# Exercise 2

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## 1

**Problem 1.** (Taylor Polynomials)

a) Compute all Taylor polynomials of $f(x) = -2x^4 + 2x^2 - 3x + 2$ around $x_0 = -1$

b) Compute the Taylor series of $g(x) = e^{1-2x}$ around $x_0 = 0$

at $x = c$

a) $$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 \ldots \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$P_4(-1) = f(-1) + f'(-1)(x+1) + \frac{1}{2}f''(-1)(x+1)^2 + \frac{1}{6}f'''(-1)(x+1)^3 + \frac{1}{24}f''''(-1)(x+1)^4$$

$f(-1) = \underline{5}$ $\quad \frac{d}{dx}f(x) = -8x^3 + 4x - 3$ $\quad \frac{d^2}{dx^2}f(x) = -24x^2 + 4$ $\quad \frac{d^3}{dx^3}f(x) = -48x$

$\Rightarrow f'(-1) = 8 - 4 - 3 = \underline{1}$ $\quad \Rightarrow f''(-1) = \underline{-20}$ $\quad \Rightarrow f'''(-1) = \underline{48}$

$\frac{d^4}{dx^4}f(x) = -48 \Rightarrow f''''(-1) = -48$

$$P_4(x) = 5 + (x+1) - 10(x+1)^2 + 8(x+1)^3 - 2(x+1)^4$$

$$P_3(x) = 5 + (x+1) - 10(x+1)^2 + 8(x+1)^3$$

$$P_2(x) = 5 + (x+1) - 10(x+1)^2$$

$$P_1(x) = 5 + (x+1)$$

$$P_0(x) = 5$$

b)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad , \quad x_0 = 0$$

$$g(0) = e \quad , \quad g'(0) = -2e \quad , \quad g''(0) = 4e \quad , \quad g'''(0) = -8e$$

$$g^{(n)}(0) = \begin{cases} 2^n e & , n \text{ even} \\ -2^n e & , n \text{ odd} \end{cases}$$

$$P_n(x) = e\left(1 - 2x + \frac{2^2x^2}{2!} - \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} \cdots \frac{2^nx^n}{n!}\right)$$

$$\boxed{P_n(x) = e \cdot \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \cdot (-1)^n}$$

## 2

a) Compute by hand the Lagrangian cardinal functions for the points

$$x_0 = -1, \qquad x_1 = 0, \qquad x_2 = 1, \qquad x_3 = 2.$$

b) Consider the function

$$f(x) = 2^{x^2-4-x}$$

and interpolate it by a polynomial of minimal degree by hand using the results above.

a)

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|
| $x$ | $-1$ | $0$ | $1$ | $2$ |
| $y$ | $y_0$ | $y_1$ | $y_2$ | $y_3$ |

$$L_0(x) = \left(\frac{x-0}{-1-0}\right)\cdot\left(\frac{x-1}{-1-1}\right)\cdot\left(\frac{x-2}{-1-2}\right) = \underline{-\frac{1}{6}x(x-1)(x-2)}$$

$$L_1(x) = (x+1)\cdot\left(\frac{x-1}{-1}\right)\cdot\left(\frac{x-2}{-2}\right) = \underline{\frac{1}{2}(1-x)(2-x)(x+1)}$$

$$L_2(x) = \left(\frac{x+1}{2}\right)\cdot x \cdot\left(\frac{x-2}{-1}\right) = \underline{\frac{1}{2}x(2-x)(x+1)}$$

$$L_3(x) = \left(\frac{x+1}{3}\right)\cdot\frac{x}{2}\cdot(x-1) = \underline{\frac{1}{6}x(x+1)(x-1)}$$

b)

$$f(x) = 2^{x^2-x-4}$$

$$y_0 = f(-1) = 2^{1-(-1)-4} = 2^{-2} = \frac{1}{4}$$

$$y_1 = f(0) = 2^{-4} = \frac{1}{16}$$

$$y_2 = f(1) = 2^{1-4-1} = 2^{-4} = \frac{1}{16}$$

$$y_3 = f(2) = 2^{4-4-2} = 2^{-2} = \frac{1}{4}$$

$$f(x) \approx P_3(x) = y_0 \cdot L_0 + y_1 \cdot L_1 + y_2 \cdot L_2 + y_3 \cdot L_3$$

$$= \frac{1}{4}\left(-\frac{1}{6}x(x-1)(x-2)\right) + \frac{1}{16}\left(\frac{1}{2}(1-x)(2-x)(x+1)\right) + \frac{1}{16}\left(\frac{1}{2}x(2-x)(x+1)\right)$$
$$+ \frac{1}{4}\left(\frac{1}{6}x(x+1)(x-1)\right)$$

$$= \underline{\frac{1}{32}\left(3x^2 - 3x + 2\right)}$$

c)

$$e(x) = \left| f(x) - P_3(x) \right| = \frac{\left| f^{(4)}(\bar{x}) \right|}{4!} \cdot \prod_{i=1}^{4} |x - x_i|$$

$$e(x) \leq \frac{\prod_{i=1}^{4} |x-x_i|}{4!} \cdot \max_{\substack{x \in [-1,2] \\ x \in [-5,5]}} \left| f^{(4)}(x) \right|$$

Vi skal vel ikke regne ut $f^{(4)}(x)$ ?

$$= 2^{x^2-4-x}(2x-1)^4 \log^4(2) + 2^{x^2-x-2}(2x-1)^2 \log^3(2) + 2^{x^2-x-1}(2x-1)^2 \log^3(2)$$
$$+ 2^{x^2-x-2} \log^2(2) + 2^{x^2-x-1} \log^2(2)$$

d)

Chebysev nodes:

$$x_i = \left(\frac{b-a}{2}\right) \cdot \cos\left(\frac{(2i+1)\pi}{2n}\right) + \frac{b+a}{2} \quad , \quad i = 1, 2, \ldots, n-1$$

$n = 3$ , $[-1, 2]$

$$x_1 = \frac{3}{2} \cos\left(\frac{3\pi}{6}\right) + \frac{1}{2}$$

$$x_2 = \frac{3}{2} \cos\left(\frac{5\pi}{6}\right) + \frac{1}{2}$$

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**Problem 4.** (Trigonometric interpolation)

In our first lectures, we saw how polynomials can be used to interpolate functions or discrete datasets. That is actually not the only possibility, and in fact we can also use trigonometric functions for interpolation. Namely, the set $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$ is a common basis for that.

Let $y = f(x) = \cos^2 x$ be the function we wish to interpolate, for which we consider three data points:

| $i$ | 0 | 1 | 2 |
|---|---|---|---|
| $x_i$ | 0 | $\pi/6$ | $\pi/2$ |
| $y_i$ | 1 | $\frac{3}{4}$ | 0 |

Our task here is to construct a trigonometric function $p(x)$ that goes through these three points. Since $f(x)$ is an even function, it suffices to use only cosine terms for the interpolation. That is, we are looking for some

$$p(x) = \alpha_0 + \alpha_1 \cos x + \alpha_2 \cos 2x$$

such that $p(x_i) = y_i$ for $i = 0, 1, 2$, with $\alpha_0$, $\alpha_1$ and $\alpha_2$ being three real coefficients to be determined.

a) Based on the data in table, compute the interpolation coefficients $\alpha_0$, $\alpha_1$ and $\alpha_2$.

b) Show that, for this particular case, we have $p(x) = f(x)$ for all $x \in \mathbb{R}$, that is, the interpolant is identical to the function being interpolated.
*Hint:* remember the trigonometric identity $\cos 2x = \cos^2 x - \sin^2 x$.

a)

$$y = f(x) = \cos^2 x$$

$$p(x) = a_0 + a_1 \cos x + a_2 \cos 2x$$

$$p(0) = 1 \Rightarrow a_0 + a_1 + a_2 = 1$$

$$p(\pi/6) = \frac{3}{4} \Rightarrow a_0 + \frac{\sqrt{3}}{2} a_1 + \frac{1}{2} a_2 = \frac{3}{4}$$

$$p(\pi/2) = 0 \Rightarrow a_0 + 0 \cdot a_1 - a_2 = 0$$

$$\Rightarrow a_0 = \frac{1}{2}$$

$$a_1 = 0 \Rightarrow \boxed{p(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)}$$

$$a_2 = \frac{1}{2}$$

b) $f(x) = \cos^2(x)$

$$p(x) = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1}{2} + \frac{1}{2}(\cos^2 x - \sin^2 x)$$

$\cos^2 x + \sin^2 x = 1$

$$= \frac{1}{2} + \frac{\cos^2 x}{2} - \frac{\sin^2 x}{2} = \frac{1}{2}(\underline{1} + \cos^2 x \underline{- \sin^2 x})$$

$\cos^2 x = 1 - \sin^2 x$

$$= \frac{1}{2} \cdot 2\cos^2 x = \cos^2 x = \underline{f(x)}$$

$$\boxed{p(x) = f(x), \ \forall \ x \in \mathbb{R}}$$