INF5620 - Mandatory Exercise 2

Ingrid Elgsaas-Vada & Simen Tennøe
ingrielg & simenten
ingrielg@math.uio.no simetenn@gmail.com

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In this project we look at the standard two-dimensional, standard, linear wave equation, with damping:

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t)$$

We have the boundary condition

$$\frac{\partial u}{\partial n} = 0$$

and the inital conditions are

$$u(x,y,0) = I(x,y)$$
$$u_t(x,y,0) = V(x,y)$$

Discretization

We start by finding the general scheme for computing $u_{i,j}^{n+1}$ at the interior spatial mesh points.

$$
\begin{aligned}
\frac{u_{i,j}^{n+1}-2u_{i,j}^{n}+u_{i,j}^{n-1}}{\Delta t^2}+b\frac{u_{i,j}^{n+1}-u_{i,j}^{n-1}}{2\Delta t} &= \frac{1}{\Delta x}\left(q_{i+\frac{1}{2},j}\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}}{\Delta x}\right)-q_{i-\frac{1}{2},j}\left(\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}\right)\right)\\
&+\frac{1}{\Delta y}\left(q_{i,j+\frac{1}{2}}\left(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}}{\Delta y}\right)-q_{i,j-\frac{1}{2}}\left(\frac{u_{i,j}^{n}-u_{i,j-1}^{n}}{\Delta y}\right)\right)+f_{i,j}^{n}\\
\frac{u_{i,j}^{n+1}-2u_{i,j}^{n}+u_{i,j}^{n-1}}{\Delta t^2} &= -b\frac{u_{i,j}^{n+1}-u_{i,j}^{n-1}}{2\Delta t}\\
&+\frac{1}{\Delta x^2}\left(\frac{1}{2}\left(q_{i+1,j}-q_{i,j}\right)\left(u_{i+1,j}^{n}-u_{i,j}^{n}\right)-\frac{1}{2}\left(q_{i,j}-q_{i-1,j}\right)\left(u_{i,j}^{n}-u_{i-1,j}^{n}\right)\right)\\
&+\frac{1}{\Delta y^2}\left(\frac{1}{2}\left(q_{i,j+1}-q_{i,j}\right)\left(u_{i,j+1}^{n}-u_{i,j}^{n}\right)-\frac{1}{2}\left(q_{i,j}-q_{i,j-1}\right)\left(u_{i,j}^{n}-u_{i,j-1}^{n}\right)\right)+f_{i,j}^{n}\\
u_{i,j}^{n+1}-2u_{i,j}^{n}+u_{i,j}^{n-1} &= \frac{-b\Delta t}{2}(u_{i,j}^{n+1}-u_{i,j}^{n-1})\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u_{i+1,j}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u_{i,j}^{n}-u_{i-1,j}^{n}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u_{i,j+1}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u_{i,j}^{n}-u_{i,j-1}^{n}\right)\right)+\Delta t^2 f_{i,j}^{n}\\
u_{i,j}^{n+1}+\frac{b\Delta t}{2}u_{i,j}^{n+1} &= 2u_{i,j}^{n}-u_{i,j}^{n-1}+\frac{b\Delta t}{2}u_{i,j}^{n-1}\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u_{i+1,j}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u_{i,j}^{n}-u_{i-1,j}^{n}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u_{i,j+1}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u_{i,j}^{n}-u_{i,j-1}^{n}\right)\right)+\Delta t^2 f_{i,j}^{n}\\
\left(1+\frac{b\Delta t}{2}\right)u_{i,j}^{n+1} &= 2u_{i,j}^{n}-\left(1-\frac{b\Delta t}{2}\right)u_{i,j}^{n-1}\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u_{i+1,j}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u_{i,j}^{n}-u_{i-1,j}^{n}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u_{i,j+1}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u_{i,j}^{n}-u_{i,j-1}^{n}\right)\right)+\Delta t^2 f_{i,j}^{n}\\
u_{i,j}^{n+1} &= \frac{1}{1+\frac{b\Delta t}{2}}(\,2u_{i,j}^{n}-\left(1-\frac{b\Delta t}{2}\right)u_{i,j}^{n-1}\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u_{i+1,j}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u_{i,j}^{n}-u_{i-1,j}^{n}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u_{i,j+1}^{n}-u_{i,j}^{n}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u_{i,j}^{n}-u_{i,j-1}^{n}\right)\right)+\Delta t^2 f_{i,j}^{n}\,)
\end{aligned}
$$

Before we can use the general scheme we need to use the initial conditions to calculate values for two steps so we can start the calculation. We calculate values for the ghost cells $u^{-1}_{i,j}$ in addition to the values for $u^0_{i,j}$ and then we can use the general scheme. We get

$$u^0_{i,j} = I(x_i, y_j)$$

directly from the first initial condition. To get the values to $n = -1$ requires some calculation

$$\begin{aligned} u_t(x,y,0) &= V(x,y) \\ \frac{u^0_{i,j} - u^{-1}_{i,j}}{\Delta t} &= V_{i,j} \\ u^{-1}_{i,j} &= u^0_{i,j} - \Delta t V_{i,j} \end{aligned}$$

For the boundaries we use the boundary condition

$$\frac{\partial u}{\partial n} = 0$$

For the end point N_x we can use this as

$$\begin{aligned} \frac{u^n_{N_x+1,j} - u^n_{N_x-1,j}}{2\Delta x} &= 0 \\ u^n_{N_x+1,j} &= u^n_{N_x-1,j} \end{aligned}$$

The same argument holds for all the other three boundaries. To calculate the values for the boundaries we can then just input these equalities in the general scheme.

Truncation Error

We start by finding the truncation error when q is constant. We write the scheme in compact notation

$$\left[D_t D_t u + b D_{2t} u = q D_x D_x u + q D_y D_y u + f\right]^n_{i,j}$$

If we now input the exact solution u_e we now know from the lecture foils that

$$\left[D_t D_t u_e + b D_{2t} u_e = q D_x D_x u_e + q D_y D_y u_e + f + R\right]^n_{i,j}$$

where the truncation error R^n is given by

$$R^n_{i,j} = \frac{1}{12} u_{e,tttt}(x_i, y_j, t_n)\Delta t^2 + \frac{1}{6} u_{e,ttt}(x_i, y_j, t_n)\Delta t^2 + \frac{1}{12} u_{e,xxxx}(x_i, y_j, t_n)\Delta x^2 + \frac{1}{12} u_{e,yyyy}(x_i, y_j, t_n)\Delta y^2 + \mathcal{O}(\Delta t^4, \Delta x^4, \Delta y^4)$$

We see from this that the error for the scheme goes as $\Delta t^2, \Delta x^2, \Delta y^2$

We now look at the case where $q(x,y)$ is a function. This gives us the scheme written in compact notation:

$$\left[D_t D_t u + b D_{2t} u = D_x \bar{q}^x D_x u + D_y \bar{q}^x D_y u + f\right]_{i,j}^n$$

We now insert the exact solution

$$\left[D_t D_t u_e + b D_{2t} u_e = D_x \bar{q}^x D_x u_e + D_y \bar{q}^y D_y u_e + f\right]_{i,j}^n$$

From the lecture notes we know that the truncation error for

$$[D_x \bar{q}^x D_x u_e]_{i,j}^n = \frac{\partial}{\partial x} q(x_i, y_j) u_{e,x}(x_i, y_j, t_n) + O(\Delta x^2)$$

So we see that the new terms for x and y also have error terms in Δx^2 and Δy^2 so the truncation error is of the second order also when $q(x,y)$ is a function.

Verification: Constant Solution

For a constant solution we must have that q is constant and for the initial conditions we get

$$u(x,y,0) = C$$
$$u_t(x,y,0) = 0$$

When we input $u = C$ into the PDE we get each term becomes zero as the derivative of a constant is zero so we get that any constant is a solution to the PDE as long as $f(x,y) = 0$

Verification: Standing, Undamped Waves

We know from the truncation analysis that the truncation error is given as $O(\Delta t^2, \Delta x^2, \Delta y^2)$ so the error only depends on the discretization parameters.
If we now define

$$\Delta t = C_1 h$$
$$\Delta x = C_2 h$$
$$\Delta y = C_3 h$$

We now get that the error can be described by $O(C_1 h, C_2 h, C_3 h)$ so we now have that the truncation error goes as Ch^2. In the code we have a slight problem, we added a convergence test get that the test goes towards 0 and not 2, as we would expect, and for the same reason we do not get a constant E/h^2.

Verification: Standing, Damped, Waves

$$u_e = (A\cos(\omega t) + B\sin(\omega t))e^{-ct}\cos(k_x x)\cos(k_y y)$$

$$\begin{aligned} u_e(x,y,0) &= (A\cos(0) + B\sin(0))e^{0}\cos(k_x x)\cos(k_y y) \\ &= A\cos(k_x x)\cos(k_y y) \end{aligned}$$

$$u_{e,t}(x,y,t) = ((-A\omega\sin(\omega t) + B\omega\cos(\omega t))e^{-ct} - c(A\cos(\omega t) + B\sin(\omega t))e^{-ct})\cos(k_x x)\cos(k_y y)$$

$$\begin{aligned} u_{e,t}(x,y,0) &= 0 \\ ((-A\omega\sin(0) + B\omega\cos(0))e^{0} - c(A\cos(0) + B\sin(0))e^{0})\cos(k_x x)\cos(k_y y) &= 0 \\ (B\omega - cA)\cos(k_x x)\cos(k_y y) &= 0 \\ B\omega - cA &= 0 \\ B &= \frac{cA}{\omega} \end{aligned}$$

Inputing this into the equation we get

$$u_e = A(\cos(\omega t) + \frac{c}{\omega}\sin(\omega t))e^{-ct}\cos(k_x x)\cos(k_y y)$$

We now calculate the various derivatives so we can input this equation into the PDE

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= Ae^{-ct}((-\omega\sin(\omega t) + c\cos(\omega t)) - c(\cos(\omega t) - \frac{c}{\omega}\sin(\omega t)))\cos(k_x x)\cos(k_y y) \\
&= Ae^{-ct} frac{-}\omega^2 - c^2 \omega sin(\omega t)\cos(k_x x)\cos(k_y y) \\
\frac{\partial^2 u}{\partial t^2} &= Ae^{-ct} frac{-}\omega^2 - c^2 \omega(\omega\cos(\omega t) - c\sin(\omega t))\cos(k_x x)\cos(k_y y) \\
\frac{\partial^2 u}{\partial x^2} &= A(\cos(\omega t) + \frac{c}{\omega}\sin(\omega t))e^{-ct}(-k_x^2)\cos(k_x x)\cos(k_y y) \\
\frac{\partial^2 u}{\partial x^2} &= A(\cos(\omega t) + \frac{c}{\omega}\sin(\omega t))e^{-ct}\cos(k_x x)(-k_y^2)\cos(k_y y)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = q\frac{\partial^2 u}{\partial x^2} + q\frac{\partial^2 u}{\partial y^2}& \\
&Ae^{-ct}\cos(k_x x)\cos(k_y y)\frac{-(\omega^2 - c^2)}{\omega}(\omega\cos(\omega t) - c\sin(\omega t) + b\sin(\omega t)) \\
&= Ae^{-ct}\cos(k_x x)\cos(k_y y)q(-k_x^2 - k_y^2)(\cos(\omega t) + \frac{c}{\omega}\sin(\omega t)) \\
&-(\omega^2 + c^2)\cos(\omega t) - \frac{\omega^2 + c^2}{\omega}(b - c)\sin(\omega t) \\
&= -q(k_x^2 + k_y^2)\cos(\omega t) - q(k_x^2 + k_y^2)\frac{c}{\omega}\sin(\omega t)
\end{aligned}
$$

We now split this PDE into two equations, one containing all the cosine parts and one containing all the sine parts

$$
\begin{aligned}
-(\omega^2 + c^2)\cos(\omega t) &= -q(k_x^2 + k_y^2)\cos(\omega t) \\
\omega^2 + c^2 &= k_x^2 q + k_y^2 q \\
\omega &= \sqrt{k_x^2 q + k_y^2 q - c^2}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\omega^2 + c^2}{\omega}(b - c)\sin(\omega t) &= -q\frac{c}{\omega}(k_x^2 + k_y^2)\sin(\omega t) \\
(\omega^2 + c^2)(b - c) &= qc(k_x^2 + k_y^2) \\
(q(k_x^2 + k_y^2) - c^2 + c^2)(b - c) &= qc(k_x^2 + k_y^2) \\
(b - c)q(k_x^2 + k_y^2) &= cq(k_x^2 + k_y^2) \\
b - c &= c \\
2c &= b \\
c &= \frac{b}{2}
\end{aligned}
$$

Here we get the same problem, the convergence test goes towards 0 instead of the expected 2, so there must be something wrong either in in the scheme or in the convergence test itself. As the results we get from the physical solution seems reasonable, we suspect the error is in the convergence test.

Verification: Manufactured Solution

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= e^{-ct}\cos(k_x x)\cos(k_y y)(-A\omega\sin(\omega t) + B\omega\cos(\omega t) - Ac\cos(\omega t) - Bc\sin(\omega)) \\
&= e^{-ct}\cos(k_x x)\cos(k_y y)(\omega(-A\sin(\omega t) + B\cos(\omega t)) - c(A\cos(\omega t) + B\sin(\omega t))) \\
\frac{\partial^2 u}{\partial t^2} &= e^{-ct}\cos(k_x x)\cos(k_y y)((-\omega^2 + c^2)(A\cos(\omega t) + B\sin(k_y y)) - 2\omega c(-A sin(\omega t) + B\cos(\omega t)) \\
\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) &= e^{-ct}(A\cos(\omega t) + B\sin(\omega t))\cos(k_y y)\frac{\partial}{\partial x}\left(q(-k_x\sin(k_x x))\right) \\
&= e^{-ct}(A\cos(\omega t) + B\sin(\omega t))\cos(k_y y)(\frac{\partial q}{\partial x}(-k_x\sin(k_x x)) - qk_x^2\cos(k_x x)) \\
\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) &= e^{-ct}(A\cos(\omega t) + B\sin(\omega t))\cos(k_x x)(\frac{\partial q}{\partial y}(-k_y\sin(k_y y)) - qk_y^2\cos(k_y y))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f \\
&(c^2 - bc - \omega^2)(A\cos(\omega t) + B\sin(\omega t)) + (b\omega - 2c\omega)(-A\sin(\omega t) + B\cos(\omega t)) \\
&= (A\cos(\omega t) + B\sin(\omega t))(-q(k_x^2 + k_y^2)\cos(k_x x)\cos(k_y y) \\
&- k_x\frac{\partial q}{\partial x}\sin(k_x x)\cos(k_y y) - k_y\frac{\partial q}{\partial y}\sin(k_y y)\cos(k_x x)) + f \\
f &= (c^2 - bc - \omega^2)(A\cos(\omega t) + B\sin(\omega t)) + (b\omega - 2c\omega)(-A\sin(\omega t) + B\cos(\omega t)) \\
&- (A\cos(\omega t) + B\sin(\omega t))(-q(k_x^2 + k_y^2)\cos(k_x x)\cos(k_y y) \\
&- k_x\frac{\partial q}{\partial x}\sin(k_x x)\cos(k_y y) - k_y\frac{\partial q}{\partial y}\sin(k_y y)\cos(k_x x)) \\
f &= (c^2 - bc - \omega^2)(A\cos(\omega t) + B\sin(\omega t)) + (b\omega - 2c\omega)(-A\sin(\omega t) + B\cos(\omega t)) \\
&+ (A\cos(\omega t) + B\sin(\omega t))(q(k_x^2 + k_y^2)\cos(k_x x)\cos(k_y y) \\
&+ k_x\frac{\partial q}{\partial x}\sin(k_x x)\cos(k_y y) + k_y\frac{\partial q}{\partial y}\sin(k_y y)\cos(k_x x)) \\
f &= (c^2 - bc - \omega^2 + q(k_x^2 + k_y^2)\cos(k_x x)\cos(k_y y) + k_x\frac{\partial q}{\partial x}\sin(k_x x)\cos(k_y y) \\
&+ k_y\frac{\partial q}{\partial y}\sin(k_y y)\cos(k_x x))(A\cos(\omega t) + B\sin(\omega t)) \\
&+ (b\omega - 2c\omega)(-A\sin(\omega t) + B\cos(\omega t))
\end{aligned}
$$

We now have the source term $f(x,y,t)$. We now need to find the corresponding I and V. To do this we use the original solution u_e as the f is only involved in the calculation of the PDE and not in finding the initial conditions.

$$\begin{aligned}
u(x,y,0) &= u_e(x,y,0) \\
&= A\cos(k_x x)\cos(k_y y) \\
u_t(x,y,0) &= u_{e,t}(x,y,0) \\
&= (\omega(-A\sin(0) + B\cos(0)) - c(A\cos(0) + B\sin(0)))e^0\cos(k_x x)\cos(k_y y) \\
&= (\omega B - cA)\cos(k_x x)\cos(k_y y)
\end{aligned}$$

In this case we have a slightly larger problem with the code, as we get an alternating source of error. But as of yet we haven't been able to figure out why.

Investigate a physical problem

We performed a series of calculations and generated several plots for different initial height of the wave, different shape of the bottom and different resolutions. As can be seen we get the behaviour we expect. The closer we are to the hill, the larger the effect of the bottom shape. Worse resolution gives more numerical noise, we also see that the sharp edges od the third bottom shape give large errors when close to the hill. One can take a look at the movies, they are very descriptive.

Visualization

The additional task we elected to do was to create a fancy 3D visualization using Matplotlib.