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Controlled contractive sets for low-complexity constrained control

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Abstract— **The design of explicit constrained control is relatively simple when a controlled contractive set is available. However, the complexity of the explicit controller will depend on the complexity of the controlled contractive set. The ability to design a low complexity controlled contractive set is therefore desirable. Most methods for finding controlled contractive sets either assume the use of a constant linear state feedback, or exploit reachable set computations. In the first case, the assumption of a constant linear state feedback is restrictive (as controllers for constrained linear systems, such as MPC, are typically piecewise affine), while in the second case the complexity of the controlled contractive set may be very high.**

Initial results on the construction of low complexity controlled contractive sets without assuming linear state feedback were reported in the recent literature. The present paper extends these results, including the ability to handle more general classes of system dynamics. The paper develops a method for finding a controlled contractive set of a specified complexity, allowing for a trade-off between the complexity of the set and its volume.

I. Introduction

Model Predictive Control (MPC) has been used in industry for decades and the area of application is wide nowadays. Standard MPC is based on online solutions of optimization problems. Due to its computational complexity, its application is limited to systems with sufficiently slow dynamics which are not safety critical [1]. Explicit MPC provides an answer to these limitations of standard MPC, by formulating the MPC problem as multi-parametric problem. Instead of solving the optimization problem online, it can be solved offline and the optimal control law can be given as piecewise affine (PWA) functions of the present state [2]. Therefore the online MPC computation is transformed to the simple evaluation of a PWA function [3]. This allows implementation on simple hardware with high sampling rate. However, as the problem size increases, the number of regions of the explicit solution and the memory required for storing the explicit solution increases rapidly. This limits the use of explicit MPC to a system with modest number of states and short prediction horizons. Thus, the complexity reduction in explicit Model Predictive Control is recognized as a big challenge.

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There has been significant research going on to find approximate solutions in order to simplify explicit solutions. A simpler solution by directly approximating the control law is formulated e.g. by polynomial functions in [4] and by patchy constructions in [5]. Vertex Control approaches can be utilized for simpler controller designs. In the vertex control approach, an admissible input for each vertex of the feasible region is used for controller design [6]. A common way to do this would be to calculate the feasible region for an MPC formulation with guaranteed stability. However, the calculation of feasible regions for MPC may also be computationally complex [7].

A simpler approach is using explicit constrained control with the help of a controlled contractive set. In this case, the complexity of the explicit solution depends on the contractive set complexity. Therefore, finding a low complexity contractive set is essential for this approach to obtain low complexity explicit constrained control. The procedure in [8] converges to the maximal controlled contractive set with a specified contraction factor. However, the complexity of the resulting set may be very high. In [1], a non-iterative procedure for obtaining low-complexity contractive set is proposed and the contractive set is used in a systematic manner to design a controller for a system with state and input constraints. An approach to handle oscillatory modes given sufficient contractive dynamics is also proposed. However, the approach is not generally applicable to systems with oscillatory modes, and is not applicable to identical modes in series. Furthermore, the contractive set obtained in [1] is of fixed complexity, without any means of trading off complexity versus the size of the contractive set. The main purpose of this paper is to describe a flexible approach to obtain a low complexity contractive sets enabling the design of low complexity explicit constrained control, which also handles identical and oscillatory modes in series and allows a trade-off between the size of the controlled contractive set and the complexity of the set.

Section 2 describes how to find a contractive set using an optimization based approach. In section 3, different solutions are merged together in order to find a larger contractive set. In section 4, two methods are introduced to simplify complex contractive sets, the first method introduces a circumscribed ellipsoid technique to simplify the contractive set while the second method directly removes hyperplanes in order to obtain a low complexity contractive set. In section 5, polytopic contractive sets are considered for controller design, as polytopic contractive sets are natural starting points for designing explicit constrained controllers. Section 6 provides examples of these approaches and conclusions are drawn in section 7.

II. Optimization based Calculation for Controlled Contractive Sets

Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

with state and input constraints given by $\mathcal{X} = \{x|H_x x \leq h_x\}$ and $\mathcal{U} = \{u|H_u u \leq h_u\}$ respectively.

$Definition$ 1: A compact polytopic set $\mathcal{P} \in \mathcal{X}$ with the origin in its interior is called controlled γ-contractive, for a given $\gamma \in [0, 1)$ if for all $x \in \mathcal{P}$ there exists an $u \in \mathcal{U}$ such that $Ax_k + Bu_k \in \gamma\mathcal{P}$.

Thus, a compact polytopic set $\mathcal{P} = \{x \in \mathcal{R}^n|Fx \leq f\}$ is contractive if $\forall x_k \in P, \exists u_k \in \mathcal{U}$ such that

$$F(Ax_k + Bu_k) \leq \gamma f \tag{2}$$

In the following an optimization based approach for the construction of contractive sets will be described. To avoid the computational complexity of having to calculate the volume of the polytopic set in the optimization, the volume of the largest ellipsoid inscribed in the polytopic set is used as the objective function. The procedure allowing the characterization of the inscribed ellipsoid of maximum volume will be addressed in the next subsection. For the volume of the inscribed ellipsoid to be bounded, the set $\mathcal{P}$ itself needs to be bounded, and additional considerations are necessary to ensure that the set is also controlled contractive with the required contraction factor. These issues will be addressed in subsequent subsections.

A. Maximum Volume Ellipsoid inscribed in a set

Consider the set $\mathcal{P}$ described by m linear inequalities as $\mathcal{P} = \{x \in \mathcal{R}^n|F_i x \leq f_i, \forall i = 1, \cdots, m\}$ and the ellipsoid Ω given as $\Omega = \{Cy + d|\|y\| \leq 1\}$, where $C = C^T > 0$, C represents shape matrix and d is the center of the ellipsoid. According to [9], the volume of the ellipsoid Ω is proportional to $det(C)$. The maximum volume ellipsoid Ω inscribed in $\mathcal{P}$ can be obtained by solving the *maxdet* optimization problem:

$$\max_{C,d,F,f} log(det(C)) \tag{3}$$

subject to

$$C = C^T > 0 \tag{4}$$

$$\begin{bmatrix} (f_i - F_i d)I & CF_i^T \\ F_i C & f_i - F_i d \end{bmatrix} \succeq 0, \forall i = 1, \cdots, m \tag{5}$$

B. Boundedness Constraints

Suppose that the maximum volume ellipsoid Ω inscribed in $\mathcal{P}$ from (3) was formulated and solved according to the previous discussion. However, this does not guarantee that contractive set obtained by such a formulation will be bounded as well. If the volume of the inscribed ellipsoid is finite (the *maxdet* problem is well posed), the set $\mathcal{P}$ is also bounded. However, we want to impose conditions to guarantee boundedness, which can be achieved by enforcing the inclusion $\mathcal{H} \supset \mathcal{P}$ for a predefined bounded set H. The inequalities defining $\mathcal{H}$ should be redundant with respect to the inequalities defining $\mathcal{P}$. The set $\mathcal{H}$ should contain the state constraints in $\mathcal{X}$, and include additional lax constraints if necessary to make $\mathcal{H}$ bounded. Let $\mathcal{H}$ be defined as:

$$\mathcal{H} = \{x \in \mathcal{R}^n|Hx \leq h\} \tag{6}$$

Then from [10] we know that $\mathcal{P}$ is contained in $\mathcal{H}$ (and hence is bounded), if and only if there exists a matrix M with non-negative elements such that

$$MF = H \tag{7}$$

$$Mf \leq h \tag{8}$$

C. Contractive Constraints

The set $\mathcal{P}$ is contractive if it fulfills Definition 1. We want to find $\mathcal{P} = \{x \in \mathcal{R}^n|Fx \leq f\}$ with a guaranteed contraction factor such that (2) is satisfied. The maximum contraction factor for a given pair (F, f) can be found by solving the following bi-level optimization problem:

$$\max_x \gamma^* \tag{9a}$$

subject to

$$Hx_k \leq h \tag{9b}$$

$$Fx_k \leq f \tag{9c}$$

$$\gamma^* = \min_{u,\gamma} \gamma \tag{9d}$$

subject to

$$F(Ax_k + Bu_k) \leq \gamma f \tag{9e}$$

$$H_u u_k \leq h_u \tag{9f}$$

Here the solution of the lower level problem imposes constraints on upper level problem. Replacing the lower-level problem by the corresponding KKT conditions [11], one obtains:

$$\max_{x,u,\gamma} \gamma^* \tag{10a}$$

subject to

$$Hx_k \leq h, Fx_k \leq f \tag{10b}$$

$$F(Ax_k + Bu_k) - \gamma f \leq 0 \tag{10c}$$

$$H_u u_k \leq h_u \tag{10d}$$

$$\lambda_a \times (F(Ax_k + Bu_k) - \gamma f) = 0 \tag{10e}$$

$$\lambda_b \times (H_u u_k - h_u) = 0 \tag{10f}$$

$$\nabla_{u,\gamma}\mathcal{L}(u, \gamma, \lambda) \leq 0 \tag{10g}$$

where $\times$ denotes element-by-element multiplication and the Lagrangian function $\mathcal{L}$ is given by

$$\mathcal{L} = \gamma^* + \lambda_a^T(FAx_k + FBu_k - \gamma h_x) + \lambda_b^T(H_u u_k - h_u) \tag{11}$$

Here λ_a and λ_b are Lagrangian multipliers for the inequality constraints. Equations (10e) and (10f) induces non-linearity in the system. We use binary variables $s \in \{0, 1\}$ to remove this non-linearity as explained in [12] to obtain following single level optimization problem:

$$\max_{x,u,\gamma,\lambda,s} \gamma^* \tag{12a}$$

subject to

$$Hx_k \leq h, Fx_k \leq f \tag{12b}$$

$$\lambda_a \geq 0, \lambda_b \geq 0 \tag{12c}$$

$$\lambda_a \leq M_a^\lambda s, \lambda_b \leq M_b^\lambda s \tag{12d}$$

$$F(Ax_k + Bu_k) - \gamma f \leq 0 \tag{12e}$$

$$F(Ax_k + Bu_k) - \gamma f \geq -M_a^u(1 - s) \tag{12f}$$

$$H_u u_k \leq h_u \tag{12g}$$

$$H_u u_k - h_u \geq -M_b^u(1 - s) \tag{12h}$$

$$\nabla_{u,\gamma}\mathcal{L}(u, \gamma) = 0 \tag{12i}$$

Here M_a^λ,M_b^λ,M_a^u and M_b^u are diagonal matrices with appropriate dimensions and sufficiently large elements on the main diagonal. Equation (12) can be solved as a mixed integer linear problem (MILP). If a candidate contractive set is given, its maximum contraction factor can be found using the formulation described above.

D. Problem Formulation

Previous subsections imposed the constraints on $\mathcal{P}$. We want to maximize the volume of an ellipsoid Ω so that it is inscribed in the contractive set $\mathcal{P}$ such that boundedness and contractive constraints are satisfied. Thus we have a following optimization problem to obtain $\mathcal{P}$:

$$\max_{C,d,F,f,M} log(det(C)) \tag{13a}$$

subject to

$$C = C^T > 0 \tag{13b}$$

$$\begin{bmatrix} (f_i - F_i d)I & CF_i^T \\ F_i C & f_i - F_i d \end{bmatrix} \succeq 0 \forall i = 1, ...m \tag{13c}$$

$$MF = H \tag{13d}$$

$$Mf \leq h \tag{13e}$$

$$\max_{x,u,\gamma,\lambda,s} \gamma^* \tag{13f}$$

subject to

$$Hx_k \leq h, Fx_k \leq f \tag{13g}$$

$$\lambda_a \geq 0, \lambda_b \geq 0 \tag{13h}$$

$$\lambda_a \leq M_a^\lambda s, \lambda_b \leq M_b^\lambda s \tag{13i}$$

$$F(Ax_k + Bu_k) - \gamma f \leq 0 \tag{13j}$$

$$\lambda(F(Ax_k + Bu_k) - \gamma f) \geq -M_a^u(1 - s) \tag{13k}$$

$$H_u u_k \leq h_u \tag{13l}$$

$$H_u u_k - h_u \geq -M_b^u(1 - s) \tag{13m}$$

$$\nabla_{u,\gamma}\mathcal{L}(u, \gamma) = 0 \tag{13n}$$

The problem formulated above can be used to find the maximum volume ellipsoid Ω and the corresponding contractive set $\mathcal{P} = \{x \in \mathcal{R}^n | Fx \leq f\}$ inside which the ellipsoid Ω resides. The advantage of this method is that the complexity of contractive set is flexible. There is a trade-off between the complexity of the contractive set and the volume of the set. The complexity of the contractive set can be selected according to the requirements of the specific application. In the formulation above, the complexity of the set is changed simply by changing the number of rows of the pair $[F \;\; f]$.

Note that the problem formulated above can be simplified by removing the parameter f and normalizing the set $\mathcal{P}$ such that $\mathcal{P} = \{x \in \mathcal{R}^n | Fx \leq \mathbf{1}\}$, where $\mathbf{1}$ is the column vector of appropriate dimensions with all elements equal to 1. If the problem formulation is symmetric, this can be exploited to further simplify the computations. Example VI-A shows the effectiveness of this algorithm by comparing the results with the method described in [8]. The contractive sets of different complexities obtained for a system with identical modes in series are shown in example VI-B.

E. Solving the Formulated Problem

For the optimization problem in (13), the possible simplifications mentioned in the previous subsection notwithstanding, is a large optimization problem with highly non-convex constraints. Solving this problem to (provable) global optimality is therefore very difficult and computationally costly. Luckily, for this problem a solution need not be globally optimal in order to be useful. In this work, Particle Swarm Optimization (PSO) is used to find good solutions to the problem in (13).

The PSO technique described in [13] (well suited to large scale optimization problems compared with the initial PSO algorithm as it avoids premature convergence) is used to solve the problem formulated in previous subsection. It is a population based stochastic method. Particles, candidate solution, are moved around search space with certain position and velocity. At each iteration step, each particle is moved in the search space with a transition law depending on its current position, its best current position obtained so far, and the current global best position of the particle swarm. The random part in this combination allows escaping from local minima, the counterpart being the lack of local optimality guarantee of the solution.

Particle Swarm Optimization is an unconstrained optimization method. In order to apply PSO, (13) is converted into an unconstrained problem by appending suitable penalty functions accounting for constraint violations to the optimization criterion. The complexity of the algorithm can easily be tuned, as it is directly linked to the number of function evaluations, equal to the size of the swarm times the number of iterations. Typically, for a problem with 20 optimization variables, population size of 20 to 30 and 200 generations are enough to obtained good suboptimal solutions. In addition, while exploring the search space, the PSO typically finds multiple feasible solutions. Such solutions can be merged to obtain an enlarged contractive set, as explained in the next section.

III. Merging of Contractive Sets

Definition 2: The Minkowski sum of two sets S_1 and S_2 is given as:

$$S_1 \oplus S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S2\} \tag{14}$$

The volume of the contractive set can be increased by finding the convex hull of different solutions obtained by the Particle Swarm Optimization. Unfortunately, in most cases, this will not only increase the volume of the contractive set but will also increase the complexity of set.

Theorem 1: The convex hull of the contractive sets S_1 and S_2 with contraction factors γ_1 and γ_2 respectively, is γ contractive with a factor $\gamma = max(\gamma_1, \gamma_2)$.

Proof: Let $x_1(k)$ and $x_2(k)$ be points on the contractive sets S_1 and S_2 respectively. Denote the convex hull of S_1 and S_2 by S_0. Assume that contraction factors for S_1 and S_2 are γ_1 and γ_2 respectively. Then for $x_1(k), \exists u_1(k) \in U$ such that $x_1(k+1) \in \gamma_1 S_1$ and for $x_2(k), \exists u_2(k) \in U$ such that $x_2(k+1) \in \gamma_2 S_2$. Let $x_0(k)$ be a point obtained outside S_1 and S_2 but inside the convex hull. Then, $x_0(k)$ can be expressed as:

$$x_0(k) = \alpha_1 x_1(k) + \alpha_2 x_2(k) \tag{15}$$

where $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 \leq 1$, for some $x_1(k) \in S_1$ and $x_2(k) \in S_2$. Applying the input,

$$u_0(k) = \alpha_1 u_1(k) + \alpha_2 u_2(k) \tag{16}$$

the system dynamics (1) results in

$$\begin{aligned} x_0(k+1) &= A(\alpha_1 x_1(k) + \alpha_2 x_2(k)) + B(\alpha_1 u_1(k) + \alpha_2 u_2(k)) \\ &= \alpha_1(Ax_1(k) + Bu_1(k)) + \alpha_2(Ax_2(k) + Bu_2(k)) \\ &= \alpha_1 x_1(k+1) + \alpha_2 x_2(k+1) \end{aligned} \tag{17}$$

As $x_1(k+1) \in \gamma_1 S_1$ and $x_2(k+1) \in \gamma_2 S_2$, therefore

$$x_0(k+1) \in \gamma(\alpha_1 S_1 \oplus \alpha_2 S_2) \subseteq \gamma S_0 \tag{18}$$

IV. Simplification of Contractive Sets

Assume that the contractive set obtained after taking the convex hull of different solutions provided by the PSO is $\mathcal{P}_c$ with m hyperplanes and nv vertices. We want to minimize the number of hyperplanes so that the complexity of the formulated contractive set is reduced. As such a convex hull for higher dimensional systems can be complex, we would like to reduce its complexity. Naturally, this complexity reduction should be obtained without reducing the volume of the contractive set substantially, and with a minimal impact on the contraction factor. We introduce two methods to obtain such simplified contractive sets. The first method obtains the new set of vertices by using a circumscribed ellipsoid while the second method removes the hyperplanes so that the set $\mathcal{P}_c$ remains contractive with required contraction factor while still fulfilling the state constraints.

A. Circumscribed Ellipsoid

This method is based on the vertex operation for complexity reduction of the set. In this method we find a minimum volume ellipsoid $\Omega_c = \{\|Cx + d\| \leq 1\}$ with center at d, circumscribing the contractive set $\mathcal{P}_c$, where C is the shape matrix. The ellipsoid Ω_c can be found by minimizing the volume of ellipsoid containing all vertices of $\mathcal{P}_c$. This results in a *maxdet* problem as follows:

$$\min_{C,d} -log(det(C)) \tag{19a}$$

subject to

$$C = C^T > 0 \tag{19b}$$

$$\|CV_i + d\| \leq 1, \forall i = 1, \cdots, nv \tag{19c}$$

Here V_i is a vertex of $\mathcal{P}_c$, and nv is the number of vertices of $\mathcal{P}_c$. Once the circumscribed ellipsoid is found, we select among the points from V_i those lying on the ellipsoidal boundary and denote this set as V_E. Let p_{f1} denote the farthest point in euclidean distance from ellipsoidal boundary such that $p_{f1} \in V_i$, then maximum volume can be added by accumulating this point in V_E. It will not only increase the volume but will also maintain the shape of the contractive set as close to the original set as possible so that there is least impact on the contraction factor. In order to do so, the ellipsoid should be rotated and scaled such that the point p_{f1} will lie on the boundary of new ellipsoid. This rotation and scaling can be done by finding a plane normal to the vector from the center of the ellipsoid to p_{f1}. Adding to symmetrically placed 'artificial' points far from the origin on this normal plane and then fitting a new ellipsoid to circumscribe this new set of points, will give an ellipsoid with the point p_f on its boundary. However, we would like this new ellipsoid to have on its boundary also other vertices far from the boundary of the original ellipsoid. To achieve this, we take the projection Ω_p of the ellipsoid Ω_c and points V_i onto this plane and then check which point's p_{f2} projection is farthest from Ω_p. We add that point p_{f2} to the current point p_{f1} and find a new plane normal to both the vector from the origin to p_{f1} and the vector from the origin to p_{f2}, and project the ellipsoid Ω_c and points V_i onto this plane. This procedure is repeated for $n-1$ times to obtain $n-1$ points. The normal to these $n-1$ points will give us a specific direction of orientation of the ellipsoid. The rotation of the ellipsoid is then accomplished by adding two artificial symmetrically placed points p_{new} along this direction of orientation to the points in V, and then find a new minimum volume ellipsoid inscribing all the points. In this way the points points p_{fi} will lie on boundary of the new ellipsoid. The points from V_i which lie on boundary of the new ellipsoid are added to V_E. The set of points V_E forms the vertices of new simplified contractive set. We repeat the rotation procedure until we find a large enough contractive set. Algorithm 1 explains how do we proceed with this procedure.

Algorithm 1 Algorithm for simplification of a polytope

Input: A complex contractive set of n-dimensions, desired volume of simplified polytope vol_d, maximal complexity of simplified polytope $comp_{max}$

Output: A simplified contractive set with volume $vol_c > vol_d$ or complexity $comp_c < comp_{max}$

```
Compute the vertices V_i of the contractive set. Select V_E = [ ].
Find the minimum volume ellipsoid containing V_i.
LOOP Process
while continue == true do
    Points from V_i which lie on boundary of the ellipsoid are added to V_E. Set j = 0, P_f = [ ].
    Compute the new simplified set with vertices V_E, calculate its volume vol_c and complexity comp_c.
    if (vol_c > vol_d or comp_c > comp_max) then
        continue = false
    end if
    while j < n - 1 do
        Select the farthest point p_f from ellipsoidal boundary such that P_f = [P_f p_f]^T, j = j + 1.
        Find a plane normal to the vectors from the center of the ellipsoid to each of the points in set P_f.
        Take the projection of the ellipsoid and points V_i on the plane.
    end while
    Place two artificial points p_new symmetrically on the plane far away from the original points V_i on the line that is normal to the vectors from the center of the ellipsoid to each of the vertices in P_f.
    Find the minimum volume ellipsoid containing all these points.
    Remove the newly added artificial points p_new.
end while
```

B. Removing a hyperplane and increasing the volume

As the method described above focuses on vertex operation for complexity reduction, there may arise a case when it increases the complexity in terms of hyperplanes. Here, we propose a method for increasing the volume and simplification of a polytope by directly operating on the hyperplanes. We remove the hyperplane if contraction factor of the simplified set doesn't exceed the required contraction constraint. We select the hyperplane H_i to be removed by checking which hyperplane has to be pushed outwards least, in order to become redundant. The hyperplane H_i can be removed if new vertices V obtained by removing H_i are contractive and also fulfills the state and boundedness constraints. If the hyperplane H_i cannot become redundant then it is pushed maximum outwards to stretch the contractive set while fulfilling the state and contractive constraints. The procedure is repeated for all the hyperplanes. As this method directly removes the hyperplanes, it guarantees complexity reduction and increase in volume of the contractive set. Results obtained with this procedure along with the formulations of previous method are illustrated in example VI-A.

V. Designing a Controller with given Contractive Set

Assume the low complexity controlled contractive set with contraction factor γ is obtained by the methods mentioned above. Then, the control formulation can be given as:

$$\min_{u_k} x_{k+1}^T Q x_{k+1} + u_k^T R u_k \tag{20a}$$

subject to

$$x_{k+1} = Ax_k + Bu_k \tag{20b}$$

$$H_u u \leq h_u \tag{20c}$$

$$Fx_{k+1} \leq \gamma\alpha f \tag{20d}$$

where

$$\alpha = max\{F_i x_{k,i}/f_i\}, \forall i = 1, \cdots, m \tag{20e}$$

The contractiveness ensures that the α will reduce by factor γ at each time step, therefore the state trajectories will converge to the origin, which ensures stability of the system. The explicit solution to (20) can be obtained by solving it parametrically, with x_k and α as the parameters. The complexity of the explicit solution (in terms of the number of critical regions obtained), is given by the number of different combinations of constraints that may be active at the optimum, when the parameters are allowed to vary throughout the given parameter region. MPC formulations with a long prediction horizon will have a high number of constraints, and thus also typically a high number of possible combinations of different constraints. The prediction horizon for the formulation in (20) is 1, and thus the complexity of the explicit solution can be expected to be low.

VI. Examples

A. Spring Mass Damper system

Consider the spring mass damper system example mentioned in [1] with state representation given as:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -7 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{21}$$

The system is discretized with sampling time of 0.01 sec,

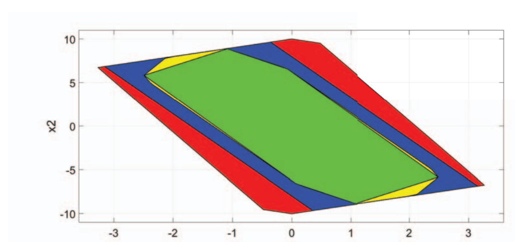

Fig. 1: Comparison of contractive sets obtained by method described in [8] and formulations explained in this paper

input and state constraints are given as $-10 \leq u \leq 10$ and $-10 \leq x_i \leq 10, \forall i = 1, 2$ respectively. First we find a contractive set by the method proposed in [8]. The set obtained after 200 iterations of the procedure described in [8] has the volume 71.7814 with 91 hyperplanes and a contraction factor $\gamma = 0.9796$ (Red set in figure 1). The set obtained by the formulations of section II has a contraction factor $\gamma = 0.9796$, the volume of 32.7378 and 6 hyperplanes (It is not shown in figure). The contractive set obtained by the method described in section III has 20 hyperplanes with volume of 40.9193 and contraction factor 0.9774 (Yellow set in figure 1). By simplifying the larger contractive set using method explained in section IV-A, the set obtained has a volume of 38.2663 with 0.9771 contraction factor and 6 hyperplanes (as shown in green in figure 1). By implementing method described in section IV-B, simplified contractive set obtained has 4 hyperplanes with volume 56.0461 and contraction factor 0.9796 (blue set in figure 1).

B. Identical Modes system

Consider a system with identical modes in series, i.e, a system with non-diagonalizable system matrix A given as:

$$x_{k+1} = \begin{bmatrix} 0.98 & 1 \\ 0 & 0.98 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0.6 & 0.9 \end{bmatrix} u \tag{22}$$

The contractive sets obtained for this system are shown

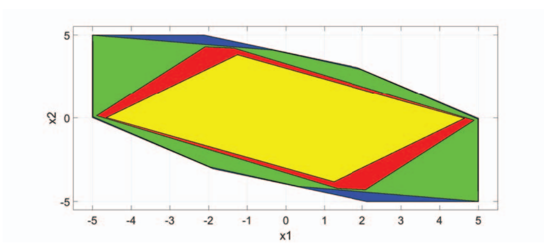

Fig. 2: Contractive sets of different complexities obtained for identical modes in series by using method described in II

in figure 2. The contractive set obtained by the method explained in [8] is shown in blue, while the green, red and yellow sets are obtained using the optimization based approach in this paper, when specifying complexities of 8,6 and 4 hyperplanes respectively. It can be clearly seen that by decreasing complexity of a set, we also lose volume. Therefore, there is a trade-off between the volume of the contractive set and its complexity.

C. Higher dimensional system with the controller design

Consider the following system:

$$x_{k+1} = \begin{bmatrix} 0.98 & 0 & 0 \\ 0.53 & 0.98 & 0 \\ -0.65 & 0.52 & 0.98 \end{bmatrix} x_k + \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \\ 0 & 0.3 \end{bmatrix} u_k \tag{23}$$

Input and state constraints are given as $-1 \leq u_i \leq 1, \forall i = 1,2$ and $-1 \leq x_j \leq 1, \forall j = 1, \cdots, 3$ respectively. The contraction factor is chosen to be 0.975. The explicit solution for a MPC with prediction horizon of 10 has 862 regions, and the total volume of the feasible region is 6.77. The contractive set obtained by the method described in [8] comprises of 98 hyperplanes with the volume of 6.4014. The contractive set for the system as explained in sections II is obtained with complexity of 12 constraints and volume 2.2924. The contractive set is then enlarged using the method explained in section III by considering 4 different solutions obtained by method of section II-E. The enlarged set has the volume of 4.3582 with 64 hyperplanes. This set is simplified and enlarged further using the technique described in section IV-B, where the simplified set has the volume of 4.7513 with 21 hyperplanes. The controller is designed by method explained in section V. By using the set obtained from section IV-B formulations, the number of regions comes out to be 84, whereas the number of regions obtained using the set formulated by method in [8] comes out to be 277. Clearly, there is a reduction in volume, but complexity (number of regions) has reduced significantly for the explicit controller.

VII. Conclusion

A novel method for finding controlled contractive sets has been described in this paper. An optimization based approach is used to find the contractive sets, subject to boundedness and contractiveness constraints. Multiple contractive sets can subsequently be merged to obtain a larger contractive set. Two techniques have been discussed to further simplify the set. The first technique utilizes a circumscribed ellipsoid to find a reduced complexity contractive set, while the second technique proposes operations on the hyperplanes to reduce the complexity and increase the volume. Finally, numerical examples shows the efficiency of the proposed method.

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