Linear Matrix Inequalities

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What are LMIs?

- A set of matrix-valued inequalities that are linear (or affine) in a set of variables
- Many design/analysis problems in control engineering can be formulated as LMIs
  - Natural connection to quadratic Lyapunov functions, energy storage functions, etc.
- Can be used to formulate problems involving several matrix variables
  - Different structures can be imposed on the matrix variables

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LMIs

- Unite many previous results in a common framework
  - H_∞ control design
  - H_2 control design
  - Mixed H_∞/ H_2 control design
  - Pole placement
  - Robust MPC
  - ...

- These will not be presented (in any detail) here, see references if interested

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Definite matrices and sign-definite quadratic functions

- A real-valued, square, symmetric matrix is said to be positive definite if

$$x^T Q x > 0 \; \forall x \neq 0$$

  - Positive semi-definite with ≥ instead of >
  - Extension to complex-valued matrices is of less interest here

- Common to write $Q > 0$ ($Q \geq 0$) to indicate that a matrix is positive (semi-)definite.
  - Occasionally ≻ (≽) is used to distinguish from sign-definiteness of the elements of Q.
  - $P > Q$ means $(P - Q) > 0$
- If Q is positive definite, $P = -Q$ is negative definite

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Definite matrices

- Any square matrix W can be decomposed into a symmetric and a skew-symmetric part

$$W = \left(\frac{W + W^T}{2}\right) + \left(\frac{W - W^T}{2}\right)$$

- Only the symmetric part contributes to the value of a quadratic function
  - I.e., when considering quadratic functions, it is non-restrictive to only consider symmetric matrices

Some LMI basics

- Basic structure of an LMI

$$F(p) = F_0 + \sum_{i=1}^{m} p_i F_i > 0$$

- Several LMIs combine into one big LMI by simple diagonal augmentation

$$F(p) > 0, G(p) > 0, \cdots$$

is equivalent to

$$\begin{bmatrix} F(p) & 0 & 0 \\ 0 & G(p) & 0 \\ 0 & 0 & \ddots \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$

Types of LMI problems

- LMI feasibility problem
  - Is $\dot{x} = Ax$ stable?
  - Valid Lyapunov function $V(x) = x^T P x$, if there exists $P > 0$ such that $A^T P + PA < 0$
  - Formulated as an LMI:

$$\begin{bmatrix} A^T P + PA & 0 \\ 0 & -P \end{bmatrix} < 0$$

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Types of LMI problems

- Convex function minimization with LMI constraints
  - LMI constraints are <u>convex</u>, giving a convex overall problem
  - Consider the system

$$\dot{x} = Ax + Bw$$
$$z = Cx + Dw$$

H_∞ norm of transfer function from w to z calculated from

$$\min \gamma$$

$$\text{s.t.} \begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

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Software for LMIs

- LMIs have appeared in control theory since the '60s or '70s
- Lots of developments in the '90s
- 'Reasonably' user friendly software have made these techniques fairly accessible since the 2000s.

- LMItools – integrated into Matlab Robust Control Toolbox
  - Some illustrations of use in Skogestad & Postlethwaite, Ch. 12.

- YALMIP
  - http://users.isy.liu.se/johanl/yalmip/
  - Integrated with Matlab
  - Need semidefinite programming solvers. Some available as freeware
    - SeDuMi http://users.isy.liu.se/johanl/yalmip/pmwiki.php?n=Solvers.SEDUMI
    - SDPT3 http://users.isy.liu.se/johanl/yalmip/pmwiki.php?n=Solvers.SDPT3
    - MOSEK https://www.mosek.com/

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LMI tricks – change of variables

- New variables may sometimes 'linearize' the problem.
- State feedback design problem. Find $u = Kx$ such that the closed loop is stable. Need to find $P > 0$ and K such that
$$(A + BK)^T P + P(A + BK) < 0$$
- Not an LMI due to nonlinear terms with products of P and K.
- Pre- and postmultiplication with $Q = P^{-1}$ does not change sign definiteness (see next slide). Pre- and postmultiplication with Q gives
$$QA^T + AQ + QK^T B^T + BKQ < 0$$
- Define $L = KQ$ to obtain
$$QA^T + AQ + L^T B^T + BL < 0$$

which is an LMI in $Q > 0$ and L. Furthermore, since Q is full rank, P and K can always be recovered from Q and L.

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LMI tricks – congruence transform

- If W is full rank, sign definiteness is not changed by pre-multiplication with W^T and post-multiplication by W. I.e.,
$$Q > 0 \Leftrightarrow W^T Q W > 0 \text{ for full rank } W$$
- Consider
$$Q = \begin{bmatrix} A^T P + PA & PBK + C^T V \\ * & -2V \end{bmatrix} < 0$$
in matrix variables $P > 0, V > 0$ and K.
- Choose the following full-rank W:
$$W = \begin{bmatrix} P^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}$$
$$W^T Q W = \begin{bmatrix} XA^T + AX & BL + XC^T \\ * & -2U \end{bmatrix}$$
which is an LMI in $X = P^{-1}, U = V^{-1}$, and $L = KV^{-1}$

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LMI tricks – Schur complement

- The following are equivalent
$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0$$
and
$$\Phi_{22} < 0$$
$$\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^T < 0$$
- Riccati inequality: Given $Q \geq 0, R > 0$ find $P > 0$ such that
$$A^T P + PA + PBR^{-1}B^T P + Q < 0$$

From the Schur complement, and noting that $-R < 0$, this is equivalent to
$$\begin{bmatrix} A^T P + PA + Q & PB \\ * & -R \end{bmatrix} < 0$$

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LMI tricks – S-procedure

- Occasionally a criterion cannot be fulfilled globally – but we need it to be fulfilled in some region defined by the sign of some function.
- For example, we may want $F_0(x) \leq 0$ whenever $F_i(x) \geq 0$. This is fulfilled, provided

$$F_{aug}(x) = F_0(x) + \sum_{i=1}^{q} \tau_i F_i(x) \leq 0\,, \qquad \tau_i \geq 0$$

- Note that $F_{aug}(x) \geq F_0(x)$ wherever $F_i(x) \geq 0, \forall i$
- $F_{aug}(x)$ is linear in τ_i.
- The S-procedure is generally conservative, except when $q = 1$.

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S-procedure example

- Find $P > 0$ such that

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ * & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} < 0$$

whenever

$$z^T z \leq x^T C^T C x \;\Leftrightarrow \begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} C^T C & 0 \\ * & -I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0$$

- From the S-procedure, we get

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \tau C^T C & PB \\ * & -\tau I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} < 0$$

which is an LMI in $P > 0$ and $\tau \geq 0$.

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LMI tricks – Projection Lemma

- In some control problems, the following type of inequalities appear

$$\Psi(X) + G(X)\Lambda H^T(X) + H(X)\Lambda^\mathrm{T} G^T(X) < 0$$

where X and Λ are the matrix variables. $\Psi(.)$, $G(.)$ and $H(.)$ are normally affine functions of X.

- The above inequality is satisfied, for some X, if and only of

$$W_{G(X)}^T \Psi(X)\, W_{G(X)}<0$$
$$W_{H(X)}^T \Psi(X)\, W_{H(X)}<0$$

Where $W_{G(x)}$ and $W_{H(x)}$ are orthogonal complements to G and H, respectively, i.e.

$$W_{G(X)}\mathrm{G(X)} = 0, \text{rank}(W_{G(X)})\text{+rank}(G) = n$$
$$W_{H(X)}\mathrm{H(x)} = 0, \text{rank}(W_{H(X)})\text{+rank}(H) = n$$

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LMI tricks – Finsler's Lemma

- In some control problems, the following type of inequalities appear

$$\Psi(X) + G(X)\Lambda H^T(X) + H(X)\Lambda^\mathrm{T} G^T(X) < 0$$

where X and Λ are the matrix variables. $\Psi(.)$, $G(.)$ and $H(.)$ are normally affine functions of X.

- This is equivalent to the following two inequalities

$$\Psi(X) - \sigma G(X)G(X)^T < 0$$
$$\Psi(X) - \sigma H(X)H(X)^T < 0$$

for some real σ

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Example, Projection Lemma

- State feedback design problem

$$(A+BK)^T P + P(A+BK) < 0, P > 0$$

transformed to

$$QA^T + AQ + L^T B^T + BL < 0, Q > 0$$

- Eliminate L using projection lemma:

$$W_B^T(AQ + QA^T)W_B < 0, Q > 0$$
$$W_I^T(AQ + QA^T)W_I < 0, Q > 0$$

- However, since the identity matrix is full rank, the second inequality is superfluous (in this case).
- First inequality has a nice intuitive interpretation: stabilization requires the system to be stable in the subspace where the inputs have no effect.

Example, Finsler's Lemma

- State feedback design problem

$$(A+BK)^T P + P(A+BK) < 0, P > 0$$

transformed to

$$QA^T + AQ + L^T B^T + BL < 0, Q > 0$$

- Eliminate L using Finsler's lemma:

$$AQ + QA^T - \sigma BB^T < 0, Q > 0$$
$$AQ + QA^T - \sigma I < 0, Q > 0$$

- However, can always find σ to fulfill second inequality, if Q, σ fulfilling first inequality is found

Projection/Finsler's lemmas

- May allow for a linear formulation
- Gives 'smaller' problems, fewer variables for computation

- However, once X is determined, this must be used in the original formulation for calculating Λ.
  - (or Q and L in the example on previous slides)
- Skogestad & Postlethwaite warns against ill-conditioning in this two-step approach. If X is poorly conditioned, the calculation of Λ might be inaccurate.

SOME APPLICATIONS

Discrete-time state feedback stabilization

- With the quadratic LF $V(x_k) = x_k^T P x_k$, for $P > 0$, the Lyapunov difference inequality tells us that the system is stable provided
$$x_k^T P x_k \text{-} x_{k+1}^T P x_{k+1} > 0$$
- This corresponds to
$$P - (\mathrm{A} + \mathrm{BK})^T P(A + BK) > 0$$
- Using the Schur complement:
$$\begin{bmatrix} P & (A + BK)^T P \\ P(A + BK) & P \end{bmatrix} > 0$$
- Congruence transform with $W = \begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix}$ gives

$$\begin{bmatrix} P^{-1} & P^{-1}(A + BK)^T \\ (A + BK)P^{-1} & P^{-1} \end{bmatrix} > 0$$

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Discrete-time ...

- Change of variables: $Q = P^{-1}$, $\mathrm{L} = KP^{-1}$

$$\begin{bmatrix} Q & QA^T + L^T B^T \\ AQ + BL & Q \end{bmatrix} > 0$$

- LMI in $Q > 0$ and L.

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Robust feedback stabilization

- Assume that the plant is not precisely known, only known to be a member of the set

$$\begin{bmatrix} A \\ B \end{bmatrix} \in Co \begin{bmatrix} A_1 & \cdots & A_q \\ B_1 & \cdots & B_q \end{bmatrix}$$

- That is, any possible A and B can be found as a convex combination of the 'extreme elements' of the set:

$$A = \sum_{i=1}^{q} a_i A_i, B = \sum_{i=1}^{q} a_i B_i, \sum_{i=1}^{q} a_i = 1, a_i \geq 0$$

- Can a controller that stabilizes all possible plants be found?

Robust feedback stabilization

- A controller that stabilizes 'all possible' plants is found if the same Q and L solves the LMI's for all the q 'extreme plants'

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

- This follows directly from the linearity of the LMI in the model matrices, and the convexity of the set of possible plants
- Remark: The controller thus found will also stabilize the system if the plant model changes arbitrarily fast inside the specified model set.

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Parameter dependent Lyapunov functions

- Above, we found a constant controller and prove with a constant LF that the system is stable for all possible uncertainties in a polytopic set.
- What if we can allow the LF to depend on the uncertain parameters?
- Will here only consider stability verification for autonomous systems (‘A matrices only’).

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Parameter dependent Lyapunov functions

- Express the model as

$$A = A_0 + \sum_{i=1}^{q} \delta_i A_i$$

with known bounds $\delta_i \in [\underline{\delta}_i \; \overline{\delta}_i]$

and rate of variation $\dot{\delta}_i \in [\underline{\lambda}_i \; \overline{\lambda}_i]$

- Define the sets

$\Delta = \{\delta | \delta_i \in [\underline{\delta}_i \; \overline{\delta}_i] \forall i\}$, with Δ_0 the set of vertices of Δ

$\Lambda = \{\dot{\delta} | \dot{\delta}_i \in [\underline{\lambda}_i \; \overline{\lambda}_i] \forall i\}$, with Λ_0 the set of vertices of Λ

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Parameter dependent LFs

- Will consider LFs of the form

$$V(\delta, x) = x^T P(\delta) x, P(\delta) = P_0 + \sum_{i=1}^{q} \delta_i P_i$$

- Note that

$$\frac{dP(\delta)}{dt} = \dot{\delta}_1 P_1 + \dot{\delta}_2 P_2 + \cdots + \dot{\delta}_q P_q = P(\dot{\delta}) - P_0$$

- And therefore

$$\dot{V}(\delta, x) = x^T \left(A^T(\delta) P(\delta) + P(\delta) A(\delta) + P(\dot{\delta}) - P_0 \right) x$$

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Parameter dependent LFs

- The system is stable if there exists matrices $P_0, \cdots, P_q$ such that

$$A^T(\delta) P(\delta) + P(\delta) A(\delta) + P(\dot{\delta}) < P_0, \forall \delta \in \Delta_0, \forall \dot{\delta} \in \Lambda_0$$
$$P(\delta) > I, \forall \delta \in \Delta_0$$
$$A_i^T P_i + P_i A_i \leq 0, i = 1, \cdots, q$$

- Can be extended to
  - State feedback design
  - LPV controller design (if δ is available online)
  - See Scherer and Weiland for details

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Robustly stable ellipsoid for system with saturated inputs

- An ellipsoid can be defined in several ways. We use
$$\mathcal{E}(P) = \{x | xP^{-1}x \le 1\}$$

- We want to find a controller K such that
  - The system with the input $u = sat(Kx)$ is stable inside $\mathcal{E}(P)$
  - The system adheres to the constraints
$$X = \{x| -1 \le F_i x \le 1\}, \forall i = 1, \cdots, n$$
  - These conditions should hold for all plants in a polytopic uncertatinty description
$$A = \sum_{i=1}^{q} a_i A_i, B = \sum_{i=1}^{q} a_i B_i, \sum_{i=1}^{q} a_i = 1, a_i \ge 0$$

- Will model saturation using a Linear Differential Inclusion approach

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LDI approach to modelling saturated inputs

- Single input, there exist β and ν such that
$$sat(u) = \beta Kx + (1-\beta)\nu$$
$$u_l \le \nu \le u_u$$
$$0 \le \beta \le 1$$

- Two inputs:
$$u_{1l} \le \nu_1 \le u_{1u}$$
$$u_{2l} \le \nu_2 \le u_{2u}$$
$$sat(u) = \beta_1 \begin{bmatrix} K_1 x \\ K_2 x \end{bmatrix} + \beta_2 \begin{bmatrix} K_1 x \\ \nu_2 \end{bmatrix} + \beta_3 \begin{bmatrix} \nu_1 \\ K_2 x \end{bmatrix} + \beta_4 \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$$
$$\sum_{i=1}^{4} \beta_i = 1, \beta_i \ge 0$$

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LDI approach...

- m inputs: Define D_m as the set of diagonal matrices whose diagonal elements are either 0 or 1. For example, for $m = 2$
$$D_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
- Each element of D_m is denoted $E_j, j = 1,2,\cdots,2^m$
- Define also $E_j^- = I - E_j$, then

$$sat(u) \in Conv\{E_j Kx + E_j^- v\}$$

for some v such that $u_{il} \leq v_i \leq u_{iu}$

- That is, there exist β_i, v_i such that
$$\beta_j \geq 0, \sum_{j=1}^{2^m} \beta_j = 1$$
$$sat(u) = \sum_{j=1}^{2^m} \beta_j \left(E_j Kx + E_j^- v\right)$$

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Robustly stable ellipsoid...

- The ellipsoid $\mathcal{E}(P) = \{x | xP^{-1}x \leq 1\}$ is invariant provided
$$\begin{bmatrix} P & \{A_i P + B_i(E_j KP + E_j^- Y)\} \\ * & P \end{bmatrix} \geq 0,$$
$$\forall i = 1,\cdots,q \text{ and } \forall j = 1,\cdots,2^m$$

For discrete time dynamics

- State constraints are fulfilled, provided
$$\begin{bmatrix} 1 & F_i P \\ * & P \end{bmatrix} \geq 0, \forall i = 1,\cdots,n$$

Maximize $\det(P)$ subject to these constraints.
-Convex problem
-Maximizes the size of the stable ellipsoid.

- Input constraints (assuming $-u_{il} = u_{iu} = u_{i,max}$):
$$\begin{bmatrix} u_{i,max}^2 & Y_i \\ * & P \end{bmatrix} \geq 0, \forall i = 1,\cdots,m$$
Where Y_i is the ith row ot the matrix Y.
- The input constraint conditions ensure that $-u_{i,max} \leq v_i \leq u_{i,max}$, and thus that the saturated input is included in the LDI, for $v_i = H_i x$, where $H_i P = Y_i$.

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Literature

- There is a lot of literature on LMI based analysis and design – including many issues not covered here. Much information is contained in the following two references (both freely available for personal use):

- Book by S. Boyd et al.: http://www.stanford.edu/~boyd/lmibook/
- Lecture notes by Carsten Scherer and Siep Weiland
  http://www.imng.uni-stuttgart.de/mst/files/LectureNotes.pdf

- The more accessible presentation of the robust design with saturated inputs is probably in the thesis by Hoai-Nam Nguyen:
  http://tel.archives-ouvertes.fr/docs/00/78/38/29/PDF/Nguyen_Hoai_Nam_these_VF.pdf
  although many of the main contributions are from other authors (see Nguyen's thesis for details).

- A quick and readable introduction to LMIs is given in Chapter 12 of
  S. Skogestad, I Postlethwaite: Multivariable Feedback Control. Analysis and Design (2nd ed.) John Wiley & Sons, 2005.

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