Ellipsoids and polytopes

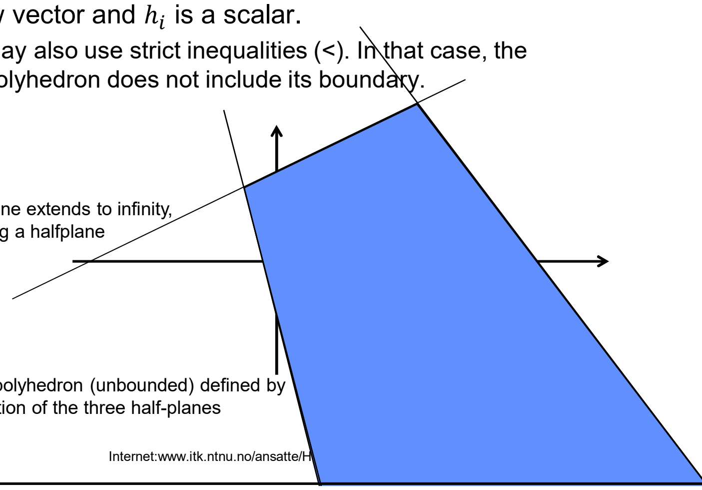
Brief notes for TK18

Ellipsoid

- Several equivalent definitions. Will typically use
  $$\mathcal{E}(P) = \{x | x^T P x < \gamma\}$$
- P symmetric, positive definite ($P > 0$)
  - Remember, anti-symmetric components do not contribute to the value of a quadratic function
  - Main axes of ellipsoid defined by eigenvectors of P.
- Specification of an ellipsoid is often normalized, such that $\gamma = 1$.
- For such a normalized ellipsoid, the n-dimensional 'volume' is proportional to $\det(P^{-1})$.
  - For that reason, polytopes are sometimes defined as $\mathcal{E}(W) = \{x | x^T W^{-1} x < \gamma\}$.
  - Volume then proportional to $\det(W)$.
- Quadratic Lyapunov functions will have ellipsoidal level sets.

Polyhedron

- A polyhedron is the intersection of a finite number of half-planes.
- A half-plane is defined by an inequality $H_i x \leq h_i$, where H_i is a row vector and h_i is a scalar.
  - May also use strict inequalities (<). In that case, the polyhedron does not include its boundary.



Each line extends to infinity, defining a halfplane

In blue the polyhedron (unbounded) defined by the intersection of the three half-planes

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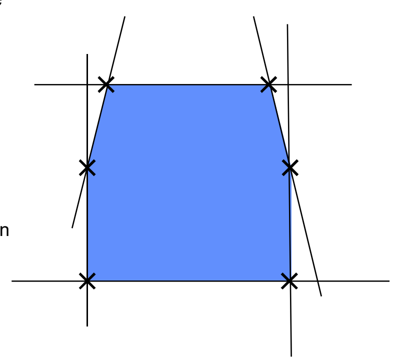
Polytope

- A polytope is a bounded polyhedron
  - See example in blue
- The polytope can be defined by its ('supporting') hyperplanes
- or, equivalently, as the *convex hull* of its *vertices*
  - Vertices marked by x in figure

For (unbounded) polyhedra, we also need *rays* and/or *lines* in addition to vertices for a complete description.
Vertices/rays/lines are called *generators* of the polytope/polyhedron



The description of a polytope may contain *redundant* half-planes or vertices.
For numerical calculations, it is usually preferable to remove such redundant half-planes/vertices

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Conversion between half-plane and vertex descriptions

- The half-plane decsription is often called the H-representation
- The vertex description is often valled the V-representation
- One may calculate the V-representation from the H-representation and *vice versa*, but
  - This can be very demanding calculations
  - Can be prone to numerical error
- Such problems are often more severe for high-dimensional polytopes.

- If the dimension of the polytope is lower than the dimension of the space, the polytope is called *degenerate*.

Faces of a polytope

- For a polytope P, an *inequality* $c^T x \leq d$ is called *valid* for P if $c^T x \leq d$ holds for all $x \in P$.
- A subset F of P is called a *face* of P if it is represented as
$$F = P \cap \{x | c^T x = d\}$$
for some valid inequality $c^T x \leq d$.
- Depending on the dimension of the face, some faces have a special name
  - Dimension 0: vertex
  - Dimension 1: edge
  - Dimension $n-2$: ridge
  - Dimension $n-1$: facet
- All points in a polytope can be expressed as a convex combination of the vertices

Complexity of representations

- For dimensions $d > 2$, there is no simple relationship between the complexity of the H- and V-representations.

- Removing a vertex from the V-representation may make the H-representation more complex
- Adding a half-plane to the H-representation may make the V-representation less complex

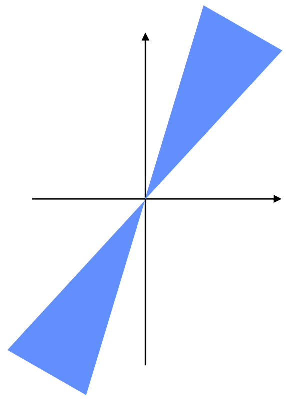
Simplex

- An n −dimensional simplex is a polytope defined by $n + 1$ vertices (which do not lie on a single plane)
  – Equivalently, an n −dimensional simplex is a non-degenerate polytope described by $n + 1$ half-planes.

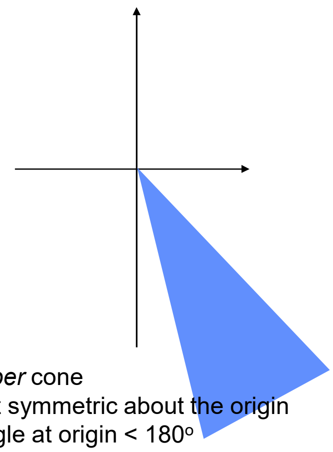
- Given a polytope and a set of points
  – Typically the vertices of the polytope, but possibly also additional points in the polytope.
- Under some assumptions on the arrangement of these points, it is then possible to find the 'best' way of partitioning the polytope into simplices
  – Avoiding 'long and thin' simplices
- This is called Delaunay triangulation
  – Used a lot in modelling of surfaces in 3d, but can be generalized to higher dimensions.

Cone

- A cone C is a set with the property that
$$x \in C \Rightarrow \lambda x \in AC, \forall \lambda \geq 0$$



A cone



A *proper* cone
- Not symmetric about the origin
- Angle at origin < 180°

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Further reading

- Much more (and much more precise) information about polyhedra/polytopes can be found in Komei Fukudas Polyhedral Computation FAQ: http://www.inf.ethz.ch/personal/fukudak/polyfaq/polyfaq.html
- The Multi-parametric toolbox for Matlab contains routines for calculations with polytopes. http://people.ee.ethz.ch/~mpt/3/
  - Freeware
  - Documentation (including that for the old version 2) contains further info/examples on polyhedral computations

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