# STAT 150 HOMEWORK #5

FALL 2023

**Due Friday, Sep 29, at 11:59 PM on Gradescope.**

Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. Pinsky and Karlin, Problem 4.3.2

   Show that: finite-state aperiodic irreducible Markov chain $\rightarrow$ regular and recurrent.

   Definition, *Period*:
   The period of a DTMC is the GCD of all integers $n_i \geq 1$ such that $P_{ii}^{(n_i)} \geq 0$

   Definition, *Aperiodic*:
   The selection mechanism of the integers is represented by the 'function' $f$ on the matrix, $A$, then the period is: $GCD(f(A))$. To be aperiodic: $GCD(f(A)) = 1$
   Which means that every state can return to itself in one step.

   Definition, *Irreducible*:
   The markov chain only has one equivalence class.
   Every state communicates with every other state in some number of finite steps.
   Formally: All $P_{ij}^{n_{ij}} > 0$ for some $n_{ij} \geq 0$

   PROOF (Regularity):
   Take $m = max\{n_{ij}\}$ for all $i, j \in A$ Since the Markov Chain is finite, $m$ is well-defined.

   Given, $P^{(m)}$, we know by an editted for of the Chapman-Kolmogorov equation, each element: $P_{ik}^{(m)}$ is positive: $P_{ik}^{(m)} = \sum_k P_{jk}^{(m-n_{ij})} P_{ij}^{(n_{ij})}$

   Edit this form to $j = k$ and we can then see why it is true: $P_{ij}^{(m)} = \sum_j P_{jj}^{(m-n_{ij})} P_{ij}^{(n_{ij})}$

   By irreducibility, $P_{ij}^{(m)} > 0$ and by aperiodicity: $P_{jj}^{(m-n_{ij})} > 0$.

   In words, we are guaranteed to get from i to j in some finite number of steps, n. Once we are there, we can stay at spot j with nonzero probability. If you stay at spot j, (m-n) times, then you will still be at spot j. This can be done for every (i,j) pair, and thus we can see that irreducibility and aperiodicity result in regularity.
   This proves that there exists an $m$ such that all the entries of $P^{(m)}$ are positive, proving that the Markov chain is regular.

PROOF (Recurrent (From Lecture)):

Using Collary 4.1 we know that if i communicates with j, and if i is recurrent, then j is recurrent. We know all states communicate with each other in a regular DTMC, so we just need to show that a single state i is recurrent.

By regularity: $P_{(i,i)}^{(n)} > 0$ but further, because $P_{(i,i)} = \pi_i$, then $P_{(i,i)}^{(n)} = \pi_i$

Thus, an infinite sum of constant term, $\pi_i$ is infinite: $\sum_{n=0}^{\infty} \pi_i = \pi_i lim_{n\to\infty} n = \infty$

Thus state i is recurrent and due to irreducibility all states j are recurrent.

2. Pinsky and Karlin, Problem 4.4.1

Consider the Markov chain on 0, 1 whose transition probability matrix is:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \beta & 1-\beta \end{array} \quad \text{where } \alpha > 0, 1 > \beta$$

(a) Verify that $(\pi_0, \pi_1) = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$ is a stationary distribution.

ALGEBRA:

$\pi_0 = \pi_0(1-\alpha) + \pi_1(\beta) \to \pi_1(\beta) = \pi_0 - \pi_0(1-\alpha) \to \pi_1 = \frac{\pi_0(\alpha)}{(\beta)}$

$1 = \pi_0 + \pi_1 = \pi_0 + \frac{\pi_0(\alpha)}{(\beta)} = \pi_0(1 + \frac{(\alpha)}{(\beta)}) = \pi_0(\frac{(\alpha+\beta)}{(\beta)}) \to \pi_0 = (\frac{(\beta)}{(\beta+\alpha)})$

$\pi_1 = 1 - (\frac{(\beta)}{(\beta+\alpha)}) = \frac{(\alpha)}{(\beta+\alpha)}$

$QED\square$

(b) Show that the first return distribution to state 0 is given by:

$f_{00}^{(1)} = (1-\alpha)$ and $f_{00}^{(n)} = \alpha\beta(1-\beta)^{n-2}$ for $n = 2, 3, \ldots$.

LOGIC: 3 cases, $n = 1, n = 2, n22$

When $n = 1$, stay at state 0 with Probability: $(1-\alpha)$

When $n = 2$, depart state 0 with Probability: $(\alpha)$ and come back immediately with Probability: $(\beta)$

When $n > 2$, depart state 0 with Probability: $(\alpha)$, stay at state 1, n-2 times with Probability: $(1-\beta)^{n-2}$

(minus two because we need to spend one turn going and one turn returning) and come back after some time with Probability: $(\beta)$

ALGEBRA:

$f_{00}^{(1)} = (P_{0,0}) = (1-\alpha)$

$f_{00}^{(2)} = (P_{0,0}^2|X_1 \neq 0) = (\alpha)(\beta)$

$f_{00}^{(3)} = (P_{0,0}^3|X_2, X_1 \neq 0) = (\alpha)(1-\beta)(\beta)$

$\vdots$

$f_{00}^{(n)} = (P_{0,0}^n|X_{n-1}, \ldots, X_1 \neq 0) = (\alpha)(1-\beta)^{n-2}(\beta)$

(c) Calculate the mean return time $m_0 = \sum_{n=1}^{\infty} n f_{00}^{(n)}$ and verify that $\pi_0 = 1/m_0$.

$m_0 = (1-\alpha) + \sum_{n=2}^{\infty} n(\alpha)(1-\beta)^{n-2}(\beta) = (1-\alpha) + \frac{(\alpha\beta)}{(1-\beta)} \sum_{n=2}^{\infty} n(1-\beta)^{n-1}$

$S = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x} \to \frac{dS}{dx} = \frac{1}{(1-x)^2}$ and thus: $m_0 = (1-\alpha) + \frac{(\alpha\beta)}{(1-\beta)}(\frac{1}{(\beta)^2} - 1) \to$ $m_0 = (1-\alpha) + \frac{(\alpha)(1-\beta^2)}{(1-\beta)(\beta)} = (1-\alpha) + \frac{(\alpha)(1-\beta)(1+\beta)}{(1-\beta)(\beta)} = \frac{(1-\alpha)(\beta)}{(\beta)} + \frac{(\alpha)(1+\beta)}{(\beta)} = \frac{(1-\alpha)(\beta)+(\alpha)(1+\beta)}{(\beta)}$

PROVED $\boxed{m_0 = \dfrac{\beta+\alpha}{\beta}}$ and $m_0 = pi_0^{-1}$

3. Pinsky and Karlin, Problem 4.4.4 (*Do not assume that the Markov chain is irreducible*)

Let $\{a_i : i = 1, 2, \ldots\}$ be a probability distribution, and consider the Markov chain whose transition probability matrix is:

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \ldots \\ \hline 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 2 & 0 & 1 & 0 & 0 & 0 & \\ 3 & 0 & 0 & 1 & 0 & 0 & \\ 4 & 0 & 0 & 0 & 1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

What condition on the probability distribution $\{a_i : i = 1, 2, \ldots\}$ is necessary and sufficient in order that a limiting distribution exist, and what is this limiting distribution? Assume $a_1 0 > 0$ and $a_1 > 0$ so that the chain is aperiodic.

CAUTION FOR GRADER: Changed indexes to start from zero

PROOF:

GIVEN: If the chain is aperiodic, then we have an irreducible class.

Two cases:
(1) The chain is irreducible.
(2) The chain is not irreducible.

(1) If the chain is irreducible, then all states communicate with each other.
A Necc and Suff condition for irreducible is: $\{i : a_i > 0\}$ where i is unbounded.
If one state is recurrent, then all states are recurrent.
Clear to see that zero is recurrent if the distribution of a's has a finite mean.

Because the return time will just be that number, this probability will be:
$m_0 = \mathbb{E}\{R_0|X_0\} = \sum_{k=1}^{\infty} a_k k < \infty$

Giving us: $\pi_i = \frac{1}{m_i}$ where all $\pi_i$ sum to 1, by Thrm 4.3

(2) If the chain is not irreducible, we know that we at least have an aperiodic, irreducible class including at least states 0 and 1. We necessarily come back to this class, meaning that we have a limiting distribution in this class, where all other states outside go to 0. For us to not be irreducible, then
$\exists n^*$ st $a_n = 0$ for all $n \geq n^*$ for some i

In this case, the distribution of a's has a finite mean and our condition is again satisfied.

The limiting distribution:

$\pi_0 = a_0\pi_0 + \pi_1$
$\pi_1 = a_1\pi_0 + \pi_2$
$\pi_2 = a_2\pi_0 + \pi_3$
$\vdots$
$\pi_n = a_n\pi_0 + \pi_{n+1}$

Rearrange

$\pi_1 = \pi_0 - a_0\pi_0 = \pi_0(1 - a_0)$
$\pi_2 = \pi_1 - a_1\pi_0 = \pi_0(1 - a_0) - a_1\pi_0$
$\pi_3 = \pi_2 - a_2\pi_0 = \pi_0(1 - a_0) - a_1\pi_0 - a_2\pi_0$
$\vdots$
$\pi_{n+1} = \pi_n - a_n\pi_0 = \pi_0(1 - \sum_{k=0}^{n} a_k)$

For us to have a distribution, these need to sum to 1:

$1 = \sum_{j=0}^{\infty} \pi_j = \pi_0 + \pi_0(1 - a_0) + \pi_0(1 - a_0) - a_1\pi_0 + \ldots + \pi_0(1 - \sum_{k=0}^{n} a_k) + \ldots$
$1 = \pi_0(1 + (1 - a_0) + (1 - a_0) - a_1 + \ldots + (1 - \sum_{k=0}^{n} a_k) + \ldots)$
$1 = \pi_0(1 + \sum_{j=0}^{\infty}(1 - \sum_{k=0}^{j} a_k)) = \pi_0(1 + \sum_{j=0}^{\infty}(\sum_{k=j+1}^{\infty} a_k)) = \pi_0(1 + \sum_{j=0}^{\infty} ja_j) \rightarrow$

$\pi_0 = \frac{1}{(1+\sum_{j=0}^{\infty} ja_j))}$

WOAHHHH: Here we have that $\pi_0$ is only positive recurrent if the mean is finite, which we can see in the denominator!!!!! This is so cool!!!

Note that the mean is $\sum_{j=0}^{\infty}(ja_j)$

4. Pinsky and Karlin, Problem 4.4.5

Let $P$ be the transition probability amtrix of a finite-state regular MC. Let $M \sim ||m_{i,j}||$ be the matrix of mean return times.

(a) use a first step argument to establish that:

$m_{i,j} = 1 + \sum_{k \neq j} P_{i,k} m_{k,j}$

METHOD: Do first step, sum across all possible first steps.
ANSWER:

$m_{ij} = \mathbb{E}\{R_j | X_0 = i\}$ where $R = min\{X_n = j, n \geq 1\}$
$m_{ij} = \sum_{k \neq j} \mathbb{E}\{R_j | X_1 = k\} P(X_1 = k | X_0 = i)$
$m_{ij} = \sum_{k \neq j} \mathbb{E}\{R_j | X_1 = k\} P_{ik}$
$m_{ij} = \sum_{k \neq j} m_{kj} P_{ik}$

(b) Then, multiply both sides of the preceding by $\pi_i$ and sum to obtain:

$\sum_i \pi_i m_{i,j} = \sum_i \pi_i + \sum_i \sum_{k \neq j} \pi_i P_{ik} m_{kj}$

Then simplify this to show:

$\pi_j m_{j,j} = 1$ or $\pi_j = \frac{1}{m_{jj}}$

ALGEBRA:

$\sum_i \pi_i m_{i,j} = \sum_i \pi_i + \sum_i \pi_i P_{ik} m_{kj} \rightarrow$
$\sum_i \pi_i m_{i,j} = 1 + \sum_{k \neq j} m_{kj} \sum_i \pi_i P_{ik} \rightarrow$
$\sum_i \pi_i m_{i,j} = 1 + \sum_{k \neq j} m_{kj} \pi_k \rightarrow$

On the left side, we can seperate into the single case where i=j.
We can change the index on the right:

$\sum_{i \neq j} \pi_i m_{i,j} + \pi_j m_{j,j} = 1 + \sum_{i \neq j} m_{ij} \pi_k$

PROVED: $\boxed{\pi_j m_{j,j} = 1}$

5. Pinsky and Karlin, Problem 4.4.8

A Markov chain on states $0, 1, ...$ has transition probabilities:

$P_{ij} = \frac{1}{i+2}$ for $j = 0, 1, 2, \ldots i, i+1$

OBJECTIVE: Find the stationary distribution
ANSWER: The distribution will be $Poisson(\lambda = 1)$

MATRIX REPRESENTATION:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & \vdots \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

$\pi_0 = (1/2)\pi_0 + (1/3)\pi_1 + (1/4)\pi_2 + (1/5)\pi_3 + \ldots$
$\pi_1 = (1/2)\pi_0 + (1/3)\pi_1 + (1/4)\pi_2 + (1/5)\pi_3 + \ldots$
$\pi_2 = (1/3)\pi_1 + (1/4)\pi_2 + (1/5)\pi_3 + (1/6)\pi_4 \ldots$
$\vdots$
$\pi_n = (1/(n+2))\pi_n + (1/(n+3))\pi_{n+1} + \ldots$

Recognize that $\pi_1 = \pi_0$ and that we can change the equation.

$\pi_0 = \pi_1$
$\pi_2 = \pi_1 - (1/2)\pi_0$
$\pi_3 = \pi_2 - (1/3)\pi_1$
$\pi_4 = \pi_3 - (1/4)\pi_2$
$\vdots$
$\pi_n = \pi_{n-1} - (1/n)\pi_{n-2}$

See that:
$\pi_2 = \pi_0 - (1/2)\pi_0 = (1/2)\pi_0$
$\pi_3 = (1/2)\pi_0 - (1/3)\pi_0 = (1/6)\pi_0$
$\pi_4 = \pi_3 - (1/4)\pi_2 = (1/6)\pi_0 - (1/8)\pi_0 = (1/12)\pi_0$
$\vdots$
$\pi_n = \frac{1}{n!}\pi_0$

$1 = \sum_{k=0}^{\infty} \pi_k = 2\pi_0 + \sum_{k=2}^{\infty} \pi_k = 2\pi_0 + \sum_{k=2}^{\infty} \frac{1}{n!}\pi_0 \rightarrow \pi_0(2 + \sum_{k=2}^{\infty} \frac{1}{n!}) = 1$

$\pi_0 = (2 + \sum_{k=2}^{\infty} \frac{1}{n!})^{-1} = (\sum_{k=0}^{\infty} \frac{1}{n!})^{-1} = e^{-1}$

$$\boxed{\pi_k = (k!)^{-1}(e)^{-1} = \frac{\lambda^x e^{-\lambda}}{k!} \text{ when } \lambda = 1}$$