III. 2.1. Finite-difference estimate of the second derivative.

taylor expansion:

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + O(h^4)$$

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + O(h^4)$$

Adding

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + h^2 f''(x_0) + O(h^4)$$

$$\therefore f''(x_0) = \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} + O(h^2)$$

III. 2.2. Second-derivative estimate on non-equidistant grids:

$h_1 = x_i - x_{i-1}$ $\quad\quad$ $h_2 = x_{i+1} - x_i$

$$f(x_i + h_2) = f(x_i) + h_2 f'(x_i) + \frac{h_2^2}{2} f''(x_i) + \frac{h_2^3}{6} f'''(x_i) + O(h_2^4) \quad \text{— I}$$

$$f(x_i - h_1) = f(x_i) - h_1 f'(x_i) + \frac{h_1^2}{2} f''(x_i) - \frac{h_1^3}{6} f'''(x_i) + O(h_1^4) \quad \text{— II}$$

$$(h_1^3 \times \text{I}) + (h_2^3 \times \text{II}) \Rightarrow$$

$$= \cancel{h_1^3 f(x_i+h_2)} = h_1^3 f(x_i) + h_1^3 h_2 f'(x_i) + \frac{h_1^3 h_2^2}{2} f''(x_i)$$

$$+ \frac{h_1^3 h_2^3}{6} f'''(x_i) + O(h_1^3 h_2^4)$$

$$+ h_2^3 f(x_i) - h_2^3 h_1 f'(x_i) + \frac{h_2^3 h_1^2}{2} f''(x_i) - \frac{h_2^3 h_1^3}{6} f'''(x_i) + O(h_2^3 h_1^4)$$

$$= (h_1^3 + h_2^3) f(x_i) + (h_1^3 h_2 - h_2^3 h_1) f'(x_i) + \left(\frac{h_1^3 h_2^2 + h_2^3 h_1^2}{2}\right) f''(x_i)$$

$$+ \left(\frac{h_1^3 h_2^3 - h_2^3 h_1^3}{6}\right) f'''(x_i) + O(\Delta x^4)$$

Epiclon

P.T.O

$$\Delta x^4 = \cancel{h_1^3 h_2^4 + h_1^4 h_2^3} = h_1^4 h_2^4 \qquad f(x_i + h_2) = f(\cancel{x_i} + x_{i+1} - \cancel{x_i}) = f(x_{i+1})$$
$$f(x_i - h_1) = f(\cancel{x_i} - \cancel{x_i} + x_{i-1}) = f(x_{i-1})$$

$$\Rightarrow f''(x_i) = \frac{2h_1^3}{h_1^3 h_2^2 + h_2^3 h_1^2} f(x_{i+1}) - \frac{2h_1^3 + 2h_2^3}{h_1^3 h_2^2 + h_2^3 h_1^2} f(x_i)$$
$$+ \frac{2h_2^3}{h_1^3 h_2^2 + h_2^3 h_1^2} f(x_{i-1}) - \frac{2h_1 h_2 (h_1^2 - h_2^2)}{h_1^3 h_2^2 + h_2^3 h_1^2} f'(x_i) + O\left(\frac{\Delta x^4}{h_1^3 h_2^2 + h_2^3 h_1^2}\right)$$

$$\Rightarrow f''(x_i) = \frac{2h_1^3}{h_1^2 h_2^2 (h_1 + h_2)} f(x_{i+1}) + \frac{2h_2^3}{h_1^2 h_2^2 (h_1 + h_2)} f(x_{i-1}) - \frac{2(h_1^3 + h_2^3)}{h_1^2 h_2^2 (h_1 + h_2)} f(x_i)$$
$$- \frac{2h_1 h_2 (h_1^2 - h_2^2) f'(x_i)}{h_1^2 h_2^2 (h_1 + h_2)} + O\left(\frac{h_1^4 h_2^4}{h_1^2 h_2^2 (h_1 + h_2)}\right)$$

$$\therefore f''(x_i) = \frac{2h_1}{h_2^2 (h_1 + h_2)} f(x_{i+1}) - \frac{2h_2}{h_1^2 (h_1 + h_2)} f(x_{i-1}) - \frac{2\cancel{(h_1 + h_2)}(h_1^2 - h_1 h_2 + h_2^2) f(x_i)}{h_1^2 h_2^2 \cancel{(h_1 + h_2)}}$$
$$- \frac{2h_1 h_2 \cancel{(h_1 + h_2)}(h_1 - h_2) f'(x_i)}{h_1^2 h_2^2 \cancel{(h_1 + h_2)}} + O\left(\frac{h_1^2 \times h_2^2}{h_1 + h_2}\right)$$

$$f''(x_i) = \frac{2h_1}{h_2^2 (h_1 + h_2)} f(x_{i+1}) - \frac{2h_2}{h_1^2 (h_1 + h_2)} f(x_{i-1}) - \frac{2(h_1^2 - h_1 h_2 + h_2^2)}{h_1^2 h_2^2} f(x_i)$$
$$- \frac{2(h_1 - h_2)}{h_1 h_2} f'(x_i) + O\left(\frac{h_1^2 h_2^2}{h_1 + h_2}\right)$$

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