Matrix Analysis Série 12: Simon J. Thür 325386 02.06.2022

1) 1:

$$\det \begin{pmatrix} -\lambda & a_1 & 0 & 0 \\ 0 & -\lambda & a_2 & 0 \\ 0 & 0 & -\lambda & a_3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = (-\lambda)^4 = 0$$

$\hookrightarrow \lambda = 0$

Eigenvector $Ax = \lambda x = 0$

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A is diagonizable $(A = PDP^{-1})$ if P is made up of orthogonal Eigenvectors which is <u>not</u> possible for A, as it only has 1 Eigenvector.

2: SVD:

$$AA^T = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{pmatrix} = \begin{pmatrix} a_1^2 & & & 0 \\ & a_2^2 & & \\ & & a_3^2 & \\ 0 & & & 0 \end{pmatrix}$$

$\hookrightarrow$ Eigenvalues AA^T: $-(a_1^2 - \lambda)(a_2^2 - \lambda)(a_3^2 - \lambda)\lambda = 0$

Eigenvalues		Singular values
$\lambda_1 = a_1^2$		$\sigma_1 = a_1$
$\lambda_2 = a_2^2$	$\Rightarrow$	$\sigma_2 = a_2$
$\lambda_3 = a_3^2$		$\sigma_3 = a_3$
$\lambda_4 = 0$		$\sigma_4 = 0$

assuming $a_1 \geq a_2 \geq a_3 \geq 0$

$$\Sigma = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & 0 \end{pmatrix}$$

Eigenvectors of AA^T: $(AA^T - \lambda_i I)e_i = 0$

case λ_1: $$\begin{pmatrix} 0 & & & \\ & a_2^2-a_1^2 & & \\ & & a_3^2-a_1^2 & \\ & & & -a_1^2 \end{pmatrix} \begin{pmatrix} e_1^1 \\ e_1^2 \\ e_1^3 \\ e_1^4 \end{pmatrix} = 0$$

$$\Rightarrow e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It can trivially be seen that $e_i = \begin{pmatrix} \delta(1-i) \\ \delta(2-i) \\ \delta(3-i) \\ \delta(4-i) \end{pmatrix}$

Which gives us $V = \begin{pmatrix} | & | & | & | \\ e_1 & e_2 & e_3 & e_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = I$

$$A^TA = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & a_1^2 & & \\ & & a_2^2 & \\ & & & a_3^2 \end{pmatrix}$$

We already know the eigenvalues
$\lambda_1 = a_1^2$
$\lambda_2 = a_2^2$
$\lambda_3 = a_3^2$
$\lambda_4 = 0$

So we can find the eigenvectors: $(A^TA - \lambda_i I)e_i = 0$

$$\Rightarrow e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad e_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us $V = \begin{pmatrix} | & | & | & | \\ e_1 & e_2 & e_3 & e_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

And so $A = U \Sigma V^T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

3: $\det \begin{pmatrix} -\lambda & a_1 & 0 & 0 \\ 0 & -\lambda & a_2 & 0 \\ 0 & 0 & -\lambda & a_3 \\ \varepsilon & 0 & 0 & -\lambda \end{pmatrix} = -\lambda(-\lambda)^3 - a_1(a_2 a_3 \varepsilon)$

$\Rightarrow (-\lambda)^4 - a_1 a_2 a_3 \varepsilon = 0$

$\Rightarrow \lambda^4 = a_1 a_2 a_3 \varepsilon$

Eigenvalues of A_ε $\begin{cases} \lambda_1 = \sqrt[4]{a_1 a_2 a_3 \varepsilon} \\ \lambda_2 = -\sqrt[4]{a_1 a_2 a_3 \varepsilon} \\ \lambda_3 = i\sqrt[4]{a_1 a_2 a_3 \varepsilon} \\ \lambda_4 = -i\sqrt[4]{a_1 a_2 a_3 \varepsilon} \end{cases}$

4: $A_\varepsilon A_\varepsilon^T = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \\ \varepsilon & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \varepsilon \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{pmatrix} = \begin{pmatrix} a_1^2 & & & 0 \\ & a_2^2 & & \\ & & a_3^2 & \\ 0 & & & \varepsilon^2 \end{pmatrix}$

Characteristic polinomial of $A_\varepsilon A_\varepsilon^T$: $(a_1^2-\lambda)(a_2^2-\lambda)(a_3^2-\lambda)(\varepsilon^2-\lambda)$

$\Rightarrow$

Eigenvalues	Singular values
$\lambda_1 = a_1^2$	$\sigma_1 = a_1$
$\lambda_2 = a_2^2$	$\sigma_2 = a_2$
$\lambda_3 = a_3^2$	$\sigma_3 = a_3$
$\lambda_4 = \varepsilon^2$	$\sigma_4 = \varepsilon$

5: $\varepsilon = 10^{-p}$

Eigenvalue:	Singular value:
$\sqrt[4]{\varepsilon} = 10^{-p/4}$	$\varepsilon = 10^{-p}$

Which shows that the singular value error is substantially smaller than that of the eigenvalue. It also shows that the singular value grows linearly with the error, whereas the eigenvalue grows exponentially.

6: Eigenvalue: $\lambda = \sqrt[n]{\varepsilon}$ Singular value $\sigma = \varepsilon$

Conclusion: The singular value error does not depend on the dimension.
The eigenvalue error gets worse with increasing dimension of the matrix.

2) 1: BB^T: The values of the diagonal represent the sum of all connections of that node
This is the same as the diagonal of $D-A$.
The value of element $i,j : i \neq j$ is -1 if there is a connection between node i and node j. and 0 otherwise.
This is exactly what the elements of $-A$ are at $i,j : i\neq j$.
and because $D_{ij} = 0$ for $i \neq j$, $D-A = BB^T$

Example:

$$BB^T = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

equal

$$D-A = \begin{pmatrix} 2 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

2:
$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 3/2 & -3/2 & 0 \\ 0 & -3/2 & 5/2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix is not full rank $\Rightarrow$ there exists an eigenvalue $\lambda = 0$.

$L = BB^T$, which means that B has eigenvalue $\sqrt{\lambda} = 0 \Rightarrow \sigma = \sqrt{0} = 0$

As a general case, one can see that the matrix is not full rank because the sum of each row is equal to zero.

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{e} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\hookrightarrow$ Eigenvector $e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

3: Eigenvector $e_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$\langle e_1, e_2 \rangle = 0 \Rightarrow e_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ etc.

$$A_c = \begin{pmatrix} n-1 & & (-1) \\ & n-1 & \\ (-1) & & \ddots \end{pmatrix} \Rightarrow A_c e_2 = \begin{pmatrix} n-1+1 \\ -1-n+1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = n e_2$$

$\hookrightarrow \lambda_{2,3,\dots,n} = n$ (with multiplicity $n-1$)

4: $x^T L x = x^T B B^T x = (B^T x)^T B^T x$

Because B^T has an equal number of $+1$ and -1 per row, at positions i,j per connection of the node k, $(i,j \in E_k)$

$(B^T x)^T B^T x = \langle B^T x, B^T x \rangle$ where the k^{th} element of $B^T x = x_i - x_j$, where i,j are the k^{th} connection of $E = \{ \underbrace{\{u,v\}}_{\text{connection } 1}, \underbrace{\{u,w\}}_{\text{connection } 2}, \dots \}$

$$\Rightarrow \langle B^T x, B^T x \rangle = \sum_{i,j \in E} (x_i - x_j)^2 = x^T L x$$

5: Let $x \in \mathbb{R}^n$ and $\|x\|_{custom}$ be defined as $\sum_{i=1}^{n} |x_i|$

(the custom norm is the sum of the absolute value of each element of x)

$cut(S) = \| B^T \mathbb{1}_S \|_{custom}$

Because the i-th element of $B^T \mathbb{1}_S$ is equal to the number of connections coming in, minus the connections going to nodes in S. This corresponds to the number of connection of the node to the outside of S, and by summing over all the values in $B^T \mathbb{1}_S$ we get the total number of connections to the outside of S.

because all elements of $B^T \mathbb{1}_S$ are 0 or ± 1 we can define $cut(S) = \| B^T \mathbb{1}_S \|_2^2$

which is equal to $\langle B^T \mathbb{1}_S, B^T \mathbb{1}_S \rangle = (B^T \mathbb{1})^T B^T \mathbb{1}_S = \mathbb{1}_S^T B B^T \mathbb{1}_S$

$$= \mathbb{1}_S^T L \mathbb{1}_S = cut(S)$$

6: $cut(S) = 0$ implies that the subset S is completly isolated from the remaining graph. If $S \neq V$, then the graph is <u>not</u> <u>conneted</u>

7: $x^T L x = x^T B B^T x = (B^T x)^T B^T x = \| B^T x \|_2^2 \geq 0$

8: (Assuming we're discussing L)

Each connected component has its own block matrix B_1, where $\lambda=0$ has multiplicity greater or equal to 1.

There being k connected components

$$\det(L) = \prod_{i=1}^{k} \det(B_i)$$

which implies that the multiplicity of $\lambda=0$ must be at <u>least k</u>.

9: It <u>cannot</u> have a multiplicity of $\lambda=0$ greater than 1.

let u be the normalized eigenvector s.t. $Lu=0$, $u=\frac{1}{n}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$.

For $\lambda=0$ to have multiplicity greater than 1, there must exist another eigenvector x such that $Lx=0$ and $x\neq ku$.

We know that for u

$$u^T \underbrace{Lu}_{=0} = \sum_{i,j\in E} \underbrace{(u_i-u_j)^2}_{=0} = 0$$

By extension, if $Lx=0$, then

$$x^T \underbrace{Lx}_{=0} = \sum_{i,j\in E} (x_i-x_j)^2 = 0$$

But since $x\neq ku$, there exists at least one relation $x_f\neq x_k$

$\Rightarrow (x_f-x_k)^2\neq 0$. This mease that $(f,k)\notin E$.

Knowing that the graph is connected, any connection $k,i\in E\neq\{\emptyset\}$

which means that any nodes i connected to k will satisfy $x_i=x_k$.

The same can be shown for all x_i connected to x_k.

So for any i which has a path to k has the same value $x_i=x_k$

however, $x_f\neq x_k$ which means x_f is not connected to the graph of k.

$\lightning$ We know that the graph is connected $\Rightarrow$ there exists only and exactly 1 eigenvector. $\Rightarrow$ $\lambda=0$ has a multiplicity of exactly 1.