TMA4135 EXERCISE 8 Simen Sandhaug

Problem 1. (Ralston's method)

Ralston's method is given by the following Butcher tableau:

$$
\begin{array}{cc}
 & \begin{array}{ccc} & n & \end{array} \\
m &
\begin{array}{rc|ccc}
0 & c_1 & \overset{a_{11}}{0} & \overset{a_{12}}{0} & \overset{a_{13}}{0} \\
1/2 & c_2 & \overset{a_{21}}{1/2} & \overset{a_{22}}{0} & \overset{a_{23}}{0} \\
3/4 & c_3 & \overset{a_{31}}{0} & \overset{a_{32}}{3/4} & \overset{a_{33}}{0} \\
\hline
 & & \underset{b_1}{2/9} & \underset{b_2}{1/3} & \underset{b_3}{4/9}
\end{array}
\end{array}
$$

a) Determine the order of this Runge–Kutta method.

b) Consider the initial value problem $y'(t) = t^2 e^{-2y}$, $y(0) = 0$. Using a time-step size $h = 0.48$, use Ralston's method to compute the *first step*, by hand. Make sure to also write down all stage derivatives k_1, k_2 and k_3.

$\boxed{1}$ a)

p	Conditions	
1	$\sum_{n=1}^{s} b_n = 1$	$\frac{2}{9} + \frac{3}{9} + \frac{4}{9} = 1$ ✓
2	$\sum_{n=1}^{s} b_n c_n = 1/2$	$0 + \frac{1}{6} + \frac{3}{4} \cdot \frac{4}{9} = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$ ✓
3	$\sum_{n=1}^{s} b_n c_n^2 = 1/3$	$0 + \frac{1}{3} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{9}{16} = \frac{1}{12} + \frac{4}{16} = \frac{1}{12} + \frac{3}{12} = \frac{1}{3}$ ✓
	$\sum_{m=1}^{s} \sum_{n=1}^{s} b_n a_{mn} c_n = 1/6$	$0 + \left\{ m=2,\ \underset{c_1=0,\ a_{22}=0,\ a_{23}=0}{\sum_{n=1}^{3} b_n a_{2n} c_n = 0} \right\} + \left\{ m=3,\ \underset{c_1=0,\ a_{33}=0}{\sum_{n=1}^{s}} = b_n a_{3n} c_n = b_2 a_{32} c_2 = \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{8} \right\} = \frac{1}{8}$ ✗

<u>The RK-method is consistent of order $O(h^2)$.</u>

b)

- $k_1 = h \cdot f(t_i, y_i)$
- $k_2 = h \cdot f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_1\right)$
- $k_3 = h \cdot f\left(t_i + \frac{3}{4}h, y_i + \frac{3}{4} h k_2\right)$

$\Rightarrow y_{i+1} = y_i + \left[\frac{2}{9}k_1 + \frac{1}{3}k_2 + \frac{4}{9}k_3\right]$

$$y'(t) = \underbrace{t^2 e^{-2y}}^{f(t,y)}, \quad y(0) = 0, \quad h = 0.48$$

1. $k_1 = 0.48 \cdot (0)^2 e^{-0} = 0$

2. $k_2 = 0.48 \cdot \left(0 + \frac{0.48}{2}\right)^2 e^0 = 0.48 \cdot 0.24^2 = 0.027648$

3. $k_3 = 0.48 \cdot \left(0 + \frac{3}{4} \cdot 0.48\right)^2 e^0 = 0.48 \cdot 0.36^2 = 0.062208$

$y_1 = 0 + \frac{2}{9} \cdot 0 + \frac{1}{3} \cdot 0.027648 + \frac{4}{9} \cdot 0.062208 = \underline{\underline{0.036864}}$

Exact:

Close

$y(t) = \frac{1}{2}\log\left(1 + \frac{1}{3}(2 \cdot t^3)\right) \Rightarrow y(0.48) \approx \underline{0.0355684}$

Problem 2. (A boundary value problem - 4D)

Given the two point boundary value problem:

$$u_{xx} + 2u_x + \pi^2 u = \cos(\pi x) - \pi(x+1)\sin(\pi x), \quad 0 \le x \le 2, \qquad u(0) = 0, \quad u(2) = 1$$

a) Verify that the exact solution is

$$u(x) = \frac{x}{2}\cos(\pi x).$$

b) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = 2/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \dots, N$.

c) Let $N = 4$ and use the above formula to find approximations $U_i \approx u(x_i)$, $i = 1, 2, 3$. (That is: Set up the system of equations, and solve it). Compare with the exact solution.

$\boxed{2}$ a) $u'' + 2u' + \pi^2 u = \cos(\pi x) - \pi\cdot(x+1)\cdot\sin(\pi x)$, $x \in [0,2]$, $u(0)=0$, $u(2)=1$.

If $u(x) = \frac{x}{2}\cos(\pi x)$:

- $\frac{d}{dx}u(x) = \frac{1}{2}\left(\cos(\pi x) - \pi x\sin(\pi x)\right) = u'(x)$

- $\frac{d^2}{dx^2}u(x) = -\frac{1}{2}\pi\left(2\sin(\pi x) + \pi x\cos(\pi x)\right) = u''(x)$

$$-\pi\sin(\pi x) - \cancel{\frac{x}{2}\pi^2\cos(\pi x)} + \cos(\pi x) - \pi x\sin(\pi x) + \cancel{\pi^2\frac{x}{2}\cos(\pi x)}$$

$$= \cos(\pi x) - \pi\left(\sin(\pi x) + x\sin(\pi x)\right)$$

$$= \underline{\underline{\cos(\pi x) - \pi(x+1)\sin(\pi x)}}$$

b)

$$u_{xx} \approx \frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2} \quad \text{and} \quad u_x \approx \frac{U_{i+1} - U_{i-1}}{2\Delta x} \quad , \quad \Delta x = \frac{2}{N} \, , \quad x_i = \Delta x i$$

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2} + \cancel{2} \frac{U_{i+1} - U_{i-1}}{\cancel{2}\Delta x} + \pi^2 U_i = \cos(\pi x_i) - \pi(x_i+1)\sin(\pi x_i)$$

$$\Longleftrightarrow \Big(\overset{\frac{N^2}{4}}{\frac{1}{(\Delta x)^2}} + \overset{\frac{N}{2}}{\frac{1}{\Delta x}} \Big) U_{i+1} + \Big(\overset{-\frac{2N^2}{4}}{\frac{-2}{(\Delta x)^2}} + \pi^2 \Big) U_i + \Big(\overset{\frac{N^2}{4}}{\frac{1}{(\Delta x)^2}} - \overset{-\frac{N}{2}}{\frac{1}{\Delta x}} \Big) U_{i-1} = \cos(\pi x_i) - \pi(x_i+1)\sin(\pi x_i)$$

For $i = 1, 2, \dots N-1$

For $i=0$, $u_0 = u(0) = 0$ and $i=N$, $u_N = u(2) = 1$.

c) For $N=4$, we get $\Delta x = \frac{2}{4} = 0.5$

$i=0$: $u_0 = u(0) = 0$.

$i=N$: $u_N = u(2) = 1$.

$i=1$: $x_i = 0.5$, $\underline{u(0.5) = 0}$ Exact

$$2\overset{0}{u_0} + (-8+\pi^2)U_1 + 6U_2 = \overset{0}{\cos\left(\frac{\pi}{2}\right)} - \frac{3}{2}\pi \overset{1}{\sin\left(\frac{\pi}{2}\right)}$$

$$\Rightarrow \underline{U_1 = \frac{-\frac{3}{2}\pi - 6U_2}{\pi^2 - 8}}$$

$i=2$: $x_i = 1$, $\underline{u(1) = -0.5}$ Exact

$$2U_1 + (\pi^2 - 8)U_2 + 6U_3 = \overset{-1}{\cos(\pi)} - 2\pi \overset{0}{\sin(\pi)}$$

$$\Rightarrow \underline{U_2 = \frac{-1 - 2U_1 - 6U_3}{\pi^2 - 8}}$$

$i = 3: \quad x_i = 1.5 \quad , \quad \underline{u(1.5) = 0}$ Exact

$$2U_2 + (\pi^2 - 8)U_3 + 6\overset{\nearrow 1}{u_4} = \overset{\nearrow 0}{\cos\left(\tfrac{3}{2}\pi\right)} - 2.5\pi \overset{\nearrow -1}{\sin\left(\tfrac{3}{2}\pi\right)}$$

$$\Rightarrow \quad U_3 = \frac{2.5\pi - 2U_2 - 6}{\pi^2 - 8}$$

solve{U_1 = (-3*pi/2 - 6*U_2)/(pi^2-8), U_2 = (-1-2*U_1-6*U_3)/(pi^2-8), U_3 = (2.5*pi - 2*U_2 - 6)/(pi^2-8)}

NATURAL LANGUAGE MATH INPUT EXTENDED KEYBOARD EXAMPLES UPLOAD RANDOM

Input interpretation

solve

$$U_1 = \frac{-3 \times \frac{\pi}{2} - 6U_2}{\pi^2 - 8}$$

$$U_2 = \frac{-1 - 2U_1 - 6U_3}{\pi^2 - 8}$$

$$U_3 = \frac{2.5\pi - 2U_2 - 6}{\pi^2 - 8}$$

Result

Enlarge Data Customize Plain Text

$U_1 = -3.07908$ and $U_2 = 0.174045$ and $U_3 = 0.80546$

$U_1 = -3.07908 \quad U_2 = 0.174045 \quad U_3 = 0.80546$ Approx

$u_1 = 0 \quad u_2 = -0.5 \quad u_3 = 0$ Exact

Problem 3. (Ralston's method - 4D)

For the ordinary differential equation

$$y'(t) = -6y(t)\,, \quad \text{with} \quad y(0) = 1\,,$$

consider *Ralston's method* given by the following Butcher tableau:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 3/4 \end{array}$$

Using the tableau and expanding the stage derivatives k_i, we can write the solution y_{n+1} in terms of the previous one, y_n, and of the time-step size $h > 0$. More precisely:

$$y_{n+1} = R(h)\,y_n\,, \quad \text{so that} \quad y_n = [R(h)]^n\,y(0)\,,$$

in which $R(h)$ is a second-degree polynomial.

a) How many stages does this Runge–Kutta method have?

b) Determine the polynomial $R(h)$.

c) Using the expression obtained for $R(h)$, determine for what range of step sizes this algorithm is stable.

$\boxed{3}$ a) 2 stages (as seen by the rows in the Butcher Tableau)

b) $k_1 = hf(t_i, y_i)$

$k_2 = hf(t_i + \frac{2}{3}h, \; y_i + \frac{2}{3}k_1)$

$y_{i+1} = y_i + \frac{1}{4}k_1 + \frac{3}{4}k_2$

For $y'(t) = -6y(t)$ and $y(0) = 1$

We have $f(t, y) = -6y$

So:

$$k_1 = \underline{-6hy_i}$$

$$k_2 = -6h\left(y_i - \tfrac{2}{3}\cdot 6hy_i\right)$$

$$= \underline{-6hy_i + 4h^2 y_i}$$

$$y_{i+1} = y_i - \tfrac{6}{4}hy_i + \tfrac{3}{4}\left(-6hy_i + 4h^2 y_i\right)$$

$$= y_i\left(1 - \tfrac{3}{4}h - h^2\right)$$

So, $\underline{\underline{R(h) = 1 - \tfrac{3}{4}h - h^2}}$

c) We need $|R(h)| \leqslant 1$ to be stable.

$$\left|1 - \tfrac{3}{4}h - h^2\right| \leqslant 1$$

$$\Rightarrow \begin{cases} 1 - \tfrac{3}{4}h - h^2 \leqslant 1 \\ 1 - \tfrac{3}{4}h - h^2 \geqslant -1 \end{cases}$$

$$\Rightarrow \begin{cases} \tfrac{3}{4}h + h^2 \geqslant 0 \\ \tfrac{3}{4}h + h^2 \leqslant 2 \end{cases}$$

$$\Rightarrow \begin{cases} h\left(\tfrac{3}{4} + h\right) \geqslant 0 \Rightarrow h \geqslant 0 \cup h \leqslant -\tfrac{3}{4} \\ h^2 + \tfrac{3}{4}h - 2 \leqslant 0 \Rightarrow \tfrac{1}{4}\left(4h^2 + 3h - 2\right) \geqslant 0 \Rightarrow 4h^2 + 3h - 2 \geqslant 0 \\ \Rightarrow \dfrac{-3 - \sqrt{137}}{8} \leqslant h \leqslant \dfrac{-3 + \sqrt{137}}{8} \end{cases}$$

If we take the intersection of these ranges we get

$$h \in \left[\frac{-3-\sqrt{137}}{8}, \frac{-3+\sqrt{137}}{8}\right] \cap \left(h \in \left(-\infty, -\frac{3}{4}\right] \cup h \in [0, \infty)\right)$$

$$\underline{\underline{h \in \left[\frac{-3-\sqrt{137}}{8}, -\frac{3}{4}\right] \cup h \in \left[0, \frac{-3+\sqrt{137}}{8}\right]}}$$

The range for which the RK method is stable.