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Computation of Piecewise Quadratic Lyapunov Functions for Hybrid Systems

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***Abstract*— This paper presents a computational approach to stability analysis of nonlinear and hybrid systems. The search for a piecewise quadratic Lyapunov function is formulated as a convex optimization problem in terms of linear matrix inequalities. The relation to frequency domain methods such as the circle and Popov criteria is explained. Several examples are included to demonstrate the flexibility and power of the approach.**

***Index Terms*— Linear matrix inequalities, Lyapunov stability, piecewise linear systems.**

I. Introduction

Construction of Lyapunov functions is one of the most fundamental problems in systems theory. The most direct application is stability analysis, but analogous problems appear more or less implicitly also in performance analysis, controller synthesis, and system identification. Consequently, methods for constructing Lyapunov functions for general nonlinear systems is of great theoretical and practical interest.

The objective of this paper is to develop a uniform and computationally tractable approach for stability analysis of a class of nonlinear systems with piecewise affine dynamics. Such systems arise naturally in many places: as hybrid control systems, as gain-scheduled systems, or as approximations of other nonlinear systems. For these systems, it is natural to consider Lyapunov functions that are piecewise quadratic. We state the search for continuous piecewise quadratic Lyapunov functions as a convex optimization problem in terms of linear matrix inequalities. The approach is extended to allow Lyapunov functions that have certain discontinuities, which turns out to be attractive for analysis of a class of hybrid systems. The use of piecewise quadratic Lyapunov functions appears to be a powerful extension of quadratic stability that also covers polytopic Lyapunov functions; see [1], [2], and the references therein. The technique presented here has been extended to treat performance analysis and optimal control problems in [3] and smooth nonlinear systems in [4]. This paper is based on [5]. Similar work has also been reported in [6].

II. Model Representation

We consider analysis of piecewise affine systems of the form

$$\dot{x}(t) = A_i x(t) + a_i, \qquad \text{for } x(t) \in X_i. \tag{1}$$

Here, $\{X_i\}_{i\in I} \subseteq \mathbf{R}^n$ is a partition of the state space into a number of closed (possibly unbounded) polyhedral cells with pairwise disjoint interior. The index set of the cells is denoted I. Let $x(t) \in \cup_{i\in I} X_i$ be a continuous piecewise C^1 function on the time interval $[0, T]$. We say that $x(t)$ is a trajectory of (1), if for every $t \in [0, T]$ such that the derivative $\dot{x}(t)$ is defined, the equation $\dot{x}(t) = A_i x(t) + a_i$ holds for all i with $x(t) \in X_i$. Note that for a given system there may be initial values such that a corresponding trajectory only exists for small T.

Let $I_0 \subseteq I$ be the index set for cells that contain origin, and let $I_1 = I \backslash I_0$. It is assumed that $a_i = 0$ for $i \in I_0$. For convenient notation, we introduce

$$\overline{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}.$$

Since the cells are polyhedra, we can construct matrices

$$\overline{E}_i = [E_i \quad e_i], \qquad \overline{F}_i = [F_i \quad f_i]$$

with $e_i = 0$ and $f_i = 0$ for $i \in I_0$ and such that

$$\overline{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \qquad x \in X_i, \ \ i \in I \tag{2}$$

$$\overline{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \overline{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}, \qquad x \in X_i \cap X_j, \ \ i, j \in I. \tag{3}$$

Here, the vector inequality $z \geq 0$ means that each entry of z is nonnegative.

Systems with switching vector fields require special attention when it comes to modeling and simulation. All simulations in this paper are performed in Omsim [7] with proper treatment of discrete events.

III. Quadratic Stability

In some cases, it is possible to prove stability of piecewise linear systems using a globally quadratic Lyapunov function $V(x) = x^T Px$. The computations are usually based on the following sufficient conditions.

Proposition 1: If there exists a matrix $P = P^T > 0$ such that $A_i^T P + PA_i < 0$, $i \in I$, then every trajectory of (1) with $a_i = 0$, $i \in I$ tends to zero exponentially. □

Quadratic stability of a family of linear system is attractive, since stability follows independently of cell partition and for a large class of switching schemes; see, e.g., [8]. Furthermore, the conditions of Proposition 1 are linear matrix inequalities in P, and verifying these conditions amounts to solving a convex optimization problem for which efficient software is publicly available [9], [10]. Current software is capable of treating several hundreds of variables in a matter of seconds.

It can sometimes be of interest to verify that no common solution P to the inequalities in Proposition 1 exists. This verification can be made by solving the following dual problem. If there exists positive definite matrices R_i, $i \in I$ satisfying

$$\sum_{i\in I} A_i^T R_i + R_i A_i > 0 \tag{4}$$

then the Lyapunov inequalities in Proposition 1 do not admit a solution $P = P^T > 0$.

IV. A Motivating Example

As a simple and illustrative example for the need to move beyond globally quadratic Lyapunov functions, consider the following nonlinear differential equation:

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \leq 0 \\ A_2 x, & \text{if } x_1 \geq 0 \end{cases}$$

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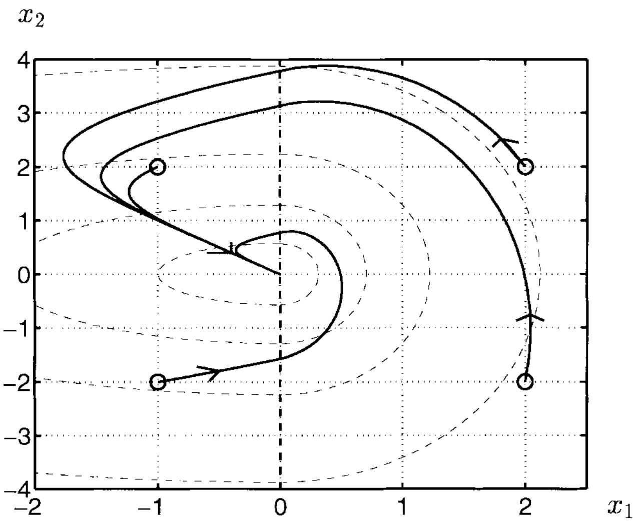


Fig. 1. Trajectories and Lyapunov function level surfaces for the motivating example. Initial values are indicated by circles.

where

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}.$$

By solving the dual problem stated in (4), one can verify that there is no globally quadratic Lyapunov function $V(x) = x^T P x$ that assures stability of the system. Still, the simulations shown in Fig. 1 indicate that the system is stable.

As an alternative to a globally quadratic Lyapunov function, it is natural to consider the following Lyapunov function candidate

$$V(x) = \begin{cases} x^T P x, & \text{if } x_1 \leq 0 \\ x^T P x + \eta x_1^2, & \text{if } x_1 \geq 0 \end{cases} \tag{5}$$

where P and $\eta \in \mathbf{R}$ are chosen so that both quadratic forms are positive definite. Note that the Lyapunov function candidate is constructed to be continuous and piecewise quadratic. The search for appropriate values of η and P can be done by numerical solution of the following linear matrix inequalities:

$$P = P^T > 0, \qquad A_1^T P + P A_1 < 0$$
$$P + \eta C^T C > 0, \qquad A_2^T (P + \eta C^T C) + (P + \eta C^T C) A_2 < 0$$

with $C = [1 \quad 0]$. One feasible solution is given by $P = \text{diag}\{1, 3\}$ and $\eta = 9$. The level surfaces of the computed Lyapunov function are indicated with dashed lines in Fig. 1.

It is instructive to compare this solution with what can be achieved using frequency domain methods, such as the circle and Popov criteria. Noting that $A_2 = A_1 - BC$ with $B = [-3 \ \ -21]^T$, we can rewrite the system equation as

$$\dot{x} = A_1 x - B\phi(Cx)$$
$$\phi(y) = \begin{cases} 0, & \text{if } y < 0 \\ y, & \text{if } y \geq 0. \end{cases}$$

Defining $G(s) = C(sI - A_1)^{-1} B$, we obtain the frequency condition $\text{Re}G(i\omega) > -1$ for the circle criterion and $\text{Re}[(1 + i\omega\eta)G(i\omega)] > -1$ for the Popov criterion. Inspection of the Nyquist and Popov plots in Fig. 2 reveals that stability follows from the Popov criterion but not from the circle criterion. The failure of the circle criterion comes as no surprise, as the circle criterion relies on the existence of a common Lyapunov function on the form $V(x) = x^T P x$ [11], which we know does not exist. The standard proof of the Popov criteria, on the other hand, uses the Lyapunov function $V(x) = x^T P x + 2\eta \int_0^{Cx} \phi(\sigma)\, d\sigma$.

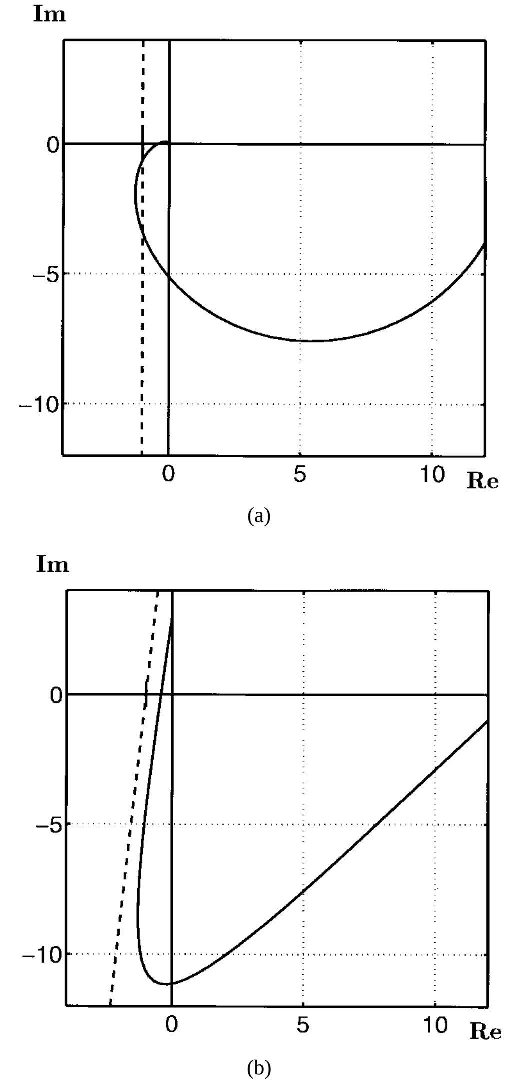


Fig. 2. The circle criterion (a) fails to prove stability. The Popov plot (b) is separated from -1 by a straight line of slope $1/\eta$. Hence stability follows.

For the simple switched system, this function evaluates to (5), used successfully in the numerical optimization above.

V. Piecewise Quadratic Stability

The conditions of Proposition 1 are stated globally. This is unnecessarily restrictive for analysis of piecewise affine systems, since the dynamics given by A_i is only valid within cell X_i. Consequently, it is sufficient to require that

$$x^T (A_i^T P + P A_i) x < 0, \qquad \text{for } x \in X_i. \tag{6}$$

One way to exploit this fact in the stability conditions is the so-called $\mathcal{S}$-procedure [12]. We thus construct matrices S_i such that $x^T S_i x \geq 0$ for $x \in X_i$ and obtain the following relaxed conditions for quadratic stability:

$$A_i^T P + P A_i + S_i < 0. \tag{7}$$

The intuition behind the method is as follows. Since $x^T S_i x \geq 0$ for $x \in X_i$, (7) clearly implies (6). If $x^T S_i x < 0$ for $x \notin X_i$, (7) may be simpler to satisfy than the Lyapunov inequalities in Proposition 1. In some cases, the $\mathcal{S}$-procedure is a nonconservative way to account for quadratic constraints [13]. Matrices for use in the $\mathcal{S}$-procedure can be constructed from the system description as follows. Let U_i

be a matrix with nonnegative entries and let $\overline{E}_i$ satisfy (2). Then, it follows that

$$x^T E_i^T U_i E_i x \geq 0, \qquad x \in X_i, \;\; i \in I_0$$
$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{E}_i^T U_i \overline{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0, \qquad x \in X_i, \;\; i \in I_1.$$

A next step in reducing conservatism is to allow the Lyapunov function to be piecewise quadratic. Consider a piecewise linear system with two cells X_j and X_k. Let the Lyapunov function be $V(x) = x^T P_j x$ for $x \in X_j$, and let the boundary between X_j and X_k be given by $f_{jk}^T x = 0$. Then, the Lyapunov function is continuous across the cell boundary if and only if the Lyapunov function has the form

$$V(x) = x^T P_k x = x^T (P_j + f_{jk} t_{jk}^T + t_{jk} f_{jk}^T) x, \qquad \text{for } x \in X_k$$

and for some $t_{jk} \in \mathbf{R}^n$. A more general parameterization of piecewise quadratic functions that are continuous across cell boundaries is $V(x) = x^T P_i x$ with

$$P_i = F_i^T T F_i$$

and F_i satisfies (3). The free parameters are now collected in the symmetric matrix T. Since the expression for P_i is linear in T, it will be possible to state the search for a piecewise quadratic Lyapunov function as a set of linear matrix inequalities. The constructed Lyapunov function will in general have the form

$$V(x) = \begin{cases} x^T P_i x, & x \in X_i, \;\; i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, & x \in X_i, \;\; i \in I_1. \end{cases} \tag{8}$$

This Lyapunov function combines the power of quadratic Lyapunov functions near an equilibrium point with the flexibility of piecewise linear functions in the large. Conditions for the existence of a piecewise quadratic Lyapunov function for the piecewise affine system (1) are formulated next.

Theorem 1: Consider symmetric matrices T and U_i and W_i, such that U_i and W_i have nonnegative entries, while

$$P_i = F_i^T T F_i, \qquad i \in I_0 \tag{9}$$
$$\overline{P}_i = \overline{F}_i^T T \overline{F}_i, \qquad i \in I \tag{10}$$

satisfy

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \qquad i \in I_0 \tag{11}$$
$$\begin{cases} 0 > \overline{A}_i^T \overline{P}_i + \overline{P}_i \overline{A}_i + \overline{E}_i^T U_i \overline{E}_i \\ 0 < \overline{P}_i - \overline{E}_i^T W_i \overline{E}_i \end{cases} \qquad i \in I_1. \tag{12}$$

Then every continuous piecewise $\mathcal{C}^1$ trajectory $x(t) \in \cup_{i \in I} X_i$, satisfying (1) for $t \geq 0$, tends to zero exponentially. □

Remark 1: The above stability conditions are linear matrix inequalities in T, U_i, and W_i. In many cases, U_i and W_i need not be full matrices. In the examples presented here, these matrices have nonzero entries only on their subdiagonals. □

The following example illustrates Theorem 1.

Example 1: Consider the piecewise linear system $\dot{x}(t) = A_i x(t)$ with the cell partition shown in Fig. 3. The system matrices are given by

$$A_1 = A_3 = \begin{bmatrix} -\epsilon & \omega \\ -\alpha\omega & -\epsilon \end{bmatrix}, \qquad A_2 = A_4 = \begin{bmatrix} -\epsilon & \alpha\omega \\ -\omega & -\epsilon \end{bmatrix}.$$

Letting $\alpha = 5$, $\omega = 1$, and $\epsilon = 0.1$, the trajectory of a simulation with initial value $x_0 = (-2, \;\; 0)^T$ moves toward the origin in a flower-like trajectory, as shown in Fig. 3. Using the enumeration of the cells indicated in Fig. 3, the vector inequalities characterizing the cells are given by the matrices

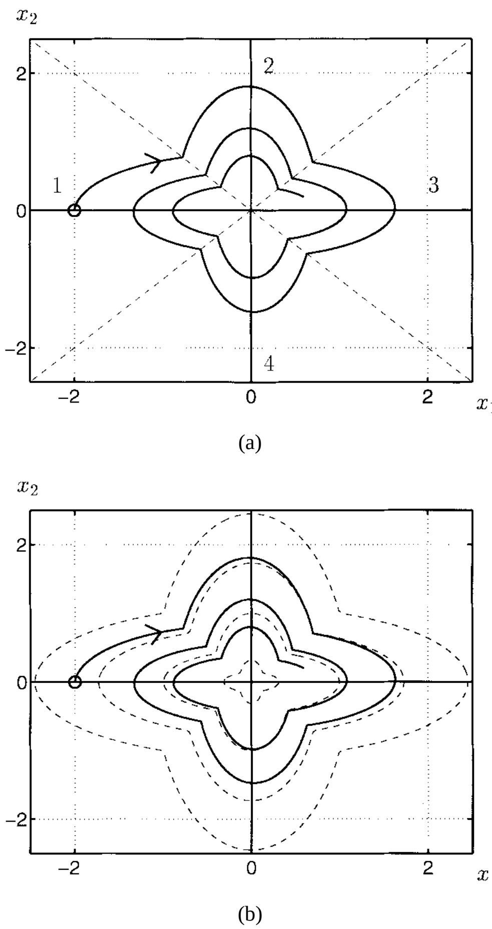


Fig. 3. (a) A simulated trajectory (full) and the cell boundaries (dashed). (b) The level curves of the Lyapunov function (dashed).

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and $F_i = [E_i^T, \;\; I]^T$, $i = 1, 2, 3, 4$. From the conditions for a continuous piecewise quadratic Lyapunov function stated in Theorem 1, we find $V(x) = x^T P_i x$ with

$$P_1 = P_3 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \qquad P_2 = P_4 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

The level surfaces of the computed Lyapunov function are shown in Fig. 3. □

VI. Hybrid Systems

Hybrid systems are systems which combine continuous dynamics and discrete events. Hybrid control systems arise when there is an interaction between logic-based devices and continuous dynamics and control. The piecewise affine systems considered so far are indeed hybrid systems, where abrupt changes in the dynamics may occur when the continuous state moves from one cell to another. In this section, we will enlarge the class of systems to include hybrid systems with hysteresis effect. These systems can be modeled by

$$\dot{x}(t) = f(x(t), i(t)), \qquad i(t) = \nu(x(t), i(t-)). \tag{13}$$

The differential equation models the continuous dynamics, while the algebraic equation models the state of the decision-making logic.

The discrete state $i(t) \in I$ is piecewise constant. The notation $t-$ indicates that the discrete state is piecewise continuous from the right. For a broad review of hybrid phenomena and associated models, we refer to the thesis [14]. So far, the discrete state has been playing the role of an index variable that keeps track of which cell the continuous state is currently evolving in. For systems with hysteresis, the discrete state will play a more crucial role. We give the following example.

Example 2: Fig. 4(a) shows a simulation of the system $\dot{x}(t) = A_{i(t)}x(t)$

$$i(t) = \begin{cases} 2, & \text{if } i(t-) = 1 \text{ and } f_{12}^T x(t) = 0 \\ 1, & \text{if } i(t-) = 2 \text{ and } f_{21}^T x(t) = 0 \end{cases} \tag{14}$$

with $i(0) = 1$, switching boundaries

$$f_{12} = [-10 \quad -1]^T, \qquad f_{21} = [2 \quad -1]^T$$

and system matrices

$$A_1 = \begin{bmatrix} -1 & -100 \\ 10 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 10 \\ -100 & 1 \end{bmatrix}.$$

The simulations shown in Fig. 4 indicate that the system is asymptotically stable. From the simulated trajectory of the system, it is also clear that it is not possible to find a Lyapunov function that disregards the influence of the discrete state. □

It is sometimes desirable to relax the requirement that the Lyapunov function should be continuous across cell boundaries. The dependence on the discrete state may even be discontinuous, if the value of the Lyapunov function decreases at the switching instants [15]. If the switching conditions for the discrete state can be expressed as affine equalities in the continuous state, it is possible to formulate the search for this type of Lyapunov functions as a linear matrix inequality (LMI) problem. This can be seen by the following simple argument. Let the Lyapunov function be $V(x) = x^T P_{i(t)} x$ and let the discrete state initially have the value j. Assume that the condition for the discrete state to change value from j to k is given by $f_{jk}^T x = 0$. Then, the requirement that the Lyapunov function should be decreasing at the switching instant, $x^T P_j x \geq x^T P_k x$, can be expressed as the linear matrix inequality in P_j, P_k, and t_{jk}

$$P_j - P_k + f_{jk} t_{jk}^T + t_{jk} f_{jk}^T \geq 0.$$

To derive stability conditions that can be solved numerically by LMI computations, we restrict the continuous dynamics of (13) to be piecewise affine, i.e.,

$$\dot{x}(t) = A_{i(t)} x(t) + a_{i(t)} \qquad \text{a.e.} \tag{15}$$

We let $I_0 \subseteq I$ be the set of indexes for which $x(t) = 0$ is admissible and let $I_1 = I \backslash I_0$. It is assumed that $a_i = 0$ for $i \in I_0$. To take into account that a discrete state may be admissible only for a subset of the continuous states, we construct matrices $E_{i(t)}$ and $\overline{E}_{i(t)}$ such that

$$E_{i(t)} x(t) \geq 0, \qquad i(t) \in I_0$$
$$\overline{E}_{i(t)} \begin{bmatrix} x(t) \\ 1 \end{bmatrix} \geq 0, \qquad i(t) \in I_1.$$

Furthermore, from the discrete dynamics of (13), we construct vectors f_{ij} and $\overline{f}_{ij}$ for $i, j \in I$ such that $f_{i,i} = 0$, $\overline{f}_{i,i} = 0$ for $i \in I$ and

$$f_{i(t-)i(t)}^T x(t) = 0 \qquad \forall t \tag{16}$$

$$\overline{f}_{i(t-)i(t)}^T \begin{bmatrix} x(t) \\ 1 \end{bmatrix} = 0 \qquad \forall t. \tag{17}$$

We let $\overline{P}_i = [I \quad 0]^T P_i [I \quad 0]$ for $i \in I_0$, and state the following result.

Theorem 2: Consider vectors t_{ij} and $\overline{t}_{ij}$, symmetric matrices U_i and W_i with nonnegative entries and symmetric matrices P_i and $\overline{P}_i$ such that

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \qquad i \in I_0 \tag{18}$$

$$\begin{cases} 0 > \overline{A}_i^T \overline{P}_i + \overline{P}_i \overline{A}_i + \overline{E}_i^T U_i \overline{E}_i \\ 0 < \overline{P}_i - \overline{E}_i^T W_i \overline{E}_i \end{cases} \qquad i \in I_1 \tag{19}$$

$$0 < \overline{P}_i - \overline{P}_j + \overline{f}_{ij} \overline{t}_{ij}^T + \overline{t}_{ij} \overline{f}_{ij}^T \qquad i \in I_1 \text{ or } j \in I_1 \tag{20}$$

$$0 < P_i - P_j + f_{ij} t_{ij}^T + t_{ij} f_{ij}^T \qquad i, j \in I_0 \tag{21}$$

where $i \neq j$. Then, every continuous, piecewise $\mathcal{C}^1$ trajectory $x(t)$ satisfying (15) tends to zero exponentially. □

Remark 2: By allowing nonstrict inequalities in (20) and (21), Theorem 1 can be seen as a special case of Theorem 2 where $\overline{f}_{ij} = \overline{f}_{ji}$, $\forall i, j \in I$. However, a formulation with nonstrict inequalities cannot be treated numerically as it stands. Inherent algebraic constraints must first be eliminated. Theorem 1 can be seen as the outcome of such an elimination. □

Theorem 2 can be applied directly to the switching system of Example 2.

Example 3: Consider again the switching system (14). We let

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} -10 & -1 \\ 2 & -1 \end{bmatrix}.$$

The LMI conditions of Theorem 2 have a feasible solution

$$P_1 = \begin{bmatrix} 17.9 & -0.89 \\ -0.89 & 179 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 739 & -38.1 \\ -38.1 & 91.8 \end{bmatrix}.$$

A simulated trajectory of the system and the corresponding value of the Lyapunov function are shown in Fig. 4. The discontinuities in the Lyapunov function occur at changes in the discrete state. □

VII. A Remark on Sliding Modes

Cell boundaries of piecewise affine systems may in general have regions where the trajectories in all adjacent cells are directed toward the boundary. With our definition, the trajectories are then well-defined only until the moment when they reach this region. Theorem 1 avoids the problem by only considering trajectories defined for all $t \geq 0$.

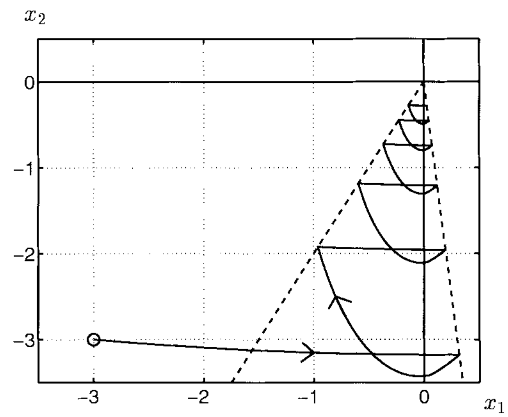
It is, however, possible to extend the trajectory definition further, using the concept sliding mode. Such definitions have been suggested by, for example, Filippov [16] and Utkin [17]. For piecewise affine systems, the approach of Filippov suggests the following definition of system trajectories. We say that a continuous piecewise $\mathcal{C}^1$ function $x(t) \in \cup_{i \in I} X_i$ is a trajectory of (1) if, for every t such that the derivative $\dot{x}(t)$ is defined, the differential inclusion

$$\dot{x}(t) \in \mathrm{Conv}_{j \in J}\{A_j x(t) + a_j\} \tag{22}$$

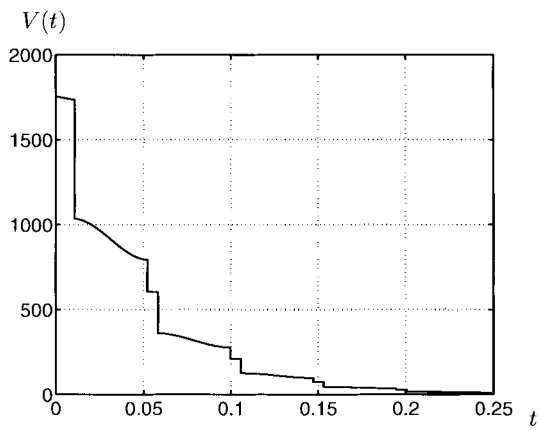
holds. Here, J is the set of all indexes j such that $x(t) \in X_j$ and Conv denotes the closed convex hull. By requiring the Lyapunov function to be decreasing for all trajectories of the form (22), the stability analysis can be extended to handle systems with sliding modes. This requirement on the Lyapunov function can be expressed as LMI's using similar arguments as before.

VIII. Conclusions

The search for piecewise quadratic Lyapunov functions for nonlinear and hybrid systems has been stated as a convex optimization problem in terms of linear matrix inequalities. The power of this



(a)



(b)

Fig. 4. (a) Sample trajectory of the hybrid system and (b) the corresponding value of the Lyapunov function.

approach appears to be very strong and the techniques can be generalized in a large number of directions, including performance analysis, global linearization, controller optimization, and model reduction. Some initial work in this direction has been reported in [3].

Appendix

Proofs

The proofs rely on the following lemma, which is standard in Lyapunov theory.

Lemma 1: Let $V(t)$ be decreasing and piecewise $\mathcal{C}^1$. If there exists α, β, and $\gamma > 0$ such that

$$\alpha|x(t)|^2 < V(t) < \beta|x(t)|^2 \tag{23}$$

$$\frac{d}{dt}V(t) \le -\gamma|x(t)|^2 \qquad \text{a.e.} \tag{24}$$

then $|x(t)|^2 \le \beta\alpha^{-1}e^{-\gamma t/\beta}|x(0)|^2$. □

Proof of Theorem 1: Consider the Lyapunov function candidate $V(x)$ defined by (8). By construction of the matrices $\overline{F}_i$ (3), P_i (9), and $\overline{P}_i$ (10), this function is continuous in x. Since $x(t)$ is assumed to be continuous and piecewise $\mathcal{C}^1$, so is $V(t)$. From the absence of affine terms in the Lyapunov function in an open neighborhood of the origin, we conclude that there exists a β such that the upper bound (23) of Lemma 1 holds. The inequalities $\alpha|x(t)|^2 < V(t)$ and $dV(t)/dt < \gamma|x(t)|^2$ follow from the conditions (11) and (12) after multiplication from the left and the right by x and $(x, 1)$, respectively.

Theorem 2: The proof of Theorem 2 follows similarly.

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