# STAT 150 HOMEWORK #2

FALL 2023

**Due Friday, Sep 15, at 11:59 PM on Gradescope.**

Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. Pinsky and Karlin, Problem 3.1.4

   The random variables $\zeta_1$, $\zeta_2$,... are independent and with the common probability mass function:

| k= | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Pr{$\zeta$ =k} | .1 | .3 | .2 | .4 |

Set $X_0 = 0$, and let $X_n = max\{\zeta_1, ..., \zeta_n\}$ be the largest $\zeta$ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

SOLUTION:

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.1 & .3 & 0.2 & 0.4 \\ 1 & 0 & 0.4 & 0.2 & 0.4 \\ 2 & 0 & 0 & 0.6 & 0.4 \\ 3 & 0 & 0 & 0 & 1 \end{array} \tag{1}$$

Once we reach a higher state, we can never go down, as the function only takes the maximums. Thus, we can then tell that the matrix must cascade upward, and when the 3rd state is reached, it will be absorbed.

2. Pinsky and Karlin, Problem 3.2.4

Suppose $X_n$ is a two-state Markov chain whose transition probability matrix is:

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \alpha & 1-\alpha \\ 1 & 1-\beta & \beta \end{array} \tag{2}$$

Then $Z_n = (X_{n-1}, X_n)$ is a Markov chain having the four states $(0,0), (0,1), (1,0),$ and $(1,1)$. Determine the transition probability matrix.

SOLUTION:

$$P = \begin{array}{c|cccc} & (0,0) & (0,1) & (1,0) & (1,1) \\ \hline (0,0) & \alpha & 1-\alpha & 0 & 0 \\ (1,0) & \alpha & 1-\alpha & 0 & 0 \\ (0,1) & 0 & 0 & 1-\beta & \beta \\ (1,1) & 0 & 0 & 1-\beta & \beta \end{array} \tag{3}$$

This matrix just adds redundant information. Whatever.

3. Pinsky and Karlin, Problem 3.2.5

A Markov chain has the transition probability matrix:

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .7 & .2 & .1 \\ 1 & .3 & .5 & .2 \\ 2 & 0 & 0 & 1 \end{array} \tag{4}$$

The Markov chain starts at time zero in state $X_0 = 0$. Let: $T = min\{n \geq 0; X_n = 2\}$ be the first time the markov chain reaches state 2. Eventually, the process will be absorbed by this state. If in some experiment we observed such a process and noted that absorption had not yet taken place, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption had not yet taken place. Determine $Pr\{X_3 = 0|X_0 = 0, T > 3\}$.

The event $\{T > 3\}$ is exactly the same as the event $\{X_3 \neq 2\} = \{X_3 = 0\} \cup \{X_3 = 1\}$.

SOLUTION:

$Pr\{X_3 = 0|X_0 = 0, X_3 \neq 2\} = \frac{Pr\{X_3=0, X_3 \neq 2|X_0=0\}}{Pr\{X_3 \neq 2|X_0=0\}} = \frac{Pr\{X_3=0|X_0=0\}}{Pr\{X_3 \neq 2|X_0=0\}}$

$$P^3 = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.457 & 0.23 & 0.313 \\ 1 & 0.345 & 0.227 & 0.428 \\ 2 & 0.0 & 0.0 & 1.0 \end{array} \tag{5}$$

$\frac{Pr\{X_3=0|X_0=0\}}{Pr\{X_3 \neq 2|X_0=0\}} = \frac{P^3(0,0)}{P^3(0,(0 \text{ or } 1))} = \boxed{\frac{.457}{.457 + .23} \sim .665}$

4. Pinsky and Karlin, Problem 3.3.7

A component in a system is placed into service, where it operates until its failure, whereupon it is replaced at the end of the period with a new component having statistically identical properties, and the process repeats. The probability that a component lasts for k periods is $a_k$, for $k = 1, 2, \ldots$ . Let $X_n$ be the remaining life of the component in service at the end-of-period n. Then, $X_n = 0$ means that $X_{n+1}$ will be the total operating life of the next component. Give the transition probabilities for the Markov chain $X_n$.

SOLUTION:

When it has no life left, we draw again from the distribution of possible lives. Because it is the end of period lifespan, we could draw a lifespam of 1 with probability $a_1$, and then be left with zero life by the end of that period. Which is why $P(0,0) = a_1$

$$M = \begin{array}{c|ccccccccc} & 0 & 1 & 2 & \ldots & n-2 & n-1 & n & \ldots \\ \hline 0 & a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n & a_{n+1} & \ldots \\ 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ 2 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n-2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ n-1 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots \\ n & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{6}$$

5. Pinsky and Karlin, Problem 3.4.1

Which will take fewer flips, on average: successively flipping a quarter until the pattern $HHT$ appears, i.e., until you observe two successive heads followed by a tails; or successively flipping a quarter until the pattern $HTH$ appears? Can you explain why these are different?

SOLUTION:

I am taking the flips to be of form $[(n-2),(n-1),(n)]$, where the most recent is the leftmost. $HHT$ and $HTH$ can both be viewed as stopping states in this model. We should view each average absorption time independently.

Expectation of HHT:

$$\begin{aligned}
H &= 1 + \frac{1}{2}T + \frac{1}{2}HH \\
T &= 1 + \frac{1}{2}H + \frac{1}{2}T \\
HH &= 1 + \frac{1}{2}HH + \frac{1}{2}HHT \\
HHT &= 0 \\
HH &= 2 \\
T &= 2 + H \\
H &= 1 + \frac{1}{2}(2+H) + \frac{1}{2}(HH) \\
H &= 6, \quad T = 8
\end{aligned}$$

solve: $HHT = 1 + \frac{1}{2}H + \frac{1}{2}T = 8$

$\boxed{\tau_{HHT} = 8, \tau_{HTH} = 10}$

Expectation of HTH:

$$\begin{aligned}
H &= 1 + \frac{1}{2}H + \frac{1}{2}HT \\
T &= 1 + \frac{1}{2}H + \frac{1}{2}T \\
HT &= 1 + \frac{1}{2}T + \frac{1}{2}HTH \\
HTH &= 0 \\
T &= 2 + H \\
H &= 2 + HT \\
HT &= 1 + \frac{1}{2}(2+H) = 1 + \frac{1}{2}(4+HT) \\
HT &= 6, \quad H = 8, \quad T = 10
\end{aligned}$$

solve: $HTH = 1 + \frac{1}{2}H + \frac{1}{2}T = 10$

$$(7) \quad P = \begin{array}{c|cccc} & \text{H} & \text{T} & \text{HH} & \text{HHT} \\ \hline \text{H} & 0 & 1/2 & 1/2 & 0 \\ \text{T} & 1/2 & 1/2 & 0 & 0 \\ \text{HH} & 0 & 0 & 1/2 & 1/2 \\ \text{HHT} & 0 & 0 & 0 & 1 \end{array}$$

$$(8) \quad P = \begin{array}{c|cccc} & \text{H} & \text{T} & \text{HT} & \text{HTH} \\ \hline \text{H} & 1/2 & 0 & 1/2 & 0 \\ \text{T} & 1/2 & 1/2 & 0 & 0 \\ \text{HT} & 0 & 1/2 & 0 & 1/2 \\ \text{HTH} & 0 & 0 & 0 & 1 \end{array}$$

6. Pinsky and Karlin, Problem 3.4.2

A zero-seeking device operates as follows: If it is in state m at time $n$, then at time $n+1$, its position is uniformly distributed over the states $0, 1, ..., m-1$. Find the expected time until the device first hits zero starting from state m. Note: This is a highly simplified model for an algorithm that seeks a maximum over a finite set of points.

Matrix Representation

$$M = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 3 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \dots \\ 4 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \dots \\ 5 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{9}$$

SOLUTION:

$$\begin{aligned} w_0 &= 0 \\ w_1 &= 1 \\ w_2 &= 1 + \frac{1}{2}w_1 + \frac{1}{2}w_0 = 1\frac{1}{2} \\ w_3 &= 1 + \frac{1}{3}w_2 + \frac{1}{3}w_1 + \frac{1}{3}w_0 = 1 + \frac{1}{2} + \frac{1}{3} \\ w_4 &= 1 + \frac{1}{4}(w_1 + w_2 + w_3) = ? \\ &\vdots \end{aligned}$$

$$\boxed{w_m = 1 + \frac{1}{m}\sum_{j=1}^{m-1} w_j} \rightarrow w_{m+1} = 1 + \frac{1}{m+1}\sum_{j=1}^{m} w_j = 1 + \frac{w_m}{m+1}\sum_{j=1}^{m-1} w_j = 1 + \frac{(w_m)(w_m-1)(m)}{m+1}$$

$$w_{m+1} = 1 + \frac{(w_m)(w_m-1)(m)}{m+1}$$

$$\boxed{w_m = 1 + (w_{m-1})(w_{m-1}-1)\frac{(m-1)}{m}}$$

The answer is thus shown as a closed-form equation in the final box.

7. Pinksy and Karlin, Problem 3.4.4

Consider the Markov chain whose transition probability matrix is given by:

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & .1 & .2 & .5 & .2 \\ 2 & .1 & .2 & .6 & .1 \\ 3 & .2 & .2 & .3 & .3 \end{array} \tag{10}$$

Starting in state $X_0 = 1$, determine the probability that the process never visits state 2. Justify your answer.

SOLUTION:

$V_i := P(\text{Never hit } 2|X_0 = i) = P(\text{hit 0 before hit } 2|X_0 = i)$

V0 = 1, V2 = 0
V1 = .1*1 +.2V1 +.5*0 + .2V3 = .1 + .2V1 +.2V3
V3 = .2*1 +.2V1 +.3*0 + .3V3 = .2 + .2V1 +.3V3

(8/10)V1 = (1/10)+ (2/10)V3 → V1 = (1/8) + (2/8)V3
(7/10)V3 = (2/10) +(2/10)V1 → V3 = (2/7) +(2/7)V1
V3 = (2/7) +(2/7)((1/8) + (2/8)V3) = (2/7) +((2/56) + (4/56)V3)
(52/56)V3 = (18/56) → V3 = (56/52)(18/56) = (9/26)
V1=(1/8)+(2/8)(9/26) =(1/8)+(18/208) = (44/208)
V1 = (44/208), V3 = (72/208)
$\boxed{V1 + V3 = 116/208 = 29/52}$

ALTERNATIVE SOLUTION: Turn state 2 into an absorbing state and find the long run probabilities of the Markov chain by taking the infinite-step transition matrix (computed by python).

$$P_\infty = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & \frac{11}{52} & 0 & \frac{41}{52} & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & \frac{9}{26} & 0 & \frac{17}{26} & 0 \end{array} \tag{11}$$

Given that we have started in a transient state (1 or 2) the probability that we are absorbed into state $\boxed{0 := (11/52) + (9/26) = (29/52)}$

8. Pinsky and Karlin, Problem 3.4.10

You have five fair coins. You toss them all so that they randomly fall heads or tails. Those that fall tails in the first toss you pick up and toss again. You toss again those that show tails after the second toss, and so on, until all show heads. Let X be the number of coins involved in the last toss. Find $Pr\{X = 1\}$.

SOLUTION:

Let's say we had n fair coins. We can then see this forms binomial probabilities. For example. With n=2, we see that with a quarter probability we will see 2 again or 0, and with probability half, we will see just 1. With n=3, we see that with probability (1/8) we will see either 0 or 3, with (3/8) we will see either 1 or 2. The general formula is: $C_k = C_0\binom{k}{0}\frac{1}{2^k} + ... + C_k\binom{k}{k}\frac{1}{2^k}$ or more cleanly: $C_k = 2^{-k}\sum_{i=0}^{k} C_i\binom{k}{i}$.

The process is the (left matrix). To answer the question, we should turn 0 and 1 into absorbing states, (right matrix).

$$P = \begin{array}{c|cccccc} & C0 & C1 & C2 & C3 & C4 & C5 \\ \hline C0 & 1 & 0 & 0 & 0 & 0 & 0 \\ C1 & 1 & 0 & 0 & 0 & 0 & 0 \\ C2 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ C3 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ C4 & \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} & 0 \\ C5 & \frac{1}{32} & \frac{5}{32} & \frac{10}{32} & \frac{10}{32} & \frac{5}{32} & \frac{1}{32} \end{array} \tag{12}$$

$$P_A = \begin{array}{c|cccccc} & C0 & C1 & C2 & C3 & C4 & C5 \\ \hline C0 & 1 & 0 & 0 & 0 & 0 & 0 \\ C1 & 0 & 1 & 0 & 0 & 0 & 0 \\ C2 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ C3 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ C4 & \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} & 0 \\ C5 & \frac{1}{32} & \frac{5}{32} & \frac{10}{32} & \frac{10}{32} & \frac{5}{32} & \frac{1}{32} \end{array} \tag{13}$$

Now we can solve the linear system on the right to get probabilities for $C_0$ and $C_1$:

$C_2 = \frac{1}{3}C_0 + \frac{2}{3}C_1$

$C_3 = \frac{1}{7}C_0 + \frac{3}{7}C_1 + \frac{3}{7}\left(\frac{1}{3}C_0 + \frac{2}{3}C_1\right) \rightarrow$
$C_3 = \frac{6}{21}C_0 + \frac{15}{21}C_1$

$C_4 = \frac{1}{15}C_0 + \frac{4}{15}C_1 + \frac{6}{15}\left(\frac{1}{3}C_0 + \frac{2}{3}C_1\right) + \frac{4}{15}\left(\frac{6}{21}C_0 + \frac{15}{21}C_1\right) \rightarrow$
$C_4 = \frac{29}{105}C_0 + \frac{76}{105}C_1$

$C_5 = \frac{1}{32}C_0 + \frac{5}{32}C_1 + \frac{10}{32}C_2 + \frac{10}{32}C_3 + \frac{5}{32}C_4 + \frac{1}{32}C_5$
$C_5 = \frac{1}{31}C_0 + \frac{5}{31}C_1 + \frac{10}{31}C_2 + \frac{10}{31}C_3 + \frac{5}{31}C_4$
$C_5 = \frac{1}{31}C_0 + \frac{5}{31}C_1 + \frac{10}{31}(\frac{1}{3}C_0 + \frac{2}{3}C_1) + \frac{10}{31}(\frac{6}{21}C_0 + \frac{15}{21}C_1) + \frac{5}{31}(\frac{29}{105}C_0 + \frac{76}{105}C_1)$

$$\boxed{C_5 = \frac{60}{217}C_0 + \frac{157}{217}C_1}$$

ALTERNATIVE METHOD: Take $P^{\infty}$ and check the probabilities.

9. Pinsky and Karlin, Problem 3.4.14 (Hint: try to reduce the size of the state space) A single die is rolled repeatedly. The game stops the first time that the sum of two successive rolls is either 5 or 7. What is the probability that the game stops at a sum of 5?

$$P = \begin{array}{c|cccccccc} & D1 & D2 & D3 & D4 & D5 & D6 & S5 & S7 \\ \hline D1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ D2 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ D3 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ D4 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ D5 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\ D6 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\ S5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ S7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \tag{14}$$

Not going to solve this one out by hand. Straight to the computational method. As discussed in section (1 and 2), (3 and 4), and (5 and 6) can be grouped together, because these states have the same transition probabilities to each other and themselves. Hoever, I do not feel comfortable using this method and I just kept the matrix as 8x8.

I asked chatGPT to transform my Latex into Python. I coded 3.92 in Pinksy here:

def computeB(Q, R):
I = np.identity(Q.shape[0])    N = np.linalg.inv(I - Q)    B = np.dot(N, R)

$$Q = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} \qquad R = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{6} \\ 0 & \frac{1}{6} \end{bmatrix}$$

B = computeB(Q, R) print(B)

Here is the output, specifying the proportions, given the starting state:

$$B = \begin{bmatrix} 0.56862745 & 0.56862745 \\ 0.56862745 & 0.56862745 \\ 0.58823529 & 0.58823529 \\ 0.58823529 & 0.58823529 \\ 0.68627451 & 0.68627451 \\ 0.68627451 & 0.68627451 \end{bmatrix}$$

Summing the rows gives us: [2.31372549,    3.68627451] which is proportionally: $\boxed{[0.386, \quad 0.614]}$
And thus the probability that the game stops at 5 is .386