Assignment 2

draft version 0.1   07/11/2025

Politecnico di Torino - Master of Science in Communications Engineering

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Assignment 2

- Exercise 1 - Properties of m-sequences: pt. X
- Exerxise 2 - Properties of Gold sequences: pt. X
- Exercise 3 - 8-PSK constellations: pt. X
- Question 1 - m-sequence property: pt. X
- Question 2 - 8-QAM: pt. X

Properties of m-sequences: pt. X

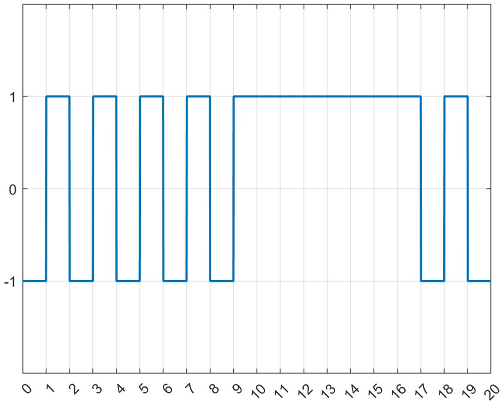
Properties of m-sequences

You must design a randomizer based on an LFSR with $m = 10$ cells.

- Choose the primitive polynomial.
- Draw the LFSR.

Sequence

- Choose a random seed and write it. (Note: I used 1010101010, use a different seed.)
- Generate the entire m-sequence of length $N = 2^m - 1$ bits.
- Plot the first 20 bits of the bipolar sequence $(0 \rightarrow -1$ and $1 \rightarrow +1)$
- Comment the result.



Note: Since your starting seed is different, your figure will vary accordingly.

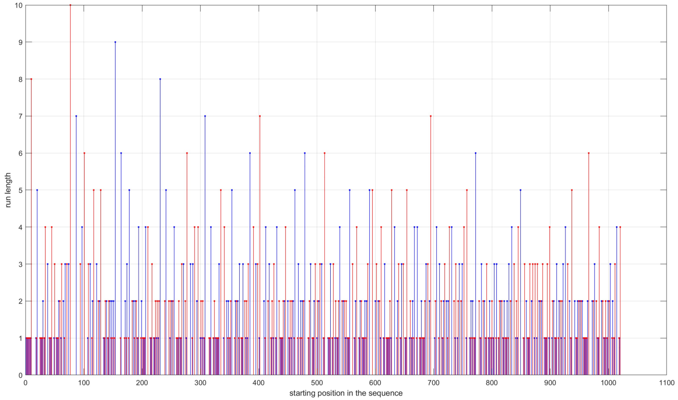
Number of zero/ones, number of transitions

Consider the entire sequence with length $N = 2^m - 1$.

- Write a table with N_1 and N_0 (number of bits equal to 1 and 0), N_T and N_{NT} (number of transitions and no transitions, view the sequence as periodic.)
- Check if the properties of an ideal binary sequences are verified and discuss the result.

Number of zero/ones, number of transitions

- Write a table with the values of $N_R(T)$ (total number of runs, $N_R(i)$ (number of runs of length i, $N_{R0}(i)$ and $N_{R1}(i)$ (number of runs of zeros and ones of length i)
- Plot the run lengths vs. their starting point in the sequence (use different colors for 0 and 1 runs)
- Compare against the run properties of an ideal binary random sequence and discuss the results.



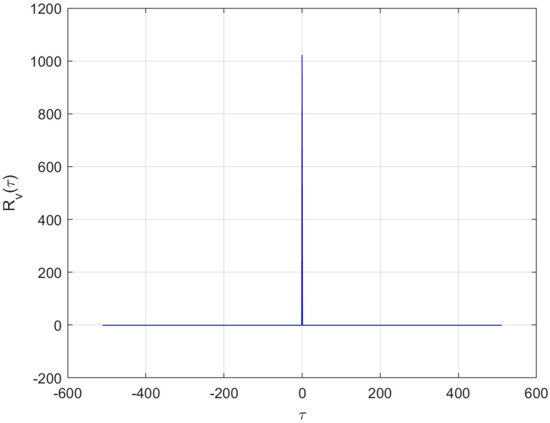
Note: Since your starting seed is different, your figure will vary accordingly.

Autocorrelation

Consider the entire bipolar sequence and compute the autocorrelation (view the sequence as periodic)

$$R_{v'}(\tau) = \sum_{i=0}^{N-1} v'(i)v'(i-\tau) \qquad -\frac{(N-1)}{2} \leq \tau \leq +\frac{(N-1)}{2}$$

- Plot the autocorrelation
- Compare its properties with those of an ideal binary random sequences.


1200
1000
800
600
400
200
0
-200
-600
-400
-200
0
200
400
600
$R_v(\tau)$
τ

Truncated sequences -MPSL

- Consider $1 \leq z \leq 200$
- For each value of z cancel the last z bits from the entire sequence to obtain a truncated sequence v_z of $N - z$ bits
- Compute the autocorrelation function of the bipolar truncated sequence v'_z
- Compute the Maximum Peak Side Lobe (MPSL) of v'_z

$$MPSL(z) = \max_{\tau \neq 0} R_{v'_z}(\tau)$$

- Plot $MPSL(z)$ vs. z
- Comment the results

Truncated sequences - seed

- Fix $z = 23$
- Consider the truncated sequence v_z of $N - z$ bits
- Compute the autocorrelation function of the bipolar truncated sequence v'_z
- Plot it
- Repeat and plot with the same z but another starting seed
- Comment the results

MATLAB

LFSR (change from 7 to 10 cells)

```
L=7; % number of cells

Nb=2^L-1; % m-sequence length

pnSequence = comm.PNSequence('Polynomial',[7 3 0], ...
        'SamplesPerFrame',Nb,'InitialConditions',[1 0 1 0 1 0 1]);
    x1 = pnSequence()';    % binary m-sequence
```

autocorrelation

```
x1b=2*x1-1; % bipolar sequence

R=ifft(fft(x1b).*conj(fft(x1b)));   %non-normalized periodic autocorr.

R=fftshift(R);
```

Properties of Gold sequences: pt. X

Gold sequences: construction

- An m-sequence c_1 generated by a Linear Feedback Shift Registers (LFSR) with m cells and primitive polynomial $p_1(D)$.
- Another m-sequence c_2 generated by a Linear Feedback Shift Registers (LFSR) with m cells and primitive polynomial $p_2(D)$ paired to $p_1(D)$.
- The set of Gold codes is made by the $(2^m + 1)$ sequences obtained as:

$$\begin{cases} c_1 \\ c_2 \\ c_1 + T^i(c_2) & 0 \leq i < 2^m - 1 \end{cases}$$

where T^i is the cyclic shift by i positions and $+$ is the binary sum (ex-or) in GF(2).

Gold sequences: properties

Basic properties:

- Sequence length (period):
$$N = 2^m - 1$$
- Number of Gold sequences:
$$2^m + 1$$
- cross-correlation property: three possible values
$$\begin{cases} -t \\ -1 \\ t-2 \end{cases}$$

$t = 2^{\frac{m+1}{2}} + 1$ for m odd, $t = 2^{\frac{m+2}{2}} + 1$ for m even

Gold sequences: Details

Let α be a primitive element of $\mathrm{GF}(2^m)$ and $p_1(D)$ the minimal polynomial of α. The paired polynomial $p_2(D)$ is the minimal polynomial of α^t where:

- $t = 2^{\frac{m+1}{2}} + 1$ for m odd
- $t = 2^{\frac{m+2}{2}} + 1$ for m even

Equivalently, the sequence c_2 is obtained by sampling the (periodic) sequence c_1 every t bits:

$$u_2(n) = u_1(nt) \qquad\qquad 0 \leq n < 2^m - 1$$

Gold sequences: Polynomials

m	N	p_1	p_2
7	127	[7 4 0]	[7 1 0]
10	1023	[10 3 0]	[10 8 3 2 0]

Gold sequences: Matlab

Matlab Gold function

```
m=7;
L=2^m-1;
goldseq = comm.GoldSequence('FirstPolynomial','x^7+x^4+1',...
'SecondPolynomial','x^7+x+1',...
'FirstInitialConditions',[0 0 0 0 0 0 1],...
'SecondInitialConditions',[0 0 0 0 0 0 1],...
'Index',1,'SamplesPerFrame',L);
x1 = goldseq()';
x1b=1-2*x1;
```

Index

Index = -2 $\Rightarrow c_1$
Index = -1 $\Rightarrow c_2$
Index = i $(0 \leq i < 2^m - 1) \Rightarrow c_1 + T^i(c_2)$

Definitions

Given a set $\{\underline{c}_i\}_{i=1}^{K}$ of K bipolar sequences of period N we define:
The cyclic autocorrelation function

$$r_{c_i'}(\tau) = \sum_{n=1}^{N} c_i'(n) c_i'(n+\tau) \quad 0 \leq \tau < N$$

The cyclic crosscorrelation function

$$r_{c_i' c_j'}(\tau) = \sum_{n=1}^{N} c_i'(n) c_j'(n+\tau) \quad 0 \leq \tau < N$$

Definitions

Given a set $\{\underline{c}'_i\}_{i=1}^{K}$ of K bipolar sequences of period N we define

- the maximal out-of-phase cyclic auto-correlation magnitude

$$r_A = \max\left\{\left|r_{c'_i}(\tau)\right| \quad 1 \le \tau \le N-1 \quad 1 \le i \le K\right\}$$

- the maximal cyclic cross-correlation magnitude

$$r_C = \max\left\{\left|r_{c'_i c'_j}(\tau)\right| \quad 0 \le \tau \le N-1 \quad 1 \le i,j \le K \quad i \neq j\right\}$$

- the maximal cyclic correlation magnitude

$$r_M = \max\{r_C, r_A\}$$

Welch bound

It holds for complex sequences of symbols with unitary magnitude:

$$r_M \geq N \sqrt[2s]{\frac{1}{KN-1}\left[\frac{KN}{\binom{N+s-1}{s}}-1\right]} \qquad \forall s \in \mathbb{Z}^+ = \{1,2,3,.....\}$$

(you can stop at s=10, if the root argument is negative set it to 0)

It is often simplified as:

$$r_M \geq N\sqrt{\frac{K-1}{KN-1}}$$

that, for large K, can be approximated to:

$$\sqrt{N}$$

(Since we are working with binary bipolar sequences, use ceil.)

Sidelnikov bound

It holds for binary bipolar sequences:

$$r_M \geq \left\lceil \sqrt{\left[(2s+1)(N-s) + \frac{s(s+1)}{2} - \frac{2^s N^{2s+1}}{K(2s)!\binom{N}{s}}\right]} \right\rceil$$

$$\forall s \in \mathbb{N} = \{0, 1, 2, 3, ...\} \qquad 0 \leq s < \frac{2N}{5}$$

(you can stop at s=10, if the root argument is negative set it to 0)

Since the Sidelnikov bound applies to binary sequences, it is tighter than the Welch bound.

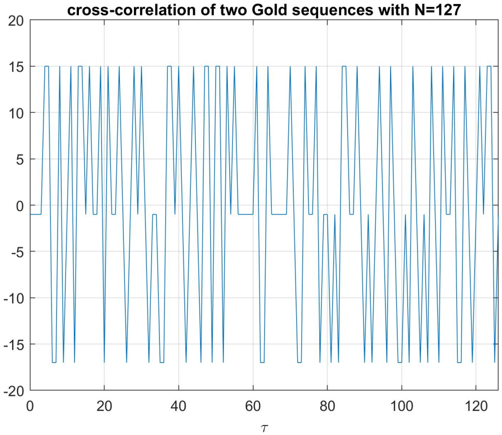
Exercise 2 - part 1

Fix $m = 7$

1. Generate a Gold sequences with Index $i \geq 0$
2. Compute and plot the cyclic auto-correlation
3. Repeat for $i < 0$
4. Comment the results

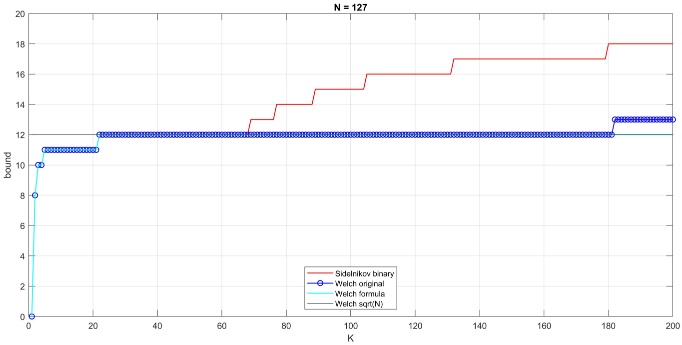
Exercise 2 - part 2

5. Generate two Gold sequences with two different (randomly extracted) Indexes $i \geq 0$
6. Compute and plot their cyclic cross-correlation
7. Generate two different m-sequences with the same length
8. Compute and plot their cyclic cross-correlation
9. Comapre the two results and comment the results


cross-correlation of two Gold sequences with N=127
20
15
10
5
0
-5
-10
-15
-20
0
20
40
60
80
100
120
τ

Exercise 2 - part 3

9. Plot the three versions of the Welch bound and the Sidelnikov bound for $N = 127$ and $1 \leq K \leq 200$
10. Consider the entire set of $K = 129$ Gold sequences and compute r_M (explain how you computed its value).
11. On the bound figure, add a point corresponding to r_M.
12. Comment the result


N = 127
bound
K
Sidelnikov binary
Welch original
Welch formula
Welch sqrt(N)
0
2
4
6
8
10
12
14
16
18
20
0
20
40
60
80
100
120
140
160
180
200

Exercise 2 - part 4

13. Starting from the Gold set of $K = 129$ sequences, reduce the length of the sequences to $N = 126$. Compute r_M (explain how you computed its value).
14. Compute and plot the bounds for $N = 126$ and $K = 129$.
15. Add r_M to the plot.
16. Comment the result.

8-PSK constellation: pt. X

8-PSK analytic curves

- Given an 8-PSK constellation, assign a Gray labeling.
- Take the signal labelled by 000 as a reference and compute the distance profile and the multiplicity values.
- In the presentation write down the expression of the bit error rate union bound and asymptotic approximation.
- Plot the bit error rate union bound and asymptotic approximation vs. E_b/N_0 between 0 and 16 dB
- Comment the result

8-PSK simulation

- Write a program to compute the 8-PSK BER by simulation (at least 100 wrong bits per simulation point)
- Add to the previous figure the BER simulated curve for E_b/N_0 between 0 and 12 dB (or less if the simulation is too slow)
- Comment the result

4-PSK and 8-PSK

- Add to the figure the 4-PSK BER exact analytic curve
- Comment the result and explain the trade-off between 4 and 8-constellations

Prove that for an m-sequence

- $N_1 = N_0 + 1$
- The longest run of zero has length $m - 1$ and the longest runs of ones has length m
- The out-of-phase autocorrelation is equal to -1 (use the linearity of the LFSR)

8-QAM: pt. X

Consider an 8-QAM constellation obtained as a subset of a 16-QAM

- Among all possible subsets, which is the best QAM in terms of BER performance? Call this constellation 8-QAM1
- Consider a sub-optimal 8-QAM constellation (8-QAM2) and analytically estimate the asymptotic gain of 8-QAM1 against 8-QAM2 (use the analytical approximation)
- Compare analytically 8-QAM1 and 8-PSK. Which constellation is better? Comment the result.

Note: no simulations are required for this question

Final version assigned on XXX
Delivery by

- XXX, 11.59 PM: +2 points
- XXX, 11.59 PM: +1 point
- XXX, 11.59 PM: 0 points
- Later: not accepted

Changes