INF5620 - Mandatory Exercise 1 - Skydiving

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a

$$F_{tot} = ma$$
$$F_{tot} = F_g + F_d^q + F_b$$

We add an extra term F_s to be used for manipulating the problem to suit a spesific solution.

$$ma = -mg - \frac{1}{2}C_D\rho A|v|v + \rho gV + F_s$$
$$m\frac{dv}{dt} = -mg - \frac{1}{2}C_D\rho A|v|v + \rho gV + F_s$$
$$\frac{dv}{dt} = -g - \frac{C_D\rho A}{2m} + \frac{\rho gV}{m} + \frac{1}{m}F_s$$

To make the expression simpler we define the constants

$$a = \frac{C_D\rho A}{2m}$$
$$b = g\left(\frac{\rho V}{m} - 1\right)$$
$$c = \frac{1}{m}$$

Using these constants we get the expression

$$v'(t) = -a|v|v + b + cF_s$$

for the ODE

b

We now need to calculate the numerical scheme for this ODE. We use the Crank-Nicolson scheme. And a, b and c are constants.

$$\frac{v^{n+1} - v^n}{\Delta t} = -a(|v|v)^{n+\frac{1}{2}} + b + cF_s^{n+\frac{1}{2}}$$

We use a geometric average to calculate $(|v|v)^{n+\frac{1}{2}}$

$$\begin{aligned} v^{n+1} - v^n &= -a\Delta t|v^n|v^{n+1} + b\Delta t + c\Delta t F_s^{n+\frac{1}{2}} \\ v^{n+1} + a\Delta t|v^n|v^{n+1} &= v^n + b\Delta t + c\Delta t F_s^{n+\frac{1}{2}} \\ v^{n+1}(1 + a\Delta t|v^n|) &= v^n + b\Delta t + c\Delta t F_s^{n+\frac{1}{2}} \\ v^{n+1} &= \frac{v^n + b\Delta t + c\Delta t F_s^{n+\frac{1}{2}}}{1 + a\Delta t|v^n|} \end{aligned}$$

c

We now assume that the ODE has a solution $\alpha t + \beta$ and we need to prove that this does not fit the discrete equation.
We can show this using a subset of linear equations where $\beta = 0$. We use the discrete equation

$$\frac{v^{n+1} - v^n}{\Delta t} = -a(|v|v)^{n+\frac{1}{2}} + b$$

If a linear equation αt is the solution to this equation then both sides will be constants. The left hand side is the discrete definition of the first order derivative and as such is obviously constant and equal to α.

$$\frac{v^{n+1} - v^n}{\Delta t} = \frac{\alpha t^{n+1} - \alpha t^n}{\Delta t} = \frac{\alpha((n+1)\Delta t - n\Delta t)}{\Delta t} = \frac{\alpha \Delta t(n + 1 - n)}{\Delta t} = \alpha$$

Now we need to see that if the same holds for the right hand side of the equation.

$$\begin{aligned} -a|v^n|v^{n+1} + b &= -a|\alpha t^n|\alpha t^{n+1} + b \\ &= -a|\alpha n\Delta t|\alpha(n + 1)\Delta t + b \end{aligned}$$

We know that both n and Δt are positive

$$= -an(n + 1)\Delta t^2 \alpha|\alpha| + b$$

So the left hand side of the equation changes with n and is therefore not constant so a linear equation is not a solution to the discrete equation.

To make a linear equation the solution to our problem we need to manipulate F_s. We still use the linear equation αt as β is the initial condition and in our problem this is 0.

$$
\begin{aligned}
-a|v^n|v^{n+1} + b + cF_s^{n+\frac{1}{2}} &= \alpha \\
-a|\alpha t^n|\alpha t^{n+1} + b + cF_s^{n+\frac{1}{2}} &= \alpha \\
cF_s^{n+\frac{1}{2}} &= \alpha + a|\alpha n\Delta t|\alpha(n+1)\Delta t - b \\
cF_s^{n+\frac{1}{2}} &= \alpha + an(n+1)\Delta t^2|\alpha|\alpha - b \\
F_s^{n+\frac{1}{2}} &= \frac{\alpha + an(n+1)\Delta t^2|\alpha|\alpha - b}{c}
\end{aligned}
$$

d

For the convergence test we use the assume the solution to the problem is $v(t) = \sin(\alpha t)$ and we now have to fit F_s to this solution

$$
\begin{aligned}
-a|v^n|v^{n+1} + b + cF_s^{n+\frac{1}{2}} &= \frac{v^{n+1} - v^n}{\Delta t} \\
-a|\sin(\alpha t^n)|\sin(\alpha t^{n+1}) + b + cF_s^{n+\frac{1}{2}} &= \cos(\alpha t^n) \\
F_s^{n+\frac{1}{2}} &= \frac{\cos(\alpha n\Delta t) + a|\sin(\alpha n\Delta t)|\sin(\alpha(n+1)\Delta t) - b}{c}
\end{aligned}
$$

We see that the convergence test fails, but are as of yet still unsure where in the code the bug is located.