SOILS AND FOUNDATIONS Vol. 14, No. 2, June 1974
Japanese Society of Soil Mechanics and Foundation Engineering

PROBABILISTIC APPROACH TO DESIGN OF EMBANKMENTS*

MINORU MATSUO** and KATSUHIKO KURODA***

ABSTRACT

Safety factors of embankments are analyzed in relation to the variabilities and the uncertainties of soil properties. It is concluded that the unconfined compression strength of a saturated homogeneous clay layer and the strength parameters c_u and $\tan\phi_u$ of an unsaturated homogeneous silty soil layer follow the theoretical normal distributions. Such physical properties as moisture content and moist density of those soil layers and moist density of an embankment just after construction are also shown to be taken as the normal random variables. A new design factor is defined and the relation between the design factor and the probability of sliding fuilure of an embankment is formulated under consideration of the randomness of the soil properties. Numerical and practical examples show that the embankment with an ordinary value of the design factor ($\bar{F}_s{}^*=1.1\sim1.5$) has an unexpected high value of the probability of sliding failure as $15\%\sim20\%$.

Key words: design, direct shear test, earth fill, partially saturated soil, safety factor, site investigation, stability analysis, statistical analysis, unconsolidated undrained shear

IGC: C0/E6/H4

INTRODUCTION

It is very important to establish a theory to treat systems of soil exploration, design and construction as a consistent system. Uncertainties are, however, the main obstacle for the systems analysis, and therefore the method to deal with the uncertainties are desired in engineering fields.

"Safety factor" or "Margin of safety" are introduced into designs of structures for covering the uncertainties. But, unfortunately, it can not explain their safety quantitatively and then it is insufficient as the measure to estimate the safety of the systems. At first uncertainties must be examined from the viewpoint of reliability which is closely connected with the concepts of system maintainability, availability and safety. In its most general form, reliability is defined as a probability of success that a device performs its purpose adequately for the period of time intended under the operating conditions encountered (Bazovsky, 1961). This definition of reliability can be applied to the structures in civil engineering.

Reliability to failure of an embankment should be investigated in relation to soil explo-

* The contents of the present paper have been partly published (Matsuo, M. and Kuroda, K., 1971, 1972).

** Associate Professor, Faculty of Engineering, Nagoya University, Furo-cho, Chigusa-ku, Nagoya.
*** Assistant Professor, Faculty of Engineering, Kyoto University, Yoshida-honmachi, Sakyo-ku Kyoto.

Written discussions on this paper should be submitted before April 1, 1975.

ration and laboratory tests, bacause it is sharply influenced by the accuracy of their informations. In other words, this accuracy is concerned with the method and the extention of soil exploration and the number of laboratory tests. These are currently decided on the basis of engineers' experiences and given costs, and then have a consequent weak point that their results can not quantitatively explain the reliability of the embankment and its confidence limit.

In this paper the authors analyze the uncertainties caused in each stage of soil exploration, design and construction of embankments, and propose a probabilistic approach to their designs.

VARIABILITY OF SOIL PROPERTIES

Factors Influencing Variations

It is well known that data of soil explorations and the results of laboratory tests show large variations. These variations depend on many kinds of factors, which cause unreliability of a design. Generally speaking, two kinds of variations are contained in those data; the one is due to the physical nature of soil, and the other is attributed to technical problems. That is, all natural soils show variations in properties from point to point in the ground because of inherent variations in composition and consistency during formation and because of variation of the underground water table. On the other hand, variations due to many kinds of technical errors are mixed up in the data. They are as follows;

(1) errors due to an engineer's misjudgement in which he regards different types of soil layers as the same one.

(2) variations due to the use of different types of samplers and test apparatuses, and due to different levels of workmanship in sampling and testing.

(3) variations due to different methods of testing and plotting the test results.

These variations have been investigated, for example, by Skempton and Sowa (1963), Noorany and Seed (1965), Nakase, et al. (1966) and the Committee of Shearing Test Method of JSSMFE (1968). Engineering efforts should be made to distinguish these variations. However, even if any effort should be made, such variations due to chance errors cannot be excluded completely. This suggests the necessity of probabilistic approaches in planning, design and construction for soils. From this point of view, variations of soil properties are investigated.

Variations in Strength

The influencial factors on the reliability of an embankment are devided into two parts; the one is the uncertainty and the variability of load, and the other is those of soil resistance. In the case of sliding failure of an embankment, the driving force is mainly caused by its own weight, and then the variability of load is concerned mostly with the non-homogeneity of embankment material and the variation of compacting energy and the moisture content. On the other hand, the soil resistance is very uncertain, because the nominal strength of soil is estimated on the basis of the results of soil tests in which various values are measured.

The subsoil profile at the Kisarazu port site in Japan is shown in Fig. 1. There is a soft clay layer between the depths of 2 and 20 m below the sea bottom. The variations of the soil properties in this clay layer are shown in Fig. 2. It is obvious from this figure that it is difficult to devide the clay layer into more than two different parts in the engineering sense. Therefore engineers can not help regarding and treating this kind of the clay layer as a homogeneous soil layer. The criteria of judgement of a homogeneous soil layer have been discussed by the authors (1972). The variation of the unconfined compression strength q_u of this clay layer are replotted in the form of the frequency distribution

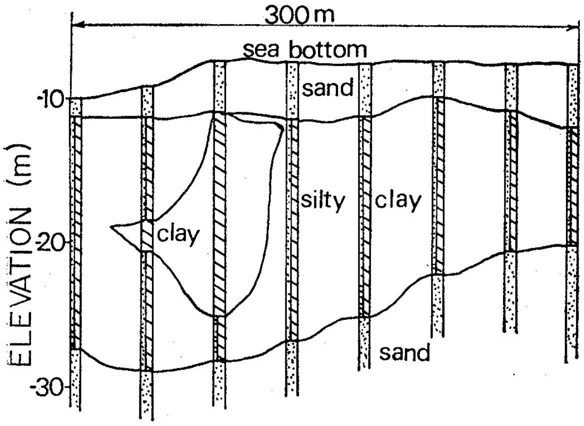


Fig. 1. Subsoil profile of Kisarazu port site

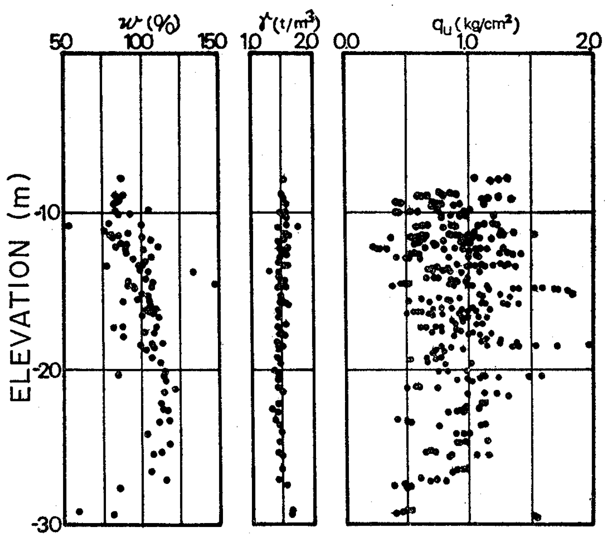


Fig. 2. Variations of soil properties of Kisarazu port site

in Fig. 3. This shows a good agreement with the theoretical normal distribution, which is tested by χ^2-test for 5% risk. Some other data of the variations of q_u of homogeneous clay layers were similarly investigated. The results of χ^2-tests of these variations are given in Table 1. Fig. 3 and Table 1 show that the unconfined compression strengths in homogeneous clay layers can be regarded as normal random variables, and it is important that the values of the coefficient of

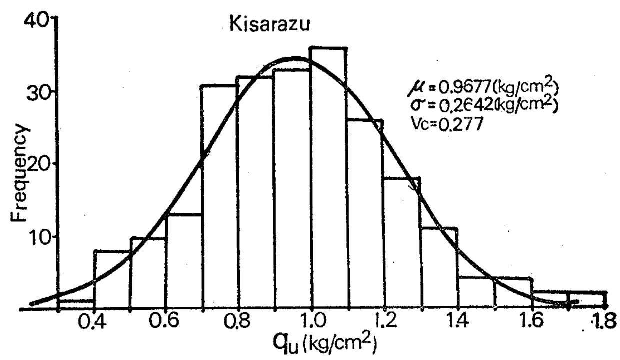


Fig. 3. Frequency distribution q_u of Kisarazu port site

Table 1. χ^2-tests of q_u

Sampling site	n	V_c	Value of χ^2	f
Horikawabashi	119	0.181	16.1034*	7
Kisarazu	231	0.273	8.0743	12
Neyagawa (1)	86	0.237	8.5055	7
Neyagawa (2)	98	0.259	12.4698	7
Yasuura	538	0.409	17.6745	12
Keihin	271	0.320	9.0946	10
Ichiba	72	0.366	10.7103*	4
Chiba	192	0.222	6.3492	4
Kinuura	127	0.373	8.8760	4

Note: n is the sample size of q_u.
f is the degree of freedom.
* was tested for $\alpha=1\%$.

variation V_c of these data are restricted in a narrow range from 0.2 to 0.4 irrespective of the large difference in their mean values μ and standard deviations σ. These values of V_c correspond with the values of $V_c=0.1\sim0.2$ (for normal) and $V_c=0.1\sim0.3$ (for lognomal), which are reported by Meyerhof (1970). These results are also supported by those of Hooper and Butler (1966), Lumb (1966) and Wu and Kraft (1967).

On the other hand, sand and silty soils in the natural state have generally both cohesion and angle of shearing resistance. Lumb (1966) has statistically investigated on the drained strength parameters c_d and $\tan\phi_d$ of those soils in Hong Kong. He shows that the parameter $\tan\phi_d$ follows the theoretical normal distribution and that it is reasonable to take the parameter c_d of silty sand as $c_d=\mu_d+\sigma_d u$, where μ_d and σ_d are constants and u is the standardized normal variable. In this paper, the undrained strength parameters c_u and $\tan\phi_u$ are investigated in order to apply these parameters to the stability analysis of embankments which are to be constructed very rapidly. The effective stress analysis for unsaturated soils is very troublesome in engineering practice, because of difficulties of estimation and prediction of the distribution of pore water pressure. Therefore the total stress analysis is applied to the usual stability analysis. In order to investigate the statistical characteristics, four kinds of unsaturated soils shown in Table 2 were tested. Soil 1 and 2 were obtained from the site near Watarase River and Soil 3 was sampled from the mountain near Agigawa River in Japan. Soil 4 was obtained from the site near the authors' laboratory. Soil 1 and Soil 2 were used for the triaxial compression tests under the unconsolidated-undrained condition. Soil 3 was used for direct shear tests where 10 specimens were tested under the conditions of the normal stresses $\sigma_N=0.4$, 0.8, 1.2 and 1.6 kg/cm². As a result, 40 sets of relations between shear strength τ_f and normal stress σ_N were obtained. In these direct shear tests, the dry density γ_d and the moisture content were fixed at 1.55 g/cm³ and 10 %, respectively. Soil 4 was tested also by the direct shear apparatus under the condition of the normal stresses $\sigma_N=0.4$, 0.8, 1.2, 1.6 and 2.0 kg/cm². Fig. 4 gives the frequency distributions of c_u and $\tan\phi_u$ of Soil 2 which were calculated from $\tau_f\sim\sigma_N$ diagrams where the confined pressure was varied from 0.5 to 5.0 kg/cm². The results of Soil 3 are shown in Fig. 5.

Table 2. Soil types used for laboratory tests

Soil No.	Soil Type
Soil 1	Clay, Silty clay
Soil 2	Silt, Sandy silt, Clayey silt, Silty sand
Soil 3	Silty sand
Soil 4	Silty sand

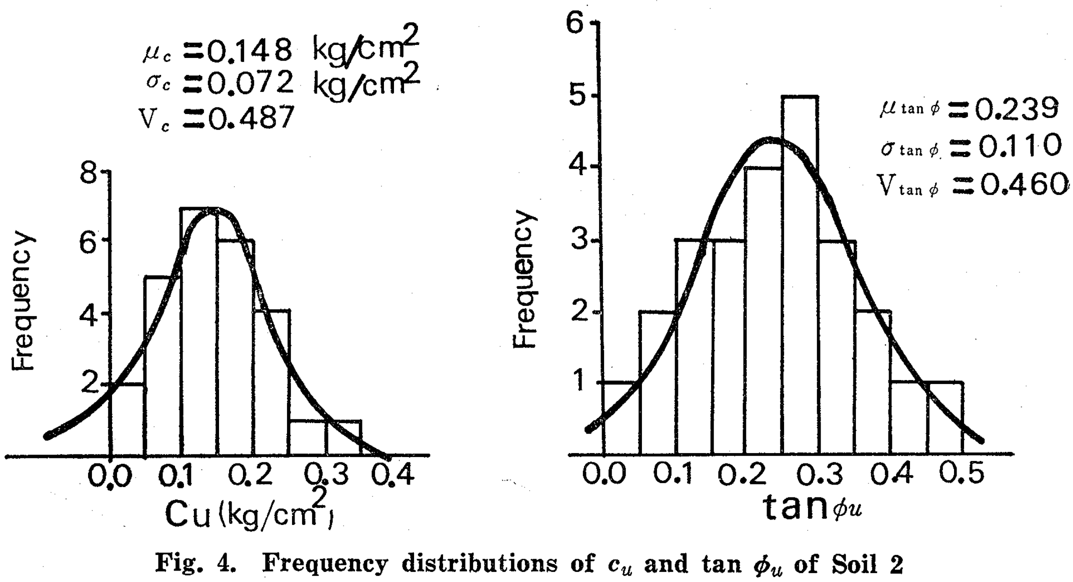


Fig. 4. Frequency distributions of c_u and $\tan\phi_u$ of Soil 2

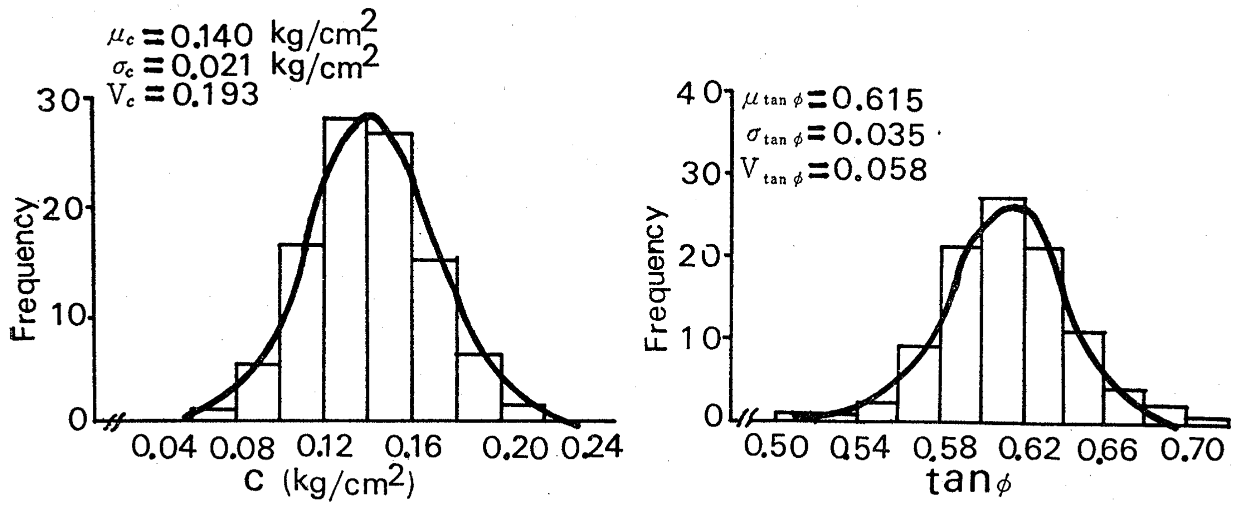


Fig. 5. Frequency distributions of c_u and $\tan\phi_u$ of Soil 3

These figures show that the undrained strength parameters c_u and $\tan\phi_u$ of the unsaturated soils can be regarded as normal random variables. It is interesting that the values of the coefficient of variation of the parameters for Soil 3 are very small as compared with those for Soil 2. This can be explained by the fact that the particle size of Soil 2 varies in a wider range than that of Soil 3 controlled in the laboratory. From Fig. 5 it can be further expected that the shearing resistance τ_f of unsaturated soils follows the normal distribution so far as soils satisfy the Mohr–Coulomb's failure criterion. Fig. 6 for Soil 4, in fact, shows the normal distribution of τ_f itself. This relationship was obtained from the results of 100 direct shear tests of Soil 4 under the fixed condition of $\gamma_d=1.5$ g/cm^3, $w=20\%$ and $\sigma_N=1.6$ kg/cm^2. In practical design, the probabilistic distributions of τ_f itself are necessary for the stability analysis. But it is impossible to investigate those of τ_f of all soils for all normal stresses, and therefore it is reasonable to investigate those of the strength parameters of soils on the assumption that the Mohr–Coulomb's failure criterion holds.

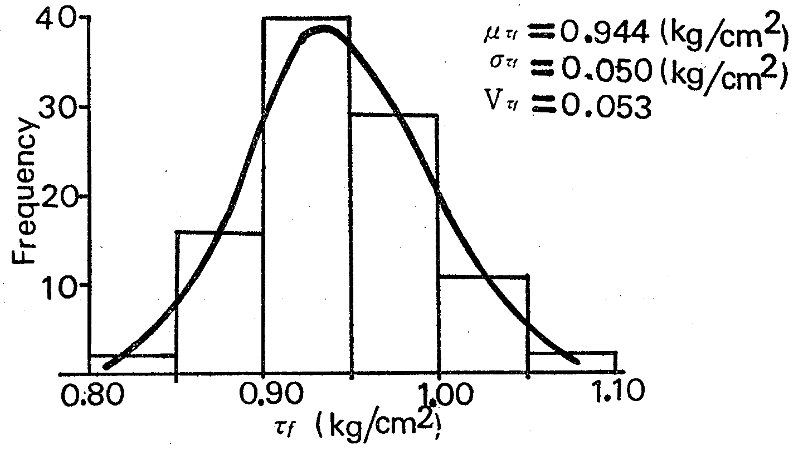


Fig. 6. Frequency distribution of the shearing resistance τ_f of Soil 4

Variations in Physical Properties

Mohr–Coulomb's failure criterion is expressed as follows:

$$\tau_f=c+\sigma_N\tan\phi \qquad (1)$$

in which c and $\tan\phi$ are the strength parameters and σ_N is the normal stress acting on the failure surface. In many engineering problems, σ_N is dependent on the dead load of

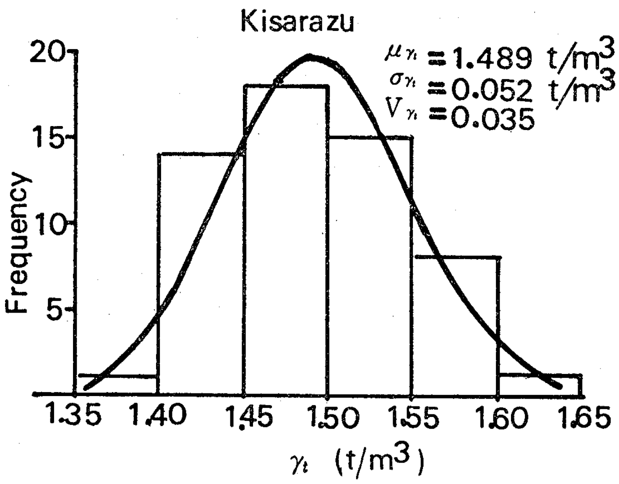


Fig. 7. Frequency distribution of γ_t

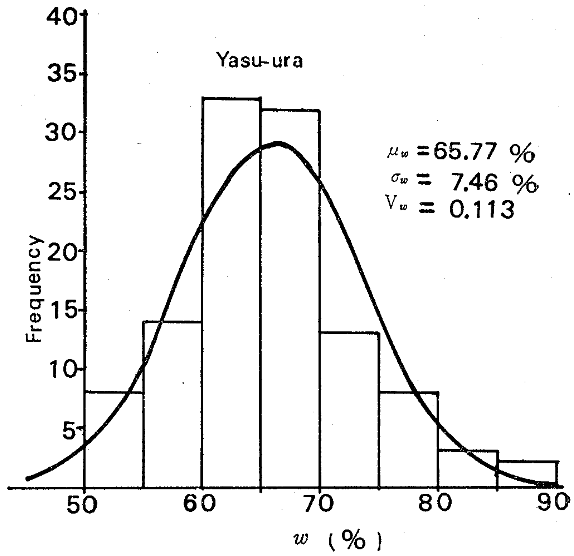


Fig. 8. Frequency distribution of w

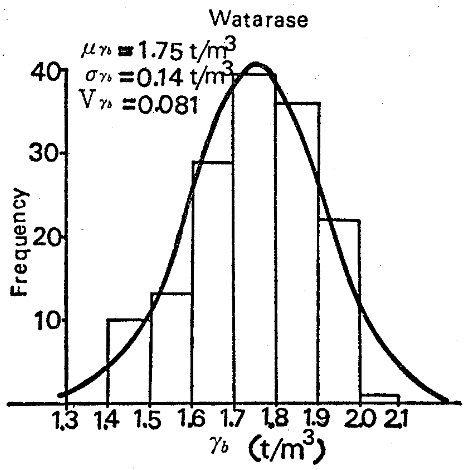


Fig. 9. Frequency distribution of γ_b

soils. Then it is important to investigate variations of their moist density γ_t and moisture content w.

Figs. 7 and 8 and Tables 3 and 4 show that γ_t and w of the homogeneous soil layers can be taken as the normal random variables. Fig. 9 gives the frequency distribution of the moist density γ_b of an embankment just after construction. From these figures and Table 3 it is concluded that the values of the coefficient of variation $V\gamma_t$ of the moist density and that of $V\gamma_b$ of the enbankment are restricted in a range from 0.01 to 0.08. It should be noticed that these values are much smaller being approximately one tenth of those of V_c. This is important in the analysis of the failure probability of an embankment as shown later.

Table 3. χ^2-tests of γ_t

Sampling site	n	V_r	Value of χ^2	f
Horikawabashi	35	0.025	1.2080	2
Kisarazu	57	0.035	3.6837	4
Neyagawa (1)	34	0.023	0.0344	2
Neyagawa (2)	43	0.026	0.6915	3
Yasuura	114	0.029	13.8227**	4
Keihin	36	0.020	0.2915	2
Ichiba	25	0.031	1.8723	3
Shiogama	27	0.037	7.7148	5

Note: n is the sample size of γ_t.
f is the degree of freedom.
** was tested for α=0.5%.

Table 4. χ^2-tests of w

Sampling site	n	V_w	Value of χ^2	f
Horikawabashi	35	0.094	0.3569	3
Neyagawa (2)	84	0.097	6.1446	5
Yasuura	113	0.113	9.5130	5
Keihin	76	0.084	2.3817	5
Ichiba	22	0.122	2.6864	4
Chiba	44	0.157	12.5922	11
Sendai	28	0.205	2.6763	8

Note: n is the sample size of w.
f is the degree of freedom.

CORRELATIONS OF SOIL PROPERTIES

It is fundamentally important for the statistical investigations of the soil properties to analyze the correlations among them. In order to investigate the correlations between strength parameters and physical properties, the results of the triaxial compression tests of Soil 2 were replotted. Soil 3 was tested by means of a direct shear apparatus. In the direct shear tests, 180 specimens were tested under the condition shown in Table 5 and then these 180 test results were statistically analyzed.

Table 5. Conditions and numbers of the direct shear tests for Soil 3

w (%) \ γ_d(g/cm^3)	1.45	1.5	1.55	1.6	1.65
5	12*	12*	12*	12*	12*
10	12*	12*	12*	12*	12*
15	12*	12*	12*	12*	12*

Note: 12* is the number of shear tests.
3 specimens were sheared under the fixed condition of σ_N, γ_d and w.

Correlation between Angle of Shearing Resistance and Density

Fig. 10 is the scattergrams of $\tan\phi_u$ against γ_d of Soil 1. This figure shows the strong positive correlation between the angle of shearing resistance and the dry density. Similar results were obtained also for Soil 2 and Soil 3. In general the linear relationship such as the one shown in Fig. 10 is mathematically formulated by the least squares method as

$$Y=AX+B \qquad (2)$$

where X and Y are symbolized by a physical property and a strength paremeter, respectively, and A and B are constants. The calculated values of A and B for these soils and the correlation coefficient r are given in Table 6. In this table S_r denotes the degree of saturation. It is evident that the $\tan\phi_u \sim \gamma_t$ relation is also expressed by Eq. (2), and their constants A and B are included in Table 6.

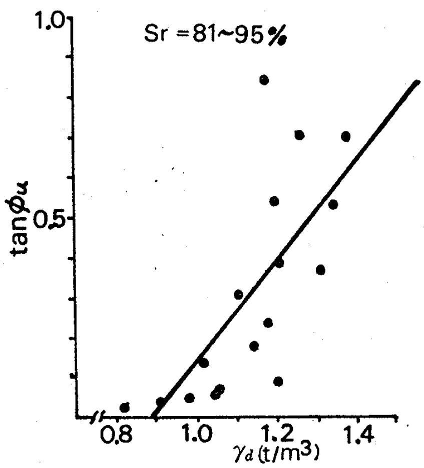


Fig. 10. Scattergram of $\tan\phi_u$ against γ_d of Soil 1

Table 6. Values of the constants A and B in Eq. (2) and the correlation coefficient r

Soil type	Properties	$Y=AX+B$		r	S_r (%)
		A	B		
Soil 1	$\tan\phi\sim w$	0.969	−1.412	−0.7125	85-95
	$\tan\phi\sim\gamma_d$	1.247	−1.108	0.7133	
	$\tan\phi\sim\gamma_t$	1.871	−2.765	0.6031	
	$c\sim w$	−0.901	0.784	−0.5529	
Soil 2	$\tan\phi\sim w$	0.673	−0.559	−0.5318	85-95
	$\tan\phi\sim\gamma_d$	0.649	−0.368	0.6557	
	$\tan\phi\sim\gamma_t$	0.896	−1.070	0.5679	
	$c\sim w$	−0.158	0.270	−0.2650	
Soil 3	$\phi\sim\gamma_d$	24.600	31.830	0.9273	16-22
	$\tan\phi\sim\gamma_d$	0.644	−0.280	0.9258	
	$\tan\phi\sim\gamma_t$	0.604	−0.263	0.9202	
Soil 3	$\phi\sim\gamma_d$	11.120	29.950	0.8620	32-44
	$\tan\phi\sim\gamma_d$	0.264	0.207	0.8592	
	$\tan\phi\sim\gamma_t$	0.237	0.212	0.8500	
Soil 3	$\phi\sim\gamma_d$	−28.600	36.350	−0.9531	49-66
	$\tan\phi\sim\gamma_d$	−0.712	1.739	−0.9431	
	$\tan\phi\sim\gamma_t$	−0.630	1.760	−0.9523	

Correlation between Cohesion and Dry Density

Fig. 11 is the scattergram of the cohesion against the dry density of Soils 1, 2 and 3. There is no correlation between them as expected.

Correlation between Cohesion and Moisture Content

It might be expected for the cohesion of soils to correlate with the moisture content. Fig. 12 is the relation between them of Soil 1 and Soil 2. Their correlation coefficients are also given in Table 6. Fig. 12 and Table 6 show the negative correlation for them, but it is not so strong as expected.

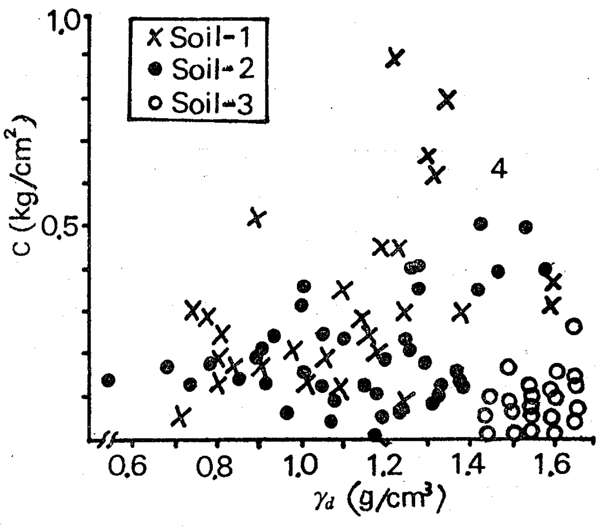


Fig. 11. Scattergram of c against γ_d

Correlation between Cohesion and Angle of Shearing Resistance

Fig. 13 shows the scattergram of the cohesion against the angle of shearing resistance of Soil 1 and Soil 2. This figure shows no correlation between c and $\tan\phi$ of unsaturated silty soils for a wide range of their natural moisture contents. In order to investigate the influence of the moisture content on the correlation, Soil 4 was tested again by means of the direct shear apparatus under the condition given in Table 7. The normal stress σ_N was varied as mentioned before, and 4 specimens were sheared under the fixed condition of γ_d,

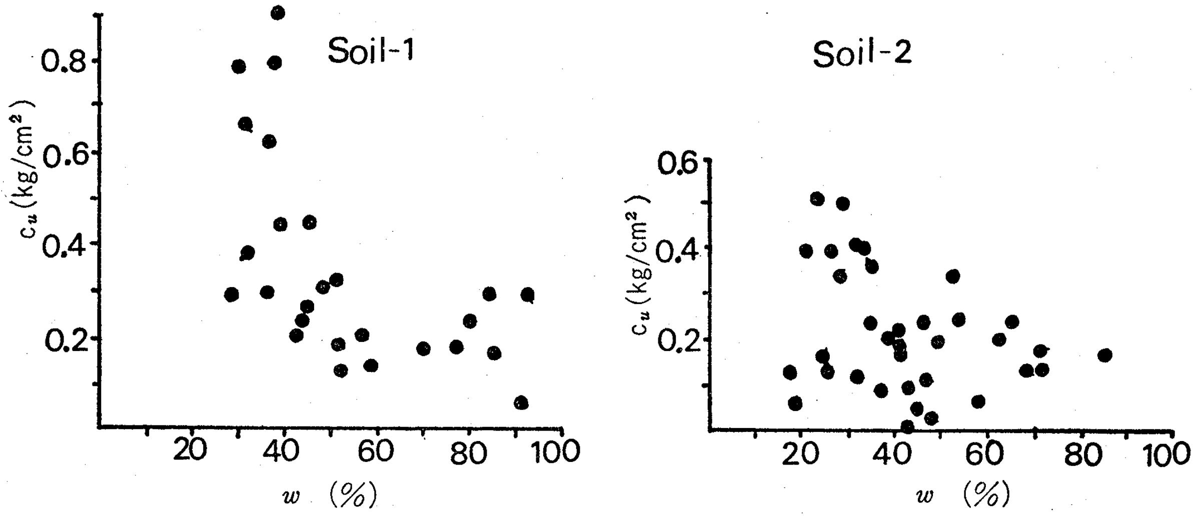


Fig. 12. Scattergram of c against w

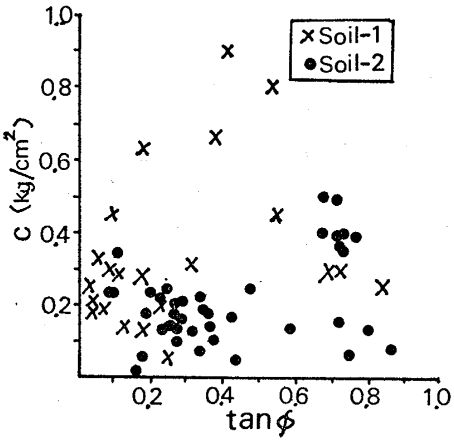


Fig. 13. Scattergram of c against tan ϕ

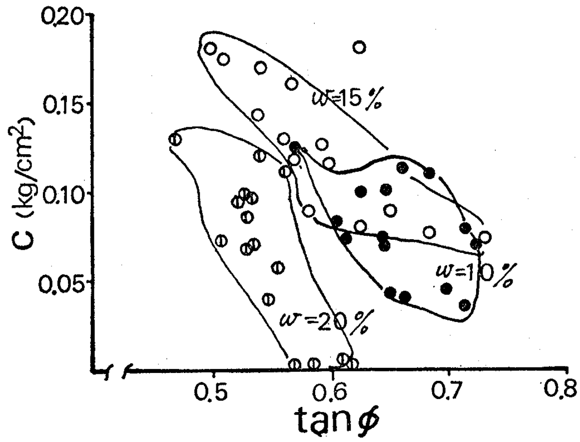


Fig. 14. Correlation between c and tan ϕ of Soil 4

Table 7. Conditions and numbers of the direct shear tests for Soil 4

γ_d(g/cm^3) / w (%)	1.45	1.50	1.55	1.60
10	20*	20*	20*	20*
15	20*	20*	20*	20*
20	20*	20*	20*	20*+100**

Note: 20* is the number of shear tests.
100** is the number of shear tests for the distribution of τ_f.

w and σ_N. Therefore 16 independent $\sigma_N \sim \tau_f$ relations were obtained for each moisture content. The correlations between c and tanϕ are shown in Fig. 14 with the moisture content as the parameter. It is shown that they have a strong negative correlation when the moisture content is fixed.

It is interesting to note in Fig. 14 that the group for $w=20\%$ represents the smaller value of tanϕ than the other groups for the same value of c. Consequently this group has the smaller value of τ_f than those for the others. This tendency appears in the range of w higher than the optimum water content w_{opt} as already explained in Lambe's discussion (1960).

RELATIONSHIP BETWEEN DESIGN FACTOR AND FAILURE PROBABILITY

Some Considerations on Safety Factors

Generally speaking, the safety factor F_s of a structure is formulated as

$$F_s=p(S_i)/q(L_j) \qquad (i,\ j=1,\ 2,\ \cdots\cdots) \tag{3}$$

where S_i is the i-th kind of resistances and L_j is the j-th kind of forces. $p(S_i)$ and $q(L_j)$ are the functions of resistances S_i and forces L_j, respectively.

For a design of an embankment, $p(S_i)$ and $q(L_j)$ correspond to the shearing resistance of soil and the load, respectively, and Eq. (3) gives the safety factor with respect to the sliding failure. The traditional safety factor F_s has been given as the ratio of $p(S_i)$ to $q(L_j)$ which is calculated deterministicaly using the representative values of S_i and L_j. But this definition of the safety factor is very ambiguous because S_i and L_j are generally uncertain. This ambiguity can be excluded if the uncertainties due to estimation, formulation and calculation are quantitatively evaluated. Fortunately as previously investigated, the variability of soil properties can be quantitatively evaluated as normal random variables. Then, from the viewpoint of reliability, the safety of structures can be discussed as the probability of performing their purposes adequately for the period of time intended under the operating conditions encountered.

Formulation of the Relation between Design Factor and Failure Probability

It is well known that the Swedish method is practical and available for the stability analysis of a slope. The reliability or safety of a slope is mainly influenced by the variability of the strength S_i and the load L_j. When the equilibrium of the whole mass such as shown in Fig. 15 is considered, the driving moment M_d is equal to the sum of the mo-

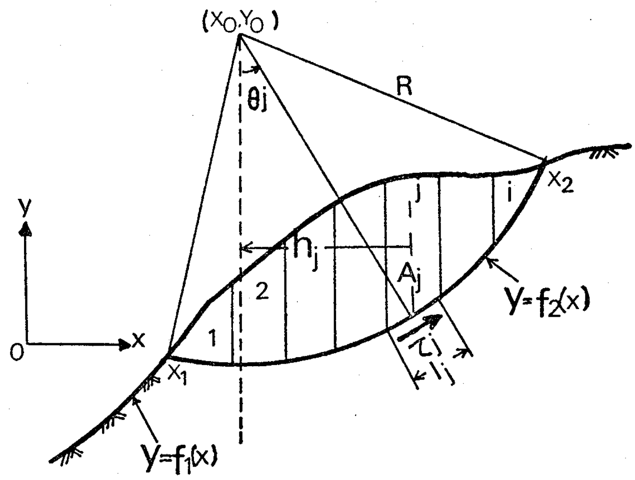


Fig. 15. Slope for stability analysis

ments of the weight of the slices. From Fig. 15, M_d is calculated as

$$M_d=\sum_{j=1}^{i}(R\cdot\sin\theta_j\cdot A_j\cdot\gamma_{tj}) \tag{4}$$

and the resisting moment M_r is given by

$$\left.\begin{aligned} M_r&=\sum_{j=1}^{i}R(\tau_j l_j)=\sum_{j=1}^{i}R\{c_j l_j+(A_j\cos\theta_j)\gamma_{tj}\tan\phi_j\}\\ &=(\sum_{j=1}^{i}Rc_j l_j)+(\sum_{j=1}^{i}RA_j\cos\theta_j\gamma_{tj}\tan\phi_j)\\ \tau_j&=c_j+\sigma_j\tan\phi_j\\ \sigma_j&=(A_j\cos\theta_j)\gamma_{tj}/l_j \end{aligned}\right\} \tag{5}$$

where A_j is the area of j-th slice, and R is radius of the slip circle.

For a fixed coordinates and for the infinite number of i, Eqs. (4) and (5) can be transformed as

$$\left.\begin{aligned} M_d &= q'(x_0, y_0, R, \gamma_t) \\ M_r &= p_1'(x_0, y_0, R, c) + p_2'(x_0, y_0, R, \gamma_t, \tan\phi) \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} p_1'(x_0, y_0, R, c) &= \int_{x_1}^{x_2} cR\sqrt{1+\left\{\frac{d}{dx}f_2(x)\right\}^2}\,dx \\ p_2'(x_0, y_0, R, \gamma_t, \tan\phi) &= \int_{x_1}^{x_2} \gamma_t \tan\phi\sqrt{R^2-(x-x_0)^2}\{f_1(x)-f_2(x)\}\,dx \\ q'(x_0, y_0, R, \gamma_t) &= \int_{x_1}^{x_2} \gamma_t (x-x_0)\{f_1(x)-f_2(x)\}\,dx \end{aligned}\right\} \tag{7}$$

where (x_0, y_0) are the coordinates of the center of the slip circle.

If the soil of the slope is homogeneous, Eqs. (6) and (7) yield the following;

$$\left.\begin{aligned} M_d &= q(x_0, y_0, R)\gamma_t \\ M_r &= p_1(x_0, y_0, R)c + p_2(x_0, y_0, R)\gamma_t \tan\phi \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} p_1(x_0, y_0, R) &= \int_{x_1}^{x_2} R\sqrt{1+\left\{\frac{d}{dx}f_2(x)\right\}^2}\,dx \\ p_2(x_0, y_0, R) &= \int_{x_1}^{x_2} \sqrt{R^2-(x-x_0)^2}\{f_1(x)-f_2(x)\}\,dx \\ q(x_0, y_0, R) &= \int_{x_1}^{x_2} (x-x_0)\{f_1(x)-f_2(x)\}\,dx \end{aligned}\right\} \tag{9}$$

In this case the functions p_1, p_2 and q are determined only from the geometrical conditions. The factor F_s is traditionally defined as

$$F_s = \frac{M_r}{M_d} = \frac{p_1'(x_0, y_0, R, c) + p_2'(x_0, y_0, R, \gamma_t, \tan\phi)}{q'(x_0, y_0, R, \gamma_t)} \tag{10}$$

for non-homogeneous soil

or

$$F_s = \frac{M_r}{M_d} = \frac{p_1(x_0, y_0, R)c + p_2(x_0, y_0, R)\gamma_t \tan\phi}{q(x_0, y_0, R)\gamma_t} \tag{11}$$

for homogeneous soil

On the basis of the former investigations, strength parameters c, $\tan\phi$ and moist density γ_t in a homogeneous soil layer can be regarded as normal random variables, which lead the factor F_s to a random variable. Therefore the probability P_F of sliding failure of the slope may be calculated. For simplicity, discussions are concentrated on an embankment rapidly constructed on a homogeneous clay layer illustrated in Fig. 16.

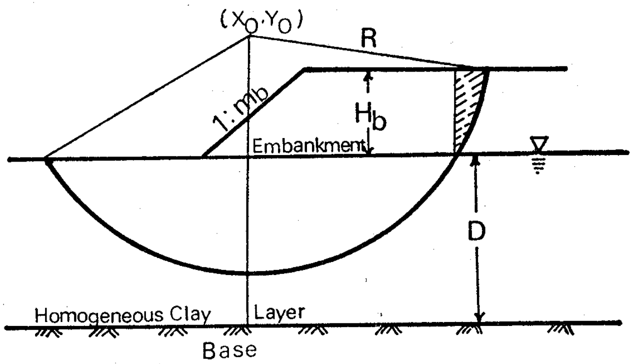


Fig. 16. Embankment on a homogeneous clay layer

Neglecting the driving moment and the shearing resistance of the dotted part, and considering $\phi=0$, and $\gamma_t=\gamma_b$, the functions p and q in Eq. (9) are given by

$$\left.\begin{aligned} p_1(x_0, y_0, R) &= 2R^2 \sin^{-1}\left(\frac{x_2-x_0}{R}\right) = 2R^2 \sin^{-1}(\sqrt{R^2-y_0^2}/R) \\ q(x_0, y_0, R) &= \frac{H_b^2}{6}(2m_bH_b-3x_0) + \frac{H_b}{2}(R^2-y_0^2) - \frac{H_b}{2}(m_bH_b-x_0)^2 \end{aligned}\right\} \quad (12)$$

and the factor F_s is rewritten as follows:

$$F_s = \frac{p_1(x_0, y_0, R)c_u}{q(x_0, y_0, R)\gamma_b} = \frac{c_u}{s} \quad (13)$$

in which s is the mobilized stress, and it is given by

$$\left.\begin{aligned} s &= h(x_0, y_0, R)\gamma_b \\ h(x_0, y_0, R) &= q(x_0, y_0, R)/p_1(x_0, y_0, R) \end{aligned}\right\} \quad (14)$$

In an engineering problem of the slope stability, the minimum value F_s* and the corresponding values (x_0*, y_0*, R*) which give the critical surface are meaningful. F_s* is obtained by the following equation:

$$F_s^* = \frac{c_u}{h^*(x_0^*, y_0^*, R^*)\gamma_b} = \frac{c_u}{s^*} \quad (15)$$

where s^* is the maximum mobilized stress and given by

$$\left.\begin{aligned} s^* &= h^*(x_0^*, y_0^*, R^*)\gamma_b \\ h^*(x_0^*, y_0^*, R^*) &= q^*(x_0^*, y_0^*, R)/p_1^*(x_0^*, y_0^*, R^*) \end{aligned}\right\} \quad (16)$$

On the other hand, c_u and γ_b can be regarded as normal random variables, which are mathematically formulated as

$$\left.\begin{aligned} c_u &= \mu_c + \sigma_c \cdot u_1 \\ \gamma_b &= \mu_{\gamma b} + \sigma_{\gamma b} \cdot u_2 \end{aligned}\right\} \quad (17)$$

in which μ_c and σ_c are the mean value and the standard deviation of c_u, respectively, and u_1 and u_2 are standardized normal random variables independent of each other. From Eqs. (16) and (17), s^* are rewritten as a normal random variable. That is,

$$s^* = h^*(x_0^*, y_0^*, R^*)\,(\mu_{\gamma b} + \sigma_{\gamma b} \cdot u_2) = \mu_s^* + \sigma_s^* \cdot u_2 \quad (18)$$

where μ_s* and σ_s* are the mean value and the standard deviation of s*. They represented by Eq. (19).

$$\left.\begin{aligned} \mu_s^* &= h^*(x_0^*, y_0^*, R^*) \\ \sigma_s^* &= h^*(x_0^*, y_0^*, R^*) \end{aligned}\right\} \quad (19)$$

Therefore, F_s* is a random variable defined as the ratio of two independent normal random variables c_u and s*, and then follows Cauchy's distribution as

$$\left.\begin{aligned} f(F_s^*) &= \frac{1}{\pi B'^2}\left(\sigma_s^*\sigma_c + \frac{1}{B'}\right)\exp\left[-\frac{1}{2}\left(\frac{A'}{B'}\right)^2\right] \\ A' &= \mu_c + \mu_s^* F_s^* \\ B'^2 &= \sigma_c^2 + (\sigma_s^* F_s^*)^2 \end{aligned}\right\} \quad (20)$$

On the other hand, the probability P_F of sliding failure of the embankment is defined as Eq. (21) or Eq. (22).

$$\begin{aligned} P_F = P\,\mathrm{rob}[F_s^* \leq 1.0] &= \int_{-\infty}^{1.0} f(F_s^*)\,dF_s^* \\ &= \int_{-\infty}^{1.0} \frac{1}{\pi B'^2}\left[\sigma_s^*\sigma_c + \frac{1}{B'}\right]\exp\left[-\frac{1}{2}\left(\frac{A'}{B'}\right)^2\right] dF_s^* \end{aligned} \quad (21)$$

$$
\begin{aligned}
P_F &= P\,\mathrm{rob}[C_u \leq s^*] = P\,\mathrm{rob}[z = s^* - c_u \geq 0] \\
&= \int_0^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_s^*\sigma_c}\exp\left[-\frac{1}{2}\left(\frac{s^*-\mu_s^*}{\sigma_s^*}\right)^2 - \frac{1}{2}\left(\frac{s^*-z-\mu_c}{\sigma_c}\right)^2\right]ds^*dz \\
&= \int_0^{\infty} \frac{1}{\sqrt{2\pi(\sigma_s^{*2}+\sigma_c^2)}}\exp\left[-\frac{1}{2}\left\{\frac{z-(\mu_s^*-\mu_c)}{\sqrt{\sigma_s^{*2}+\sigma_c^2}}\right\}^2\right]dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{K_c}^{\infty}\exp\left[-\frac{t^2}{2}\right]dt
\end{aligned} \tag{22}
$$

In Eq. (22), t is the standardized normal ramdom variable and K_c is the lower limit of the integral of the standardized normal curve and given by

$$K_c = (\mu_c - \mu_s^*)/\sqrt{\sigma_s^{*2}+\sigma_c^2} \tag{23}$$

It has been discussed by the authors (1972) that the calculated results of Eqs. (21) and (22) are nearly equal. For futher discussion, denoting the coefficients of variation of c_u and s^* as $V_c=\sigma_c/\mu_c$ and $V_s=\sigma_s^*/\mu_s^*$, K_c is transformed as follows:

$$K_c = \frac{\mu_c/\mu_s^* - 1}{\sqrt{V_s^2+(V_c\mu_c/\mu_s^*)^2}} = \frac{\bar{F}_s^* - 1}{\sqrt{V_s^2+(V_c\bar{F}_s^*)^2}} \tag{24}$$

where $\bar{F}_s^*$ is given by the ratio of the mean values μ_c and μ_s^* as

$$\bar{F}_s^* = \mu_c/\mu_s^* \tag{25}$$

Eq. (24) shows that the factor $\bar{F}_s^*$ corresponds uniquely to the failure probaility P_F under the given values of V_s and V_c. From Eqs. (22) and (24), $\bar{F}_s^*>1.0$ does not always mean the non-failure of the embankment. Therefore, $\bar{F}_s^*$ is insufficient to indicate the safety of the embankment, and then, the authors designate it as a design factor. It is, however, impossible to know the population means μ_c and $\mu_{\gamma b}$. In engineering practice it is rational to use their estimators which can be statistically calculated from the data of the soil exploration and laboratory tests with a given confidence defined as probability. That is, instead of the population mean μ_c of c_u, the following confidence lower limit $\mu_{cL}(n)$ for $\alpha\%$ probability should be used:

$$\text{if } \sigma_c \text{ is known} \qquad \mu_{cL}(n) = \mu_c(n) - \frac{\sigma_c}{\sqrt{n}}Z_\alpha \tag{26}$$

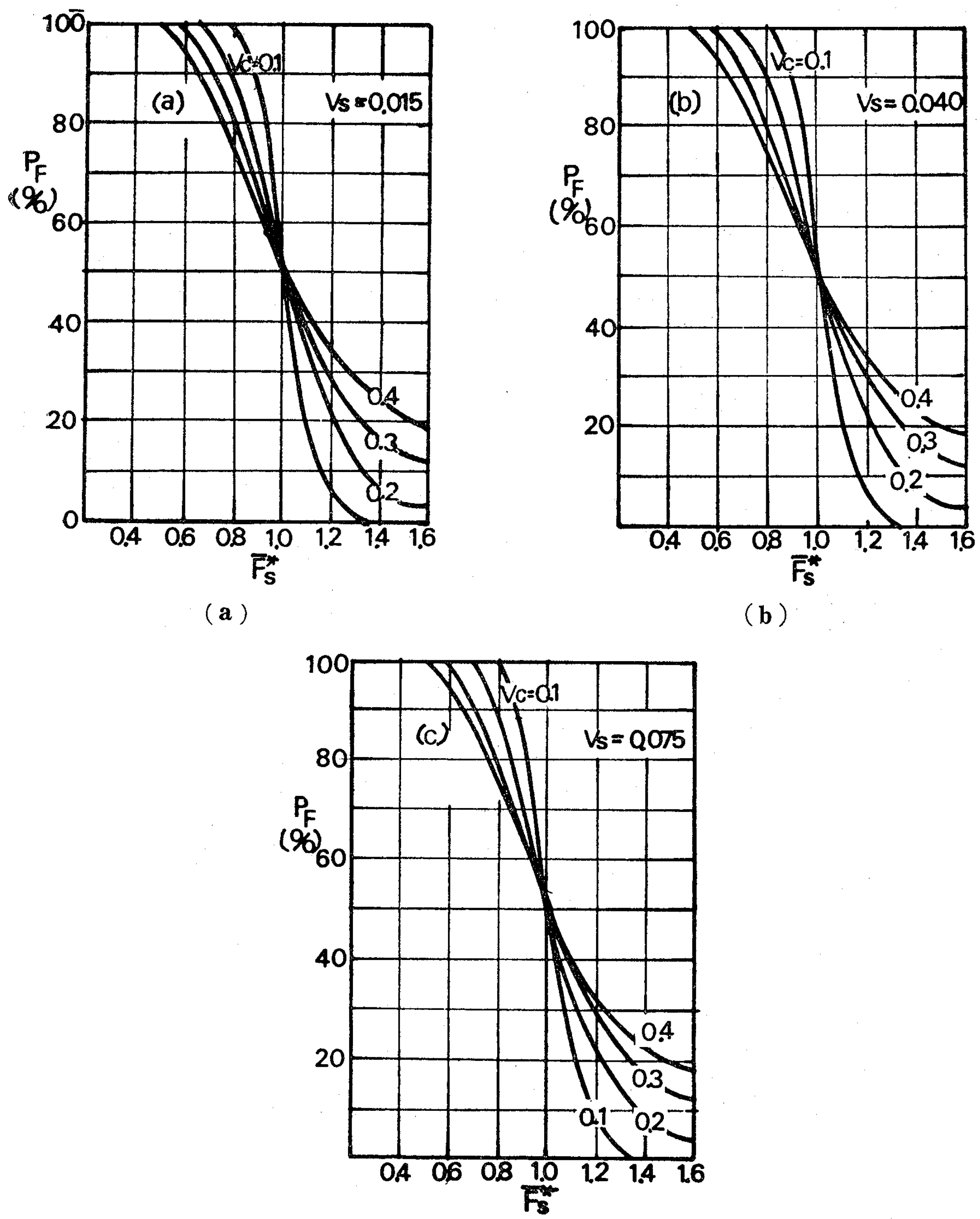
$$\text{or, if } V_c \text{ is known} \qquad \mu_{cL}(n) = \mu_c(n)\left(1 - \frac{V_c}{\sqrt{n}}Z_\alpha\right) \tag{27}$$

where n is the number of the undrained shear tests, and Z_α is the $\alpha\%$ point on the abscissa of the standardized normal curve. $\mu_{\gamma b}$ might depend on the soil and compacting conditions.

Some Numerical and Practical Examples

Some of the numerical results calculated from Eq. (22) are shown in Figs. 17 (a), 17 (b) and 17 (c) with the parameters V_c and V_s. The values of V_c were selected as 0.1, 0.2, 0.3 and 0.4 on the basis of the results given in Table 1 and Fig. 3. The moist density γ_b of an embankment might depend on the conditions of the soils and the compacting method. On the basis of considerations under some practical cases, $V_s=0.015$, 0.040 and 0.075 are therefore adopted as typical values for numerical examples.

From Fig. 17 it should be noticed that the failure probability corresponding to the ordinary values of $\bar{F}_s^*(=1.0\sim1.5)$ in design is unexpectedly great. This figure also shows that the value of V_c sharply influences the relation between $\bar{F}_s^*$ and P_F comparing with the value of V_s, and that the failure probability increases extremely in the range of $\bar{F}_s^*$ less than unity. Bishop and Bjerrum (1960) analyzed many practical failures of slopes by the $\phi_u=0$ method, and made it clear that their safety factors concentrated in the range from



(a) (b) (c)

Fig. 17 (a), (b), (c). Relations between the failure probability P_F and the design factor $\bar{F}_s$*

0.85 to 1.15. This is supported by the results shown in Fig. 17.

A practical example as shown in Fig. 18 was analyzed. This embankment was constructed on a soft saturated clay layer extending to 12 m in depth below the ground surface. It was rapidly constructed up to 4.0 m in height and its safety factor F_s* had been calculated by the $\phi_u=0$ method as 1.13 under the conditions given in the figure. The embankment, however, slided. According to the following considerations, this can be explained in the following manner.

The undrained shear strength of this clay layer was replotted in the form of the frequency distribution as shown in Fig. 19. This can be regarded as the normal distribution whose sample mean $\mu_c(n)$ and the sample standard deviation $\sigma_c(n)$ are 0.159 kg/cm^2 and 0.060 kg/cm^2, respectively. Therefore, the confidence lower limit $\mu_{cL}(n)$ for $\alpha=5\%$ was calculated as 0.1476 kg/cm^2 from Eq. (26). $\sigma_c(n)$ is nearly equal to the population standard

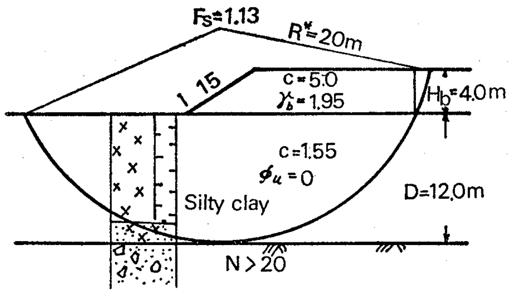


Fig. 18. Profile of practical embankment

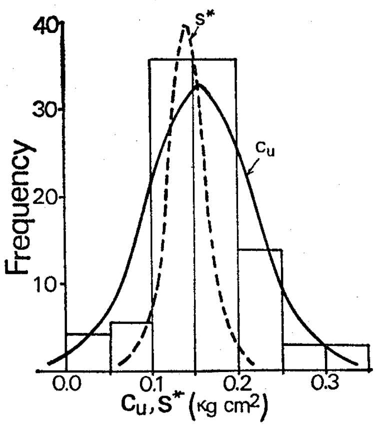


Fig. 19. Frequency distributions of c_u and s^* for the practical case

deviation σ_c because of the large number of tests $n=100$. The distribution of the moist density γ_b of this embankment was determined under the following assumptions; $\mu_{\gamma b}$ was equal to 1.95 t/m³ and V_{γ_b} ($=V_s$) was equal to 0.081 referring to Fig. 9. Using these conditions and Eqs. (22) and (24), the failure probability was calculated as 40%. Then it is obvious that this embankment had a low reliability when the load was rapidly applied.

CONCLUSIONS AND DISCUSSIONS

There are two kinds of variations in the data of soil exploration and laboratory tests: the one is due to the physical nature of soils and the other is attributed to the technical problems. Engineering efforts should be made in order to decrease the variations caused by the technical problems, but usually it is very difficult to separate these two kinds of variations. Therefore it is important to investigate the total variabilities containing two kinds of variations and the method to treat them. The concluding remarks obtained from this point of view are as follows;

(1) The undrained strength parameters c_u and $\tan\phi_u$ can be regarded as normal random variables in a uniform soil layer in the engineering sense, and the values of the coefficient of variation range from 0.1 to 0.4.

(2) The physical properties γ_t, γ_d and w can be also regarded as normal random variables, and their values of the coefficient of variation range from 0.01 to 0.08.

(3) The moist density γ_b of an embankment can be considered as normal random variable.

(4) The angle of shearing resistance of soils has a positive correlation with their dry density γ_d and has a negative one with their moisture content w.

(5) The cohesion c of soils has a negative correlation against the moisture content w, but has no correlation with the dry density γ_d.

(6) The cohesion c of in-situ soils has no correlation with the angle of shearing resistance ϕ, but has a strong negative correlation when the range of the value of moisture content is restricted.

The traditional safety factor can not fully explain the safety of structures because the variations of soil properties are not taken into consideration. It is rational to introduce the probability of failure into design according to the randomness of the soil properties. In this paper, the new design factor $\bar{F}_s^*$ is defined and the failure probability P_F of an embankment is formulated in relation to $\bar{F}_s^*$, and some numerical and practical examples are discussed. From these formulation and examples, the following concluding remarks are obtained:

(7) The failure probability of an embankment constructed on a homogeneous clay layer is determined by $\bar{F}_s^*$, V_s and V_c.

(8) The failure probability of an embankment is more sharply influenced by V_c rather than V_s, and then the accuracy of soil exploration and tests is very important.

ACKNOWLEDGEMENT

The authors acknowledge the cooperation given by the parties concerned with the soil exploration and laboratory tests for offering their data, and express their gratitude to Mr. Kenichi Okumura of the Ministry of Transportation and to Mr. Takuto Obayashi of Japanese Port and Harbour Consultants Corporation for their carrying out the laboratory tests.

NOTATION

A, B = constants
c = apparent cohesion
c_d = cohesion of silty sand under drained condition
c_u = undrained shear strength
F_s = safety factor
$\bar{F}_s^*$ = design factor defined in this paper
H_b = height of embankment
m_b = slope gradient of embankment
M_d = driving moment
M_r = resisting moment
P_F = probability of sliding failure of an embankment
q_u = unconfined compression strength
s = mobilized shear stress
s^* = maximum mobilized shear stress
S_r = degree of saturation
τ_f = shear strength
u, u_1, u_2 = normal random variables
V_c = coefficient of variation of undrained shear strength
V_{q_u} = coefficient of variation of unconfined compression strength
V_s = coefficient of variation of mobilized shear stress
$V_{\tan\phi}$ = coefficient of variation of tangent of angle of shearing resistance
V_w = coefficient of variation of moisture content
V_{γ_b} = coefficient of variation of moist density of embankment
V_{γ_t} = coefficient of variation of moist density of in-situ soil
w = moisture content
σ_N = normal stress
γ_b = moist density of embankment
γ_d = dry density
γ_t = moist density
ϕ = apparent angle of shearing resistance
ϕ_d = angle of shearing resistance under drained condition
ϕ_u = angle of shearing resistance under unconsolidated undrained condition
σ_c = standard deviation of undrained shear strength

$\sigma_c(n)$ = sample standard deviation of undrained shear strength
σ_d = standard deviation of cohesion of silty sand under drained condition
$\sigma_s{}^*$ = standard deviation of maximum mobilized shear strength S^*
σ_{q_u} = standard deviation of unconfined compression strength
$\sigma_{\tan\phi}$ = standard deviation of tangent of angle of shearing resistance
σ_w = standard deviation of moisture content
μ_c = mean of undrained shear strength
$\mu_c(n)$ = sample mean of undrained shear strength
$\mu_{cL}(n)$ = confidence lower limit of the mean of undrained shear strength
μ_d = mean of cohesion of silty sand under drained condition
μ_{q_u} = mean of unconfined compression strength
$\mu_s{}^*$ = mean of maximum mobilized shear stress
$\mu_{\tan\phi}$ = mean of tangent of angle of shearing resistance
μ_w = mean of moisture content
μ_{γ_b} = mean of moisture density of an embankment

REFERENCES

1) Bazovsky, I. (1961): Reliability Theory and Practice, Maruzen Asian Edition, p. 11.
2) Bishop, A. W. and Bjerrum, L. (1960): "The relevance of the triaxial test to the solution of stability problems," Proc. of ASCE, Research Conf. on Shear Strength of Cohesive Soils, pp. 437-501.
3) Committee of Japanese Soc. of SMFE. (1968): A Basic Study on the Shearing Test Method of Soils (in Japanese).
4) Hooper, J. A. and Butler, F. G. (1966): "Some numerical results concerning the shear strength of London clay," Geotechnique, Vol. 16, pp. 282-304.
5) Lambe, T. W. (1960): "Compacted clay; Structure," Trans. of ASCE, Vol. 125, pp. 682-756.
6) Lumb, P. (1966): "The variability of natural soils," Canadian Geotechnical Journal, Vol. 3, No. 2, pp. 74-97.
7) Matsuo, M. and Kuroda, K. (1971): "Study on the soil exploration system for stability analysis of embankment," Trans. of JSCE, Vol. 3, part 2, pp. 218-219, 1972, and originally published in Proc. of JSCE., No. 196, pp. 75-86 (in Japanese).
8) Matsuo, M. and Kuroda, K. (1972): "A stochastic study on some properties and failure probability for unsaturated soils," Trans, of JSCE, Vol. 4, Part 3, pp. 22-23, 1972 and originally published in Proc. of JSCE, No. 208, pp. 65-75 (in Japanese).
9) Meyerhof, G. G. (1970): "Safety factors in soil mechanics," Canadian Geotechnical Journal, Vol. 7, No. 4, pp. 349-355.
10) Nakase, A. (1966): "Contribution to the $\phi_u=0$ analysis of stability," Report No. 1 of Soil Division of Port and Harbour Research Institure.
11) Noorany, L. and Seed, H. B. (1965): "In-situ strength characteristics of soft clays," Proc. of ASCE, Vol. 91, No. SM-2, pp. 49-80.
12) Skempton, A. W. and Sowa, V. (1963): "The behaviour of saturated clays during sampling and testing," Geotechnique, Vol. 13, No. 4, pp. 269-290.
13) Wu, T. H. and Kraft, L. M. (1967): "The probability of foundation safety," Proc. of ASCE, Vol. 93, No. SM-5, pp. 213-231.

(Received September 3, 1973)