Counting Geodesics

Using curve shortening flow to show existence of closed geodesics on compact Riemannian surfaces

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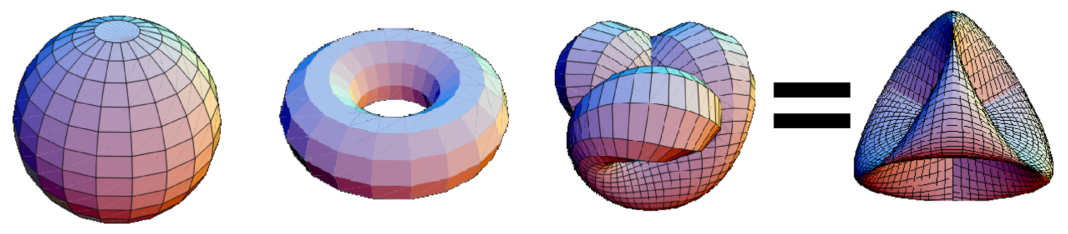
1. Introduction

Geodesics are a special type of curve on a surface that can be thought of as an equivalent to "straight lines" in curved space. Despite being originally motivated by calculus of variations, they reveal a lot about the internal structure of a surface, so are worthwhile to study in their own right. Much like straight lines, there exists a geodesic from any point to any other point - however, there might be more than one, and we might find geodesics which close back up on themselves, to look like a circle or a loop. These are rarer, so provide more insight.

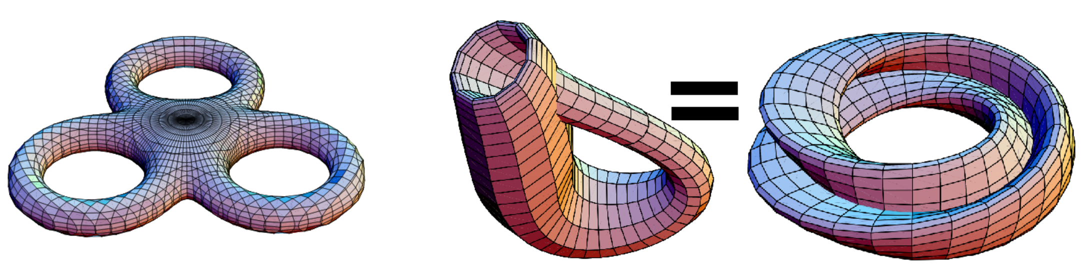
We can ask several natural questions about how many closed geodesics a surface has, and what we can say about them. It turns out that curve shortening flow provides lower bounds on the number of geodesics given just the topology of a surface, and in certain cases, it can guarantee that infinitely many exist. We consider both embedded geodesics (which are not allowed to self-intersect), as well as the more general case.

2. Compact surfaces

A **surface** is a space that is locally 2-dimensional, like a sphere or a hollow cylinder. We are mostly interested in **compact** surfaces, which don't extend infinitely and have no edge. There are 3 basic compact surfaces: the sphere S^2; the torus T^2; and the projective plane $\mathbb{RP}^2$ - a disc with antipodal points glued together.



We can glue these together to form other surfaces, like the 3-holed torus and Klein bottle K^2.

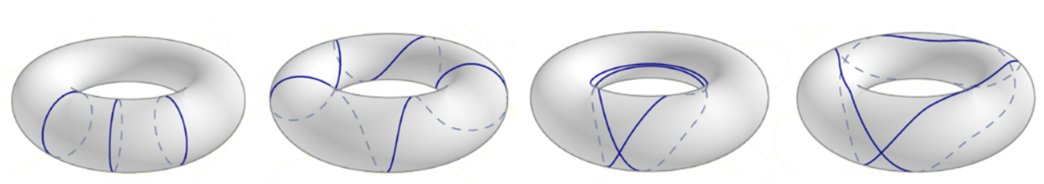


Any surface can be "stretched" into such a surface - we call this its **topology**. To fix the geometry of the surface, we add a [*Riemannian*] **metric**, which defines the length of any curve. The two $\mathbb{RP}^2$s and two K^2s above each have the same topology, but different geometries.

3. Closed geodesics

A **closed geodesic** is a closed curve/loop γ on a surface S that is shorter than all "nearby" curves - slightly perturbing the curve always makes it longer. This is equivalent to the acceleration vector γ'' being normal to the surface. If it has no self-intersections, we call it **embedded**.

The geodesics of a unit sphere are the great circles. Some geodesics of a ring torus are shown.



4. Curve Shortening Flow

For a curve γ on S, we define its **curvature** k to to be the length of the projection of γ'' onto the tangent plane to S. This measures how much γ curves relative to the surface - note that k depends on the metric, and $k = 0$ for geodesics. A family γ_t of curves evolves by CSF if $\frac{\partial \gamma}{\partial t} = k\mathbf{N}$ for some normal $\mathbf{N}$. This has the effect of shrinking the initial curve γ_0 as fast as possible. Under fairly light constraints:

- γ_t exists up until some time $T > 0$;
- If $T = \infty$, then $k \to 0$, and there is a subsequence γ_{t_n} converging to a geodesic;
- Else, γ_t has finitely many singularites, and if γ_0 is smooth, it shrinks to a point;
- the number of self-intersections of γ_t decreases with t.

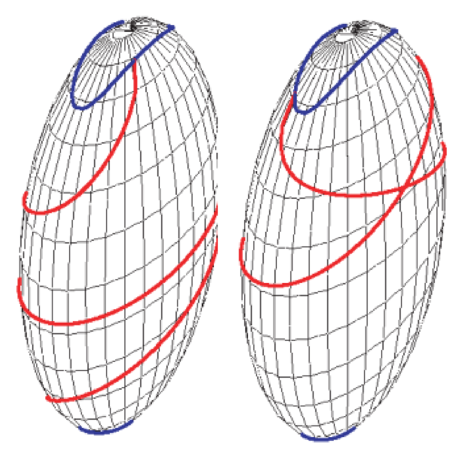
5. Existence theorems and lower bounds

It is a classical result that any surface contains at least one closed geodesic. On any surface other than S^2 we can choose a curve that cannot contract to a point - e.g., the curves in §3. Letting this flow under CSF, we must find a geodesic. In fact, we can always choose an embedded curve, getting us an embedded geodesic. [*The case of S^2 is harder - we define $E = \int |\dot{\gamma}|$, find a family $\mathcal{A}$ of flows that minimize $\inf_{\gamma_t \in \mathcal{A}} \sup_t E(\gamma_t)$, and show this infimum is achieved at a geodesic.*]

We can then ask the general question of how many closed geodesics there will be. This is still open in general, but restricting to embedded geodesics, we can give lower bounds from *just the topology*.

Topology	Min #
S^2	3
$\mathbb{RP}^2$	3
K^2	≥ 5
Else	∞

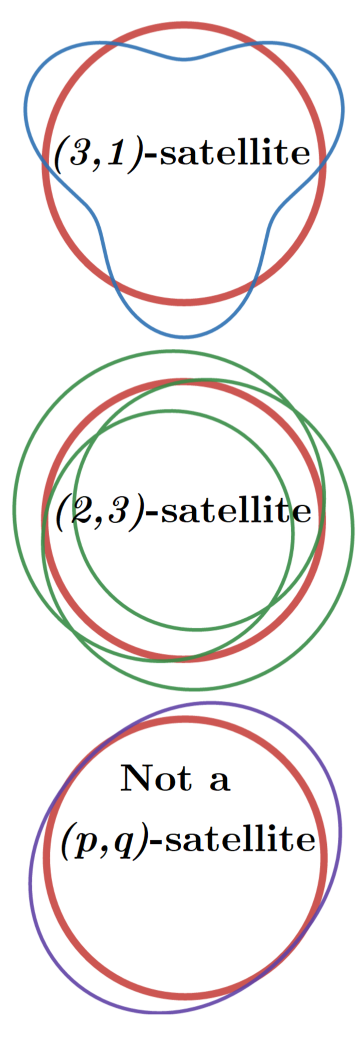
The easiest case is a set of tori glued together. Choose some hole, and a closed curve that loops through it n times before coming back to the start - e.g., the second curve in §3, with $n = 5$. This can't contract to a point, so flowing under CSF gives us a distinct geodesic for each n. A similar trick [*non-trivial embedded homotopy classes*] works for gluing more than two $\mathbb{RP}^2$s - for exactly two copies (K^2) this only gives 5 embedded geodesics.



S^2 is much harder. We begin by considering the unit sphere and the space Λ of smooth embedded loops on it. Letting these shrink under CSF, we consider only the loops with length $< 2\pi + \varepsilon$, which (mostly) fall into three classes. [*Let Λ_0 be the constant loops, so CSF gives a strong deformation retract $(\Lambda, \Lambda_0) \to (\Lambda^{2\pi+\varepsilon}, \Lambda_0) \simeq (\mathbb{RP}^3 \setminus D^3, \partial)$, which has $\mathbb{Z}_2$-homology classes of dimension 1,2,3.*] In general, these loops don't collapse to a point, so we get three distinct geodesics by applying CSF again. Although the calculations require us to pick a geometry, the classes of loops are topological, so the number of geodesics just relies on the topology of S^2.

$\mathbb{RP}^2$ can also be thought of S^2 with antipodal points identified, so the minimal number follows from S^2. Further, for S^2 (and so $\mathbb{RP}^2$) a near-spherical ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ gives an example of a metric with exactly 3 embedded geodesics - the ellipses $x = 0, y = 0, z = 0$. The diagram right shows how other geodesics spiral, so are not embedded.

6. (p, q)-satellites and S^2



For S^2 only, we can say that there are infinitely many closed geodesics for any metric, and say quite a lot about what they look like.

Choose any fixed curve γ on S^2. Then another curve α is called a (p, q)**-satellite** of γ if it looks like a sine curve with frequency p/q wrapped q times around γ. Each metric that makes γ into a geodesic also assigns it a number ρ to do with the curvature of the surface along γ. We split the metrics into two non-resonant classes: $p/q < \rho$ and $p/q > \rho$. Angenent (2005) then establishes the following:

- If γ_t converges to γ under CSF, it eventually becomes a (p, q)-satellite of γ for some fixed p, q;
- for fixed curve γ and fixed p, q, either all of the metrics with $p/q < \rho$ or all those with $p/q > \rho$ have a geodesic that is also a (p, q)-satellite.

So let γ be the equator of the unit sphere, which has $\rho = 1$. This has no (p, q)-satellites that are geodesics for $p/q \neq 1$. So for any metric that makes γ a geodesic, any p/q strictly between 1 and ρ gives a (p, q)-satellite of γ that is a geodesic in that metric. In particular, if $\rho \neq 1$, we get infinitely many closed geodesics. The case with $\rho = 1$ was actually known first, but currently requires a different approach.

7. Research directions

Over my project, I examined several open problems in the area:

- Is there a metric on K^2 giving exactly 5 closed, embedded, geodesics?
- Which surfaces other than S^2 can Angenent's approach cover? What about $\rho = 1$?
- Can (p, q)-satellites be used to prove other conjectures/theorems about geodesics in a more unified theory?