STAT 150 HOMEWORK #7

FALL 2023

Due Friday, Oct 20th, at 11:59 PM on Gradescope.

Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. Pinsky and Karlin, Problem 5.3.6

   Customers arrive at a holding facility at random according to a Poisson process having rate λ. The facility processes in batches of size Q.
   That is, the first $(Q-1)$ customers wait until the arrival of the Qth customer.
   Then, all are passed simultaneously, and the process repeats.
   Service times are instantaneous.
   Let $N(t)$ be the number of customers in the holding facility at time t.
   Assume that:
   $N(0)$ =0 and let $T = min\{t \geq 0 : N(t) = Q\}$ be the first dispatch time.
   Show that:
   $E[T] = \frac{Q}{\lambda}$ and $E[\int_0^T N(t)dt] = [1+2+3+4+5+\ldots+(Q-1)]/\lambda = (Q)(Q-1)]/2\lambda$

   SOLUTION:

   Using the fact that waiting times for Poisson Processes: W_Q are $Gamma(Q, \lambda)$
   (We know that the sume of n exponentials with the same rate λ is $Gamma(n, \lambda)$)
   $E[W_Q] = E[Gamma(Q, \lambda)] = [\frac{Q}{\lambda}]$ then we have readily proved the first part.
   $W_Q = T$ and thus $\boxed{E[T] = [\frac{Q}{\lambda}]}$

   Integrating the total number of customers in the holding facility, $N(t)$ across time is equivalent to taking the total sum of time waited by the entire population until T. The first customer arrives at $S_1 = \frac{1}{\lambda}$ and waits $T - \frac{1}{\lambda}$ units of time. The second customer arrives at $S_2 = \frac{2}{\lambda}$ and waits $T - \frac{2}{\lambda}$ units of time. So on and so forth, until the second to last customer waits $T - \frac{Q-1}{\lambda}$ units of time, and the last customer does not wait at all. This sum is:

   $E[\int_0^T N(t)dt] = E[\sum_{k=1}^{Q}(T - S_k)] = E(T - \frac{1}{\lambda}) + E(T - \frac{2}{\lambda}) + \ldots + E(T - \frac{Q-1}{\lambda}) =$
   $(Q-1)E(T) - (1+2+\ldots+Q-1)/\lambda = \frac{(Q)(Q-1)}{\lambda} - \frac{(Q)(Q-1)}{2\lambda} = \frac{(Q)(Q-1)}{2\lambda}$

   $$\boxed{E[\int_0^T N(t)dt] = \frac{(Q)(Q-1)}{2\lambda}}$$

2. Pinsky and Karlin, Problem 5.3.8 (Note: do not simply cite the density of a beta distribution. The point is to derive the density from properties of the Poisson distribution.)

   Consider a Poisson process with parameter λ. Given that $X(t) = n$ events occur in time t, find the density function for W_r, the time of occurrence of the rth event. Assume that $r \leq n$.

   SOLUTION:

   In the notation below, T_i represents inter-arrival times.

   $W_r = T_1 + T_2 + \ldots + T_r$ and $t = T_1 + T_2 + \ldots + T_n$

   $P(W_r > w|N(t) = n) =$
   $P(N(w) < r|N(t) = n) =$
   $P(N(w) < r \text{ and } N(t) = n)/P(N(t) = n) =$

   We proved last week that this is binomial (PK in 5.1.6) The only difference this time is that we are taking the CDF across the variable r.

   $P(W_r > w|N(t) = n) = \sum_{k=0}^{r-1} \binom{n}{k} (\frac{w}{t})^k (1 - \frac{w}{t})^{n-k}$
   $P(W_r = w|N(t) = n) = \frac{d}{dw}(1 - P(W_r > w|N(t) = n)) = -\frac{d}{dw} \sum_{k=0}^{r-1} \binom{n}{k} (\frac{w}{t})^k (1 - \frac{w}{t})^{n-k} =$

   By the product rule:

   $-\sum_{k=0}^{r-1} \frac{n!}{(n-k)!(k)!} [\frac{k}{t}(\frac{w}{t})^{k-1}(t) \cdot (1 - \frac{w}{t})^{n-k} + (\frac{w}{t})^k \cdot (-\frac{1}{t})(n-k)(1 - \frac{w}{t})^{n-k-1}]$

   $\sum_{k=0}^{r-1} \frac{n!}{(n-k)!(k)!} [(\frac{w}{t})^k \cdot (\frac{1}{t})(n-k)(1 - \frac{w}{t})^{n-k-1} - \frac{k}{t}(\frac{w}{t})^{k-1}(t) \cdot (1 - \frac{w}{t})^{n-k}]$

   This is a telescoping sum where the first (negative) term is zero. So we can just take the positive element of the last term $k = (r-1)$. To see the telescoping in action, you really have to take apart the combiantorics term, so I won't do this.

   $\frac{n!}{(n-(r-1))!(r-1)!} [(\frac{w}{t})^{r-1} \cdot (\frac{1}{t})(n-(r-1))(1 - \frac{w}{t})^{n-(r-1)-1}] =$
   $\frac{n!}{(n-(r-1)-1)!(r-1)!} [(\frac{w}{t})^{r-1} \cdot (\frac{1}{t})(1 - \frac{w}{t})^{n-(r-1)-1}] =$
   $\frac{n!}{(n-r)!(r-1)!} [(\frac{w}{t})^{r-1}(\frac{1}{t})(1 - \frac{w}{t})^{n-r}] =$
   $[\frac{(n)!}{(n-r)!(r-1)!}(\frac{w}{t})^{r-1}(1 - \frac{w}{t})^{n-r}] \cdot [(\frac{1}{t})] = \boxed{f_{W_r|X(t)=n}(w) \sim Beta(r, n-r+1) \cdot [(\frac{1}{t})]}$

   This is just a scaled Beta Distribution. Imagine an interval t and you're placing objects on the interval. Place $(r-1)$ objects in $[0, w)$, 1 object in $[w, w + \Delta w)$ and $n - r$ objects in $[t, w + \Delta w)$. These are unordered objects, so you can work out the combinatorics after.

3. Pinsky and Karlin, Problem 5.4.1 (Note: this requires the assigned reading I mentioned on the schedule, PK 2.4. Alternatively, if you understand conditional densities, then you will be fine.)

   Let $W_1, W_2, ...$ be the event times in a Poisson process $X(t); t \geq 0$ of rate λ. Suppose it is known that $X(1) = n$. For $k < n$, what is the conditional density function of $W_1, ..., W_{k-1}, W_{k+1}, ..., W_n$, given that $W_k = w$?

   SOLUTION:

   Use the previous part and the knowledge of conditioning on densities:

   $f_{(W_1 \ldots W_{k-1}, W_{k+1} \ldots W_n | W_k, N(t)=n)}(w_1 \ldots w_{k-1}, w_{k+1} \ldots w_n) = f_{(W_1 \ldots W_n | N(t)=n)}(w_1 \ldots w_n) / f_{(W_k | N(t)=n)}(w_k)$

   $f = \frac{n!}{t^n} \cdot (1/(Beta(k, n-k+1) \cdot [(\frac{1}{t})]))$

   $f = \frac{n!}{t^n} \cdot [\frac{(n-k)!(k-1)!}{(n)!}(\frac{t}{w})^{k-1}(\frac{t}{t-w})^{n-k}] \cdot [t]$

   $f = \frac{1}{t^{n-1}} \cdot [(n-k)!(k-1)!(\frac{t}{w})^{k-1}(\frac{t}{t-w})^{n-k}]$

   $f = [(n-k)!(k-1)!(\frac{1}{w})^{k-1}(\frac{1}{t-w})^{n-k}]$

   Here, t = 1, so we can treat w as a probability, $w \in (0, 1)$

   $$f = \boxed{[(n-k)!(k-1)!(\frac{1}{w})^{k-1}(\frac{1}{1-w})^{n-k}]}$$

   We can keep messing with it to try to get something interesting, but there is no need. Stick with the boxed answer I guess.

   $f = (n-1)![\binom{n-1}{k-1}(w)^{k-1}(t-w)^{n-k}]^{-1}$

   $f = (n-1)!/Binomial(w, n-1)$

4. Pinsky and Karlin, Problem 5.4.4
Let $W_1, W_2, \ldots$ be the waiting times in a Poisson process $(X(t); t \geq 0)$ of rate λ. Independent of the process, let $Z_1, Z_2, \ldots$ be independent and identically distributed random variables with common probability density function $f(x), 0 < x < \infty$. Determine $Pr(Z > z)$, where $Z = min(W_1 + Z_1, W_2 + Z_2, \ldots)$
SOLUTION:

Define: $Z(t) = min(W_1 + Z_1, W_2 + Z_2, \ldots W_t + Z_t)$

$Pr(Z(t) > z) = \sum_{n=0}^{\infty} Pr(Z(t) > z | N(t) = n) Pr(N(t) = n)$

$Pr(Z(t) > z) = \sum_{n=0}^{\infty} Pr((W_1 + Z_1, W_2 + Z_2, \ldots W_t + Z_t) > z | N(t) = n) Pr(N(t) = n)$

Because ordering is not used, we can replace W_i with U_i.

$Pr(Z(t) > z) = \sum_{n=0}^{\infty} Pr((U_1 + Z_1, U_2 + Z_2, \ldots U_t + Z_t) > z | N(t) = n) Pr(N(t) = n)$

$Pr(Z(t) > z) = \sum_{n=0}^{\infty} (Pr(U_i + Z_i > z))^n Pr(N(t) = n)$

$U \sim Uniform(0, t)$ and $\xi \sim i.i.d.$

$Pr(Z(t) > z) = \sum_{n=0}^{\infty} \left(Pr(U + \xi > z)\right)^n \frac{e^{\lambda t} (\lambda t)^n}{n!}$

$Pr(Z(t) > z) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n P(U + \xi \geq z)^n}{n!}$

$Pr(Z(t) > z) = exp\{-\lambda t\} \cdot \sum_{n=0}^{\infty} \frac{(\lambda t) \cdot P(U + \xi \geq z)}{n!}$

By the Taylor expansion of e^x:

$Pr(Z(t) > z) = exp\{-\lambda t\} \cdot exp\{(\lambda t)^n \cdot P(U + \xi \geq z)\}$

$1 - F(z - u)$ is the survival function (The variable is $> z - u$)

$Pr(Z(t) > z) = e^{-\lambda t} \cdot \exp\left[\lambda t \int_0^t \frac{1}{t}(1 - F(z - u)) du\right]$
$Pr(Z(t) > z) = \exp\left[-\lambda t + \lambda t \int_0^t \frac{1}{t}(1 - F(z - u)) du\right]$
$Pr(Z(t) > z) = \exp\left[-\lambda (t - t \int_0^t \frac{1}{t}(1 - F(z - u)) du)\right]$
$Pr(Z(t) > z) = \exp\left[-\lambda (t - \int_0^t (1 - F(z - u)) du)\right]$
$v = z - u,$
$dv = -du,$
$u = 0 \rightarrow v = z,$
$u = t \rightarrow v = z - t$

$$\boxed{Pr(Z(t) > z) = exp\{-\lambda \int_{z-t}^{z} F(v) dv\}}$$

5. Pinsky and Karlin, Problem 5.4.5

Let $W_1, W_2, \ldots$ be the waiting times in a Poisson process $\{N(t); t \geq 0\}$ of rate λ. Determine the limiting distribution of W_1, under the condition that $N(t) = n$ as $n \to \infty$ and $t \to \infty$ in such a way that $\frac{n}{t} = \beta > 0$.

SOLUTION:

$Pr(W_1 \leq w|X(t) = n) =$
$Pr(U_{(1)} \leq w|X(t) = n) =$
$Pr(min\{U_1, U_2, U_3 \cdots U_n\} \leq w|X(t) = n) =$
$Pr(min\{U_1, U_2, U_3 \cdots U_n\} \leq w|X(t) = n) =$
$1 - Pr(min\{U_1, U_2, U_3 \cdots U_n\} > w|X(t) = n) =$
$1 - Pr(all\{U_1, U_2, U_3 \cdots U_n\} > w|X(t) = n) =$
$1 - \Pi_{i=1}^{n}(1 - \frac{w}{t}) =$
$1 - (1 - \frac{w}{t})^n =$
Because: $\beta = \frac{n}{t}$
$1 - (1 - \frac{\beta w}{n})^n =$
From calculus: $(1 - \frac{a}{n})^n = exp(-a)$
$Pr(W_1 \leq w|X(t) = n) = 1 - exp(-\beta w)$

Differentiate PDF to get PDF

$\boxed{Pr(W_1 = w|X(t) = n) = \beta \cdot exp\{(-\beta w)\}}$

$\boxed{Pr(W_1 = w|X(t) = n) \sim Exponenetial(\beta)}$

6. Pinsky and Karlin, Problem 5.4.6

Customers arrive at a service facility according to a Poisson process of rate λ customers/hour. Let $X(t)$ be the number of customers that have arrived up to time t. Let $W_1, W_2, \ldots$ be the successive arrival times of the customers.

(a) Determine the conditional mean $E[W_1|X(t) = 2]$.
(b) Determine the conditional mean $E[W_3|X(t) = 5]$.
(c) Determine the conditional PDF for W_2, given that $X(t) = 5$.

SOLUTION:(All of these use the results of 5.3.8)

(a) $E(W_1|X(t) = 2) =$
$\int_0^t (w)f(w)dw =$
$\int_0^t (w)\frac{2!}{0!1!}\frac{w}{t}^0\frac{1}{t}(1-\frac{w}{t})^1 dw \int_0^t (w)\frac{2!}{0!1!}\frac{w}{t}^0\frac{1}{t}(1-\frac{w}{t})^1 dw = \boxed{\frac{t}{3}}$

(b)
Iterate same thing for (b). Did it on paper, redundant to transfer directly.
$\boxed{\frac{t}{2}}$

(c)
This one is the conditional, so it is actually easier.

$f_{W2|X(t)=s}(w) = \frac{5!}{(2-1)!(5-2)!}(\frac{w}{t})^{2-1}(\frac{1}{t})^1(1-\frac{w}{t})^{5-2}$
$\boxed{20(\frac{w}{t^2})(1-\frac{w}{t})^3}$

7. Pinsky and Karlin, Problem 5.4.7

   Let $W_1, W_2, \cdots$ be the event times in a Poisson process $\{X(t); t \leq 0\}$ of rate λ, and let $f(w)$ be an arbitrary function. Verify that:

   $E[\sum_{i=1}^{X(t)} f(W_i)] = \lambda \int_0^t f(w)dw$

   SOLUTION:

   Condition:

   $\sum_{i=1}^{\infty} E[f(W_i)|N(t) = i] \cdot Pr(N(t) = i) =$

   $\sum_{i=1}^{\infty} E[f(W_i)|N(t) = i] \cdot \frac{(-\lambda t)^i exp\{(-\lambda t)\}}{i!} =$

   $E[f(U)] \sum_{i=1}^{\infty} (i) \frac{(-\lambda t)^i exp\{(-\lambda t)\}}{i!} =$

   $(\lambda t) \cdot \int_0^t \frac{1}{t} f(v)dv = \boxed{(\lambda) \cdot \displaystyle\int_0^t f(v)dv}$

8. Pinsky and Karlin, Problem 5.4.9

Customers arrive at a service facility according to a Poisson process of rate λ customers per hour. Let $N(t)$ be the number of customers that have arrived up to time t, and let $W_1, W_2, \dots$ be the successive arrival times of the customers. Determine the expected value of the product of the waiting times up to time t.
(Assume that $(W_1, W_2, \dots, W_{N(t)}) = 1$ when $N(t) = 0$.)

SOLUTION

Condition:

$E[(W_1, W_2, \dots, W_{N(t)})] =$

$\sum_{i=0}^{\infty} E[(W_1, W_2, \dots, W_{N(t)})|N(t) = i] \cdot Pr(N(t) = i) =$

Ordering doesn't matter, they are all being multiplied anyway:

$\sum_{i=1}^{\infty} E[(U_1, U_2, \dots, U_{N(t)})|N(t) = i] \cdot Pr(N(t) = i) =$
$\sum_{i=1}^{\infty} E[(U_1, U_2, \dots, U_{N(t)})|N(t) = i] \cdot \frac{(\lambda t)^i e^{-(\lambda t)}}{i!} =$
$\sum_{i=1}^{\infty} E[(U_1, U_2, \dots, U_i)] \cdot \frac{(\lambda t)^i e^{-(\lambda t)}}{i!} =$
$\sum_{i=1}^{\infty} E[(U_1] \cdot E[(U_2], \dots, E[(U_i])] \cdot \frac{(\lambda t)^i e^{-(\lambda t)}}{i!} =$
$\sum_{i=1}^{\infty} (\frac{t}{2})^i \cdot \frac{(\lambda t)^i e^{-(\lambda t)}}{i!} =$
$\sum_{i=1}^{\infty} (\frac{\lambda t^2}{2})^i \cdot \frac{e^{-(\lambda t)}}{i!} =$
$e^{-(\lambda t)} \sum_{i=1}^{\infty} (\frac{\lambda t^2}{2})^i \cdot \frac{1}{i!} =$
$exp\{-(\lambda t)\} \cdot exp\{(\frac{\lambda t^2}{2})\} =$

$$\boxed{exp\{-(\lambda t)(1 - \frac{t}{2})\}}$$