Bayesian estimation of ETAS models using Rstan

(applied to seismic recurrence in Ecuador 2016)

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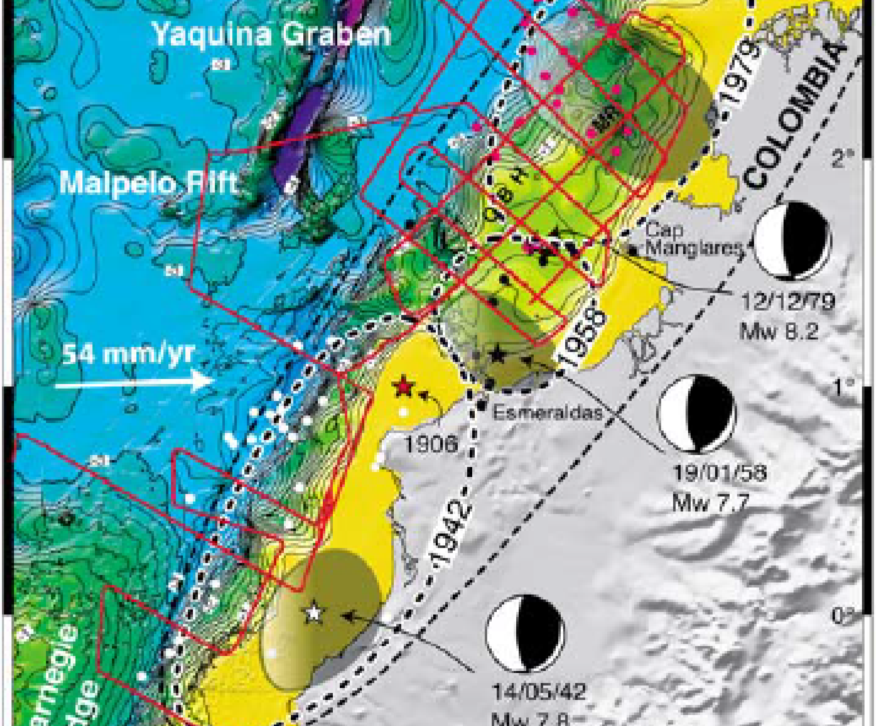
StanCon2018

Introduction

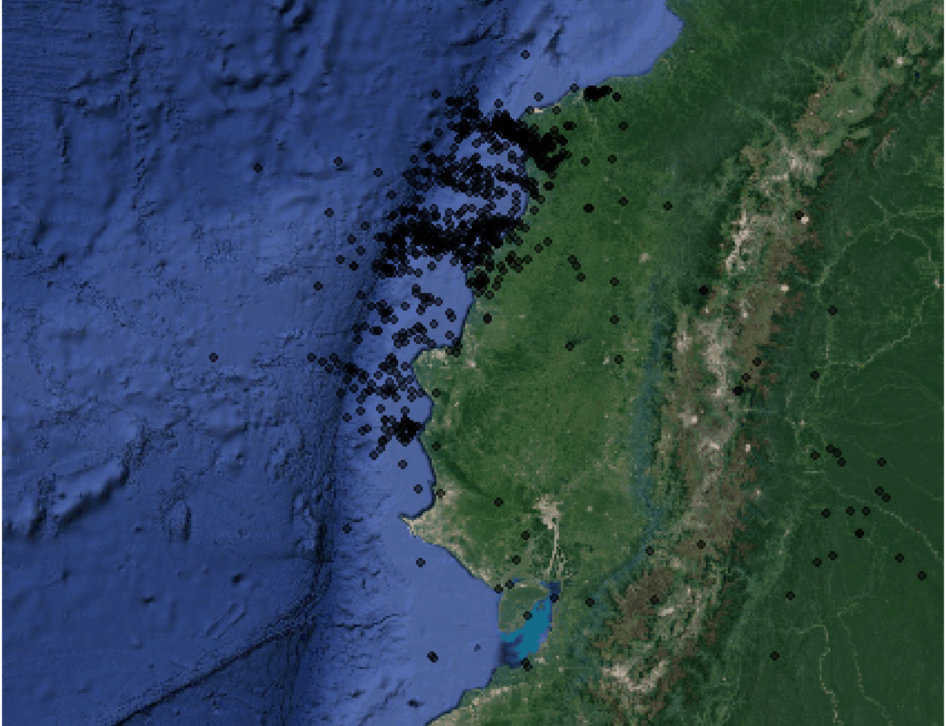
There were four mega earthquakes in the Ecuadorian Coast in the past century:

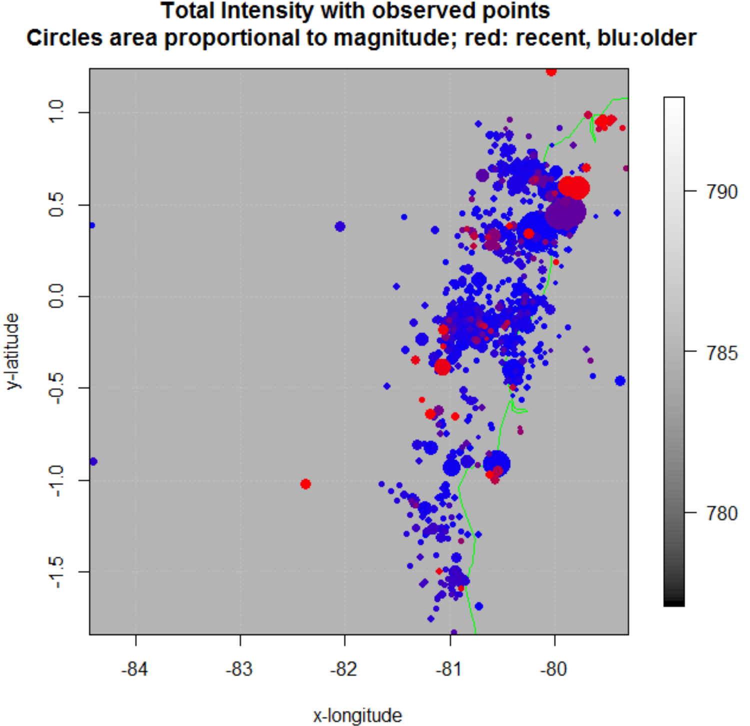
- 1906(magnitude 8.8)
- 1942(magnitude 7.8)
- 1958(magnitude 7.7)
- 1979(magnitude 8.2)

And then... April 16th, 2016 a quake of magnitude 7.8 according to USGS (magnitude 7.4 according to Seismology Institute - EPN)



Yaquina Graben
Malpelo Rift
54 mm/yr
COLOMBIA
1979
1958
1942
1906
Cap Manglares
Esmeraldas
12/12/79
Mw 8.2
19/01/58
Mw 7.7
14/05/42
2°
1°
0°





Total Intensity with observed points
Circles area proportional to magnitude; red: recent, blu:older
y-latitude
x-longitude
1.0
0.5
0.0
-0.5
-1.0
-1.5
-84
-83
-82
-81
-80
790
785
780

Methods

Omori law

The empirical law of Omori and the law of Omori-Utsu (also called Modified Law of Omori) Utsu1995 describe the decreasing frequency of aftershocks over time after an earthquake:

$$N(t) = \frac{K}{(t + c)} \tag{1}$$

$$N(t) = \frac{K}{(t + c)^p} \tag{2}$$

Where $N(t)$ is the occurrence rate of events, t is the time since the earthquake and K, c, p are constants.

Methods

Gutenberg-Ritcher law

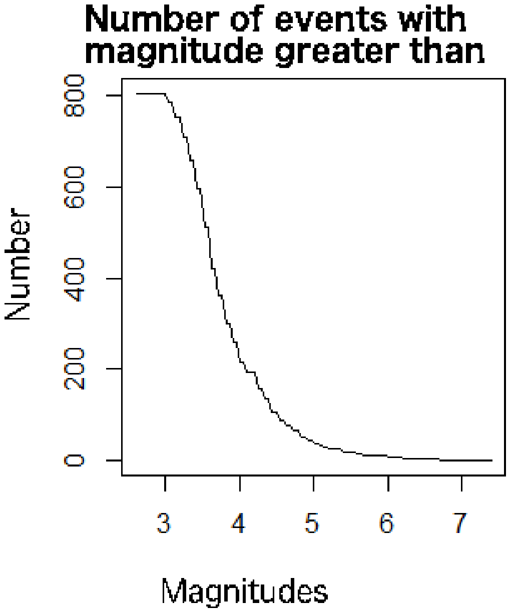
... relates the magnitude to the number of earthquakes with magnitudes greater than M:

$$log_{10} N(\geq M) = a - bM \tag{3}$$

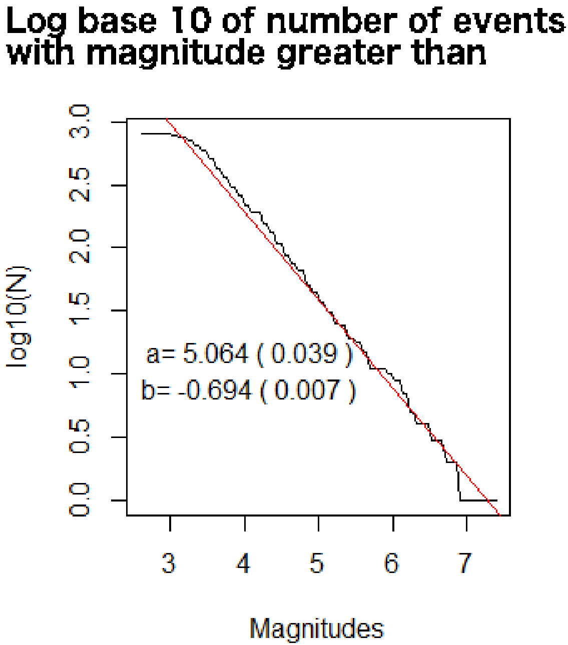
That is, the number of events of magnitude greater than a threshold, decreases exponentially with the increase of that threshold value by a power law.
So, the distribution of magnitudes by the Gutenberg-Richter law $s(m) = \beta e^{-\beta m}$

Gutenberg-Richter law

Gutenberg-Richter law: linear regression $log(N) = a + bM$ and then $a = 5.064(sd = 0.039)$ and $-0.694(sd = 0.007)$. From these values, b is 0.694 and $\beta = bln(10) = 1.598$.



Number of events with
magnitude greater than
Number
0
200
400
600
800
3
4
5
6
7
Magnitudes


Log base 10 of number of events
with magnitude greater than
log10(N)
Magnitudes
a= 5.064 (0.039)
b= -0.694 (0.007)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3
4
5
6
7

Methods

An earthquake T is represented by a tuple (x_i, y_i, z_i, M_i)

Temporal ETAS model

The simplest ETAS model is the temporal model with constant background seismicity: $\lambda(t|H_t) = \mu + \sum_{j:t_j<t} \frac{K(p-1)c^{p-1}}{(t-t_j+c)^p}$
where μ is the background intensity that is assumed to be constant (measured in events / day) and

$$g(t) = \frac{(p-1)c^{p-1}}{(t+c)^p} \tag{4}$$

is the probability density function of the occurrence times of the events triggered by previous earthquakes.

Methods

Temporal ETAS model

If the background intensity is not constant, but depends only on the longitude and latitude x, y we have:
$\lambda(t|H_t) = \mu(x,y) + \sum_{j:t_j<t} \frac{K(p-1)c^{p-1}}{(t-t_j+c)^p}$
where $\mu(x,y)$ is now measured in events per day per unit of longitude and per unit of latitude. In this case, it is generally assumed that $\mu(x,y) = \mu u(x,y)$ where μ on the right side is a constant.

Methods

ETAS model with magnitudes

Considering the magnitudes of earthquakes we have the model

$$\lambda(t|H_t) = \mu + \sum_{j:t_j<t} \frac{K(p-1)c^{p-1}Ae^{\alpha(M_j-M_0)}}{(t-t_j+c)^p}$$

Methods

ETAS models stability

The event rate in the ETAS models may explode. Stability depends on the branching ratio n =expected number of descendants of a parent event. We have n$=\int_0^\infty dt \int_{M_0}^{M_{max}} s(m)\lambda(t)dm$
with $s(m) = \beta e^{-\beta m}$ the distribution of magnitudes by the Gutenberg-Richter law and $\lambda(t)$ the branching term.
For the temporal ETAS model with magnitudes and $p > 1$ we have
$n = \frac{Kc^{1-p}\beta}{(p-1)(\beta-\alpha)} \frac{1-e^{-(\beta-\alpha)(M_{max}-M_0)}}{1-e^{-\beta(M_{max}-M_0)}}$

Methods

For the temporal ETAS model with magnitudes and $p > 1$ we have $n = \frac{Kc^{1-p}\beta}{(p-1)(\beta-\alpha)} \frac{1-e^{-(\beta-\alpha)(M_{max}-M_0)}}{1-e^{-\beta(M_{max}-M_0)}}$ Assuming $M_{max} = \infty$ the previous formula can be reduced to: $n = \frac{Kc^{1-p}\beta}{(p-1)(\beta-\alpha)}$
Sornette2005 y Touati2011

and n is infinite if $p < 1$ or if $\alpha > \beta$.

Methods

ETAS models stability

If each event induces another event: $n = 1$ then the process propagates indefinitely. This justifies normalizing the functions that appear in the sum over the preceding events. For example $\int_0^\infty \frac{K}{(t+c)^p} dt = 1$ implies that we need to add $(p-1)c^{p-1}$ to the constant K and similarly $\int_{M_0}^{M_{max}} \beta e^{-\beta(m)} dm = 1$ implies we need to add
$1/(exp(-\beta M_0) - exp(-\beta M_{max}))$ to the constant β .

Methods

For temporal models: $\log\text{L}(\theta) = \sum_j log \lambda(t_j|H_t) - \int_0^{T_{max}} \lambda(t)dt$
The closed forms for temporal and temporal with magnitudes models: $\log\text{L}(\mu, k, p, c) = \sum_{i=1}^{N} log(\lambda(t_i)) - \mu T_{max} - k \sum_{i=1}^{N}(1 - \frac{c^{p-1}}{(T_{max}-t_i+c)^{p-1}})$
$\log\text{L}(\mu, k, p, c, A, \alpha) = \sum_{i=1}^{N} log(\lambda(t_i)) - \mu T_{max} - kA \sum_{i=1}^{N} e^{\alpha(M_i - M_0)}(1 - \frac{c^{p-1}}{(T_{max}-t_i+c)^{p-1}})$

Listing 1: Temporal Rstan model with constant background seismicity

```
functions{
  real loglikelihood(int N,
                     real mu,
                     real k,
                     real p,
                     real c,
                     vector times_diff,
                     real tmax){
    real seismic_rate[N];
    real integral_of_rate[N];
    real integral_mu;
    real log_likelihood;
    seismic_rate[N]=log(mu);
    integral_of_rate[N]=0;
    for(j in 1:(N-1)){
      vector[N-j] y;
      int start;
      int end;
      start =N*(j-1)-(j*(j-1))/2 + 1;
      end =j*N-(j*(j+1))/2;
      y=times_diff[start:end];
      y=(k*(p - 1) * c^(p-1))*exp(-p*log(y+c));
      seismic_rate[j]=log(mu+sum(y));
      integral_of_rate[j]=(k)*(1-c^(p-1)/
        ((times_diff[j]+c)^(p-1)));
    }
    integral_mu=mu*tmax;
    log_likelihood=sum(seismic_rate)-integral_mu-sum(integral_of_rate);
    return(log_likelihood);
  }
}
data{
  int<lower=0> N;
  real<lower=0> max_time;
  vector[N*(N-1)/2] times_diff;
}
parameters{
  real<lower=0> mu;
  real<lower=0> k;
  real<lower=1.000005> p;
  real<lower=0.00005> c;
}
model{
 mu~exponential(2.8);
k~exponential(2.8);
p~exponential(0.3);
c~exponential(2.8);
 increment_log_prob(loglikelihood(N,mu,k,
   p,c,times_diff,max_time));
}
```

Listing 2: Temporal with magnitudes Rstan model with constant background seismicity

```
functions{
  real loglikelihood(int N,
                     real mu,
                     real k,
                     real p,
                     real c,
                     real alpha,
                     real A,
                     vector t,
                     vector magnitudes,
                     vector times_diff,
                     real max_time,
                     real magnitude0){
    real seismic_rate[N];
    real integral_of_rate[N];
    real integral_mu;
    real log_likelihood;
    seismic_rate[N]= log(mu);
    integral_of_rate[N]=0;
    for(j in 1:(N-1)){
      vector[N-j] y;
      int start;
      int end;
      start = N*(j - 1)-(j*(j-1))/2 + 1;
      end =j*N-(j*(j+1))/2;
      y=times_diff[start:end];
      y=(k*A*(p-1)*c^(p-1))*exp(alpha*(magnitudes[(j+1):]-magnitude0))
        .*exp(-p*log(y+c));
      seismic_rate[j]=log(mu+sum(y));
      integral_of_rate[j]=(k*A)*exp(alpha*
        (magnitudes[j+1]-
        magnitude0))*(1-c^(p-1)/
        ((times_diff[j]+c)^(p-1)));
    }
    integral_mu =mu*max_time;
    log_likelihood=sum(seismic_rate)-
      integral_mu-sum(integral_of_rate);
    return(log_likelihood);
  }
}
data{
  int<lower=0> N;
  vector[N] times;
  real<lower=0> max_time;
  vector[N] magnitudes;
  vector[N*(N-1)/2] times_diff;
  real<lower=0> threshold_magnitude;
}
parameters{
  real<lower=0> mu;
  real<lower=0> k;
  real<lower=1.000005> p;
  real<lower=0.00005> c;
  real<lower=0> alpha;
  real<lower=0> A;
}
model{
 mu~exponential(2.8);
 k~exponential(2.8);
 p~exponential(0.3);
 c~exponential(2.8);
  alpha~exponential(2.8);
```

```
A~exponential(2.8);
increment_log_prob(loglikelihood(N,
      mu,k,p,c
,alpha,A,times,magnitudes,times_diff
,
max_time,threshold_magnitude));
}
```

Methods

Isotropic spatio temporal ETAS model

In 1998 Ogata proposed a modified version of the spatio-temporal ETAS model:

$\lambda(t, x, y) = \mu + \sum_{j:t_j<t} g(t - t_j, x - x_j, y - y_j, M_j|H_t)$

where

$g(t, x, y, M)\frac{Ke^{\alpha(M-M_0)}}{(t+c)^p}\{\frac{x^2+y^2}{e^{\gamma(M-M_0)}} + d\}^{-q}$ and the background intensity μ is constant. We can normalize:

$g(t, x, y, M) = \frac{K(p-1)c^{p-1}(q-1)d^{q-1}\alpha e^{(\alpha-\gamma)(M-M_0)}}{\pi(t+c)^p}\{\frac{x^2+y^2}{e^{\gamma(M-M_0)}} + d\}^{-q}$

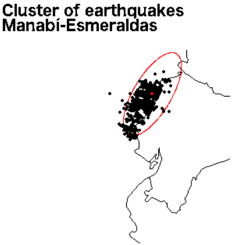
Methods

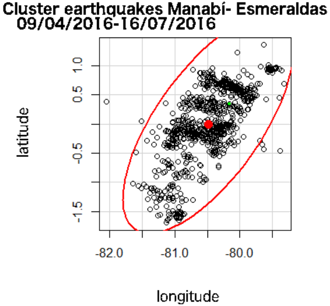
Anisotropic spatio temporal ETAS model

$\lambda(t, x, y) = \mu + \sum_{j:t_j<t} g(t - t_j, x - x_j, y - y_j, M_j | H_t)$

where $g(t, x, y, M) = \frac{Ke^{\alpha(M-M_0}}{(t+c)^p}\{\frac{pS_j p^T}{e^{\alpha(M_j-M_0)}} + d\}^{-q}$

and $p = (x - x_j, y - y_j)$ is a row vector, x_j, y_j are the coordinates of the earthquake j preceding the earthquake with epicenter x, y (both in the same cluster) and S_j $(j = 1, 2, ...)$ are positive definite symmetric matrices representing the normalized covariance matrix of the earthquake cluster obtained by applying MBC or Magnitude Based Cluster algorithm


Cluster of earthquakes
Manabí-Esmeraldas


Cluster earthquakes Manabí- Esmeraldas
09/04/2016-16/07/2016
latitude
longitude
1.0
0.5
-0.5
-1.5
-82.0
-81.0
-80.0

Methods

Magnitude Based Cluster MBC

This method is based on selecting the greatest magnitude earthquake(with magnitude M_j) between those that are not in any cluster yet (if there are two with equal magnitude the oldest one is chosen) and then the earthquakes of the cluster associated with the previous selected earthquake, are those with latitude and longitude $\pm\ 3.33 * 10^{0.5 M_j - 2}$ km (Utsu spatial distance) from the latitude and longitude of the selected earthquake and with a time difference therefrom (towards the future) of
$max(100, 10^{0.5 M_j - 1})$ days Ogata1998. Then the process is repeated with earthquakes that are not yet in any cluster until all earthquakes belong to a cluster.

Bivariate normal distribution

For the anisotropic model, four normal bivariate models

Ogata1998: $\mathrm{N}(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} \tilde{\sigma}^2 & 0 \\ 0 & \tilde{\sigma}^2 \end{pmatrix})$

$\mathrm{N}(\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \begin{pmatrix} \hat{\sigma}^2 & 0 \\ 0 & \hat{\sigma}^2 \end{pmatrix})$

$\mathrm{N}(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} \tilde{\sigma_1}^2 & \tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 \\ \tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 & \tilde{\sigma_2}^2 \end{pmatrix})$

$\mathrm{N}(\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \begin{pmatrix} \hat{\sigma_1}^2 & \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 \\ \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 & \hat{\sigma_2}^2 \end{pmatrix})$

$\mathrm{N}(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} \tilde{\sigma_1}^2 & \tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 \\ \tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 & \tilde{\sigma_2}^2 \end{pmatrix})$

$\mathrm{N}(\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \begin{pmatrix} \hat{\sigma_1}^2 & \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 \\ \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 & \hat{\sigma_2}^2 \end{pmatrix})$

where (x_1, y_1) is the position of the cluster's main earthquake, $(\bar{x}, \bar{y})$ and centroid of the cluster.
(Ogata 1998)

Bivariate normal distribution

And $\tilde{\sigma}^2 = [\sum_j (x_j - x_1)^2 + \sum_j (y_j - y_1)^2]/(2n)$
$\hat{\sigma}^2 = [\sum_j (x_j - \bar{x})^2 + \sum_j (y_j - \bar{y})^2]/(2n)$
$\tilde{\sigma_1}^2 = [\sum_j (x_j - x_1)^2]/n$ $\tilde{\sigma_2}^2 = [\sum_j (y_j - y_1)^2]/n$
$\tilde{\rho} = [\sum_j (x_j - x_1)(y_j - y_1)]/(n\tilde{\sigma_1}\tilde{\sigma_2})$
$\tilde{\sigma_1}^2 = [\sum_j (x_j - x_1)^2]/n$
$\tilde{\sigma_2}^2 = [\sum_j (y_j - y_1)^2]/n$ $\tilde{\rho} = [\sum_j (x_j - x_1)(y_j - y_1)]/(n\tilde{\sigma_1}\tilde{\sigma_2})$
$\hat{\sigma_1}^2 = [\sum_j (x_j - \bar{x})^2]/n$
$\hat{\sigma_2}^2 = [\sum_j (y_j - \bar{y})^2]/n$
$\hat{\rho} = [\sum_j (x_j - \bar{x})(y_j - \bar{y})]/(n\hat{\sigma_1}\hat{\sigma_2})$
(Ogata 1998)

Methods

Bivariate normal

We select the model with the lowest $AIC = -nln(det(S)) + 2k$ where S is the variance covariance matrix for each of the four models and k is the corresponding number of parameters. Ogata1998.
Then the selected matrix is normalized:
$(\frac{1}{\sqrt{(1-\rho^2)}})\begin{pmatrix} \sigma_2/\sigma_1 & -\rho \\ -\rho & \sigma_1/\sigma_2 \end{pmatrix}$

Methods

Hypocentral ETAS model
Guo2015 introduces a modification of the spatio temporal ETAS model that includes the depths of earthquakes
$\lambda(t,x,y) = \mu + \sum_{j:t_j<t} g(t-t_j, x-x_j, y-y_j, M_j|H_t)$
where $g(t,x,y,M) = \frac{Ke^{\alpha(M-M_0}}{(t+c)^p}\{\frac{pS_jp^T}{e^{\alpha(M_j-M_0)}} + d\}^{-q} h(z-z_i, z_i)$ and
$h(z,z^{'}) = \frac{(\frac{z}{Z})^{\eta\frac{z^{'}}{Z}}(1-\frac{z}{Z})^{\eta(1-\frac{z^{'}}{Z})}}{ZB(\eta\frac{z^{'}}{Z}+1,\eta(1-\frac{z^{'}}{Z})+1)}$
with Z the thickness of the seismogenic layer and
$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}$ is the Beta function.

Methods

For spatio temporal models
$logL(\theta) = \sum_j log\lambda(t_j, x_j, y_j | H_t) - \int_0^{T_{max}} \int \int_S \lambda(t, x, y) dxdydt$

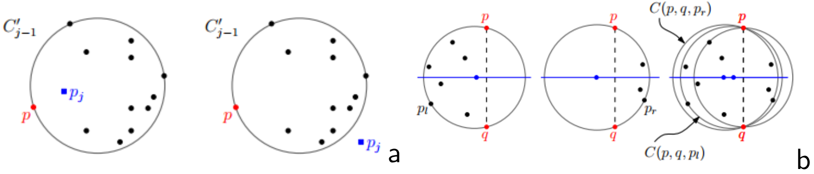
For the hypocentral ETAS model, the logarithm of the likelihood is
$logL(\theta) = \sum_j log\lambda(t_j, x_j, y_j | H_t) - \int_0^{T_{max}} \int \int_S \int_0^Z \lambda(t, x, y) dzdxdydt$

In spatio temporals models, the log likelihood does not have a closed form.

Methods

We approximate the log likelihood for spatio temporals models using polar coordinates and S the minimum covering circle

Minimum Covering Circle Welzl 1991



Methods(Cont)

Approximation

$$\text{logL}(\mu, k, p, c, d, \alpha, \gamma) \approx \sum_{i=1}^{N} log(\lambda(t_i)) - \mu T_{max} \pi r^2 - k\alpha \sum_{i=1}^{N} e^{\alpha(M_i - M_0)} \left(1 - \frac{c^{p-1}}{(T_{max} - t_i + c)^{p-1}}\right)\left(1 - \frac{d^{q-1}}{\left(\frac{r_i^2}{e^{\alpha(M_i - M_0)}} + d\right)^{q-1}}\right)$$

where r is the radius of the minimum covering circle, r_i is the greatest distance between earthquake i and the previous earthquakes according to the metric defined in the cluster.

Methods

For the anisotropic model, the Euclidean distance is changed by the metric defined by the standardized variance covariance matrix S_j where now r_i is calculated as the maximum distance (according to the previous metric) between each earthquake and the previous earthquakes in the cluster.

Methods

Pre-processing

We sorted the data in reverse chronological time and calculated the time, latitude, and longitude differences for each earthquake j for which we know the initial and final positions where the differences of its time, latitude, and longitude with respect to the previous earthquakes are:

$$start = N * (j-1) - (j * (j-1))/2 + 1$$
$$end = j * N - (j * (j+1))/2$$

Listing 3: Anisotropic spatio-temporal Rstan model with constant background seismicity

```
functions{
  real loglikelihood(int N,
                     real mu,
                     real k,
                     real p,
                     real c,
                     real q,
                     real d,
                     real alpha,
                     real gamma,
                     vector t,
                     vector magnitudes,
                     vector times_diff,
                     vector latitudes,
                     vector longitudes,
                     vector quadratic_factorAni,
                     vector quadratic_factorIso,
                     real tmax,
                     real magnitude0,
                     real lat_min,
                     real lat_max,
                     real long_min,
                     real long_max,
                     real radius
                     ){
    real seismic_rate[N];
    real integral_of_rate[N];
    real integral_mu;
    real log_likelihood;
    seismic_rate[N]=log(mu);
    integral_of_rate[N]=0;
    for(j in 1:(N-1)){
      vector[N-j] y;
      vector[N-j] z;
      vector[N-1] x;
      real temp;
      real temp1;
      real temp2;
      int start;
      int end;
      start =N*(j-1)-(j*(j-1))/2 + 1;
      end =j*N-(j*(j+1))/2;
      y=times_diff[start:end];
      z=exp(-q*log(quadratic_factorAni[start:end]./
      (exp(gamma*(magnitudes[(j+1):]-magnitude0)))+d));
      y=(k*alpha*(p-1)*c^(p-1)*(q-1)*d^(q-1)*(1/pi()))
       *exp((alpha-gamma)*(magnitudes[(j+1):]-
       magnitude0))
       .*exp(-p*log(y+c));
      y=y .* z;
      seismic_rate[j]=log(mu+sum(y));
      temp=exp(alpha*(magnitudes[j+1]-magnitude0));
      temp1=exp(gamma*(magnitudes[j+1]-magnitude0));
      temp2=max(quadratic_factorAni[start:end]);
      integral_of_rate[j]=k*alpha*temp
```

```
            *(1-c^(p-1)/
        ((times_diff[j]+c)^(p-1)))*(1-d
              ^(q-1)/
        ((temp2/(temp1)+d)^(q-1)));
      }
      integral_mu=mu*tmax*pi()*radius^2;
      log_likelihood=sum(seismic_rate)-
        integral_mu-sum(integral_of_rate
             );
      return(log_likelihood);
    }
  }
  data{
    int<lower=0> N;
    vector[N] times;
    real<lower=0> max_time;
    vector[N] magnitudes;
    vector[N*(N-1)/2]
         quadratic_factorAni;
    vector[N*(N-1)/2]
         quadratic_factorIso;
    vector[N*(N-1)/2] times_diff;
    real<lower=0> threshold_magnitude;
    vector[N] latitudes;
    vector[N] longitudes;
    real lat_min;
    real lat_max;
    real long_min;
    real long_max;
    real<lower=0> radius;
  }
  parameters{
    real<lower=0> mu;
    real<lower=0> k;
    real<lower=1.000005> p;
    real<lower=0> c;
    real<lower=0> d;
    real<lower=1.00005> q;
    real<lower=0> alpha;
    real<lower=0> gamma;
  }
  model{
    mu~exponential(2.8);
    k~exponential(2.8);
    p~exponential(2.8);
    c~exponential(2.8);
    d~exponential(2.8);
    q~exponential(2.8);
    gamma~exponential(2.8);
    alpha-gamma~exponential(5);
    increment_log_prob(loglikelihood(N,
         mu,k,p,c,q,d,
    alpha,gamma,times,magnitudes,
         times_diff,
    latitudes,longitudes,
         quadratic_factorAni,
    quadratic_factorIso,max_time,
         threshold_magnitude,
    lat_min,lat_max,long_min,long_max,
         radius));
  }
```

Listing 4: Anisotropic spatio temporal hypocentral Rstan model with constant background seismicity

```
functions{
  real loglikelihood(int N,
                     real mu,
                     real k,
                     real p,
                     real c,
                     real q,
                     real d,
                     real alpha,
                     real gamma,
                     real eta,
                     vector t,
                     vector magnitudes,
                     vector times_diff,
                     vector depths_diff
                     vector latitudes
                     vector longitudes,
                     vector depths,
                     vector quadratic_factorAni,
                     vector quadratic_factorIso,
                     real tmax,
                     real magnitude0,
                     real lat_min,
                     real lat_max,
                     real long_min,
                     real long_max,
                     real radius,
                     real layer_depth
                     ){

    real seismic_rate[N];
    real integral_of_rate[N];
    real integral_mu;
    real log_likelihood;
    seismic_rate[N]=log(mu);
    integral_of_rate[N]=0;
    for(j in 1:(N-1)){
      vector[N-j] y;
      vector[N-j] z;
      vector[N-j] x;
      vector[N-j] w;
      real temp;
      real temp1;
      real temp2;
      real temp3;
      int start;
      int end;
      start =N*(j-1)-(j*(j-1))/2 + 1;
      end =j*N-(j*(j+1))/2;
      y=times_diff[start:end];
      z=exp(-q*log(quadratic_factorAni[start:end]
        ./(exp(gamma*(magnitudes[(j+1):]-
        magnitude0)))+d));
      y=(k*alpha*(p-1)*c^(p-1)*(q-1)*d^(q-1)*
        (1/pi()))*exp((alpha-gamma)*
        (magnitudes[(j+1):]-magnitude0
```

```
            ))
      .*exp(-p*log(y+c));
    x=y .* z ;
    z=(1/layer_depth) *
        depths_diff[start:end];
    y=(1/layer_depth) *
        depths[(j+1):N];
    for(i in 1:(N-j)){
       w[i]=(1/(layer_depth * exp(
             lbeta(eta *
        ( y[i]) + 1,eta - eta * ( y[
             i])+ 1)))) *
        (z[i] ^ (eta * (y[i]))) *
             ((1 - z[i]) ^
        (eta-eta * (  y[i])));
    }
    seismic_rate[j]=log(mu+sum(x .*
          w));
    temp=exp(alpha*(magnitudes[j+1]-
          magnitude0));
    temp1=exp(gamma*(magnitudes[j
          +1]-magnitude0));
    temp2=max(quadratic_factorAni[
          start:end]);
    integral_of_rate[j]=k*alpha*temp
          *(1-c^(p-1)
     /((times_diff[j]+c)^(p-1)))*(1-
          d^(q-1)
     /((temp2/(temp1)+d)^(q-1)));
  }
  integral_mu=mu*tmax*pi()*
       layer_depth*
     (radius^2);
  log_likelihood=sum(seismic_rate)-
    integral_mu-sum(integral_of_rate
          );
  return(log_likelihood);
 }
}
data{
  int<lower=0> N;
  vector[N] times;
  real<lower=0> max_time;
  vector[N] magnitudes;
  vector[N*(N-1)/2]
       quadratic_factorAni;
  vector[N*(N-1)/2]
       quadratic_factorIso;
  vector[N*(N-1)/2] times_diff;
  vector[N*(N-1)/2] depths_diff;
  real<lower=0> threshold_magnitude;
  vector[N] latitudes;
  vector[N] longitudes;
  vector[N] depths;
  real lat_min;
  real lat_max;
  real long_min;
  real long_max;
  real<lower=0> radius;
  real<lower=0> layer_depth;
}
parameters{
  real<lower=0> mu;
  real<lower=0> k;
  real<lower=1.000005> p;
  real<lower=0> c;
  real<lower=0> d;
  real<lower=1.00005> q;
  real<lower=0> alpha;
  real<lower=0> gamma;
  real<lower=0> eta;
}
model{
  mu~exponential(2);
```

```
  k~exponential(2);
  p~exponential(2);
  c~exponential(2);
d~exponential(2);
  q~exponential(2);
  eta~exponential(2);
  gamma~exponential(2);
  alpha-gamma~exponential(5);
  increment_log_prob(loglikelihood(N,
       mu,k,p,c,q,d,
    alpha,gamma,eta,times,magnitudes,
         times_diff,
    depths_diff,latitudes,longitudes,
    depths,quadratic_factorAni,
    quadratic_factorIso,max_time,
    threshold_magnitude,lat_min,
         lat_max,long_min,
    long_max,radius,layer_depth));
}
```

Listing 5: Anisotropic Rstan model with variable background seismicity

```
functions{
real loglikelihood(int N,
                   real mu,
                   real k,
                   real p,
                   real c,
                   real q,
                   real d,
                   real alpha,
                   real gamma,
                   vector t,
                   vector magnitudes,
                   vector times_diff,
                   vector latitudes,
                   vector longitudes,
                   vector background_seismic_rates,
                   vector quadratic_factorAni,
                   vector quadratic_factorIso,
                   real tmax,
                   real magnitude0,
                   real lat_min,
                   real lat_max,
                   real long_min,
                   real long_max,
                   real radius
                       ){
real seismic_rate[N];
real integral_of_rate[N];
real integral_mu;
real log_likelihood;
seismic_rate[N]=log(mu);
integral_of_rate[N]=0;
for(j in 1:(N-1)){
  vector[N-j] y;
  vector[N-j] z;
  vector[N-1] x;
  real temp;
  real temp1;
  real temp2;
  int start;
  int end;
  start =N*(j-1)-(j*(j-1))/2 + 1;
  end =j*N-(j*(j+1))/2;
  y=times_diff[start:end];
  z=
    exp(-q*log(quadratic_factorAni[start:end]./
    (exp(gamma*(magnitudes[(j+1):]-magnitude0)))+d);
  y=(k*alpha*(p-1)*c^(p-1)*(q-1)*d^(q-1)*
    (1/pi()))*exp((alpha-gamma)*(magnitudes[(j+1):]
   -magnitude0)).*exp(-p*log(y+c));
  y=y .* z;
  seismic_rate[j]=
    log(mu*background_seismic_rates[j+1]+sum(y));
  temp=exp(alpha*(magnitudes[j+1]-magnitude0));
  temp1=exp(gamma*(magnitudes[j
```

```
            +1]-magnitude0));
        temp2=max(quadratic_factorAni[
            start:end]);
        integral_of_rate[j]=k*alpha*temp1
            *(1-c^(p-1)/
          ((times_diff[j]+c)^(p-1)))*
          (1-d^(q-1)/((temp2/(temp1)+d)^(
            q-1)));
      }
      integral_mu=mu*sum(
          background_seismic_rates)*
        tmax*pi()*radius^2;
      log_likelihood=sum(seismic_rate)-
        integral_mu-sum(integral_of_rate
            );
      return(log_likelihood);
    }
}
data{
  int<lower=0> N;
  vector[N] times;
  real<lower=0> max_time;
  vector[N] magnitudes;
  vector[N*(N-1)/2]
      quadratic_factorAni;
  vector[N*(N-1)/2]
      quadratic_factorIso;
  vector[N*(N-1)/2] times_diff;
  real<lower=0> threshold_magnitude;
  vector[N] latitudes;
  vector[N] longitudes;
  vector[N] background_seismic_rates;
  real lat_min;
  real lat_max;
  real long_min;
  real long_max;
  real<lower=0> radius;
}
parameters{
  real<lower=0> mu;
  real<lower=0> k;
  real<lower=1.000005> p;
  real<lower=0> c;
  real<lower=0> d;
  real<lower=1.00005> q;
  real<lower=0> alpha;
  real<lower=0> gamma;
}
model{
  mu~exponential(2.8);
  k~exponential(2.8);
  p~exponential(2.8);
  c~exponential(2.8);
 d~exponential(2.8);
  q~exponential(2.8);
  gamma~exponential(2.8);
  alpha - gamma ~ exponential(5);
  increment_log_prob(loglikelihood(N,
      mu,k,p,c,q,d,
    alpha,gamma,times,magnitudes,
        times_diff,
    latitudes,longitudes,
        background_seismic_rates,
    quadratic_factorAni,
        quadratic_factorIso,
    max_time,threshold_magnitude,
    lat_min,lat_max,long_min,long_max,
        radius));
}
```

Methods

We can estimate the probability that a given event is spontaneous or is triggered by others Kagan1980Zhuang2002. The contribution of the spontaneous seismicity rate to the occurrence of an event i can be taken as the probability that the event i is spontaneousZhuang2008:
$\phi(i) = \frac{\mu(x_i,y_i)}{\lambda(t_i,x_i,y_i)}$
Similarly, the probability that the event j is produced by the event i is $\rho_{ij} = \frac{\kappa(M)g(t_j-t_i)f(x_j-x_i,y_j-y_i,m_i)}{\lambda(t_i,x_i,y_i)}$
We can also obtain the expected number of direct aftershocks from the earthquake i as $\sum_j \rho_{ij}$ Zhuang2008

Methods

Inter-event times

The probability density function of the recurrence times (time between two successive events) τ

$H(\tau) \approx \lambda f(\lambda\tau)$

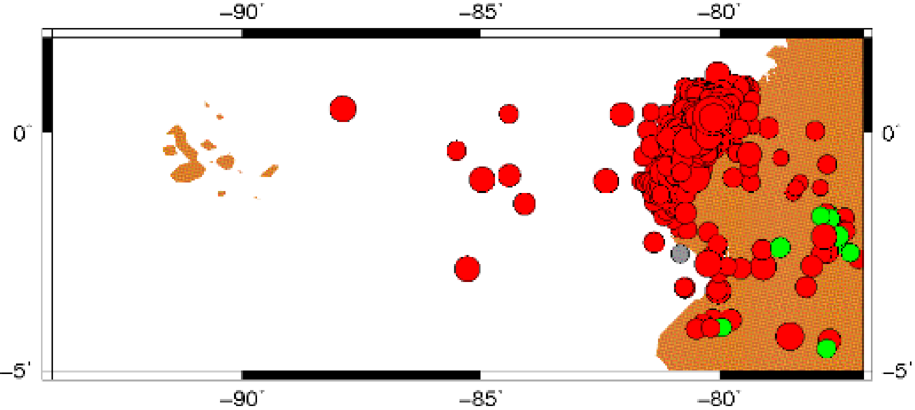
where the function $f(x)$ has been found practically the same in different regions and λ is the average rate of events observed in the analyzed region. Saichev2007

The scaling factor of times between earthquakes is taken as the inverse of its mean.

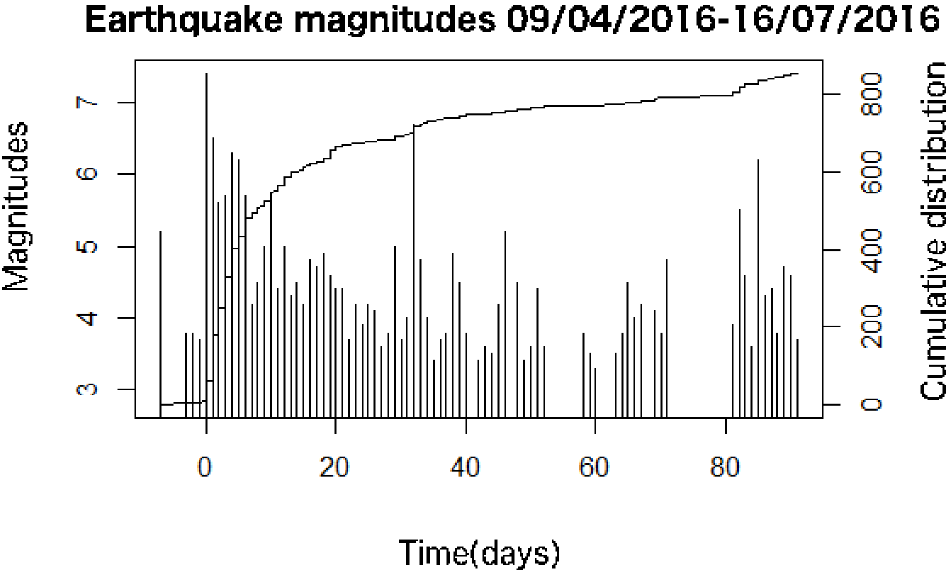
The form of the function $f(x)$ which is demonstrated in Saichev2007 is
$f(x) = (n\epsilon^{\theta}x^{-1-\theta} + [1 - n + n\epsilon^{\theta}x^{-\theta}]^2) *$
$\varphi(x, \epsilon)$

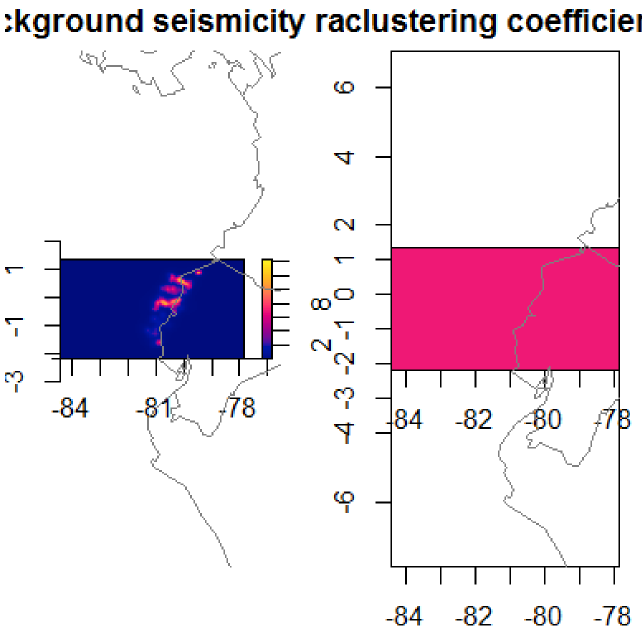
Table 1: Earthquake distribution in Ecuador 18/03/2016-16/07/2016

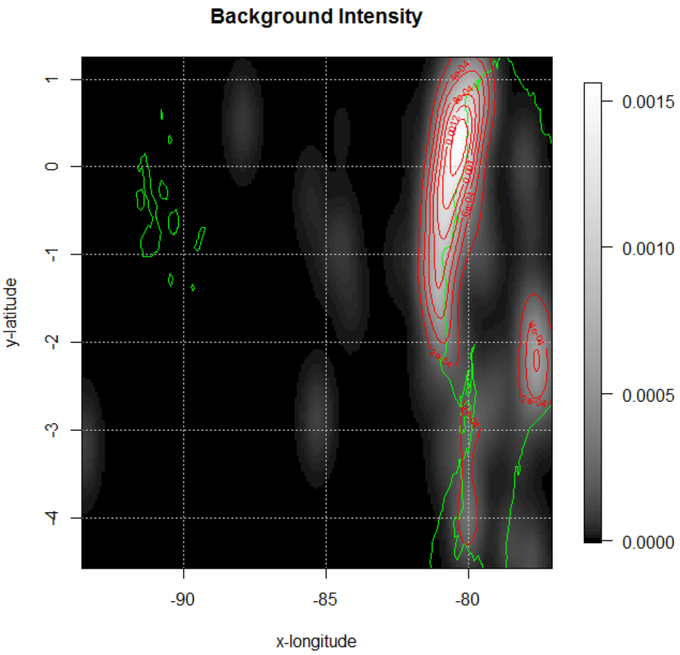
Closest province	March	April	May	June	July	Total
Azuay			1			1
Bolivar		4				4
Canar	1	4				5
Carchi	1	1				2
Cotopaxi	1	1				2
El Oro	2	3	4	2		11
Esmeraldas		171	47	16	35	269
Galapagos		3				3
Guayas		3	5			8
Imbabura	1	1				2
Loja	1	3	1	1		6
Los Rios		1		1		2
Manabi	1	399	92	27	22	541
Los Rios		1		1		2
Morona Santiago		6	7	2	1	16
Napo		1	1			2
Pastaza		4	1	1		6
Pichincha	2	5				7
Santa Elena		3	3	1		7
Sto Domingo de los Tsachilas		4				4
Tungurahua		5	1			6
Zamora Chinchipe		2	2			4
Total	10	624	165	51	58	908



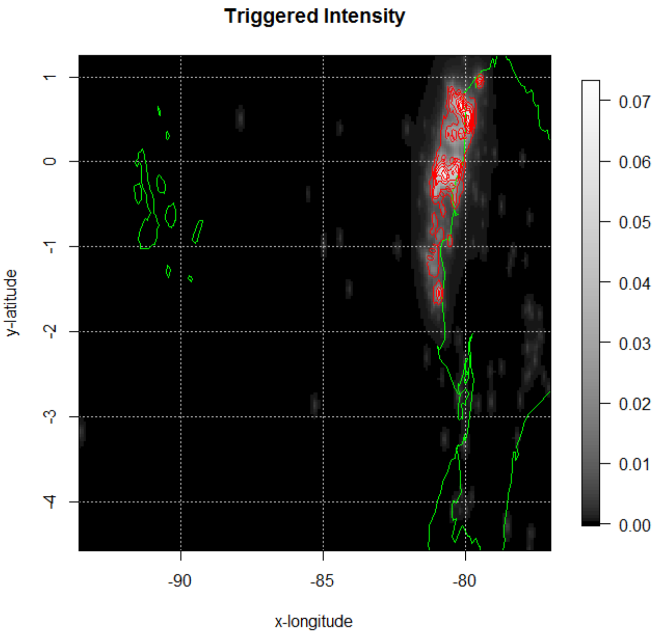
−90˚
−85˚
−80˚
0˚
0˚
−5˚
−5˚
−90˚
−85˚
−80˚

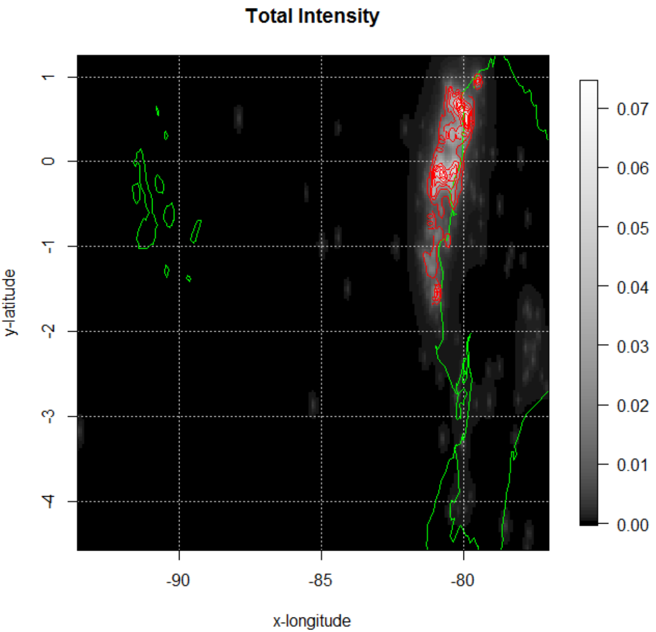

Earthquake magnitudes 09/04/2016-16/07/2016
Magnitudes
Cumulative distribution
Time(days)
3
4
5
6
7
0
200
400
600
800
0
20
40
60
80


:kground seismicity raclustering coefficient
1
-1
-3
-84
-81
-78
6
4
2
1
0
-1
-2
-3
-4
-6
-84
-82
-80
-78
-84
-82
-80
-78
8
2

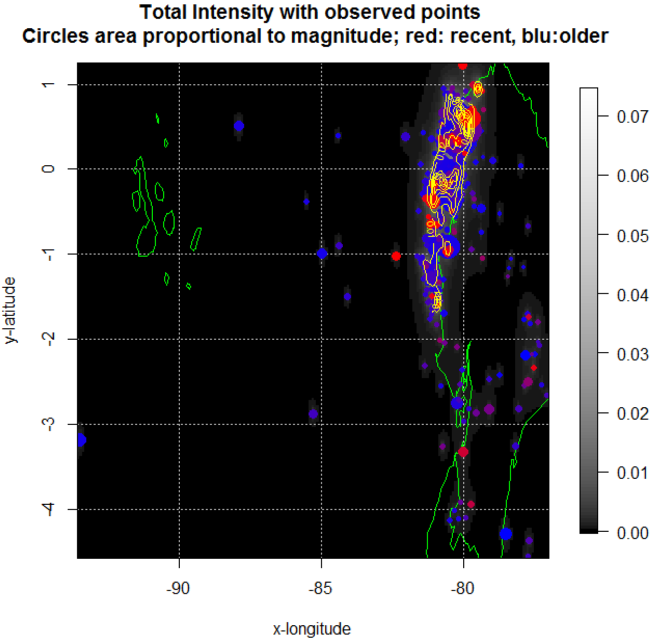


Background Intensity
y-latitude
x-longitude
1
0
-1
-2
-3
-4
-90
-85
-80
0.0015
0.0010
0.0005
0.0000


Triggered Intensity
y-latitude
x-longitude
1
0
-1
-2
-3
-4
-90
-85
-80
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00

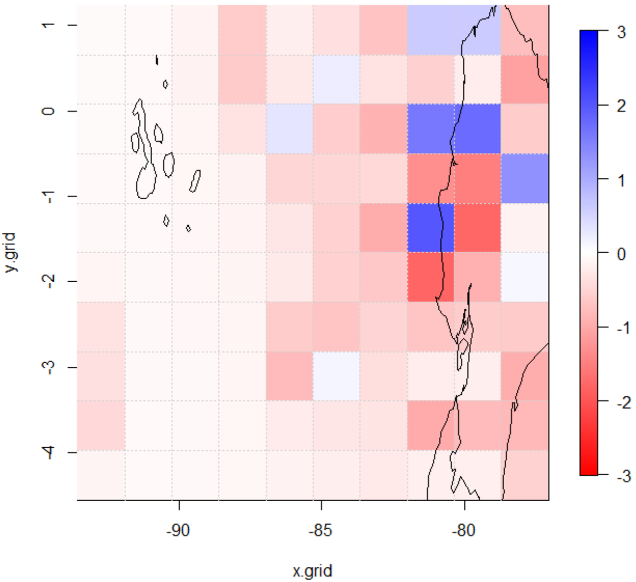


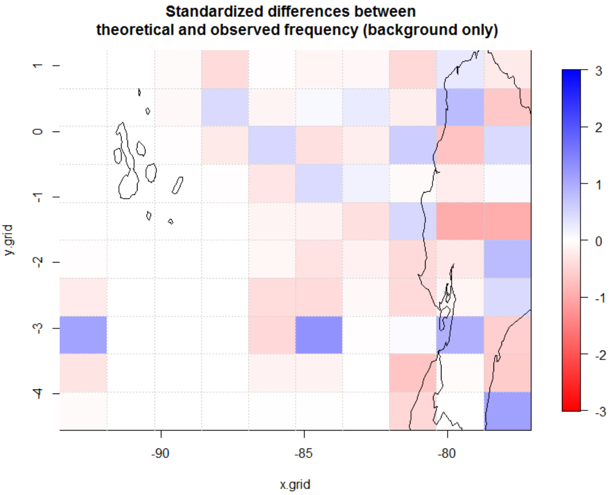
Total Intensity
y-latitude
x-longitude
1
0
-1
-2
-3
-4
-90
-85
-80
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00



Total Intensity with observed points
Circles area proportional to magnitude; red: recent, blu:older
y-latitude
x-longitude
1
0
-1
-2
-3
-4
-90
-85
-80
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00

Standardized differences between theoretical and observed frequency (whole model)




Standardized differences between
theoretical and observed frequency (background only)
y.grid
1
0
-1
-2
-3
-4
x.grid
-90
-85
-80
3
2
1
0
-1
-2
-3

Results

Anisotropic Distribution

The covariance matrix for the cluster associated with the April 16th big earthquake (804 earthquakes in the cluster) is

$$\begin{pmatrix} 1.469 & -0.696 \\ -0.696 & 0.709 \end{pmatrix}$$

and corresponds to the fourth bivariate model fitted.

Table 12: Etas models comparison

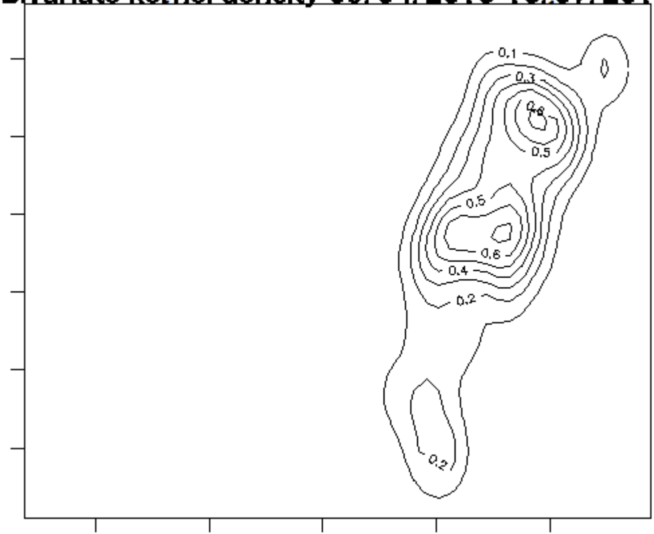
Etas Model	ln(L)	Int. 95% ln(L)		# param.	AIC	int.95% AIC	
Temp. sis. cst	1790.7	1787.3	1792.8	4	-3573.5	-3577.6	-3566.5
Temp. seis. var.	1773.3	1769.7	1775.0	4	-3538.6	-3542.0	-3531.4
Magn. seis. cst	1832.8	1689.2	1842.8	6	-3653.7	-3673.7	-3366.5
Magn. seis. var.	1800.8	1797.4	1802.9	6	-3589.5	-3593.8	-3582.7
Spa. temp. iso. seis. cst	1394.4	1387.4	1398.1	8	-2772.7	-2780.1	-2758.7
Spa. temp. ani. seis. cst	1857.1	1852.7	1859.6	8	-3698.3	-3703.2	-3689.4
Spa. temp. ani. seis. var.	1799.9	1794.6	1801.9	8	-3583.8	-3936.8	-3573.3
Hypo. ani. seis. cst.	1974.1	1969.4	1977.0	9	-3930.2	-3936.1	-3920.8
Hypo. ani. seis. var.	1923.3.1	1919.4	1926.1	9	-3828.6	-3834.2	-3821.4

Results

Kernel bivariate density

Kernel bivariate density using the R package kde.

Bivariate kernel density 09/04/2016-16/07/2016



Priors

Weakly informative priors

For the ETAS parameters(must be positive) we used exponential priors and for p and q we use a minimum value close to 1:1.000005.

The radius of the minimum covering circle using Welzl algorithm was, $r = 1.70479$.

Results

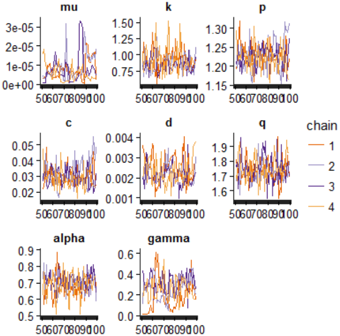


Figure: Chains for anisotropic ETAS model with constant background seismicity

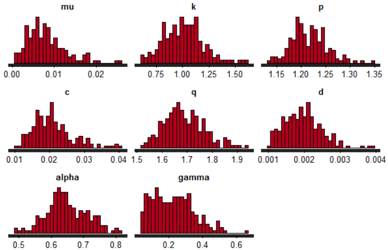


Figure: *A posteriori* parameter distributions for anisotropic ETAS model with constant background seismicity

Results

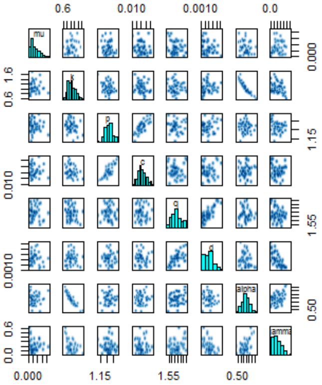


Figure: Correlation of parameter values in the chains for anisotropic ETAS model with constant background seismicity

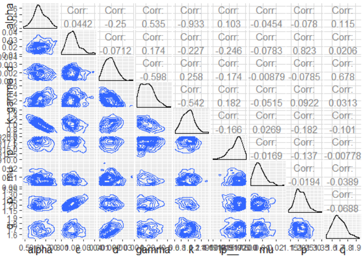


Figure: Correlation of parameter values in the chains for anisotropic ETAS model with constant background seismicity

Results

Residuals

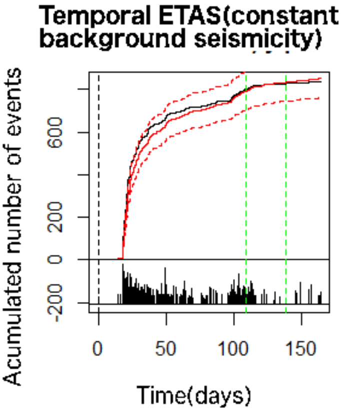


Figure: Acumulated number of events for temporal Etas model with constant background seismicity

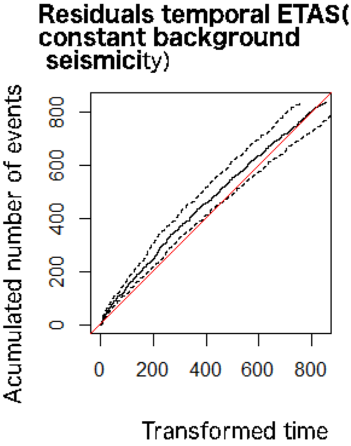


Figure: Residuals for Etas temporal temporal Etas model with constant background seismicity

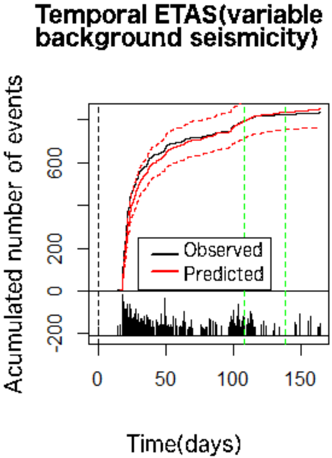


Figure: Accumulated number of events for temporal ETAS model with variable background seismicity

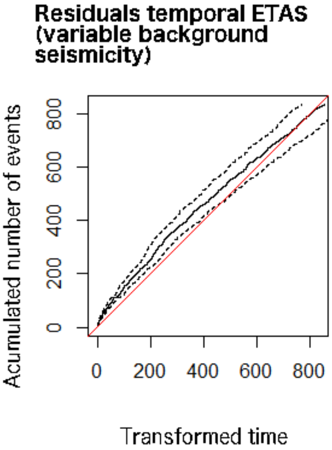


Figure: Residuals for temporal ETAS model with variable background seismicity

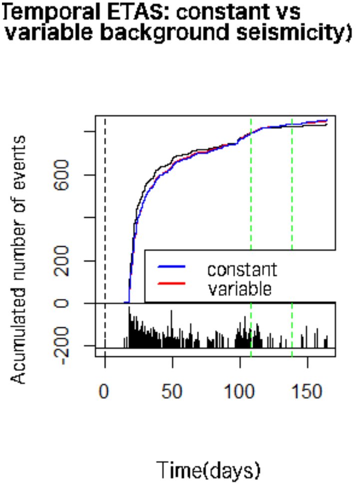


Figure: Accumulated number of events for temporal ETAS model with constant background seismicity vs variable background seismicity

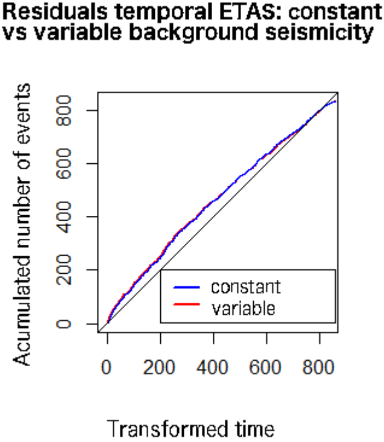


Figure: Residuals for temporal ETAS model with constant background seismicity vs variable background seismicity

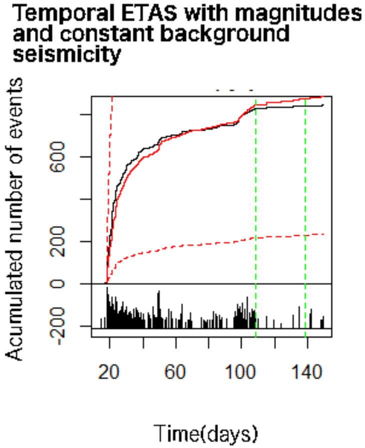


Figure: Accumulated number of events for temporal ETAS with magnitudes model and constant background seismicity

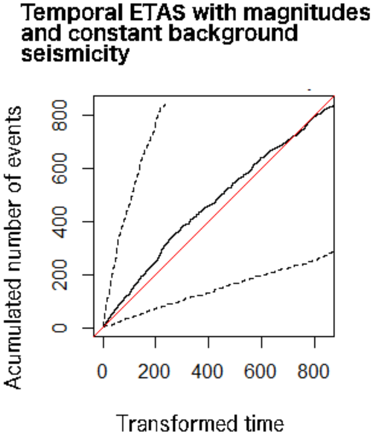


Figure: Residuals for temporal ETAS with magnitudes model and constant background seismicity

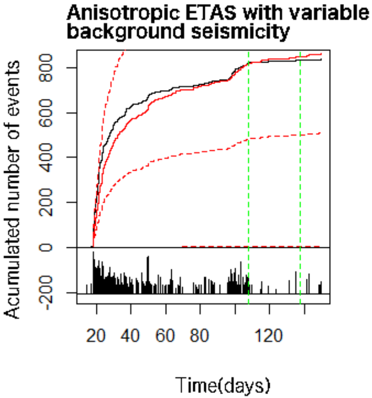


Figure: Accumulated number of events for spatio temporal anisotropic ETAS model and variable background seismicity

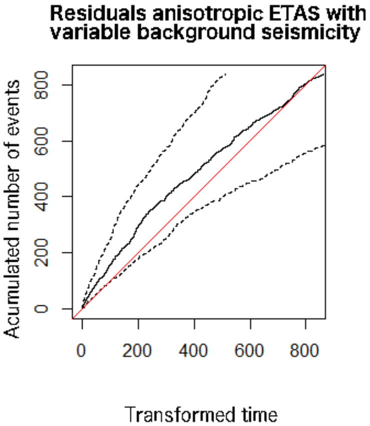
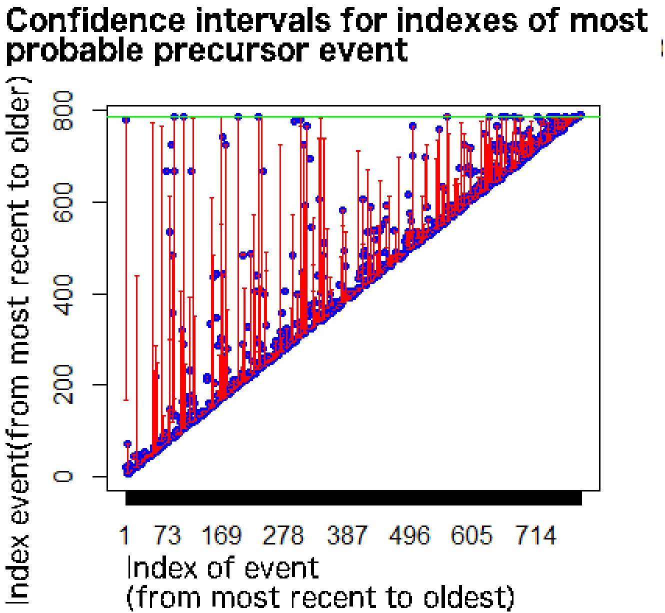


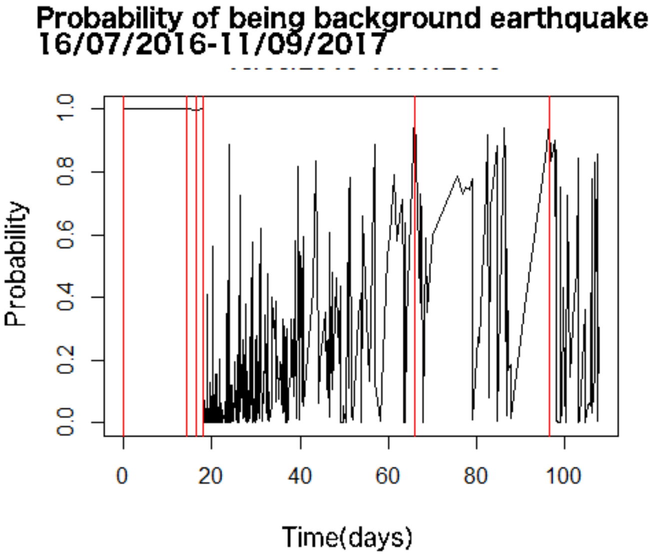
Figure: Residuals for spatio temporal anisotropic ETAS model and variable background seismicity

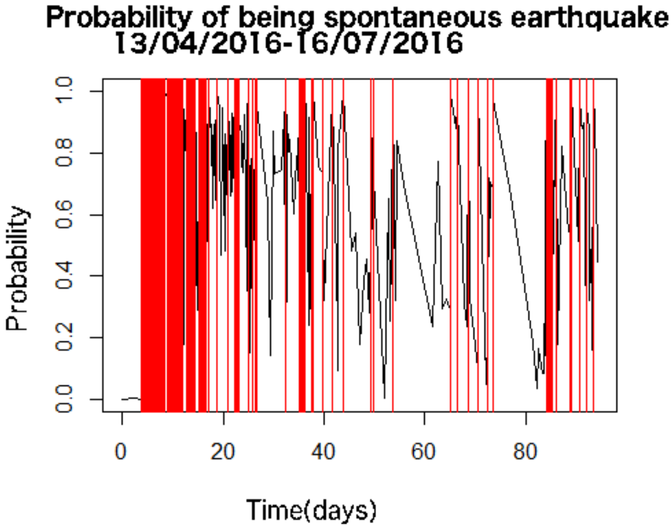
Results

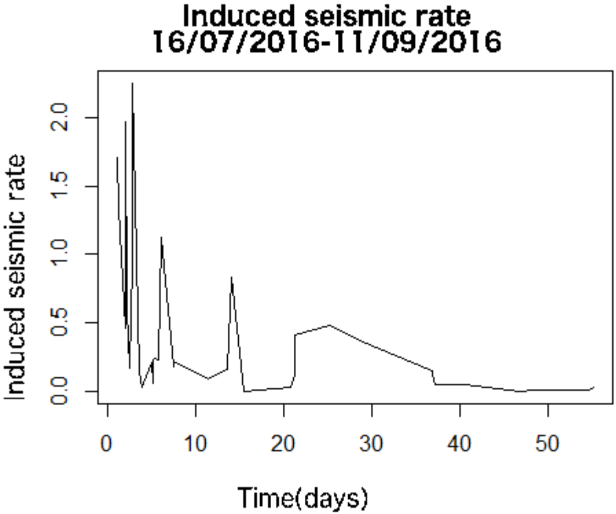
Parent earthquakes

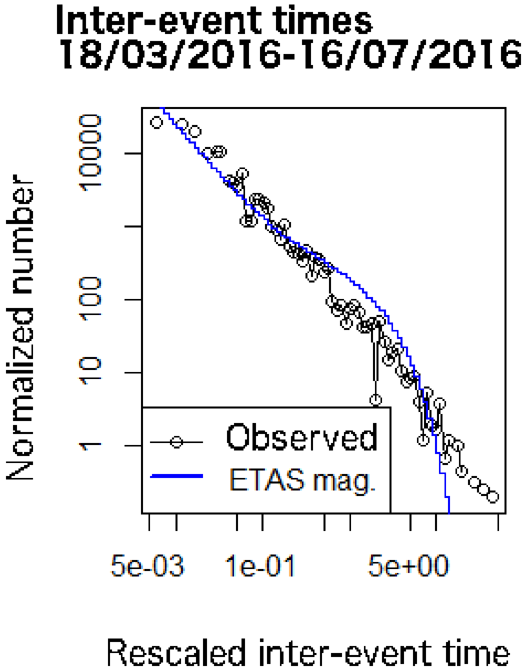
95% credibility intervals for the index of the most probable parent earthquake with 1000 draws from the posterior distributions.
The green line is the big earthquake on April 16 and blue points are the medians of the credibility intervals using the anisotropic spatio temporal model with variable background seismicity.

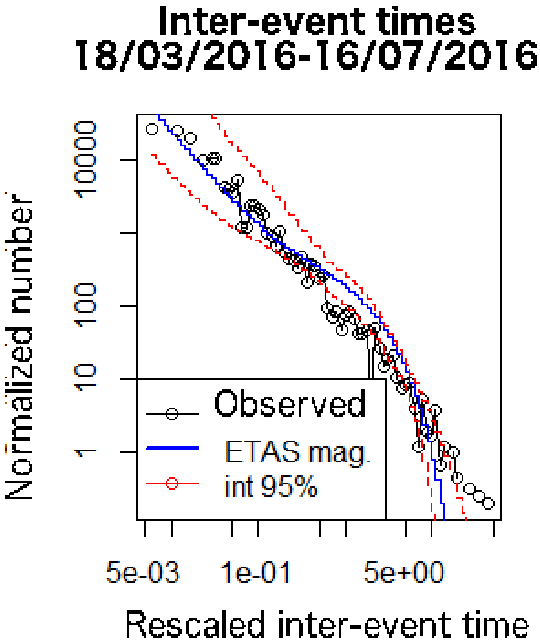

Confidence intervals for indexes of most probable precursor event
Index event(from most recent to older)
0
200
400
600
800
1
73
169
278
387
496
605
714
Index of event
(from most recent to oldest)


Probability of being background earthquake
16/07/2016-11/09/2017
Probability
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100
Time(days)


Probability of being spontaneous earthquake
13/04/2016-16/07/2016
Probability
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
Time(days)


Induced seismic rate
16/07/2016-11/09/2016
Induced seismic rate
0.0
0.5
1.0
1.5
2.0
0
10
20
30
40
50
Time(days)


Inter-event times
18/03/2016-16/07/2016
Normalized number
10000
100
10
1
Observed
ETAS mag.
5e-03
1e-01
5e+00
Rescaled inter-event time



Inter-event times
18/03/2016-16/07/2016
Normalized number
10000
100
10
1
5e-03
1e-01
5e+00
Rescaled inter-event time
Observed
ETAS mag.
int 95%

Thanks