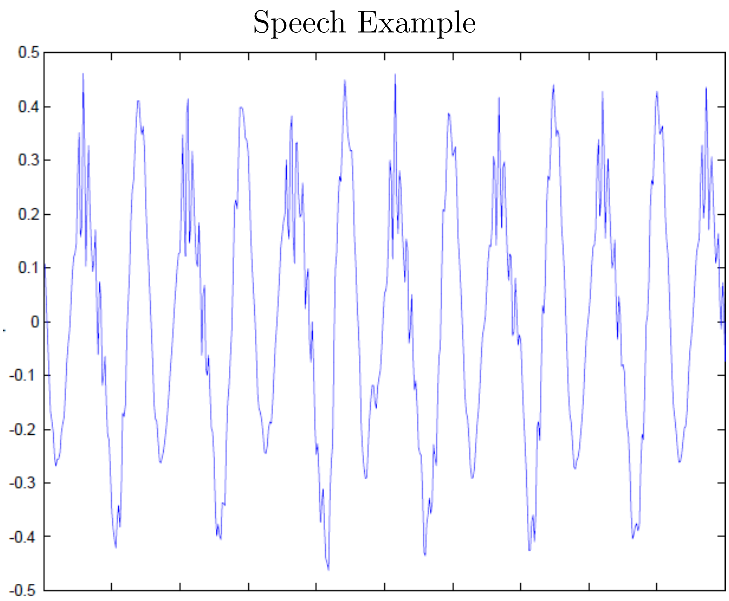
A **Signal** is

- *A physical quantity that carries information.*
- Any physical quantity that varies with one or more independent variable(s)

- A function of ______________________________ .

Examples :

- ______________________________
- ______________________________
- ______________________________
- ______________________________




A speech signal's amplitude relates to tiny air pressure variations. Shown is a recording of the vowel "e" (as in "speech").

**Terminology**

| horizontal axis | vertical axis |
|---|---|
| • x-axis | • y-axis |
| • ______________ | • ______________ |
| • ______________ | • ______________ |
| • ______________ | • ______________ |
| • ______________ | • ______________ |
| Examples | |
| • ______________ | • ______________ |
| • ______________ | • ______________ |
| • ______________ | • ______________ |
| • ______________ | • ______________ |

## 1.b Sampling and Quantization

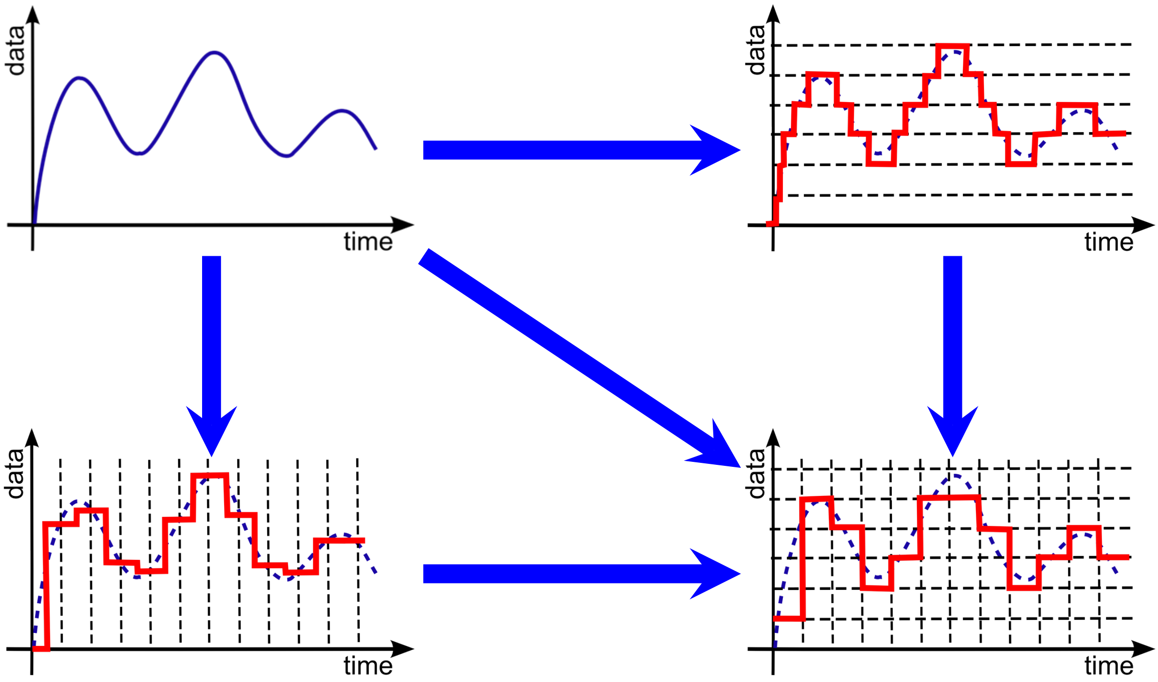
Passing a signal through a digital sensor forces each of the 2 parameters to be *digitized* to have a finite number of values.

Example:
A sensor measuring the voltage signal at a 1000 samples per second rate and storing it with a 12 bits per sample resolution:

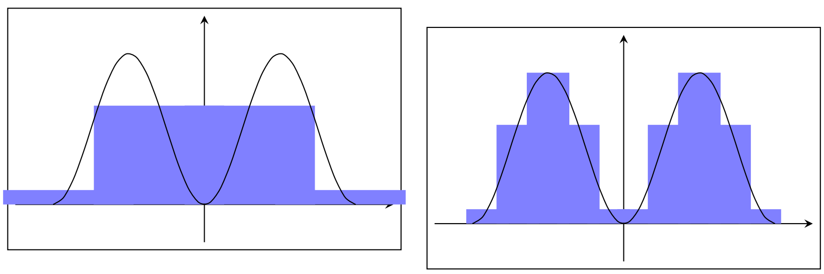
- ________ possible discrete levels ;
- ________ time increment.

**Sampling** converts the independent variable (*e.g. time*) from continuous to a discrete space.

**Quantization** converts the dependant variable (*the amplitude*) from continuous to discrete space.




**Theorem I.1** (Sampling Theorem / Nyquist-Shanon Theorem)**.** *An analog signal that has been sampled can be perfectly reconstructed from the samples if the sampling rate exceeds* $2 \times F_c$ *where* $F_c$ *is the highest frequency of the original signal.*

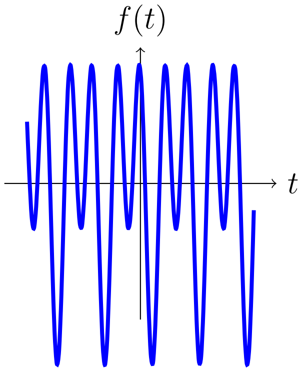


*Proof.* See Chapter III.5 □

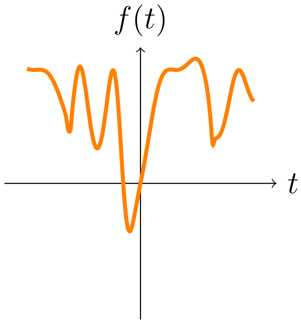
# 2 Signal Classification

In this section, we present some ways of classifying signals. Signal classification allows to determine common properties of signals from different sources and thus ease their processing and analysis. In the following section examples are given as continuous signals, but the classification also works for discrete signals.

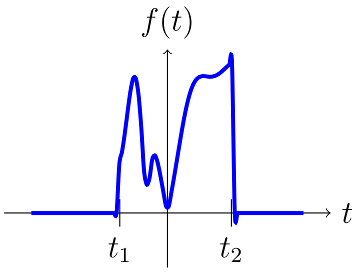
Periodic Signal



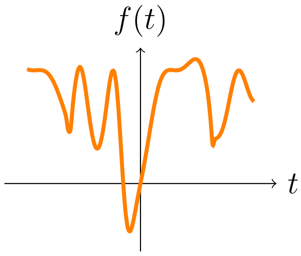

Aperiodic signal



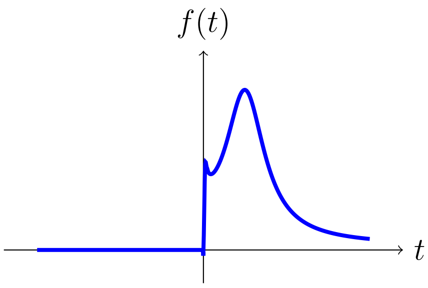

Finite-length Signal



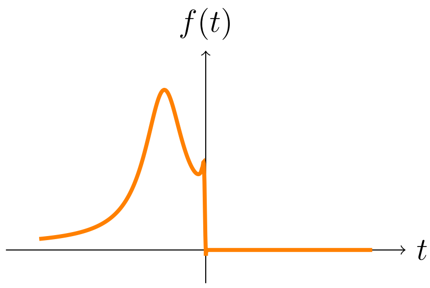

Infinite-length Signal

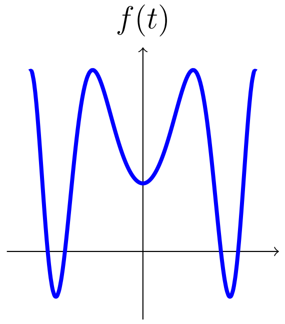
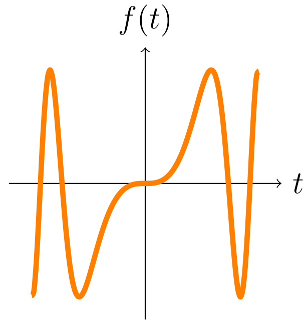
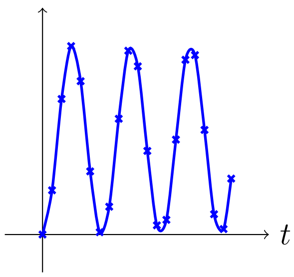
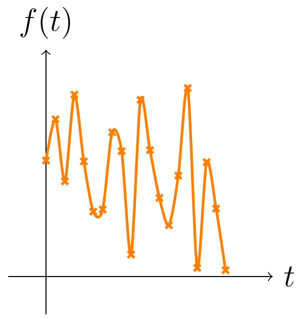



Causal Signal




Anticausal Signal



| Even Signal | Odd Signal |
|---|---|
|   |   |
| **Deterministic Signal** | **Random Signal** |
| $f(t) = sin^2(t)$   Each value of the signal is fixed and can be determined by a mathematical expression, rule or table. Future values of the signal can be calculated from past values with complete confidence. |   Lot of uncertainty about its behavior. Values of a random signal cannot be accurately predicted and can usually only b guessed based on the averages of tests of signals. |

## Examples

$$f(t) = \begin{cases} sin(2\pi t)/t & t \geq 1 \\ 0 & t < 1 \end{cases}$$

- continuous / analog
- discrete-time
- discrete-value
- periodic
- aperiodic
- finite-length
- infinite-length
- causal
- anticausal
- non-causal
- even
- odd
- deterministic
- random

$$x[k] = \begin{cases} 1 & if k = 0 \\ 0 & \forall k \in \mathbb{Z}^* \end{cases}$$

- continuous / analog
- discrete-time
- discrete-value
- periodic
- aperiodic
- finite-length
- infinite-length
- causal
- anticausal
- non-causal
- even
- odd
- deterministic
- random

$$x[k] = \frac{1}{2\pi} e^{-k^2/2}$$

- continuous / analog
- discrete-time
- discrete-value
- periodic
- aperiodic
- finite-length
- infinite-length
- causal
- anticausal
- non-causal
- even
- odd
- deterministic
- random

# 3 Examples of Simple Signals

In the following, We will see that any signal can be expressed as a combination of very simple signals. In the spirit of *divide and conquer*, as the study of simple signals is, as their name suggests *simple*, the study of complex signals will be eased by considering them as combinations of simple signals.

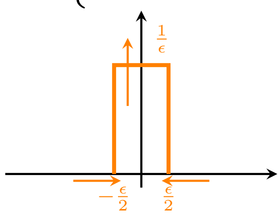
In the following section, the continuous and discrete signals are represented side by side.

## 3.a Unit Impulse

Continuous Dirac Delta Function

$$\delta(t) = lim_{\epsilon\to 0}\left\{\begin{array}{ll}\frac{1}{\epsilon} & -\frac{\epsilon}{2} < t < \frac{\epsilon}{2}\\ 0 & otherwise\end{array}\right.$$

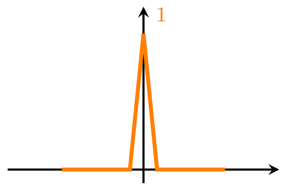
$$\delta(t) = \left\{\begin{array}{ll}\infty & t = 0\\ 0 & otherwise\end{array}\right.$$




$$\int_{t=-\infty}^{+\infty}\delta(t)dt = 1$$

Discrete Unit Impulse

$$\delta[k] = \left\{\begin{array}{ll}1 & \text{if } k = 0\\ 0 & otherwise\end{array}\right.$$




$$\sum_{k=-\infty}^{+\infty}\delta[k] = 1$$

Properties of continuous and discrete unit impulse

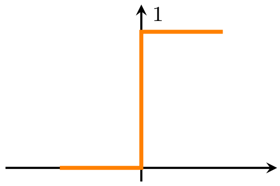
## 3.b Unit Step

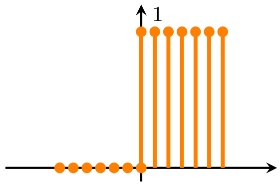
Continuous Unit Step

$$u(t) = \left\{\begin{array}{ll}1 & if t \geq 0\\ 0 & if t < 0\end{array}\right.$$

Discrete Unit Step

$$u[k] = \left\{\begin{array}{ll}1 & \text{if } k \geq 0\\ 0 & \text{if } k < 0\end{array}\right.$$



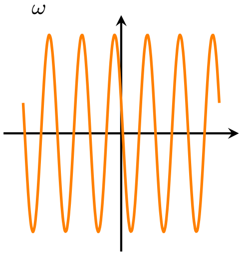

$\frac{\partial u(t)}{\partial t} =$ ______ $\quad u[k] - u[k-1] =$ ______

## 3.c Sinusoids

**Continuous Sinusoid**

$$f(t) = A\cos(\omega t + \phi)$$

- $A$ : amplitude of the signal
- $\omega$ : angular frequency
- $\phi$ : phase
- $T = \frac{2\pi}{\omega}$

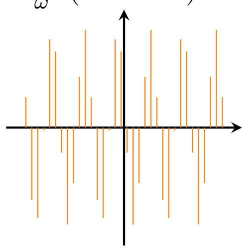


Sinusoid with $A = 1$ $\omega = 3 \times 2\pi$ $\phi = 0.4\pi$.

**Discrete-Time Sinusoid**

$$x[k] = A\cos(\omega k + \phi)$$

- $A$ : amplitude of the signal
- $\omega$ : angular frequency
- $\phi$ : phase
- $T = \frac{2\pi}{\omega}$ (rational)



Sinusoid with $A = 1$ $\omega = 3 \times 2\pi$ $\phi = 0.4\pi$.

## 3.d Complex Exponentials

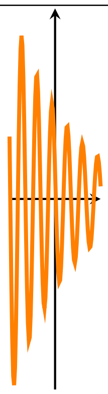
**Continuous Complex Exponential**

$$f(t) = Ae^{(i\omega+\sigma)t+i\phi}$$
$$Re\,(f(t)) = Ae^{\sigma t}\cos(\,\omega t + \phi)$$
$$Im\,(f(t)) = Ae^{\sigma t}\sin(\,\omega t + \phi)$$

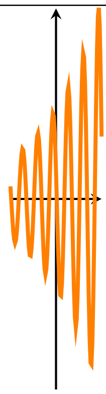
**Discrete Complex Exponential**

$$x[k] = Ae^{(i\omega+\sigma)k}$$
$$Re\,(x[k]) = Ae^{\sigma n}\cos(\,\omega k + \phi)$$
$$Im\,(x[k]) = Ae^{\sigma n}\sin(\,\omega k + \phi)$$

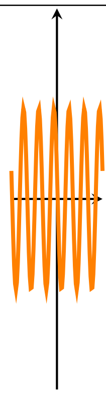
Examples of Real Parts of Complex Exponential Signals



$A = 2$, $\omega = 3 \times 2\pi$, $\phi = 0.4\pi$, $\sigma = -0.8$



$A = 2$, $\omega = 3 \times 2\pi$, $\phi = 0.4\pi$, $\sigma = +0.8$



$A = 2$, $\omega = 3 \times 2\pi$, $\phi = 0.4\pi$, $\sigma = 0.0$

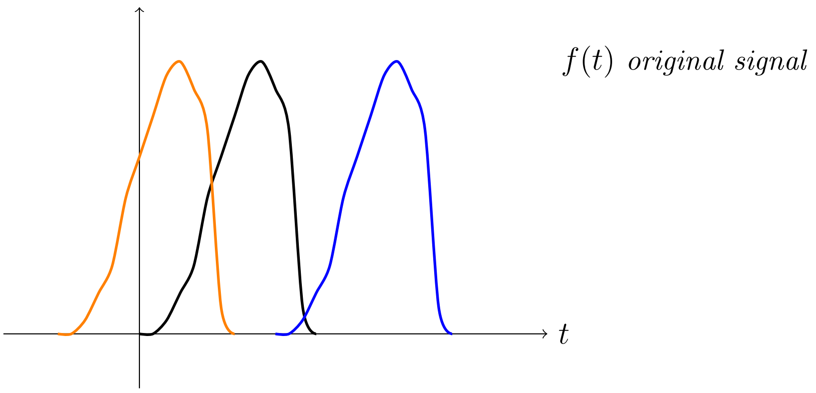
# 4 Signal Operations

## 4.a Manipulating the Time Parameter

### Time Shifting

Time shifting is performed by adding or subtracting a quantity to the independent variable (*time*).

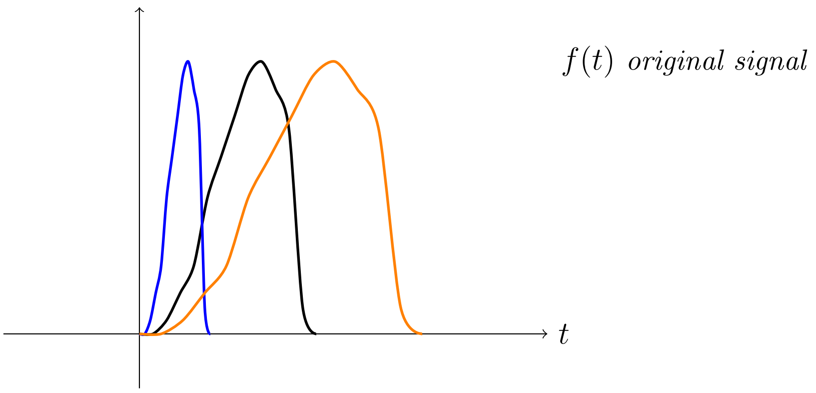
- Subtract a positive quantity $\implies$ shift the signal to the right (*delay*)
- Add a positive quantity $\implies$ shift the signal to the left (*advance*).




### Time Scaling

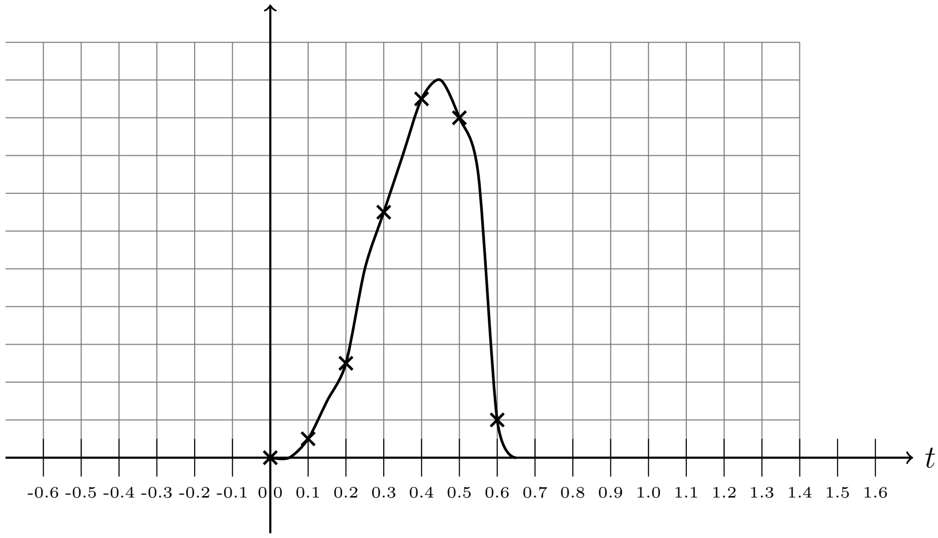
Time scaling compresses or dilates a signal by multiplying a time variable by some quantity ($a$)

- $a > 1 \implies$ the signal becomes narrower (*compression*)
- $a < 1 \implies$ the signal becomes wider (*dilation*)



**Example**

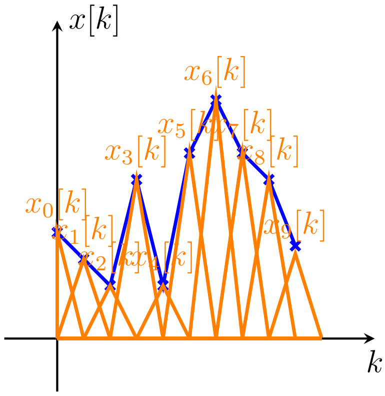
Given the continuous signal $f(t)$, sketch $f(0.5 \times t + 0.3)$.

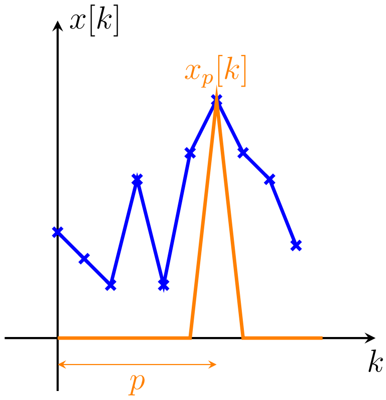


## 4.b Signal Decomposition

### Impulse Decomposition

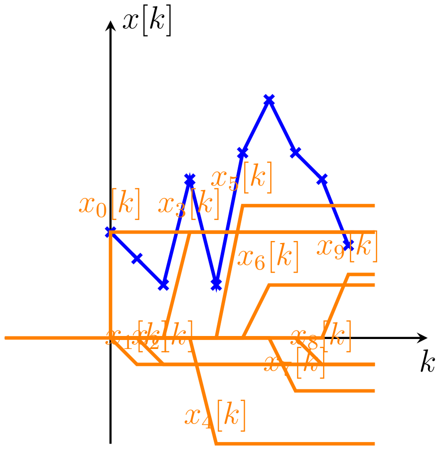
Impulse decomposition breaks a N samples signal into N component signals. Each component signal is a N samples (non unit) impulse.

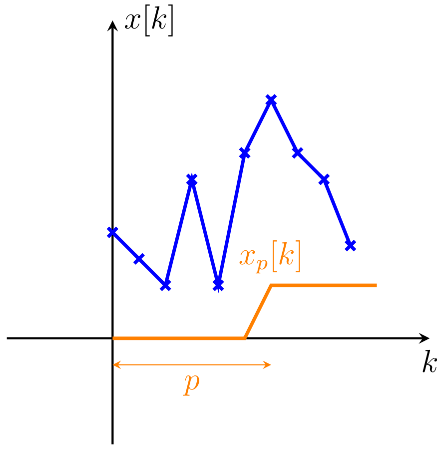


### Step Decomposition

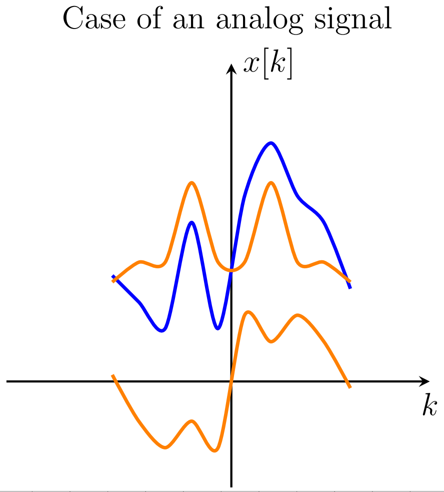
Step decomposition breaks a N samples signal into N component signals. Each component signal is a N samples (non unit) step.

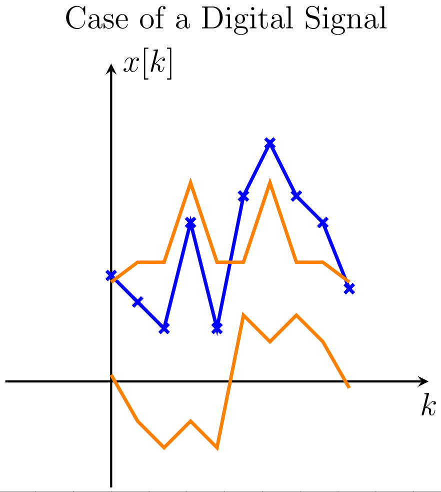


**Even / Odd Decomposition**

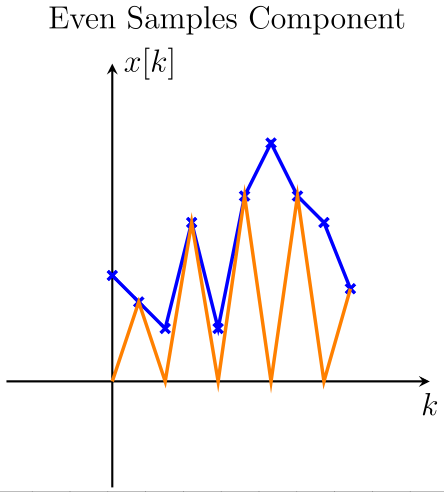
The even / odd decomposition breaks a N samples signals int 2 component signals, one having even symmetry, the other one having dd symmetry.

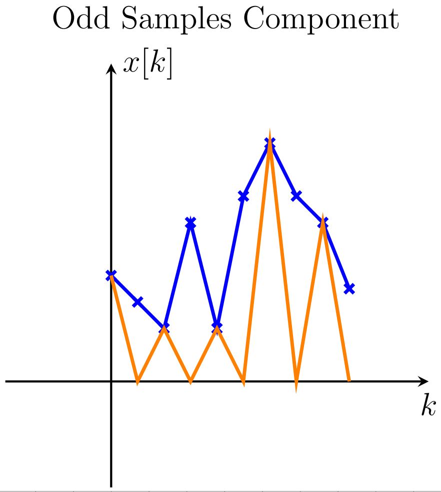


**Interlaced Decomposition**

The interlaced decomposition breaks the signal into 2 component signals, an even sample signal and an odd sample signal.



# 5 Stochastic Signals Characterization

Unlike deterministic signals, **stochastic** or **random** signals cannot be characterized by a simple, well-defined mathematical equation and their future values cannot be predicted. Rather, we must use probability and statistics to analyze their behavior. Also, because of their randomness, average values from a collection of signals are usually studied rather than analyzing one individual signal.

## 5.a Random Variable

**Definition I.1** (Random Variable)**.** *A* **random variable** *is a function from a probability space* $\Omega$*, to the real numbers, which is measurable. Intuitively, a random variable is a numerical description of the outcome of an experiment.*

**Example I.1.** *Let us consider a random variable* $\chi$ *describing the process of rolling a die. The set* $\Omega$ *of its possible outcomes will be:* $\Omega = \{\_\_\_, \_\_\_, \_\_\_, \_\_\_, \_\_\_, \_\_\_\}$.

$$\chi(\omega) = \begin{cases} \_\_\_ & \text{, if 1 is rolled,} \\ \_\_\_ & \text{, if 2 is rolled,} \\ \_\_\_ & \text{, if 3 is rolled,} \\ \_\_\_ & \text{, if 4 is rolled,} \\ \_\_\_ & \text{, if 5 is rolled,} \\ \_\_\_ & \text{, if 6 is rolled,} \end{cases}$$

**Example I.2.** *Let us now consider a continuous random variable* $\chi$ *equidistributed between the values a and b (continuous case of the die, with a instead of* 1 *and b instead of* 6*).*
*The set* $\Omega$ *of possible outcomes will be:*
$\Omega = \underline{\hspace{4cm}}$ .

**Definition I.2** (Probability Distribution Function)**.** *The* **Probability Distribution Function** *is used to identify the probability that a random variable* $\chi$ *is less than or equal to a given number.*

$$F(x) = Probability[\chi \leq x]$$

### Properties of Probability Distribution Function

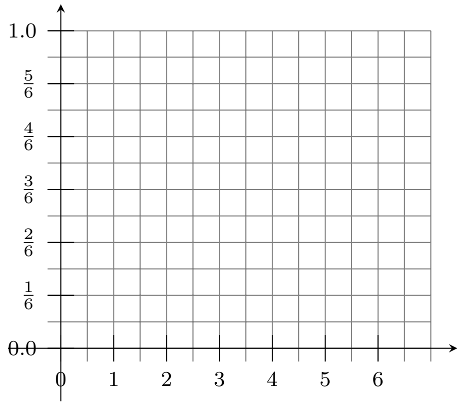
- $\forall x1 < x2, Probability[x1 \leq \chi(\omega) < x2] = F(x2) - F(x1)$
- $lim_{x \to \infty} F(x) = 1$
- $lim_{x \to -\infty} F(x) = 0$

**Definition I.3** (Probability Density Function (pdf))**.** *The* **Probability Density Function (pdf)** *is the derivative of the probability distribution function:*

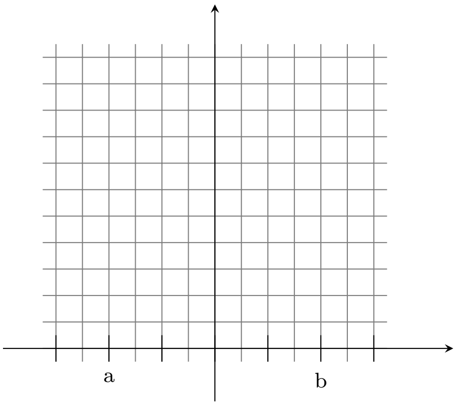
$$f(x) = \frac{\partial F(x)}{\partial x}$$

$$f(x)dx = Probability[x \leq \chi(\omega) < x + dx]$$

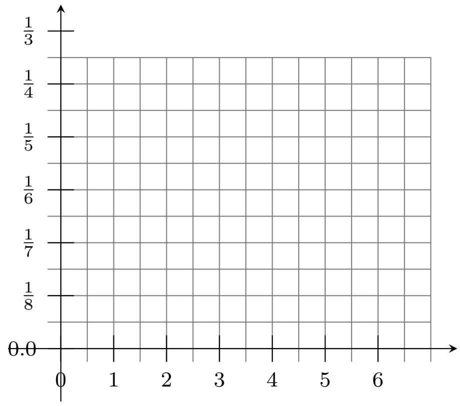
**Example I.3.** *In the case of the 6-faces die (Exemple I.1), we obtain the following* Probability Distribution Function*:*



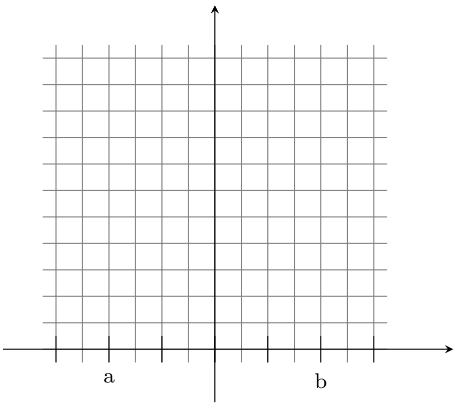

**Example I.4.** *In the case of the equidistributed continuous variable (Example I.2), we obtain the following* Probability Distribution Function*:*




**Example I.5.** *In the case of the 6-faces die (Exemple I.1), we obtain the following* Probability Density Function*:*




**Example I.6.** *In the case of the equidistributed continuous variable (Example I.2), we obtain the following* Probability Density Function*:*




**Definition I.4** ($k^{th}$ Moment)**.** *The $k^{th}$* **moment** *of a random variable $\chi$ is defined as:*

$$E[\chi^k] = \int_{-\infty}^{\infty} x^k dF(x) = \int_{-\infty}^{\infty} x^k p(x)dx$$

**Definition I.5** (Mathematical Expectation)**.** *The* **mathematical expectation** *of a* continuous *random process, called the average of the process is the* $1^{st}$ *moment:*

$$E[\chi] = \int_{-\infty}^{\infty} x dF(x) = \int_{-\infty}^{\infty} x p(x) dx$$

*In the case of a* Discrete variable, *this writes:*

$$E[\chi] = \sum_{\alpha \in \Omega} \alpha Probability[\chi = \alpha]$$

**Example I.7.** *In the case of the 6-faces die (Example I.1),* $E[\chi]$ *writes:*

**Example I.8.** *In the case of a continuous random variable* $\chi$ *equidistributed between values a and b (Example I.2),* $E[\chi]$ *writes:*

**Example I.9.** *If you play the roulette in a casino, you have 1 chance out of 37 (the numbers go from 0 to 36) to win 36 times your bet. Suppose you bet* 10\$, *then the probability to loose* 10\$ *is* 1 *and the probability to win* $36 \times 10\$$ *is* $\frac{1}{37}$. *The mathematical expectation is thus* $-10 + \frac{10 \times 36}{37} = -0.27$. *The mathematical expectation is negative for the player. This is why in average, the casino always wins...*

### Properties of the mathematical expectation

Given a constant value $\alpha$,

- $E[\alpha] =$ ____________________
- $E[\chi + \alpha] =$ ____________________
- $E[\alpha \chi] =$ ____________________
- $E[\chi 1 + \chi 2] =$ ____________________

**Definition I.6** (Variance)**.** *The* **Variance** *of a random variable is its centered* $2^{nd}$ *moment:*

$$Var_\chi = E\left[(\chi - E[\chi])^2\right] = \int_{-\infty}^{\infty} (x - E[\chi])^2 p(x) dx$$

**Properties of Variance**

- $Var(\chi) = \sigma^2$ where $\sigma$ is called standard deviation
- $Var(\chi) = E[\chi^2] - (E[\chi])^2$

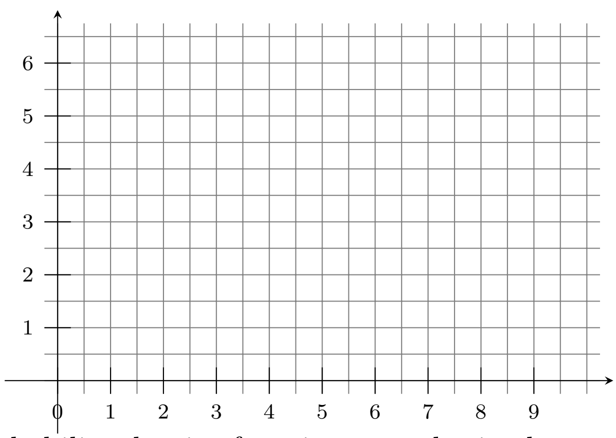
**Example I.10.** *In the case of the 6-faces die (Example I.1), $Var_\chi$ writes:*

**Example I.11.** *In the case of a continuous random variable equidistributed between values a and b (Example I.2), $Var_\chi$ writes:*

## 5.b Random Process

**Definition I.7** (Random Process)**.** *A family or ensemble of signals that correspond to every possible outcome of a certain signal measurement. Each signal in this collection is referred to as a* **realization** *or* **sample function** *of the process.*

**Example I.12.** *Example of the 6-faces die. Let us note $X = \{x[0], x[1], ..., x[9]\}$ a* **realization** *of the random variable of example I.1. $X$ is a 10-samples signal.*



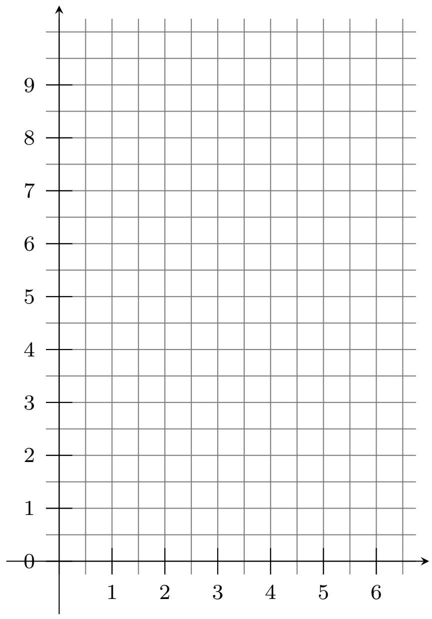

The probability distribution and probability density functions are obtained on a random variable, that is to say, a theoretical model of the random signal we generally measure. During a measure one generally gets a digital signal, with a finite number of values.

On can approximate the pdf of the underlying random variable by the histogram of the measured signal.

**Definition I.8** (Histogram)**.** *The* **histogram** $H = \{H_i, i \in \Omega\}$ *of a signal* $X = \{x[k], k = 0..N-1\}$ *displays the number of samples there are in the signal that have each possible values.*

$$\forall i \in \Omega H_i = card\,\{k/x[k] = i\}$$

The histogram of the realisation of the measure of the 6-faces die example (Example I.12 is as follows:




**Definition I.9** (Mean of a measured signal)**.** *The* **mean** *of a measured discrete signal is:*

$$\mu = \frac{1}{N}\sum_{k=0}^{N-1} x[k]$$

**Example I.13.** *In the case of the signal of Example I.12, the mean $\mu$ writes:*

**Definition I.10** (Variance and Standard Deviation of a measured signal)**.** *In the same way, the standard deviation $\sigma$ writes :*

$$\sigma^2 = \frac{1}{N-1}\sum_{k=0}^{N-1}(x[k]-\mu)^2$$

In some situations, the mean describes what is being measured, while standard deviation represents the noise and other interferences. In these cases, $\sigma$ is not interesting in itself, but in comparison with $\mu$. We define:

**Definition I.11** (Signal to Noise Ratio)**.**

$$SNR = \frac{\mu}{\sigma}$$

Because many signals have a very wide dynamic range, signals are often expressed using the logarithmic decibel scale. this gives:

**Definition I.12** (Signal to Noise Ration (dB))**.**

$$SNR_{dB} = 10log_{10}(SNR)$$

We can thus express the variance of the signal itself to the variance of the noise thanks to SNR:

$$\frac{Var(Signal)}{Var(noise)} = 10^{\frac{SNR_{dB}}{10}}$$

### Precision vs Accuracy

Every system that measures, estimates or predicts produces errors.

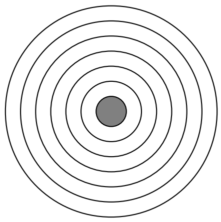
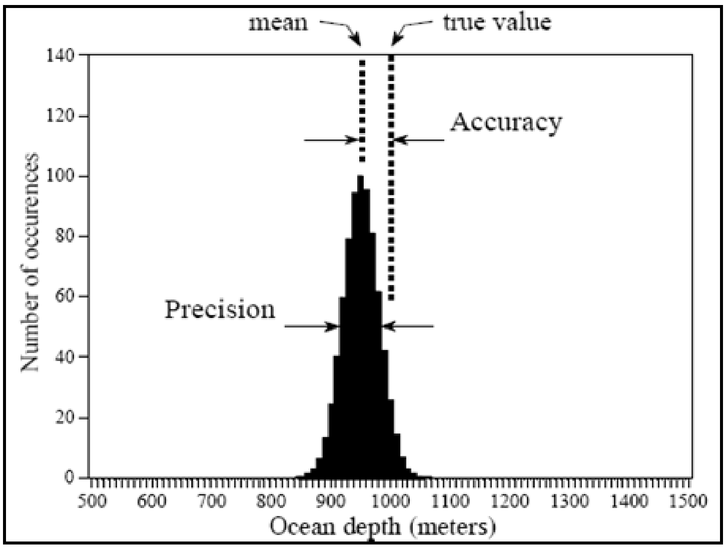


Figure I.1: Darts targets.

**accuracy** is the difference between the true value and the mean of the underlying process generating the data.

**precision** is the spread of the values, specified by standard deviation, signal to noise ratio or c.v.




**Example I.14.** *In the case of the 6-faces die (Examples I.1 and its realization I.12), what are the precision and accuracy of the realization?*

# 6 Notes

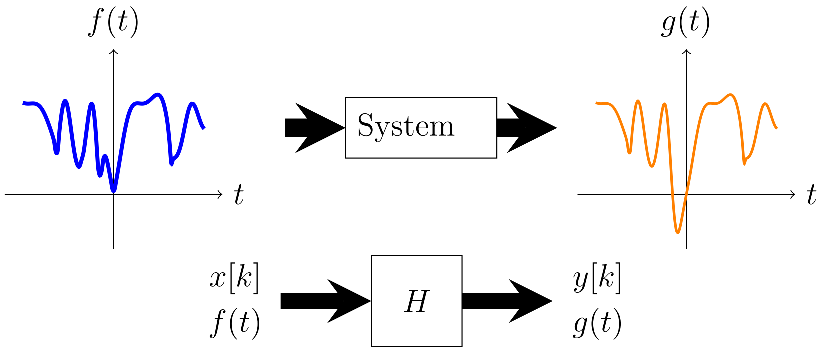
*Chapter II*

# ***Systems***

## Contents

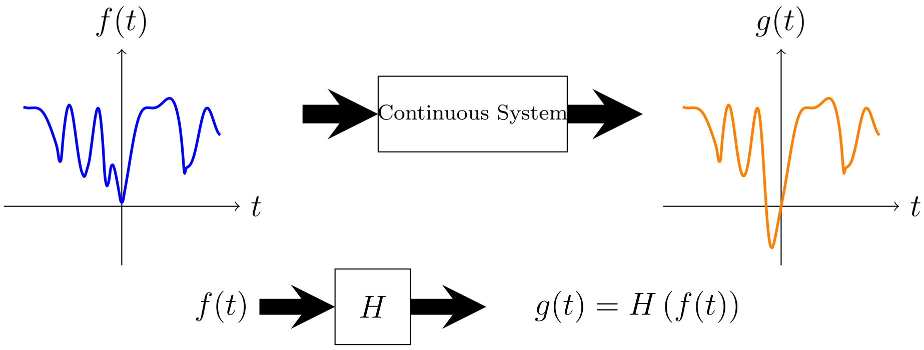
# 1 Definitions / Classifications

**Definition II.1** (System)**.** *A **system** is any process that produces an output signal in response to an input signal.*

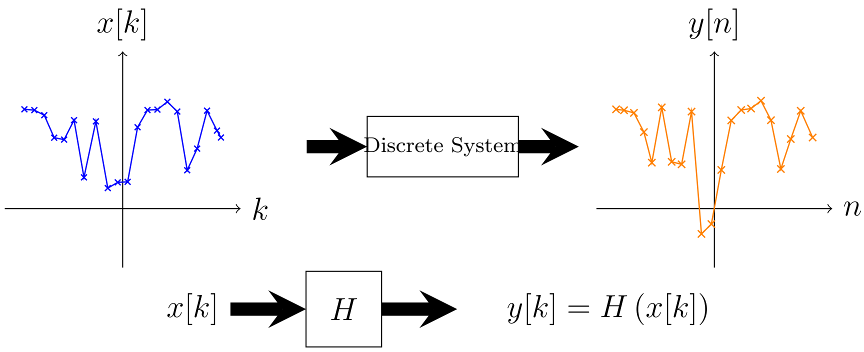



## 1.a Continuous vs. Discrete Systems

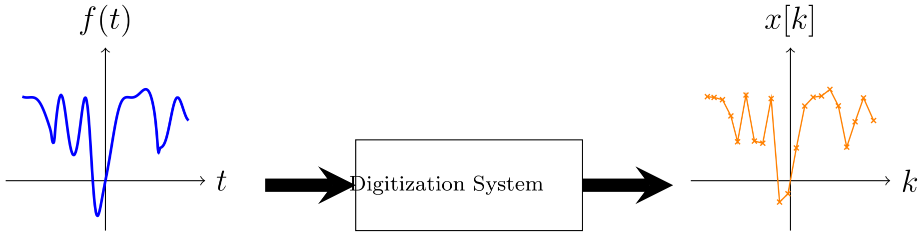
**Definition II.2** (Continuous Systems)**.** *A system in which the input signal and output signal both have continuous domain is said to be a **continuous** system.*

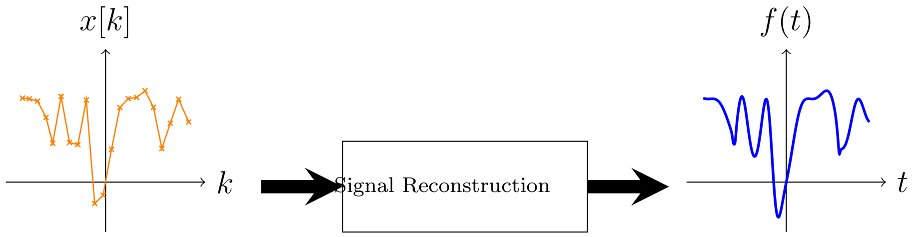



**Definition II.3** (Discrete Systems)**.** *A system in which the input signal and the output signal both have discrete domain is said to be a **discrete** system.*



**Other Systems**




## 1.b Time Invariant vs. Time Varying Systems

**Definition II.4** (Parameter/Time Shift Operator)**.** $S_T : S_T\left(f(t)\right) = f(t - T)$

**Definition II.5** (Time Invariant System)**.** *A system is said to be time invariant if it commutes with the parameter shift operator:*

$$HS_T = S_T H$$

*Suppose* $H(f) = g$*, i.e.* $H\left(f(t)\right)(s) = g(s)$

$$H\left(S_T(f(t))\right) = H\left(f(t - T)\right) = S_T\left(H(f)\right) = S_T(g) = g(s - T)$$

## 1.c Causal vs. Non-causal

**Definition II.6** (Causal System)**.** *A* ***causal*** *system is a system in which the output depends only on current or past inputs, but not future inputs.*

**Definition II.7** (Anticausal System)**.** *Similarly an* ***anticausal*** *system is a system in which the output depends on current or future inputs, but not past inputs.*

**Definition II.8** (Non-causal System)**.** *A* ***non-causal*** *system is a system in which the output depends on both past and future inputs.*

***Note:*** A *real time* system must be *causal* since it cannot have future inputs available while processing.
***Note:*** Non-causal system are often encountered in image processing (where the independent variable is not time).

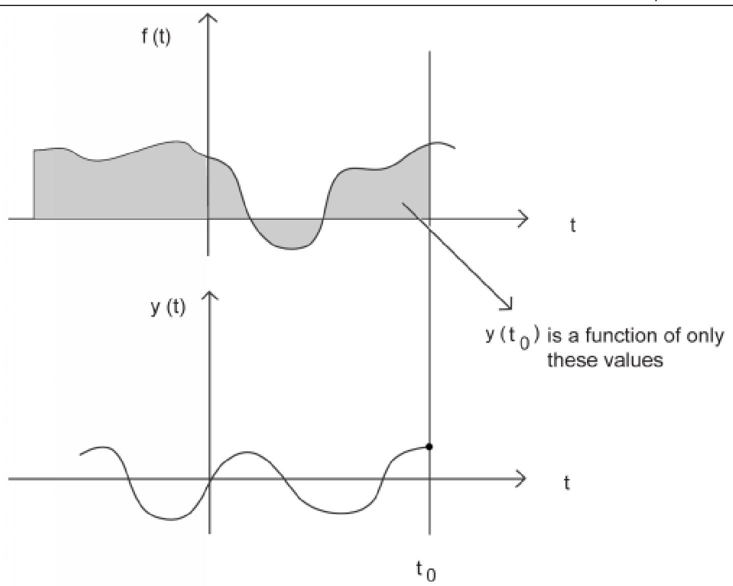



Figure II.1: Non-causal system (up) and Causal system (down)

## 1.d Stable vs. Unstable

There are several definitions of stability for a system, but the one that will be used here is *Bounded Input Bounded Output* (BIBO) stability.

**Definition II.9** (Stable System)**.** *A **stable** system is one in which the output is bounded if the input is also bounded.*

$$|f(t)| \leq M_f < \infty \Longrightarrow |g(t)| \leq M_g < \infty$$

**Definition II.10** (Unstable System)**.** *An **unstable** system is one in which at least one bounded input produces an unbounded output.*

$$\exists f \textit{ such that } |f(t)| \leq M_f < \infty \textit{ and } H(f) = g(t) \to \infty$$

## 1.e Linear vs. Non-Linear

In this course, we only study linear systems.
In practice, there is only one major strategy for analysing systems that are not linear: *make it resemble to a linear system.*

- ignore non-linearity (if the non-linearity is small enough)
- keep small amplitudes
- apply a linearizing transform
  ex: $a[n] = b[n] \times c[n] \Longrightarrow log(a[n]) = log(b[n]) + log(c[n])$
  *(homeomorphic signal processing)*

# 2 Linear Time Invariant Systems

Linearity and time invariance are 2 system properties that greatly simplify the study of systems that exhibit them. In our study of signals and systems, we will be especially interested in systems that demonstrate both of these properties, which together allow the use of some of the most powerful tools of signal processing.

## 2.a Linear Systems

**Definition II.11** (Linear System)**.** *A system is linear if and only if it verifies the following properties:*

- *Homogeneity / Scaling*
- *Additivity / Superposition*

**Definition II.12** (Homogeneity / Linear Scaling)**.** *A system is* **homogeneous***, if when its input is scaled by a constant value, then its output is scaled by the same value.*

$$f(t) \longrightarrow \boxed{H} \longrightarrow g(t) \implies \forall \alpha \in \mathbb{C},\ \alpha f(t) \longrightarrow \boxed{H} \longrightarrow \alpha g(t)$$

$$\forall \alpha \in \mathbb{C},\ H(\alpha f) = \alpha H(x) = \alpha g$$

**Definition II.13** (Additivity / Superposition)**.** *If two inputs are added together and passed through a linear system, the output will be the sum of the individual input's outputs.*

$$f_1 \longrightarrow \boxed{H} \longrightarrow g1 \quad and \quad f_2 \longrightarrow \boxed{H} \longrightarrow g2$$
$$f_1 + f_2 \longrightarrow \boxed{H} \longrightarrow g1 + g2$$

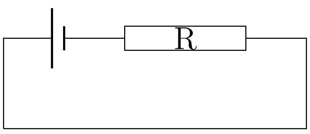
$$H\left(f_1 + f_2\right) = H(f_1) + H(f_2) = g1 + g2$$

Thus, for a linear system, we have:

$$f_1 \longrightarrow \boxed{\text{H}} \longrightarrow g1 \quad \text{and} \quad f_2 \longrightarrow \boxed{\text{H}} \longrightarrow g2$$
$$\forall \alpha, beta \in \mathbb{C},\ \alpha f_1 + \beta f_2 \longrightarrow \boxed{\text{H}} \longrightarrow \alpha g1 + \beta g2$$

$$H\left(\sum_{k=0}^{N-1} \alpha_k f_k\right) = \sum_{k=0}^{N-1} \alpha_k H(f_k) = \sum_{k=0}^{N-1} \alpha_k g_k$$

**Example**




- $v(t)$: voltage
- $i(t)$: intensity
- $p(t)$: power

a) $H_1$: Ohm law
$v(t) = H_1(i(t)) =$ ______

$$i(t) \longrightarrow \boxed{H_1} \longrightarrow v(t)$$

b) $H_2$: Electrical power:
$p(t) = H_2(i(t)) =$ ______

$$i(t) \longrightarrow \boxed{H_2} \longrightarrow p(t)$$

c) $H_3$: Other system:
$H_3(f(t)) = tf(t)$

$$f(t) \longrightarrow \boxed{H_3} \longrightarrow tf(t)$$

Are these systems linear?

## 2.b Time Invariant Systems

**Definition II.14** (Time Invariant System)**.** *A* **time-invariant system** *has the property that a certain input will always give the same output (up to timing), without regard to when the input was applied to the system.*

$$f(t) \longrightarrow \boxed{H} \longrightarrow g(t) \implies f(t-T) \longrightarrow \boxed{H} \longrightarrow g(t-T)$$

**Example**

Are the previous system examples time invariant ?

### 2.c Linear Time Invariant Systems

A linear time invariant system has the following properties:

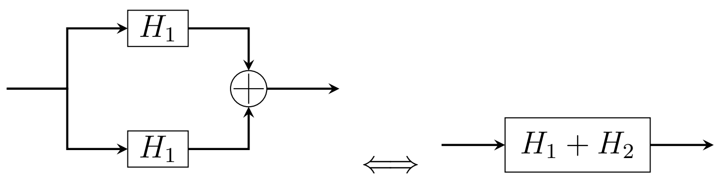
$$f_1 \longrightarrow \boxed{\text{LTI}} \longrightarrow g1 \quad \text{and} \quad f_2 \longrightarrow \boxed{\text{LTI}} \longrightarrow g2$$

$$\alpha f_1(t - T_1) + \beta f_2(t - T_2) \longrightarrow \boxed{\text{H}} \longrightarrow \alpha g1(t - T_1) + \beta g2(t - T_2)$$

This has for consequences that the order of cascaded linear time invariant systems (LTI) can be interchanged without changing the overall effect.

$$\longrightarrow \boxed{H_1} \longrightarrow \boxed{H_2} \longrightarrow \quad \Longleftrightarrow \quad \longrightarrow \boxed{H_2} \longrightarrow \boxed{H_1} \longrightarrow$$

Moreover, when two or more LTI systems are in parallel with one another, an equivalent system is one that is defined as the sum of these individual systems:




# 3 Time Domain Analysis of Linear Systems

### 3.a Superposition / Decomposition: Divide and Conquer

**Definition II.15** (Superposition / Synthesis)**.** *The process of combining signals through scaling and additions is called **synthesis** or **superposition**.*

**Definition II.16** (Decomposition)**.** ***Decomposition** is the inverse operation of synthesis, where a single signal is broken into 2 or more additive components.*

Consider an input signal $x[k]$ passing through a LTI system resulting in an output signal $y[k]$. The input signal can be decomposed into a group of simpler signals $x_0[k], x_1[k], x_2[k], etc.$ They will be called ***input signal components***. Each input signal component can be individually passed through the system, resulting in a set of output signal components $y_0[k], y_1[k], y_2[k], etc.$. These output signal components can then be synthesized into the output signal $y[k]$.

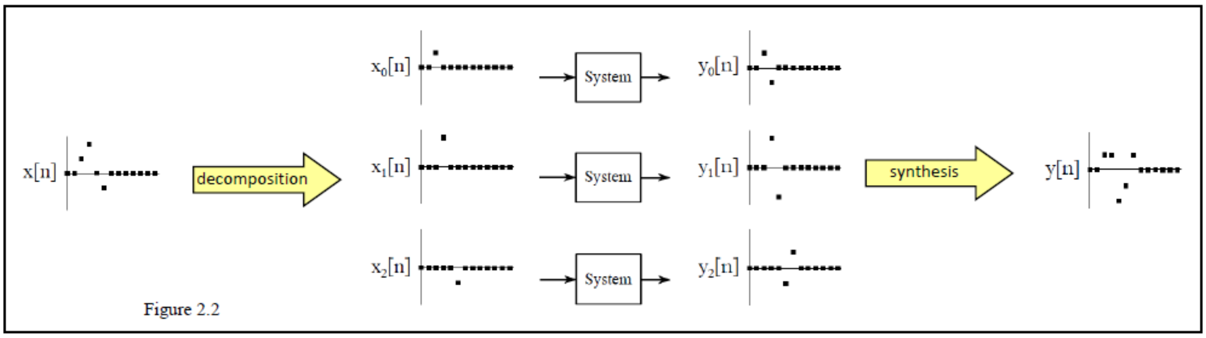



Figure II.2: Signal decomposition and synthesis through a linear system.

## 3.b Kernel / Impulse Response of a System

**Definition II.17** (Impulse response)**.** *The* ***impulse response*** *of a system is the signal that exits the system when the input signal is a delta function* $\delta[k]$*. It is generally denoted* $h[k]$*.*

$$\delta[k] \longrightarrow \boxed{\textit{Linear System}} \longrightarrow h[k]$$

In the following, when we will use systems to filter signals / images, the system will be called ***filter*** and $h$ ***filter kernel***.

## 3.c Recall: Impulse Decomposition

## 3.d Signal Decomposition and Systems

Given a system / filter with a kernel $h[k]$, the output $y[k]$ of the system when the input is $x[n]$ is:

$$x[k] \longrightarrow \boxed{\begin{matrix} S \\ h[n] \end{matrix}} \longrightarrow y[k]$$

# 4 Convolution

## 4.a Definition and Properties

**Definition II.18** (Discrete time convolution)**.** ***Discrete time convolution*** *is an operation on two discrete time signals defined by the sum:*

$$(x * y)[k] = \sum_{l=-\infty}^{+\infty} x[l]y[k-l]$$

*for all signals $x$, $y$ defined on $\mathbb{Z}$.*

**Definition II.19** (Continuous time convolution)**.** ***Continuous time convolution*** *is an operation on two continuous time signals defined by the sum:*

$$(f * g)(t) = \int_{\tau=-\infty}^{+\infty} f(\tau)g(t-\tau)dt$$

*for all signals $f$, $g$ defined on $\mathbb{R}$.*

### Property of Commutativity

$$x * y = y * x$$

**Property of Associativity**

$$x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$$

**Property of Distributivity**

$$x_1 * (x_2 + x_3) = (x_1 * x_2) + (x_1 * x_3)$$

**Property of Multi-linearity**

$$\alpha(x * y) = (\alpha x) * y = x * (\alpha y)$$

**Property of Conjugation**

$$\overline{x * y} = \overline{x} * \overline{y}$$

**Property of Time Shift**

$$S_T(x * y) = (S_T x) * y = x * (S_T y)$$

**Impulse Convolution**

$$x * \delta = x$$

## 4.b Convolution on Causal Signals

Convolution is defined on infinite signals ($\int_{\infty}^{\infty}$, $\sum_{k=-\infty}^{\infty}$). However in the case of discrete signals, we often encounter finite-length signals.

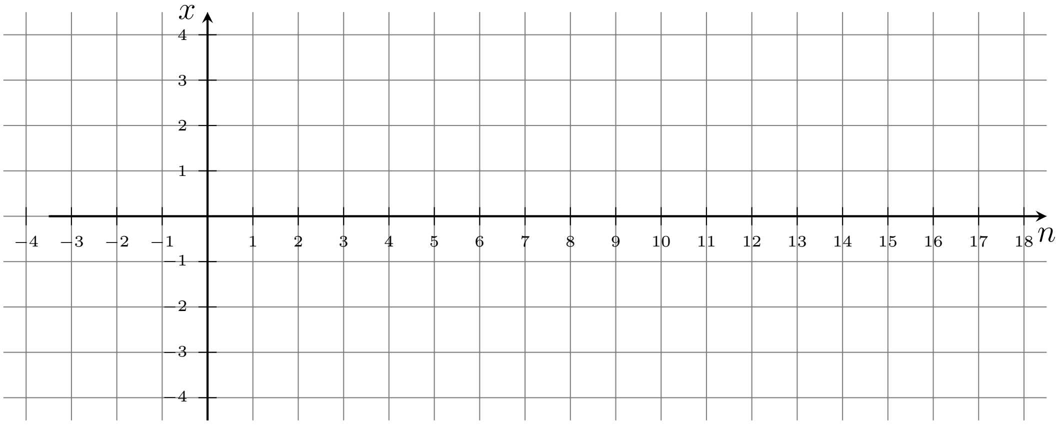
**Definition II.20** (Definition)**.** *Suppose $x[k]$ and $y[k]$ are discrete, causal signals. Discrete time convolution on causal signals is defined by:*

$$(x * y)[k] = \sum_{l=-\infty}^{\infty} x[l][k-l] \quad \text{(II.1)}$$

$$= \sum_{l=0}^{k} x[l][k-l] \quad \text{(II.2)}$$

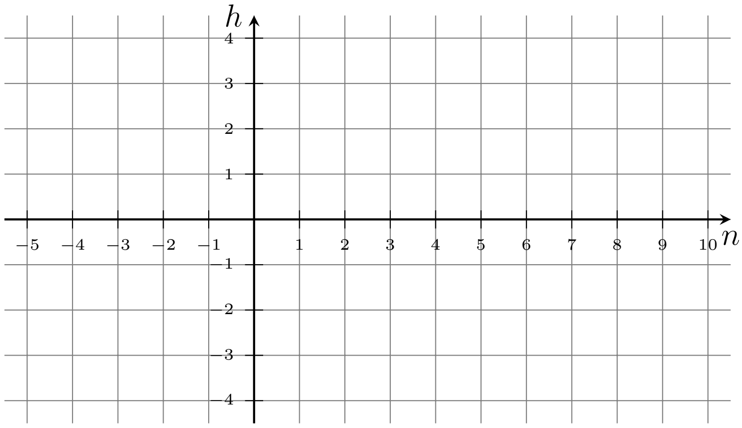
**Example II.1.** *Let us consider the signal $(x[n], n = 0..15)$ defined by:*

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x[n]$ | 1 | 0 | 2 | 3 | 2 | 1 | $-1$ | $-2$ | $-1$ | 0 | 2 | 3 | 3 | 2 | 1 | 1 |
| *index n* | *0* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* | *10* | *11* | *12* | *13* | *14* | *15* |



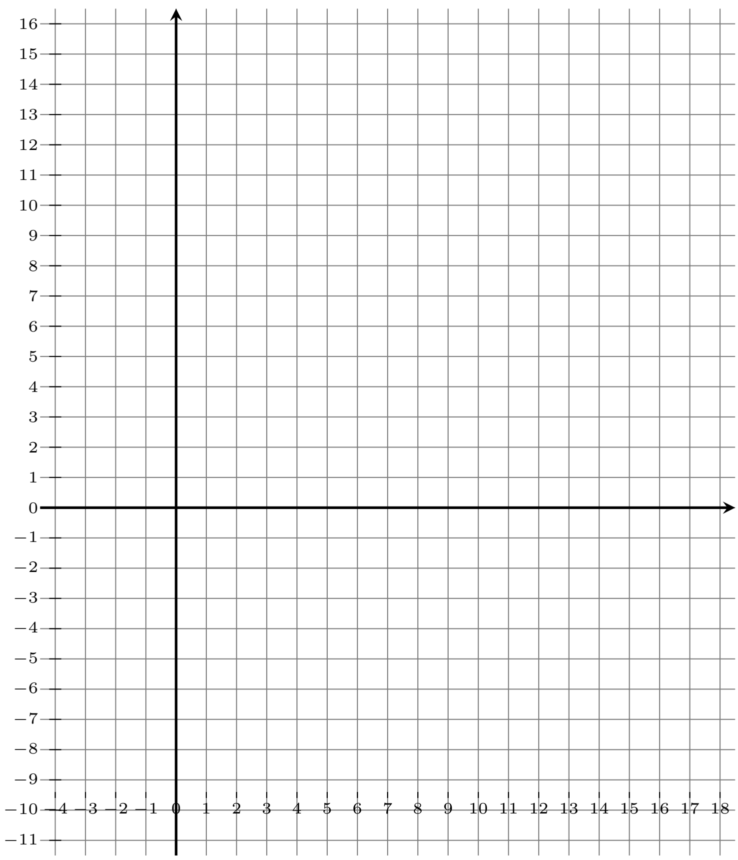
*and the impulse response ($h[n], n = 0..4$) defined by:*

| $h[n]$ | 2 | 2 | $-1$ | $-1$ | 3 |
|---|---|---|---|---|---|
| *index n* | *0* | *1* | *2* | *3* | *4* |



*To compute $y[n] = (x * h)[n]$, we obtain for example:*
$y[5] = \sum_{k=0}^{4} x[5]h[5-k] =$

*And finally:*



# 5 Notes

*Chapter III*

# ***Fourier Transform***

## Contents



Born in 1768, François Joseph Fourier was a scientist, a politician and an economist. In 1811, as he was prefect of Isère (French disctric where Grenoble is) he entered a competition on heat equations sponsored by the French Academy of Sciences. The paper he submitted described a novel analytical technique that we today call the Fourier Transform, and he won the competition ; but the prize jury declined to publish it, criticizing the sloppiness of Fourier's reasoning. Now, hovever, his name is everywhere. The Fourier Transform is a way to decompose a signal into its constituent frequencies, and versions of it are used to generate and filter cell-phone and Wi-Fi transmissions, to compress audio, image and video files so that they take up less bandwidth, as well as to solve differential equations, among other things.

# 1 Definition and Properties

**Definition III.1** (Continuous time Fourier transform). *For a given function $f$ such that $\int_{-\infty}^{\infty} |f(x)|dx < \infty$, the* Fourier transform *of $f$ is defined, for each real number $\nu$ by*

$$\mathcal{F}(f)(\nu) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\nu x}dx \tag{III.1}$$

The idea behind this defnition is that, for each value of $\nu$, the complex value of $\mathcal{F}(f)(\nu)$ captures the component at the frequency $\nu$ (and period $\frac{1}{\nu}$), in $f$.

**Example III.1** (Fourier Transform of a Gaussian Function). *Let $gauss(x) = e^{-Ax^2}$, for some positive constant $A > 0$. Then we have*

$$\mathcal{F}gauss(\nu) = \sqrt{\frac{\pi}{A}}e^{-\frac{\pi^2\nu^2}{A}} \tag{III.2}$$

**Lemma III.1.** *For $A > 0$, we have $\int_{-\infty}^{\infty} e^{-Ax^2}dx = \sqrt{\frac{\pi}{A}}$*

*Proof.* Let us square the integral:

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-Ax^2}dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-Ax^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-Ax^2}dx\right) \\
&= \left(\int_{-\infty}^{\infty} e^{-Ax^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-Ay^2}dy\right) \\
&= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{-Ax^2}e^{-Ay^2}dxdy \\
&= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{-A(x^2+y^2)}dxdy \\
& x = r\cos(\theta), y = r\sin(\theta) \text{ (polar coordinates )} \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-Ar^2}rdrd\theta \\
&= \int_{\theta=0}^{2\pi}\left(\lim_{b\to\infty}\int_{r=0}^{b} re^{-Ar^2}dr\right)d\theta \\
&= \int_{\theta=0}^{2\pi}\left(\lim_{b\to\infty}[\tfrac{-1}{2A}e^{-Ax^2}]_0^b\right)d\theta \\
&= \int_{\theta=0}^{2\pi}\left(\lim_{b\to\infty}\tfrac{1-e^{-Ab^2}}{2A}\right)d\theta \\
&= \int_{\theta=0}^{2\pi}\tfrac{1}{2A}d\theta \\
&= \tfrac{\pi}{A}
\end{aligned} \tag{III.3}$$

□

Proof of equation III.2:

The Fourier transform of a Gauss function with a standard deviation $\sigma$ ($gauss(t) = e^{\frac{-t^2}{2\sigma^2}}$) is a ________________________________

## 1.a Additivity

$$\mathcal{F}(f+g)(\nu) = \mathcal{F}f(\nu) + \mathcal{F}g(\nu) \tag{III.4}$$

## 1.b Homogeneity

$$\mathcal{F}(\alpha \cdot f)(\nu) = \alpha \mathcal{F}f(\nu) \tag{III.5}$$

## 1.c Time Shifting

$$\mathcal{F}(S_\alpha f)(\nu) = e^{-i2\pi\nu\alpha}\mathcal{F}f(\nu) \tag{III.6}$$

## 1.d Frequency Shifting

$$\mathcal{F}(e^{2i\pi\nu_0 t}f(t))(\nu) = \mathcal{F}f(\nu - \nu_0) \tag{III.7}$$

## 1.e Time scaling

For a given function $f$ and a fixed real number $\alpha \neq 0$, let $\phi(t) = f(\alpha t)$, then

$$\mathcal{F}(\phi)(\nu) = \frac{1}{|\alpha|}\mathcal{F}f\left(\frac{\nu}{\alpha}\right) \tag{III.8}$$

Suppose $\alpha > 0$, and then $\alpha < 0$

## 1.f Even and Odd functions

Recall:

$$\begin{aligned} \mathcal{F}f(\nu) &= \int_{t=-\infty}^{\infty} f(t)e^{-i2\pi\nu t}dt \\ &= \int_{t=-\infty}^{\infty} f(t)\cos(2\pi\nu t)dt - i \cdot \int_{t=-\infty}^{\infty} f(t)\sin(2\pi\nu t)dt \end{aligned} \tag{III.9}$$

If $f$ is *even*

- $f(-t) = f(t)$
- $f(t)\sin(2\pi\nu t)$ ________
- $f(t)\cos(2\pi\nu t)$ ________
- ____________________
- $\mathcal{F}f(\nu)$ is ________

If $f$ is *odd*

- $f(-t) = -f(t)$
- $f(t)\sin(2\pi\nu t)$ ________
- $f(t)\cos(2\pi\nu t)$ ________
- ____________________
- $\mathcal{F}f(\nu)$ is ________

## 1.g Transform of the complex conjugate

For a complex-number-valued function $f$ defined on **reals**, the complex conjugate of $f$ is the function $\overline{f}$ is defined by $\overline{f}(t) = \overline{f(t)}$

$$\mathcal{F}\overline{f}(\nu) = \overline{\mathcal{F}f}(-\nu) \tag{III.10}$$

## 1.h Transform of a Dirac function

$$\mathcal{F}\delta(\nu) = 1 \tag{III.11}$$

## 2 Inversion of Fourier Transform

**Definition III.2** (Inverse Fourier Transform). *For a function $g$ for which $\int_{-\infty}^{\infty} |g(\nu)| d\nu < \infty$, the* Inverse Fourier Transform *of $g$ is defined, of each real number $t$ by:*

$$\mathcal{F}^{-1} g(t) = \int_{-\infty}^{\infty} g(\nu) e^{i2\pi\nu t} d\nu \tag{III.12}$$

**Theorem III.1** (Fourier Inversion Theorem). *If $f$ is* continuous, *defined on* real *and $\int_{-\infty}^{\infty} |f(t)| dt < \infty$, then*

$$\mathcal{F}^{-1}(\mathcal{F}f)(t) = f(t) \tag{III.13}$$

The demonstration of this theorem is behind the scope of this lecture.

## 3 Canonical Examples

- $\mathcal{F}(\delta)(\nu) = 1$

  Consequently, $\mathcal{F}^{-1}(1)(t) = \delta(t)$.

- $\mathcal{F}\left(e^{i2\pi\nu_0 t}\right)(\nu) = \delta(\nu - \nu_0)$

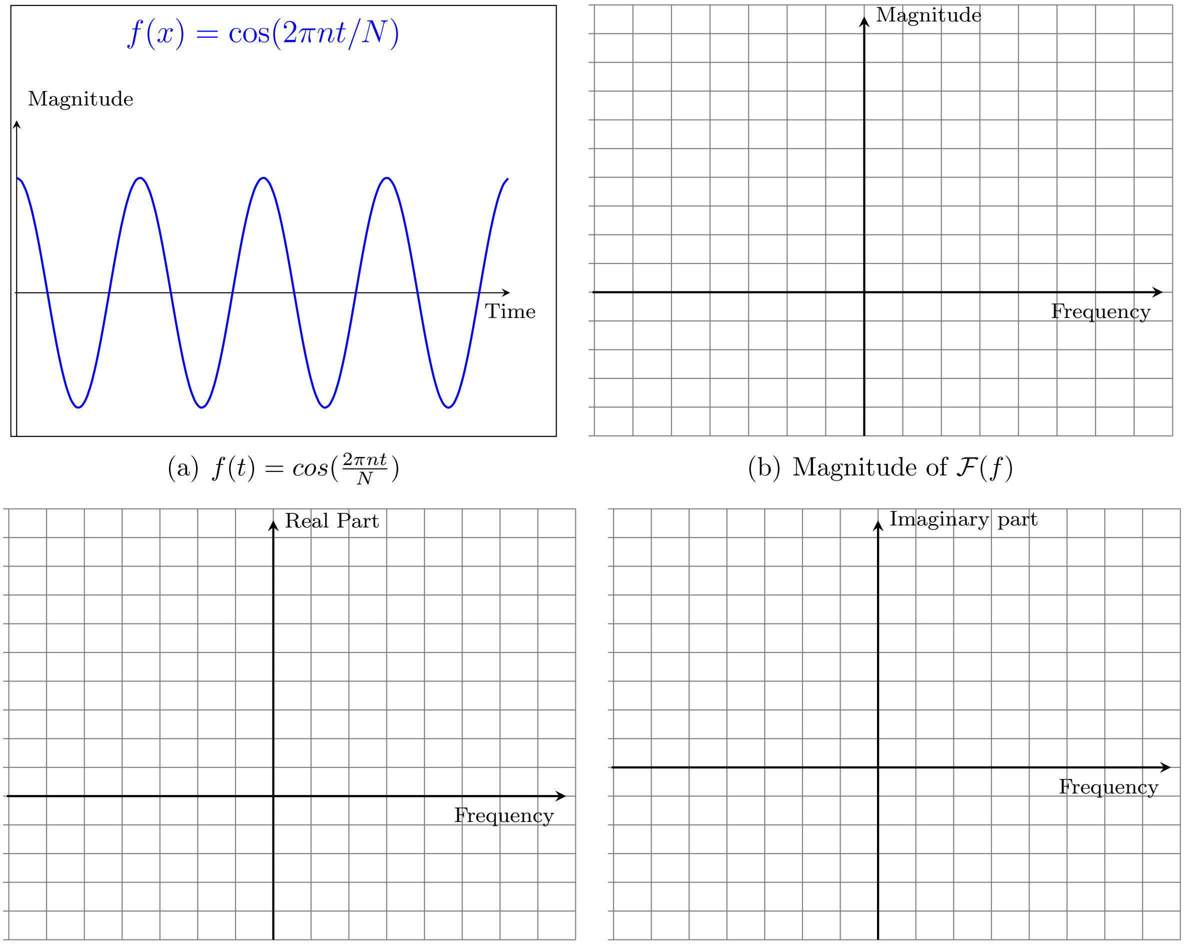
- $\mathcal{F}\left(cos(2\pi\nu_0 t)\right)(\nu) = \frac{1}{2}\left(\delta(\nu+\nu_0)+\delta(\nu-\nu_0)\right)$

- $\mathcal{F}\left(sin(2\pi\nu_0 t)\right)(\nu) = \frac{i}{2}\left(\delta(\nu+\nu_0)-\delta(\nu-\nu_0)\right)$

## 3.a Fourier Transform of Cosine / Sine

Consider the function $f(t) = cos(\frac{2\pi n t}{N})$(see Figure III.1 (a)).
$f$ is periodic and its period is __________ .
Thus, its frequency is __________ .

$$\begin{aligned}
\mathcal{F}(f)(\nu) &= \mathcal{F}\left(cos(\tfrac{2\pi n t}{N})\right)(\nu) \\
&= \rule{6cm}{0.4pt} \\
&= \rule{6cm}{0.4pt}
\end{aligned}$$



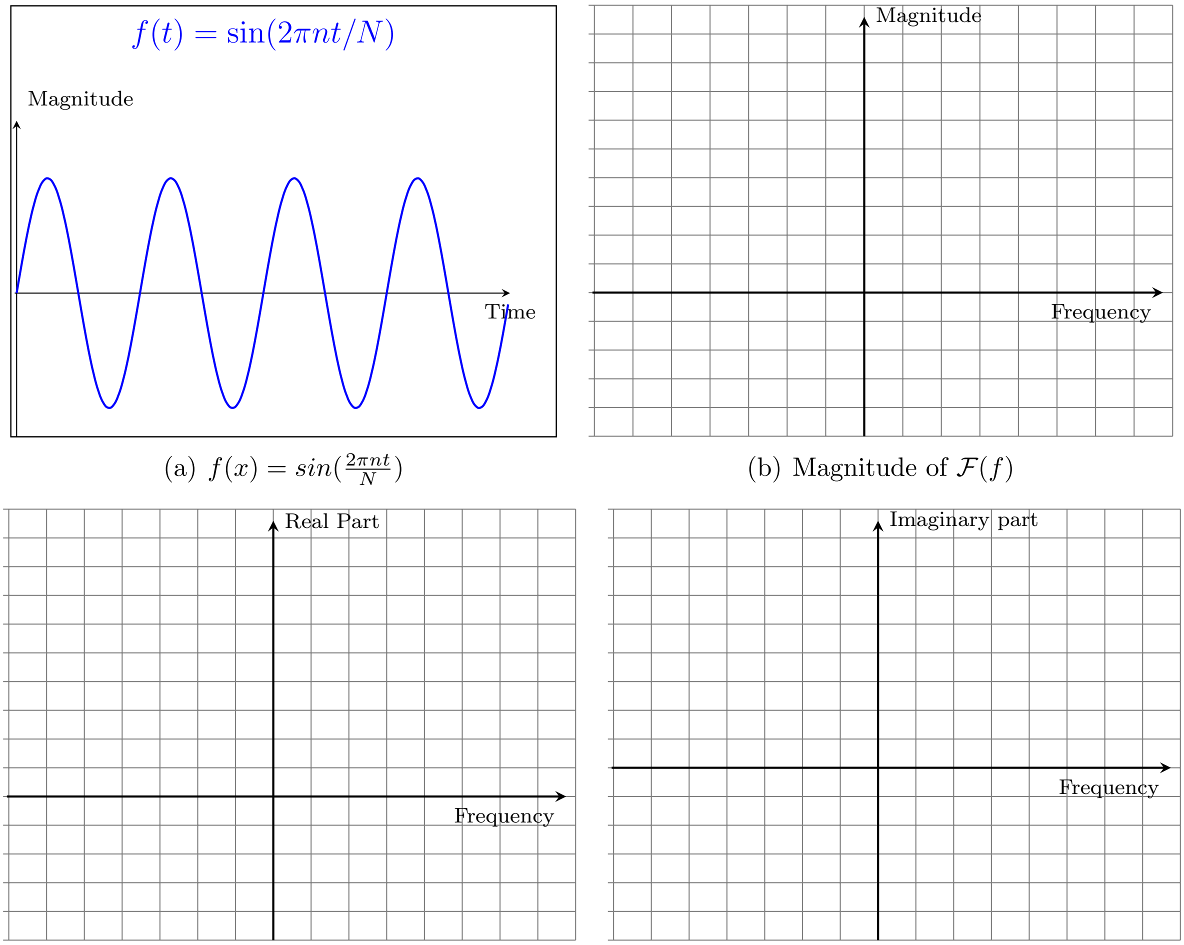

(a) $f(t) = cos(\frac{2\pi n t}{N})$ (b) Magnitude of $\mathcal{F}(f)$

(c) Real part $\mathcal{F}(f)$ (d) Imaginary part of $\mathcal{F}(f)$

Figure III.1: Representation of a cosine function an its Fourier transform.

Consider the function $f(t) = sin(\frac{2\pi nt}{N})$ see Figure III.2 (a)).
$f$ is periodic and its period is ________ .
Thus, its frequency is ________ .

$$\begin{aligned}\mathcal{F}(f)(\nu) &= \mathcal{F}\left(sin(\tfrac{2\pi nt}{N})\right)(\nu)\\ &= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\\ &= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\end{aligned}$$

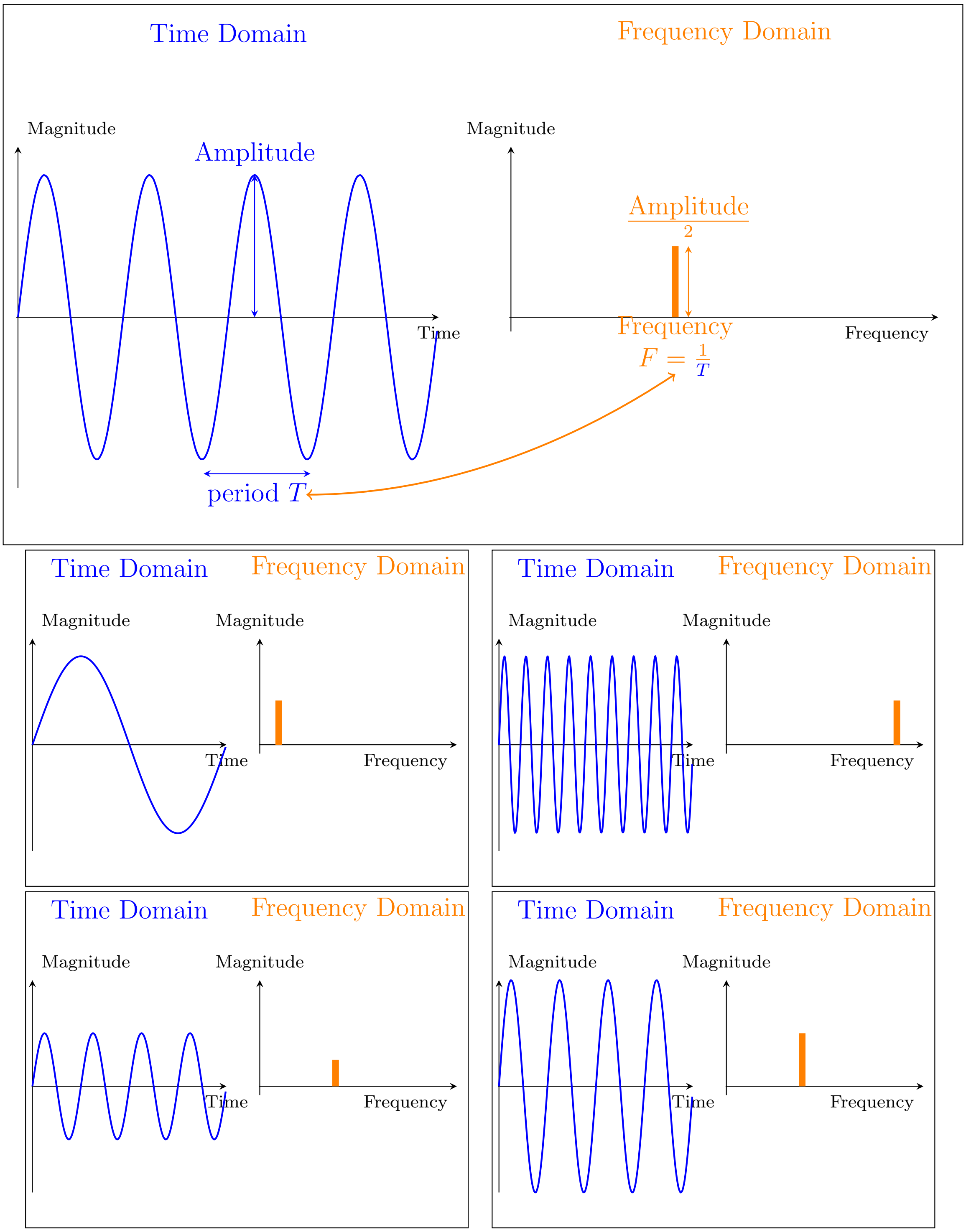


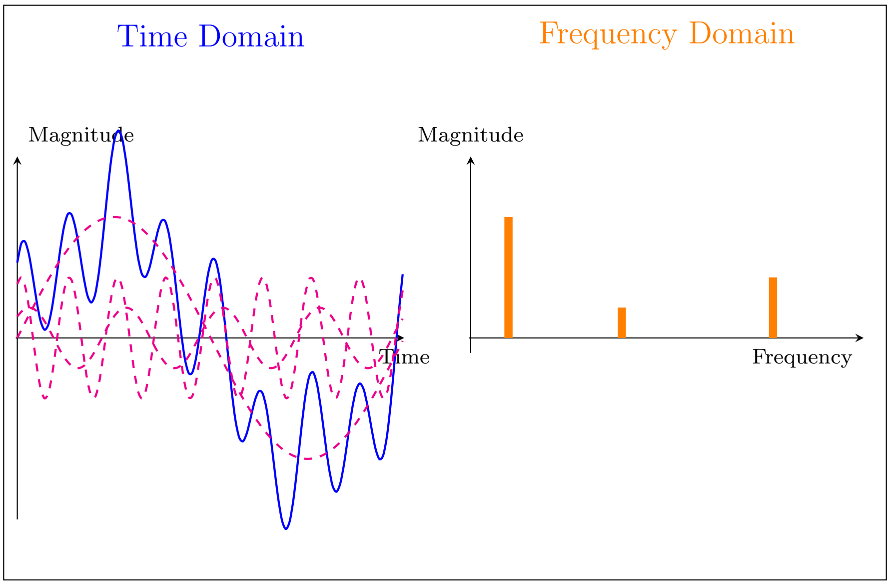

(a) $f(x) = sin(\frac{2\pi nt}{N})$ (b) Magnitude of $\mathcal{F}(f)$

(c) Real part $\mathcal{F}(f)$ (d) Imaginary part of $\mathcal{F}(f)$

Figure III.2: Representation of a sine function an its Fourier transform.

## 3.b Fourier Transform of a Periodic Signal






Fourier postulated around 1807 that any periodic signal of period $T = T_0$ (frequency $\nu = \nu_0$ ($f_{T_0}(t)$) can be built up as an infinite linear combination of harmonic sinusoidal waves. This result is provided by Fourier Series study. This summation can be extended for complex signals, and we have :

$$f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} C_n e^{\frac{2\pi int}{T_0}} \tag{III.14}$$

$$C_n = \frac{1}{T_0} \int_{\frac{-T_0}{2}}^{\frac{-T_0}{2}} f_{T_0}(x) e^{\frac{-2\pi int}{T_0}} dt \tag{III.15}$$

The coefficients $C_n, n \in \mathbf{Z}$ are the coefficients of the Fourier serie for the function $f_{T_0}(t)$. Then the Fourier spectrum of a periodic function is a *line spectrum* :

$$\mathcal{F}(f_{T_0})(\nu) = \sum_{n=-\infty}^{+\infty} C_n \delta(\nu - n\nu_0) \tag{III.16}$$

$C_0$ denotes the average value, $C_1$, the contributionn of the component at the fundamental frequency $\nu_0$, and $C_n$, the contributionn of the component at the nth harmonic frequency $n\nu_0$.

# 4 Fourier Transform and Convolution

The relation between convolution product and simple product through Fourier Transform is a major theorem in signal processing.

**Theorem III.2** (Convolution Theorem)**.**

$$\mathcal{F}(f * g) = \mathcal{F}(f) \times \mathcal{F}(g) \tag{III.17}$$

*and*

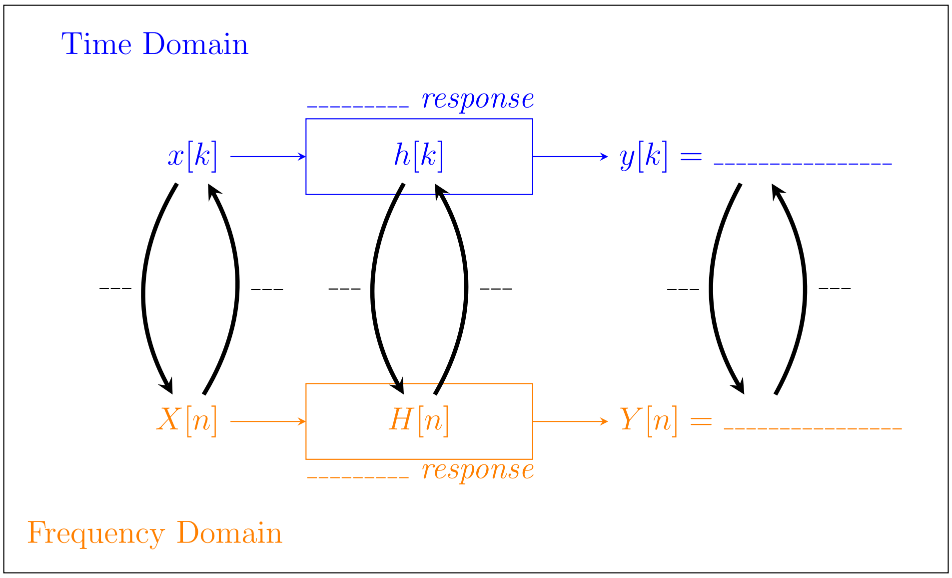
$$\mathcal{F}(f \times g) = \mathcal{F}(f) * \mathcal{F}(g) \tag{III.18}$$

*Proof.* of equation III.17

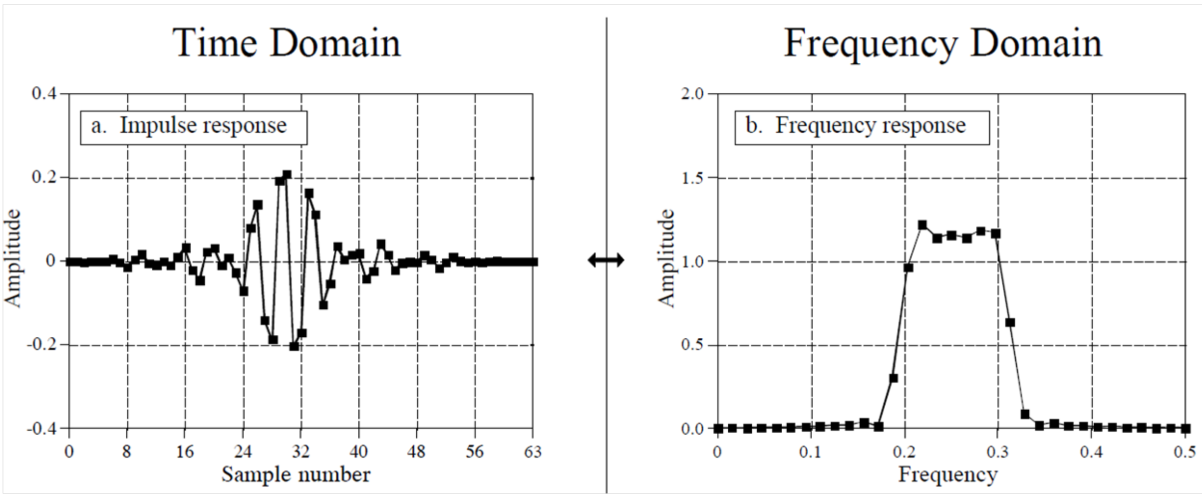
□

*Proof.* of equation III.18

□




## 4.a Example: Interpretation of the action of a filter



# 5 Fourier Transforms and Sampling

## 5.a Band-limited Signal

Heuristically we might consider a non periodic signal as a signal with an infinite period, and apply the Fourier Series results on this signal. Then we might think of a signal as consisting of a compilation of sine or cosine waves of various frequencies and amplitudes. The narrowest *bump* in this compilation is due to the wave that has the shortest period and also constitutes the smallest feature that is present in the signal. The reciprocal of the shortest period present in the signal is the maximum frequency that is present in the Fourier Transform of the signal.

In the signal presented at the figureIII.3, there is a perturbation in high frequency superposed with a slowest waveform in low frequency. In the Fourier domain, these two contributions are are easily distinguishable.

In all the physical signals (recorded, computed, etc. from the physical world) the highest frequency in the signal is not infinite. The signal is a *band-limited signal*.

Let us notice $\nu_M$ this maximum frequency : $\mathcal{F}(f)(\nu) = 0$ if $|\nu| > \nu_M$.
In practice, the equality with 0 means negligible.

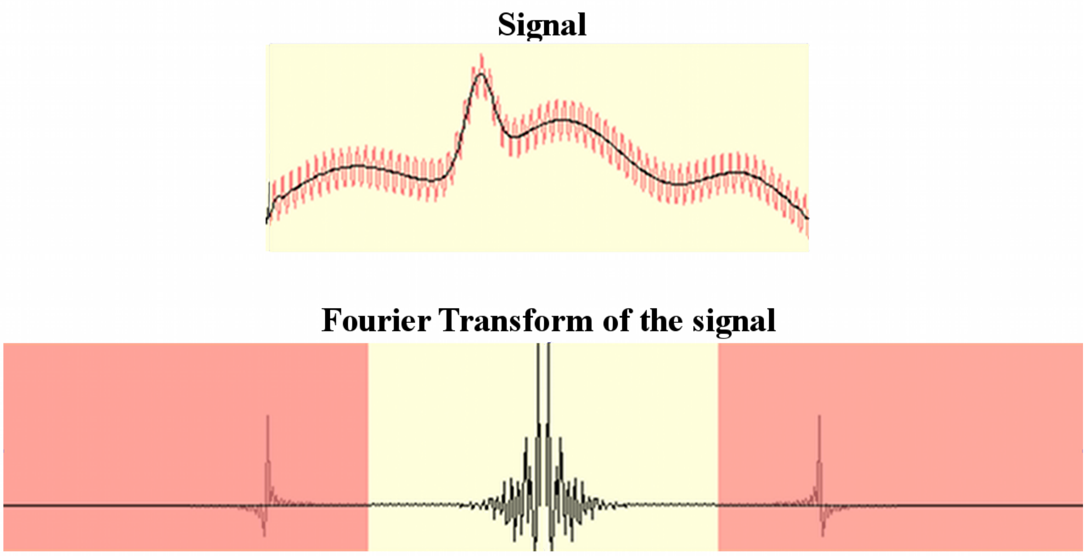



Figure III.3: Example of a signal which high frequencies have been removed.

## 5.b Sampling

Recall: The term *sampling* refers to the situation where the values of a function that presumably is defined on the whole real line are known or are computed only at a discrete set of points.
Let us consider sample values of $f$. This means for $k \in \mathbf{Z}$, $t = k.\Delta_T$, with $\Delta_T$ the sampling period (the time between two adjacent samples. After sampling, we denote the sampled waveform as the signal $f^\star(t)$ as:

$$f^\star(t) = f(t) \times s(t; \Delta_T) \tag{III.19}$$

$$s(t;\Delta_T) = \sum_{-\infty}^{+\infty} \delta(t - k.\Delta_T) \tag{III.20}$$

$s(t;\Delta_T)$ is the ideal sampling function called the *Dirac comb sampling function*, consisting of a set of delayed Dirac $\delta(T)$ functions. This Dirac comb function is $\Delta_T$-periodic function whose the spectrum is a *line spectrum* :

$$\mathcal{F}(s(t;\Delta_T))(\nu) = \frac{1}{\Delta_T} \sum_{n=-\infty}^{+\infty} \delta(\nu - \frac{n}{\Delta_T}) \tag{III.21}$$

## 5.c Shannon-Nyquist's theorem

The question is now : given a set of samples $f(k\Delta_T); k \in \mathbf{Z}$, can we unambiguously reconstruct $f(t)$? The answer is no, while there are an infinity of candidate functions to "fill in" $f(t)$ between consecutive samples. Nevertheless the theorem states that the reconstruction can uniquely defined if the original $f(t)$ signal is a band-limited signal and if the sampling frequency is greater than twice the highest frequency $\nu_M$ in $f(t)$.

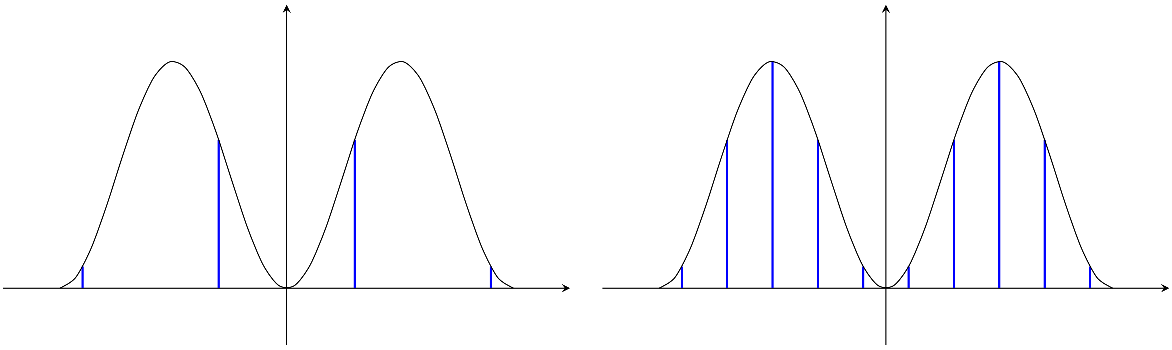
Therefore, *the maximum sample spacing is equal to half of the size of the smallest detail present in the signal.*

$$\Delta_T < \frac{1}{2\nu_M} \tag{III.22}$$

This result is known the Nyquist-Shannon's theorem and the minimum sampling rate to satisfy this sampling theorem, $2\nu_M$ is known as the Nyquist rate.

**Remark.** *Before sampling, the signal must be filtered by a low-pass filter to cut the frequencies above half of the sampling frequency. This analogic filter is called the anti-aliasing filter. This will retain spectral components in the range* $-\frac{1}{2\Delta_T} < \nu < \frac{1}{2\Delta_T}$ *and reject all other frequencies.*

## 5.d Interpretation



(a) Sampling with too low frequency

(b) Sampling with correct frequency

Figure III.4: Illustration of the sampling theorem

## 5.e Reconstruction

An easy way to understand how $f$ can be reconstructed from its sampled version $f^\star$ is to considered the periodic spectrum of the signal $f^\star$ and to filter it with an ideal low-pass filter. The cut-off frequency of this filter must be equal to half of the sampling frequency. This ideal filter is the same as the anti-aliasing filter. Its transfert function is $H(\nu)$ defined as :

$$H(\nu) = \Delta_T; |\nu| < \frac{1}{2\Delta_T} \tag{III.23}$$

$$H(\nu) = 0; otherwise \tag{III.24}$$

The output of the convolution filter can be expressed in the Fourier domain but also in the time domain such. We have :

$$H(\nu)\mathcal{F}(f^\star)(\nu) = \mathcal{F})(\nu) \tag{III.25}$$

and then

$$\mathcal{F}^{-1}(H(\nu)\mathcal{F}(f^\star)(\nu))(t) = f(t) \tag{III.26}$$

Consequently

$$f(t) = f^\star(t) * h(t) \tag{III.27}$$

$$f(t) = \sum_{k=-\infty}^{+\infty} f(k\Delta_T).\frac{\sin(\pi(t - k\Delta_T)/\Delta_T)}{\pi(t - k\Delta_T)/\Delta_T} \tag{III.28}$$

this last equation is known as the cardinal reconstruction function. It is a superposition of shifted sinc functions. We can easily see that at $t = k\Delta_T$ the reconstructed value is the kth sample $f(k\Delta_T) = f^\star(k\Delta_T)$.

# 6 Discrete Time Fourier Transform

Let us consider now a time sampled signal $x(t)$ which is now $x(k\Delta_T) = x[k]$ with $\Delta_T$ the sampling period and $1/\Delta_T$ the sampling frequency. To respect the sampling theorem, the highest frequency ($\nu_M$) in the signal $x(t)$ must be lower than $\frac{1}{(2\Delta_T)}$.

**Definition III.3** (Discrete Time Fourier Transform)**.** *For a given discrete signal $x[k]$ such as $\sum_{k=-\infty}^{+\infty} |x[k]| < \infty$, the DTFT (*Discrete Time Fourier Transform*) of $x[n]$ is defined, for each real number $\lambda$ by:*

$$\mathcal{F}_{DT}(x)(\lambda) = \sum_{k=-\infty}^{+\infty} x[k] e^{-2i\pi\lambda k} \tag{III.29}$$

*where $\lambda$ is the* normalized frequency*: $\lambda = \nu\Delta_T$*

***Note:*** $\mathcal{F}_{DT}$ is a 1-periodic function with $\lambda$: $\mathcal{F}_{DT}(x)(\lambda+1) = \mathcal{F}_{DT}(x)(\lambda)$

**Definition III.4** (Inverse Discrete Time Fourier Transform)**.** *For a given 1-periodic continuous function $F$, the IDTF (*Inverse Discrete Time Fourier Transform*) of $F(\lambda)$ is defined, for each relative number $k$ by:*

$$\mathcal{F}_{DT}^{-1}(F)[k] = \int_{\lambda=-\frac{1}{2}}^{+\frac{1}{2}} F(\lambda)e^{+2i\pi k\lambda} \tag{III.30}$$

**Remark.** *The Fourier Transform of a sampled signal $x[k]$ is a periodic function in the frequency domain. When this function is expressed in frequency ($\nu$), the period is $\frac{1}{\Delta_T}$ (the sampling frequency). When this spectrum is expressed with the normalized frequency ($\lambda = \nu\Delta_T$), the period is 1.*

**Remark.** *The computation of this spectrum function ($\mathcal{F}_{DT}(x)(\lambda)$) requires a summation over an infinite number of samples, and this is a continuous function with $\lambda$. To use the Fourier Transform for digital signal processing, the number of samples must be finite, and the frequency variable ($\nu$, or $\lambda$)must be also sampled. We will see that these two pooints are linked in the definition of the Discrete Fourrier Transform (DFT).*

# 7 Discrete Fourier Transform

Let us consider a finite number $N$ of samples. That means the signal is recorded during $N\Delta_T$ seconds, with $\Delta_T$ the sampling period. The Discrete Time Fourier Transform is then:

$$\mathcal{F}_{DT}(x)(\lambda) = \sum_{k=0}^{N-1} x[k]e^{-2\pi ik\lambda}$$

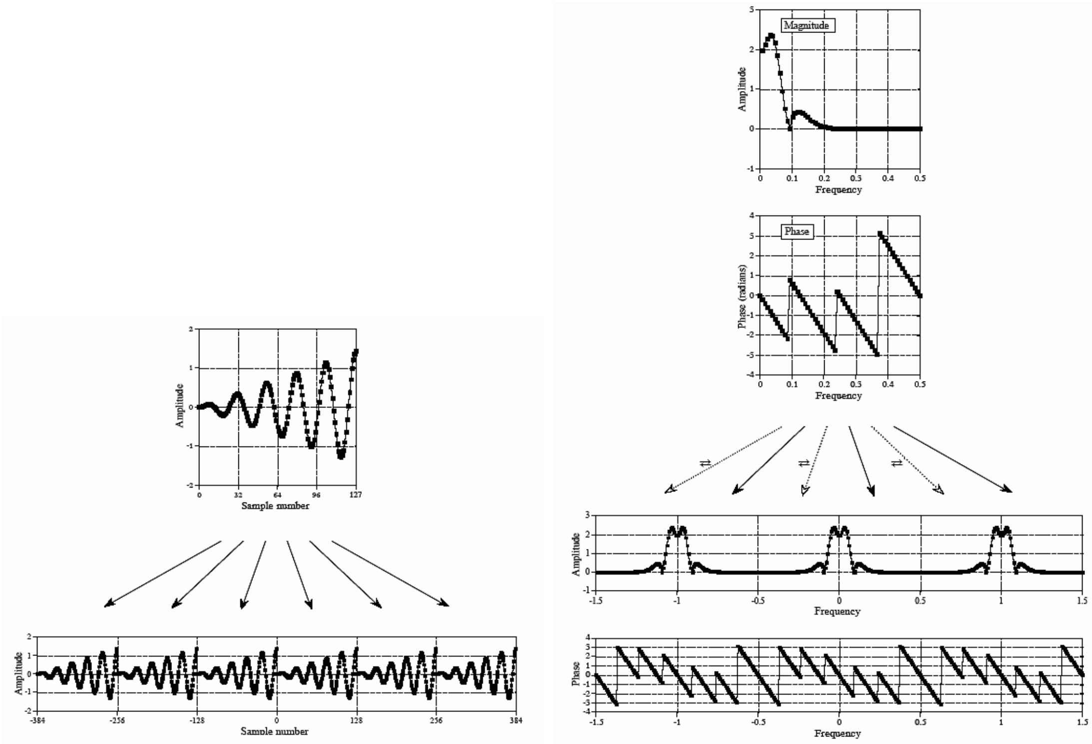
To obtain the **Discrete Fourier Transform**, this spectrum function is sampled on $N$ frequency samples on each frequency period. The frequency period is 1, there are $N$ frequency samples, the lag between each of them is $\frac{1}{N}$. We have $\lambda = \frac{n}{N}$. Thus, the Discrete Time Fourier Transform writes:

$$\mathcal{F}_{DT}(x)\left[\frac{n}{N}\right] = \sum_{k=0}^{N-1} x[k]e^{-2\pi i\frac{nk}{N}}$$

Considering the index $n$, we can write, and take the following notation:

$$\mathcal{F}_{DT}(x)\left[\frac{n}{N}\right] = X[n] \tag{III.31}$$

$X$ is a $N$-periodic signal which exactly corresponds to the Discrete Time Fourier Transform of a $N$-periodic signal $x$ in time. This signal is built by the infinite concatenation of the $N$ recorded samples, as illustrated figure III.5.




(a) Time-Domain signal

(b) Frequency-Domain signal

Figure III.5: Finite-length Signal and DFT viewed as periodic signals.

**Definition III.5** (Discrete Fourier Transform)**.** *The* Discrete Fourier Transform*, denoted $\mathcal{F}_\mathcal{D}$, transforms an $N$-periodic discrete signal $x$ into another $N$-periodic discrete signal $\mathcal{F}_D(x)$ (also denoted $X$) defined by:*

$$\forall n \in [0..N-1],\ \left(\mathcal{F}_D(x)\right)[n] = X[n] = \sum_{k=0}^{N-1} x[k] e^{-i2\pi \frac{kn}{N}} \tag{III.32}$$

for other values of $n$, $X[n]$ is defined by periodicity.

**Remark.** *As $\mathcal{F}_D(x) = X$ is periodic, we can replace $\sum_{k=0}^{N-1}$ by $\sum_{k=M}^{M+N-1}$ for any number $M$.*

**Definition III.6** (Inverse Discrete Fourier Transform)**.** *For a $N$-periodic discrete signal $X$, the* Discrete Inverse Fourier Transform *of $X$ is the $N$-periodic signal defined by*

$$\forall n \in [0..N-1],\ \left(\mathcal{F}_D^{-1}(X)\right)[k] = x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[n] e^{i2\pi \frac{kn}{N}} d\lambda \tag{III.33}$$

for other values of $n$, $X[n]$ is defined by periodicity.

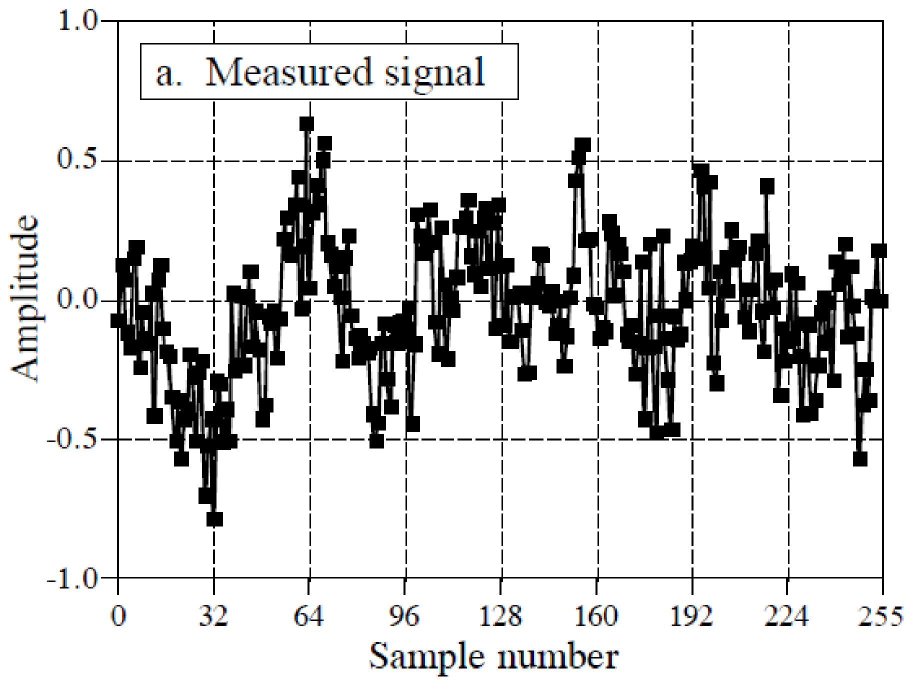
**Remark.** *As $\mathcal{F}_D(x) = X$ is periodic, we can replace $\sum_{k=0}^{N-1}$ by $\sum_{k=M}^{M+N-1}$ for any number $M$.*

***Note:*** For sake of simplicity, we drop the $D$ notation using [] for *Discrete*. In the following, the Fourier Transform is implicitly the Discrete Fourier Transform.

# 8 Example of Spectral Analysis of a Signal

## 8.a Measured Signal

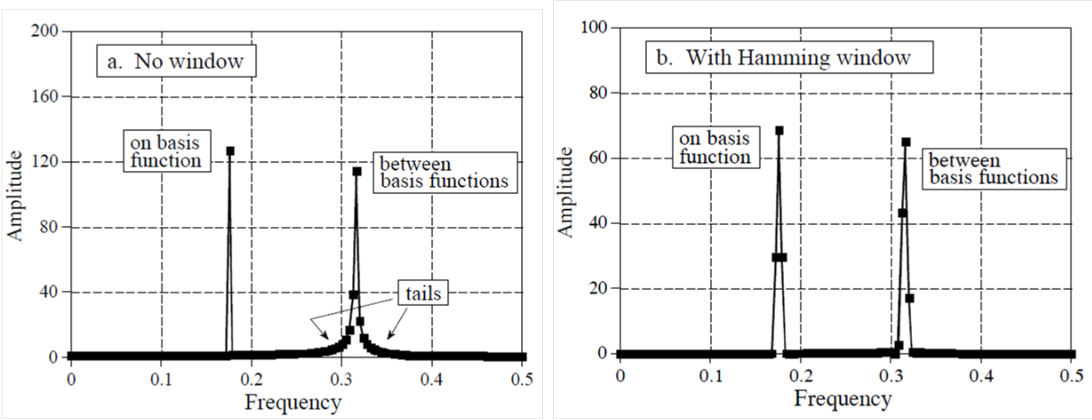
- A microphone is placed in the water
- The resulting signal
  - is amplified to a reasonable level (few volts)
  - an analog low-pass filter is used to remove all frequencies above $80Hz$
  - is digitized a 160 samples per second.

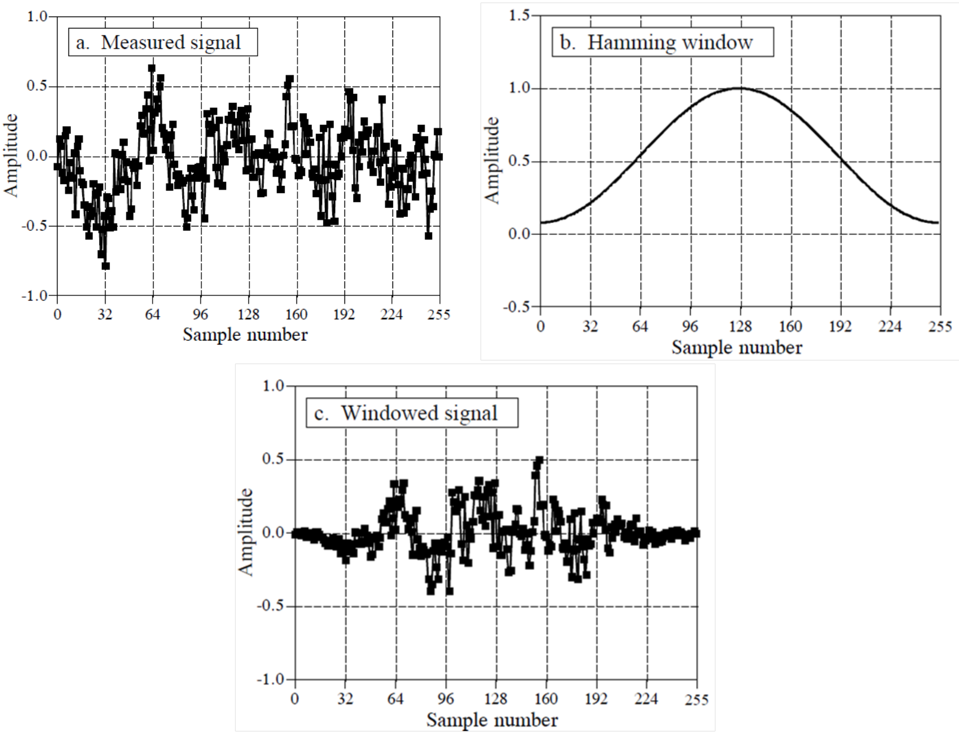



## 8.b Digression: Hamming Window

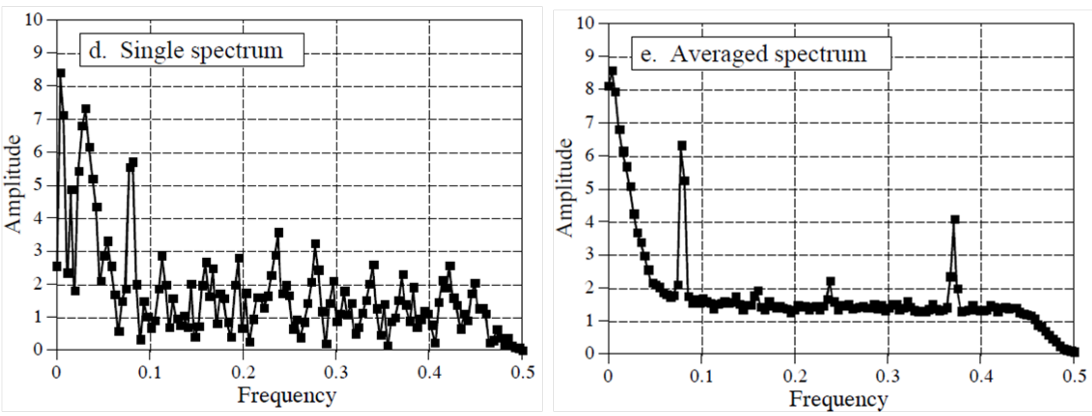
Recall: $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nt/T}$ *basis functions*
What happens in the DFT when the input signal contains a sinusoid with a frequency *between* 2 discrete basis functions ?
► tails on corresponding DFT peaks.

► Convolution with a *Hamming Window*

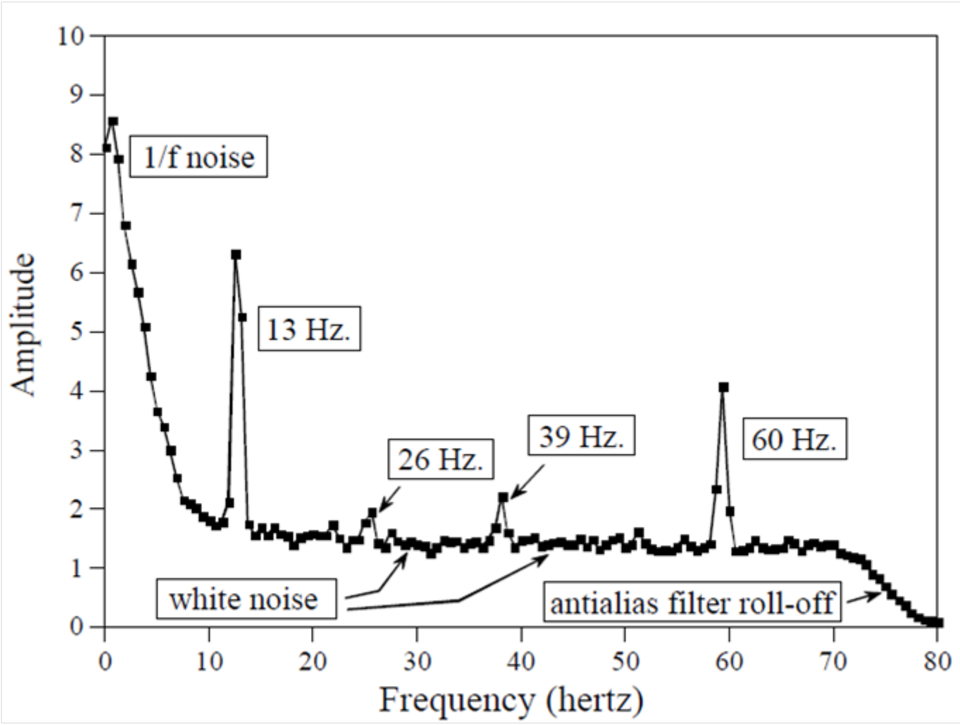



## 8.c Discrete Fourier Transform and convolution with 64- average filter



## 8.d Spectral Analysis




1. Step 1: Ignoring sharp peaks

   - Between 10 and 70$Hz$
     If you ignore sharp peaks: **White Noise**
   - Above 70$Hz$
     Result of the analog low-pass filter: as it is not perfect, the frequencies are not cut perfectly
   - Below 10$Hz$
     Rapidly increasing noise: $\frac{1}{f}$ noise (*actually unexplained*)

2. Step 2: Sharp peaks

60$Hz$ Result of electromagnetic interference from commercial electrical power.

13$Hz$, 26$Hz$ and 39$Hz$ Frequency spectrum of a non-sinusoidal periodic waveform
*note:* $26 = 2 \times 13$, $39 = 3 \times 13$

13$Hz$ fundamental frequency
26$Hz$ second harmonic
39$Hz$ third harmonic

52$Hz$, 65$Hz$, 78$Hz$, etc. are buried in the white noise

▶ submarine's 3-bladed propeller turning at 4.33 revolutions per second

This is the basis of passive sonar: identifying undersea sounds by their frequency and harmonic content.

# 9 Fast Fourier Transform

## 9.a Discrete Fourier Transform Algorithm

Given the Fourier Transform definition of a discrete $N$-samples signal $x[k], k = 0...N-1$, the Fourier transform $X[n]$ of $x$ is a $N$-samples signal where for each $n \in [0, N-1]$, $X[n] = \sum_{k=0}^{N-1} x[n]e^{-i2\pi\frac{kn}{N}}$.

A simple straightforward algorithm to compute $X[n], n = 0..N$ is:

The complexity, i.e. the number of operations of this algorithm is: ______ .

In 1965, IBM researcher Jim Cooley and Princeton faculty member John Tukey developed what is now known as the *Fast Fourier Transform* (FFT). It is an algorithm for computing DFT that has a complexity of $O\left(N \log_2(N)\right)$ for certain length inputs. Now when the length of data doubles, the spectral computational time will not quadruple as with the DFT algorithm ; instead, it approximately doubles. Later research showed that no algorithm for computing the DFT could have a smaller complexity than the FFT. Surprisingly, historical work has shown that Gauss in the early nineteenth century developed the same algorithm, but did not publish it !

**Example III.2.** *Consider a signal which Fourier Transform computation takes* $1ms$. *We want to calculate a transform of a signal that is 10 times longer.*

- *The DFT computation would take* ______
- *The FFT computation would take* ______

## 9.b Fourier Transform Decomposition

To derive the FFT, we assume that the signal's length is a power of 2: $N = 2^L$.

$$X[n] = \sum_{k=0}^{N-1} x[k] e^{-i2\pi \frac{kn}{N}}$$

A $N$-samples Fourier Transform is a sum of 2 $\frac{N}{2}$-samples Fourier Transforms : an interlaced decomposition (i.e. an even-samples signal and an odd-samples signal) of the original signal with a factor $e^{-i\frac{2\pi n}{N}}$.

**Example III.3.** *Example with a 8-samples signal:* $x[n], n = 0..7$.
*Let us denote* $\omega_N^k = e^{\frac{2\pi kn}{N}}$. *For each* $k \in [0..N-1]$, *we have:*

$(X[n])$

$$x[0].\omega_N^0 + x[1].\omega_N^1 + x[2].\omega_N^2 + x[3].\omega_N^3 + x[4].\omega_N^4 + x[5].\omega_N^5 + x[6].\omega_N^6 + x[7].\omega_N^7$$

$(G[n])$

$$x[\ ].\omega + x[\ ].\omega + x[\ ].\omega + x[\ ].\omega$$

$$+e^{-\frac{i\pi k}{N/2}}\Big($$

$(H[n])$

$$x[\ ].\omega + x[\ ].\omega + x[\ ].\omega + x[\ ].\omega\Big)$$

$(L[n])$

$$x[\ ].\omega + x[\ ].\omega$$

$$+e^{-\frac{i\pi k}{N/4}}\Big($$

$(M[n])$

$$x[\ ].\omega + x[\ ].\omega\Big)$$

$(O[n])$

$$x[\ ].\omega + x[\ ].\omega$$

$$+e^{-\frac{i\pi k}{N/4}}\Big($$

$(P[n])$

$$x[\ ].\omega + x[\ ].\omega\Big)$$

$$x[\ ]\quad x[\ ]\quad x[\ ]\quad x[\ ]\quad x[\ ]\quad x[\ ]\quad x[\ ]\quad x[\ ]$$

This decomposition is called a *decimation in time* because the time samples are rearranged in alternating groups, and a *radix-2* algorithm because there a two groups.

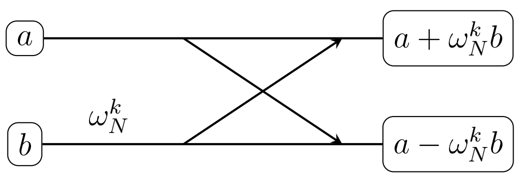
For each value of $n$, we compute two $N/2$-length Fourier Transform ($O(\frac{N^2}{4})$), multiply one by a complex and add the result. Because $N = 2^L$, each of the half-length transform can be reduced to 2 quarter-length transforms, each of these to eighth-length transforms, and so on... This decomposition continues until 2-length transforms. The number of stages, i.e. the number of times $N$ can be divided by 2 is $\log_2(N)$, thus, the complexity for computing $X[n]$ **for one** $n$ is ________

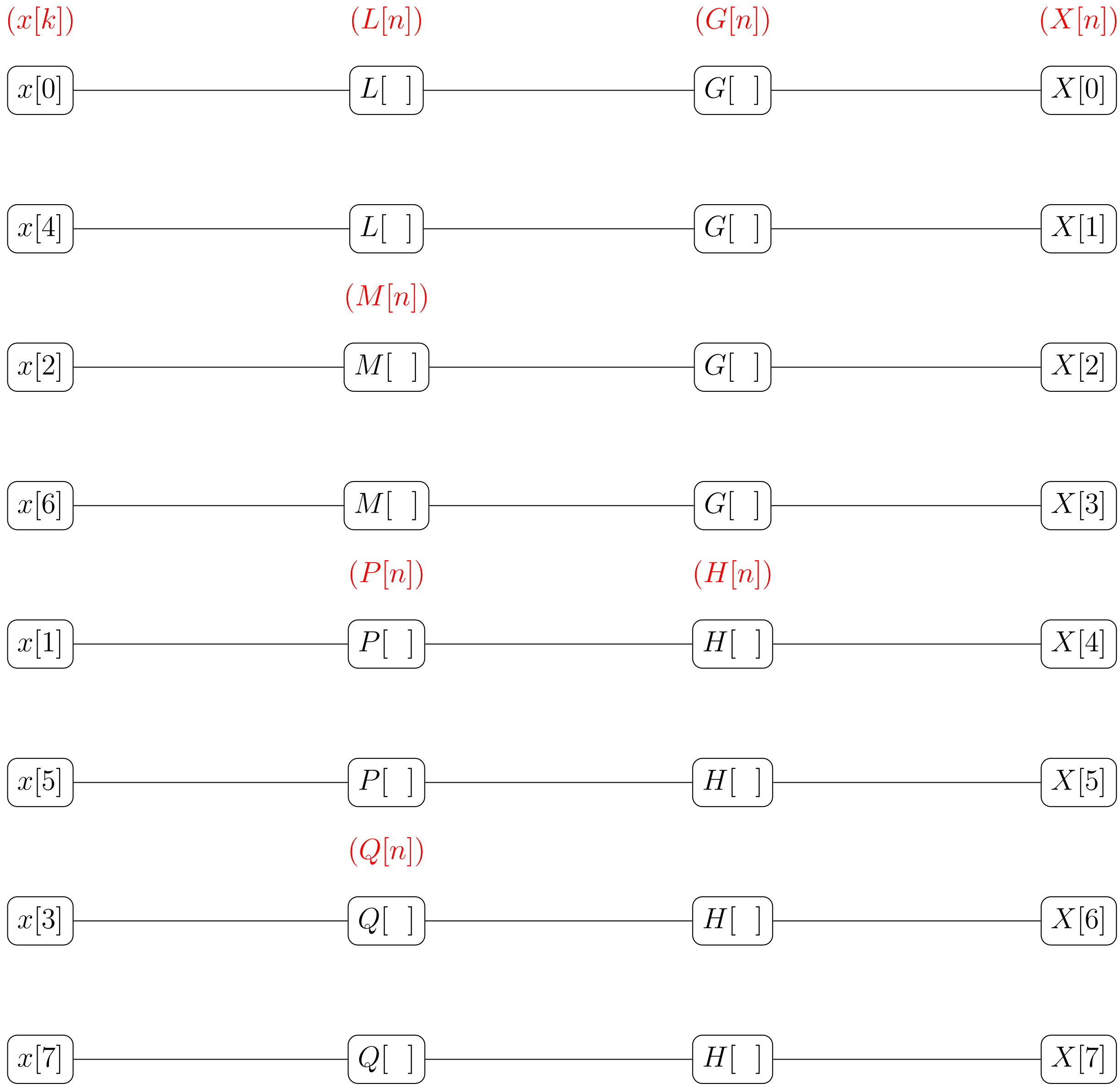
Problem:

## 9.c The butterfly operation

Recall: Let us denote $M = N/2$, we have $X[n] = G[n] + e^{-i\frac{n\pi}{L}} H[n]$.

The butterfly operation consists in taking 2 complex numbers $a$ and $b$ and compute $a + e^{-i\frac{\pi k}{N}} b$ and $a - e^{-i\frac{\pi k}{N}} b$.



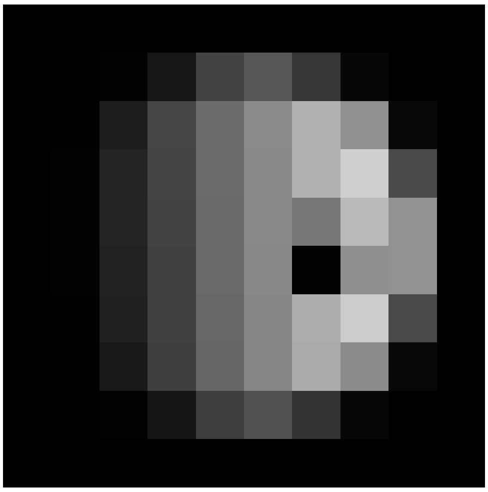



Each butterfly operation requires 1 complex multiplication and 2 complex additions. Each step, i.e. each decomposition requires ______ butterfly operations. The number of stages, i.e. the number of times $N$ can be divided by 2 is ______ . Thus, the whole Fourier Transform of $x$ can be computed in __________ operations.

# 10 2D Discrete Fourier Transform

## 10.a Image viewed as a 2D Signal

$$f : \begin{array}{ccc} [0..K-1] \times [0..L-1] & & [0..N-1] \\ x \times y & \to & f[x,y] \end{array}$$



(a) Image seen as an image

$$\begin{array}{cc}
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 21 & 62 & 81 & 51 & 7 & 0 & 0 \\
0 & 0 & 25 & 63 & 102 & 133 & 171 & 140 & 8 & 0 \\
0 & 1 & 33 & 65 & 103 & 134 & 173 & 204 & 73 & 0 \\
0 & 2 & 34 & 65 & 105 & 136 & 3 & 143 & 146 & 0 \\
0 & 2 & 36 & 67 & 106 & 137 & 119 & 186 & 146 & 0 \\
0 & 2 & 36 & 68 & 107 & 138 & 177 & 207 & 74 & 0 \\
0 & 0 & 29 & 69 & 108 & 139 & 177 & 144 & 8 & 0 \\
0 & 0 & 2 & 23 & 67 & 86 & 54 & 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} & L \\
K &
\end{array}$$

(b) Image seen as a 2D matrix

Figure III.6: Detail of a digitized 2D image

## 10.b Discrete Fourier Transform of a 2D signal

$$f : \begin{array}{ccc} [0..K-1] \times [0..L-1] & & [0..N-1] \\ x \times y & \to & f[x,y] \end{array}$$

$$\mathcal{F}(f)[u,v] = \sum_{y=0}^{L-1} \sum_{x=0}^{K-1} f[x,y] e^{-i2\pi(\frac{ux}{K} + \frac{vy}{L})} \tag{III.34}$$

$$f[x,y] = \mathcal{F}^{-1}(F)[x,y] = \frac{1}{KL} \sum_{v=0}^{L-1} \sum_{u=0}^{K-1} F[u,v] e^{+i2\pi(\frac{ux}{K} + \frac{vy}{L})} \tag{III.35}$$

## 10.c Time Shifting

- in Time Domain

$$\begin{tikzcd} f[x-x_0, y-y_0] \arrow[r, bend left, "\mathcal{DFT}"] & \mathcal{F}(f)[u,v]\text{\_\_\_\_\_\_\_\_\_\_\_\_} \arrow[l, bend left, "\mathcal{IDFT}"] \end{tikzcd}$$

- in Frequency Domain

$$\begin{tikzcd}
f[x,y]\underline{\hspace{2cm}} \arrow[r, bend left, "\mathcal{DFT}"] & \mathcal{F}(f)[u-u_0, v-v_0] \arrow[l, bend left, "\mathcal{IDFT}"]
\end{tikzcd}$$

Particular cases

- in Time Domain

$$\begin{tikzcd}
f[x-\frac{K}{2}, y-\frac{L}{2}] \arrow[r, bend left, "\mathcal{DFT}"] & \mathcal{F}(f)[u,v]\underline{\hspace{2cm}} \arrow[l, bend left, "\mathcal{IDFT}"]
\end{tikzcd}$$

- in Frequency Domain

$$\begin{tikzcd}
f[x,y]\underline{\hspace{2cm}} \arrow[r, bend left, "\mathcal{DFT}"] & \mathcal{F}(f)[u-\frac{K}{2}, v-\frac{L}{2}] \arrow[l, bend left, "\mathcal{IDFT}"]
\end{tikzcd}$$

## 10.d Time Scaling

$$\begin{tikzcd}
f[\alpha x, \beta y] \arrow[r, bend left, "\mathcal{DFT}"] & \frac{1}{|\alpha\beta|}\mathcal{F}(f)[\frac{u}{\alpha}, \frac{v}{\beta}] \arrow[l, bend left, "\mathcal{IDFT}"]
\end{tikzcd}$$

## 10.e Distributivity and Scaling

- Distributivity

$$\begin{tikzcd}
f_1[x,y] + f_2[x,y] \arrow[r, bend left, "\mathcal{DFT}"] & \mathcal{F}(f_1)[u,v] + \mathcal{F}(f_2)[u,v] \arrow[l, bend left, "\mathcal{IDFT}"]
\end{tikzcd}$$

- Scaling

$$\begin{tikzcd}
\alpha f[x,y] \arrow[r, bend left, "\mathcal{DFT}"] & \alpha \mathcal{F}(f)[u,v] \arrow[l, bend left, "\mathcal{IDFT}"]
\end{tikzcd}$$

## 10.f Conjugate Symmetry

If $f[x,y]$ is real (i.e. its imaginary part is null), its Fourier Transform is conjugate symmetric:

$$\mathcal{F}(f)[u,v] = \overline{\mathcal{F}(f)[-u,-v]}$$

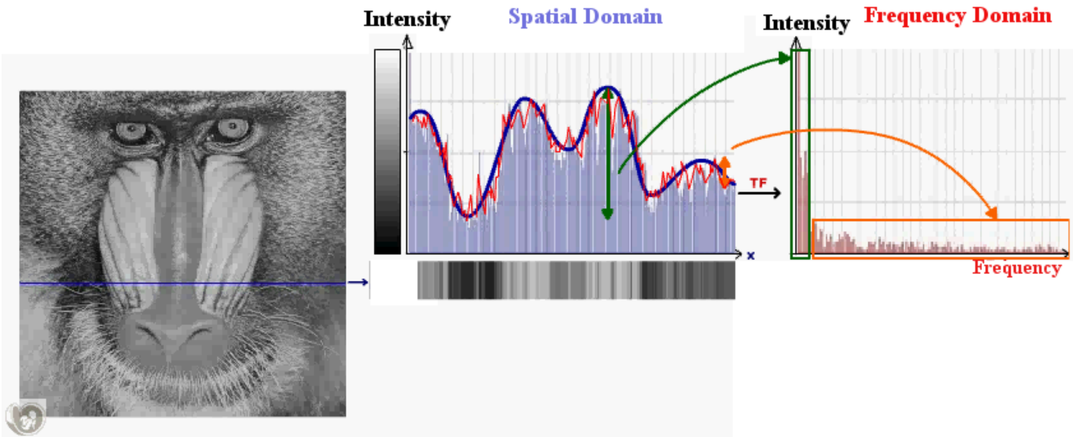
**Remark.** $\mathcal{F}(f)[0,0] = ??$

## 10.g Separability

The Discrete Fourier Transform can be expressed in separable form.

Let us consider a 1D Fourier transform on a 2D image. For example, we compute Fourier transforms on each line of the image i.e. we fix one value of $y$ and compute the Fourier transform on the variable $x$.

$$\mathcal{F}(f)[u,y] = \sum_{x=0}^{K-1} f[x,y]e^{-i2\pi\frac{ux}{K}}$$




This can be done for any $y$ in $[0, L-1]$ (i.e. for each line). We can then sum $\sum_{y=0}^{L-1} \mathcal{F}(f)[u,y]e^{-2i\pi\frac{vy}{L}}$

We obtain:

$$\begin{aligned}
\sum_{y=0}^{L-1} \mathcal{F}(f)[u,y]e^{-2i\pi\frac{vy}{L}} &= \sum_{y=0}^{L-1} \left(\sum_{x=0}^{K-1} f[x,y]e^{-i2\pi\frac{ux}{K}}\right) e^{-2i\pi\frac{vy}{L}} \\
&= \sum_{y=0}^{L-1} \sum_{x=0}^{K-1} f[x,y]e^{-i2\pi\frac{ux}{K}} e^{-2i\pi\frac{vy}{L}} \\
&= \sum_{y=0}^{L-1} \sum_{x=0}^{K-1} f[x,y]e^{-i2\pi(\frac{ux}{K}+\frac{vy}{L})} \\
&= \mathcal{F}(f)[u,v]
\end{aligned}$$

To sum up:

$$\boxed{f[x,y]} \xrightarrow[DFT]{\text{1D (rows)}} \boxed{\mathcal{F}(f)[u,y]} \xrightarrow[DFT]{\text{1D (cols)}} \boxed{\mathcal{F}(f)[u,v]}$$

$$\mathcal{F}(f)[u,y] = \sum_{x=0}^{K-1} f[x,y]e^{-2i\pi\frac{ux}{K}}$$

$$\mathcal{F}(f)[u,v] = \sum_{y=0}^{L-1} \mathcal{F}(f)[u,y]e^{-2i\pi\frac{vy}{L}}$$

The same result holds if we reverse the order of computation (columns first, followed by rows).

## 10.h Centering DFT

For images, generally, the 2D Fourier Transform is *centered* around zero. This means than instead of considering the Fourier Transform from $n = 0$ to $n = N - 1$, we consider it from $n = -\frac{N}{2} + 1$ to $n = \frac{N}{2}$. This is possible as we can consider a finite length signal as an infinitely long periodic signal (see Figre **??**). This can be obtained by symmetry.

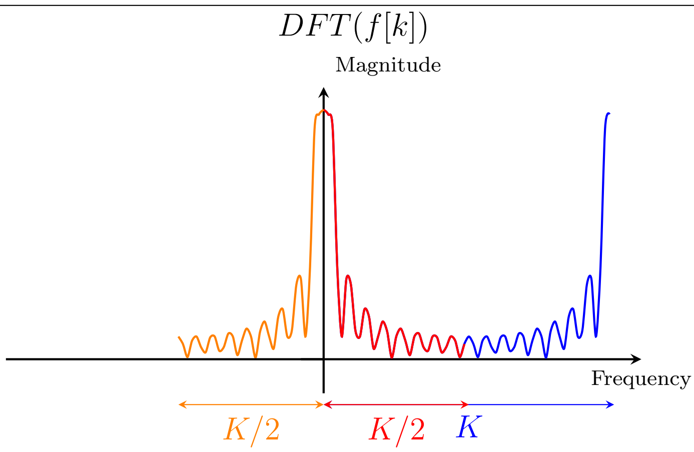



Figure III.7: Centering a 1D Fourier Transform around zero.

Indeed, $|\mathcal{F}(f)[-u,-v]| = |\mathcal{F}(f)[x,y]|$ as $\overline{\mathcal{F}(f)[-u,-v]} = \mathcal{F}(f)[x,y]$. But the best way to directly obtain a centered Fourier Transform is ________________________ .

## 10.i Examples of 2D Fourier Transform representation

**Example III.4.**

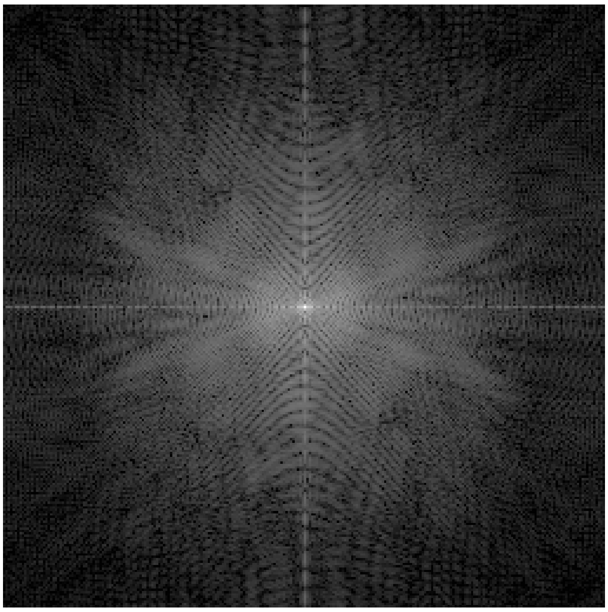


Figure III.8: Example of 2D Fourier Transform Magnitude of an image.

**Example III.5.**

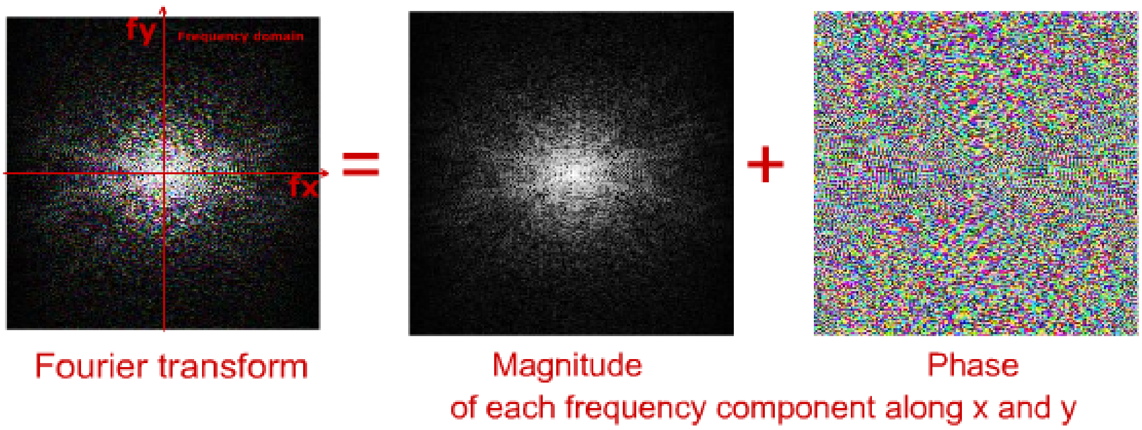



Figure III.9: Example of 2D Fourier Transform Magnitude and Phase of an image.

# 11 Notes

*Chapter IV*

# *An example of ISP application: Computed Tomography*

## Contents

# 1 Introduction

In 1979, the Nobel Prize for Medicine and Physiology was awarded jointly to Allan McLeod Cormack and Godfrey Newbold Housfield, the two pioneering scientist-engineers primarily responsible for the development, in the 1960s and early 1970s, of computerized actial tomography, popularly known as the CAT or CT scan. In his papers, Cormack, then Professor at Tufts University, in Massachusetts, developed certain mathematical algorithms that, he envisioned, could be used to create an image from X-ray data. Working completely independently of Cormack and at about the same time, Housfield, a research scientist at EMI Central Research Laboratories in the United Kingdom, designed the first operational CT scanner as well as the first commercially available model.

Since 1980, the number of CT scans performed each year in the United States has risen from about 3 million to over 67 million. What few people who have had CT scans probably realize is that the fundamental problem behind this procedure is essentially mathematical: If we know the values of the integral of a two- or three-dimensional function along all possible cross-sections, then how can we reconstruct the function itself?

This particular example of what is kinown as an *inverse problem* was studied by Johann Radon, an Austrian mathematician, in the early part of the twentieth century. Radon's work incorporated a sophisticated use of the theory of transforms and integral operators, and, by expanding the scope of that theory, contributed to the development of the rich and vitrant mathematical field of functional analysis. Cormack essentially rediscovered Radon's ideas, but did so at a time when technological applications werer actually conceivable.


http://www.orau.org/ptp/museumdirectory.htm

**XIXs** Photography

**1895** X-Rays discovery by William K. RÃ¶ntgen

**1896** First Fluoroscope by Thomas Edison

**1900** X-Rays popularized

**1901** RÃ¶ntgen recive the first Physics Nobel Prize

**1914-18** Use of X-Rays for First World War Injured

**1917** Tomography Bases by Johan Radon

**1960** Allan M. Cormac publishes the idea of CT

**1972** First Medical X-ray scanner by Hounsfield (EMI Central Research Laboratories)

**1979** Hounsfield and Mac Cormack receive the Nobel Prize of Medicine

# 2 X-Rays

X-Rays are electromagnetic waves of high frequencies.

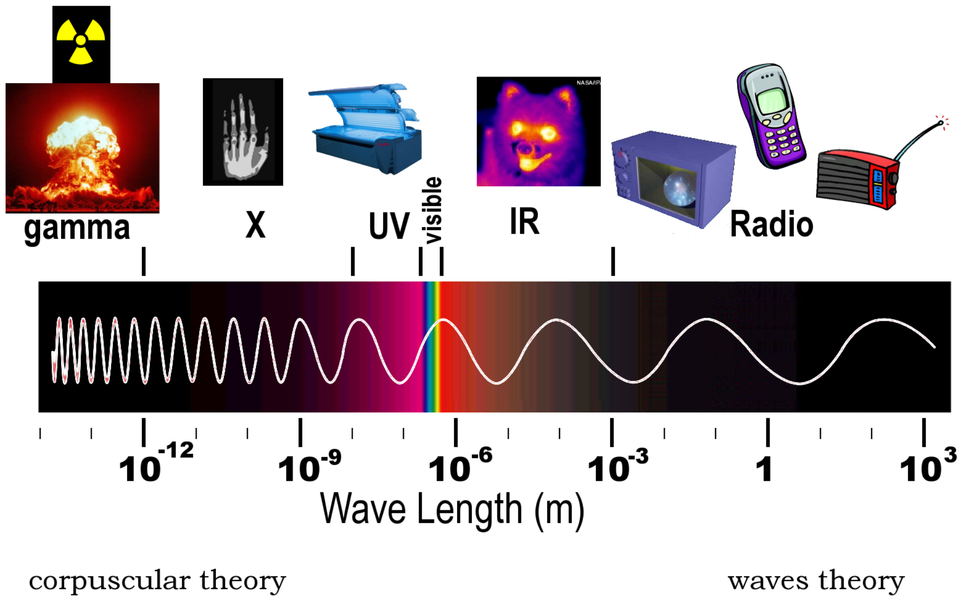



Figure IV.1: X-Rays are electromagnetic waves

## 2.a X-Rays Production

X-rays are produced when rapidly moving electrons that have been accelerated through a potential difference of order 1 kV to 1 MV strikes a metal target.

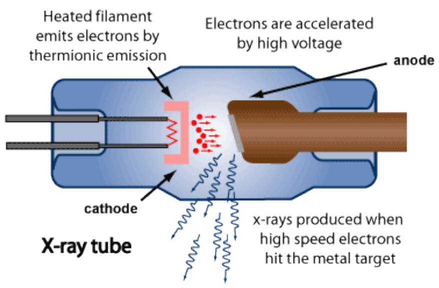
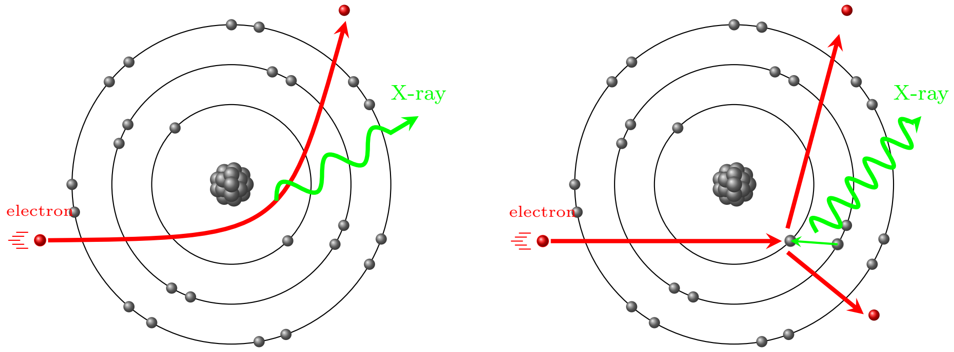



Figure IV.2: X-Ray tube

The incoming electrons release x-rays as they slow down in the target (braking radiation or bremsstrahlung). The x-ray photons produced in this manner range in energy from near zero up to the energy of the electrons. An incoming electron may also collide with an atom in the target, kicking out an electron and leaving a vacancy in one of the atom's electron shells. Another electron may fill the vacancy and in so doing release an x-ray photon of a specific energy (a characteristic x-ray). The x-ray spectrum shown in the picture is a plot of the number of photons against the photon energy.




(a) Braking Radiation (b) Characteristic Radiation

Figure IV.3: Twoo modes of X-Rays production

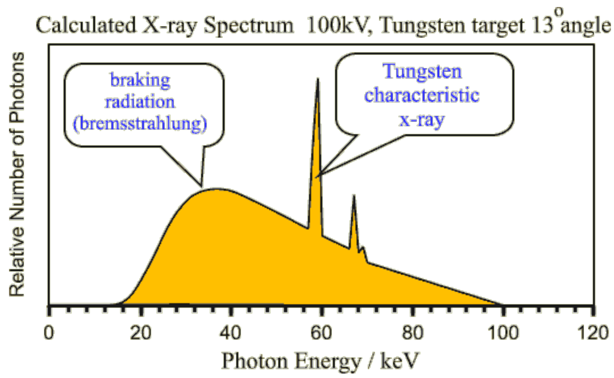
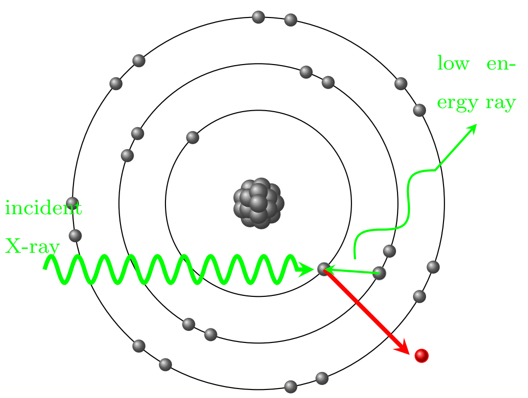



Figure IV.4: The x-ray spectrum :number of photons against photon energy.

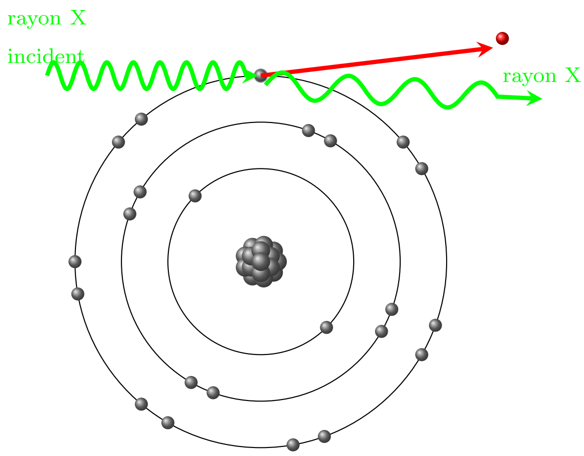
## 2.b X-ray / matter interaction

Photoelectric effect




When a X-ray interacts with matter by photoelectric effect, it transmits all its energy to an electron (of the inner shells) which is ejected from the atom.

Compton Effect




When an X-ray interacts with matter by Compton Effect, it transmits a part of its energy to an electron (of the outer shells) which is ejected from the atom. The incoming X-ray is deflected and has lower energy (lower frequency).

## 2.c X-Ray behavior and Beer-Lambet's law

To simplify the analysis, we will make some assumptions that present an idealized view of what an X-ray is and how it behaves. Specifically, in thinking of an X-ray beam as being composed of photons, we will assume that the beam is *monochromatic*. That is, each photon has the same energy level $E$ and the beam propagates at a constant frequency, with the same number of photons per second passing through every centimeter of the path of the beam. If $N(x)$ denotes the number of photns per second passing throuhg a point $x$, then the intensity of the bean at the point $x\ is I(x) = E \cdot N(x)$.

We also assume that X-ray beam has *zero width* and that it is not subject to refraction or diffraction. That is, X-rays beams are not bent by the medium nor do they spread out as they propagate.

Every substance absorbs a certain proportion of the photons that pass through it. This proportion, which is specific to the substance, is called the *attenuation coefficient* of the material. The units of the attenuation coefficient are something like *proportion of photons absorbed per millimeter of the medium.* In general, the attenuation coefficient is nonnegative and its value depends on the substance involved. Bone has a very high attenuation coefficient, air has a low coefficient, and water is somewhere in between. Different soft tissues have slightly different attenuation coefficients associated with them.

Now suppose an X-ray beam passes through some medium located between the position $x$ and the position $x+\Delta x$, and suppose that $\mu(x)$ is the attenuation coefficient of the medium located there. Then the proportion of all photons that will be absorbed in the interval $[x, x+\Delta x$ is $p(x) = \mu(x)\cdot\Delta x$. Thus, the number of photons that will be absorbed per second by the medium located in the interval $[x, x+\Delta x$ is $p(x)\cdot N(x) = \mu(x)\cdot N(x)\cdot\Delta x$. If we multiply both sides by the energy level $E$ of each phonon, we see that the corresponding *loss of intensity* of the X-ray beam over this interval is:

$$\Delta I \approx -\mu(x)\cdot I(x)\cdot\Delta x$$

Let $\Delta x \longleftarrow 0$ to get the differential equation known as *Beer-Lambert's law*:

$$\frac{dI}{dx} = -\mu(x)\cdot I(x) \tag{IV.1}$$

The differential equation IV.1 is separable and can be written as

$$\frac{dI}{I} = -mu(x)dx$$

If the beam startas at location $x_0$ with initial intensity $I_0 = I(0)$ and is detected, after passing through the medium, at the location $x_1$ with final intensity $I_1 = I(x_1)$ then we get

$$\int_{x_0}^{x_1} \frac{dI}{I} = -\int_{x_0}^{x_1} \mu_x dx$$

fom which it follows that

$$\ln(I(x_1)) - \ln(I(x_0)) = -\int_{x_0}^{x_1} \mu_x dx$$

Multiplying both sides by -1 yields the result

$$\int_{x_0}^{x_1} \mu_x dx = \ln\left(\frac{I_0}{I_1}\right) \tag{IV.2}$$

### 2.d Hounsfield Units

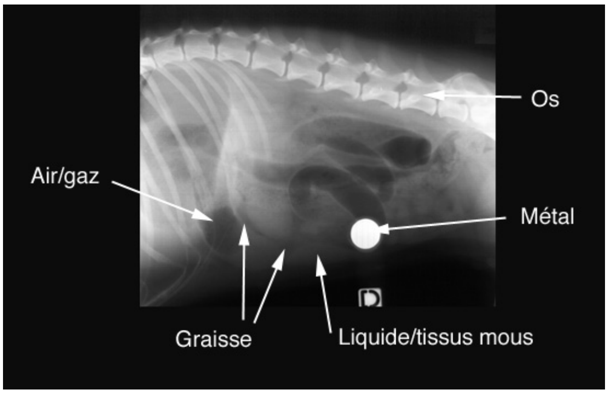
Radiologists actually use a variant of the attenuation coefficient in their work. Developed by Godfrey Hounsfield, the *Hounsfield unit* associated with a medium is a number that represents a comparison of attenuation coefficient of the medium with that of water. Specifically, the Hounsfield unit of a medium is:

$$H_{medium} = \frac{\mu_{medium} - \mu_{water}}{\mu_{wahter}} \tag{IV.3}$$

where $\mu$ denotes the true attenuation coefficient.

| Tissue | $HounsfieldUnits$ |
|---|---|
| Air | -1000 |
| Fat | -100 to -50 |
| Water | 0 |
| Cerebrospinal Fluid | +15 |
| Kideny | 30 |
| Myscke | 10 to 40 |
| Blood | 40 |
| Bone | +1000 |

Table IV.1: Approximate Hounsfield units for certain organic substances.


## 3 Radiography / X-Ray photography

For an X-Ray photography, the X-Rays are aare measured by sensors after having gone through tissues. Thus, depending on the thickness, the density, the atomic number ($Z$) of the tissues and their energy, they can be either:

- not affected, creating the darkest parts of the image
- stopped by the photoelectric effect, which conditions the gray levels on the image

- deviated through Compton scattering, which produces a uniform shadow on the image.

Figures IV.5 and IV.6 show several example of X-Ray radiography.

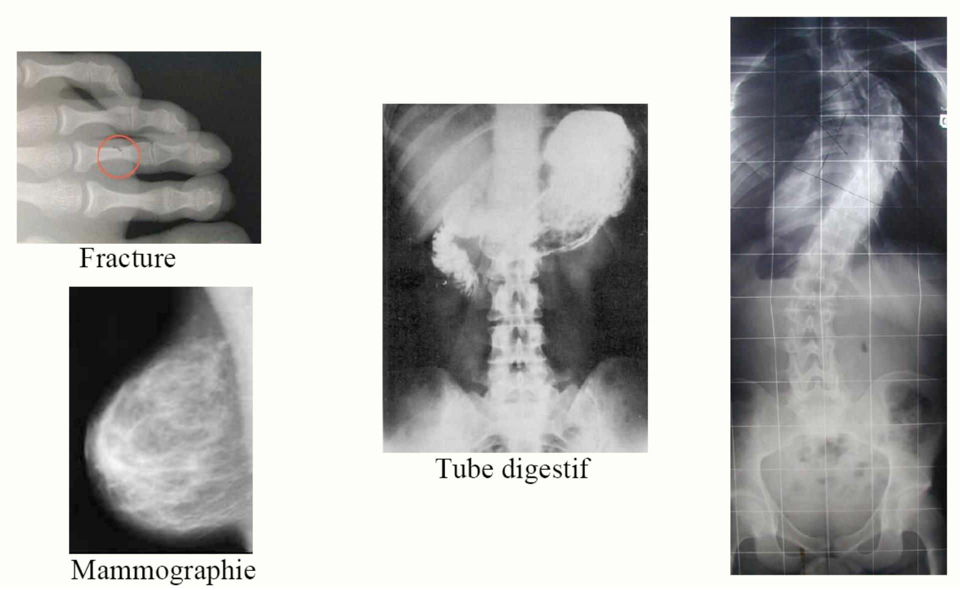



Figure IV.5: X-Ray radiography examples

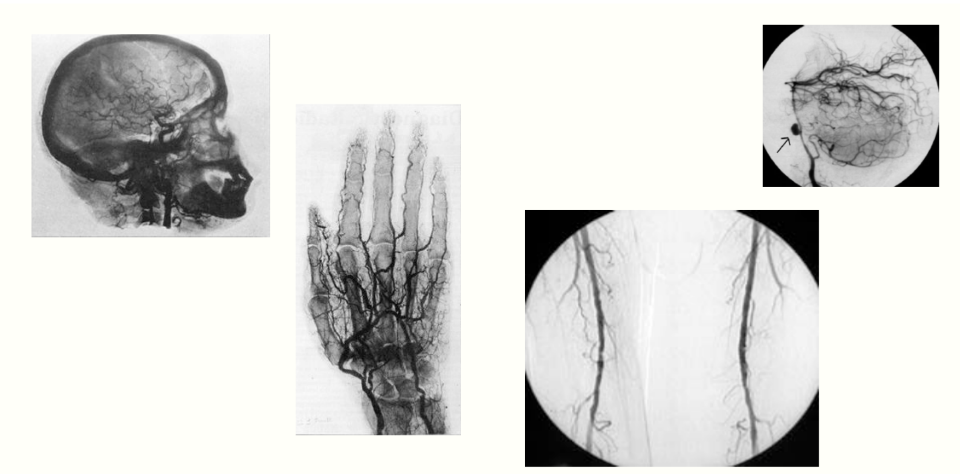


Figure IV.6: Angiography examples

# 4 Radon Transform

## 4.a Hypothesis

We consider the 2D plane $\mathbb{R}^2$ and denote :

$$f: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ p & \longmapsto & f(p) \end{array}$$

The support $\Omega$ of $f$ is bounded and normalized to 1:

$$\Omega = \{p \in \mathbb{R}, |p| < 1\}$$

The axis of rotation of the scanner goes through the oringin $O$ aof $\Omega$. We use the canonical unit vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ We consider an X-Ray beam, which source is the point $S$ and an X-Ray detector at the point $D$. Figure **??** sums up the hypothesis.

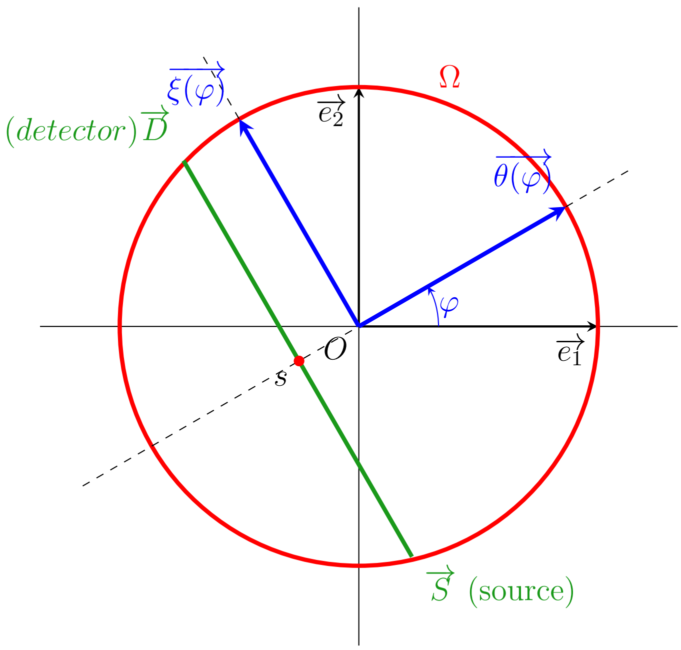



Figure IV.7: Conditions of Randon Transform Definition

## 4.b Definition

The *Radon Transform* of a function $f$ is defined as:

$$\mathcal{R}f(\varphi, s) = \int_{-\infty}^{\infty} f\left(s \cdot \overrightarrow{\theta(\varphi)} + t \cdot \overrightarrow{\xi(\varphi)}\right) dt \tag{IV.4}$$

## 4.c Properties

$\mathcal{R}$ is linear.

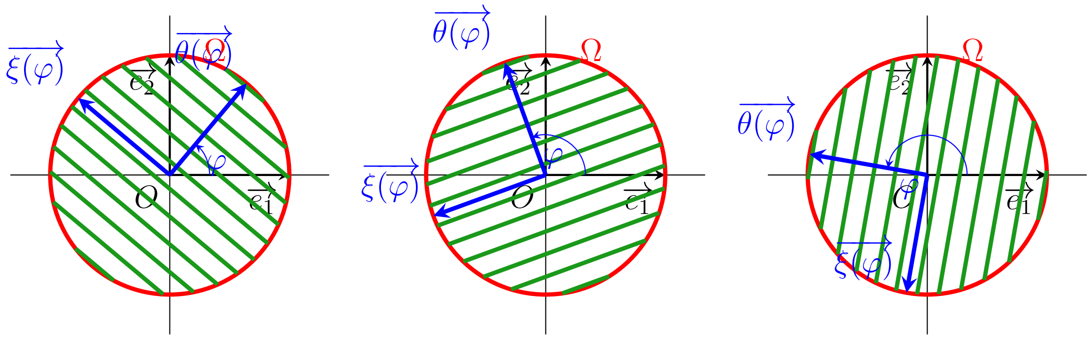
## 4.d Projection Slice Theorem

$$\mathcal{F}\left(\mathcal{R}f\right)(\sigma) = \mathcal{F}f(\sigma\overrightarrow{\theta(\varphi)}) \tag{IV.5}$$

### 4.e Filterd Back-Projection formula

Let $f$ be *sufficiently smooth*, then:

$$f(\overrightarrow{x}) = \int_0^{\pi} \int_{\mathbb{R}} \mathcal{F}\left(\mathcal{R}_{\overrightarrow{theta}} f(\sigma)\right) |\sigma| e^{2i\pi\sigma \overrightarrow{x} \cdot \overrightarrow{\theta}} \, d\sigma d\varphi \tag{IV.6}$$




# 5 Back to CT

The principle of Computed Tomography Scanner is to measure 1-Dimensional X-Ray attenuation functions across the patient for several orientations.

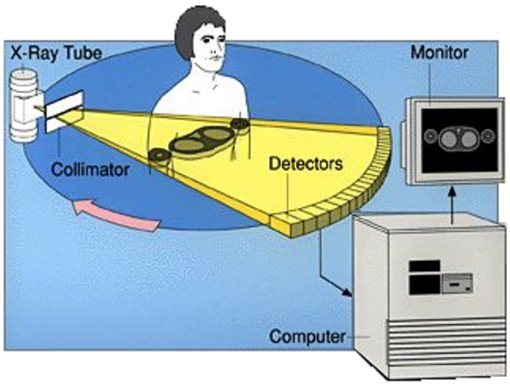



Figure IV.8: The principle of Computed Tomography Scanner is to measure 1-Dimensional X-Ray attenuation functions across the patient for several orientations.

The CT scanner actually acquires a *sinogramm* of the 2D patient's X-Ray attenuation function. The *sinogramm* is a way to record the Radon transform of a function.

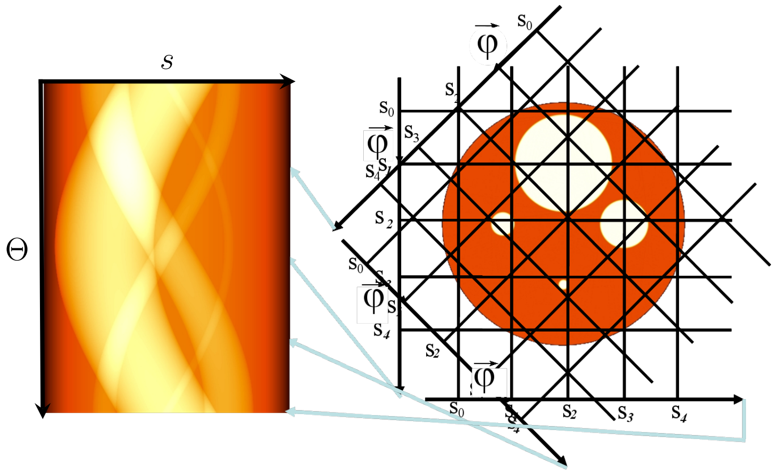
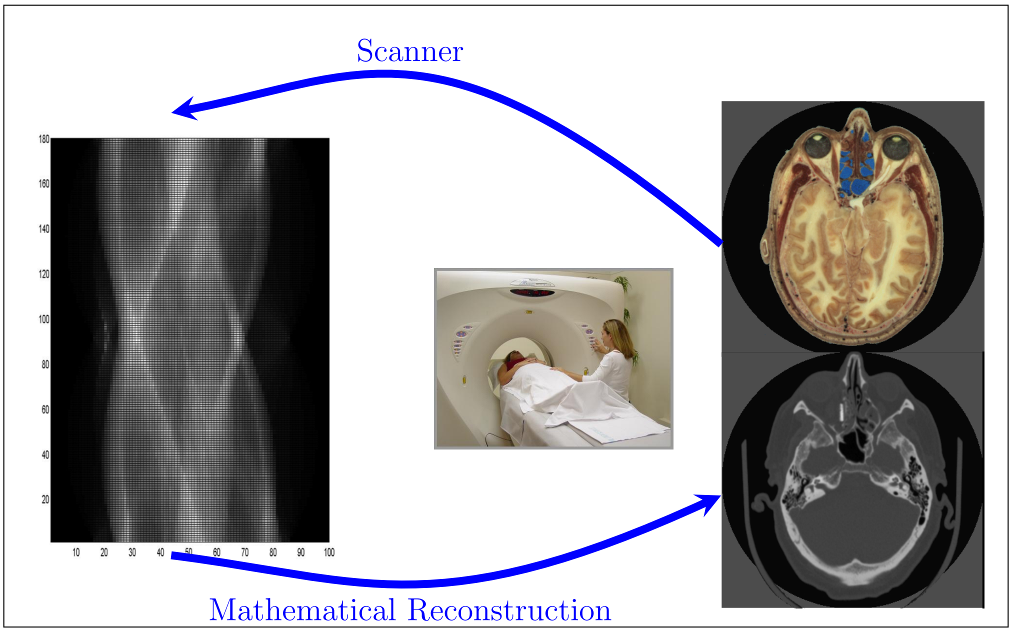



Figure IV.9: The *sinogramm* is a way to record the Radon transform of a function.


Scanner
180
160
140
120
100
80
60
40
20
10 20 30 40 50 60 70 80 90 100
Mathematical Reconstruction

# 6 Notes