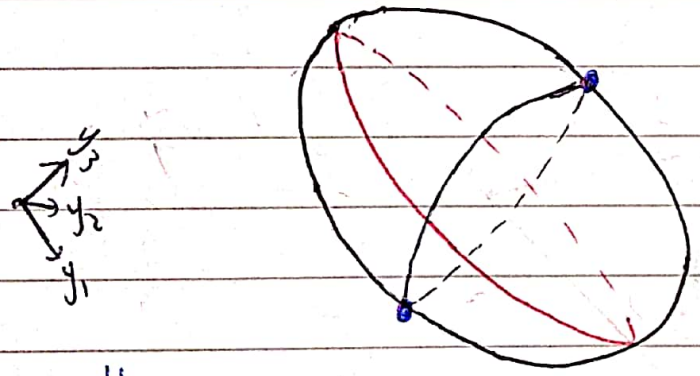
4) b) Find the maximum and minimum values of $x^T x$ for those x, such that $x^T A x = 1$. How many x achieve each of the maximum and minimum values?

From (a), A has eigenvalues $4, 4, 8$, so by the Spectral Theorem, there is an orthogonal matrix P such that $P^T A P = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$. Letting $P^T x = y = (y_1, y_2, y_3)$, we get

$$x^T A x = x^T P D P^T x = y^T D y = 4y_1^2 + 4y_2^2 + 8y_3^2 = 1.$$

Thus, these x lie on an ellipse flattened in the y_3-direction.



red = max, blue = min

Hence, the only many vectors x in the y_1y_2-plane achieve the maximum $x^T x = \frac{1}{4}$, while only the two vectors on the y_3-axis achieve the minimum $x^T x = \frac{1}{8}$.