Operations Research I

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Grenoble, 2015-2016

Duality

Plan

Duality

New concept in Linear Programming

Primal

- input A, b, c
- minimize

Dual

- same input A, b, c
- maximize

Plan

1. Economical Illustration
2. How to prove optimality?
3. Write the dual
4. Properties

Plan

Primal problem ($\mathcal{P}$)

A family uses 6 food products
as a source of vitamin A and C

	products (units/kg)						demand
	1	2	3	4	5	6	(units)
vitamin A	1	0	2	2	1	2	9
vitamin C	0	1	3	1	3	2	19
Price per kg	35	30	60	50	27	22	

Objective : minimize the total cost

Modelization

Modelization of the primal problem ($\mathcal{P}$)

- **variables**
  x_i: quantity (kg) of food product i bought

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- **constraints** (demands)

$$\begin{array}{ccccccccccccc} x_1 & & & + & 2x_3 & + & 2x_4 & + & x_5 & + & 2x_6 & \geq & 9 \\ & & x_2 & + & 3x_3 & + & x_4 & + & 3x_5 & + & 2x_6 & \geq & 19 \end{array}$$

$x_i \geq 0$ pour tout $i = 1, 2 \ldots 6$

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  $x_i \geq 0$ pour tout $i = 1, 2 \ldots 6$
- **objective** (cost)
  minimizer $z = 35x_1 + 30x_2 + 60x_3 + 50x_4 + 27x_5 + 22x_6$

Dual problem $(\mathcal{D})$ associated to $(\mathcal{P})$

A producer of tablets of synthetic vitamin wants to convince the family to by his vitamins.

What are the selling prices w_A and w_C ?

- to be competitive
- and maximize the profit

Modelization

Modelization of the dual problem ($\mathcal{D}$)

- **variables** (selling price)

  w_A = prix of one unit of synthetic A

  w_C = prix of one unit of synthetic C

Modelization of the dual problem $(\mathcal{D})$

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- **constraints** (remain competitive)

$$
\begin{array}{rcrcll}
w_A & & & \leq & 35 & \text{(product 1)} \\
 & & w_C & \leq & 30 & \text{(product 2)} \\
2w_A & + & 3w_C & \leq & 60 & \text{(product 3)} \\
2w_A & + & w_C & \leq & 50 & \text{(product 4)} \\
w_A & + & 3w_C & \leq & 27 & \text{(product 5)} \\
2w_A & + & 2w_C & \leq & 22 & \text{(product 6)} \\
\multicolumn{5}{l}{w_A \geq 0 \qquad w_C \geq 0} & \text{(no present)}
\end{array}
$$

Modelization of the dual problem $(\mathcal{D})$

- **variables** (selling price)
  $w_A =$ prix of one unit of synthetic A
  $w_C =$ prix of one unit of synthetic C

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2w_A & + & 2w_C & \leq & 22 & \text{(product 6)} \\
w_A \geq 0 & & w_C \geq 0 & & & \text{(no present)}
\end{array}
$$

- **objective** (profit)
  maximize $v = 9w_A + 19w_C$

matrix modelling

Primal problem

the family: buy food products at the minimum price to satisfy the need in vitamin A and C.

Matrix modelling

matrix modelling

Primal problem

the family: buy food products at the minimum price to satisfy the need in vitamin A and C.

$$(\mathcal{P}) \begin{cases} \min \quad z = (35, 30, 60, 50, 27, 22) \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \\ \text{s.t.} \\ \quad \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 2 \\ 0 & 1 & 3 & 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \geq \begin{pmatrix} 9 \\ 19 \end{pmatrix} \\ \quad x_i \geq 0 \end{cases}$$

matrix modelling

Primal problem

the family: buy food products at the minimum price to satisfy the need in vitamin A and C.

Matrix modelling

Dual problem

Synthetic vitamin producer: be competitive with the food products as a source of vitamin and maximize the sale profit

Matrix modelling

Matrix modelling

$$(\mathcal{P})\begin{cases} \min \quad z = (35, 30, 60, 50, 27, 22)\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \\ \text{s.t.} \\ \quad \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 2 \\ 0 & 1 & 3 & 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \geq \begin{pmatrix} 9 \\ 19 \end{pmatrix} \\ \quad x_i \geq 0 \end{cases}$$

$$(\mathcal{D})\begin{cases} \max \quad v = (w_A, w_C)\begin{pmatrix} 9 \\ 19 \end{pmatrix} \\ \text{s.t.} \\ (w_A, w_C)\begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 2 \\ 0 & 1 & 3 & 1 & 3 & 2 \end{pmatrix} \leq (35, 30, 60, 50, 27, 22) \\ w_i \geq 0 \end{cases}$$

Generalization of the economic illustration

	resource i	demand j
product j	a_{ij}	c_j
cost i	b_i	

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Primal problem (product purchaser): What quantity of resource i has to be bought to satisfy the demand at minimum cost?

$$\min \sum_i b_i x_i \qquad s.t. \quad \sum_i a_{ij} x_i \geq c_j \quad \forall j$$

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$$\min \sum_i b_i x_i \qquad s.t. \quad \sum_i a_{ij} x_i \geq c_j \quad \forall j$$

Dual problem (seller of the product): What price should product j be proposed at to maximize the profit while remaining competitive?

$$\max \sum_j c_j w_j \qquad s.t. \quad \sum_j a_{ij} w_j \leq b_i \quad \forall i$$

Plan

How to prove optimality?

Objective: prove the optimality of a solution

$$
\begin{array}{l}
\max z = x_1 + x_2 \\
\begin{array}{rcrcr}
4x_1 & + & 5x_2 & \leq & 20 \\
2x_1 & + & x_2 & \leq & 6 \\
 & & x_2 & \leq & 2
\end{array} \\
x_1, x_2 \geq 0
\end{array}
$$

Idea: find a valid combination of the constraints that bounds each term of the objective function

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Idea: find a valid combination of the constraints that bounds each term of the objective function

$$
\begin{array}{rcrcll}
4x_1 & + & 5x_2 & \leq & 20 & \times 1 \\
2x_1 & + & x_2 & \leq & 6 & \times 0 \\
 & & x_2 & \leq & 2 & \times 0 \\
\hline
4x_1 & + & 5x_2 & \leq & 20 &
\end{array}
$$

How to prove optimality?

$\max z = x_1 + x_2$

$$
\begin{array}{rcrcll}
4x_1 & + & 5x_2 & \leq & 20 & \times y_1 \\
2x_1 & + & x_2 & \leq & 6 & \times y_2 \\
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(4y_1 + 2y_2)x_1 & + & (5y_1 + y_2 + y_3)x_2 & \leq & 20y_1 + 6y_2 + 2y_3 &
\end{array}
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 & & & \uparrow & & \\
 & & & \multicolumn{2}{l}{y_1, y_2, y_3 \geq 0} &
\end{array}
$$

$y_1 = 1 \quad y_2 = 0 \quad y_3 = 0 \quad \Rightarrow \quad x_1 + x_2 \leq 4x_1 + 5x_2 \leq 20$

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\begin{array}{rcrcll}
4x_1 & + & 5x_2 & \leq & 20 & \times y_1 \\
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$y_1 = 0 \quad y_2 = 1 \quad y_3 = 0 \quad \Rightarrow \quad x_1 + x_2 \leq 2x_1 + 1x_2 \leq 6$

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$y_1 = 0 \quad y_2 = 0 \quad y_3 = 1 \quad \Rightarrow \quad x_2 \leq 2$ no conclusion

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\begin{array}{rcrcll}
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$y_1 = 0 \quad y_2 = 0 \quad y_3 = 1 \quad \Rightarrow \quad x_2 \leq 2$ no conclusion

$y_1 = 0 \quad y_2 = 0.5 \quad y_3 = 0.5$

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\begin{array}{rcrcll}
4x_1 & + & 5x_2 & \leq & 20 & \times y_1 \\
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$y_1 = 0 \quad y_2 = 0 \quad y_3 = 1 \quad \Rightarrow \quad x_2 \leq 2$ no conclusion

$y_1 = 0 \quad y_2 = 0.5 \quad y_3 = 0.5 \quad \Rightarrow \quad x_1 + x_2 \leq x_1 + x_2 \leq 4$

How to prove optimality?

$\max z = x_1 + x_2$

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4x_1 & + & 5x_2 & \leq & 20 & \times y_1 \\
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 & & & \uparrow & & \\
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\end{array}
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$y_1 = 0 \quad y_2 = 0.5 \quad y_3 = 0.5 \quad \Rightarrow \quad x_1 + x_2 \leq x_1 + x_2 \leq 4$

Yet $x_1 = 2$ and $x_2 = 2$ verify all the constraints and achieve this bound. Hence, it is an **optimal solution**.

How to prove optimality?

$\max z = x_1 + x_2$

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\begin{array}{rcrcll}
4x_1 & + & 5x_2 & \leq & 20 & \times y_1 \\
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(4y_1 + 2y_2)x_1 & + & (5y_1 + y_2 + y_3)x_2 & \leq & 20y_1 + 6y_2 + 2y_3 & \\
 & & & \uparrow & & \\
 & & & y_1, y_2, y_3 \geq 0 & &
\end{array}
$$

Finally,

$$\min \quad 20y_1 \quad + \quad 6y_2 \quad + \quad 2y_3$$ (minimal upper bound)

How to prove optimality?

$\max z = x_1 + x_2$

$$
\begin{array}{rcrcll}
4x_1 & + & 5x_2 & \leq & 20 & \times y_1 \\
2x_1 & + & x_2 & \leq & 6 & \times y_2 \\
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(4y_1 + 2y_2)x_1 & + & (5y_1 + y_2 + y_3)x_2 & \leq & 20y_1 + 6y_2 + 2y_3 & \\
 & & & \uparrow & & \\
 & & & \multicolumn{2}{l}{y_1, y_2, y_3 \geq 0} &
\end{array}
$$

Finally,

$$
\begin{array}{lrcrcrcl}
\min & 20y_1 & + & 6y_2 & + & 2y_3 & & \\
\text{s.t.} & & & & & & & \\
 & 4y_1 & + & 2y_2 & & & \geq & 1 \\
 & 5y_1 & + & y_2 & + & y_3 & \geq & 1 \\
 & y_i & \geq & 0 & & & &
\end{array}
$$

(minimal upper bound)

(bound each term of the objective fu

Plan

Canonical form of Duality

Input A, b, c

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{cases}$$

$$(\mathcal{D}) \quad \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \geq 0 \end{cases}$$

Other equivalent forms

$$(\mathcal{P}) \quad \left\{ \begin{array}{ll} \min & z = cx \\ s.t. & Ax \leq b \\ & x \geq 0 \end{array} \right.$$

write the dual

Other equivalent forms

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \leq b \\ & x \geq 0 \end{cases}$$

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & [-A]x \geq [-b] \\ & x \geq 0 \end{cases}$$

Other equivalent forms

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \leq b \\ & x \geq 0 \end{cases}$$

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & [-A]x \geq [-b] \\ & x \geq 0 \end{cases}$$

$$\Rightarrow \text{canonical dual} \quad (\mathcal{D}) \quad \begin{cases} \max & v = w[-b] \\ s.t. & w[-A] \leq c \\ & w \geq 0 \end{cases}$$

Other equivalent forms

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \leq b \\ & x \geq 0 \end{cases}$$

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$$\Rightarrow \text{canonical dual} \quad (\mathcal{D}) \quad \begin{cases} \max & v = w[-b] \\ s.t. & w[-A] \leq c \\ & w \geq 0 \end{cases}$$

replace $-w$ by w

$$(\mathcal{D}) \quad \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \leq 0 \end{cases}$$

Other equivalent forms

$$(\mathcal{P}) \qquad \begin{cases} \min & z = cx \\ s.t. & Ax = b \\ & x \geq 0 \end{cases}$$

write the dual

Other equivalent forms

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax = b \\ & x \geq 0 \end{cases}$$

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & \begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \end{bmatrix} \\ & x \geq 0 \end{cases}$$

Other equivalent forms

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax = b \\ & x \geq 0 \end{cases}$$

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & \begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \end{bmatrix} \\ & x \geq 0 \end{cases}$$

$$\Rightarrow \text{dual canonique}(\mathcal{D}) \quad \begin{cases} \max & v = (w^+, w^-) \begin{bmatrix} b \\ -b \end{bmatrix} \\ s.t. & (w^+, w-) \begin{bmatrix} A \\ -A \end{bmatrix} \leq c \\ & w^+, w^- \geq 0 \end{cases}$$

Other equivalent forms

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax{=}b \\ & x \geq 0 \end{cases}$$

$$\Rightarrow (\mathcal{D}) \quad \begin{cases} \max & v = (w^+ - w^-)b \\ s.t. & (w^+ - w^-)A \leq c \\ & w^+, w^- \geq 0 \end{cases}$$

set $w = w^+ - w^-$

Other equivalent forms

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax = b \\ & x \geq 0 \end{cases}$$

$$\Rightarrow (\mathcal{D}) \quad \begin{cases} \max & v = (w^+ - w^-)b \\ s.t. & (w^+ - w^-)A \leq c \\ & w^+, w^- \geq 0 \end{cases}$$

set $w = w^+ - w^-$

$$(\mathcal{D}) \quad \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \ \textbf{of arbitrary sign} \end{cases}$$

Sign table

min	**max**
primal	dual
dual	primal
variable ≥ 0	constraint $\leq$
variable $\lesseqgtr 0$	constraint $=$
variable ≤ 0	constraint $\geq$
constraint $\leq$	variable ≤ 0
constraint $=$	variable $\lesseqgtr 0$
constraint $\geq$	variable ≥ 0

Sign table

min	max
primal	dual
dual	primal
variable ≥ 0	constraint $\leq$
variable $\lesseqgtr 0$	constraint $=$
variable ≤ 0	constraint $\geq$
constraint $\leq$	variable ≤ 0
constraint $=$	variable $\lesseqgtr 0$
constraint $\geq$	variable ≥ 0

Writing the dual is automatic:

- the variables
- the objective function
- the constraints

Write the dual

Write the dual program of

$\max z = 4x_1 + 5x_2 + 2x_3$

$$
\begin{array}{ccccccc}
2x_1 & + & 4x_2 & & & = & 3 \\
 & & & & 2x_3 & \geq & 2 \\
3x_1 & + & x_2 & + & x_3 & \leq & 2 \\
 & & x_2 & + & x_3 & \leq & 1
\end{array}
$$

$x_1 \geq 0 \quad x_2 \leq 0 \quad x_3 \geq 0$

Write the dual

Write the dual program of

$\max z = 4x_1 + 5x_2 + 2x_3$

$$
\begin{array}{ccccccc}
2x_1 & + & 4x_2 & & & = & 3 \\
& & & & 2x_3 & \geq & 2 \\
3x_1 & + & x_2 & + & x_3 & \leq & 2 \\
& & x_2 & + & x_3 & \leq & 1
\end{array}
$$

$x_1 \geq 0 \quad x_2 \leq 0 \quad x_3 \geq 0$

$\min = 3w_1 + 2w_2 + 2w_3 + w_4$

$$
\begin{array}{ccccccccc}
2w_1 & & & + & 3w_3 & & & \geq & 4 \\
4w_1 & & & + & w_3 & + & w_4 & \leq & 5 \\
& & 2w_2 & + & w_3 & + & w_4 & \geq & 2
\end{array}
$$

$w_1 \lesseqgtr 0 \quad w_2 \leq 0 \quad w_3 \geq 0 \quad w_4 \geq 0$

Plan

Properties

Property

The dual of the dual is equivalent to the primal

Properties

Property

The dual of the dual is equivalent to the primal

verify on an example

$$\max z = 2x_1 + 3x_2 + 4x_3$$

$$\begin{array}{ccccccc} 2x_1 & + & x_2 & & & \leq & 3 \\ & & & & x_3 & \geq & 2 \\ 3x_1 & + & x_2 & + & x_3 & \leq & 2 \\ & & x_2 & & & \leq & 1 \end{array}$$

$$x_1, x_2 \geq 0, \qquad x_3 \leq 0$$

Properties

$$\max z = 2x_1 + 3x_2 + 4x_3$$

$$\begin{array}{rcrcl}
2x_1 + x_2 & & & \leq & 3 \\
& & x_3 & \geq & 2 \\
3x_1 + x_2 & + & x_3 & \leq & 2 \\
x_2 & & & \leq & 1
\end{array}$$

$$x_1, x_2 \geq 0, \qquad x_3 \leq 0$$

Properties

$$\max z = 2x_1 + 3x_2 + 4x_3$$

$$\begin{array}{rcrcl} 2x_1 + x_2 & & & \leq & 3 \\ & & x_3 & \geq & 2 \\ 3x_1 + x_2 & + & x_3 & \leq & 2 \\ x_2 & & & \leq & 1 \end{array}$$

$$x_1, x_2 \geq 0, \qquad x_3 \leq 0$$

$$\min = 3w_1 + 2w_2 + 2w_3 + w_4$$

$$\begin{array}{lcccccl} 2w_1 & + & w_3 & & & \geq & 2 \\ w_1 & + & w_3 & + & w_4 & \geq & 3 \\ w_2 & + & w_3 & & & \leq & 4 \end{array}$$

$$w_1 \geq 0; w_2 \leq 0; w_3 \geq 0; w_4 \geq 0$$

Properties

$$\max z = 2x_1 + 3x_2 + 4x_3$$

$$\begin{array}{rcrcl}
2x_1 + x_2 & & & \leq & 3 \\
 & & x_3 & \geq & 2 \\
3x_1 + x_2 & + & x_3 & \leq & 2 \\
x_2 & & & \leq & 1
\end{array}$$

$$x_1, x_2 \geq 0, \qquad x_3 \leq 0$$

$$\min = 3w_1 + 2w_2 + 2w_3 + w_4$$

$$\begin{array}{lclcll}
2w_1 & + w_3 & & & \geq & 2 \\
w_1 & + w_3 & + & w_4 & \geq & 3 \\
w_2 & + w_3 & & & \leq & 4
\end{array}$$

$$w_1 \geq 0; w_2 \leq 0; w_3 \geq 0; w_4 \geq 0$$

$$\max z = 2y_1 + 3y_2 + 4y_3$$

$$\begin{array}{rcrcl}
2y_1 + y_2 & & & \leq & 3 \\
 & & y_3 & \geq & 2 \\
3y_1 + y_2 & + & y_3 & \leq & 2 \\
y_2 & & & \leq & 1
\end{array}$$

$$y_1, y_2 \geq 0, \qquad y_3 \leq 0$$

Properties

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{cases} \qquad (\mathcal{D}) \quad \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \geq 0 \end{cases}$$

Weak duality theorem

For each pair of feasible solutions x of $(\mathcal{P})$ and w of $(\mathcal{D})$

$z = cx \geq wb = v$

Properties

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{cases} \qquad (\mathcal{D}) \quad \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \geq 0 \end{cases}$$

Weak duality theorem

For each pair of feasible solutions x of $(\mathcal{P})$ and w of $(\mathcal{D})$

$z = cx \geq wb = v$

fast proof

Properties

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{cases} \qquad (\mathcal{D}) \quad \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \geq 0 \end{cases}$$

Weak duality theorem

For each pair of feasible solutions x of $(\mathcal{P})$ and w of $(\mathcal{D})$

$z = cx \geq wb = v$

$$\begin{array}{ccccc} cx & \geq & wAx & \geq & wb \\ & \uparrow & & \uparrow & \\ & wA \leq c & & Ax \geq b & \\ & x \geq 0 & & w \geq 0 & \end{array}$$

Properties

$$(\mathcal{P}) \quad \left\{ \begin{array}{ll} \min & z = cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{array} \right. \qquad (\mathcal{D}) \quad \left\{ \begin{array}{ll} \max & v = wb \\ s.t. & wA \leq c \\ & w \geq 0 \end{array} \right.$$

Weak duality theorem

For each pair of feasible solutions x of $(\mathcal{P})$ and w of $(\mathcal{D})$

$z = cx \geq wb = v$

Consequence: what if one of them is not bounded?

And optimality?

Optimality certificate

If

$$z = cx = wb = v$$

for feasible solutions x of $(\mathcal{P})$ and w of $(\mathcal{D})$, then x and w are optimal.

And optimality?

Optimality certificate

If

$$z = cx = wb = v$$

for feasible solutions x of $(\mathcal{P})$ and w of $(\mathcal{D})$, then x and w are optimal.

Strong duality theorem

If $(\mathcal{P})$ has solutions and $(\mathcal{D})$ has solutions, then

$$cx^* = w^* b$$

Complementary slackness property

For the vitamins example

- write the primal with the slack variables (s_i)
- write the dual with the slack variables (t_i)
- find a primal optimal solution
- find a dual optimal solution
- write the pairs of variables (s_i, w_i) and (x_j, t_j)
- can you notice something?

Complementary slackness property

$$
\begin{array}{lllllllllllll}
\min z = 35x_1 + 30x_2 + 60x_3 + 50x_4 + 27x_5 + 22x_6 \\
x_1 & + & 2x_3 & + & 2x_4 & + & x_5 & + & 2x_6 & \geq & 9 & \color{red}{s_1} \\
x_2 & + & 3x_3 & + & x_4 & + & 3x_5 & + & 2x_6 & \geq & 19 & \color{red}{s_2} \\
x_i \geq 0 \text{ for all } i = 1, 2 \ldots 6
\end{array}
$$

Complementary slackness property

$$\min z = 35x_1 + 30x_2 + 60x_3 + 50x_4 + 27x_5 + 22x_6$$

$$\begin{array}{ccccccccccccl}
x_1 & + & 2x_3 & + & 2x_4 & + & x_5 & + & 2x_6 & \geq & 9 & & \color{red}{s_1} \\
x_2 & + & 3x_3 & + & x_4 & + & 3x_5 & + & 2x_6 & \geq & 19 & & \color{red}{s_2}
\end{array}$$

$x_i \geq 0$ for all $i = 1, 2 \ldots 6$

$$\max v = 9w_A + 19w_C$$

$$\begin{array}{cccccl}
w_A & & & \leq & 35 & \color{red}{t_1} \\
 & & w_C & \leq & 30 & \color{red}{t_2} \\
2w_A & + & 3w_C & \leq & 60 & \color{red}{t_3} \\
2w_A & + & w_C & \leq & 50 & \color{red}{t_4} \\
w_A & + & 3w_C & \leq & 27 & \color{red}{t_5} \\
2w_A & + & 2w_C & \leq & 22 & \color{red}{t_6}
\end{array}$$

$w_A \geq 0 \qquad w_C \geq 0$

Complementary slackness property

$$
\begin{array}{lcccccccccccl}
\min z = 35x_1 + 30x_2 + 60x_3 + 50x_4 + 27x_5 + 22x_6 \\
x_1 & + & 2x_3 & + & 2x_4 & + & x_5 & + & 2x_6 & \geq & 9 & & \color{red}{s_1} \\
x_2 & + & 3x_3 & + & x_4 & + & 3x_5 & + & 2x_6 & \geq & 19 & & \color{red}{s_2} \\
\end{array}
$$

$x_i \geq 0$ for all $i = 1, 2 \ldots 6$

$$
\begin{array}{lcccccc}
\max v = 9w_A + 19w_C \\
w_A & & & \leq & 35 & & \color{red}{t_1} \\
 & & w_C & \leq & 30 & & \color{red}{t_2} \\
2w_A & + & 3w_C & \leq & 60 & & \color{red}{t_3} \\
2w_A & + & w_C & \leq & 50 & & \color{red}{t_4} \\
w_A & + & 3w_C & \leq & 27 & & \color{red}{t_5} \\
2w_A & + & 2w_C & \leq & 22 & & \color{red}{t_6} \\
\end{array}
$$

$w_A \geq 0 \qquad w_C \geq 0$

primal solution: $x_5 = 5$; $x_6 = 2$; $z = 27 \times 5 + 22 \times 2 = 179$

dual solution: $w_A = 3$; $w_C = 8$; $z = 9 \times 3 + 19 \times 8 = 179$

Complementary slackness property

$$
\begin{array}{lllllllllllll}
\min z = 35x_1 + 30x_2 + 60x_3 + 50x_4 + 27x_5 + 22x_6 \\
x_1 & + & 2x_3 & + & 2x_4 & + & x_5 & + & 2x_6 & \geq & 9 & s_1 = 0; w_A = 3 \\
x_2 & + & 3x_3 & + & x_4 & + & 3x_5 & + & 2x_6 & \geq & 19 & s_2 = 0; w_C = 8 \\
x_i \geq 0 \text{ for all } i = 1, 2 \ldots 6
\end{array}
$$

$$
\begin{array}{llllllll}
\max v = 9w_A + 19w_C \\
w_A & & & \leq & 35 & t_1 = 32; & x_1 = 0 \\
 & & w_C & \leq & 30 & t_2 = 22; & x_2 = 0 \\
2w_A & + & 3w_C & \leq & 60 & t_3 = 30; & x_3 = 0 \\
2w_A & + & w_C & \leq & 50 & t_4 = 36; & x_4 = 0 \\
w_A & + & 3w_C & \leq & 27 & t_5 = 0; & x_5 = 5 \\
2w_A & + & 2w_C & \leq & 22 & t_6 = 0; & x_6 = 2 \\
w_A \geq 0 & & w_C \geq 0
\end{array}
$$

primal solution: $x_5 = 5$; $x_6 = 2$; $z = 27 \times 5 + 22 \times 2 = 179$

dual solution: $w_A = 3$; $w_C = 8$; $z = 9 \times 3 + 19 \times 8 = 179$

Complementary slackness property

Complementary slackness property

x^* optimal for $(\mathcal{P})$ and w^* optimal for $(\mathcal{D})$ verify

- the slack variable of a constraint of $(\mathcal{P})$ is zero
  OR
- the variable associated with this constraint in w^* is zero

likewise in the other way round

$$x_j t_j = 0 \text{ and } s_i w_i = 0$$

Proof

Complementary slackness property

$$x_j t_j = 0 \text{ and } s_i w_i = 0$$

$$z = cx = wAx = wb = v$$

1. $cx = wAx$ or $\underbrace{(c - wA)}_{\geq 0} \underbrace{x}_{\geq 0} = 0$

   $\sum \underbrace{(c - wA)_j}_{t_j \geq 0} \underbrace{x_j}_{x_j \geq 0} = 0$ or $x_j t_j = 0$

2. $wAx = wb$ or $\underbrace{w}_{\geq 0} \underbrace{Ax - b}_{\geq 0} = 0$

   $\sum \underbrace{w_j}_{w_j \geq 0} \underbrace{(Ax - b)_j}_{s_j \geq 0} = 0$ or $w_j s_j = 0$

Complementary slackness property

interest Knowing an optimal solution x^* of $(\mathcal{P})$, then y^* can be found by applying the complementary slackness property (thus proving optimality of x^*)

Complementary slackness property

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try on an example

$\max z = x_1 + x_2$

$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$

$x_1, x_2 \geq 0$

with $x_1 = 2$ and $x_2 = 2$

Complementary slackness property

$\max z = x_1 + x_2$

$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$

with $x_1 = 2$ and $x_2 = 2$

Complementary slackness property

$\max z = x_1 + x_2$

$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$

with $x_1 = 2$ and $x_2 = 2$

$\min 20y_1 + 6y_2 + 2y_3$

$$\begin{array}{rcrcrcl} 4y_1 & + & 2y_2 & & & \geq & 1 \\ 5y_1 & + & y_2 & + & y_3 & \geq & 1 \end{array}$$

Complementary slackness property

$\max z = x_1 + x_2$

$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$

$\min 20y_1 + 6y_2 + 2y_3$

$$\begin{array}{rcrcrcl} 4y_1 & + & 2y_2 & & & \geq & 1 \\ 5y_1 & + & y_2 & + & y_3 & \geq & 1 \end{array}$$

with $x_1 = 2$ and $x_2 = 2$

$\Rightarrow s_1 \neq 0 \qquad s_2 = 0 \qquad s_3 = 0 \qquad t_1 = 0 \qquad t_2 = 0$

Complementary slackness property

$\max z = x_1 + x_2$

$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$

$\min 20y_1 + 6y_2 + 2y_3$

$$\begin{array}{rcrcrcl} 4y_1 & + & 2y_2 & & & \geq & 1 \\ 5y_1 & + & y_2 & + & y_3 & \geq & 1 \end{array}$$

with $x_1 = 2$ and $x_2 = 2$

$\Rightarrow s_1 \neq 0 \qquad s_2 = 0 \qquad s_3 = 0 \qquad t_1 = 0 \qquad t_2 = 0$

$\Rightarrow y_1 = 0 \qquad 2y_2 = 1 \qquad y_2 + y_3 = 1$

Complementary slackness property

$\max z = x_1 + x_2$

$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$

$\min 20y_1 + 6y_2 + 2y_3$

$$\begin{array}{rcrcrcl} 4y_1 & + & 2y_2 & & & \geq & 1 \\ 5y_1 & + & y_2 & + & y_3 & \geq & 1 \end{array}$$

with $x_1 = 2$ and $x_2 = 2$

$\Rightarrow s_1 \neq 0 \qquad s_2 = 0 \qquad s_3 = 0 \qquad t_1 = 0 \qquad t_2 = 0$

$\Rightarrow y_1 = 0 \qquad 2y_2 = 1 \qquad y_2 + y_3 = 1$

$\Rightarrow y_1 = 0 \qquad y_2 = 0.5 \qquad y_3 = 0.5$

A small philosophy of Duality

What is the interest of the three theorems of duality

- Weak duality: to make the **proof of optimality**
- Complementary slackness: to find an optimal solution of the dual knowing an optimal solution of the primal
- Strong duality: **guaranties** that an optimality proof is feasible (using duality)