Solutions week 1

Simon Elias Schrader

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1 Discussion exercises

- Q1.1 According to quantum mechanics, the Hydrogen atom has a "ground state" energy, which is not negative infinity - this is related to the Hydrogen atom having quantized, nonzero angular momentum in Bohr's theory of the atom - as opposed to classical mechanics, where the electron could be arbitrarily close to the nucleus and the system having "negative infinite" energy. In addition, there is an upper limit for the atom to become ionized, at which point the electron "escapes" from the proton. Thus, there are bounds on both the upper limit and the lower limit of the energy a bound electron can have, and that gap is the largest photon energy that can be observed.

- Q1.5 Even at very low intensities of the incident light, we can observe the ejection of photoions. That is, even with intensities so small that all energy incident on the solid surface is only slightly greater than the threshold energy required to yield a single photoelectron, an electron can be emitted. This observation indicates that the light that liberates the photoelectron is not uniformly distributed over the surface. If such uniformity were the case, no individual electron could receive enough energy to escape into the vacuum. The experiment shows that all incident light energy can be concentrated in a single electron excitation. This observation supports the hypothesis that light can be described as a spatially localized packet or photon, and this type of behaviour is usually associated with particles.

- Q1.9 If a single particle is being sent, then yes - because it shows that the particle passes through both slits at the same time, indicating wave-like behaviour. However, if many particles are sent, then both the interpretation "the particles pass through one of the slits, both of them equally likely" and "the particles pass through both slits simultaneously" is a correct interpretation based on the given experiment.

- Q1.10 The classical theory fails at large frequencies ω. This behaviour becomes apparent as the frequency increases into the ultraviolet range - hence, "ultraviolet catastrophe".

- Q1.12 Two possible explanations:
  - This behaviour has been observed with single-electron experiments, i.e. interaction with other electrons is ruled out.
  - Each electron has a random phase angle with respect to every other electron. Consequently, two electrons can never interfere with one another to produce a diffraction pattern. One electron gives rise to the diffraction pattern.

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(3) We use the pq formula and find

$$x_{1,2} = -\left(\frac{-2}{2}\right) \pm \sqrt{\left(\frac{-2}{2}\right)^2 - 4}$$
$$= 1 \pm \sqrt{-3}$$
$$= 1 \pm \sqrt{3}\, i$$

(4) For any fixed x,

we can write $f(x) = g(x) + ih(x)$

But that is just a fixed complex number,

so we can write

$$f(x) = z = e^{i\theta(x)} r(x)$$

(5) As $f(x) = r(x)\, e^{i\theta(x)}$, we have

$$f^*(x) f(x) = \left(r(x)\, e^{i\theta(x)}\right)^* \left(r(x)\, e^{i\theta(x)}\right)$$
$$= r^2(x)\, \underbrace{e^{-i\theta(x)}\, e^{i\theta(x)}}_{=1}$$
$$= r^2(x)$$

as $r(x)$ is real, so is $r^2(x)$.

(6) We remember: If $z = re^{i\theta}$, then $|z| = r$

also, $z = r(\cos(\theta) - i\sin(\theta))$

so $|z| - z = r((1-\cos(\theta)) + i\sin(\theta))$

we need $1-\cos(\theta) = 0$, so $\theta = 0$

but then

$|z| - z = 0$ because $\sin(\theta) = 0$

so no such number can exist.

(7)
$$\int_0^{2\pi} e^{ix+k}\,dx$$
$$= e^k \int_0^{2\pi} e^{ix}\,dx$$
$$= e^k\left[\int_0^{2\pi} \cos(x)\,dx + i\int_0^{2\pi} \sin(x)\,dx\right]$$
$$= e^k\left(\left[\sin(x)\right]_{2\pi}^{0} - i\left[\cos(x)\right]_{\cancel{0}2\pi}^{\cancel{0}}\right)$$
$$= 0$$

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(1) For example:
$$\sin(kx),\ \cos(kx),\ e^{ikx},\ e^{-ikx}$$

(2) We have $\frac{d^2}{dx^2} e^{-ax^2} = 2ae^{-ax^2}(2ax^2-1)$

Set $a = 0{,}5$
$$\Rightarrow f''(x) = e^{-\frac{1}{2}x^2}(x^2-1)$$
So $x^2 f(x) - f''(x) = e^{-1/2\,x^2}$

So $E = 1$

(3)
$$\left(x^2 - \frac{d^2}{dx^2}\right)^2 e^{-1/2\,x^2} = \underbrace{\left(x^2 - \frac{d^2}{dx^2}\right)\underbrace{\left(x^2 - \frac{d^2}{dx^2}\right)e^{-1/2\,x^2}}_{= e^{-1/2\,x^2}}}_{= e^{-1/2\,x^2}}$$

So $f(x) = e^{-1/2\,x^2}$!

④ We get

$$\left(kx^2 - \frac{1}{2}\frac{d^2}{dx^2}\right)e^{-ax^2}$$

$$= kx^2 e^{-ax^2} - \frac{1}{2}\left(2a\, e^{-ax^2}\left(2ax^2 - 1\right)\right)$$

$$= \left(kx^2 - \left(2a^2x^2 - a\right)\right)e^{-ax^2}$$

$$= \left(\left(k - 2a^2\right)x^2 + a\right)e^{-ax^2}$$

For this to be an eigenfunction, we need

$k - 2a^2 = 0$ so the x^2 term cancels

so $k = 2a^2$

$$\pm\sqrt{\frac{k}{2}} = a$$

then we get

$$\left(kx^2 - \frac{1}{2}\frac{d^2}{dx^2}\right)e^{-\left(\pm\sqrt{\frac{k}{2}}\,x^2\right)}$$

$$= \pm\sqrt{\frac{k}{2}}\, e^{-\left(\pm\sqrt{\frac{k}{2}}\,x^2\right)}$$

so $E = \pm\sqrt{\frac{k}{2}}$, depending on the solution we choose.

(5) We want

$$\int_{-\infty}^{\infty} N^2 \left| e^{-ax^2} \right|^2 dx = 1$$

~~[illegible]~~,

$$1 = N^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx$$

if $a \leq 0$, we have that the integrand is larger than 1 for all x, and

$$1 = N^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = \infty$$

, ~~so no such~~ which is impossible.

However, if $a > 0$, we get

$$1 = N^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = \frac{\sqrt{\pi/2}}{\sqrt{a}} \cdot N^2$$

So $N^2 = \frac{\sqrt{a}}{\sqrt{\pi/2}} \Rightarrow N = \left(\frac{2a}{\pi}\right)^{1/4}$