Approximate Explicit MPC using Bilevel Optimization

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Abstract— **A linear quadratic model predictive controller (MPC) can be written as a parametric quadratic optimization problem whose solution is a piecewise affine (PWA) map from the state to the optimal input. While this 'explicit solution' can offer several orders of magnitude reduction in online evaluation time in some cases, the primary limitation is that the complexity can grow quickly with problem size. In this paper we introduce a new method based on bilevel optimization that allows the direct approximation of the non-convex receding horizon control law. The ability to approximate the control law directly, rather than first approximating a convex cost function leads to simple control laws and tighter approximation errors than previous approaches. Furthermore, stability conditions also based on bilevel optimization are given that are substantially less conservative than existing statements.**

I. INTRODUCTION

This paper considers the implementation of an MPC controller for a linearly constrained linear system with a quadratic performance index. Standard practice is to compute the optimal control action in this case by solving a quadratic program at each time instant for the current value of the state. It was shown in [1]–[3] that this quadratic program can be posed as a parametric problem (pQP), where the parameter is the state of the system and that this pQP results in a piecewise affine function that maps the state to the optimal input; the so-called 'explicit solution'.

The motivation for computing the explicit solution is that the resulting piecewise affine function can be much faster and simpler to evaluate online than solving a quadratic program, which can in some cases lead to several orders of magnitude reduction in computation time making MPC applicable to very high-speed applications. The primary limitation of this approach is that the complexity, or number of affine pieces, of this explicit solution can grow very quickly with problem size.

In this paper, we propose an approximation approach that generates a low-complexity piecewise affine function directly from the optimal MPC formulation (i.e. without computing the optimal solution first). The approach is simple in that it proceeds by choosing a small number of states and then interpolates the optimal control action at these points. The key questions to be answered are then: Which points to select, how to certify that the sub-optimal controller is stabilizing and how to define and compute the resulting level of sub-optimality.

Several authors have proposed similar interpolated approximations that operate by exploiting the convexity of the optimal cost function. Given a finite set of sampled points X, these methods differ primarily in the manner in which they choose which points to interpolate to define the approximate control law, or equivalently how they partition the feasible set. In [4], the authors propose a recursive simplical partition, in [5], [6] the partition is formed as a result of incremental convex hull algorithms and in [7], [8] a box decomposition scheme is used. The common feature amongst these proposals is the method of choosing the set X, which is done in an incremental fashion by inserting at each step the state for which the error between the approximate cost function and the optimal one is the greatest. The main motivation for this is that these points can be found by solving convex problems. However, these approaches require that the entire optimal input sequence over the prediction horizon is approximated, rather than just the first step, which defines the control law in a receding-horizon controller. This requirement of approximating the entire optimal sequence leads to a very conservative test for approximation error.

This paper introduces a new method that approximates only the first step of the optimal control sequence and leaves the remainder defined implicity as the result of a secondary parametric optimization problem. The result is a decrease in the conservatism of the approximation error, which in general results in a significant decrease in approximation complexity. The cost of this improvement is that the optimization problems to be solved are no longer convex, but are indefinite *bilevel* quadratic optimization problems. Bilevel problems are those in which some of the optimization variables are constrained to be optimizers of a secondary optimization and are, even in the simplest case, NP-hard to solve. We show that the indefinite bilevel QPs required for this approach can be re-written as mixed-integer linear programs (MILPs) and hence solved using very efficient methods. The paper also introduces an improved test for stability for the resulting interpolated control laws over those given previously, which is also based on bilevel optimization.

Proposals have also been made in the literature to derive simpler approximate explicit control laws using methods other than interpolation. The reader is referred to the recent survey [9] for a complete review.

NOTATION

A *polyhedron* is the intersection of a finite number of halfspaces and a *polytope* is a bounded polyhedron. If V is a subset of $\mathbb{R}^d$, then the convex hull of V, $\operatorname{conv}(V)$ is the intersection of all convex sets containing V. If $V = \{v_0, \ldots, v_n\}$ is a finite set, then $\operatorname{conv}(V) = \{\sum_{i=0}^n v_i\lambda_i \mid \lambda_i \geq 0,\ \sum\lambda_i = 1\}$. The bold symbol $\mathbf{x}$ is used to describe an ordered sequence $\mathbf{x} := (x_0, \ldots, x_N)$, where x_i is the i^{th} element of $\mathbf{x}$.

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II. Background

The goal is to control the linear system

$$x^+ = Ax + Bu \ , \tag{1}$$

where the state x and input u are constrained to lie in the polytopic sets $\mathbb{X} = \{x \mid Fx \le f\} \subset \mathbb{R}^n$ and $\mathbb{U} = \{u \mid Gu \le g\} \subset \mathbb{R}^m$ respectively.

Consider the following semi-infinite horizon optimal control problem:

$$\begin{aligned} J^\star(x) := \min\ & J(\mathbf{x},\mathbf{u}) \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \ , \qquad \forall i = 0,\dots,N-1 \\ & x_i \in \mathbb{X} \ , \ u_i \in \mathbb{U} \ , \qquad \forall i = 0,\dots,N-1 \\ & x_N \in \mathbb{X}_N \ , \ x_0 = x \ , \end{aligned} \tag{2}$$

where $\mathbb{X}_N = \{x \mid Hx \le h\} \subset \mathbb{X}$ is a polytopic invariant set for the system $x^+ = Ax + B\nu(x)$ for some given linear control law $\nu : \mathbb{R}^n \mapsto \mathbb{R}^m$. We define $\mathscr{X} \subset \mathbb{R}^n$ to be the set of states x for which there exists a feasible solution to (2). The quadratic cost function J is defined as

$$J(\mathbf{x},\mathbf{u}) := \frac{1}{2} x_N^T Q_N x_N + \frac{1}{2}\sum_{i=0}^{N-1} u_i^T R u_i + x_i^T Q x_i \tag{3}$$

where the matrices $Q_N \in \mathbb{R}^{n\times n} \ge 0$ and $Q \subset \mathbb{R}^{n\times n} \ge 0$ and $R \in \mathbb{R}^{m\times m} > 0$ define the cost function.

If $\mathbf{u}^\star(x)$ is the optimal input sequence of (2) for the state x, and $u_0^\star(x)$ is the resulting receding horizon control law, then $J^\star$ is a Lyapunov function for the system $x^+ = Ax + Bu_0^\star(x)$ under the assumption that $V_N(x) = x^T Q_N x$ is a Lyapunov function for the system $x^+ = Ax + B\nu(x)$ and that the decay rate of V_N is greater than the stage cost $l(x,u) = u^T R u + x^T Q x$ within the set $\mathbb{X}_N$ [10].

The goal in this paper is to compute a PWA function of low complexity that approximates the control law $u_0^\star(x)$ as closely as possible while still guaranteeing stability.

III. Interpolated Control

We define an interpolated control law by choosing a finite set of distinct feasible states and then interpolating amongst the optimal control action at each of these states. In order to formalize this idea, we must first define the regions over which this interpolation will occur.

Definition 1 (Triangulation): A triangulation of a finite set of points $V \subset \mathbb{R}^n$ is a finite collection $T_V := \{S_0,\dots,S_L\}$ such that

- $S_i = \operatorname{conv}(V_i)$ is an $n-$dimensional simplex for some $V_i \subset V$
- $\operatorname{conv}(V) = \cup S_i$ and $\operatorname{int} S_i \cap \operatorname{int} S_j = \emptyset$ for all $i \ne j$
- If $i \ne j$ then there is a common (possibly empty) face F of the boundaries of S_i and S_j such that $S_i \cap S_j = F$.

There are various triangulations possible, most of which are compatible with the proposed approach. For example, the recursive triangulation developed in [4], [11] has the strong property of generating a simple hierarchy that can significantly speed online evaluation of the resulting control law. The Delaunay triangulation [12], which has the nice property of minimizing the number of skinny triangles, or those with small angles is a common choice for which incremental update algorithms are well-studied and readily available (i.e. computation of $T_{V\cup\{v\}}$ given T_V). A particularly suitable Delaunay triangulation can be defined by using the optimal cost function as a weighting term, which causes the resulting triangulation to closely match the optimal partition [5].

Given a discrete set of states V, we can now define an interpolated control law $\mu_{T_V} : \mathbb{R}^n \mapsto \mathbb{R}^m$.

Definition 2: If $V \subset \mathscr{X} \subset \mathbb{R}^n$ is a finite set and $T = \{\operatorname{conv}(V_1),\dots,\operatorname{conv}(V_L)\}$ is a triangulation of V, then the *interpolated control law* $\mu_T : \operatorname{conv}(V) \mapsto \mathbb{R}^m$ is

$$\mu_T(x) := \sum_{v\in V_j} u_0^\star(v)\lambda_v \ , \ \text{if } x \in \operatorname{conv}(V_j) \tag{4}$$

where $\lambda_v \ge 0$, $\sum \lambda_v = 1$ and $x = \sum_{v\in V_j} v\lambda_v$.

The following lemma describes the relevant properties of interpolated control laws.

Lemma 1: If $V \subset \mathscr{X} \subset \mathbb{R}^n$ is a finite set and T_V is a triangulation of V, then the interpolated control law μ_{T_V} is:

1) Feasible for all $x \in \operatorname{conv}(V)$:
   $\exists$ $\mathbf{x}$, $\mathbf{u}$ feasible for (2) such that $x_0 = x$ and $u_0 = \mu_{T_V}(x)$
2) An affine function in each simplex $\operatorname{conv}(V_i)$
3) Continuous

Our goal is to compute an interpolated control law μ_{T_V} for the system (1) that is as close to the optimal $u_0^\star$ as possible by sampling a set of points V such that T_V is of a pre-specified complexity. As proposed in various papers [4]–[8], we do this in an incremental fashion beginning from any inner approximation $\operatorname{conv}(V)$ of the feasible set $\mathscr{X}$ of (2) and the resulting initial triangulation $T_V = \{V_0,\dots,V_l\}$.

At each iteration of the algorithm, we maintain a point set V and the resulting triangulation T_V. This is then used to compute a state x_{T_V} that is a maximizer for some function $\gamma : \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ that measures the error between the optimal control law $u_0^\star$ and the interpolated one μ_{T_V}.

$$x_{T_V} := \underset{x\in\operatorname{conv}(V)}{\operatorname{arg\,max}}\ \gamma(\mu_{T_V}(x), u_0^\star(x)) \ . \tag{5}$$

The point set $V' := V \cup \{x_{T_V}\}$ and triangulation $T_{V'}$ are then updated to include this worst-fit point. This simple procedure repeats until some specified approximation error ε is achieved, or the complexity of the triangulation has exceeded some bound. The general method is given as Algorithm 1 below.

Remark 1: An initial inner approximation of the feasible set $\mathscr{X}$ can be computed by, for example, using the projection approximation approach proposed in [6].

Remark 2: Several methods are available to compute $T_{V\cup\{x_{T_V}\}}$ given the previous triangulation T_V in an incremental fashion [12].

The key questions we seek to answer here are how best to define the function γ and how to compute the maximization (5).

Algorithm 1 Construction of an interpolated control law

Require: A finite subset V of the feasible set of (2), an error function $\gamma : \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ and an approximation error ε or a complexity specification $COMP$.

Ensure: A point set V such that $|T_V| \leq COMP$ or $\max_x \gamma(\mu_{T_V}(x), u_0^\star(x)) \leq \varepsilon$.

1: Compute T_V
2: **repeat**
3: Compute a point $x_{T_V} \in \arg\max_x \gamma(\mu_{T_V}(x), u_0^\star(x))$
4: Set $err \leftarrow \gamma(\mu_{T_V}(x_{T_V}), u_0^\star(x_{T_V}))$
5: Update $V \leftarrow V \cup \{x_{T_V}\}$
6: Compute T_V
7: **until** $|T_V| \geq COMP$ or $err \leq \varepsilon$
8: Return V and T_V

IV. Error Computation

In this section we propose to define the error function as

$$\gamma(x) := \|u_0^\star(x) - \mu_{T_V}(x)\|_\infty \ , \tag{6}$$

which is natural for measuring the worst-case fit between two functions. The key requirement of using such a function is that we be able to solve the optimization problem (5). One immediate method of doing this is to first compute the optimal solution $u_0^\star(x)$ to the pQP (2) using one of a number of standard methods available [9]. This would provide an explicit PWA representation of $u_0^\star(x)$ and would then make the computation of (5) straightforward. However, it is likely that this optimal control law is too complex to be computed directly, and so here we aim to find an *implicit* representation of $u_0^\star(x)$. This section will outline how this can be done by writing (5) as a bilevel optimization problem, which can in turn be solved using a mixed-integer linear solver.

Bilevel Optimization

Bilevel optimization problems have been extensively studied in the literature, and the reader is referred to the recent survey [13] for background details. Bilevel problems are hierachical in that the optimization variables are split into upper y and lower z parts, with the lower level variables constrained to be an optimal solution to a secondary optimization problem.

$$\begin{aligned}
\min_{y} \quad & V_U(y,z) \\
\text{s.t.} \quad & G_U(y,z) \leq 0 \\
& z = \arg\min_{z} \quad \{V_L(y,\hat{z}) \ | \ G_L(y,\hat{z}) \leq 0\}
\end{aligned} \tag{7}$$

Remark 3: The bilevel formulation given here makes the implicit assumption that the optimizer of the lower-level problem is unique. This assumption clearly does not hold in general and there is a large literature available on how to deal with non-unique optimizers. However, for the purpose of computing (5), this assumption is valid in the following since the optimizer will be unique as long as the matrix R in (3) is positive definite.

Solution methods: Bilevel optimization problems are in general very difficult to solve. Even the simplest case where all functions are linear is NP-hard [14]. Several computational methods have been proposed for various types of bilevel optimization problems (see [13] for a survey), but for the purposes of this paper, the most relevant is that originally given in [15] for quadratic bilevel problems. The key observation is that if the lower level problem is convex and regular, then it can be replaced by its necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions, yielding a standard single-level optimization problem.

$$\begin{aligned}
\min_{y,z,\lambda} \quad & V_U(y,z) & \text{(8a)}\\
\text{s.t.} \quad & G_U(y,z) \leq 0 \\
& G_L(y,z) \leq 0 \\
& \lambda \geq 0 \\
& \lambda^T G_L(y,z) = 0 & \text{(8b)}\\
& \nabla_z \mathscr{L}(y,z,\lambda) = 0
\end{aligned}$$

where $\mathscr{L}(y,z,\lambda) := V_L(y,z) + \lambda^T G_L(y,z)$ is the Lagrangian function associated with the lower-level problem. For the special case of linear constraints and a quadratic cost, all constraints of (8) are linear and the complementarity condition (8b) is a set of disjunctive linear constraints, which can be described using binary variables, and thus leads to mixed-integer linear constraints.

Error Computation via Bilevel Optimization

To compute the maximum of the function γ given in (6) while maintaining an implicit representation of the optimal control law $u_0^\star$, we set the upper level cost V_U to γ and the lower level V_L to J. If the triangulation defining the interpolated control law is $T_V := \{S^0, \ldots, S^L\}$ and the interpolated control law in simplex S^i is $\mu_{S^i} = T^i x + t^i$, then for each simplical region S^i, we can compute the maximum of the error function γ with the following bilevel optimization:

$$\begin{aligned}
\gamma_i := \max \quad & \|\mu - u_0\|_\infty \\
\text{s.t.} \quad & x \in S^i \\
& \mu = T^i x + t^i \\
& u_0 = \arg\min \quad \frac{1}{2} x_N^T Q_N x_N + \frac{1}{2} \sum_{i=0}^{N-1} u_i^T R u_i + x_i^T Q x_i \\
& \qquad \text{s.t.} \quad x_{i+1} = A x_i + B u_i \\
& \qquad \qquad F x_i \leq f \ , \quad G u_i \leq g \ , \quad H x_N \leq h \\
& \qquad \qquad x_0 = x
\end{aligned}$$

The lower-level optimization problem is clearly strictly convex, and can therefore be solved by replacing it with its KKT conditions, which results in the disjunctive optimization problem (9).

$$
\begin{aligned}
\gamma_i := \max \quad & \|\mu - u_0\|_\infty \\
\text{s.t.} \quad & \text{Upper-level constraints} && \text{(9a)} \\
& \left\lfloor \begin{array}{l} x \in S^i \\ \mu = T^i x + t^i \end{array} \right. \\
& \text{Primal constraints} && \text{(9b)} \\
& \left\lfloor \begin{array}{l} x_{i+1} = Ax_i + Bu_i \\ \quad x_0 = x \\ Fx_i \le f \;\;,\quad Gu_i \le g \;\;,\quad Hx_N \le h \end{array} \right. \\
& \text{Dual constraints} && \text{(9c)} \\
& \left\lfloor \begin{array}{l} \lambda_i^x \ge 0 \;\;,\quad \lambda_i^u \ge 0 \;\;,\quad \lambda_N^x \ge 0 \\ \nu_i \text{ free} \end{array} \right. \\
& \text{First order optimality } \nabla \mathscr{L} = 0 && \text{(9d)} \\
& \left\lfloor \begin{array}{l} 0 = Qx_i + A^T \nu_i - \nu_{i-1} + F^T \lambda_i^x \\ 0 = Ru_i + B^T \nu_i + G^T \lambda_i^u \\ 0 = Q_N x_N + H^T \lambda_N^x - \nu_{N-1} \end{array} \right. \\
& \text{Complementarity conditions} && \text{(9e)} \\
& \left\lfloor \begin{array}{l} \lambda_j^{x_i} = 0 \quad \text{or} \quad F_j x_i = f_j \\ \lambda_j^{u_i} = 0 \quad \text{or} \quad G_j u_i = g_j \\ \lambda_j^{x_N} = 0 \quad \text{or} \quad H_j x_N = h_j \end{array} \right.
\end{aligned}
$$

Remark 4: The maximization of a convex infinity norm $\|t\|_\infty$ (as in (9)) can be accomplished by using a standard technique (e.g. [16]), in which we introduce binary variables n_i, p_i for each element of t and add the condition that the binary variable p_i is one if $\|t\|_\infty = t_i$ and n_i is one if it is $\|t\|_\infty = -t_i$. The method adds only linear and binary conditions to (9) and therefore it remains a MILP.

Remark 5: As written, (9) is a mixed integer linear programming problem (MILP) with logic constraints. Standard techniques exist to convert such logical constraints to linear ones with binary variables, and the reader is referred to e.g. [16] for details. While the resulting MILP is NP-hard to solve, there are both free and commercial solvers available that can tackle very large problems.

Remark 6: It is also possible to write (9) with a quadratic cost in place of the infinity norm. This would result is a bilevel indefinite QP, which can also be solved as a mixed-integer LP as discussed in Section V-A, although this would result in a larger number of binary variables.

V. STABILITY

If the matrices of the cost function (3) and the set $\mathbb{X}_N$ are defined appropriately, then the optimal control law $u_0^\star$ is stabilizing for (1) and the optimal cost function $J^\star$ is a Lyapunov function for the resulting closed-loop system [10]. In this section, we seek verifiable conditions under which $J^\star$ is also a Lyapunov function for the approximate closed-loop system $x^+ = Ax + B\mu_V(x)$. We begin by giving a minor modification of the standard condition for an approximate control law to be stabilizing [17], which requires only that the approximate control law be specified, rather than an entire approximate input sequence.

Theorem 1: Let $J^\star$ be the optimal solution of (2) and a Lyapunov function for $x^+ = Ax + Bu_0^\star(x)$. If $\mu(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a control law defined over the set S, then $J^\star$ is also a Lyapunov function for $x^+ = Ax + B\mu(x)$ if for all $x_0 \in S$, there exists a feasible state/input sequence $(\mathbf{x}, \mathbf{u})$ to (2) such that $u_0 = \mu(x_0)$ and

$$
J(\mathbf{x}, \mathbf{u}) - J^\star(x_0) \le \frac{1}{2} x_0^T Q x_0 + \frac{1}{2} u_0^T R u_0 \tag{10}
$$

Proof: If the input/state sequence $(\mu(x_0), u_1, \ldots, u_{N-1}, x_0, \ldots, x_N)$ is feasible for (2) and satisfies condition (10), then the shifted sequence $(u_1, \ldots, u_{N-1}, \nu(x_N),\ x_1, \ldots, x_N, x_{N+1})$ is also feasible, where $x_{N+1} := Ax_N + B\nu(x_N)$. Define the stage and terminal costs as $l(x,u) := \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u$ and $V_N(x) := \frac{1}{2}x^T Q_N x$ respectively. Evaluating the cost function at this shifted sequence gives

$$
\begin{aligned}
J^\star(x_1) &\le V_N(x_{N+1}) + \sum_{i=1}^{N} l(x_i, u_i) \\
&= V_N(x_{N+1}) - V_N(x_N) + l(x_N, u_N) && \text{(11a)} \\
&\quad + V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) && \text{(11b)} \\
&\quad - l(x_0, \mu(x_0))
\end{aligned}
$$

Equation 11a is negative by the assumption that V_N decreases faster than the stage cost l in the set $\mathbb{X}_N$ and (11b) is less than or equal to $J^\star(x_0) + l(x_0, \mu(x_0))$ by the assumption (10). It follows that $J^\star(x_1) - J^\star(x_0) \le 0$ and therefore $J^\star$ is a Lyapunov function for the approximate system $x^+ = Ax + B\mu(x)$. ■

Remark 7: The condition given in Theorem 1 is essentially the same as that used in several other papers on approximate MPC [17]. The standard approach is to define a function $\hat{J}$, which is the interpolation of the optimal cost at the vertices of the triangulation and then test this cost under condition (10). This test is efficient because the function $\hat{J}$ is piecewise affine and hence (10) can be evaluated as a series of convex optimizations. Rather than taking a linear interpolation, we here assume that the candidate Lyapunov function $\tilde{J}$ is given implicity by the optimization $\tilde{J}(x) := \min\{J(\mathbf{x}, \mathbf{u}) \mid (2),\ x_0 = x,\ u_0 = \mu(x)\}$, which makes $\tilde{J}$ a convex piecewise quadratic function. Clearly, the condition $\tilde{J} \le \hat{J}$ holds, which makes the condition given in Theorem 1 less conservative than previous proposals and often significantly so. The cost is that condition (10) can no longer be verified by solving convex problems.

Theorem 1 gives a condition under which the optimal cost function $J^\star$ is a Lyapunov function for the closed-loop system under the interpolated control law $\mu_{T_V}(x)$. This condition is not trivial to test, but can be confirmed by solving a series of bilevel programs, which we demonstrate below.

Let $V \subset \mathscr{X}$ be a finite set that defines the interpolated control law μ_{T_V} over the triangulation $T_V := \{S_1, \ldots, S_L\}$. Define ξ_i to be the optimal cost of the bilevel optimization problem (12) for each $i = 1, \ldots, L$, where $\mu := T^i x + t^i$ is the affine control law in the simplical region S_i.

$$
\begin{aligned}
\xi_i := \min \quad & \frac{1}{2}\tilde{x}_0^T Q\tilde{x}_0 + \frac{1}{2}\tilde{u}_0^T R\tilde{u}_0 + J(\mathbf{x},\mathbf{u}) - J(\tilde{\mathbf{x}},\tilde{\mathbf{u}}) && \text{(12a)}\\
\text{s.t.} \quad & x_0 \in S_i \\
& \text{Constraints (2) on } \mathbf{x},\mathbf{u} \\
& (\tilde{\mathbf{x}},\tilde{\mathbf{u}}) = \arg\min \quad J(\tilde{\mathbf{x}},\tilde{\mathbf{u}}) && \text{(12b)}\\
& \qquad \text{s.t. Constraints (2) on } \tilde{\mathbf{x}},\tilde{\mathbf{u}} \\
& \qquad\quad \tilde{x}_0 = x_0 \\
& \qquad\quad \tilde{u}_0 = T^i x_0 + t^t
\end{aligned}
$$

One can see from (12) that the conditions of Theorem 1 are met if and only if $\max\{\xi_i\}$ is negative.

Corollary 1: $J^\star$ is a Lyapunov function for the system $x^+ = Ax + B\mu_{T_V}(x)$ if $\max\{\xi_i\} \le 0$.

A Lyapunov function is insufficient to prove stability for a constrained system; the system must also be invariant. As discussed in [6], since level sets of Lyapunov functions are invariant, it is possible to determine an invariant subset of $\operatorname{conv}(V)$ given the vertices of each region S_i without further processing.

A. Computation of Stability Criterion

The bilevel optimization problem (12) differs from that tackled in the previous section in that the upper level is an indefinite quadratic program, while the lower is a convex QP. In the following, we demonstrate that this class of problems can also be re-formulated as a disjunctive LP and hence solved using standard MILP software.

We begin with the following lemma, which demonstrates that an indefinite QP can be written as a mixed-integer LP.

Lemma 2: Consider the following indefinite QP

$$
\begin{aligned}
J^\star := \min_z \quad & \frac{1}{2} z^T D z && \text{(13)}\\
\text{s.t.} \quad & Bz \le b \ , \\
& Cz = c \ ,
\end{aligned}
$$

where $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{l\times n}$ and assume that Slater's condition holds. If $(z^\star,\lambda,\gamma)$ is an optimal solution of the MILP (14), then $z^\star$ is an optimizer of (13) and $\check{J} = J^\star$.

$$
\begin{aligned}
\check{J} = \min_{x,\lambda,\gamma} \quad & -\frac{1}{2}\left(b^T\lambda + c^T\gamma\right) && \text{(14)}\\
\text{s.t.} \quad & Bz \le b \ , \quad Cz = c \ , && \text{Primal feasibility}\\
& \nabla_z \mathscr{L} = Dz + B^T\lambda + C^T\gamma = 0 \ , && \text{Stationarity}\\
& \lambda \ge 0 \ , \quad \gamma \text{ free} \ , && \text{Dual feasibility}\\
& \lambda_i = 0 \quad \text{or} \quad B_i z = b_i \ , && \text{Complementarity}
\end{aligned}
$$

Proof: The constraints of (14) are precisely the KKT conditions of (13), which are necessary but not sufficient because the problem is indefinite. We gain sufficiency by minimizing the cost function of (13) while enforcing the necessary optimality conditions, which leads to an optimal solution. We have now to show that the linear cost function of (14) is in fact equivalent to the indefinite cost of (13).

We begin by taking the inner product of the stationarity condition and the primal optimization variable z

$$
z^T \nabla_z \mathscr{L} = 0 = z^T D z + z^T B^T \lambda + z^T C^T \gamma
$$

The complementarity conditions $\lambda^T(Bz - b) = 0$ and $\gamma^T(Cz - c) = 0$ then give the result

$$
z^T D z = -b^T\lambda - c^T\gamma
$$

■

We can now show that an optimizer of (12) can be computed by solving a mixed-integer linear program.

Theorem 2: Consider the following quadratic bilevel optimization problem:

$$
\begin{aligned}
J^\star = \min \quad & \frac{1}{2} y^T S y + \frac{1}{2} z^{\star T} T z^\star && \text{(15)}\\
\text{s.t.} \quad & Ay \le b \\
& z^\star = \arg\min \left\{ \frac{1}{2} z^T D z \ \middle| \ Ez + Fy \le g \right\}
\end{aligned}
$$

where D is positive definite and S and T are indefinite matrices. If $(y,z,\beta^L,\beta^U,\lambda^L,\lambda^U,\lambda^{UL},l)$ is an optimal solution of (16), then (y,z) is an optimal solution to (15).

$$
\begin{aligned}
\min \quad & -\frac{1}{2}\left(b^T\lambda^U + g^T\lambda^{UL}\right) && \text{(16a)}\\
\text{s.t.} \quad & \beta_i^L \in \{0,1\} \ , \ \beta_i^U \in \{0,1\} \\
& \text{Upper level} \\
& \quad \text{Primal and dual feasibility} \\
& \quad\quad Ay \le b \ , \quad \lambda^U \ge 0 \\
& \quad \text{Stationarity} \\
& \quad\quad \nabla_y \mathscr{L}^U = 0 = Sy + A^T\lambda^U + F^T\lambda^{UL} \\
& \quad\quad \nabla_z \mathscr{L}^U = 0 = Tz + D^T\gamma^U + E^T\lambda^{UL} \\
& \quad\quad \nabla_{\lambda^L} \mathscr{L}^U = 0 = E\gamma^U - l \\
& \quad \text{Complementarity} \\
& \quad\quad \beta_i^L = 1 \Rightarrow \lambda_i^{UL} = 0 \ , \quad \beta_i^L = 0 \Rightarrow l_i = 0 \\
& \quad\quad \beta_i^U = 1 \Rightarrow \lambda_i^U = 0 \ , \quad \beta_i^U = 0 \Rightarrow A_i y = b_i \\
& \text{Lower level} && \text{(16b)}\\
& \quad \text{Primal and dual feasibility} \\
& \quad\quad Ez + Fy \le g \ , \quad \lambda^L \ge 0 \\
& \quad \text{Stationarity} \\
& \quad\quad \nabla_z \mathscr{L}^L = 0 = Dz + E^T\lambda^L = 0 \\
& \quad \text{Complementarity} \\
& \quad\quad \beta_i^L = 1 \Rightarrow \lambda_i^L = 0 \ , \quad \beta_i^L = 0 \Rightarrow E_i z + F_i y = g_i
\end{aligned}
$$

Proof: The matrix D is positive definite and so its KKT conditions are both necessary and sufficient for optimality of the lower level problem. As a result, we can replace the lower level problem with these conditions in order to get an equivalent single level problem with mixed-integer constraints. We introduce the binary variable β^L, which encodes the complementarity conditions of the lower level problem and define the following optimization problem as a function of this variable:

$$
\begin{aligned}
J(\beta^L) := \min_{y,z,\lambda^L} \quad & \frac{1}{2}y^T S y + \frac{1}{2}z^T T z \qquad (17)\\
\text{s.t.} \quad & Ay \le b \\
& \text{Lower level optimality conditions (16b)}
\end{aligned}
$$

For each β^L, (17) is a single-level indefinite quadratic program, which can be written as a MILP using Lemma 2. This gives the optimization problem (16), where we introduce appropriate dual variables λ^U, λ^{UL}, l and binaries β^U to represent the upper-level complementarity conditions.

Finally, we have $J^\star = \min_{\beta^L} \left\{ J(\beta^L) \mid (17) \text{ feasible} \right\}$, which gives the desired result. ■

Remark 8: Note that the structure of the problem (15) differs slightly from that required to solve the stability criteria (12) in that it does not include any equality constraints. These constraints were left out of Theorem 2 for reasons of clarity and space restrictions, but the proof can be readily extended to this case.

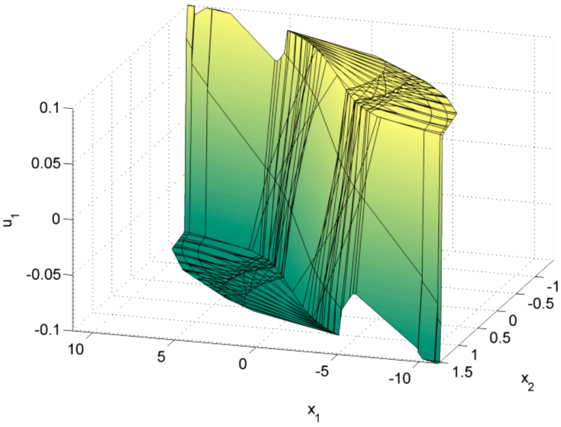
VI. Illustrative Example

Consider the following simple two-state, two-input example:

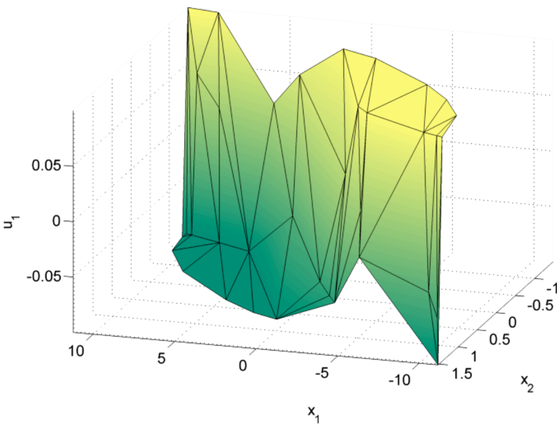
$$
x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.42 & 0.90 \\ 0.38 & 0.67 \end{bmatrix} u \ ,
$$

with the input and state constraints $\|u\|_\infty \le 0.1$, $|x_1| \le 40$, $|x_2| \le 10$ and a horizon N of length 10 with the stage cost taken to be $l(x,u) := x'x + 30u'u$. The optimal control law in this case requires $1,155$ regions and can be seen in Figure 1(a).

We here approximate this optimal control law with one interpolated from 36 points, which results in 52 simplical regions (Figure 1(b)). The error γ between the optimal and approximate control laws is only 0.06 and the maximum error between the optimal and suboptimal cost functions is 0.75. The approximate system is stable, as verified by solving MILP (16).



(a) Optimal control law $u_1(x)$; $1,155$ regions.



(b) Approximate control law $u_1(x)$; 52 regions.

Fig. 1. Optimal and sub-optimal control laws for Example VI.

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