TMA 4135 | Exercise 6 | Simen Sandhaug

Problem 2. (Dirac function rhs)

Consider the Dirac function $\delta(t-a)$, with $a \geq 0$. In this exercise, you are asked to *use the Laplace transform* to solve various initial value problems containing Dirac inputs.

a) Solve $y'' + 4y = \delta(t-\pi)$, with $y(0) = 8$ and $y'(0) = 0$.

b) Solve $y'' + 3y' + 2y = 10(\sin t + \delta(t-1))$, with $y(0) = 1$ and $y'(0) = -1$

c) Solve $y'' + +2y' + 5y = 25t - 100\delta(t-\pi)$, with $y(0) = -2$ and $y'(0) = 5$.

$\boxed{2}$ a)

$$y'' + 4y = \delta(t-\pi)$$

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)]$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = e^{-\pi s}$$

$$Y(s) = \frac{e^{-\pi s} + 8s}{s^2+4} = e^{-\pi s}\cdot\frac{1}{2}\cdot\underbrace{\frac{2}{s^2+2^2}}_{f(t-a) = \frac{\sin(2(t-a))}{2}} + 8\frac{s}{s^2+4}$$

$\pi \to a$

$+\ 8\cdot\cos(2t)$

$$\mathcal{L}^{-1}[e^{-as}\cdot F(s)] = u(t-a)\cdot f(t-a)$$

$$\mathcal{L}^{-1}[Y(s)] = u(t-\pi)\cdot\frac{\sin(2t-2\pi)}{2} + 8\cos(2t) = u(t-\pi)\cdot\frac{\sin(2t)}{2} + 8\cos(2t)$$

$$y(t) = u(t-\pi)\cdot\sin t\cdot\cos t + 8\cdot\cos(2t)$$

b) $y'' + 3y' + 2y = 10(\sin t + \delta(t-1))$, $y(0) = 1$, $y'(0) = -1$.

$$s^2 Y(s) - s + 1 + 3(s \cdot Y(s) - 1) + 2Y(s) = 10 \cdot \frac{1}{s^2+1} + 10e^{-s}$$

$$Y(s)(s^2 + 3s + 2) - s - 2 = 10 \cdot \frac{1}{s^2+1} + 10e^{-s}$$

$$Y(s) = \frac{10}{(s^2+3s+2)(s^2+1)} + 10e^{-s} \cdot \frac{1}{s^2+3s+2} + \frac{s}{s^2+3s+2} + \frac{2}{s^2+3s+2}$$

$s^2 + 3s + 2 = (s+2)(s+1)$

$$Y(s) = \frac{1}{s^2+1} - 3 \cdot \frac{s}{s^2+1} + \frac{5}{s+1} - \frac{2}{s+2} + 10e^{-s}\overset{a=1}{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)} + \frac{2}{s+2} - \frac{1}{s+1} + \frac{2}{s+1} - \frac{2}{s+2}$$

$$y(t) = \sin(t) - 3\cos(t) + 5e^{-t} \cancel{- 2e^{-2t}} + 10\,u(t-1)e\left(e^{-t} - e^{-2t}\right) \cancel{+ 2e^{-2t}} - e^{-t} + 2e^{-t} - 2e^{-2t}$$

$$\underline{\underline{y(t) = \sin(t) - 3\cos(t) + 6e^{-t} - 2e^{-2t} + 10\,u(t-1)e\left(e^{-t} - e^{-2t}\right)}}$$

c) $y'' + 2y' + 5y = 25t - 100\,\delta(t-\pi)$, $y(0) = -2$, $y'(0) = 5$.

$$s^2 Y(s) + 2s - 5 + 2(sY(s) + 2) + 5Y(s) = \frac{25}{s^2} - 100e^{-\pi s}$$

$$Y(s) = \frac{25}{s^2(s^2+2s+5)} - 100e^{-\pi s}\left(\frac{1}{s^2+2s+5}\right) - \frac{2s}{s^2+2s+5} + \frac{5}{s^2+2s+5}$$

$s^2 + 2s + 5 = (s+1)^2 + 4$

$$= \frac{2s}{\cancel{(s+1)^2} + 4} - \frac{1}{(s+1)^2+4} + \frac{5}{s^2} - \frac{2}{s} - 100e^{-\pi s} \cdot \frac{1}{2} \cdot \left(\frac{2}{(s+1)^2 + 2^2}\right) - \frac{2s}{\cancel{(s+1)^2} + 4} + \frac{5}{(s+1)^2+4}$$

$$= 5 \cdot \frac{1}{s^2} - 2 \cdot \frac{1}{s} - 50e^{-\pi s}\left(\frac{2}{(s+1)^2 + 2^2}\right) + 2 \cdot \left(\frac{2}{(s+1)^2 + 2^2}\right)$$

$$\underline{\underline{y(t) = 5t - 2 - 50e^{\pi - t} \cdot u(t-\pi) \cdot \sin(2t)}}$$

Problem 3. (Numerical differentiation - 4D only)

Consider the three-point finite-difference formula below:

$$\frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} \approx u'(x).$$

a) Compute the convergence order p of the approximation, i.e., the exponent p such that

$$\frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} = u'(x) + O(h^p).$$

Hint: you can consider the Taylor expansion of $u(x+h)$ and $u(x+2h)$ up to the h^3 term.

b) If a computer with machine accuracy $\varepsilon > 0$ is used, the approximation of u' will have, in addition to the above truncation error $O(h^p)$, a rounding error dependent on ε and h. These two errors will have comparable orders of magnitude when

$$h = O(\varepsilon^k), \quad k > 0. \tag{1}$$

Determine the value of k.

a)

$$\frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} = u'(x) + O(h^p)$$

$$\frac{\cancel{-3u(x)} + 4\left[\cancel{u(x)} + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x)\right] - \left[\cancel{u(x)} + 2hu'(x) + 2h^2u''(x) + \frac{4}{3}h^3u'''(x)\right]}{2h}$$

$$\frac{\cancel{2}\cancel{h}u'(x) - \overset{\frac{3}{4}}{\cancel{\frac{3}{2}}}h^{\cancel{2}}u''(x) - \overset{\frac{7}{12}}{\cancel{\frac{7}{6}}}h^{\overset{2}{\cancel{3}}}u'''(x)}{\cancel{2}\cancel{h}} = u'(x) + O(h^p)$$

$$u'(x) - c_1 hu''(x) - c_2 h^2 u'''(x) = u'(x) + O(h^p)$$

$$O(h^p) = c_1 hu''(x) + c_2 h^2 u'''(x) = O(h^2)$$

$$\underline{\underline{p = 2}}$$

b)

$$h^p = \frac{\varepsilon}{h} \quad \text{or} \quad h^{p+1} = \varepsilon$$

$$p = 2: \quad h^3 = \varepsilon \Rightarrow h = O(\varepsilon^{1/3})$$

$$\underline{k = \frac{1}{3}}$$