STAT 150 HOMEWORK #9

FALL 2023

Due Friday, Nov 3rd, at 11:59 PM on Gradescope.

Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

REVIEW:

Poisson Processes are monotonically increasing and have $Poisson(\lambda)$ distributed number of arrivals in time t. In other words, for an arbitrary time t, there exists $Poisson(\lambda t)$ arrivals.

The distribution for the waiting time for the n-th arrival is $Gammar(n, \lambda)$. Thus, each inter-arrival time is $Exponential(\lambda)$.

Conditional on the number of arrivals in a period of time, the location of arrivals is uniformly distributed on the interval $[0, t)$ with density $\frac{1}{t}$

The joint density of occurrence times, conditional on the number of arrivals is distributed by the density function: $f_{W_1 \ldots W_n | N(t)=n}(w_1 \ldots w_n) = \frac{n!}{t^n}$

The marginal density of the k-th order statistics conditional on the number of arrivals, is $Beta(k, n+1-k)$

First Moment Integral:
$E[X] = \int_0^\infty (1 - F(x))\, dx$

Law of Total Expectation / Tower Property / Iterated Law of Expectations:
$E[X] = E[E[X|F]]$

Memoryless Property:
$Pr(X > \tau + x | X > \tau) = Pr(X > x)$

1. Pinsky and Karlin, EXERCISE 7.5.1

Jobs arrive at a certain service system according to a Poisson process of rate λ. The server will accept an arriving customer only if it is idle at the time of arrival. Potential customers arriving when the system is busy are lost.

Suppose that the service times are independent random variables with mean service time μ. Show that the long run fraction of time that the server is idle is $\frac{1}{(1+\lambda\mu)}$. What is the long run fraction of potential customers that are lost?

SOLUTION:

(a) Customer's order tasks (send jobs to the server). Consider X_i an inter-arrival time for tasks sent to the server (customers). We know that the waiting time for a task to arrive is $Exp(\lambda)$, and $\mathbb{E}[X_i] = \frac{1}{\lambda}$

After that, we know the amount taken to complete a task is a RV, lets call it Y_i which takes an average of μ time. Thus, $\mathbb{E}[Y_i] = \mu$. Thus the whole cycle can be represented by the sum: $T_i = X_i + Y_i$

The Renewal-Reward Theorem can be applied in this context by considering each served customer as a 'renewal' and the idle time immediately following the service as the 'reward'. The theorem states that the long-run average reward rate (the rate of reward per unit time) is equal to the expected reward per cycle divided by the expected length of a cycle. In this case, the reward (idle time) is just the time between the departure of a customer and the arrival of the next one, which is exponentially distributed with parameter λ

$$\boxed{\text{Long-Run Fraction Server is Idle } = \frac{\mathbb{E}[X_i]}{\mathbb{E}[T_i]} = \frac{\mathbb{E}[X_i]}{\mathbb{E}[X_i] + \mathbb{E}[Y_i]} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \mu} = \frac{1}{1 + \lambda\mu}}$$

(b) The next part asks what is the long run fraction of potential customers that are lost. Thus, we need to find the amount of customers R_i that would usually show up in the occupied time,Y_i in a period T_i.

$$\boxed{\text{Long-Run Fraction Lost Customers } = \frac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]} = \frac{\mathbb{E}[\mathbb{E}[R_i|Y_i]]}{\mathbb{E}[T_i]} = \frac{\lambda\mathbb{E}[Y_i]}{\frac{1}{\lambda} + \mu} = \frac{\lambda^2\mu}{1 + \lambda\mu}}$$

2. Pinsky and Karlin, Problem 7.5.2

   The random lifetime X of an item has a distribution function F(x). What is the mean total life $E[X|X > x]$ of an item of age x?

   SOLUTION:

   An idea is to use the fact that:

   $$Pr(X > y|X > x) = \begin{cases} 1 & \text{when } x < y \\ \frac{1-F(y)}{1-F(x)} & \text{when } 0 < x < y \end{cases}$$

   The first equality (second term) comes from the first moment integral (Integral representation of Expectation using the CDF)

   The second equality (third term) comes from the fact that:
   $Pr(X > y|X > x) = 1$ for $y < x$.

   $E[X|X > x] = \int_0^\infty \frac{1-F(y)}{1-F(x)} dy = \int_0^x 1dy + \int_x^\infty \frac{1-F(y)}{1-F(x)} dy = x + \int_x^\infty \frac{1-F(y)}{1-F(x)} dy$

   $$\boxed{E[X|X > x] = x + \frac{1}{1-F(x)} \int_x^\infty 1 - F(y)dy}$$

3. Pinsky and Karlin, Problem 7.5.4

A lazy professor has a ceiling fixture in his office that contains two light- bulbs. To replace a bulb, the professor must fetch a ladder, and being lazy, when a single bulb fails, he waits until the second bulb fails before replacing them both. Assume that the length of life of the bulbs are independent random variables.

(a) If the lifetimes of the bulbs are exponentially distributed, with the same parameter, what fraction of time, in the long run, is our professor's office half lit?

(b) What fraction of time, in the long run, is our professor's office half lit if the bulbs that he buys have the same uniform (0, 1) lifetime distribution?

BOTH (a) and (b):

We must define 'rewards' and 'cycles' so that we can use the reward-renewal theorem. Here, we can claim that the reward period is the period of time between the first and second failure R_i. The cycle time, T_i is the time till the second failure. Note that we have not specified which light fails first or second.

$R_i = w_2 - w_1 = s_2$
$T_i = w_2$

(SOLUTION A)

Proved many times before: The minimum of 2 random, exponentially distributed variables is distributed exponentially with added rates of the two exponentials.
Here we have that both are $Exp(\lambda)$ so that the first arrival is distributed with $Exp(2\lambda)$

After the first arrival, there is only one lightbulb.
Because of the memoryless property, the remaining life of the lightbulb still working is: $Exp(\lambda)$

First inter-arrival: $W_1 = S_1 \sim Exp(2\lambda)$
Second inter-arrival: $S_2 \sim Exp(\lambda)$
Second arrival $S_1 + S_2 = W_2 \sim Exp(2\lambda) + Exp(1\lambda)$

$\mathbb{E}[R_i] = \mathbb{E}[S_2] = \mathbb{E}[Exp(\lambda)] = \frac{1}{\lambda}$
$\mathbb{E}[T_i] = \mathbb{E}[W_2] = \mathbb{E}[Exp(\lambda)] + \mathbb{E}[Exp(2\lambda)] = \frac{1}{\lambda} + \frac{1}{2\lambda}$

Thus, $\boxed{\dfrac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]} = \dfrac{1/\lambda}{3/(2\lambda)} = \dfrac{2}{3}}$

(SOLUTION B)
I have done this in two ways, here is the first.

(b)(SOLUTION 1)
The minimum of 2 random, uniformly distributed variables is the first order statistic, and the maximum is the second order statistic. We know that the distribution of the k-th order statistic of n draws is:
$Beta(k, n+1-k)$ thus, the expectation is: $= \frac{k}{k+n+1-k} = \frac{k}{n+1}$.
Thus the expectation of the first light outage is $t = \frac{1}{3}$ and the expectation of the second light outage is $t = \frac{2}{3}$.
After the second outage, the cycle resets, so $\mathbb{E}[T_i] = \frac{2}{3}$
The time between the first and second outage is $\mathbb{E}[R_i] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

Thus, $\boxed{\dfrac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]} = \dfrac{1/3}{2/3} = \dfrac{1}{2}}$

(b)(SOLUTION 2)

This alternative approach is actually the same thing and not that different.
Notice, this time U_i's are not order statistics, just regular draws.

$R_i = max(U_1, U_2) - min(U_1, U_2)$
$T_i = max(U_1, U_2)$

$F_{max(U_1,U_2)}(u_1, u_2) = P(max(U_1, U_2) < u) = P(U_1, U_2 < u) =$
$P(U_1 < u)P(U_2 < u) = [P(U_1 < u)]^2 = u^2$

$F_{max(U_1,U_2)}(u_1, u_2) = u^2$
$f_{max(U_1,U_2)}(u_1, u_2) = \frac{d}{du}u^2 = 2u$

$\mathbb{E}[max] = \int_0^1 2u^2 du = \frac{2}{3}$

$F_{min(U_1,U_2)}(u_1, u_2) = P(min(U_1, U_2) < u) = 1 - P(min(U_1, U_2) > u) =$
$1 - P(U_1, U_2 > u) = 1 - [P(U_1 > u)P(U_2 > u)] = 1 - [P(U_1 > u)]^2 = 1 - (1-u)^2$

$F_{min(U_1,U_2)}(u_1, u_2) = 1 - (1-u)^2$
$f_{min(U_1,U_2)}(u_1, u_2) = \frac{d}{du}[1 - (1-u)^2] = 2(1-u)$

$\mathbb{E}[min] = \int_0^1 2u(1-u)du = \frac{1}{3}$

By linearity of expectation:
$\mathbb{E}[R_i] = \mathbb{E}[max] - \mathbb{E}[min]$
$\mathbb{E}[T_i] = \mathbb{E}[max]$

By the Reward Renewal Theorem: $\boxed{\dfrac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]} = \dfrac{1/3}{2/3} = \dfrac{1}{2}}$

4. Durrett, Exercise 3.5

In front of terminal C at the Chicago airport is an area where hotel shuttle vans park. Customers arrive at times of a Poisson process with rate 10 per hour looking for transportation to the Hilton hotel nearby. When seven people are in the van it leaves for the 36-minute round trip to the hotel. Customers who arrive while the van is gone go to some other hotel instead.

(a) What fraction of the customers actually go to the Hilton?

(b) What is the average amount of time that a person who actually goes to the Hilton ends up waiting in the van?

(a) SOLUTION

Define a cycle as the time it takes for 7 people to line up + 36 minutes:
$T_i = W_7 + (\frac{3}{5} Hours) = Gamma(7, \lambda = (\frac{\text{10 people}}{\text{per hour}})) + (\frac{3}{5} Hours)$

Define the 'total reward' R_i as all the people who arrive, and define the 'Hilton reward H_i as the people that arrive at the hilton.

$R_i = 7 + PP((t = \frac{3}{5} hours)(\lambda = (\frac{\text{10 people}}{\text{per hour}})))$
$H_i = 7$

Caution to not calculate $\frac{\mathbb{E}[H_i]}{\mathbb{E}[R_i]}$ directly. Although this yields the correct answer, and makes sense intuitively. we must understand that this result comes from the reward renewal theorem.

Instead, consider the total number of people who arrive per cycle, $\frac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]}$ and also the number of Hilton-goers per cycle, $\frac{\mathbb{E}[H_i]}{\mathbb{E}[T_i]}$. Because both of these results are given by the RR Theorem, we can combine them to yield:

(Hilton-goers per cycle)(Total # per cycle)$^{-1} = \frac{\mathbb{E}[H_i]}{\mathbb{E}[T_i]} \frac{\mathbb{E}[T_i]}{\mathbb{E}[R_i]} = \frac{\mathbb{E}[H_i]}{\mathbb{E}[R_i]}$

$\mathbb{E}[H_i] = 7$
$\mathbb{E}[R_i] = \mathbb{E}[7] + \mathbb{E}[PP((t = \frac{3}{5} hours)(\lambda = (\frac{\text{10 people}}{\text{per hour}})))] = 7 + \mathbb{E}[Poisson(\lambda = 6)] = 13$

$$\boxed{\frac{\mathbb{E}[H_i]}{\mathbb{E}[R_i]} = 13}$$

(b) SOLUTION

Between all 7 people, the seventh person does not wait at all, because the shuttle leaves instantly after the 7th person arrives, thus the sum below ranges from the first person to the sixth person.

The third equality (fourth term) comes from the fact that we can sum the waiting mass vertically. In other words, between the first and second arrival there is 1 person waiting, between the 2rd and 3rd arrival, there are 3 people waiting, etc etc

The fourth equality (fifth term) which includes: $(w_{i+1} - w_i)$ is an inter-arrival time. This means it is exponentially distributed with $\boxed{\text{mean } = \frac{1}{\lambda} = \frac{1 \text{ hour}}{\text{per 10 people}} = \frac{6 \text{ minutes}}{\text{per person}}}$

(Average Wait Time) $=$
$\frac{1}{7}\sum_1^7 (w_7 - w_i) =$
$\frac{1}{7}\sum_1^6 (w_7 - w_i) =$
$\frac{1}{7}\sum_1^6 (i)(w_{i+1} - w_i) =$
$\frac{1}{7}\sum_1^6 (i \text{ people})(\frac{1}{10})(\text{ hour / person}) =$
$(\frac{1}{7})(\frac{1}{10} \text{ hour})(\frac{(N{=}7)(N{-}1{=}6)}{2}) =$

$\frac{21}{7 \cdot 10}$

$$\boxed{\text{(Average Wait Time)} = \frac{3}{10}}$$

5. Durrett, Exercise 3.9

A cocaine dealer is standing on a street corner. Customers arrive at times of a Poisson process with rate λ. The customer and the dealer then disappear from the street for an amount of time with distribution G while the transaction is completed. Customers that arrive during this time go away never to return.

(a) At what rate does the dealer make sales?

(b) What fraction of customers are lost?

SOLUTION (a)

The dealer needs to wait some time distributed by $Exp(\lambda)$ to receive a customer, in addition to μ_G amount of time to complete the transaction. Thus the rate of transactions is:

$$\boxed{\frac{1}{((1/\lambda)+\mu_G)}\frac{\text{sale}}{\text{units of time}}}$$

SOLUTION (b)

This is a reward renewal type of question, where the total cycle time is an entire transaction, and the 'reward period' is the amount of time that the cocaine dealer cannot process orders.

The coke dealer is absent for amount of time distributed by G:
$\mathbb{E}[R_i] = \mu_G$

The full cycle time from the first moment that the dealer is available to the end of a sale is the same cycle time in part (a)'s denominator:
$\mathbb{E}[T_i] = ((1/\lambda) + \mu_G)$

Thus, $\boxed{\frac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]} = \frac{\mu_G}{((1/\lambda)+\mu_G)}}$

6. Durrett, Exercise 3.12

A worker has a number of machines to repair. Each time are pair is completed a new one is begun. Each repair independently takes an exponential amount of time with rate μ to complete. However, independent of this, mistakes occur according to a Poisson process with rate λ. Whenever a mistake occurs, the item is ruined and work is started on a new item. In the long run how often are jobs completed?

Preiminary thoughts

We should expect that completed jobs are independent of the mistakes. We will go through the math to 'show' this. But we know that the Poisson Processes are independent anyway.

Preparation

We can also think of this problem as a processes of coin flipping. Every time we start fixing a box, it is a Bernoulli trial with $p \sim \frac{\mu}{\mu+\lambda}$. This is an argument we have made many times before by super-positioning two PP's. When we condition on the number of arrivals in a set period of time, given that an arrival has the possibility of being one of two types with p and $1-p$.

In this setting, a coin is flipped depending on whether either PP arrives first. All this means is that inter-arrival times, T_i, are : $min(Exp(\lambda), Exp(\mu))$
which we have established is: $Exp(\mu + \lambda)$

SOLUTION

$$R_i = \begin{cases} 0 & \text{if } T_i \text{ is a mistake with probability: } \frac{\lambda}{\mu+\lambda} \\ 1 & \text{if } T_i \text{ is a repair with probability: } \frac{\mu}{\mu+\lambda} \end{cases}$$

$\mathbb{E}[R_i] = \frac{\mu}{\mu+\lambda}$

$\mathbb{E}[T_i] = \mathbb{E}[Exp(\mu + \lambda)] = \frac{1}{(\mu+\lambda)}$

$$\boxed{\frac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]} = \mu}$$

7. Durrett, Exercise 3.18 (a) (no need to do part (b) since the roots of the derivative cannot be easily found)

   A scientist has a machine for measuring ozone in the atmosphere that is located in the mountains just north of Los Angeles. At times of a Poisson process with rate 1, storms or animals disturb the equipment so that it can no longer collect data. The scientist comes every L units of time to check the equipment. If the equipment has been disturbed, then she can usually fix it quickly so we will assume the repairs take 0 time.

   (a) What is the limiting fraction of time the machine is working?

   SOLUTION (a)

   This can be set up using the RR Theorem. Essentially, we should decide on a total cycle time, and a total reward time. Here, the total cycle time is L because we know that the machine will necessarily be working instantaneously after L. Also, the total reward time will be the amount of time that no animals have disturbed the machine between the start of the cycle and L.

   Because the process' inter-arrival times are each $Exp(\lambda)$, and that we are trying to find the survival function of the reward (to calculate its expectation):

   $T_i = L$
   $\mathbb{E}[T_i] = L$

   $R_i = Exp(\lambda)$
   $\mathbb{E}[R_i] = \int_0^\infty P(R_i > r)dr$

   $$P(R_i > r) = \begin{cases} e^{-r} & \text{if } r \leq L \\ 0 & \text{if } r > L \end{cases}$$

   - If $r \leq L$ then we have the survival function of the exponential with rate 1.

   - If $r > L$ Though it is possible for the machine to work for longer than L time, is is not possible for the machine to work longer than L time per cycle, because each cycle is defined to have L time. Which is why $P(R_i > r) = 0$ if $r > L$.

   $\mathbb{E}[T_i] = L$
   $\mathbb{E}[R_i] = \int_0^\infty P(R_i > r)dr = \int_0^L e^{-r}dr = 1 - e^{-L}$

   Thus, $\boxed{\dfrac{\mathbb{E}[R_i]}{\mathbb{E}[T_i]} = \dfrac{1 - e^{-L}}{L}}$

8. Durrett, Exercise 3.24.

Suppose that the limiting age distribution in (3.9) is the same as the original distribution. Conclude that $F(x) = 1 - e^{-\lambda x}$ for $\lambda > 0$

SOLUTION:

Let $g(z) = \frac{P(t_i > z)}{\mathbb{E}[t_i]} = f_t(z)$ where $f_t(z)$ is the original density of t.

$P(t_i > z) = f_t(z)\mathbb{E}[(t_i)]$

$1 - P(t_i < z) = f_t(z)\mathbb{E}[(t_i)]$

$1 - f_t(z)\mathbb{E}[(t_i)] = P(t_i < z)$

$F(z) = 1 - f_t(z)\mathbb{E}[(t_i)]$

$\frac{d}{dz}F(z) = \frac{d}{dz}[1 - f_t(z)\mathbb{E}[(t_i)]]$

$f_t(z) = -\frac{d}{dz}[f_t(z)] \cdot \mathbb{E}[(t_i)]$

$f_t'(z) = \frac{-f_t(z)}{\mathbb{E}[(t_i)]}$

Thus, $\boxed{f_t(z) = exp\{-\dfrac{z}{E(t_i)}\}}$

If $E(t_i) = \lambda$ then we have $f_t(z) = ce^{-\frac{z}{\lambda}}$

However, for $f_t(z)$ to be a probability density, $c = \frac{1}{\lambda}$

So $f_t(z) = \frac{1}{\lambda}e^{-\frac{z}{\lambda}}$ which is the exponential (λ) density

$\boxed{F(z) = 1 - e^{-\frac{z}{\lambda}}}$ is the CDF of $\exp(\lambda)$.