TMA4135

EXERCISE 5

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Problem 1.

Compute the Laplace transform of the following functions.

a) $f(t) = (t-2)^4$.

b) $f(t) = te^{-t}$.

c) $f(t) = e^{-5t}\sin(t)$.

d) $f(t) = e^{-2t}\cos^2(3t) - 3t^2e^{3t}$

a) $f(t) = (t-2)^4 = (t-2)(t-2)(t-2)(t-2)$

$$\mathcal{L}[f(t)] = \mathcal{L}[(t^2-4t+4)(t^2-4t+4)]$$

$$= \mathcal{L}[t^4 - 4t^3 + 4t^2 - 4t^3 + 16t^2 - 16t + 4t^2 - 16t + 16]$$

$$= \mathcal{L}[t^4 - 8t^3 + 24t^2 - 32t + 16]$$

$$= \mathcal{L}[t^4] - 8\mathcal{L}[t^3] + 24\mathcal{L}[t^2] - 32\mathcal{L}[t] + 16\mathcal{L}[1]$$

$$= \frac{4!}{s^5} - 8\cdot\frac{3!}{s^4} + 24\cdot\frac{2!}{s^3} - 32\cdot\frac{1}{s^2} + 16\cdot\frac{1}{s}$$

$$= \frac{24}{s^5} - \frac{48}{s^4} + \frac{48}{s^3} - \frac{32}{s^2} + \frac{16}{s}$$

b) $\mathcal{L}[te^{-t}] = \mathcal{L}[te^{at}]$, $a=-1$

$$\Rightarrow \mathcal{L}[te^{-t}] = F(s+1) \quad , \quad f(t) = t.$$

$$\Rightarrow \mathcal{L}[t] = \frac{1}{s^2} = F(s)$$

$$\Rightarrow \mathcal{L}[te^{-t}] = F(s+1) = \frac{1}{(s+1)^2} = \underline{\underline{\frac{1}{s^2+2s+1}}}$$

c) $$\mathcal{L}[e^{-5t}\sin t] = F(s+5) \quad , \text{ where } \quad F(s) = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}[e^{-5t}\sin t] = \frac{1}{(s+5)^2+1} = \underline{\underline{\frac{1}{s^2+10s+26}}}$$

d) $$f(t) = e^{-2t}\cos^2(3t) - 3t^2e^{3t}$$

$$\mathcal{L}[e^{-2t}\cos^2(3t) - 3t^2e^{3t}] = \mathcal{L}[e^{-2t}\cos^2(3t)] - 3\mathcal{L}[t^2e^{3t}]$$

$$\cos(2\omega t) = 2\cos^2(\omega t) - 1$$

$$\cos^2(\omega t) = \frac{\cos(2\omega t)+1}{2}$$

$$\mathcal{L}[\cos^2(\omega t)] = \frac{1}{2}\mathcal{L}[\cos(2\omega t)] + \frac{1}{2}\mathcal{L}[1] = \frac{1}{2}\left(\frac{s}{s^2+4\omega^2}\right) + \frac{1}{2s}$$

$$\Rightarrow \mathcal{L}[e^{-2t}\cos^2(3t)] = F(s+2) \quad , \quad F(s) = \mathcal{L}[\cos^2(3t)] = \frac{1}{2}\left(\frac{s}{s^2+36}\right) + \frac{1}{2s}$$

$$F(s+2) = \frac{1}{2}\left(\frac{s+2}{(s+2)^2+36}\right) + \frac{1}{2(s+2)} = \mathcal{L}[e^{-2t}\cos^2(3t)]$$

$$\mathcal{L}[e^{-2t}\cos^2(3t)] - 3\mathcal{L}(t^2 e^{3t}) = \frac{1}{2}\left(\frac{s+2}{(s+2)^2+36}\right) + \frac{1}{2(s+2)} - \left(\frac{6}{(s-3)^3}\right)$$

$$= \frac{1}{2}\left(\frac{s+2}{(s+2)^2+36} - \frac{12}{(s-3)^3} + \frac{1}{s+2}\right)$$

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Problem 2.

Find the inverse Laplace transform of the following functions.

a) $F(s) = \dfrac{2s}{s^2-3}$.

b) $F(s) = \dfrac{s^2+s+1}{s^3+s}$.

c) $F(s) = \dfrac{1}{(s-1)^2(s+1)}$.

a)

$$\mathcal{L}[f(t)] = \frac{2s}{s^2-3} = 2\cdot\frac{s}{s^2-3}$$

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2-3}\right] = 2\cdot\mathcal{L}^{-1}\left[\frac{s}{s^2-3}\right] = \underline{\underline{2\cdot\cosh(\sqrt{3}t)}}$$

b)

$$\mathcal{L}[f(t)] = \frac{s^2+s+1}{s^3+s} = \frac{s^2+s+1}{s(s^2+1)} = \frac{s+\frac{1}{s}+1}{s^2+1}$$

$$= \frac{s}{s^2+1} + \frac{\frac{1}{s}}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{s+\frac{1}{s}}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{\frac{s^2}{s}+\frac{1}{s}}{s^2+1} + \frac{1}{s^2+1} = \cancel{\frac{s^2+1}{s^2+1}}\left(\frac{1}{s}\right) + \frac{1}{s^2+1} = \underline{\frac{1}{s^2+1} + \frac{1}{s}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}+\frac{1}{s}\right] = \underline{\underline{\sin t + 1}}$$

c) $\mathcal{L}[f(t)] = \dfrac{1}{(s-1)^2(s+1)}$

$$\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+1} = \frac{1}{(s-1)^2(s+1)}$$

$$A(s-1)(s+1)+B(s+1)+C(s-1)^2 = 1$$

$s=-1$:

$$C(-2)^2 = 1$$

$$\underline{C = \frac{1}{4}}$$

$s=1$:

$$\underline{B = \frac{1}{2}}$$

$s=0$:

$$-A = -B - C + 1$$

$$A = B + C - 1 = \frac{1}{2}+\frac{1}{4}-1 = \frac{2}{4}-\frac{3}{4} = -\frac{1}{4}$$

$$\underline{A = -\frac{1}{4}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2(s+1)}\right] = -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$= -\frac{e^t}{4} + \frac{e^t t}{2} + \frac{e^{-t}}{4}$$

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Problem 3.

Decide for each of the following statements whether it is true or false. Explain your answer.

a) If f and g are two functions for which the Laplace transform exists, then $\mathcal{L}(f-g) = \mathcal{L}(f) - \mathcal{L}(g)$. TRUE

b) If f and g are two functions, for which the Laplace transform exists, then $\mathcal{L}(f \cdot g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$. FALSE

c) If the function f satisfies $0 \leq f(t)$ for all $t \geq 0$, then $\mathcal{L}(f)(s) \geq 0$ for all s for which $\mathcal{L}(f)(s)$ exists. TRUE

d) If the function f is continuous and satisfies $0 \leq f(t) \leq 1$ for all $t \geq 0$, then $\mathcal{L}(f)(s)$ exists for all $s > 0$. TRUE

TRUE

a) $$\mathcal{L}[f-g] = \int_0^\infty e^{-st}(f-g)\,dt = \int_0^\infty e^{-st}f - e^{-st}g\,dt = \int_0^\infty e^{-st}f(t)\,dt - \int_0^\infty e^{-st}g(t)\,dt$$

$$= \mathcal{L}[f(t)] - \mathcal{L}[g(t)]$$ QED. (It follows from linearity)

b) FALSE

* Proof by counterexample:

$f(t) = g(t) = e^{-t}$ then $\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = \frac{1}{s+1}$

so $\mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)] = \frac{1}{(s+1)^2}$

but $\mathcal{L}[f(t) \cdot g(t)] = \int_0^\infty e^{-2t}e^{-st}\,dt = \frac{1}{s+2} \neq \frac{1}{(s+1)^2}$ QED.

c) TRUE since: $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

we know $f(t) \geq 0 \quad \forall\, t \in [0, \infty)$

and $e^{-st} \geq 0 \quad \forall \quad s, t \in \mathbb{R}$

so $e^{-st} \cdot f(t)$ is non-negative $\forall\, t \in [0, \infty)$. The integral of a non-negative function over a range is itself non-negative, so:

$\mathcal{L}[f(t)] \geq 0 \quad \forall\, t \in [0, \infty)$ QED.

d) TRUE Since:

$f(t) = c \;, c \in [0, 1] \quad \forall \quad t \in [0, \infty)$

So $\mathcal{L}[f(t)] = c \cdot \int_0^\infty e^{-st}\, dt \qquad \forall \quad t \in [0, \infty)$

Since $\int_0^\infty e^{-st}\, dt$ converges for all $s > 0$, we have

$\exists\, \mathcal{L}[f(t)] \quad \forall \quad s > 0$ QED.

(If $f(t)$ is bounded it can never outgrow e^{-st} so the integral will always converge for positive s)

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Problem 4.

Use the Laplace transform in order to solve the following initial value problems:

a) $-y'' + 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = 2$.

b) $y'' - 3' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$.

c) $y'' - 10y' + 9y = 5t$, $y(0) = -1$, $y'(0) = 2$.

From lecture: $\boxed{Y(s) = \dfrac{F(s) + ay_0 s + by_0 + av_0}{as^2 + bs + c}}$ where $y_0 = y(0)$ and $v_0 = y'(0)$

$\rightarrow F(s) = 0$

a) $a=-1$, $b=2$, $c=-3$, $f(t)=0$, $y_0=1$, $v_0=2$.

$$Y(s) = \frac{0 - s + 2 - 2}{-s^2 + 2s - 3} = \frac{-s}{-s^2+2s-3} = \frac{s-1}{(s-1)^2+2} + \frac{1}{(s-1)^2+2}$$

$$\mathcal{L}^{-1}[Y(s)] = \underline{\underline{e^t\left(\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right) = y(t)}}$$

b) $a=1$, $b=-3$, $c=2$, $f(t)=e^{3t} \Rightarrow F(s) = \dfrac{1}{s-3}$, $y_0 = 0$, $v_0 = 0$

$$Y(s) = \frac{\frac{1}{s-3} + (s-3)}{s^2 - 3s + 2} = \underset{(1)}{\frac{1}{(s-3)(s-2)(s-1)}} + \underset{(2)}{\frac{s}{(s-2)(s-1)}} - \underset{(3)}{\frac{3}{(s-2)(s-1)}}$$

(1): $$\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} = \frac{1}{(s-3)(s-2)(s-1)}$$

$$A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) = 1$$

$s=1 \Rightarrow A(-1)(-2)=1 \Rightarrow \underline{A=\frac{1}{2}}$

$s=2 \Rightarrow B(1)(-1)=1 \Rightarrow \underline{B=-1}$

$s=3 \Rightarrow C(2)(1)=1 \Rightarrow \underline{C=\frac{1}{2}}$

$$\frac{1}{(s-3)(s-2)(s-1)} = \frac{1}{2(s-1)} - \frac{1}{s-2} + \frac{1}{2(s-3)}$$

(2) :
$$\frac{A}{s-1} + \frac{B}{s-2} = \frac{s}{(s-2)(s-1)}$$

$A(s-2) + B(s-1) = s$

$s=1 \Rightarrow -A=1 \Rightarrow \underline{A=-1}$

$s=2 \Rightarrow \underline{B=2}$

$$\frac{s}{(s-2)(s-1)} = -\frac{1}{s-1} + \frac{2}{s-2}$$

(3) : $A(s-2) + B(s-1) = -3$

$s=1 \Rightarrow \underline{A=3}$

$s=2 \Rightarrow \underline{B=-3}$

$$-\frac{3}{(s-2)(s-1)} = \frac{3}{s-1} - \frac{3}{s-2}$$

$$Y(s) = \frac{5}{2}\cdot\frac{1}{s-1} - 2\cdot\frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-3}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$$

c) $a=1$, $b=-10$, $c=9$, $f(t)=5t \Rightarrow F(s)=\frac{5}{s^2}$, $y_0=-1$, $V_0=2$

$$Y(s) = \frac{\frac{5}{s^2}+s+12}{s^2-10s+9} = \frac{5}{s^2(s^2-10s+9)} + \frac{s-5}{s^2-10s+9} + \frac{17}{s^2-10s+9}$$

$(s-5)^2-16$

(1) (2) (3)

(2) $$\mathcal{L}^{-1}\left[\frac{s-5}{(s-5)^2-16}\right] = e^{5t}\cosh(4t)$$

(1) $$\frac{5}{s^2(s-1)(s-9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-9}$$

$$5 = A(s)(s-1)(s-9) + B(s-1)(s-9) + C(s^2)(s-9) + D(s^2)(s-1)$$

$s=0 \Rightarrow 9B=5 \Rightarrow B=\frac{5}{9}$

$s=1 \Rightarrow -8C=5 \Rightarrow C=-\frac{5}{8}$

$s=9 \Rightarrow 8\cdot 81D=5 \Rightarrow D=\frac{5}{648}$

$s=2 \Rightarrow -14A-7B-28C+4D=5$

Solve for A:

$$A = \frac{5 + 7B + 28C - 4D}{-14} = \frac{50}{81}$$

$$\frac{5}{s^2(s-1)(s-9)} = \frac{50}{81}\cdot\frac{1}{s} + \frac{5}{9}\cdot\frac{1}{s^2} - \frac{5}{8}\cdot\frac{1}{s-1} + \frac{5}{648}\cdot\frac{1}{s-9}$$

$$\mathcal{L}^{-1}\left[\frac{5}{s^2(s-1)(s-9)}\right] = \frac{50}{81} + \frac{5}{9}t - \frac{5}{8}e^{t} + \frac{5}{648}e^{9t}$$

(3) $$\frac{17}{(s-1)(s-9)} = \frac{A}{s-1} + \frac{B}{s-9}$$

$$A(s-9) + B(s-1) = 17$$

$$s = 1 \Rightarrow A = -\frac{17}{8}$$

$$s = 9 \Rightarrow B = \frac{17}{8}$$

$$\mathcal{L}^{-1}\left[\frac{17}{(s-1)(s-9)}\right] = -\frac{17}{18}e^{t} + \frac{17}{18}e^{9t}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \underbrace{\mathcal{L}^{-1}\left[\frac{17}{(s-1)(s-9)}\right]}_{(3)} + \underbrace{\mathcal{L}^{-1}\left[\frac{s-5}{(s-5)^2-16}\right]}_{(2)} + \underbrace{\mathcal{L}^{-1}\left[\frac{5}{s^2(s-1)(s-9)}\right]}_{(1)}$$

$$= \frac{50}{81} + \frac{5}{9}t + \frac{617}{648}e^{9t} - \frac{113}{72}e^{t} + e^{5t}\cosh(4t)$$