Operations Research I

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Linear Programming

Plan

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Linear Programming

Framework

Linear Programming

finite number of real variables, linear constraints, linear objective

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Real variables $x_1, x_2 \ldots x_n$

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Constraint (i):

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

Linear Programming

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Linear Programming

finite number of real variables, linear constraints, linear objective

Real variables $x_1, x_2 \ldots x_n$

Constraint (i):

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

Objective function (maximize / minimize):

$$f(x_1, x_2 \ldots x_n) = \sum_{j=1}^{n} c_j x_j$$

Linear Programming

Example: Cucumber and onions culture

Constraints about the quantities of fertilizer and anti-parasite

- 8ℓ of fertilizer A available
  $\rightarrow 2\ell/m^2$ for cucumbers, $1\ell/m^2$ for onions
- 7ℓ of fertilizer B available
  $\rightarrow 1\ell/m^2$ for cucumbers, $2\ell/m^2$ for onions
- 3ℓ of anti-parasites available
  $\rightarrow 1\ell/m^2$ for onions

Objective: produce the maximum weight of vegetables knowing that the yield is $4kg/m^2$ for cucumbers, $5kg/m^2$ for onions

Linear Programming

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- x_c: area of cucumbers
- x_o: area of onions

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- $x_c + 2x_o \leq 7$ (fertilizer B)
- $x_o \leq 3$ (anti-parasites)
- $x_c \geq 0$ and $x_o \geq 0$

Linear Programming

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General optimization problem with constraints

$\Rightarrow$ **NO GENERAL solution method!!**

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$\Rightarrow$ general and efficient solution methods

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Those methods are efficient in theory and in practice
$\Rightarrow$ existence of numerous solution softwares:
Excel, CPLEX, Mathematica, LP-Solve. . .

Linear Programming

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Restrictive framework

- real variables
- linear constraints
- linear objective

Linear Programming

In extenso representation

- $\max 4x_c + 5x_o$
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matrix representation

$$\max \quad \begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} x_c \\ x_o \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_c \\ x_o \end{pmatrix} \leq \begin{pmatrix} 8 \\ 7 \\ 3 \end{pmatrix}$$

$$x_c \geq 0 \qquad x_o \geq 0$$

Linear Programming

in extenso representation

$$
\begin{array}{llcll}
\max z = & \sum_j c_j x_j & & & \\
s.t. & \sum_j a_{ij} x_j & \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} & b_i & i = 1, 2 \ldots m \\
 & x_j & \geq & 0 & j = 1, 2 \ldots n
\end{array}
$$

Linear Programming

- second member $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

- n decision variables $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

- matrix $m \times n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- cost (or profit) $c = (c_1, c_2 \dots c_n)$

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Matrix representation

$$\begin{array}{lll} \max z & = cx & \\ s.t. \quad Ax & \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} & b \\ x & \geq & 0 \end{array}$$

Linear Programming

Vocabulary

- x_i **decision variable** of the problem
- $x = (x_1, \ldots, x_n)$ **feasible solution**
  iff it satisfies all constraints
- set of all feasible solutions = **admissible region**
- $x = (x_1, \ldots, x_n)$ **optimal solution**
  iff it is feasible and it optimizes the objective function

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- **constraints** linear equalities or inequalities
  - $a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n \leq b_1$
  - $a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n \geq b_2$
  - $a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n = b_3$

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- linear **objective function** (or economical function)
  - max / min $c_1x_1 + c_2x_2 \ldots + c_nx_n$

Programmation linéaire

Applications

Exercice sheet: Linear programming

- Wine production
- Advertizing
- Olive oil production
- Bergamote

Caseine: Lab Linear Programming

- Perfumes
- Wines Q1 et Q2
- Dairy Products

Linear Programming

Canonical form of an LP

- maximization
- all variables are non negative
- all constraints are inequalities of the type "$\leq$"

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$$\max z = \sum_j c_j x_j$$

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- matrix form

$$\begin{array}{rccl} \max z = cx & & & \\ s.t. \quad Ax & \leq & b \\ x & \geq & 0 \end{array}$$

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Changing form

- equality $\rightarrow$ inequality

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$$ax = b \quad \Longleftrightarrow \quad \begin{cases} ax \leq b \\ ax \geq b \end{cases}$$

- max $\leftrightarrow$ min

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- equality $\rightarrow$ inequality

$$ax = b \quad \Longleftrightarrow \quad \begin{cases} ax \leq b \\ ax \geq b \end{cases}$$

- max $\leftrightarrow$ min $\qquad \max f(x) = -\min -f(x)$
- inequality $\rightarrow$ equality:

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- max $\leftrightarrow$ min $\qquad \max f(x) = -\min -f(x)$
- inequality $\to$ equality: add a slack variable

$$\begin{array}{lcll} ax \leq b & \iff & ax + s = b, & s \geq 0 \\ ax \geq b & \iff & ax - s = b, & s \geq 0 \end{array}$$

- unconstrained variable $\to$ positive variable

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$$x \lesseqgtr 0 \iff \begin{cases} x = x^+ - x^- \\ x^+, x^- \ge 0 \end{cases}$$

Linear Programming

Changing form

Exercices sheet: Linear Programming

- Linear and canonical forms

Linear Programming

Linearize a non linear formulation

e_i: linear expression of the decision variables

- **obj:** $\min \max\{e_1, e_2 \dots e_n\}$

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$$|e| = \max(e, -e) \quad \begin{cases} \min y \\ y \geq e_1 \\ y \geq -e_1 \end{cases} \qquad \begin{cases} \min e^+ + e^- \\ e_1 = e^+ - e^- \\ e^+, e^- \geq 0 \end{cases}$$

Linear Programming

Linearize a non linear formulation

Exercices sheet: Linear Programming

- Linearization

Linear Programming

A little history

- 30's-40's: Kantorovitch, soviet economist
  $\Rightarrow$ linear models for production planning and optimization
- 40's-50's: Dantzig, american mathematician
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- historical application
  - Operation Vittles and Plainfare for supply of the trizone during the blockade of Berlin by airlift (June 23, 1948 has – May 12, 1949)
  - simplex executed by hand (thousands of variables), up to 12 000 tons of hardware each day!

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- 1975: Kantorovitch has the Nobel price in economy
- XXIème century: software with LP available everywhere, use of LP in all industrial domains...

Plan

Geometrical interpretation

Example: Cucumber and onions culture

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Objective function $\max 4x_c + 5x_n$

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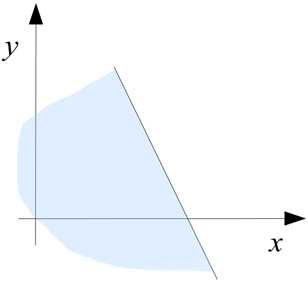
Geometrical interpretation

Interpret the constraints cucumbers and onions

- $2x + y \leq 8 \Rightarrow$ half-plane of $\mathbb{R}^2$



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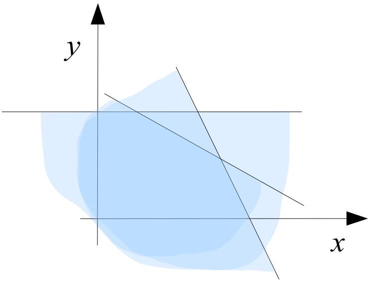

Geometrical interpretation

Interpret the constraints cucumbers and onions

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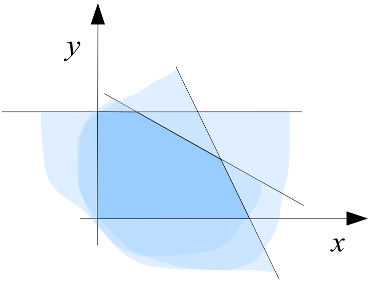

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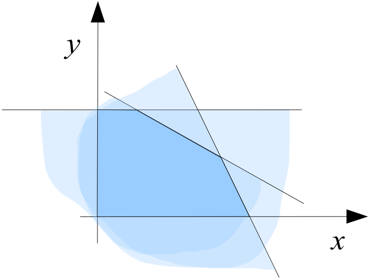
Set of feasible solutions = intersection of half-planes: **polyedron**



Geometrical interpretation

Optimize the objective

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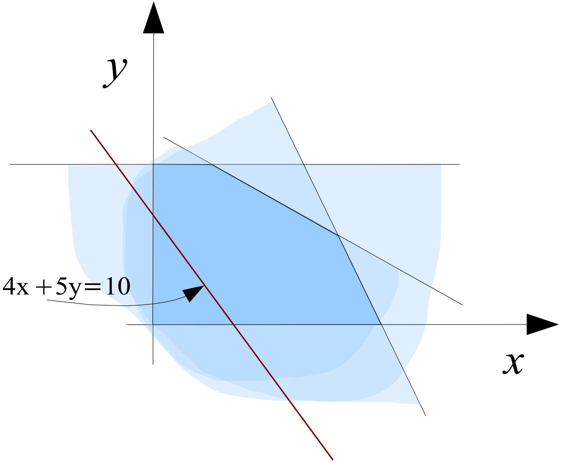

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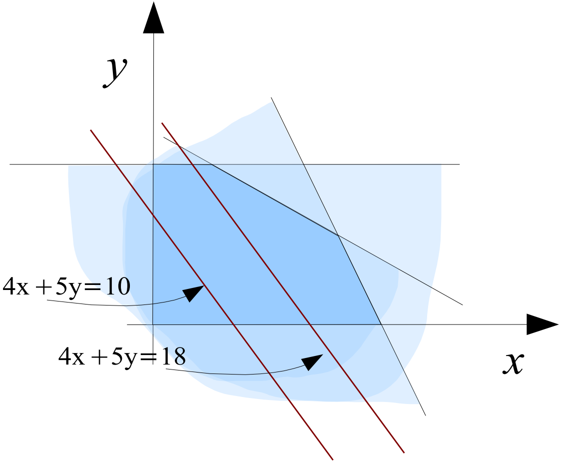

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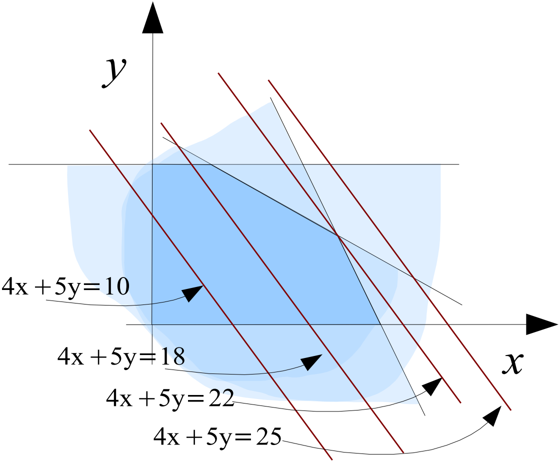

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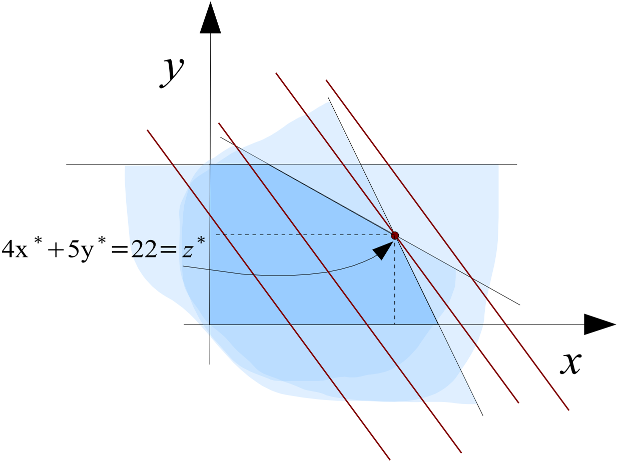
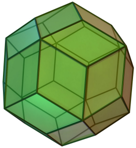


Geometrical interpretation

Geometry of a PL

The set of feasible solutions is always a **polyhedron** (intersection of half-spaces)



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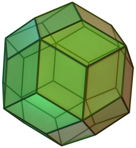
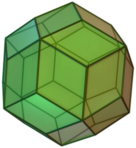


The level lines $\{f = \text{constant}\}$ of the objective function f are **affine hyperplanes** ($n = 2 \Rightarrow$ line, $n = 3 \Rightarrow$ plan...)

Geometrical interpretation

Geometry of a PL



Optimum is reached on the edge

The optimum of the objective function, if it exists, is reached on (at least) one of the **vertices** of the polyhedron

Mathematical justification:
the partial derivatives of $f(x) = c.x$ are never zero,
and the domain $\{x \mid \sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1, \ldots, m\}$ is compact
$\Rightarrow$ the optimum is reached on the edge...

Linear Programming

Solutions of an LP

The feasible region can be

- empty
- non empty, bounded
- non empty, unbounded

Propose examples of LP for each case

Linear Programming

Solutions of an LP

The feasible region can be

- empty
  - nb of optimal solutions: 0
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  - nb of optimal solutions: 1 or ∞
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Propose examples of LP for each case

Exercice sheet: Linear programming

- **Graphical solution**

Plan

Basis and extreme points

Recall

$$\begin{array}{lrcl} \max & z & = & cx \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

- A matrix $m \times n$
- $x = (x_1\, x_2 \ldots x_n)$
- $b = (b_1\, b_2 \ldots b_m)$
- $c = (c_1\, c_2 \ldots c_n)$

- The constraints define a polyhedron
- An optimal solution is a vertex of the polyhedron

How to enumerate the vertices of a polyhedron?

Basis and extreme points

Change to the standard form

Standard form

Add **slack variables**:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \;\Leftrightarrow\; \sum_{j=1}^{n} a_{ij}x_j + e_i = b_i, e_i \geq 0$$

Standard LP:

$$\begin{array}{rccc} \max & z(x) & = & c.x \\ \text{s.c} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

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We work on a space of higher dimension but all constraints are equalities

- Easier algebraic manipulations

Basis and extreme points

Change to the standard form

$$
\begin{aligned}
\max z &= 4x + 5y \\
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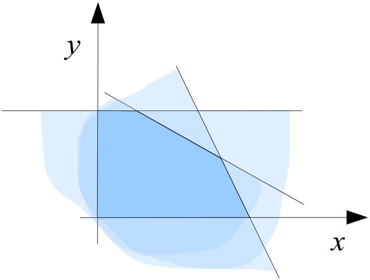

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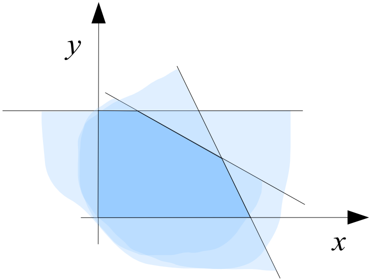

$$\begin{aligned} \max z &= 4x + 5y \\ \text{s.t.} \quad & 2x + y + e_1 = 8 \\ & x + 2y + e_2 = 7 \\ & y + e_3 = 3 \\ & x, y, e_1, e_2, e_3 \geq 0 \end{aligned}$$

9 interesting points
(intersection of constraints)

5 feasible points

enumeration of those 9 points as solutions of the standard form (basic solution)

Basis and extreme points

$$
\begin{array}{lcccccccccccc}
\text{s.t.} & 2x & + & y & + & e_1 & & & & & = & 8 \\
& x & + & 2y & & & + & e_2 & & & = & 7 \\
& & & y & & & & & + & e_3 & = & 3 \\
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x	y	e_1	e_2	e_3	basic solution	admiss.	extreme pt

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& x, & & y, & & e_1, & & e_2, & e_3 & \geq 0
\end{array}
$$

x	y	e_1	e_2	e_3	basic solution	admiss.	extreme pt
$\underline{0}$	$\underline{0}$	8	7	3	✔	✔	(0,0)

Basis and extreme points

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$\underline{0}$	8	$\underline{0}$	-9	-5	✔	✘	
$\underline{0}$	3.5	4.5	$\underline{0}$	-0.5	✔	✘	
$\underline{0}$	3	5	1	$\underline{0}$	✔	✔	(0,3)
4	$\underline{0}$	$\underline{0}$	3	3	✔	✔	(4,0)
7	$\underline{0}$	-6	$\underline{0}$	3	✔	✘	
	$\underline{0}$			$\underline{0}$	✘	✘	
3	2	$\underline{0}$	$\underline{0}$	1	✔	✔	(3,2)
2.5	3	$\underline{0}$	-1.5	$\underline{0}$	✔	✘	
1	3	3	$\underline{0}$	$\underline{0}$	✔	✔	(1,3)

Basis and extreme points

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$$\{\textbf{extreme points}\} \iff \{\textbf{feasible basic solutions}\}$$

Basis and extreme points

- Linear system $Ax = b$
- A format $m \times n$, rank $A = m \leq n$
- **Basis** of A: invertible submatrix $B(m \times m)$ of A
  $A = (B, N)$

$$(B, N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \qquad \text{or} \qquad Bx_B + Nx_N = b$$

$$\Rightarrow \quad x_B = B^{-1}b - B^{-1}Nx_N$$

- **Basic solution** associated to B:
  - $x_N = 0$ non-basic variables
  - $x_B = B^{-1}b$ basic variables

Basis and extreme points

Applications

Exercice sheet: Linear programming

-
- Exercice Bases *2

Basis and extreme points

Basis and basic solutions

$$\begin{cases} 2x + y + e_1 = 8 \\ x + 2y + e_2 = 7 \\ y + e_3 = 3 \\ x, y, e_1, e_2, e_3 \geq 0 \end{cases}$$

Initial basis ? $\{e_1, e_2, e_3\}$ for example:

Basis and extreme points

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$$\begin{cases} 2x + y + e_1 = 8 \\ x + 2y + e_2 = 7 \\ y + e_3 = 3 \end{cases} \Leftrightarrow \begin{cases} e_1 = 8 - 2x - y \\ e_2 = 7 - x - 2y \\ e_3 = 3 - y \end{cases}$$

$e_1, e_2, e_3 =$ basic variables, $x, y =$ non-basic variables

Basis and extreme points

Basis and basic solutions

$$\begin{cases} e_1 = 8 - 2x - y \\ e_2 = 7 - x - 2y \\ e_3 = 3 - y \end{cases}$$

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- basic variables are then calculated

Basis and extreme points

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Basis and extreme points

- $Ax = b, \qquad x \geq 0$
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Basis and extreme points

- $Ax = b, \qquad x \geq 0$
- $(x_B, 0)$ associated to B is a **feasible basic solution** if $x_B \geq 0$
- $\{$**extreme points** of the polyhedron$\}$ $\Longleftrightarrow$ $\{$feasible basic solutions of the corresponding linear system$\}$
- number of extreme points $\approx C_n^m = \frac{n!}{m!(n-m)!}$
- basic degenerated solutions: some basic variables are zero
- if A is invertible: a single basic solution

Basis and extreme points

Neighboring basis and pivoting

Neighboring basis

Two neighboring vertices correspond to two bases B and B' such that a variable of B is replaced to obtain B'

Basis and extreme points

Neighboring basis and pivoting

Neighboring basis

Two neighboring vertices correspond to two bases B and B' such that a variable of B is replaced to obtain B'

- pass to a neighboring vertex: change basis (neighboring basis)

pivoting principle

Basis and extreme points

Which variable enters the basis?

Let's try with y: what is the max value for y?

Basis and extreme points

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▶ candidate for a new basis: $\{e_1, e_2, e_3\} \cup \{y\} \setminus \{e_3\} = \{e_1, e_2, y\}$

$(x, y, e_1, e_2, e_3) = (0, 3, 5, 1, 0)$

Plan

The simplex algorithm

Towards a solution algorithm

▶ A naive solution method: enumerate all vertices, calculate f on these points, choose the vertex with optimal f:

- works: finite number of vertices
- limitation: this number can be very large in general...

The simplex algorithm (G. B. Dantzig 1947) iterative algorithm allowing to solve a linear program.

The simplex algorithm

Local improvement principle

From a current vertex, find a neighboring vertex that improves the objective.

Local improvement principle (maximization):

Let x_0 be a non optimal vertex. Then, there exists x, a **neighboring** vertex of x_0, such that $f(x) > f(x_0)$.

The simplex algorithm

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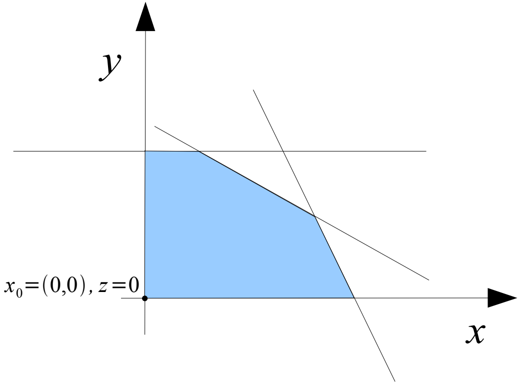
Remark: we change from a **continuous** problem (real variables) to a **discrete** problem (finite number of vertices)...

The simplex algorithm

2D Illustration: Cucumber and onions $z = 4x + 5y$

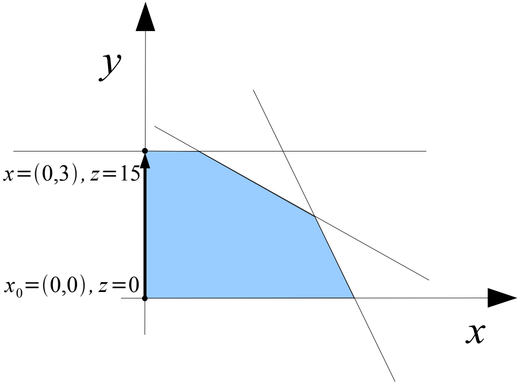
$x_0 = (0, 0), z = 0$



The simplex algorithm

2D Illustration: Cucumber and onions $z = 4x + 5y$

$x_0 = (0, 0), z = 0 \rightarrow x = (0, 3), z = 15$



The simplex algorithm

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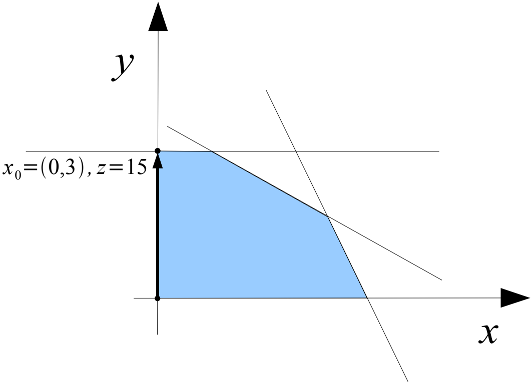
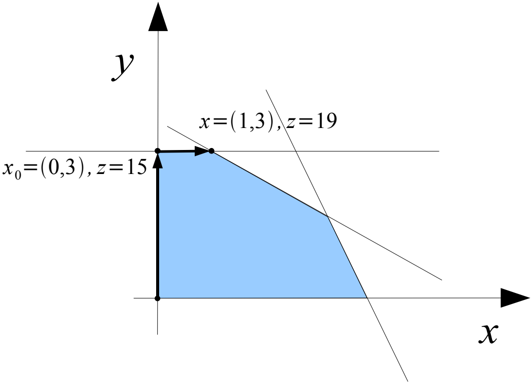


The simplex algorithm

2D Illustration: Cucumber and onions

$z = 4x + 5y$

$x_0 = (0, 0), z = 0 \to x = (0, 3), z = 15$
$x_0 = (0, 3), z = 15 \to x = (1, 3), z = 19$



The simplex algorithm

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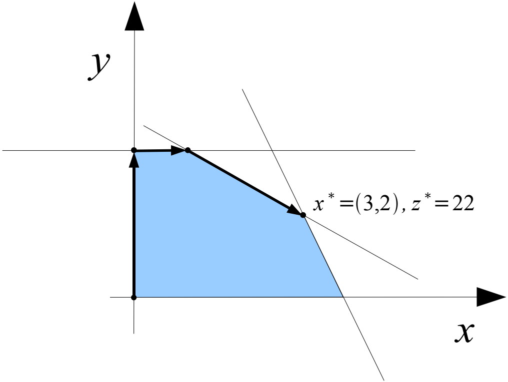
$x_0 = (0, 0), z = 0 \rightarrow x = (0, 3), z = 15$
$x_0 = (0, 3), z = 15 \rightarrow x = (1, 3), z = 19$
$x_0 = (1, 3), z = 19 \rightarrow x = (3, 2), z = 22$



- no more local improvement is possible $\Rightarrow$ optimum

The simplex algorithm

Concrete illustration

- Standardization:

$$\text{Maximize } z = 4x + 5y$$

$$\text{s.t.} \begin{cases} 2x + y \leq 8 \\ x + 2y \leq 7 \\ y \leq 3 \\ x, y \geq 0 \end{cases}$$

The simplex algorithm

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The simplex algorithm

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e_1, e_2, e_3 = basic variables, x, y = non-basic variables

The simplex algorithm

Associated basic solution

- set the non-basic variables to 0
- We deduce:
  - value of the basic variables
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The simplex algorithm

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here: $x = y = 0 \Rightarrow \begin{cases} e_1 = 8 - 2x - y = 8 \\ e_2 = 7 - x - 2y = 7 \\ e_3 = 3 - y = 3 \end{cases}$ and $z = 4x + 5y = 0$

The simplex algorithm

Basis change

Essential observation: $z = 4x + 5y = 0 \Rightarrow$ we can improve z if x or y enters in the basis.

The simplex algorithm

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The simplex algorithm

New basis $\{e_1, e_2, y\}$

$$\begin{cases} e_1 = 8 - 2x - y \\ e_2 = 7 - x - 2y \\ e_3 = 3 - y \end{cases} \Rightarrow \begin{cases} e_1 = 8 - 2x - y = 5 - 2x + e_3 \\ e_2 = 7 - x - 2y = 1 - x + 2e_3 \\ y = 3 - e_3 \end{cases}$$

The simplex algorithm

New basis $\{e_1, e_2, y\}$

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Describe z with the non-basic variables

- $z = 4x + 5y = 15 + 4x - 5e_3$

The simplex algorithm

New basis $\{e_1, e_2, y\}$

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Describe z with the non-basic variables

- $z = 4x + 5y = 15 + 4x - 5e_3$

Associated basic solution

$$x = e_3 = 0 \Rightarrow \begin{cases} e_1 = 5 - 2x + e_3 = 5 \\ e_2 = 1 - x + 2e_3 = 1 \\ y = 3 - e_3 = 3 \end{cases} \quad \text{and} \quad z = 15$$

The simplex algorithm

Iteration

$z = 15 + 4x - 5e_3$ can still increase if x enters the basis

The simplex algorithm

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Max value for x:

The simplex algorithm

Iteration

$z = 15 + 4x - 5e_3$ can still increase if x enters the basis

If x enters, which variable leaves the basis?
Max value for x:

- $e_1 = 5 - 2x + e_3 \geq 0 \Rightarrow x \leq 2.5$
- $e_2 = 1 - x + 2e_3 \geq 0 \Rightarrow x \leq 1$
- $y = 3 - e_3 \geq 0 \Rightarrow$ no constraint on x

The simplex algorithm

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If x enters, which variable leaves the basis?
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Finally: $x_{\max} = 1$ and e_2 leaves the basis
New basis $\{e_1, y, x\}$

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New basis $\{e_1, y, x\}$

$$\begin{cases} e_1 = 3 + 2e_2 - 3e_3 \\ x = 1 - e_2 + 2e_3 \\ y = 3 - e_3 \\ z = 19 - 4e_2 + 3e_3 \end{cases}$$

The simplex algorithm

Iteration (cont.)

$z = 19 - 4e_2 + 3e_3$ can still increase if e_3 enters the basis

The simplex algorithm

Iteration (cont.)

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The simplex algorithm

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- $x = 1 - e_2 + 2e_3 \geq 0 \Rightarrow$ no constraint on e_3
- $y = 3 - e_3 \geq 0 \Rightarrow e_3 \leq 3$

The simplex algorithm

Iteration (cont.)

$z = 19 - 4e_2 + 3e_3$ can still increase if e_3 enters the basis

If e_3 enters, which variable leaves the basis?
Max value for e_3:

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- $y = 3 - e_3 \geq 0 \Rightarrow e_3 \leq 3$

Finally: $e_{3_{\max}} = 1$, e_1 leaves. New basis $\{e_3, y, x\}$:

The simplex algorithm

Iteration (cont.)

$z = 19 - 4e_2 + 3e_3$ can still increase if e_3 enters the basis

If e_3 enters, which variable leaves the basis?
Max value for e_3:

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- $y = 3 - e_3 \geq 0 \Rightarrow e_3 \leq 3$

Finally: $e_{3_{\max}} = 1$, e_1 leaves. New basis $\{e_3, y, x\}$:

$$\begin{cases} e_3 = 1 + 2/3e_2 - 1/3e_1 \\ x = 3 + 1/3e_2 - 2/3e_1 \\ y = 2 - 2/3e_2 + 1/3e_1 \\ z = 22 - 2e_2 - e_1 \end{cases}$$

The simplex algorithm

Termination

One has $z = 22 - 2e_2 - e_1$, hence $z^* \leq 22$

The simplex algorithm

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but the basic solution $x = 3, y = 2, e_3 = 1$ leads to $z = 22$

- optimum

The simplex algorithm

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One has $z = 22 - 2e_2 - e_1$, hence $z^* \leq 22$

but the basic solution $x = 3, y = 2, e_3 = 1$ leads to $z = 22$

- optimum

The termination conditions concern the coefficients of z expressed with the non-basic variables

The simplex algorithm

Reduced costs

B, a basis of $Ax = b$

the objective function

$$
\begin{aligned}
z &= cx = c_B x_B + c_N x_N \\
&= c_B B^{-1} b - (c_B B^{-1} N - c_N) x_N \\
&= z_0 - \sum_{j=1}^{n} (c_B B^{-1} a^j - c_j) x_j \\
&= z_0 - \sum_{j=1}^{n} (z_j - c_j) x_j
\end{aligned}
$$

$z_j - c_j = c_B B^{-1} a^j - c_j$ are the reduced costs of the non-basic variable x_j

The simplex algorithm

At each iteration

			x_N	x_B
z	$=$	z_0	reduced costs	0
x_B	$=$	$\oplus$	$\ddots$	

For the optimum

			x_N	x_B
z	$=$	z_0^*	$\ominus$	0
x_B	$=$	$\oplus$	$\ddots$	

The simplex algorithm

Heuristic: the variable with the higher coefficient enters the basis

Which variable leaves the basis?

Minimal quotient principle

pivot column x_1	right member ≥ 0	quotient
$a_1 \leq 0$	b_1	-
$a_2 > 0$	b_2	$\frac{b_2}{a_2}$
$a_3 > 0$	b_3	$\frac{b_3}{a_3}$
$a_4 = 0$	b_4	-

row r $\quad \frac{b_r}{a_r} = \min\left\{\frac{b_i}{a_i} | a_i > 0\right\}$

The simplex algorithm

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The simplex algorithm

Pivoting

$$\begin{array}{rccc} & c & \cdots & a \\ & \vdots & & \vdots \\ \text{pivot row} - & p & \cdots & b \\ & | & & \\ & \begin{array}{c}\text{pivot}\\ \text{column}\end{array} & & \end{array} \implies a \to a - \frac{b}{p} c$$

The simplex algorithm

Phase II

Input: a linear program and a feasible basic solution
Output: an optimal feasible solution or declare "non-bounded linear program"

1. Choose an entering column (variable)
   - choose a non-basic variable x_j (column) with a negative reduced cost
   - if there is no entering variable: STOP, the basic solution is optimal
2. Choose a leaving row (variable)
   - choose a row r minimizing the quotient
   - if there is no leaving row: STOP the current linear program is unbounded
3. Update the basis and the program
   - pivot around a_{rj} and go to (1)

The simplex algorithm

- A basic solution is degenerated if at least one basic variable is zero (in this case, there isn't a bijection between the feasible basic solutions and the extreme points)
- If all basic solutions are non degenerated, then the simplex algorithm terminates after a finite number of iterations

The simplex algorithm

Phase I

- Exercice on Chamilo