Cone-Copositive Piecewise Quadratic Lyapunov Functions for Conewise Linear Systems

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***Abstract*—In this technical note, cone-copositive piecewise quadratic Lyapunov functions (PWQ-LFs) for the stability analysis of conewise linear systems are proposed. The existence of a PWQ-LF is formulated as the feasibility of a cone-copositive programming problem which is represented in terms of linear matrix inequalities. A constructive procedure for its solution is provided. Examples show the effectiveness of the approach, also in the case of uncertain conewise linear systems.**

***Index Terms*—Conewise linear systems, LMIs, piecewise quadratic Lyapunov functions, stability analysis, switched systems, uncertain systems.**

I. Introduction

Piecewise linear systems represent a particular class of hybrid systems characterized by a partition of the state space into regions where system dynamics can be described by linear models [1]. We consider piecewise linear systems where the state partition consists of convex polyhedral cones and in each cone the dynamics are linear time-invariant. Such systems are called conewise linear systems [2], and can be viewed as a particularization of piecewise linear differential inclusions [3], [4], state-dependent switched linear systems [5], or linear parameter varying systems [6]. Despite their apparent modeling simplicity, the stability analysis of conewise linear systems is a hard issue. The simplest way to tackle the problem consists in employing a common Lyapunov function [7]. To reduce conservativeness, one could use the multiple Lyapunov functions approach, i.e., to combine Lyapunov functions defined over different regions of the state space, see [5], [8]. In particular, when the regions of the state space are convex polyhedra, piecewise quadratic Lyapunov functions (PWQ-LFs) can be used for solving the stability problem in terms of linear matrix inequalities (LMIs). The LMIs are obtained by applying the $\mathcal{S}$-procedure with quadratic forms representing regions that include the polyhedra [9]–[11].

In this technical note, we propose a new approach which exploits the particular structure of conewise linear systems and explicitly considers the conic constraints by formulating a so called cone-copositive problem, thus obtaining less conservative results. Recently, several papers have appeared in the mathematical literature dealing with copositive programming, see [12], [13] and the references therein. However, these specialist papers focus on searching conditions for which a given matrix is cone-copositive. The stability problem posed in this technical note is more challenging because it consists in finding a set of *unknown* cone-copositive matrices that defines the PWQ-LF. The search of a PWQ-LF for the conewise linear system is expressed as a feasibility problem of cone-copositive programming. Necessary and sufficient conditions for the existence of such a PWQ-LF are derived and a corresponding algorithm is given. The extension to uncertain conewise linear systems is also considered. The effectiveness of the proposed approach is illustrated by analyzing three examples: the exponential stability of a fourth order conewise linear system, the robust stability of a linear system subject to a parametric time-varying uncertainty, and the absolute stability of a Lur'e system with asymmetric sector conditions.

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II. Cone-Copositive and Copositive Problems

In this section, it is shown how cone-copositive problems can be solved by translating them into equivalent copositive problems. To this aim some preliminary definitions are recalled.

Given λ points $v_\ell \in \mathbb{R}^n$, $\ell \in \{1,\ldots,\lambda\}$, a conical hull is the set of points $v \in \mathbb{R}^n$ such that $v = \sum_{\ell=1}^{\lambda} \theta_\ell v_\ell$, with $\theta_\ell \in \mathbb{R}_+$. A convex hull $\overline{\mathrm{co}}\{v_1, v_2, \ldots, v_\lambda\}$ is a conical hull with $\sum_{\ell=1}^{\lambda} \theta_\ell = 1$. Given λ points $v_\ell \in \mathbb{R}^n$, $\ell \in \{1,\ldots,\lambda\}$, which are affinely independent, i.e., the $\lambda-1$ points $v_2 - v_1, \ldots, v_\lambda - v_1$ are linearly independent, a $(\lambda-1)$-simplex is the convex hull $\overline{\mathrm{co}}\{v_1, v_2, \ldots, v_\lambda\}$. The simplex *vertices* are the vectors v_ℓ, $\ell \in \{1,\ldots,\lambda\}$. Clearly in order to define a simplex in $\mathbb{R}^n$ we need $\lambda \le n+1$. A set $\mathcal{C} \subset \mathbb{R}^n$ is a simplicial cone if it is the conical hull of n linearly independent points. Polyhedral cones with nonempty interiors (proper polyhedral cones) can be always partitioned into a finite number of simplicial cones [14]. Hereinafter, without loss of generality, simplicial cones are considered. Given a simplicial cone $\mathcal{C} \subset \mathbb{R}^n$ there exists a nonsingular matrix $R \in \mathbb{R}^{n\times n}$, such that for any $v \in \mathcal{C}$ one can write $v = R\theta$ where $\theta \in \mathbb{R}^n_+$. The matrix R identifies the so-called $\mathcal{V}$*-representation* of the cone and its columns r_j, $j \in \{1,\ldots,n\}$, are the *extremal rays* of the cone. Each extremal ray is uniquely defined up to a positive multiple.

Given a set $X \subseteq \mathbb{R}^n$ and a finite positive integer η, a *partition* of X is the family $\mathcal{P} = \{X_h\}_{h=1}^{\eta}$ of sets satisfying $X = \cup_{h=1}^{\eta} X_h$, with $\mathrm{int}(X_h) \neq \emptyset$ for all h and $\mathrm{int}(X_h) \cap \mathrm{int}(X_m) = \emptyset$ for $h \neq m$. The particular case $\eta = 1$, i.e., $\mathcal{P} = \{X\}$, is called *trivial partition* of X. If the sets X_h are simplices the partition is called a *simplicial partition* of X [15]. We denote by $\mathcal{V}(X_h)$ the set of vertices of the simplex X_h and by V_h the matrix having those vertices as columns. We can define a measure of the "fineness" of the simplicial partition $\mathcal{P}$ as

$$\delta(\mathcal{P}) \triangleq \max_{X_h \in \mathcal{P}} \max_{u,v \in \mathcal{V}(X_h)} \|u - v\|. \tag{1}$$

Consider the set $\mathcal{B}_1 = \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$, $\|\cdot\|_1$ being the 1-norm of a vector, i.e., the sum of the absolute values of the vector components. The simplex $\mathcal{S} = \mathbb{R}^n_+ \cap \mathcal{B}_1$ is called *standard simplex*. It is always possible to find a simplicial partition of the standard simplex with $(n-1)$-simplices [15]. If the sets X_h are simplicial cones the partition is called a *simplicial conic partition* of X. We denote by

$\{R_h\}_{h=1}^{\eta}$ the set of extremal ray matrices defining the cones of the simplicial conic partition. Note that given a simplicial conic partition of $\mathbb{R}_+^n$ the intersections of its cones with $\mathcal{B}_1$ uniquely provide a simplicial partition of the standard simplex $\mathcal{S} = \mathbb{R}_+^n \cap \mathcal{B}_1$, say $\mathcal{P}_\mathcal{S}$, and also the converse holds.

A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is cone-copositive with respect to a cone $\mathcal{C} \subseteq \mathbb{R}^n$ if and only if $v^\mathsf{T} M v \geq 0$ for any $v \in \mathcal{C}$. A cone-copositive matrix will be denoted by $M \succcurlyeq^{\mathcal{C}} 0$. If the equality only holds for $v = 0$, then M is *strictly cone-copositive* and the notation is $M \succ^{\mathcal{C}} 0$. In the particular case $\mathcal{C} = \mathbb{R}_+^n$, a (strictly) cone-copositive matrix is called (strictly) *copositive*, i.e., $(M \succ^{\mathbb{R}_+^n} 0)$ $M \succcurlyeq^{\mathbb{R}_+^n} 0$. The notation $M \succcurlyeq 0$, i.e., without any superscript on the inequality, indicates that M is positive semidefinite. The cone-copositivity evaluation of a symmetric matrix M on a simplicial cone $\mathcal{C}$ can be always transformed into an equivalent copositive problem, as stated by the following result [12, Corollary 2.21].

Lemma 1: Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathcal{C} \subset \mathbb{R}^n$ a simplicial cone. Then $M \succ^{\mathcal{C}} 0$ if and only if

$$R^\mathsf{T} M R \succ^{\mathbb{R}_+^n} 0 \tag{2}$$

where $R \in \mathbb{R}^{n\times n}$ is the matrix of extremal rays of any $\mathcal{V}$-representation of $\mathcal{C}$.

The strict copositivity can be checked through LMIs by using the simplicial partition of the standard simplex, so as shown below.

Lemma 2: Let $\widetilde{M} \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $\mathcal{P}_\mathcal{S} = \{X_h\}_{h=1}^{\eta}$ be a simplicial partition of the standard simplex $\mathcal{S}$, and $\{V_h\}_{h=1}^{\eta}$, with $V_h \in \mathbb{R}^{n\times n}$, the matrices of the simplices vertices. If there exist symmetric (entrywise) positive matrices $\{N_h\}_{h=1}^{\eta}$ such that

$$V_h^\mathsf{T} \widetilde{M} V_h - N_h \succcurlyeq 0, \quad \forall h \in \{1,\ldots,\eta\}, \tag{3}$$

then $\widetilde{M} \succ^{\mathbb{R}_+^n} 0$.

Proof: The proof follows the arguments of Theorem 2.1 in [13]. Say x a vector belonging to the standard simplex $\mathcal{S}$ and X_h a simplex in $\mathcal{P}_\mathcal{S}$ containing x. Then there exists $\theta_h \in \mathbb{R}_+^n$ such that $x = V_h\theta_h$ and

$$x^\mathsf{T} \widetilde{M} x = (V_h\theta_h)^\mathsf{T} \widetilde{M} (V_h\theta_h) = \theta_h^\mathsf{T} V_h^\mathsf{T} \widetilde{M} V_h \theta_h.$$

From (3) it is $V_h^\mathsf{T} \widetilde{M} V_h - N_h = Q_h$ with $Q_h \succcurlyeq 0$. Then

$$x^\mathsf{T} \widetilde{M} x = \theta_h^\mathsf{T} (Q_h + N_h) \theta_h > 0. \tag{4}$$

Lemma 1.a in [16] shows that $x^\mathsf{T} \widetilde{M} x > 0$ with $x \in \mathcal{S}$ is equivalent to $x^\mathsf{T} \widetilde{M} x > 0$ with $x \in \mathbb{R}_+^n - \{0\}$. Then from (4) one can conclude that $\widetilde{M}$ is strictly copositive. ∎

The next lemma guarantees the search on a finite number of positive semidefinite conditions in order to obtain the strictly copositive result.

Lemma 3: Let $\widetilde{M} \in \mathbb{R}^{n\times n}$ be a symmetric strictly copositive matrix, i.e., $\widetilde{M} \succ^{\mathbb{R}_+^n} 0$. Then there exists a $\bar{\delta} > 0$ such that for any simplicial partition $\mathcal{P}_\mathcal{S} = \{X_h\}_{h=1}^{\eta}$ of the standard simplex $\mathcal{S}$ with $\delta(\mathcal{P}_\mathcal{S}) \leq \bar{\delta}$ the inequalities (3) hold with $\{V_h\}_{h=1}^{\eta}$ being the matrices of the simplices vertices and $\{N_h\}_{h=1}^{\eta}$ some symmetric positive matrices.

Proof: From Theorem 2 in [16], since $\widetilde{M}$ is strictly copositive, then there exists a $\bar{\delta} > 0$ such that for any simplicial partition $\mathcal{P}_\mathcal{S} = \{X_h\}_{h=1}^{\eta}$ of the standard simplex $\mathcal{S}$ with $\delta(\mathcal{P}_\mathcal{S}) \leq \bar{\delta}$ the matrices $V_h^\mathsf{T} \widetilde{M} V_h$ are entrywise positive. Then, for each h there exist a $Q_h \succcurlyeq 0$ and a symmetric positive matrix N_h such that $V_h^\mathsf{T} \widetilde{M} V_h = Q_h + N_h$, which implies (3). ∎

By exploiting the above lemmas we can prove the following theorem.

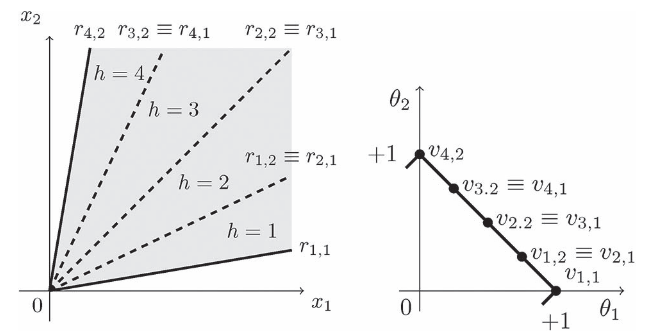

Fig. 1. A simplicial cone in $\mathbb{R}^2$ (gray area in the left figure) with the refinement induced by a simplicial partition of the standard simplex (right), i.e., $R = [r_{1,1}\ r_{4,2}]$, $R_h = [r_{h,1}\ r_{h,2}]$, $V_h = [v_{h,1}\ v_{h,2}]$, $h = 1,\ldots,4$.

Theorem 1: Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathcal{C}$ a simplicial cone. Then

$$M \succ^{\mathcal{C}} 0 \tag{5}$$

if and only if there exist a simplicial conic partition of $\mathcal{C}$ with corresponding extremal ray matrices $\{R_h\}_{h=1}^{\eta}$, with $R_h \in \mathbb{R}^{n\times n}$, and symmetric positive matrices $\{N_h\}_{h=1}^{\eta}$ such that

$$R_h^\mathsf{T} M R_h - N_h \succcurlyeq 0, \quad \forall h \in \{1,\ldots,\eta\}. \tag{6}$$

Proof: Assume that (6) hold for a simplicial conic partition of $\mathcal{C}$. Choose

$$V_h = R^{-1} R_h \Gamma_h, \quad h \in \{1,\ldots,\eta\} \tag{7}$$

where $R \in \mathbb{R}^{n\times n}$ is the matrix of extremal rays of a $\mathcal{V}$-representation of the cone $\mathcal{C}$ and Γ_h is the diagonal matrix with positive diagonal elements which ensures the columns of V_h having unitary 1-norm, i.e., the (j,j)-th element is given by $1/\|R^{-1} r_{h,j}\|_1$ for $j \in \{1,\ldots,n\}$. Then the columns of V_h are the vertices of the simplex X_h, defining for $h \in \{1,\ldots,\eta\}$ a simplicial partition of the standard simplex. By using (7), the conditions (6) become

$$\Gamma_h^{-\mathsf{T}} V_h^\mathsf{T} R^\mathsf{T} M R V_h \Gamma_h^{-1} - N_h \succcurlyeq 0, \quad \forall h \in \{1,\ldots,\eta\}. \tag{8}$$

Since $\Gamma_h^\mathsf{T} N_h \Gamma_h$ are symmetric positive matrices, by using Lemma 2 and then Lemma 1 one can conclude that $M \succ^{\mathcal{C}} 0$.

Assume that $M \succ^{\mathcal{C}} 0$. From Lemma 1 it follows $R^\mathsf{T} M R \succ^{\mathbb{R}_+^n} 0$ where $R \in \mathbb{R}^{n\times n}$ is the matrix of extremal rays of a $\mathcal{V}$-representation of $\mathcal{C}$. Then, by applying Lemma 3, there exists a sufficiently fine simplicial partition of the standard simplex such that

$$V_h^\mathsf{T} R^\mathsf{T} M R V_h - N_h \succcurlyeq 0, \quad \forall h \in \{1,\ldots,\eta\} \tag{9}$$

for some positive matrices N_h. Choose

$$R_h = R V_h, \quad h \in \{1,\ldots,\eta\}. \tag{10}$$

The invertibility of R ensures that (10) define a simplicial conic partition of $\mathcal{C}$. Then (6) hold on that simplicial conic partition and the proof is complete. ∎

Fig. 1 illustrates a four cones *refinement* of a cone $\mathcal{C} \subset \mathbb{R}^2$ and the corresponding simplicial partition of the standard simplex $\mathcal{S}$ through (10). In other words in order to check (5) one could choose a partition of the standard simplex (which provides V_h), then compute R_h by using (10) with R already defined by $\mathcal{C}$, and searching matrices N_h which allow to satisfy the LMIs (6). A possible approach for refining the partition of $\mathcal{C}$ consists in using (10) with V_h determined by applying the bisection along the longest edge technique [17] to the simplicial partition of the standard simplex.

A condition for a matrix of being not cone-copositive can be simply derived.

Lemma 4: Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathcal{C}$ a simplicial cone. Then M is not cone-copositive with respect to $\mathcal{C}$ if and only if there exists a simplicial conic partition of $\mathcal{C}$, with corresponding extremal ray matrices $\{R_h\}_{h=1}^{\eta}$, with $R_h \in \mathbb{R}^{n\times n}$, such that

$$r_{h,j}^{\mathsf{T}} M r_{h,j} < 0 \tag{11}$$

holds for some extremal ray $r_{h,j}$ of some R_h.

Proof: The lemma can be demonstrated by using the above Lemma 1 and Lemma 2.3 in [13]. ■

III. Continuous PWQ Functions on Conic Partitions

The stability result provided in this technical note is based on the use of a continuous PWQ-LF defined over a conic partition of the state space. In this section conditions will be provided for guaranteeing the continuity of PWQ functions.

Let us consider a partition of $\mathbb{R}^n$, say $\mathcal{P}_{\mathbb{R}^n}$, into a family of λ simplicial cones, i.e., a *simplicial complete fan* in $\mathbb{R}^n$. The cones are represented by

$$\mathcal{C}_i = \{R_i\theta, \theta \in \mathbb{R}_+^n\}, \quad i \in \{1,\ldots,\lambda\} \tag{12}$$

where $R_i \in \mathbb{R}^{n\times n}$ is a nonsingular matrix whose columns are the extremal rays of a $\mathcal{V}$-representation of $\mathcal{C}_i$. For the simplicial conic partition $\mathcal{P}_{\mathbb{R}^n}$ one can define a set of so-called continuity matrices [3].

Definition 1 (Continuity Matrices Set): A set of nonzero matrices $F_i \in \mathbb{R}^{q\times n}$ with $q \geq n$, say $\mathcal{F} = \{F_i\}_{i=1}^{\lambda}$, is a set of continuity matrices for the simplicial conic partition $\mathcal{P}_{\mathbb{R}^n}$ given by (12) if

$$F_i x = F_j x \tag{13}$$

for any pair $F_i, F_j \in \mathcal{F}$ and any $x \in \mathcal{C}_i \cap \mathcal{C}_j$.

The elements of the set $\mathcal{F}$ are not unique. For instance one could simply choose all F_i equal to the identity matrix or one could find others by appending identity matrix blocks.

In the following, a construction procedure of the continuity matrices useful for our purposes is provided. The procedure also shows that it is always possible to define continuity matrices whose number of rows is equal to the number of distinct extremal rays of the considered partition.

Define the matrix $\widetilde{R} \in \mathbb{R}^{n\times q}$, with $q > n$, whose columns r_j are all distinct extremal rays of the partition $\mathcal{P}_{\mathbb{R}^n}$. For each simplicial cone $\mathcal{C}_i$ represented as in (12), define an extraction matrix $E_i \in \mathbb{R}^{q\times n}$ as follows. The j-th row of E_i is zero for all j such that $r_j \notin \mathcal{C}_i$ and the remaining rows of E_i are equal to the rows of the n-dimensional identity matrix. Then the extremal ray matrix of $\mathcal{C}_i$ is $R_i = \widetilde{R}E_i$. As shown in [9], the choices

$$F_i = E_i(\widetilde{R}E_i)^{-1}, \quad i \in \{1,\ldots,\lambda\} \tag{14}$$

provide matrices F_i which satisfy (13).

Define the PWQ function

$$V(x) = x^{\mathsf{T}} P_i x \quad \text{if } x \in \mathcal{C}_i, \quad i \in \{1,\ldots,\lambda\} \tag{15}$$

where $\{P_i\}_{i=1}^{\lambda}$ are symmetric matrices. The function (15) is not guaranteed to be continuous across the cones boundaries, in general. A parametrization through the continuity matrices allows to obtain a continuous PWQ function, according to the following lemma.

Lemma 5: The PWQ function (15) defined over a simplicial conic partition (12) is continuous if and only if there exists a symmetric matrix $T \in \mathbb{R}^{q\times q}$ such that

$$P_i = F_i^{\mathsf{T}} T F_i, \quad \forall i \in \{1,\ldots,\lambda\} \tag{16}$$

for $\{F_i\}_{i=1}^{\lambda}$ having the expression (14).

Proof: If the P_i in (15) are chosen as in (16) with (14), then the PWQ function (15) is continuous because F_i are continuity matrices. The necessary part directly follows from the application of Lemma 1 in [11] with the continuity matrices as in (14). ■

IV. Stability Results

A conewise linear system is described by a finite set of linear time-invariant dynamics as follows:

$$\dot{x} = A_i x \quad \text{if } x \in \mathcal{C}_i, \quad i \in \{1,\ldots,\lambda\} \tag{17}$$

where $x \in \mathbb{R}^n$, $\{A_i\}_{i=1}^{\lambda}$ are known real matrices and $\{\mathcal{C}_i\}_{i=1}^{\lambda}$ are simplicial cones given by (12) which define a simplicial conic partition of R^n.

Assumption 1: Herein, it is assumed that (17) has at least one absolutely continuous solution $x(t)$, $t \in \mathbb{R}_+$, for any initial condition x_0.

The assumption above implies that the results of the technical note cannot deal with systems with sliding behavior.

The stability of the zero solution of (17) is analyzed by using Lyapunov arguments. Consider a PWQ function in the form

$$V(x) = x^{\mathsf{T}} P_i x \quad \text{if } x \in \mathcal{C}_i, \quad i \in \{1,\ldots,\lambda\}. \tag{18}$$

If there exist $\{P_i\}_{i=1}^{\lambda}$ matrices such that (18) is continuous, strictly positive and strictly decreasing in time along all solutions of (17), then the system is globally exponentially stable [3] and the function $V(x)$ is called a PWQ-LF for (17).

Since $x(t)$ is assumed to be absolutely continuous, the function $V(x(t))$ is also piecewise differentiable with respect to time, i.e., it can be non-differentiable only on the cones boundaries. By taking the time derivative of $V(x(t))$ and by using (17), the search for a PWQ-LF can be formulated as the following problem.

Problem 1: Find $\{P_i\}_{i=1}^{\lambda}$ with $P_i \in \mathbb{R}^{n\times n}$ such that (18) is continuous and

$$P_i \succ^{\mathcal{C}_i} 0 \tag{19a}$$

$$-\left(A_i^{\mathsf{T}} P_i + P_i A_i\right) \succ^{\mathcal{C}_i} 0 \tag{19b}$$

for all $i \in \{1,\ldots,\lambda\}$.

From Lemma 5, it follows that the continuity of (18) is equivalent to consider $P_i = F_i^{\mathsf{T}} T F_i$, $i \in \{1,\ldots,\lambda\}$, and $\{F_i\}_{i=1}^{\lambda}$ given by (14). Therefore, Problem 1 can be rewritten as follows.

Problem 2: Find $T \in \mathbb{R}^{q\times q}$ such that

$$F_i^{\mathsf{T}} T F_i \succ^{\mathcal{C}_i} 0 \tag{20a}$$

$$-\left(A_i^{\mathsf{T}} F_i^{\mathsf{T}} T F_i + F_i^{\mathsf{T}} T F_i A_i\right) \succ^{\mathcal{C}_i} 0 \tag{20b}$$

for all $i \in \{1,\ldots,\lambda\}$, with $\{F_i\}_{i=1}^{\lambda}$ given by (14).

The solution of (20) can be obtained by solving a suitable set of corresponding LMIs without introducing any conservatism.

Lemma 6: Problem 2 has a solution if and only if there exists a simplicial conic partition for each $\mathcal{C}_i$ with corresponding extremal ray matrices $\{R_{i,h}\}_{h=1}^{\eta_i}$, with $R_{i,h} \in \mathbb{R}^{n\times n}$, such that the set of LMIs

$$R_{i,h}^\top F_i^\top T F_i R_{i,h} - N_{P_{i,h}} \succcurlyeq 0 \quad (21a)$$

$$-R_{i,h}^\top \left(A_i^\top F_i^\top T F_i + F_i^\top T F_i A_i\right) R_{i,h} - N_{Q_{i,h}} \succcurlyeq 0 \quad (21b)$$

for all $i \in \{1,\ldots,\lambda\}$, for all $h \in \{1,\ldots,\eta_i\}$, with $\{F_i\}_{i=1}^{\lambda}$ given by (14), has a solution $\{T, N_{P_{i,h}}, N_{Q_{i,h}}\}_{i=1,h=1}^{\lambda,\eta_i}$ with $T \in \mathbb{R}^{q\times q}$ symmetric matrix, $N_{P_{i,h}}$ and $N_{Q_{i,h}}$ symmetric positive matrices.

Proof: The proof follows by applying Theorem 1 to all conditions (20). ■

Lemma 6 allows to derive a sufficient condition for the exponential stability of (17).

Theorem 2: The conewise linear system (17) is globally exponentially stable if there exists a simplicial conic partition for each $\mathcal{C}_i$ with corresponding extremal ray matrices $\{R_{i,h}\}_{h=1}^{\eta_i}$, with $R_{i,h} \in \mathbb{R}^{n\times n}$, such that the set of LMIs (21), with $\{F_i\}_{i=1}^{\lambda}$ given by (14), has a solution $\{T, N_{P_{i,h}}, N_{Q_{i,h}}\}_{i=1,h=1}^{\lambda,\eta_i}$ with $T \in \mathbb{R}^{q\times q}$ symmetric matrix, $N_{P_{i,h}}$ and $N_{Q_{i,h}}$ symmetric positive matrices.

Proof: From Lemma 6, if (21) have a solution then T is also a solution for (20). Therefore the function (18) with with $\{P_i\}_{i=1}^{\lambda}$ given by (16) is continuous, strictly positive and strictly decreasing along the system trajectories. Then it is a PWQ-LF for (17) which is globally exponentially stable. ■

The practical interest of the above result resides in the fact that any solution of the LMIs (21) directly provides the matrices of a specific PWQ-LF.

In order to define an algorithm for the construction of a PWQ-LF by (21), it is convenient to provide simple necessary conditions for the existence of a PWQ-LF which can be used as exit conditions for the algorithm iterations. To this aim one can consider (19) without invoking the continuity of (18).

Lemma 7: The conewise linear system (17) does not admit a PWQ-LF if there exists a simplicial conic partition of some $\mathcal{C}_i$ with corresponding extremal ray matrices $\{R_{i,h}\}_{h=1}^{\eta_i}$, with $R_{i,h} \in \mathbb{R}^{n\times n}$, such that the inequalities

$$r_{i,h,j}^\top P_i r_{i,h,j} > 0 \quad (22a)$$

$$-r_{i,h,j}^\top \left(A_i^\top P_i + P_i A_i\right) r_{i,h,j} > 0 \quad (22b)$$

have no solution P_i for some column $r_{i,h,j}$ of some $R_{i,h}$.

Proof: The proof follows from the definition of strict copositivity applied to (19). ■

With Theorem 2 the problem of the existence of a PWQ-LF is simplified as to find a single matrix T and suitable symmetric positive matrices $\{N_{P_{i,h}}, N_{Q_{i,h}}\}_{i=1,h=1}^{\lambda,\eta_i}$ such that the set of LMIs (21) is satisfied. In other words, if Problem 2 has a solution then it is always possible to find for each $\mathcal{C}_i$ a refinement consisting of a simplicial conic partition such that the set of LMIs (21) is feasible. By using Lemma 7 if the set of inequalities (22) has no solution for some refinement of some $\mathcal{C}_i$, then Problem 2 does not admit a solution with that set of cones $\mathcal{C}_i$.

A key point of Theorem 2 consists of finding a suitable refinement for each cone $\mathcal{C}_i$. The proof of Theorem 1 shows that such a search corresponds to finding a sufficiently fine simplicial partition $\mathcal{P}_\mathcal{S}$ of the standard simplex $\mathcal{S}$. Without loss of generality, by using the result of Lemma 3, the same partition $\mathcal{P}_\mathcal{S}$ of the standard simplex can be considered for all cones. Starting from the trivial partition of the standard simplex, the partition can be conveniently refined by using the bisection along the longest edge technique [17]. This technique guarantees that the fineness $\delta(\mathcal{P}_\mathcal{S})$ goes to zero as the refinement steps proceed. The procedure is implemented in Algorithm 1.

Algorithm 1: Algorithm for finding a PWQ-LF

```
Data: {A_i, R_i}_{i=1}^λ , η_max, i.e., the conewise linear system
      and the maximum number of elements for P_S
Result: "No PWQ-LF function exists" or
      "A PWQ-LF (18) exists with {P_i}_{i=1}^λ found" or
      "η_max has been reached"
begin
  n ← size(A_i);
  η ← 1, V_1 ← I_n;
  {F_i}_{i=1}^λ ← (14);
  repeat
    for i ← 1 to λ do
      for h ← 1 to η do
        R_{i,h} ← R_i V_h;
      end
    end
    necCond ← feasible (22);
    if necCond then
      suffCond ← feasible (21);
      if suffCond then
        /* Compute the solution */
        {T, N_{P_{i,h}}, N_{Q_{i,h}}}_{i=1,h=1}^{λ,η} ←
        getSolution (21);
        {P_i}_{i=1}^λ ← {F_i^T T F_i}_{i=1}^λ ;
      else
        /* Refinement */
        {h̄, v_{h̄,l}, v_{h̄,m}} ←
        maxDistanceColumn ({V_h}_{h=1}^η);
        w ← (1/2)(v_{h̄,l} + v_{h̄,m});
        V'_h̄ ← [v_{h̄,1} ··· v_{h̄,l-1} w v_{h̄,l+1} ··· v_{h̄,n}];
        V''_h̄ ←
        [v_{h̄,1} ··· v_{h̄,m-1} w v_{h̄,m+1} ··· v_{h̄,n}];
        {V_h}_{h=1}^{η+1} ←
        {V_1, ..., V_{h̄-1}, V'_h̄, V''_h̄, V_{h̄+1}, ..., V_η};
        η ← η + 1;
      end
    end
  until (NOT(necCond) OR suffCond OR η > η_max);
end
```

Note that given the system order n, the number of cones λ, the number of extremal rays q and the number of elements of the cones partitions η_i, the $2\sum_{i=1}^{\lambda}\eta_i$ LMIs (21) of order $n\times n$ have $0.5[q^2 + q + 2(n^2+n)\sum_{i=1}^{\lambda}\eta_i]$ unknown variables to be found.

Remark 1: Theorem 2 can be used also if the cones of the conewise linear system are polyhedral but not simplicial. In particular, each polyhedral cone can be partitioned into simplicial cones and, in order to define (17), the same dynamic matrix can be considered for all cones of that partition. The PWQ-LF must be intended with respect to the state space simplicial partition.

Remark 2: If the system (17) is shown not to admit a PWQ-LF with the given state space partition one can restart Algorithm 1 by using a new state space partition with a larger number of cones. The same approach can be used if the existence of a PWQ-LF cannot be established, i.e., Algorithm 1 reaches $\eta_{\max}$. In the latter case one could alternatively increase $\eta_{\max}$ by maintaining the same state space simplicial partition. Of course the increase of λ and/or $\eta_{\max}$ is paid with the increase of the computational burden, especially for high order systems.

The cone-copositive stability approach described above applies straightforwardly to the case of uncertain conewise linear systems. In particular, consider the generalization of (17) represented by the system

$$\dot{x} \in \overline{\mathrm{co}}_{k\in\{1,\ldots,\mu_i\}}\{A_{i,k}\}x \quad \text{if } x \in \mathcal{C}_i, \quad i \in \{1,\ldots,\lambda\} \tag{23}$$

where $\overline{\mathrm{co}}_{k\in\{1,\ldots,\mu_i\}}\{A_{i,k}\}$ is the convex hull of μ_i known matrices called the *extreme matrices* of the i-th cone. The stability analysis of (23) can be performed by extending Theorem 2 and by adapting Algorithm 1 accordingly. In details the corresponding stability conditions consist of (21a) and μ_i LMIs in the form (21b), one for each extreme matrix, with the same continuity matrix F_i but with possibly different symmetric positive matrices, say $\{N_{Q_{i,k,h}}\}_{i=1,k=1,h=1}^{\lambda,\mu_i,\eta_i}$.

V. Examples

The numerical results have been obtained by using `Matlab` and `yalmip` with a PC Intel Dual Core processor at 2.8 GHz. All inequalities have been also *a posteriori* checked to be strictly verified with a lowest bound equal to 10^{-8} (note that the LMIs (21) are not strict).

A. Example 1

Let us consider the fourth-order system given by Example 7 in [18]. It is a two mode switched linear system and the dynamics are given by the following system matrices A_1 (left) and A_2 (right)

$$\begin{pmatrix} -1 & -23 & 12 & -2 \\ -0.5 & 8.5 & -6 & 0.5 \\ 0.5 & 26 & -9.5 & 5 \\ -3 & -35 & 12 & -6 \end{pmatrix}, \begin{pmatrix} -1.4 & -18.6 & 8 & -1.6 \\ -0.3 & 8.3 & -4 & 1.3 \\ 1.7 & 20.6 & -5.7 & 3.6 \\ -3.4 & -28.6 & 8 & -4.6 \end{pmatrix}$$

with $\dot{x} = A_1 x$ for $g_1(x)\cdot g_2(x) \geq 0$ and $\dot{x} = A_2 x$ for $g_1(x)\cdot g_2(x) \leq 0$, where $g_1(x) = x_1 + 0.5x_2 + 1.5x_3 + 0.5x_4$ and $g_2(x) = x_1 - 0.5x_2 + 0.5x_3 - 0.5x_4$. As reported in [18], with this partition no PWQ-LF exists for this system. Anyway, we can represent the system as (17) by partitioning the state space into simplicial cones corresponding to the 16 different state space regions determined by the signum of $g_1(x)$, $g_2(x)$ and the two further linear combinations $g_3(x) = x_1$ and $g_4(x) = x_2$. By applying Algorithm 1 with such simplicial conic partition we proved the exponential stability of the system according to Theorem 2. The algorithm provides a positive answer in 1.3 s.

B. Example 2

Let us consider the second order system $\dot{x} \in \overline{\mathrm{co}}\{A_1, A_2\}x$ where $A_1 = [0\ 1; -2\ \ -1]$, $A_2 = [0\ 1; (-2-p)\ \ -1]$, with p a time-varying uncertain parameter. In the robust stability literature, the typical numerical problem analyzed for this example consists of finding the maximum positive real number, say p^*, such that the system is exponentially stable for any $p \in [0, p^*]$. In [19] it is derived that a theoretical upper bound for p^* is 6.98513. Assume the state space being uniformly partitioned into λ simplicial cones. By applying the extension of Algorithm 1 to the uncertain case, we obtained p^* equal to 6.850, 6.950, 6.982, 6.984 with λ equal to 40, 80, 420, 580, respectively and $\eta = 2$ different from 1 only for $p^* = 6.982$. The computational times required for solving the LMIs of the sufficient conditions are 2.3 s, 5.1 s, 86.8 s, 113.2 s, respectively. To the best of our knowledge our approach provides the largest value of p^* with respect to other PWQ-LFs approaches in the literature, e.g., [11], [20].

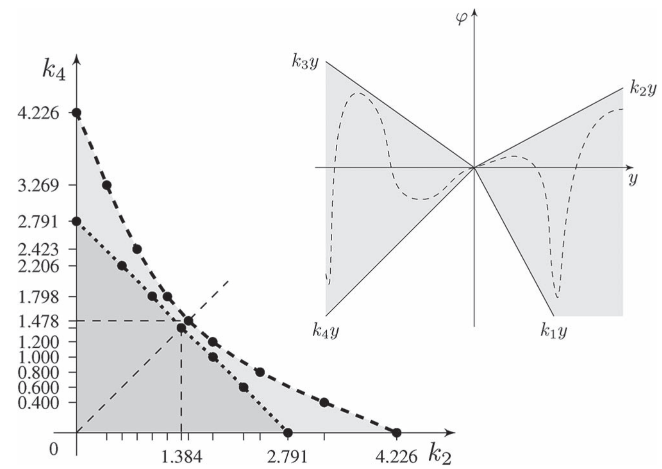

Fig. 2. Right: A generic feedback characteristic $\varphi(y)$ (dashed line) belonging to the uncertainty set (gray area). Left: The estimated region of absolute stability for the Lur'e system with $k_1 = k_3 = 0$ and $k_2, k_4 \geq 0$; the points correspond to numerically computed values; the dark gray area is for $\lambda = 8$, the union of light and dark gray areas is for $\lambda = 24$.

C. Example 3

Let us consider a Lur'e feedback system $\dot{x} = Ax + bu$, $y = c^{\mathsf{T}}x$ and $u = -\varphi(y)$, where (A, b, c^{T}) is a minimal realization in the observable canonical form. The uncertain characteristic φ is assumed to belong to the possibly asymmetric region given by the sector (k_1, k_2) with $k_1 \leq k_2$ when $y \geq 0$ and the sector (k_3, k_4) with $k_3 \leq k_4$ when $y \leq 0$, see Fig. 2. This will be indicated by the quadruple (k_1, k_2, k_3, k_4). By using global linearization [19] the system can be represented as

$$\dot{x} \in \overline{\mathrm{co}}\left\{A - bk_1c^{\mathsf{T}}, A - bk_2c^{\mathsf{T}}\right\}x \quad \text{if } c^{\mathsf{T}}x \geq 0 \tag{24a}$$

$$\dot{x} \in \overline{\mathrm{co}}\left\{A - bk_3c^{\mathsf{T}}, A - bk_4c^{\mathsf{T}}\right\}x \quad \text{if } c^{\mathsf{T}}x \leq 0 \tag{24b}$$

and can be written as (23) by considering the natural simplicial conic partition of $\mathbb{R}^n$ into the 2^n orthants. In particular, we consider

$$A = \begin{pmatrix} -5 & 1 & 0 \\ -7 & 0 & 1 \\ -11 & 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 20 \\ 20 \end{pmatrix}$$

and $c^{\mathsf{T}} = (1\ 0\ 0)$. By applying the circle criterion, which can only deal with symmetric domains (sectors), in the specific case $k_1 = k_3 = 0$ and $k_2 = k_4$ one gets the absolute stability for characteristics belonging to the symmetric sector (0, 1.14, 0, 1.14).

By applying the PWQ approach in [3], we were able to prove the stability with the asymmetric domain (0.1, 1.4, 0, 0.2) by partitioning the state space into the eight octants. However by maintaining the same state space partition we were not able to prove the absolute stability in the asymmetric domain (0, 1.4, 0, 0.2).

By using the proposed PWQ-LF approach, we proved the absolute stability in the domain $(0, k_2, 0, k_4)$ with the pairs (k_2, k_4) represented in Fig. 2. The computational time for getting each solution is always less than 2.5 s. As a by-product, we enlarged the absolute stability sector determined by the circle criterion to (0, 1.478, 0, 1.478). Note that the symmetry in Fig. 2 is due to the fact that the absolute stability with respect to the quadruple (k_1, k_2, k_3, k_4) is equivalent to the absolute stability with respect to the quadruple (k_3, k_4, k_1, k_2). In order to prove that, it is sufficient a simple change of variables applied to the state space representation of the Lur'e system.

VI. Conclusion

Conditions for the existence of a piecewise quadratic Lyapunov function and hence for the exponential stability of conewise linear systems have been provided. A corresponding algorithm based on cone-copositive programming allows to obtain less conservative results with respect to the existing approaches based on piecewise quadratic Lyapunov functions. Examples show that the proposed algorithm can be effectively used also for uncertain systems. Directions for future research could be to consider other refinement strategies and to exploit the results for control design problems.

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