Hidden Markov Models with Stan

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HMMs in Stan? Absolutely!

Posted by Bob Carpenter on 7 February 2017, 2:20 pm

I was having a conversation with Andrew that went like this yesterday:

Andrew:

> Hey, someone's giving a talk today on HMMs (that someone was Yang Chen, who was giving a talk based on her *JASA* paper Analyzing single-molecule protein transportation experiments via hierarchical hidden Markov models).
>
> Maybe we should add some specialized discrete modules to Stan so we can fit HMMs. Word on the street is that Stan can't fit models with discrete parameters.

Me:

> Uh, we can already fit HMMs in Stan. There's a section in the manual that explains how (section 9.6, Hidden Markov Models).


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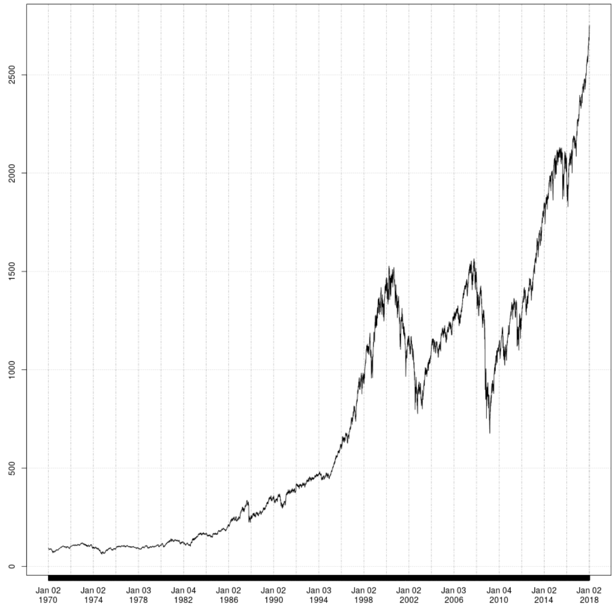
Hidden Markov Models



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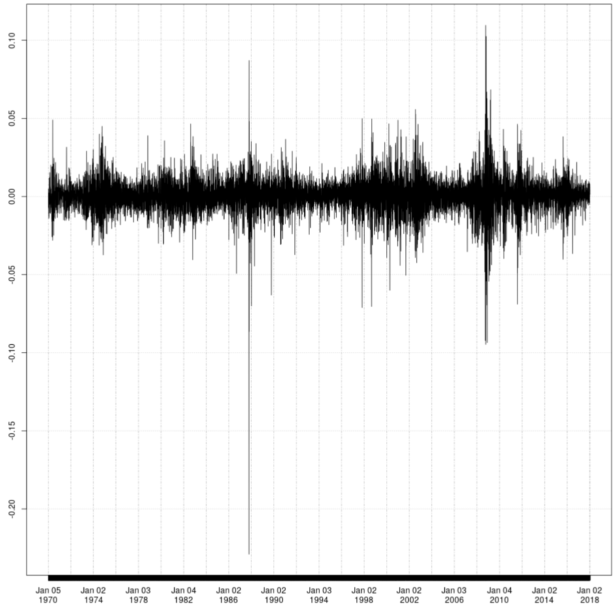

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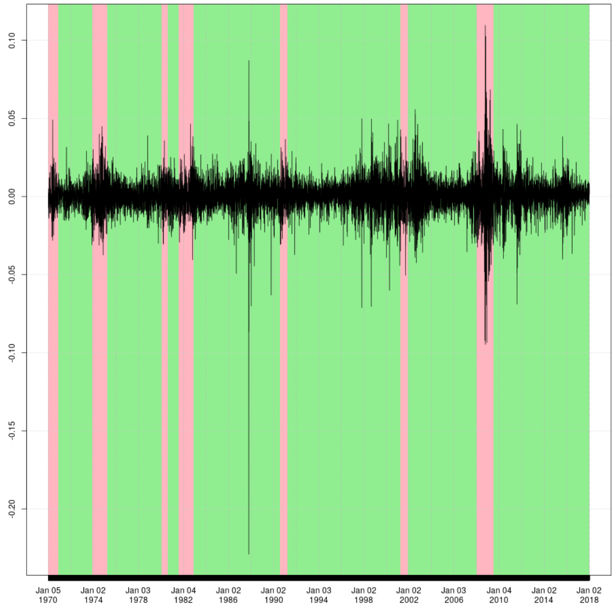
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Complex dynamics can be (mostly) captured by simpler underlying structure

Suppose we have a sequence of observations $\{Y_t\}$ with complex dynamics

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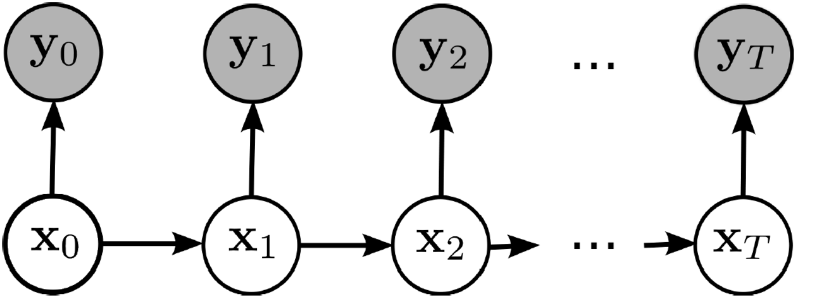
A *Hidden Markov Model* assumes that $\{Y_t\}$ is driven by an unobserved (hidden) sequence $\{X_t\}$ which has Markovian dynamics



Formally, we have:

- Y_t depends only on X_t: $\mathrm{Law}[Y_t|Y_1, \ldots, Y_{t-1}, X_1, \ldots, X_t] = \mathrm{Law}[Y_t|X_t]$
- X_t is Markovian: $\mathrm{Law}[X_t|X_1, \ldots, X_{t-1}] = \mathrm{Law}[X_t|X_{t-1}]$

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Sweet spot of tractability (underlying Markovian dynamics) and expressive power / interpretability

Important: $\{Y_t\}$ is **not** necessarily Markovian

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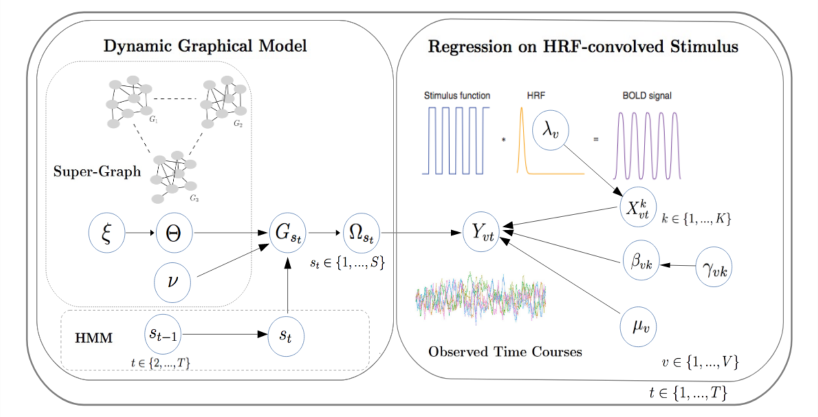
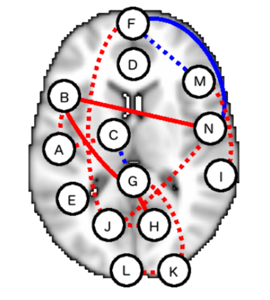
Can embed in larger models without loosing HMM dynamics:

- Direct or indirect observations of $\{X_t\}$
- $\{Y_t\}$ not observed directly


Dynamic Graphical Model
Super-Graph
G_1
G_2
G_3
ξ
Θ
ν
G_{s_t}
Ω_{s_t}
$s_t \in \{1, ..., S\}$
HMM
s_{t-1}
s_t
$t \in \{2, ..., T\}$
Regression on HRF-convolved Stimulus
Stimulus function
HRF
BOLD signal
*
=
λ_v
X^k_{vt}
$k \in \{1, ..., K\}$
Y_{vt}
β_{vk}
γ_{vk}
μ_v
Observed Time Courses
$v \in \{1, ..., V\}$
$t \in \{1, ..., T\}$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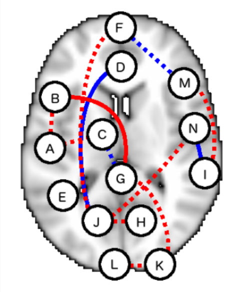
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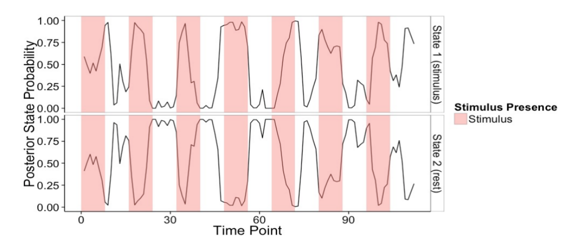
Posterior State Probability
1.00
0.75
0.50
0.25
0.00
State 1 (stimulus)
State 2 (rest)
0
30
60
90
Time Point
Stimulus Presence
Stimulus

Bayesian Inference in HMMs

Bayesian Inference

Bayesian inference is “easy:”

- Formalize priors and likelihood
- `Stan` happens
- Posterior inference

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Difficulty is in the likelihood:

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In general, essentially intractable; two commonly-used special cases:

- Everything is linear and Gaussian $\implies$ state-space models, Kalman Filtering
- $\mathcal{X}$ is finite – integral becomes a sum


A LONG TIME AGO IN A GALAXY FAR FAR AWAY...
STAR WARS
A NEW HOPE
TWENTIETH CENTURY FOX LUCASFILM LTD. STAR WARS EPISODE IV A NEW HOPE
MARK HAMILL HARRISON FORD CARRIE FISHER
PETER CUSHING ALEC GUINNESS
GEORGE LUCAS GARY KURTZ JOHN WILLIAMS
DOLBY DIGITAL
THX
www.STAR WARS.com

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Key idea: Markov property allows us to only consider previous state

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Dynamic programming algorithm which keeps computations tractable – linear instead of exponential in T

Key idea: Markov property allows us to only consider previous state

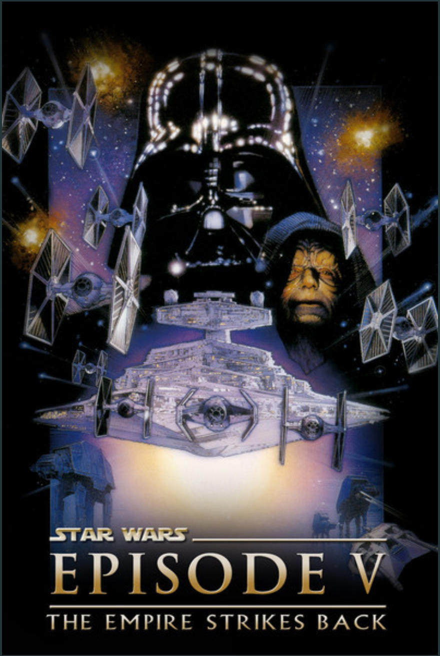
For each state $1, \ldots, K$, we sum probability of coming from each of $1, \ldots, K$ states at previous observation: $\mathcal{O}(K \times K)$

"Roll forward" through all T observations to get $\mathcal{O}(TK^2)$

Likelihood falls out of normalizability requirements

Summary of the hidden quantities and their corresponding inference algorithm.

Name	Hidden Quantity	Availability at	Algorithm	Complexity
Filtering	$p(z_t \mid \mathbf{y}_{1:t})$	t (online)	Forward	$O(K^2T)$ $O(KT)$ if left-to-right
Smoothing	$p(z_t \mid \mathbf{y}_{1:T})$	T (offline)	Forward-backward	$O(K^2T)$ $O(KT)$ if left-to-right
Fixed lag smoothing	$p(z_{t-\ell} \mid \mathbf{y}_{1:t}), \ell \geq 1$	$t + \ell$ (lagged)	Forward-backward	$O(K^2T)$ $O(KT)$ if left-to-right
State prediction	$p(z_{t+h} \mid \mathbf{y}_{1:t}), h \geq 1$	t		
Observation prediction	$p(y_{t+h} \mid \mathbf{y}_{1:t}), h \geq 1$	t		
MAP Estimation	$\text{argmax}_{\mathbf{z}_{1:T}}\, p(\mathbf{z}_{1:T} \mid \mathbf{y}_{1:T})$	T	Viterbi decoding	$O(K^2T)$
Log likelihood	$p(\mathbf{y}_{1:T})$	T	Forward	$O(K^2T)$ $O(KT)$ if left-to-right


STAR WARS
EPISODE V
THE EMPIRE STRIKES BACK

Posterior distribution of HMMs can be tricky to sample from:

The Posterior Strikes Back: Multimodality and Identifiability

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- Identifiability:
  - Strong identifiability (symmetry breaking) – use the `ordered` or `positive_ordered` type
  - Weak identifiability – use a strong prior and good initialization

The Posterior Strikes Back: Multimodality and Identifiability

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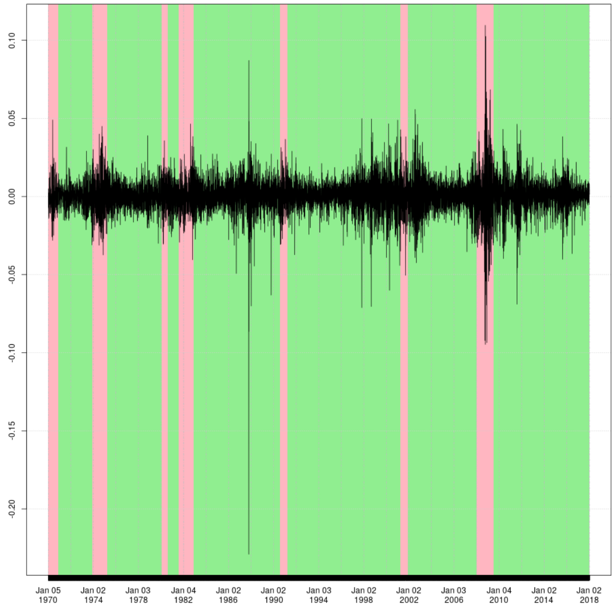
- Identifiability:
    - Strong identifiability (symmetry breaking) – use the `ordered` or `positive_ordered` type
    - Weak identifiability – use a strong prior and good initialization
- Multimodality:
    - Use strong priors or the semi-supervised strategy discussed in the Stan manual

A Regime Switching GARCH Model



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$$Y_t | X_t = k \sim \text{GARCH}(\alpha_k, \beta_k) \tag{GARCH}$$

$$X_t | X_{t-1} = k \sim \text{Categorical}(\mathbf{p}_k) \tag{HMM}$$

Stan Implementation

luisdamiano/stancon2018/stan/hmm_garch.stan

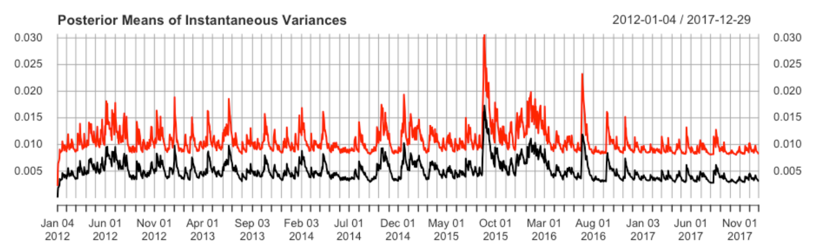
```
transformed parameters {
...

  // GARCH Component
  // ---------------

  // Initialize at unconditional variances
  sigma_t[1, 1] = alpha0[1] / (1 - alpha1[1] - beta1[1]); // Low-vol
  sigma_t[1, 2] = alpha0[2] / (1 - alpha1[2] - beta1[2]); // High-vol

  // GARCH dynamics rolling forward
  for(t in 2:T){
    for(i in 1:2){
      sigma_t[t, i] = sqrt(alpha0[i] +
                           alpha1[i] * pow(y[t-1], 2) +
                           beta1[i] * pow(sigma_t[t-1, i], 2));
    }
  }
}
```

Stan Implementation

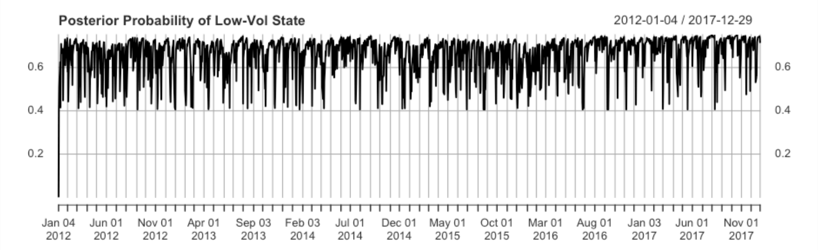
luisdamiano/stancon2018/stan/hmm_garch.stan

```
transformed parameters {
...
  // HMM Component
  // -------------
  // Calculate log p(state at t = j | history up to t) recursively
  // Markov property allows us to do one-step updates

  real accumulator[2];

  // Assume initial equal distribution among two states
  // Better model would be to weight by HMM stationary distribution
  log_alpha[1, 1] = log(0.5) + normal_lpdf(y[1] | 0, sigma_t[1, 1]);
  log_alpha[1, 2] = log(0.5) + normal_lpdf(y[1] | 0, sigma_t[1, 2]);

  for(t in 2:T){
    for(j in 1:2) { // Current state
      for(i in 1:2) { // Previous state
        accumulator[i] = log_alpha[t-1, i] + // Probability from previous obs
                         log(A[i, j]) + // Transition probability
                         // (Local) likelihood / evidence for given state
                         normal_lpdf(y[t] | 0, sigma_t[t-1, i]);
      }
      log_alpha[t, j] = log_sum_exp(accumulator);
    }
```

MSGARCH: Results



MSGARCH: Results



"Gain" from using the MLE instead of posterior inference

	Stocks			Indices			Exchange rates		
	QL 1%	QL 5%	wCRPS	QL 1%	QL 5%	wCRPS	QL 1%	QL 5%	wCRPS
Panel A: Markov–switching GARCH models									
GARCH $\mathcal{N}$	-3.65	-3.26	-2.24	-0.33	-0.58	-0.25	-1.33	0.99	-2.02
GARCH sk$\mathcal{N}$	-3.58	-2.93	-0.60	-1.56	-2.33	-1.04	-0.82	-1.24	-1.04
GARCH $\mathcal{S}$	-2.20	-5.78	-5.55	0.77	-0.17	-0.85	-0.78	0.29	0.35
GARCH sk$\mathcal{S}$	-5.04	-6.88	-7.04	1.13	-0.52	-0.58	-1.54	-1.64	-2.98
GJR $\mathcal{N}$	-1.91	-2.66	-3.22	-1.21	-2.95	-2.08	-1.09	-1.38	-3.61
GJR sk$\mathcal{N}$	-1.83	-3.12	-2.06	-1.11	-0.84	-1.40	0.06	-0.32	-1.17
GJR $\mathcal{S}$	-1.07	-3.11	-4.48	-1.29	-1.56	-4.11	-1.75	-2.40	-4.19
GJR sk$\mathcal{S}$	-3.10	-3.90	-5.28	-2.95	-2.02	-3.48	-1.59	-0.38	-2.50

Acknowledgements

10.6. Hidden Markov Models

A hidden Markov model (HMM) generates a sequence of T output variables y_t conditioned on a parallel sequence of latent categorical state variables $z_t \in \{1, \ldots, K\}$. These “hidden” state variables are assumed to form a Markov chain so that z_t is conditionally independent of other variables given z_{t-1}. This Markov chain is parameterized by a transition matrix θ where θ_k is a K-simplex for $k \in \{1, \ldots, K\}$. The probability of transitioning to state z_t from state z_{t-1} is

$$z_t \sim \mathsf{Categorical}(\theta_{z[t-1]}).$$

Thank You

Questions?