Counting Geodesics with Curve Shortening Flow

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November 23, 2022

Smooth surfaces

Intuition

A **smooth surface** is a subset $M \subseteq \mathbb{R}^n$ that locally looks like $\mathbb{R}^2$. Similar to the Prelims definition, except that we don't have to miss random points/lines, and we aren't stuck in $\mathbb{R}^3$.

Definition

A **smooth surface** is a subset $M \subset \mathbb{R}^n$ with a family of maps $\varphi_i : V_i \to U_i$ between open subsets $V_i \subseteq \mathbb{R}^2$ and $U_i \subseteq M$ such that and for all i,

- the U_i cover M: $\bigcup_i U_i = M$;
- φ_i, φ_i^{-1} are continuous (so M locally looks like $\mathbb{R}^2$);
- φ_i is smooth (has partial derivatives of all orders);
- at each $(x_0, y_0) \in V_i$, the derivatives $\partial_x \varphi_i(x_0, y_0)$ and $\partial_y \varphi_i(x_0, y_0)$ are linearly independent.

Topology vs Geometry

Definitions?

Two ways of talking about "shapes" and "space":

- A space with a **geometry** has a notion of *distance* - we can measure the lengths of paths.
- A space with a **topology** just has a notion of *closeness* – we can define *neighbourhoods* of points.
- In particular, topological spaces have a concept of open sets, and geometric spaces are also topological.
- Continuous maps preserve "closeness".

Topology vs Geometry

Equivalence

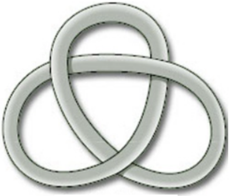
Two geometric spaces are equivalent if there is an **isometry**:

$$\exists \text{ bijection } \varphi : X \to Y \text{ that preserves distance.}$$

Two topological spaces are equivalent if there is a **homeomorphism**:

$$\exists \text{ bijection } \varphi : X \to Y \text{ with } \varphi, \varphi^{-1} \text{ continuous.}$$

Think of geometric spaces as rigid, topological spaces as stretchy.



Smooth surfaces

Remarks

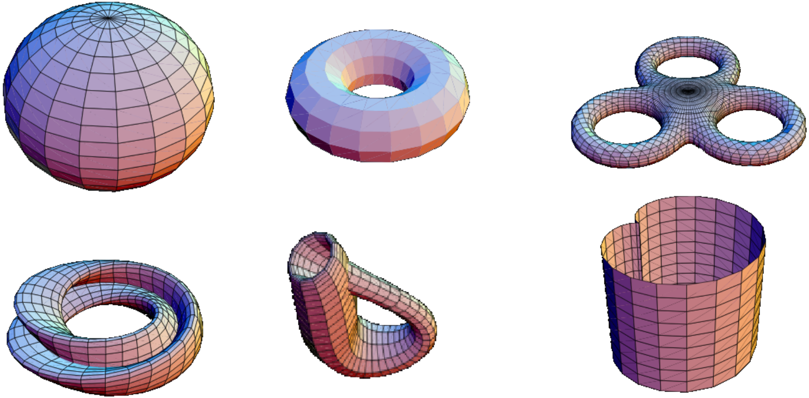
For ease, we will be restricting to surfaces which are:

- **compact** - are bounded in $\mathbb{R}^n$ and have no boundary;
- **orientable** - have an "inside" and an "outside".

Theorem

Let M be a compact, orientable, smooth surface. Then M is topologically equivalent to either a sphere S^2, or to a torus with g holes.

Smooth surfaces



Geodesics

Intuition

Geodesics on M are locally length-minimising curves, and play the roles of "straight lines" on curved surfaces.

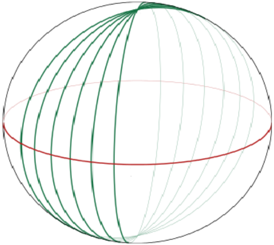
Definition

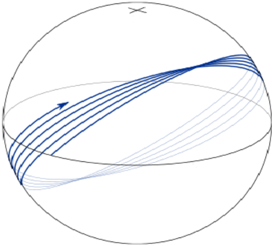
Let $\gamma(x)$ be a curve in M, and $\mathbf{n}(x)$ a vector tangent to M and orthogonal to $\gamma'(x)$. The **geodesic curvature** of γ is $\kappa := \mathbf{n} \cdot \gamma''$, so that $\kappa\mathbf{n}$ is the component of γ'' tangent to M.
γ is a **geodesic** is $\kappa = 0$, so γ'' is normal to the surface.

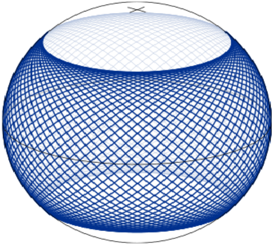
Fact

At any point $p \in M$, and any tangent vector v at p, there is a unique geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$.

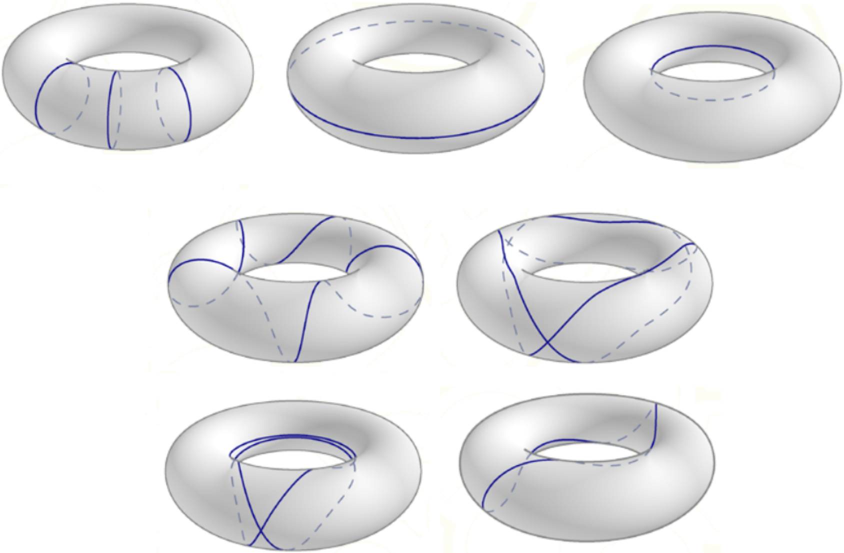
Geodesics







Geodesics



Geodesics

Definition

If γ is injective, so does not self-intersect, we call it **embedded**. Sometimes a geodesic γ will be periodic. Then we call it **closed**, and think of it as a function $S^1 \to M$.

Definition

I will say **loop** to mean a smooth map $\gamma : S^1 \to M$ with $\gamma' \neq 0$, considered up to orientation-preserving reparameterisation.

- Basically oriented circles on M.
- Each loop γ has a unique constant-speed parameterisation.
- A closed geodesic is a loop that is also a geodesic.

Curve Shortening Flow

Definition

Let M be an orientable surface.
A family of loops $\gamma(x,t) : S^1 \times [0,T) \to M$ evolves by **curve shortening flow** (CSF) if:

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{n}.$$

We typically write $\gamma_t := \gamma(\cdot, t)$ and $\gamma = \gamma_0$.

Facts and Observations

CSF decreases the length of γ_0 as fast as possible.
Geodesics are stationary.
Recalling $\kappa \mathbf{n} = \text{proj}\left(\frac{\partial^2 \gamma}{\partial x^2}\right)$, CSF is a heat-type equation.

Curve Shortening Flow

Well-posedness

CSF is well-posed: for any initial γ_0, there is some $T_{\max} > 0$ such that γ_t exists for $t \in [0, T_{\max})$. Further, this evolution is unique, and continuous.

Theorems (Grayson, 1989)

Let γ_t be a curve evolving under CSF. Then:

- The number of self-intersections of γ_t decreases with t.
- If $T_{\max} < \infty$, γ_t converges to a point.
- if $T_{\max} = \infty$, there is a subsequence γ_{t_n} converging to a closed geodesic.

Embedded geodesics on the torus

Theorem

Let M be a surface homeomorphic to a torus. Then there are infinitely many closed, embedded geodesics on M.

Sketch of Proof

Fix any $n \geq 1$, and choose a curve $\gamma^{(n)}$ that wraps through the "hole" n times without self-intersecting. Let it evolve under CSF.
$\gamma_t^{(n)}$ can't collapse to a point, so there must be a subsequence converging to a geodesic $\Gamma^{(n)}$.
Similarly, $\gamma_t^{(n)}$ can't "unwind" itself, so $\Gamma^{(n)}$ must also wrap through the "hole" n times.
CSF keeps embedded curves embedded, so for each $n \geq 1$, we get a distinct closed, embedded geodesic $\Gamma^{(n)}$ on M.

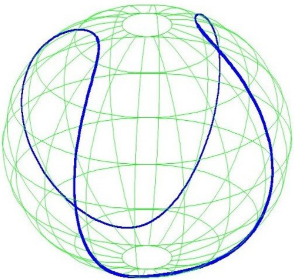
Tennis Ball Theorem

Definition

A **point of inflection** of a curve γ is a point where κ changes sign.

Theorem (Weiner, 1977)

Any embedded curve on the unit sphere $S \subset \mathbb{R}^3$ that divides the sphere into two parts of equal area has at least 4 inflection points, or is a great circle.



Tennis Ball Theorem, Proof

Part 1 - notation and key fact

Let $\gamma = \gamma_0$ be the curve, and γ_t its evolution under CSF. Let $R(t)$ be one of the components of $S \setminus \gamma_t$, and $A(t)$ its area.
Assume γ is not a great circle. Under CSF, the number of inflection points cannot increase. *Intuition - CSF makes curves less wiggly.*

Part 2 - γ_t converges to a great circle

Then $A(t) = 2\pi$ for all t. Using Gauss-Bonnet, we could write:

$$A'(t) = -\int_{\gamma_t} \gamma''(t) \cdot \mathbf{n}\, ds = -\int_{\gamma_t} \kappa\, ds = \iint_{R(t)} K\, dA - 2\pi = A(t) - 2\pi$$

Then γ_t cannot collapse to a point, so exists for all time.
We can show that on S, γ_{t_n} converges $\implies$ γ_t converges. The only geodesics of S are great circles.

Tennis Ball Theorem, Proof

Part 3 - Linearising PDEs

We can assume γ_t converges to the equator $z = 0$.
For large enough t, we can use cylindrical coordinates (z, φ), and γ_t will be given by $z = u(\varphi, t)$.
Under this change of coordinates, CSF becomes:

$$\frac{\partial u}{\partial t} = \lambda(u, u_\varphi)\left(\frac{\partial^2 u}{\partial \varphi^2} + u\right)$$

for some smooth λ with $\lambda(0,0) = 1$.
Our solution converges to $u = 0$, so we linearise by $\lambda = 1$ and solve:

$$\implies u(\varphi, t) = Ce^{(1-n^2)t}\cos\left(n(\varphi - \varphi_0)\right) + o\left(e^{(1-n^2)t}\right)$$

for some $n \geq 2$.
Graphing this, we have at least $2n \geq 4$ inflection points.

Theorem of Three Geodesics

Theorem (Lyusternik-Schnirelmann, Ballman, Grayson, ...)

Any surface M homeomorphic to S^2 has at least 3 closed, embedded geodesics.

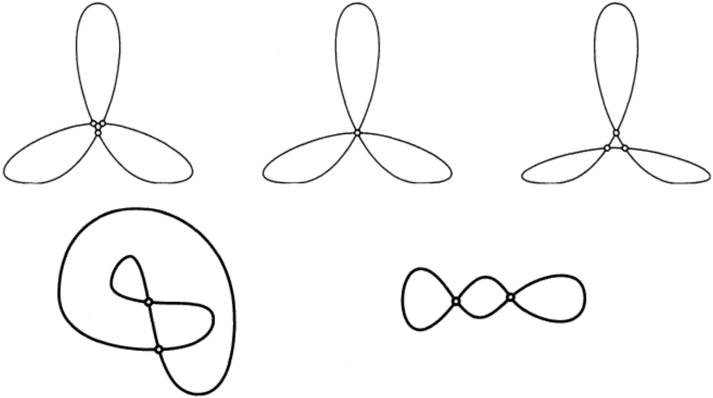
More specifically, we have one of the following three cases:

- There are exactly three closed embedded geodesics, each with different length.
- There is a one-parameter family of closed embedded geodesics which cover M, all of the same length, and one embedded closed geodesic of a different length.
- Every geodesic is closed and embedded, and they all have the same length.

Flat Knots

Definition

A loop α on M is a **flat knot** if it has no self-tangencies.
Two flat knots α, β are **equivalent** if they can be continuously deformed into each other while staying flat knots.



Relative Flat Knots

Definition

Let $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$ be a collection of loops on M. α is a flat knot **relative to** Γ if it has no self-tangencies and is never tangent to a $\gamma_i \in \Gamma$. Two flat knots α, β rel. Γ are **equivalent** if they can be continuously deformed into each other while staying flat knots rel. Γ.

Notation

Let Ω be the topological space of loops on M.
For fixed Γ, let Δ be such that $\Omega \setminus \Delta$ is the flat knots rel. Γ.
Then α, β are equivalent if they are in the same path-component of $\Omega \setminus \Delta$.
We call these path-components **flat knot types**, and write $\mathcal{B}_\alpha$.

p, q-satellites

Intuition

For p, q coprime integers, a p, q-**satellite** of $\gamma_i \in \Gamma$ is a curve in $\Omega \setminus \Delta$ that looks like a sine curve with frequency p/q wrapped q times around γ_i.

Definition

Fix p, q, and define a curve $\alpha_{p,q}$ on the cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ to be a sine curve with frequency p/q wrapped q times:

$$\alpha_{p,q}(x) = (x, \sin(2\pi px/q)), \qquad x \in \mathbb{R}/q\mathbb{Z}$$

As in the Tennis Ball Theorem, give a small strip around γ_i coordinates (φ, z) from the cylinder, so that γ_i has $z = 0$.
Under this identification, for a small enough strip, $\alpha_{p,q} \in \Omega \setminus \Delta$.
We say a curve is a p, q-satellite if it is equivalent to $\alpha_{p,q}$ or has the reverse orientation. Write $\mathcal{B}_{p,q}$.

p, q-satellites

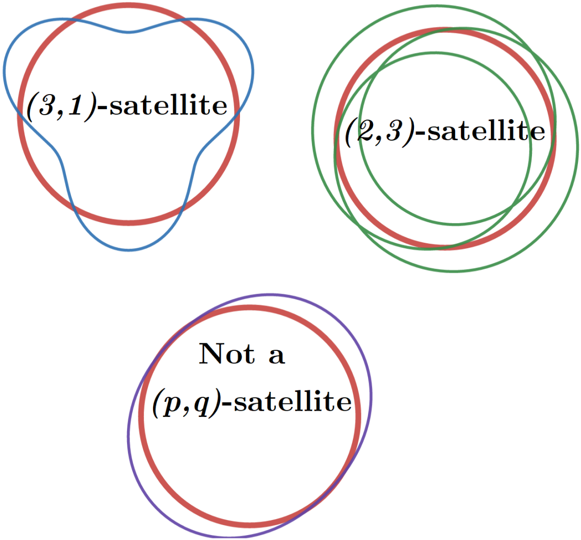
Lemma

Let γ_t evolve under curve shortening, and suppose γ_t converges to a geodesic γ_∞. Then there are p, q such that, for large enough t, $\gamma_t \in \mathcal{B}_{p,q}(\gamma_\infty)$.

Interpretation

CSF traces out paths in Ω. If the path converges, it approaches from within a component of $\Omega \setminus \Delta$ corresponding to p, q-satellites.

p, q-satellites



Conley Index

Final construction

1. Consider M as a topological surface, and fix some curves $\Gamma = \{\gamma_1, \ldots, \gamma_N\} \subset \Omega$, and special curve $\alpha \in \Omega$.
2. Consider geometries such that the γ_i are geodesics and $\alpha \in \Omega \setminus \Delta$. Order γ_i so that α is a p_i, q_i-satellite for $1 \leq i \leq n$, and not for $i > n$.
3. Calculate the **rotation number** ρ_i for each γ_i, and assume $\rho_i \neq p_i/q_i$. Split the geometries into 2^N classes based on $I := \{i : \rho_i < p_i/q_i\} \subseteq \{1, \ldots, N\}$.
4. Define the **Conley index** h^I. Roughly, the closure flat knot type $\mathcal{B}_\alpha$, but with exit points glued.
   This is mostly independent of the geometry, and depends only on I.
5. Key Result 1: If there is a geodesic in $\mathcal{B}_\alpha$, h^I is non-trivial (not contractible).
6. Key Result 2: If $I \subset J$, h^I and h^J can't both be trivial.

Closed geodesics on the sphere

Theorem (Angenent, 2005)

Let M be a surface homeomorphic to the sphere, and γ some geodesic on M with $\rho \neq 1$. Then for each p/q strictly between 1 and ρ, there is a geodesic that is a p, q-satellite of γ.

Sketch of proof

By changing geometry, we can assume that γ on M corresponds to the equator ζ of the unit sphere S. Note that $\rho(\zeta) = 1$.
So let $\Gamma = \{\zeta\}$, and α a curve in $\mathcal{B}_{p,q}$ with $p/q \neq 1$. Note there is no closed geodesic in $\mathcal{B}_{p,q}$.
If $p/q < 1$, $I = \{1\}$ on S, so $h^{\{1\}}(\mathcal{B}_{p,q})$ is trivial. So if $\rho(\gamma) < p/q$, $I = \varnothing$ on M, and there is a geodesic in $\mathcal{B}_{p,q}$.
If $p/q > 1$, $I = \varnothing$ on S, so $h^{\varnothing}(\mathcal{B}_{p,q})$ is trivial. So if $\rho(\gamma) > p/q$, $I = \{1\}$ on M, and there is a geodesic in $\mathcal{B}_{p,q}$.