STAT 150 HOMEWORK #3

FALL 2023

Due Friday, Sep 22nd, at 11:59 PM on Gradescope.

Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. Pinsky and Karlin, Problem 3.8.2

   Let $Z = \sum_{i=0}^{\infty} X_n$ be the total family size in a branching process whose offspring distribution has a mean $\mu = E[\zeta] < 1$. Assuming that $X_0 = 1$, show that: $E[Z] = \frac{1}{(1-\mu)}$.

   SOLUTION:

   From logic, or if you want me to cite the textbook, Pinsky (3.98), the given mean at time n is: $M(n) = \mu^n$. Thus we have: $X_n = \mu^n$

   $$E[Z] = E[\sum_{i=0}^{\infty} X_n] = \sum_{i=0}^{\infty} E[X_n] = \sum_{i=0}^{\infty} \mu^n = \frac{1}{1-\mu}$$

   This means that we are counting the total population that has ever lived.
   With $\mu < 1$ we then have the infinite geometric sum, which converges to: $\frac{1}{1-\mu}$

2. Pinsky and Karlin, Problem 3.9.5

At time 0, a blood culture starts with one red cell. At the end of 1 min, the red cell dies and is replaced by one of the following combinations with the probabilities as indicated:

2 Red	1 Red, 1 White	2 White
1/4	2/3	1/12

Each red cell lives for 1 min and gives birth to offspring in the same way as the parent cell. Each white cell lives for 1 min and dies without reproducing. Assume that individual cells behave independently.

(a) At $time = n + \frac{1}{2}$ min after the culture begins, what is the probability that no white cells have yet appeared?

When $n = \frac{1}{2}$ there is only one red cell. When $n = \frac{3}{2}$ the first cycle has taken place. The possible scenarios can be grouped by:

0 White	Either 1 or 2 White
1/4	3/4

If we have no white, then the next cycle, we will have two red, both of which can only have probability of not ending the cycle of $\frac{1}{4}$. So for both to not produce a white, the probability is $(\frac{1}{4})^2$. Iterative with this same fashion leaves us with the probability of no whites appearing at time n being:

$$\boxed{\Pi_{i=1}^{n}(1/4)^{2^{i-1}}}$$

(b) What is the probability that the entire culture eventually dies out entirely?

SOLUTION:

Using the fixed point method, we can set up the problem as a an equality using the generating function and the probability of (eventual) extinction:

$$\phi(u_\infty) = u_\infty$$
$$\phi(u) = u = \frac{1}{12} + \frac{2}{3}u + \frac{1}{4}u^2$$
$$0 = \frac{1}{12} - \frac{1}{3}u + \frac{1}{4}u^2$$
$$s = \frac{\frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{1}{12}}}{\frac{1}{2}} = 2[\frac{1}{3} \pm \sqrt{\frac{1}{36}}] = 2[\frac{1}{3} \pm \frac{1}{6}] = 1, \frac{1}{3}$$
$$\boxed{u_\infty = min\{1, \frac{1}{3}\} = \frac{1}{3}}$$

3. Pinsky and Karlin, Problem 3.9.8

   Consider a branching process whose offspring follow the geometric distribution: $p_k = (1-c)c^k$ for $k = 0, 1, ...,$ where $0 < c < 1$.
   Determine the probability of eventual extinction.

$$\phi(s) = \sum_{k=0}^{n} s^k p_k = \sum_{i=0}^{n} s^k (1-c)c^k = (1-c)\sum_{i=0}^{n}(sc)^k = \frac{(1-c)}{1-sc}$$

$$s = \frac{(1-c)}{1-sc} \rightarrow 0 = ((1-c) - s + s^2 c))$$

$$\frac{1 \pm \sqrt{1-(4(1-c)c)}}{2c} = \frac{1 \pm \sqrt{4c^2 - 4c + 1}}{2c} = \frac{1 \pm 2c - 1}{2c} = \boxed{\frac{1-c}{c}, 1}$$

$$\boxed{u_\infty = \begin{cases} \frac{1-c}{c} & \text{if } c > \frac{1}{2} \\ 1 & \text{if } c \leq \frac{1}{2} \end{cases}}$$

4. Pinsky and Karlin, Problem 3.9.10

Suppose that in a branching process the number of offspring of an initial particle has a distribution whose generating function is $f(s)$. Each member of the first generation has a number of offspring whose distribution has generating function $g(s)$. The next generation has generating function f , the next has g, and the distributions continue to alternate in this way from generation to generation.

(a) Determine the extinction probability of the process in terms of $f(s)$ and $g(s)$.

SOLUTION:
Extinction probabilities are given by: $u_{n+1} = \phi(u_n)$.
For n here we know that:

$$u_{2n+2} = fg(u_{2n})$$

Because an additional two generations gives us: fg(previous generation))
So we take the limit:

$$lim_{n\to\infty}(u_{2n+2} = fg(u_{2n}))$$
$$lim_{n\to\infty}(u_{2n+2}) = lim_{n\to\infty}(fg(u_{2n}))$$

And we take the smallest solution to:

$$u_\infty = fg(u_\infty)$$

(b) Determine the mean population size at generation n.

SOLUTION:

$$\frac{d}{ds}(fgfgfg\ldots fg(s))|_{s=1} = f'(s)g'(s)\ldots f'(s)g'(s)|_{s=1}$$

When n is even, we have equal f's and g's.
When n is odd, we have one more f than we have a g.

$$f'(1)^{\lfloor\frac{n}{2}\rfloor+(n\%2)}g'(1)^{\lfloor\frac{n}{2}\rfloor}$$

(c) Would any of these quantities change if the process started with the $g(s)$ process and then continued to alternate?

Yes,
(a) would become: $u_\infty = gf(u_\infty)$
(b) would become: $g'(1)^{\lfloor\frac{n}{2}\rfloor+(n\%2)}f'(1)^{\lfloor\frac{n}{2}\rfloor}$