LMIs for PWA systems

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Piecewise affine systems

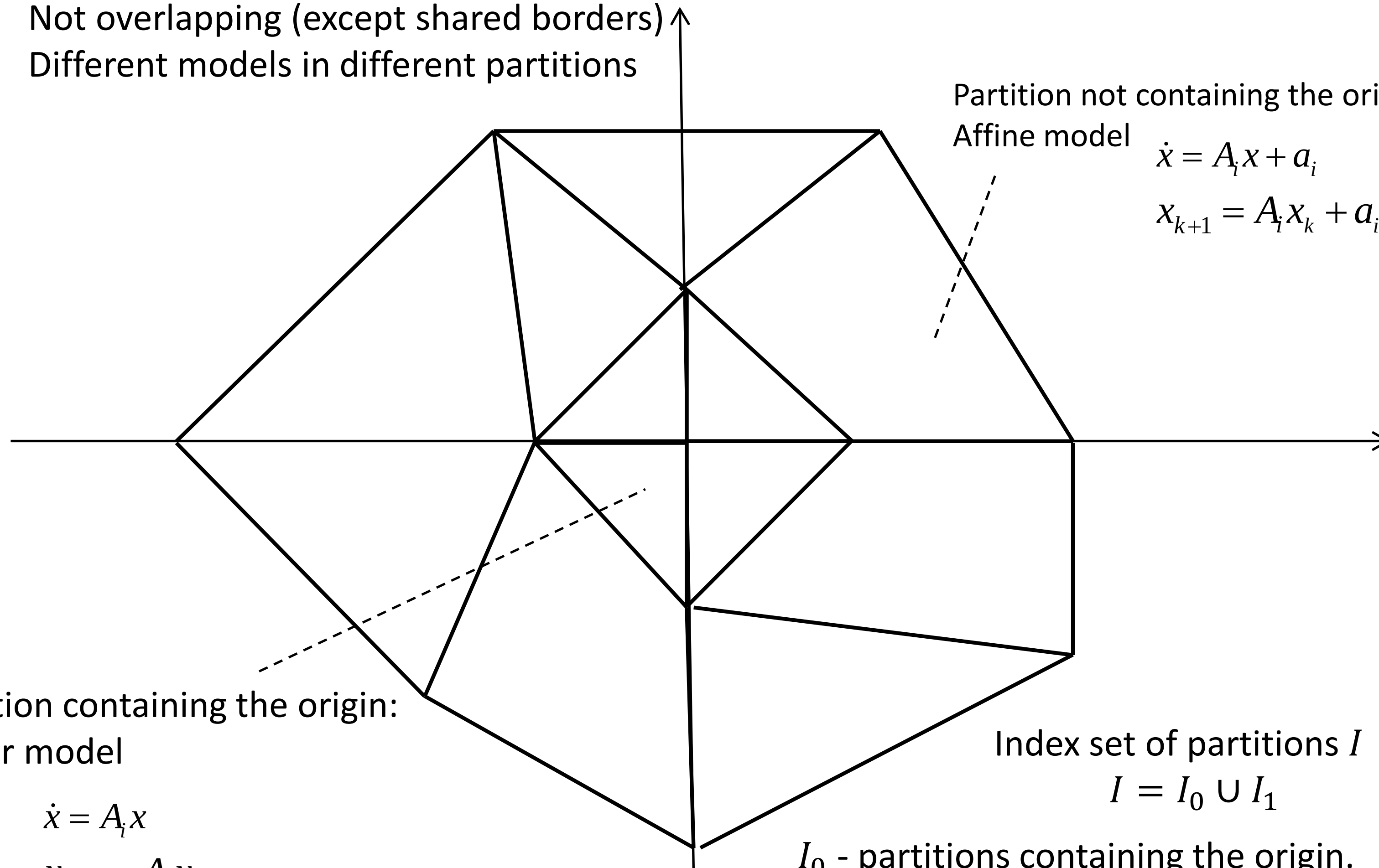
State space (or subset thereof) X divided into partitions X_i

- Not overlapping (except shared borders)
- Different models in different partitions

Partition not containing the origin:
Affine model

$$\dot{x} = A_i x + a_i$$

$$x_{k+1} = A_i x_k + a_i$$



Partition containing the origin:
Linear model

$$\dot{x} = A_i x$$

$$x_{k+1} = A_i x_k$$

Index set of partitions I

$$I = I_0 \cup I_1$$

I_0 - partitions containing the origin.
I_1- partitions not containg the origin

Stability of continuous time PWA systems

- LMIs naturally connected to quadratic Lyapunov functions
- Common Lyapunov function for PWL systems:
  - If $a_i = 0 \forall i$, and there exists a matrix $P = P^T > 0$ such that $A_i^T P + PA_i < 0 \forall i \in I$, then every trajectory tends to the origin exponentially.
  - If $a_i = 0 \forall i$, and there exists matrices $R_i > 0 \forall i \in I$ such that $\sum_{i \in I} \left(A_i^T R_i + R_i A_i \right) > 0$, then a common Lyapunov function cannot be found.

- If a common Lyapunov function cannot be found – or we are unable to find one, then we need to consider the possibility of allowing the LF to depend on the location in the state space.

Additional notation

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}$$

Since the individual partitions are polyhedra, we can construct

$$\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}, \bar{\mathrm{F}}_i = \begin{bmatrix} F_i & f_i \end{bmatrix},$$

With $e_i = 0$ and $f_i = 0 \forall i \in I_o$ such that

$$\overline{E_i} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \forall x \in X_i \forall i \in I$$

$$\overline{F_i} \begin{bmatrix} x \\ 1 \end{bmatrix} = \overline{F_j} \begin{bmatrix} x \\ 1 \end{bmatrix} \forall x \in X_i \cap X_j, \forall i, j \in I$$

Egentlig gjelder $e_i = 0$ for individuelle halvplan, ikke hele partisjonen.
Dvs. dersom en ‘grense mellom to partisjoner’ går gjennom origo er $e_i = 0$

Lyapunov function

$$V(x) = \begin{cases} x^T P_i x, & x \in X_i, I \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, & x \in X_i, I \in I_1 \end{cases}$$

Where $P_i = F_i^T T F_i, \quad \overline{P}_i = \overline{F}_i^T T \overline{F}_i$

With the specifications on F_i and $\overline{F}_i$, this parametrization ensures that the Lyapunov function is continuous across partition boundaries.

Relaxing the Lyapunov function

The LF has to be positive everywhere – but this does not have to hold for the P_i 's, they only have to be positive inside the corresponding partition X_i.
This is captured with the S-procedure, with relaxations of the form

$$x^T E_i^T U_i E_i x, \quad x \in X_i, I \in I_0$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{E}_i^T U_i \overline{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in X_i, I \in I_1$$

With the U_i constrained to have only non-negative elements, these relaxations are positive over their respective partitions X_i.
Similarly, the condition for the decrease (negative time derivative) of the LF only has to hold inside the partition in question.

Overall LF design

Find symmetric matrices T, U_i, and W_i, with U_i, and W_i having non-negative elements, such that

$$\begin{cases} A_i^T P_i + P_i A_i + E_i^T U_i E_i < 0 \\ P_i - E_i^T W_i E_i > 0 \end{cases} \quad i \in I_0$$

$$\begin{cases} \overline{A}_i^T \overline{P}_i + \overline{P}_i \overline{A}_i + \overline{E}_i^T U_i \overline{E}_i < 0 \\ \overline{P}_i - \overline{E}_i^T W_i \overline{E}_i > 0 \end{cases} \quad i \in I_1$$

Remember $\quad P_i = F_i^T T F_i, \quad \overline{P}_i = \overline{F}_i^T T \overline{F}_i$

Determining $\bar{E}_i$ and $\bar{F}_i$

Presentation on previous slides taken from
M. Johansson and A. Rantzer, IEEE Transactions on Automatic Control, Vol 43, No. 4, pp 555 – 559, 1998.

These results are further extended in
A. Rantzer and M. Johansson, IEEE Transactions on Automatic Control, Vol 45, No. 4, pp. 629 – 637, 2000

-Gives explicit formulas for calculating $\bar{E}_i$ and $\bar{F}_i$ when each polytopic partition is sub-partitioned into simplices
-Shows also how to handle unbounded partitions, extending to infinity in some directions
- Also provides some results on bounds on quadratic transient integrals and piecewise linear quadratic optimal control

Stability of discrete-time PWA systems

- Main differences from continuous time:
  - LF does not take continuous values anyway, do not have to impose continuity of the LF
  - Lyapunov difference inequality instead of a Lyapunov differential inequality
  - Have to track which partitions the state (can) go to in the next timestep

- For PWL systems, no common Lyapunov function can be found if there exists matrices $R_i > 0$ such that

$$\sum_{i \in I} (A_i^T R_i A_i - R_i) > 0$$

Additional notation

- Let $f(X_l)$ denote a quadratic function that is positive over the partition X_l
  - Particular choices are those proposed by Johansson and Rantzer

$$f(X_i) = \mathrm{x}^T\, E_i^T U_i E_i x, \quad x \in I_0$$

$$f(X_i) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{E}_i^T U_i \overline{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in I_1$$

  - Will, with some abuse of notation, use $f(X_l)$ both to denote the quadratic function as specified above, and the corresponding entry in the LMIs (i.e., without pre- and postmultiplication by x).
- Let X_{ij} denote the subset of partition X_i where the state in the next timestep moves to partition X_j.

Additional notation

- Define $\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \bar{x}_{k+1} = \bar{A}_i \bar{x}_k$.
- $\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 1 \end{bmatrix}, x_k \in X_i, i \in I_1, x_{k+1} \in X_j, j \in I_1$
- $\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, x_k \in X_i, i \in I_1, x_{k+1} \in X_j, j \in I_0$
- $\bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, x_k \in X_i, i \in I_0, x_{k+1} \in X_j, j \in I_0$
- Implicit in the above is the assumption that the region $\cup_{i \in I_0} X_i$ is positively invariant. If the system is stable near the origin, it is always possible to sub-partition the state space to fulfill this assumption
  - And the auxiliary state is changed from 1 to 0 as soon as the state enters this region

Stability criteria

$$P_i - f(X_i) > 0, \quad \forall i$$

$$P_i - \overline{A}_i^T P_j \overline{A}_i - f(X_{ij}) > 0, \ \forall (i, j) \text{ with nonempty } X_{ij}$$

Note that $i = j$ may be possible

Alternative relaxations

Let R denote a bounded polyhedron in state space,
with $V(R)$ the set of vertices of R. The following quadratic function is positive over R:

$$f(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T H \begin{bmatrix} x \\ 1 \end{bmatrix}$$

where

$$H = \bar{H} + C,$$
$$\bar{H} < 0,$$
$$C = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix},$$
$$c > 0 \text{ (scalar)},$$
$$\begin{bmatrix} v_i \\ 1 \end{bmatrix}^T [\bar{H} + C] \begin{bmatrix} v_i \\ 1 \end{bmatrix} > 0, \forall v_i \in V(R)$$

Alternative relaxations – unbounded partitions

For unbounded partitions we first have to embed these in a proper polyhedral cone – and if no proper embedding cone exists we have to sub-partition the unbounded partition.

Let r_m be a ray in the interior of the embedding polyhedral cone, chosen such that there is a set of $n-1$ rays $R_p(r_m)$, where each member of R_p is

- outside the embedding polyhedral cone
- orthogonal to r_m
- orthogonal to all other members of R_p

Define further the set $R_e = \{r_{e1} \quad \cdots \quad r_{e\rho}\}$ the set of generators for the embedding cone

Alternative relaxations – unbounded partitions

An indefinite quadratic function $f(x) = x^T F x$ is positive over the embedding polyhedral cone (except at the origin) provided

It is positive over all $n-1$-dimensional faces of the embedding cone, and

$$r_{pi}^T F r_{pi} \leq 0, \forall r_{pi} \in R_p(r_m)$$

Have to start from 1-dimensional faces (corresponding to $r_{ei}^T F r_{ei} \geq 0, \forall r_{ei} \in R_e$), and then derive conditions for higher-dimensional faces.

Alternative relaxations

Numerical experience indicates that the alternative relaxations are preferable for $n_x > 2$.

Example 1

- Open loop unstable system
$$x_{k+1} = 1.1x_k + u_k$$
- Input constraints $-2 \leq u_k \leq 2.$
- Quadratic MPC with
  - state weight $Q = 10$
  - Input weight $R = 0.1$
  - Prediction horizon: 5
- Corresponding LQR controller $u = -1.089x$
- Unconstrained region $-1.836 \ \leq x \leq 1.863$
- Feasible region for MPC $-8.722 \ \leq x \leq 8.722$
  - Eleven regions/partitions for the explicit solution

- Trivial inspection shows that the system can be stabilized for $-20 < x < 20$

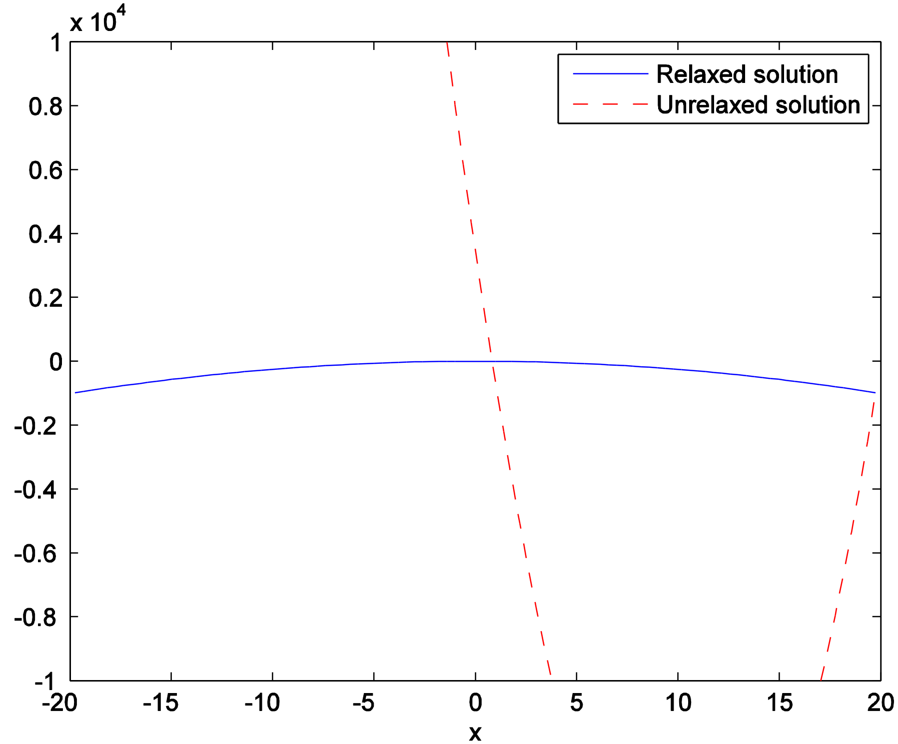
Example 1

- LMI solution: three regions
  - Unconstrained region (region 0)
  - Input at upper constraint limit (region 1)
  - Input at lower constraint limit (region 2)

- Easy to show stability for nearly the entire stabilizable region
  - Numerical problems occur when including states very close to ± 20.

Example 1

- Importance of relaxation
- Unrelaxed solution holds (decreases) locally, but fails globally
- Relaxation reduces decrease inside region 1, increases it elsewhere.



Lyapunov function difference inequality for states starting in region 1 and staying in region 1 for the next timestep (the region $3.487 < x < 20$)

Example 2

- Taken from Hovd & Olaru, MIC, vol 31, pp. 45-53, (example 3)
- Three state example, single input, single output. Constraints on output and input.
- Trying to achieve an approximation to explicit MPC by
  - Solving the MPC problem for a number of points in the state space
  - Use Delaunay tesselation on these points to partition the region concerned into simplices
    - Remember: simplices = ‘hypertriangles’
  - Linear interpolation on the input inside each simplex gives
    - Affine state feedback $u_k = K_i x_k + k_i$
    - Controller is continuous accross simplex borders
- But how do we prove stability
  - Not ensured even if original MPC is stable
- Try to prove stability using LMIs

Example 2

- Initial tesselation gives 199 regions
  - Reduced to 147 after merging neighbouring regions with the same affine controller
  - After merging, there are 1478 possible transitions between regions (including ‘transitions’ where $i = j$)
- Using conventional relaxations
  - LF positivity constraint $P_i - f(X_i) > 0$ fails for 86 regions
  - LF decrease constraint $P_i - A_i^T P_j A_i - f(X_{ij}) > 0$ fails for 1353 transitions
- Using alternative relaxations
  - LF decrease constraint fails for 4 transitions, all of which have $i = j$. Closer inspection reveals a fixed point in each of these regions. These regions are two pairs of neighbours.
  - Two extra points are added to the set of points used in the Delaunay tesselation
  - Stability proven with 155 regions and 1768 transitions
- Explicit MPC solution has 472 regions, can be reduced to 274 by merging.

Relaxations involving cone-copositivity

- The following results are taken from

R. Iervolino, F. Vasca, L. Ianelli: Cone-copositive Piecewise Quadratic Lyapunov Functions for Conewise Linear Systems.

IEEE Transactions on Automatic Control, Vol 60, No 11, 2015, pp. 3077 – 3082.

Recap on cones

- A cone is a set $\mathcal{C}$ such that $x \in \mathcal{C} \Longrightarrow \alpha x \in \mathcal{C} \forall \alpha \geq 0$
  - ($\alpha > 0$ for an open cone)
- A *proper* cone has a vertex at the origin
  - And thus a proper cone cannot have both $x \in \mathcal{C}$ and $-x \in \mathcal{C}$ for $x \neq 0$
- A polyhedral cone is a cone such that one can intersect the cone with a halfplane, and the intersection has the shape of a polyhedron.
- A simplicial cone is a proper polyhedral cone where the intersection is a simplex of dimension $n - 1$.
- Any proper polyhedral cone can be partitioned into a set of simplicial cones
- The results of this section apply only to simplicial cones
  - A (sufficiently fine) partitioning of polyhedral cones is assumed

Extreme rays and cone-copositivity

- A simplicial cone $\mathcal{C} \in \mathbb{R}^n$ is defined by n *extreme rays* v_i (the 'generators of the cone') such that

$$\mathcal{C} = \{x | x = R\theta, \theta \in \mathbb{R}_+^n\}$$

I.e., all elements of the vector θ are non-negative

where $R = [v_1 \cdots v_n]$

- A symmetric matrix $M \in \mathbb{R}^{n \times n}$ is cone-copositive with respect to the cone $\mathcal{C} \in \mathbb{R}^n$ if $x^T M x \ \geq 0 \forall x \in \mathcal{C}$.
- This is denoted $M \geq^{\mathcal{C}} 0$
- It is strictly cone-copositive if $x^T M x = 0 \Longrightarrow x = 0$.
  - This is denoted $M >^{\mathcal{C}} 0$
  - (Some publications insist on using 'curved' inequality signs)

Testing cone-copositivity

- $M \geq^{\mathcal{C}} 0$ if there exists a symmetric $\mathrm{N} \in \mathbb{R}^{n \times n}$ with non-negative elements such that $R^T M R - N \geq 0$

Continuous time PWQ LF design

Find symmetric matrices $T, N_{G,i}, N_{H,i}, \overline{N}_{G,i}, \overline{N}_{G,i}$ such that

$$\begin{cases} \mathrm{R}_i^T (A_i^T P_i + P_i A_i) R_i + N_{G,i} < 0 \\ R_i^R P_i R_i - N_{P,i} > 0 \end{cases} \quad i \in I_0$$

$$\begin{cases} \overline{R}_i^T (\overline{A}_i^T \overline{P}_i + \overline{P}_i \overline{A}_i) \overline{R}_i + \overline{N}_{G,i} < 0 \\ \overline{R}_i^T \overline{P}_i \overline{R}_i - \overline{N}_{H,i} > 0 \end{cases} \quad i \in I_1$$

Appears to be less conservative than the original relaxations by Johansson and Rantzer

where $T > 0$ and $N_{G,i}, N_{H,i}, \overline{N}_{G,i}, \overline{N}_{G,i}$ have non-negative elements, and $\bar{R}_i$ is the matrix of vertices of partition i in the ‘lifted’ space (with an extra dimension). Rays in unbounded directions will be independent of the augmented state.

Remember $\quad P_i = F_i^T T F_i, \quad \overline{P}_i = \overline{F}_i^T T \overline{F}_i$

Discrete time PWQ LF design

- For bounded partitions, relaxations by Hovd & Olaru appear less conservative and are straight forward to use
- For unbounded partitions, relaxations by Iervolino et al. are much simpler to use

Discrete time PWQ LF design

- For unbounded X_i and X_{ij} only, use Iervolino et al.

$$\overline{R}_i^T P_i \overline{R}_i - N_{H,i} > 0, \quad \forall i$$

$$\overline{R}_{ij}^T (P_i - \overline{A}_i^T P_j \overline{A}_i) \overline{R}_{ij} - N_{D,ij} > 0, \ \forall (i, j) \text{ with nonempty } X_{ij}$$

where $\bar{R}_i$ and $\bar{R}_{ij}$ are matrices of rays of the corresponding partition in the lifted space.

Comments

- For continuous-time systems, the possibility of 'sliding modes' have not been considered in the methods presented here.
- The publication by Iervolino et al. deals with piecewise *linear* systems defined on a conical partition of the state space. Extension to piecewise affine systems requires 'lifting' (introduction of an augmented 'state'). Application of the method of Iervolino et al. requires some care in defining the appropriate conical partitions in the augmented state space.