TMA4135 # EXERCISE 9 Simen Sandhaug

3.

a) $f(x) = \frac{1}{2} + \cos(2x) - 4\sin(4x)$.

$$\left.\begin{aligned} \cos(2x) &\Rightarrow \frac{2\pi}{2} = \underline{\pi} \\ \sin(4x) &\Rightarrow \frac{2\pi}{4} = \frac{\pi}{2} \end{aligned}\right\} \text{LCM}\left(\pi, \frac{\pi}{2}\right) = \pi$$

$\underline{p = \pi}$ and $2L = p$ so $L = p/2 = \underline{\frac{\pi}{2}}$.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{\pi/2}\right) + b_n \sin\left(\frac{n\pi x}{\pi/2}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(2nx) + b_n \sin(2nx)\right]$$

we can see that $\underline{\underline{a_0 = \frac{1}{2}, \quad a_1 = 1, \quad b_2 = -4}}$

b) $f(x) = |\cos(3\pi x)|$

$2L' = \frac{2\pi}{3\pi} = \frac{2}{3}$ but since we have absolute value $2L = \frac{2L'}{2} = \underline{\frac{1}{3}}$ and

$\underline{L = \frac{1}{6}}$

$f(x)$ is even:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0.$$

$\cos(3\pi x) \geq 0, \quad -\frac{1}{6} \leq x \leq \frac{1}{6}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 3\int_{-1/6}^{1/6} |\cos(3\pi x)|\,dx = 3\int_{-1/6}^{1/6} \cos(3\pi x)\,dx = 3\cdot\left[\frac{1}{3\pi}\sin(3\pi x)\right]_{-1/6}^{1/6}$$

$$= 3\left[\frac{1}{3\pi}\cdot 1 - \left(\frac{1}{3\pi}\cdot -1\right)\right] = \cancel{3}\cdot\frac{2}{\cancel{3}\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 6\cdot\int_{-1/6}^{1/6} \cos(3\pi x)\cdot\cos(6n\pi x)\,dx$$

$$= \frac{12\cos(\pi n)}{3\pi - 12\pi n^2} = \frac{12}{3\pi(1-4n^2)}\cdot(-1)^n = \frac{4}{\pi - 4\pi n^2}\cdot(-1)^n$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi - 4\pi n^2}\cos(6n\pi x)$$

c)

$$f(x) = 1 - \sin^2(x) \qquad \left(\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}\right)$$

$$f(x) = \frac{\cos(2x)+1}{2} = \frac{1}{2} + \frac{\cos(2x)}{2}, \quad p = \frac{2\pi}{2} = \pi$$

$$2L = p \Rightarrow L = \frac{\pi}{2}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{\pi n x}{\pi/2}\right)\right] = a_0 + \sum_{n=1}^{\infty} a_n\cos(2nx)$$

$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}$

d)

$f(x) = |x-1| \quad , \quad -1 \leq x \leq 1 \qquad T = 2L = 2.$

$$a_0 = \frac{1}{2}\int_{-1}^{1} |x-1|\, dx = \frac{1}{2}\int_{-1}^{1} (1-x)\, dx = \frac{1}{2}\left[x - \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}\left[\left(1-\frac{1}{2}\right) - \left(-1-\frac{1}{2}\right)\right] = \frac{\frac{1}{2}+\frac{3}{2}}{2} = \underline{1}$$

$$a_n = \int_{-1}^{1} |x-1| \cos(n\pi x)\, dx = \underline{\frac{2\sin(n\pi)}{n\pi}}$$

$$b_n = \int_{-1}^{1} |x-1| \sin(n\pi x)\, dx = \underline{\frac{2n\pi\cos(n\pi) - 2\sin(n\pi)}{(\pi n)^2}}$$

4.

a) odd: $-f(x) = f(-x)$, let $g(x) = f^2(x)$

$$\Rightarrow g(-x) = f^2(-x) = (f(-x))^2 = (-f(x))^2 = f^2(x) = g(x). \quad \underline{\text{even } g(-x) = g(x)} \quad \square$$

b) $f(x)$ odd $\Rightarrow f(-x) = -f(x)$, $g(x)$ even $\Rightarrow g(-x) = g(x)$

$$h(-x) = g(-x)f(-x) = -f(x)\cdot g(x) = -h(x). \quad \underline{\text{odd} \quad h(-x) = -h(x)} \quad \square$$

c) $$\int_{-L}^{L} f(x)\, dx = \int_{-L}^{0} \overset{\text{odd}}{f(x)}\, dx + \int_{0}^{L} \overset{\text{odd}}{f(x)}\, dx$$

Let $u = -x \qquad \frac{du}{dx} = -1 \Rightarrow du = -dx$

$$\int_{-L}^{0} f(x)\, dx = -\int_{L}^{0} f(-u)\, du = \int_{0}^{L} f(-u)\, du.$$

f odd

$$\Rightarrow \int_0^L f(-u)\,du = -\int_0^L f(u)\,du = -\int_0^L f(x)\,dx$$

So
$$\int_{-L}^{L} f(x)\,dx = \int_0^L f(x)\,dx - \int_0^L f(x)\,dx = \underline{\underline{0}}. \square$$

d)

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx$$

Let $u = -x$ $\quad du = -du$

$$\int_{-L}^{0} f(-x)\,dx = -\int_L^0 f(u)\,du = \int_0^L f(u)\,du = \int_0^L f(x)\,dx$$

$$\int_{-L}^{L} f(x)\,dx = \int_0^L f(x)\,dx + \int_0^L f(x)\,dx = \underline{\underline{2\int_0^L f(x)\,dx}}$$