STAT 150 HOMEWORK #6

FALL 2023

Due Friday, Oct 13th, at 11:59 PM on Gradescope.

Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. Pinsky and Karlin, Problem 5.1.6

   Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. For $s, t > 0$, determine the conditional distribution of $X(t)$, given that $X(t+s) = n$.

   SOLUTION:

   $P(X(t) = k | X(t{+}s) = n) = \frac{P(X(t)=k, X(t+s)=n)}{P(X(t+s)=n)} = \frac{P(X(t)=k, X(s)=n-k)}{P(X(t+s)=n)} = \frac{P(X(t)=k)P(X(s)=n-k)}{P(X(t+s)=n)} =$
   $\frac{e^{-\lambda t}(\lambda t)^k}{k!} * \frac{e^{-\lambda s}(\lambda s)^{(n-k)}}{(n-k)!} * \frac{n!}{e^{-\lambda(t+s)}(\lambda(t+s))^n} = \frac{n!}{k!(n-k)!} * \frac{e^{-\lambda t}e^{-\lambda s}}{e^{-\lambda(t+s)}} * \frac{\lambda^k \lambda^{n-k}}{\lambda^n} * \frac{(t)^k (s)^{n-k}}{(t+s)^n} =$
   $\frac{n!}{k!(n-k)!} * (\frac{t}{t+s})^k * (\frac{s}{t+s})^{n-k} = \boxed{\binom{n}{k}(p)^k(1-p)^{n-k}}$ where $p = \frac{t}{t+s}$

   Thus, the conditional distribution of $X(t)$ is binomial. This makes sense because if we predefine a number of events, and there are two possible buckets that the events might end up in, all we should care about is the relative size of the buckets. Which are governed by the relative time periods, s and t.

2. Pinsky and Karlin, Problem 5.1.7

Shocks occur to a system according to a Poisson process of rate λ. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time t?

SOLUTION:

Define $E(t)$ as the chance of being alive at time t.

$P(E(t)|N(t) = k) = \alpha^k$
$P(E(t)) = \sum_{k=0}^{\infty}(P(E(t)|N(t) = k)P(N(t) = k))$
$P(E(t)) = \sum_{k=0}^{\infty}(\alpha^k \frac{e^{-\lambda t}(\lambda t)^k}{k!}) = e^{-\alpha}e^{\alpha} \sum_{k=0}^{\infty}(\alpha^k \frac{e^{-\lambda t}(\lambda t)^k}{k!}) = e^{\alpha} \sum_{k=0}^{\infty}(\frac{e^{-\lambda t \alpha}(\lambda t \alpha)^k}{k!}) =$ $\boxed{e^{\alpha}}$

3. Pinsky and Karlin, Problem 5.1.10 (Hint: the facility dispatches even if 0 arrivals) Customers arrive at a facility at random according to a Poisson process of rate λ. There is a waiting time cost of c per customer per unit time. The customers gather at the facility and are processed or dispatched in groups at fixed times $T, 2T, 3T, \ldots$ There is a dispatch cost of K. The process is depicted in the following graph.
(a) What is the total dispatch cost during the first cycle from time 0 to time T?
(b) What is mean total customer (waiting)cost during the 1st cycle?
(c) What is mean total customer (waiting+dispatch)cost/unit time during 1st cycle?
(d) What value of T minimizes this mean cost per unit time?

SOLUTION (a): The dispatch happens at times nT.
Thus, it only happens once between 0 and T. The answer is just $\boxed{\text{K}}$

SOLUTION (b): Sum up all the possible costs of every possible occurrence. Call w_i the occurrences. Thus, the cost for the first person is $c(T - w_1)$ the cost for the second is $c(T - w_2)$ and so on. We can represent this as: $c\sum_{i=1}^{N(t)}(T - w_i)$.

Though T is deterministic, $N(T)$ is random. We can get around this by taking the expectation and conditioning on $N(T) = n$. We want: $E(h(N(t))) = E(E(h(n)))$

$h(n) = E(c\sum_{i=1}^{N(T)}(T - w_i)|N(T) = n) = E(c\sum_{i=1}^{n}(T - w_i))$

We can use the fact that: $(w_1 \ldots w_{N(t)}|N(T) = n) \stackrel{d}{=} (u_{(1)} \ldots u_{(n)})$
which are order statistics from Uniform $[0, T]$
$E(c\sum_{i=1}^{N(T)}(T - w_i)|N(T) = n) = E(c\sum_{i=1}^{n}(T - w_i)) = E(c\sum_{i=1}^{n}(T - u_{(i)}))$

But the ordered version is just one possible of any possible ordering of the sum, so we can just take the generic, not ordered version, denoted by u_i.
Then, denote $v_i = T - w_i$ and we have that v_i is also Uniform $[0, T]$

$E(c\sum_{i=1}^{N(T)}(T - w_i)|N(t) = n) = E(c\sum_{i=1}^{n}(T - u_i)) = E(c\sum_{i=1}^{n}(v_i)) = c\sum_{i=1}^{n}E(v_i)$

The expectation is $\frac{T}{2}$ and thus: $\boxed{h(n) = \frac{cnT}{2}}$

$N(T) \stackrel{d}{=} Poisson(\lambda T)$ so we can find the final answer: $\boxed{E(h(n)) = E(\frac{cN(T)T}{2}) = \frac{c\lambda T^2}{2}}$

SOLUTION (c): The total is the sum of the two terms: $\frac{K}{T} + \frac{c\lambda T^2}{2T} = \boxed{\frac{K}{T} + \frac{c\lambda T}{2}}$

SOLUTION (d): Take the first order condition: $\frac{d}{dT}(\frac{K}{T} + \frac{c\lambda T}{2}) = 0$

$\boxed{\frac{K}{T^2} = \frac{c\lambda}{2} \to T = \sqrt{\frac{2K}{c\lambda}}}$

We know this is a minimum because: $\frac{d}{dT}(\frac{-K}{T^2} + \frac{c\lambda}{2}) = \frac{2K}{T^3} = 2K * \left(\sqrt{\frac{c\lambda}{2K}}\right)^3 > 0$

4. Pinsky and Karlin, Problem 5.1.11

Assume that a device fails when a cumulative effect of k shocks occurs. If the shocks happen according to a Poisson process of parameter λ, what is the density function for the life T of the device?

SOLUTION:

This is the exact setting where waiting times of Poisson Processes are distributed by the Gamma distribution. We can use Theorem 5.4

The waiting time W_n has the gamma distribution whose probability density function is: $f_{W_k}(t) = \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}$

The probability that the k-th shock appears before time t is equal to the probability that the poisson process at time t has had more than k arrivals.

$F_{W_k}(t) = Pr(W_k \leq t) = Pr(X(t) > k)$
$F_{W_k}(t) = \sum_{k=n}^{\infty} \frac{e^{-\lambda t} \lambda^{kt}}{k!}$
$f_{W_k}(t) = \frac{d}{dt} F_{W_n}(t) = \frac{d}{dt}(1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t} \lambda^{kt}}{k!})$
$f_{W_k}(t) = -\frac{d}{dt}\left(\sum_{n=0}^{k-1} \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right)$
$f_{W_k}(t) = -\sum_{n=0}^{k-1} \frac{d}{dt}\left(\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right)$
$f_{W_k}(t) = -\sum_{n=0}^{k-1} \frac{1}{n!} \frac{d}{dt}\left((\lambda t)^n e^{-\lambda t}\right)$
$f_{W_k}(t) = -\sum_{n=0}^{k-1} t^n \frac{1}{n!} \frac{d}{dt}\left(\lambda^n e^{-\lambda t}\right)$
$f_{W_k}(t) = -\sum_{n=0}^{k-1} t^n \frac{1}{n!}\left[n\lambda^{n-1} e^{-\lambda t} - \lambda e^{-\lambda t} \lambda^n\right]$
$f_{W_k}(t) = -\sum_{n=0}^{k-1} t^n \frac{1}{n!} e^{-\lambda t}\left[n\lambda^{n-1} - \lambda^{n+1}\right]$

Distribute the (-)

$f_{W_k}(t) = \sum_{n=0}^{k-1} t^n \frac{1}{n!} e^{-\lambda t}\left[\lambda^{n+1} - n\lambda^{n-1}\right]$

At zero, there is no term. We use telescoping, except this time only the largest term remains. Substitute $k = (n-1)$ to the final positive term.

$f_{W_k}(t) = t^{k-1} \frac{1}{(k-1)!} e^{-\lambda t} \lambda^n$

$$\boxed{f_{W_k}(t) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \sim Gamma(k, \lambda)}$$

5. Pinsky and Karlin, Problem 5.2.11

Let X and Y be jointly distributed random variables and B an arbitrary set. Fill in the details that justify the inequality: $|Pr\{X \in B\} - Pr\{Y \in B\}| \leq Pr\{X \neq Y\}$.

SOLUTION:

Obvious from total law of probability:

$$Pr(X \in B) = Pr(X \in B, X = Y) + Pr(X \in B, X \neq Y)$$
$$Pr(Y \in B) = Pr(Y \in B, X = Y) + Pr(Y \in B, X \neq Y)$$

If $X = Y$ then $Pr(X \in B, X = Y) = Pr(Y \in B, X = Y)$ so that:

$$Pr(X \in B) - Pr(Y \in B) = Pr(X \in B, X \neq Y) - Pr(Y \in B, X \neq Y)$$

Because probabilities are non-negative:

$$Pr(X \in B) - Pr(Y \in B) = Pr(X \in B, X \neq Y) - Pr(Y \in B, X \neq Y)$$

$$and Pr(X \in B, X \neq Y) - Pr(Y \in B, X \neq Y) \leq Pr(X \in B, X \neq Y)$$

Because: $P(A \cap B) \leq P(A)$ we thus have:

$$\boxed{Pr(X \in B) - Pr(Y \in B) \leq Pr(X \neq Y)}$$

THIS FINISHES THE PROOF

Because the statement is symmetric in X and Y, and we have already proved: $Pr(X \in B) - Pr(Y \in B) \leq Pr(X \neq Y)$ nothing really stops us from re-doing the proof in the other direction.

6. Pinsky and Karlin, Problem 5.2.10
(this is not a typo, I mean for you to do Problem 5.2.11 first)

Establish: $|Pr(S_n \in I) - Pr(X(\mu) \in I)| \leq \sum_{i=1}^{n}(p_i)^2$

For any set of non-negative integers I.

Preliminary Idea (THIS IS WRONG, GRADER SKIP THIS PART):

We have proved: $|Pr(S_n \in I) - Pr(X(\mu) \in I)| \leq Pr(S_n \neq X(\mu))$

So proving: $\sum_{i=1}^{n}(p_i)^2 \leq Pr(S_n \neq X(\mu))$ Is sufficient.
(This is wrong. IGNORE THIS SECTION)

GIVEN:
$S_n = \sum_{i=1}^{n} \varepsilon_i$, where $\varepsilon_i \sim$ Bernoulli(p)
$X(\mu) = \sum_{i=1}^{n} z_i$, where $z_i \sim$ Poisson$(\frac{\mu}{n})$

SOLUTION:

Show: $S_n \neq X(\mu) \subseteq \bigcup_{i=1}^{n}\{\varepsilon_i \neq z_i\}$

Because we can define the joint distribution ('coupling technique') such that:

$P\{\varepsilon_i \neq z_i\} \leq p^2$

We have $A \subseteq B$ so $\rightarrow B^c \subseteq A^c$
Below, LHS is true because if each component $\varepsilon_i = z_i$ and $S_n = X(\mu)$ because each is the sum of the individual components.

$$(S_n = X(\mu)) \supseteq \bigcap_{i=1}^{n}\{\varepsilon_i = z_i\} \rightarrow (S_n \neq X(\mu)) \subseteq \bigcup_{i=1}^{n}\{\varepsilon_i \neq z_i\}$$

Thus

$$|P[S_n \in I] - P[X(\mu) \in I]| \leq \sum_{i=1}^{n} P\{\varepsilon_i \neq z_i\}$$

But if each individual piece satisfies the inequality
then the sum also obeys the inequality

$$\boxed{|P[S_n \in I] - P[X(\mu) \in I]| \leq \sum_{i=1}^{n} p^2}$$