

Solutions week 2

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1 Classroom exercises

- P3.17 This property is called "linearity". Example of a non-linear operators are \sqrt{x}, x^2, e^x : $(f(x) + g(x))^2 \neq f^2(x) + g^2(x)$.
- Q3.5 If \hat{A} is Hermitian, that means that expectation values are real:

$$\langle A \rangle = \int \psi^*(x) [\hat{A}\psi(x)] dx.$$

Complex conjugating this expression gives

$$\langle A \rangle = \int \psi(x) [\hat{A}\psi(x)]^* dx$$

as integration and complex conjugation commute (i.e. can be interchanged). If you don't see this, write $\psi(x) = f(x) + ig(x)$ and do the derivation yourself, remembering that $f(x)$ and $g(x)$ are real functions.

- Q3.9 This question opens up a rabbit hole that I think is not intended by the book. I think the intended answer is 'If you know that it is an eigenstate, and you measure your observable of interest, then you can say "it is in the state that corresponds to the eigenstate XXX"'. That statement is correct. You can also say "You know that it is in an eigenstate, because if you repeatedly prepare the system in the same way and measure your operator of interest, you always get the same result". Or perhaps "You don't change the wave function if you measure the system in an eigenstate". However, when you measure anything else other than the your operator of which your wave function is an eigenfunction, you might collapse the wave function. Then you get something else. From measuring your observable of interest, you don't "get" the state as a whole: You don't get $\Psi(x)$. So even though it is an eigenstate, you cannot just "get" the wave function, which itself is not an observable. To get the wave function, or get an idea of what it looks like, you would have to do many complicated measurements (this is referred to as "quantum state tomography"), each of them requiring that you prepare the state anew due to wave function collapse.
- Q4.15 The ground state energy (or zero-point energy) of a particle in a box is

$$E_1 = \frac{\hbar^2 \pi^2}{2L^2 m}$$

where L is the length of the box and m the particle's mass. As a Helium atom is much heavier than an electron, the ground state energy is lower.

- Q4.6 Purely mathematically speaking, it is not. However, for a box of size L , the probability density of the n 'th eigenstate will have n peaks (all at the same height) and $n + 1$ evenly spaced nodes (or zeroes) ($n - 1$ on the inside and 2 at the boundary). So the nodes get closer and closer. For very large n , they are so close that any type of equipment will not be able to give a fine enough resolution to show that there are peaks and nodes, and the "washed out" probability distribution will look clean and constant.

- Q5.5 Make a box of size $L_1 \times L_1 \times L_2$, where L_1 is very large (of the magnitude of a meter or so) and L_2 very small (of the magnitude of a nanometer or less) and put a particle in it. As the energy gap between the eigenstates goes like $1/L^2$ in the box size L , along the first two dimensions, the spacing between the energy levels will be so small that it is practically speaking continuous, while it will be discrete in the third dimension. See also the next exercise.
- P4.19 We have that

$$\begin{aligned}
\hat{H}\Psi_{n_x, n_y, n_z}(x, y, z) &= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Psi_{n_x, n_y, n_z}(x, y, z) \\
&= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi_{n_x, n_y, n_z}(x, y, z) - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} \Psi_{n_x, n_y, n_z}(x, y, z) - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \Psi_{n_x, n_y, n_z}(x, y, z) \\
&= E_{n_x} \Psi_{n_x, n_y, n_z}(x, y, z) + E_{n_y} \Psi_{n_x, n_y, n_z}(x, y, z) + E_{n_z} \Psi_{n_x, n_y, n_z}(x, y, z) \\
&= (E_{n_x} + E_{n_y} + E_{n_z}) \Psi_{n_x, n_y, n_z}(x, y, z)
\end{aligned}$$

so $E_{n_x, n_y, n_z} = E_{n_x} + E_{n_y} + E_{n_z}$. We used the fact that

$$\begin{aligned}
-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi_{n_x, n_y, n_z}(x, y, z) &= N \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \sin\left(\frac{n_x \pi x}{a}\right) \right) \sin\left(\frac{n_y \pi y}{b}\right) \sin\left(\frac{n_z \pi z}{c}\right) \\
&= N \left(\frac{\hbar^2 \pi^2 n_x^2}{2ma^2} \sin\left(\frac{n_x \pi x}{a}\right) \right) \sin\left(\frac{n_y \pi y}{b}\right) \sin\left(\frac{n_z \pi z}{c}\right) \\
&= E_{n_x} \Psi_{n_x, n_y, n_z}(x, y, z)
\end{aligned}$$

and equally for the partial derivatives with respect to y and z . The total energy hence reads

$$E_{n_x, n_y, n_z} = E_{n_x} + E_{n_y} + E_{n_z} = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right)$$

- P4.35 We have that

$$\hat{H}\psi(x) = \sqrt{\frac{2}{a}} (E_n \sin(n\pi x/a) + E_m \sin(m\pi x/a)).$$

So unless $n = m$ or either n or m are zero, we do **not** have that

$$\hat{H}\psi(x) = \sqrt{\frac{2}{a}} (E_n + E_m) (\sin(n\pi x/a) + \sin(m\pi x/a)) = (E_n + E_m)\psi(x)$$

so a linear combination of eigenstates is NOT an eigenstate, unless they have the same energy.