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Relaxing PWQ Lyapunov stability criteria for PWA systems[☆]

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ABSTRACT

The note addresses the calculation of piecewise quadratic (PWQ) Lyapunov functions for PWA (piecewise affine) systems. New LMI relaxations are proposed. These relaxations have been found to be effective in cases where the traditional relaxations fail.

1. Introduction

Piecewise linear (PWL) and piecewise affine (PWA) systems are natural extensions from linear toward a non-linear framework (Sontag, 1981) in control theory. Linear predictive control with constraints is also known to result in PWA closed-loop dynamics (Bemporad, Morari, Dua, & Pistikopoulos, 2002). Recent work on hybrid systems has given further emphasis to PWA systems as an important class of models (Heemels, De Schutter, & Bemporad, 2001).

The present paper deals with the asymptotic stability of the origin for discrete-time PWA dynamics. For PWA systems, it is clearly very restrictive to use the same quadratic Lyapunov function for all local system dynamics. Starting from this observation, the use of PWQ Lyapunov functions calculated using the LMI approach was introduced by Johansson and Rantzer (1998). The conservatism of the initial PWQ approach was ameliorated in Rantzer and Johansson (2000) by the introduction of relaxations to the LMI formulation. The same relaxation principles have been adapted to the discrete-time particularities (see, e.g., Feng, 2002). Motivated by failure in finding PWQ Lyapunov functions in certain cases, alternative relaxations have been sought. The main contribution of the present paper is to propose alternative relaxations based on the *generators of the polyhedral representations* for the regions in the PWA system description.

Due to space limitations we refer to standard literature such as Blanchini and Miani (2008) and Boyd and Vandenberghe (2004) for definitions of terms such as *(proper, polyhedral) cone, polyhedron, etc.*, and concepts such as $\mathcal{H}-$ (halfspace) and $\mathcal{V}-$ (vertex) representations of polytopes. We will assume that all cones or conic hulls are *proper*. We further assume that the reader is familiar with the concepts of *positive (definite)* and *indefinite quadratic functions*, as well as *invariant sets*, and has a basic knowledge of Lyapunov stability theory.

Definition 1. A conic hull of a set $\mathcal{X} \subset \mathbb{R}^n$ is defined as:

$$\mathcal{C}(\mathcal{X}) = \left\{x \in \mathbb{R}^n | x = \lambda y, \lambda \in \mathbb{R}_{\geq 0}, y \in \mathcal{X}\right\}. \quad (1)$$

2. PWL systems analysis

2.1. System description

Definition 2. A discrete-time piecewise linear (PWL) dynamical system is described by a finite set of maps:

$$x_{k+1} \in f_{pwl}(x_k) = \{A_i x_k, \text{ for } i \text{ such that } x_k \in \mathcal{X}_i\} \quad (2)$$

where $\mathcal{X} = \cup_{i \in I} \mathcal{X}_i \subset \mathbb{R}^n$, each set $\mathcal{X}_i$ is a closed subset with a non-empty interior, $0 \in \mathcal{X}_i$ for at least one i, and $i \in I \subset \mathbb{N}$ is the index of the regions with I a finite subset of $\mathbb{N}$.

Excepting the closedness and non-empty interior properties, there are no particular convexity, boundedness or nonoverlapping properties for the regions $\mathcal{X}_i$ in Definition 2. In the case of overlapping regions with different dynamics, stability will have to be addressed for *all possible dynamics* in the overlapping regions. Thus, the stability problem is well posed even if the system state at any

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given instant does not uniquely determine the future trajectory of the state, as it is described by the difference inclusion (2).

Assumption 1. The set $\cup_{i\in I}(\mathcal{X}_i)$ is invariant with respect to (2).

2.2. Lyapunov stability: piecewise quadratic criteria and existing relaxation techniques

Following a classical Lyapunov function description for PWL systems (Feng, 2002), for each region of the state space one can define

$$V_i : \mathcal{X}_i \to \mathbb{R} \quad \text{with} \quad V_i(x) = x^T P_i x. \tag{3}$$

Let us also define the sets $X_{ij} = \{x | x \in \mathcal{X}_i, A_i x \in \mathcal{X}_j\}$. The origin of system (2) will then be stable provided

$$V_i(x_i) > 0, \quad \forall x_i \in \mathcal{X}_i; \qquad V_i(x_i) > V_j(f_{pwl}(x_i)), \quad \forall x_i \in \mathcal{X}_{ij},$$

which is fulfilled provided

$$P_i - F_i > 0, \qquad A_i^T P_j A_i - P_i + G_{ij} < 0 \tag{4}$$

where G_{ij} and F_i are symmetric matrices defining quadratic functions as follows:

$$f_i(x) = x^T F_i x; \quad f_i(x) \geq 0, \forall x \in \mathcal{X}_i$$
$$g_{ij}(x) = x^T G_{ij} x; \quad g_{ij}(x) \geq 0, \forall x \in \mathcal{X}_{ij}.$$

Note that if a quadratic function in R^n is positive over a set $\mathcal{X}_i$, it is also positive over the conic hull $\mathcal{C}(\mathcal{X}_i)$ of the set. The relaxation techniques available in the literature (Feng, 2002; Rantzer & Johansson, 2000) can be adapted to the stability analysis of PWL systems (2) by employing polyhedral cone overapproximations of the conic hulls $\mathcal{C}(\mathcal{X}_i)$ and $\mathcal{C}(\mathcal{X}_{ij})$:

$$\mathcal{C}^P(\mathcal{X}_i) = \{x | E_i x \geq 0\} \supset \mathcal{C}(\mathcal{X}_i) \tag{5}$$

$$\mathcal{C}^P(\mathcal{X}_{ij}) = \left\{x | E_{ij} x \geq 0\right\} \supset \mathcal{C}(\mathcal{X}_{ij}). \tag{6}$$

Thus, F_i and G_{ij} of the form

$$F_i = E_i^T U_i E_i, \qquad G_{ij} = E_{ij}^T W_{ij} E_{ij},$$

where U_i and W_{ij} are symmetric, non-negative matrices, result in valid quadratic relaxation functions.

2.3. A novel relaxation for PWL systems defined on polytopic cones

Alternative relaxations have been sought in order to exploit the generators representation of the polyhedral cones. In higher dimensions these relaxations can prove to be interesting alternatives to the existing relaxations. The following properties of a proper cone $\mathcal{C} \in \mathbb{R}^n$ will be used:

- There exists a ray r_m in the interior of the cone ($\lambda r_m \in Int(\mathcal{C})$, $\forall \lambda \in \mathbb{R}_{>0}$), with a corresponding set of mutually orthogonal rays $r_{pj}, j \in \mathbb{N}_{[1,n-1]}$ that are outside the cone and are also orthogonal to r_m. The set of these rays r_{pj} is denoted $\mathbf{R}_p(r_m)$.
- There exists a proper polyhedral cone over-approximation embedding $\mathcal{C}$:

$$\mathcal{C}^P = cone(\left\{\mathbf{r}_{e1}, \ldots, \mathbf{r}_{e\rho}\right\}) \tag{7}$$

$$= \left\{x | x = \sum_{j=1}^{\rho} \beta_j \mathbf{r}_{ej}, \beta_j \geq 0\right\} \tag{8}$$

satisfying $\mathcal{C}^P \supset \mathcal{C}$ and $\mathbf{R}_p(r_m) \cap \mathcal{C}^P = \emptyset$.

Proposition 1. *Let $\mathcal{C} \in \mathbb{R}^n$ be a proper cone. Let $r_m \in Int(\mathcal{C})$ be a ray with the associated set of mutually orthogonal rays $\mathbf{R}_p(r_m)$ and an embedding polyhedral cone $\mathcal{C}^P = cone(\mathbf{R}_e) \supset \mathcal{C}$ with $\mathbf{R}_e = \left\{\mathbf{r}_{e1}, \ldots, \mathbf{r}_{e\rho}\right\}$ its finite set of generators. An indefinite quadratic function $x^T Fx$ is positive over the proper cone $\mathcal{C}$ (except at the origin) provided that*

(1) *It is positive over all $n-1$-dimensional faces of $\mathcal{C}^P$,*
(2) $r_{pi}^T F r_{pi} \leq 0, \quad \forall r_{pi} \in \mathbf{R}_p(r_m)$.

Proof. Any ray $r_s \in Int(\mathcal{C})$ together with one of the rays r_{pi} forms a two-dimensional plane, called $S(r_s, r_{pi})$. The frontier of $\mathcal{C}$ cuts this two-dimensional plane along two rays, forming a proper cone $\mathcal{C} \cap S(r_s, r_{pi})$ on the two-dimensional plane. Clearly, any purely quadratic function $f(x) = x^T Fx$ is symmetric about the origin, and can be defined by a symmetric matrix F. We observe that the restriction of the quadratic function $f(x)$ on the two-dimensional plane $S(r_s, r_{pi})$ inherits the symmetry property, since $S(r_s, r_{pi})$ goes through the origin. Positivity along the rays forming the proper cone in the two-dimensional plane $S(r_s, r_{pi})$ combined with negativity of the quadratic function along $\pm r_{pi}$, then ensures that the quadratic function is positive everywhere inside the proper two-dimensional cone $\mathcal{C} \cap S(r_s, r_{pi})$. For any ray $r_s \in Int(\mathcal{C})$ such a proper two-dimensional cone can be constructed, and thus the quadratic function has been proven to be positive over the entire cone $\mathcal{C}$. □

For a polyhedral cone $\mathcal{C}^P$, let $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}^P)$ denote a set of cones consisting of the cone $\mathcal{C}^P$ itself, and all r-dimensional faces of $\mathcal{C}^P$, $r \geq 2$. For each $Z_i \in \mathcal{Z}_{\mathcal{C}}(\mathcal{C}^P)$, let $r_{pi}(Z_i)$ be a set of mutually orthogonal rays that are also orthogonal to a ray $r_{mi}(Z_i) \in RelInt(Z_i)$. Proposition 1 provides conditions for the positivity of a function over a proper cone $\mathcal{C}$ based on positivity over lower-dimensional faces of the embedding polyhedral cone $\mathcal{C}^{\mathcal{P}}$. The introduction of the set $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}^{\mathcal{P}})$ allows the succinct expression of these conditions for the relaxation of an LMI, as stated in Lemma 1.

Lemma 1. *Consider a generic $n \times n$ LMI problem $\mathcal{L} > 0$. The associated inequality $x^T \mathcal{L} x > 0$ holds $\forall x \in \mathcal{C}$ (except the origin), with $\mathcal{C}$ a proper cone, if the following LMI (termed the* relaxation *of $\mathcal{L} > 0$ on the cone $\mathcal{C}$) is feasible:*

$$Relax(\mathcal{L} > 0 | \mathcal{C}) : \begin{cases} r_{ej}^T F r_{ej} \geq 0 \quad \forall r_{ej} \in \mathcal{R}_e \\ r_{pi}^T(Z_i) F r_{pi}(Z_i) \leq 0, \\ \qquad \forall r_{pi}(Z_i), \forall Z_i \in \mathcal{Z}_{\mathcal{C}}(\mathcal{C}^P) \\ \mathcal{L} - F > 0 \end{cases} \tag{9}$$

where $\mathcal{C}^P$ is an embedding polyhedral cone $\mathcal{C}^P = cone(\mathbf{R}_e) \supset \mathcal{C}$ with $\mathbf{R}_e = \left\{\mathbf{r}_{e1}, \ldots, \mathbf{r}_{e\rho}\right\}$.

Proof. The feasibility of the first two inequalities in (9) assure according to Proposition 1 the existence of a quadratic relaxation function over the cone $\mathcal{C}$. The third matrix inequality make use of this relaxation in conjunction with the original LMI. □

2.4. Effective LMI stability criteria

Assumption 2. The polyhedral cone over-approximations for the conic hulls of the regions in the discrete-time PWL system definition (2) result in proper cones $\mathcal{C}^P(\mathcal{X}_i)$.

Theorem 1. *Let the PWL system (2) and associated conic hulls satisfy Assumptions 1 and 2. The origin is asymptotically stable over $\cup_{i\in I} \mathcal{X}_i$ if the following LMI problem is feasible:*

$$\begin{cases} Relax(P_i > 0 | \mathcal{C}^P(\mathcal{X}_i)), \forall i \in I \\ Relax(A_i^T P_j A_i - P_i < 0 | \mathcal{C}^P(\mathcal{X}_{ij})), \forall \mathcal{X}_{ij} \neq \emptyset. \end{cases} \tag{10}$$

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Proof. Follows directly from (4) and Lemma 1. □

3. PWA systems analysis

3.1. PWA system dynamics

Definition 3. A discrete-time piecewise affine (PWA) dynamical system is described by a finite set of maps:

$$x_{k+1} \in f_{pwa}(x_k) = \{A_i x_k + a_i, \text{ for } i \text{ s.t. } x_k \in \mathcal{X}_i\} \tag{11}$$

where $\mathcal{X} = \cup_{i \in I} \mathcal{X}_i$, each $\mathcal{X}_i$ is a closed subset of $\mathbb{R}^n$ with a non-empty interior and $i \in I \subset \mathbb{N}$ the index of the regions with I a finite subset of $\mathbb{N}$.

Assumption 1 will be considered to hold also in the PWA case.In order to allow a well-posed stability analysis problem, the following natural assumption is made.

Assumption 3. The origin is a fixed point for (11).

Partitioning the finite set $I = I_0 \cup I_1 \subset \mathbb{N}$ into two subsets, I_0 representing the cells containing the origin and I_1 representing all other cells, the assumption can be translated into the condition $a_i = 0, \forall i \in I_0$.

Standing Assumption. *The set $\cup_{i \in I_0} \mathcal{X}_i$ is invariant.*

The fulfillment of this Standing Assumption can be linked to the stability of PWL systems treated in the previous section. The invariance properties in the Standing Assumption allow the use of a "lifting" from the n-dimensional system dynamics (11) to a piecewise linear dynamics in an enlarged state space of dimension $n+1$ for a homogeneous treatment. However, to be able to use standard Lyapunov stability criteria on the enlarged state space, we must clearly be able to set the auxiliary state to zero at the origin. This implies that what linear dynamics to use will depend not only on which region the state is in at present, but also what region the state transits to at the next timestep. The approach in Johansson and Rantzer (1998) can be employed to simplify the description of PWA systems by translating any initial conditions $x \longrightarrow \bar{x} = \begin{bmatrix} x^T & 1 \end{bmatrix}^T$. The correspondence will be preserved for the definition of the regions in the extended state space:

$$\mathcal{X}_i \longrightarrow \bar{\mathcal{X}}_i = \left\{ \bar{x} | \begin{bmatrix} I_n & \mathbf{0}_{n,1} \end{bmatrix} \bar{x} \in \mathcal{X}_i \right\}. \tag{12}$$

Then, the PWA dynamics (11) satisfying Assumption 3 can be expressed as

$$\begin{aligned} \bar{x}_{k+1} &\in \bar{f}_{pwa}(\bar{x}_k) = \{\bar{A}_i \bar{x}_k, \text{ with } i \text{ s.t. } \bar{x}_k \in \bar{\mathcal{X}}_i\} \\ &= \left\{ \begin{array}{l} \begin{bmatrix} A_i & a_i \\ 0 & 1 \end{bmatrix} \bar{x}_k \text{ for } \bar{x}_k \in \bar{\mathcal{X}}_i, \bar{x}_{k+1} \in \bar{\mathcal{X}}_j, \forall j \in I_1 \\ \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix} \bar{x}_k \text{ for } \bar{x}_k \in \bar{\mathcal{X}}_i, \bar{x}_{k+1} \in \bar{\mathcal{X}}_j, \forall j \in I_0 \end{array} \right\}. \end{aligned} \tag{13}$$

3.2. Lyapunov stability — piecewise quadratic criteria

Similar to (3), PWQ Lyapunov functions can be considered for each region of the extended state space:

$$V_i : \bar{\mathcal{X}}_i \to \mathbb{R} \text{ with } V_i(\bar{x}) = \bar{x}^T \bar{P}_i \bar{x}. \tag{14}$$

The PWA system (11) is stable provided:

$$V_i(\bar{x}) > 0, \quad \forall \bar{x} \in \bar{\mathcal{X}}_i, \bar{x} \neq 0 \tag{15}$$

$$\begin{aligned} V_i(\bar{x}) > \sup\{V_j(f_{pwa}(\bar{x}))\}, \quad &\forall \bar{x} \in \bar{\mathcal{X}}_i, \\ &\forall j \text{ such that } f_{pwa}(\bar{x}) \in \bar{\mathcal{X}}_j. \end{aligned} \tag{16}$$

Define $\bar{\mathcal{X}}_{ij} = \left\{ \bar{x} \in \bar{\mathcal{X}}_i | \bar{A}_i \bar{x} \in \bar{\mathcal{X}}_j \right\}$, and consider the generic region $\mathcal{R} = \{x | Ex \geq e\}$. This corresponds to $\bar{\mathcal{R}} = \left\{ \bar{x} | \bar{E}\bar{x} \geq 0 \right\}$ with $\bar{E} = \begin{bmatrix} E & -e \end{bmatrix}$. Then, the LMI solution for $\bar{F} = \bar{E}^T U \bar{E}$ where U is a symmetric, non-negative matrix will provide a relaxations of the form:

$$\bar{h}(x) = \bar{x}^T \bar{F} \bar{x}; \tag{17}$$

as long as $h(\bar{x}) > 0, \forall \bar{x} \in \bar{\mathcal{R}}$ and the generic region $\bar{\mathcal{R}}$ can be replaced by $\bar{\mathcal{X}}_i$ or $\bar{\mathcal{X}}_{ij}$ as appropriate.

3.3. A novel LMI relaxation over polytopes

The basic idea behind the novel relaxations is given by the next result.

Lemma 2. *Consider a polytope $\mathcal{R} \in \mathbb{R}^n$ and $\mathbf{V}(\mathcal{R})$ the set of its vertices. Any feasible pair $(\bar{H}, c)$ of the LMI problem:*

$$\begin{cases} \bar{H} = \bar{H}^T < 0 \\ \begin{bmatrix} v_i^T & 1 \end{bmatrix} \bar{H} \begin{bmatrix} v_i \\ 1 \end{bmatrix} + c > 0, \quad \forall v_i \in \mathbf{V}(\mathcal{R}) \end{cases} \tag{18}$$

defines a positive function over $\mathcal{R}$ in the form:

$$h(x) = \begin{bmatrix} x^T 1 \end{bmatrix} \bar{H} \begin{bmatrix} x^T & 1 \end{bmatrix}^T + c > 0, \quad \forall x \in \mathcal{R}. \tag{19}$$

Proof. The proof is immediate by observing that due to $\bar{H}$ being negative definite, the level sets of $h(x)$ are convex, while the second condition in (18) ensures positivity at the vertices of $\mathcal{R}$. □

Proposition 2. *Consider a generic $(n+1) \times (n+1)$ LMI problem $\mathcal{L} > 0$. The associated inequality: $\begin{bmatrix} x^T & 1 \end{bmatrix} \mathcal{L} \begin{bmatrix} x^T & 1 \end{bmatrix}^T > 0$ holds $\forall x \in \mathcal{R}$, with $\mathcal{R}$ a bounded polyhedral region if the following LMI (termed the* relaxation of *$\mathcal{L} > 0$ on the polyhedron $\mathcal{R}$) is feasible:*

$$Relax(\mathcal{L} > 0 | \mathcal{R}) : \begin{cases} \bar{H} = \bar{H}^T < 0 \\ \begin{bmatrix} v_i \\ 1 \end{bmatrix}^T \bar{H} \begin{bmatrix} v_i \\ 1 \end{bmatrix} + c > 0, \quad \forall v_i \in \mathbf{V}(\mathcal{R}) \\ \mathcal{L} - \bar{H} + \bar{C} > 0, \quad \bar{C} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \end{cases}. \tag{20}$$

Proof. The feasibility of the first two inequalities in (20) assure according to Lemma 2 the existence of a relaxation function which is positive over the polyhedral region $\mathcal{R}$. The third matrix inequality make use of this relaxation in conjunction with the original LMI. □

3.4. Stability criteria for bounded PWA systems

The LMI based stability criterion based on piecewise quadratic Lyapunov function is a natural application of the relaxation techniques.

Theorem 2. *Let the PWA system (11) defined over a bounded region $\mathcal{X} = \cup_{i \in I} \mathcal{X}_i$ satisfy Assumptions 1 and 3 as well as the Standing Assumption, with polytopic approximations of the regions $\mathcal{R}_i \supset \mathcal{X}_i$ and $\mathcal{R}_{ij} \supset \mathcal{X}_{ij}$. The fixed point 0 is asymptotically stable over $\mathcal{X}$ if the following LMI problem is feasible:*

$$\begin{cases} Relax(\bar{P}_i > 0 | \bar{\mathcal{R}}_i), \quad \forall i \in I \\ Relax(\bar{A}_i^T \bar{P}_j \bar{A}_i - \bar{P}_i < 0 | \bar{\mathcal{R}}_{ij}), \quad \forall \mathcal{X}_{ij} \neq \emptyset. \end{cases} \tag{21}$$

4. Conclusions

New LMI relaxations based on the generators description of the polyhedral regions have been proposed to refine those available in the literature. For a high-complexity example of the use of the new

PWA relaxations where the traditional relaxations prove of little use, readers are referred to Hovd and Olaru (2010). This, as well as a series of other numerical tests, show that the novel relaxations offer interesting alternatives for the stability analysis of PWL/PWA systems using PWQ Lyapunov functions.

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