Exercises week 1

Simon Elias Schrader

August 23, 2024

1 Exercises on complex numbers

1. Write the following numbers in "algebraic" form (e.g. $a + ib$):

$$z = \frac{i-4}{2i-3} \qquad v = (1+i)^6 \qquad w = i^{17}$$

2. Write the following numbers of the form $re^{i\theta}$, $r \geq 0$:

$$z = -2 \qquad v = (1+i)^{30} \qquad u = (\cos(2\pi/3) + i\sin(2\pi/3))^3 \qquad w = -3+5i$$

3. Let $f(x) = x^2 - 2x + 4$. Find the points x_1, x_2 such that $f(x_1) = f(x_2) = 0$.

4. A function $f(x)$ taking a real number and returning a complex number $f : \mathbb{R} \to \mathbb{C}$ can always be written as $f(x) = g(x) + ih(x)$, where $g(x), h(x)$ are real functions e.g. $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$. Explain why we can also write $f(x)$ as $f(x) = e^{i\phi(x)}r(x)$ where $r(x), \phi(x)$ are real functions $r : \mathbb{R} \to [0, \infty)$, $\phi : \mathbb{R} \to [0, 2\pi)$.

5. Show that $|f(x)|^2 = f^*(x)f(x)$ is real for all complex functions $f : \mathbb{R} \to \mathbb{C}$.

6. Show that there is no complex number z such that $|z| - z = i$. Hint: Write $z = re^{i\theta}$.

7. Solve the integral for an arbitrary complex number k

$$\int_0^{2\pi} e^{ix+k} dx \tag{1}$$

2 Eigenfunctions

Eigenfunctions play an important role in quantum mechanics, and solving the Schrödinger equation essentially means to find such an eigenfunction. While we will not look at the Schrödinger equation yet, getting accustomed to the idea of eigenfunctions will make it easier to understand the calculations necessary to solve the Schrödinger equation.

1. Find some functions $f(x)$ such that $\frac{d^2}{dx^2}f(x) = -f(x)$. *Hint: I do not expect you to do any calculations. Put differently: Which functions are identical to their second derivative, except for a sign change?*
   $f(x)$ is called an eigenfunction of the *differential operator* $\frac{d^2}{dx^2}$ with eigenvalue -1.

2. We consider now the so-called "harmonic oscillator" model. We will not (yet) go through the physical implications of this model, but it is a very useful model in quantum mechanics.
   We want to find a function $f(x)$ such that

$$\left(x^2 - \frac{d^2}{dx^2}\right) f(x) = Ef(x). \tag{2}$$

   Show that $f(x) = e^{-\frac{1}{2}x^2}$ is such a function, and find E. *Hint: The notation* $\left(x^2 - \frac{d^2}{dx^2}\right) f(x)$ *means* $x^2 f(x) - f''(x)$

3. We consider now the more complicated eigenvalue problem

$$\left(x^2 - \frac{d^2}{dx^2}\right)^2 f(x) = Ef(x). \tag{3}$$

   Find at least one function $f(x)$ that satisfies this equation ($E \neq 0$). *Hint: No calculations required.*

4. We consider now a modified eigenvalue problem

$$\left(kx^2 - \frac{1}{2}\frac{d^2}{dx^2}\right) f(x) = Ef(x) \tag{4}$$

   where $k > 0$. Using the ansatz $f(x) = e^{-ax^2}$, find a as function of k such that $f(x)$ is eigenfunction of the eigenvalue problem in eq. 4. What is the value of E?

5. Finally, we want $f(x)$ to be *normalized*. That means that we want to to multiply $f(x)$ by some constant N (real, positive) and define $g(x) = Nf(x)$. N is chosen such that

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} N^2 |f(x)|^2 dx = 1 \tag{5}$$

   Find N as function of k. Feel free to use numerical software such as Wolframalpha to solve the integral, it is nasty to do this by hand. *Normalization is important in quantum mechanics. It essentially means that there is one particle in total, not more or less, but you'll learn more about this later.*

3 Extra: Math exercises from chapter 1

1. (P1.2 in the book) Show that the energy density radiated by a blackbody

$$\frac{E_{\text{total}}(T)}{V} = \int_0^\infty \rho(\nu, T) d\nu = \int_0^\infty \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu$$

   depends on the temperature as T^4. (Hint: Make the substitution of variables $x = h\nu/kT$.) Use the definite integral $\int_0^\infty \left[x^3/\left(e^x - 1\right)\right] dx = \pi^4/15$. Using your result, calculate the energy density radiated by a blackbody at 1100K and 6000K.