1 The Question

BMO1 2015: Problem 4. James has a red jar, a blue jar, and a pile of 100 pebbles. A move consists of moving a pebble from the pile into one of the jars or returning a pebble from one of the jars to the pile. The numbers of pebbles in the red and blue jars determine the *state* of the game. The following conditions must be satisfied:

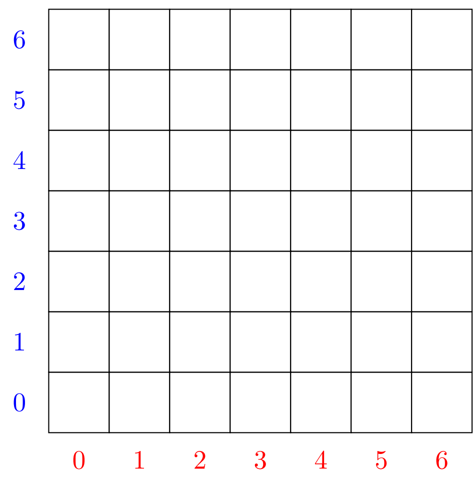
(a) The red jar may never contain fewer pebbles than the blue jar;

(b) The game may never be returned to a previous state.

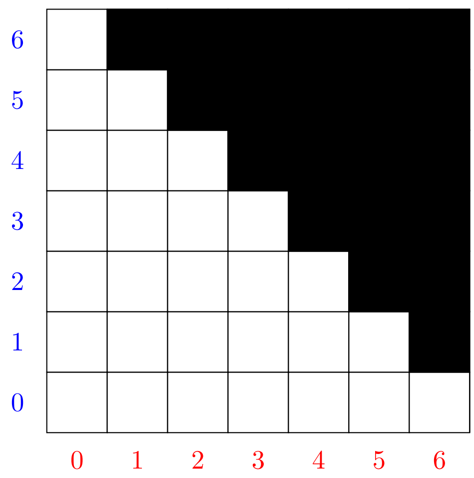
What is the maximum number of moves that James can make?

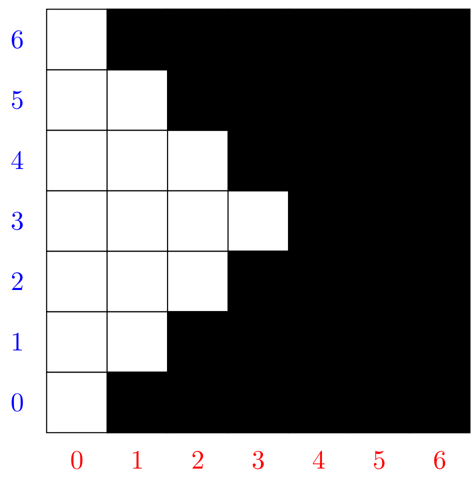
2 The Solution

[Spent 10 mins before hints.] We can represent the possible states on an $n \times n$ grid, where the square (i, j) (counting from 0) represents having i pebbles in the red jar, and j in the blue jar. For the problem, $n = 101$, but for illustrative purposes, I will draw $n = 7$; that is, a game of 6 pebbles.



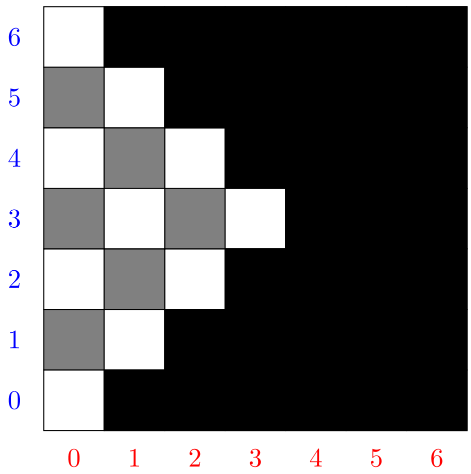
Now, the fact that there a total of n pebbles means we can't access the squares in the top-right triangle, as illustrated. Condition (a) further cuts out the squares where $i < j$.





Condition (b) now allows us to reframe the question. Recall that each square in the grid refers to a state of the game. Then each move in the game refers to a move on the grid, either right/left if a red pebble is added/removed, or up/down if it is a blue pebble. Hence the question becomes: what is the longest path through the grid that never visits a square twice? [Wait for several ideas.]

After some playing around, we hit upon the idea of a chessboard colouring. Then each move moves from a white square to a grey square, or vice versa.



As each move involves a grey square, and each grey square can be used at most twice (entering and leaving), we can make at most $2g$ moves, where there are g grey squares. So what is g? Well, as, in the game, there are 101 rows, there are 51 white squares, and 50 grey squares in the first column. Similarly, there are 49 grey squares in the second. Repeating the pattern gives

$$g = 50 + 49 + \cdots + 2 + 1 = \frac{(50)(51)}{2} = 1275.$$

Thus, we can make *at most* $2 \times 1275 = 2550$ moves. But can we make *exactly* 2550 moves? [Wait for them to find the path.] It turns out we can: simply zig-zagging up and down means that we pass through every grey square, so have made the most possible moves.