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A note on the weighted matching with penalty problem

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Abstract

Given a weight on each edge and a penalty on each vertex of a bipartite graph, the bipartite weighted matching with penalty problem is to find a matching that minimises the sum of the weights of the matched edges and the penalties of the unmatched vertices. It is shown that this problem can be reduced easily to the standard bipartite weighted matching problem. The method easily generalises to graphs that are not necessarily bipartite.

Keywords: Weighted matching; Bipartite graph; General graph

The bipartite weighted matching problem (BWMP) is to find a minimum weight maximum matching in a bipartite graph. In some problems, notably character recognition, we wish to modify the restriction that the matching be maximum. Cheng et al. (1989) modify the BWMP by introducing dummy vertices and show that their problem can be reduced to the assignment problem (AP). Hsieh et al. (1995a,b) modify the BWMP by introducing penalties on unmatched vertices to get the bipartite weighted matching with penalty problem (BWMPP). They show that it can be reduced to the AP, and apply it to the problem of recognising handwritten Chinese characters. This note will describe a simple reduction of the BWMPP to a bipartite maximum weight matching problem.

Let G be a graph. Let $w(uv)$ denote the weight on the edge uv, and let $s(v)$ denote the penalty on vertex v if it is unmatched. Write $v \sim M$ if the vertex v is not matched by the matching M. Define the *weighted matching with penalty problem* (WMPP) to be that of finding a matching M of G minimising

$$\omega(M) := \sum_{uv \in M} w(uv) + \sum_{v \sim M} s(v).$$

Note that if G is a complete bipartite graph, the WMPP is the same as the BWMPP defined by Hsieh et al. (1995b). We will write BWMPP to refer to the WMPP on any (not necessarily complete) bipartite graph.

Consider the WMPP above. Define $w^*(uv) = w(uv) - s(u) - s(v)$ for each edge uv of G. Construct a new graph G' with the same vertices as G and with an edge uv whenever uv is an edge of G with $w^*(uv) \leqslant 0$. Define weights on the edges of G' by putting $w'(uv) = -w^*(uv)$ for each edge uv, and define $\omega'(M) := \sum_{uv \in M'} w'(uv)$ for each matching M of G'.

Proposition 1. *A matching M of G is a solution to the WMPP if and only if it is a matching of G' that maximises $\omega'(M)$.*

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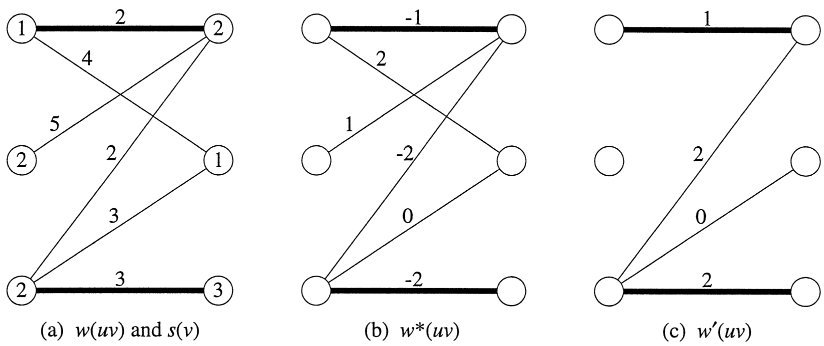

Fig. 1. Reduction of BWMPP to maximum weight matching problem.

Proof. Let T be the sum of the penalties of all the vertices of G. For any matching M of G, $\omega^*(M) := \sum_{uv \in M} w^*(uv) = \sum_{uv \in M} w(uv) - T + \sum_{v \sim M} s(v) = \omega(M) - T$. Thus, M is a solution of the WMPP if and only if it minimises $\omega^*(M)$. If a matching M minimises $\omega^*(M)$ then clearly it cannot contain an edge uv with $w^*(uv) > 0$. Thus we need only consider matchings of G that are also matchings of G'. A matching M of G' minimises $\omega^*(M)$ if and only if it maximises $-\omega^*(M) = \omega'(M)$. Hence a matching M is a solution to the WMPP if and only if it maximises $\omega'(M)$ over all matchings of G'. □

Note that if G is bipartite, then G' is also bipartite; so the BWMPP can be reduced to the bipartite maximum weight matching problem of maximising $\omega'(M)$ over all matchings of G'.

Consider as an example the bipartite graph G of Fig. 1(a) where the weights $w(uv)$ are shown against their edges and the penalties $s(v)$ on the vertices. Fig. 1(b) shows the same graph with $w^*(uv)$ against each edge uv. Fig. 1(c) shows the graph G' with $w'(uv)$ against each edge uv. In this example, $T = 1 + 2 + 2 + 2 + 1 + 3 = 11$, and a maximum weight matching M in G' and the same matching in G are shown by the thick edges. Here we have $\omega'(M) = 3$, $\omega^*(M) = -\omega'(M) = -3$ and $\omega(M) = \omega^*(M) + T = 8$.

Fig. 2 shows an example of the reduction of the (general) WMPP to a maximum weight matching problem. The example is worked out exactly as the previous example, and we have $T = 20$, $\omega'(M) = 7$, $\omega^*(M) = -\omega'(M) = -7$ and $\omega(M) = \omega^*(M) + T = 13$.

The reduction to a maximum weight matching problem is not only easier than the reduction of Hsieh et al. (1995b) to an AP, but also produces a smaller problem. Bipartite maximum weight matching, BWMP and AP are all solved by essentially the same method. Galil (1986) describes efficient algorithms for finding maximum weight matchings in graphs. Balas and Pulleybank (1989) use a method similar to the one described above to show how to find a matching that maximises the sum of the

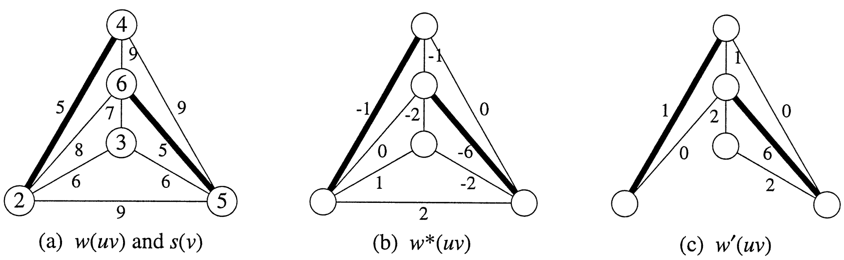

Fig. 2. Reduction of WMPP to maximum weight matching problem.

weights of matched vertices in a graph with vertex weights but without edge weights.

References

Balas, E., Pulleybank, W.R., 1989. The perfectly matchable subgraph polytope of an arbitrary graph. Combinatorica 9 (4), 321–337.

Cheng, F.H., Hsu, W.H., Chen, C.A., 1989. Fuzzy approach to solve the recognition problem of handwritten Chinese characters. Pattern Recognition 22, 133–141.

Galil, Z., 1986. Efficient algorithms for finding maximal matchings in graphs. Computing Surveys 18 (1), 23–38.

Hsieh, A.-J., Fan, K.-C., Fan, T.-I., 1995a. Bipartite weighted matching for on-line handwritten Chinese character recognition. Pattern Recognition 28, 143–151.

Hsieh, A.-J., Ho, C.-W., Fan, K.-C., 1995b. An extension of the bipartite weighted matching problem. Pattern Recognition Lett. 16, 347–353.