STAT 150 HOMEWORK #4

FALL 2023

Due Friday, Sep 22nd, at 11:59 PM on Gradescope.

Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

ACKNOWLEDGEMENTS: I use ChatGpt to generate equations, matrices and tables.

1. Pinsky and Karlin, Problem 4.1.3

   A Markov chain has the transition probability matrix (1)
   Determine the long run probability of being in state 0

$$M = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \tag{1}$$

   To find the long-run probability of being in state 0, we need to find the stationary distribution, $\vec{\pi}$, of this Markov chain. The stationary distribution is a row vector that satisfies:

$$\vec{\pi} P = \vec{\pi}$$

   where P is the transition probability matrix. Additionally, the elements of $\vec{\pi}$ must sum to 1:

$$\sum_{i=0}^{5} \pi_i = 1$$

   We can set up the following system of linear equations to find the stationary distribution $\vec{\pi} = [\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5]$:

$$\begin{aligned} \pi_0\alpha_1 + \pi_1 &= \pi_0 \\ \pi_0\alpha_2 + \pi_2 &= \pi_1 \\ \pi_0\alpha_3 + \pi_3 &= \pi_2 \\ \pi_0\alpha_4 + \pi_4 &= \pi_3 \\ \pi_0\alpha_5 + \pi_5 &= \pi_4 \\ \pi_0\alpha_6 &= \pi_5 \end{aligned}$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$$

The long-run probability of being in state 0 will be the value of π_0

$$\pi_0\alpha_1 + \pi_0\alpha_2 + \pi_0\alpha_3 + \pi_0\alpha_4 + \pi_0\alpha_5 + \pi_0\alpha_6 = \pi_0$$
$$\pi_0\alpha_2 + \pi_0\alpha_3 + \pi_0\alpha_4 + \pi_0\alpha_5 + \pi_0\alpha_6 = \pi_1$$
$$\pi_0\alpha_3 + \pi_0\alpha_4 + \pi_0\alpha_5 + \pi_0\alpha_6 = \pi_2$$
$$\pi_0\alpha_4 + \pi_0\alpha_5 + \pi_0\alpha_6 = \pi_3$$
$$\pi_0\alpha_5 + \pi_0\alpha_6 = \pi_4$$
$$\pi_0\alpha_6 = \pi_5$$
$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$$

We then have:

$$\begin{aligned}\pi_0 + \pi_0\alpha_2 + \pi_0\alpha_3 + \pi_0\alpha_4 + \pi_0\alpha_5 + \pi_0\alpha_6 +&\\ \pi_0\alpha_3 + \pi_0\alpha_4 + \pi_0\alpha_5 + \pi_0\alpha_6 +&\\ \pi_0\alpha_4 + \pi_0\alpha_5 + \pi_0\alpha_6 +&\\ \pi_0\alpha_5 + \pi_0\alpha_6 +&\\ \pi_0\alpha_6 &= 1\end{aligned}$$

We then have:

$$\pi_0(\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6) = 1$$

And finally:

$$\boxed{\pi_0 = \frac{1}{(\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6)}}$$

2. Pinsky and Karlin, Problem 4.1.4

A finite-state regular Markov chain has transition probability matrix $P = ||P_{(i,j)}||$ and limiting distribution $\pi = ||\pi_{(i)}||$. In the long run, what fraction of the transitions are from a prescribed state k to a prescribed state m?

SOLUTION:

Logically, the solution should me the long run probability we spend at k multiplied by the probability we move to m from that state. This probability is: $\pi_k P_{(k,m)}$
But we should show this more formally.

Review from the book:
...recall that if a sequence $a0, a1, ...$ of real numbers converges to a limit a, then the averages of these numbers also converge in the manner.

Using zero-indexing:

$lim_{n\to\infty} \frac{1}{l} \sum_{k=0}^{l-1} a_k = a$

We apply this result to the convergence of the distribution:

$lim_{\to\infty} \frac{1}{l} \sum_{k=0}^{l-1} P^k_{(i,j)} = \pi_k$

And thus:

$\frac{1}{1} \sum_{k=0}^{l-1} \mathbb{1}\{X_k = j\}$

Starting the problem:

We want this quantity:

$\mathbb{E}\{lim_{l\to\infty}\{\frac{1}{l} \sum_{n=0}^{l-1} \mathbb{1}\{X_{n+1} = m, X_n = k\}\}\} =$

$lim_{l\to\infty} \frac{1}{l} \sum_{n=0}^{l-1} \mathbb{P}(X_{n+1} = m, X_n = k | X_0 = i) =$

$lim_{l\to\infty} \frac{1}{l} \sum_{n=0}^{l-1} \mathbb{P}(X_{n+1} = m | X_n = j, X_0 = i)\mathbb{P}(X_n = k | X_0 = i) =$

$lim_{l\to\infty} \frac{1}{l} \sum_{n=0}^{l-1} \mathbb{P}(X_{n+1} = m | X_n = j)\mathbb{P}(X_n = k) =$

$\boxed{P_{(k,m)}\pi_k}$

3. Pinsky and Karlin, Problem 4.1.6

Determine the following limits in terms of the transition probability matrix $P = ||P_{(i,j)}||$ and limiting distribution $\pi = ||\pi_{(k)}||$ of a finite-state regular Markov chain X_n:

(a) $lim_{n\to\infty} Pr\{X_{n+1} = j | X_0 = i\}$

Take: $m = n + 1$

$$\boxed{lim_{n\to\infty} Pr\{X_{n+1} = j | X_0 = i\} = lim_{m\to\infty} Pr\{X_m = j | X_0 = i\} = \pi_j}$$

CAUTION FOR GRADER:
In both (b) and (c), I skip the step of un-conditioning
(Moving the initial probability to the denominator, etc etc)
I hope this is fairly obvious, and also slightly unnecessary, because it clear.

(b) $lim_{n\to\infty} Pr\{X_n = k, X_{n+1} = j | X_0 = i\}$

This is the same thing as HW4 P2 (Pinsky Problem 4.1.4)

$lim_{n\to\infty} Pr\{X_{n+1} = j, X_n = k | X_0 = i\} =$
$lim_{n\to\infty} Pr\{X_{n+1} = j | X_n = k, X_0 = i\} Pr\{X_n = k | X_0 = i\} =$

$$\boxed{P_{k,j}\pi_k}$$

(c) $lim_{n\to\infty} Pr\{X_{n-1} = k, X_n = j | X_0 = i\}$.

This is the same thing as (b), with a time shift.

$lim_{n\to\infty} Pr\{X_n = j, X_{n-1} = k | X_0 = i\} =$
$lim_{n\to\infty} Pr\{X_n = j | X_{n-1} = k, X_0 = i\} Pr\{X_{n-1} = k | X_0 = i\} =$

$$\boxed{P_{k,j}\pi_k}$$

4. Pinsky and Karlin, Problem 4.1.10

Consider a Markov chain with transition following probability matrix, then find the limiting distribution.

$$\begin{bmatrix} p_0 & p_1 & p_2 & \cdots & p_N \\ p_N & p_0 & p_1 & \cdots & p_{N-1} \\ p_{N-1} & p_N & p_0 & \cdots & p_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_0 \end{bmatrix}$$

SOLUTION:

We know the chain is irreducible because all cells communicate with each other (Every $P_{(i,j)}$ is positive)
We know the chain is aperiodic because each state can return to itself in $N+1$ steps (All $P_{(i,i)}$ is positive)

Thus, the chain has a limiting distribution.

Another observation is that the matrix is doubly-stochastic
(Each column contains the same probabilities as each row due to the shifting)

In the limit, each probability is thus: $\boxed{\pi_i = \dfrac{1}{N+1}}$

Proof at: (Pinsky 4.1.1)

5. Pinsky and Karlin, Problem 4.1.13

A Markov chain has the transition probability matrix (*)
After a long period of time, you observe the chain and see that it is in state 1.
What is the conditional probability that the previous state was state 2?

$$(*) = \left[\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.4 & 0.4 & 0.2 \\ 1 & 0.6 & 0.2 & 0.2 \\ 2 & 0.4 & 0.2 & 0.4 \end{array}\right]$$

OBJECTIVE:
Find the following: $lim_{n\to\infty}\mathbb{P}\{X_{n-1} = 2|X_n = 1\}$

$lim_{n\to\infty}\mathbb{P}\{X_{n-1} = 2|X_n = 1\} = \frac{\mathbb{P}\{X_{n-1}=2,X_n=1\}}{\mathbb{P}\{X_n=1\}} = (\pi_{s2} \cdot P_{(s2,s1)})/(\pi_{s1})$

Now time for the tedious system of linear equations:

$\pi_0 = 4/10\pi_0 + 6/10\pi_1 + 4/10\pi_2$
$6/10\pi_0 = 6/10\pi_1 + 4/10\pi_2$
$\pi_0 = \pi_1 + 4/6\pi_2$
$\pi_1 = 4/10\pi_0 + 2/10\pi_1 + 2/10\pi_2$
$8/10\pi_1 = 4/10\pi_0 + 2/10\pi_2$
$\pi_1 = 4/8\pi_0 + 2/8\pi_2$

$\pi_0 = 1/2\pi_0 + 1/4\pi_2 + 2/3\pi_2$
$\pi_0 = 6/12\pi_2 + 16/12\pi_2$
$\pi_0 = 11/6\pi_2$

$\pi_1 = 4/8\pi_0 + 2/8\pi_2$
$\pi_1 = 4/8(\pi_1 + 4/6\pi_2) + 2/8\pi_2$
$\pi_1 = 32/48\pi_2 + 1/2\pi_2$
$\pi_1 = 7/6\pi_2$

$\pi_2 = 1 - \pi_0 - \pi_1$
$\pi_2 = 1 - 11/6\pi_2 - 7/6\pi_2$
$\pi_2 = 1 - 3\pi_2$
$4\pi_2 = 1$

$$\boxed{\pi_2 = \frac{1}{4}, \pi_1 = \frac{7}{24}, \pi_0 = \frac{11}{24}}$$

Give: $P_{(s2,s1)} = \frac{2}{10}$ and $lim_{n\to\infty}\mathbb{P}\{X_{n-1} = 2|X_n = 1\} = (\pi_{s2} \cdot P_{(s2,s1)})/(\pi_{s1})$

We have: $\boxed{\frac{\frac{1}{4}\frac{2}{10}}{\frac{7}{24}} = \frac{6}{35}}$

6. Durrett 1.38 (on page 85 of the published version of the Durrett textbook)

An individual has three umbrellas, some at her office, and some at home. If she is leaving home in the morning (or leaving work at night) and it is raining, she will take an umbrella, if one is there. Otherwise, she gets wet. Assume that independent of the past, it rains on each trip with probability $\frac{2}{10}$. To formulate a Markov chain, let X_n be the number of umbrellas at her current location.

(a) Find the transition probability for this Markov chain.
(b) Calculate the limiting fraction of time she gets wet.

COMMENT: There was a very similar problem in Data 140. Lol.

(a) MATRIX:

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & .8 & .2 \\ 2 & 0 & .8 & 0.2 & 0 \\ 3 & .8 & 0.2 & 0 & 0 \end{array}$$

(b)SOLUTION:

Getting wet requires that she is in state 0, and then it rains.
The probability of rain is constant, $\frac{2}{10}$
The long run probability of her being in state 0 is π_0

Thus,
$\mathbb{P}\{(\text{She has no umbrella}) \cup (\text{it rains})\} = \frac{2\pi_0}{10}$

$$\boxed{\mathbb{P}\{(\text{She is at state } 0) \cup (\text{it rains})\} = \frac{2\pi_0}{10}}$$

$\pi_0 = .8\pi_3$
$\pi_1 = .8\pi_2 + .2\pi_3$
$\pi_2 = .8\pi_1 + .2\pi_2$
$\pi_3 = 1\pi_0 + .2\pi_1$

$\pi_0 = .8(1\pi_0 + .2\pi_1) \to \pi_0 = .8\pi_0 + 16/100\pi_1 \to$
$\pi_0 = 8/10\pi_0 + 16/100\pi_1 \to 2/10\pi_0 = 16/100\pi_1 \to \pi_0 = 16/20\pi_1$

$\pi_2 = \pi_3$
$\pi_2 = \pi_1$
$\pi_0 = 1 - 3\pi_1 \to \pi_0 = 1 - \frac{15}{4}\pi_0 \to \frac{19}{4}\pi_0 = 1 \to \boxed{\pi_0 = \frac{4}{19}}$

$$\boxed{\mathbb{P}\{(\text{She is at state } 0) \cup (\text{it rains})\} = \frac{2\pi_0}{10} = \frac{2 \cdot 4}{10 \cdot 19} = \frac{4}{95}}$$