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Local characterization of strong surfaces within strongly separating objects

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Abstract

In (Bertrand and Malgouyres, 1996), a characterization of discrete surfaces of $\mathbb{Z}^3$ is proposed which is called *strong surfaces*. However, strong surfaces are defined by global properties and the question of their local characterization remains. We propose a local characterization, within the separating and thin objects, of strong surfaces. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the framework of Digital Topology and 3D thinning algorithms with surface skeletons, the search for a definition of surfaces as subsets of $\mathbb{Z}^3$ has been developed since the early 80's (see (Morgenthaler and Rosenfeld, 1981)). The present paper follows a first paper (Bertrand and Malgouyres, 1996) where we proposed a topological characterization for surfaces of $\mathbb{Z}^3$. This approach can be called *graph theoretical* since we consider the classical adjacency relations between points and the derived notions of connectivity, homotopy (see (Kong and Rosenfeld, 1989; Bertrand and Malandain, 1994)), and strong homotopy (see (Bertrand, 1995)). The originality of our approach is that, instead of defining surfaces directly by local conditions, we characterize in (Bertrand and Malgouyres, 1996) surfaces by ''simple'' *global properties*, and now try to find an equivalent *local characterization* which might not have any semantic content. This paper is a significant step in this direction.

Now let us introduce the global properties on which is based our characterization of surfaces:

We say that a subset X of $\mathbb{Z}^3$ satisfies the property Π_1 if and only if the complement of X has two 6-connected components A and B. Then the set X is said to satisfy the property Π_2 if any point of X is 6-adjacent to both A and B. We say that the set X is *strongly separating* if it satisfies Π_1 and Π_2.

It is proved in (Malgouyres, 1996) that the class of strongly separating objects can not be *locally* characterized, so that we must impose another property for characterizing surfaces. A strongly separating set X is said to satisfy the property Π_3' if A is homotopic to $A \cup X$ and similarly for B. It is also shown in Bertrand and Malgouyres, 1996 that the class of strongly separating sets satisfying Π_3' can not be locally characterized. So we introduce in (Bertrand and Malgouyres, 1996) a stronger property

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Π_3: a strongly separating set X is called a *strong surface* if A is strongly homotopic to $A \cup X$ and similarly for B. Note that knowing whether the class of continuous objects (in $\mathbb{R}^3$) satisfying properties analogous to Π_1, Π_2 and Π_3 corresponds to the class of topological surfaces is still a conjecture.

In (Bertrand and Malgouyres, 1996), it is proved that the Morgenthaler's 26-surfaces defined in (Morgenthaler and Rosenfeld, 1981) and the surfaces defined in (Malgouyres, 1997); are particular cases of strong surfaces. Besides, we introduced some necessary and sufficient conditions for a strongly separating set to be a strong surface. These properties are based on the notion of *geodesic neighbourhoods* (see (Bertrand, 1994)). However, these conditions are still not local and the purpose of this paper is to propose some local properties which characterize those of the strongly separating set which are strong surfaces. We emphasize that these properties are the first proposed local necessary conditions for strong surfaces.

2. Basic notions and notations

For $X \subset \mathbb{Z}^3$ we denote by $\overline{X}$ the complement of X. For $x = (i,j,k) \in \mathbb{Z}^3$, we consider the three following neighbourhoods:

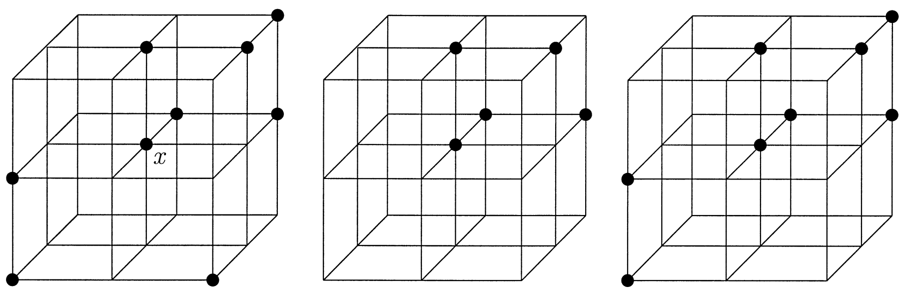
- $N_{26}(x) = \{x' = (i',j',k') \in \mathbb{Z}^3 \mid \max(|i'-i|,|j'-j|,|k'-k|) = 1\}$,
- $N_6(x) = \{x' = (i',j',k') \in \mathbb{Z}^3 \mid |i'-i| + |j'-j| + |k'-k| = 1\}$,
- $N_{18}(x) = N_{26}(x) \cap \{x' = (i',j',k') \in \mathbb{Z}^3 \mid |i'-i| + |j'-j| + |k'-k| \le 2\}$.

For $n = 6,18,26$, two points x and y are said to be *n-adjacent* if $y \in N_n(x)$.

For $X \subset \mathbb{Z}^3$ and $x \in \mathbb{Z}^3$ we denote by X^x the set $X \cap N_{26}(x)$. An *n-path* from a point x to a point y is a finite sequence $x = P_1, \ldots, P_k = y$ with $k \ge 2$ such that for $i = 1, \ldots, k-1$ the point P_i is n-adjacent to P_{i+1}. The *length* of such an n-path is the number k of points of the n-path (including the extremities). Two points $x,y \in X$ are said to be *in the same n-connected component* of X if there is an n-path from x to y which is included in X. This defines an equivalence relation on points of X and the corresponding equivalence classes are called *n-connected components* of X. X is said to be *n-connected* if it has a single n-connected component. We denote by $C_n(X)$ the set of n-connected components of X. The set of all n-connected components of X which are n-adjacent to a given point x is denoted by $C_n^x(X)$. Observe that $C_n(X)$ and $C_n^x(X)$ are sets of subsets of X and not sets of points. We denote by $card(C_n(X))$ and $card(C_n^x(X))$ respectively the cardinality of $C_n(X)$ and the cardinality of $C_n^x(X)$.

As in the 2 case, if we use n-connectivity for X, we have to use another $\overline{n}$-connectivity for $\overline{X}$. In the following, $(n,\overline{n}) \in \{(26,6),(18,6^+)\}$, where a 6^+-notion (connectivity, adjacency, etc.) for $\overline{X}$ will stand for the corresponding 6-notion when associated with 18-connectivity.

As far as *homotopy preservation* is concerned, we refer to (Kong and Rosenfeld, 1989) and (Bertrand, 1994). An *n-simple point* in an object $X \subset \mathbb{Z}^3$ is a point the deletion of which preserves the homotopy type of the object X when X is analyzed with n-connectivity. An object $Y \subset X$ is said to be (*lower*) *n-homotopic* to the set X if and only if Y can be obtained from X by deleting *sequentially* n-simple points.



(a) The set X (b) $G_6(x,X)$ $G_{18}(x,X)$

Fig. 1.

To introduce the characterization of n-simple points given in (Bertrand and Malandain, 1994), and later the characterization of strong surfaces obtained in (Bertrand and Malgouyres, 1996), we need the notion of a *geodesic neighbourhood*. Let X be an object of $\mathbb{Z}^3$ and $x \in \mathbb{Z}^3$. The *geodesic n-neighbourhood of order k of x inside X* is the set $N_n^k(x,X)$ defined recursively by:

$$N_n^0(x,X) = \emptyset,$$

$$N_n^1(x,X) = N_n(x) \cap X, \quad \text{and}$$

$$N_n^k(x,X) = \bigcup_{y \in N_n^{k-1}(x,X)} (N_n(y) \cup \{y\}) \cap N_{26}(x) \cap X.$$

In other words, $N_n^k(x,X)$ is the set composed of all points y of $N_{26}(x) \cap X$ such that there exists an n-path π from x to y of a length less than or equal to $k+1$, all points of π, except possibly x, belonging to $N_{26}(x) \cap X$.

Definition 1. Let $X \subset \mathbb{Z}^3$ and $x \in \mathbb{Z}^3$.

The *geodesic neighbourhoods* $G_n(x,X)$ are defined by:

$$G_6(x,X) = N_6^2(x,X),$$

$$G_{6^+}(x,X) = N_6^3(x,X),$$

$$G_{18}(x,X) = N_{18}^2(x,X),$$

$$G_{26}(x,X) = N_{26}^1(x,X).$$

The *topological numbers* $T_n(x,X)$ are defined by:

$$T_6(x,X) = card(C_6[G_6(x,X)]),$$

$$T_{6^+}(x,X) = card(C_6[G_{6^+}(x,X)]),$$

$$T_{18}(x,X) = card(C_{18}[G_{18}(x,X)]),$$

$$T_{26}(x,X) = card(C_{26}[G_{26}(x,X)]).$$

Fig. 1 shows an example of a point x in a set X and the corresponding sets $G_6(x,X)$ and $G_{18}(x,X)$.

The following proposition (Bertrand, 1994) illustrates the strong relationship between topological numbers and topology preservation:

Proposition 2. Let $X \subset \mathbb{Z}^3$ and $x \in X$. Then x is an n-simple point if and only if $T_n(x,X) = 1$ and $T_{\bar{n}}(x,\overline{X}) = 1$.

3. Strong homotopy and strong surfaces

The purpose of this section is to set the definition of a strong surface originally introduced in (Bertrand and Malgouyres, 1996). First, we must recall the notions of *strong homotopy* following Bertrand (1995).

Definition 3. Let $X \subset \mathbb{Z}^3$ and $Y \subset X$. The set Y is *strongly* (*lower*) *n-homotopic* to X if for any $Y \subset Z \subset X$, the set Z is (lower) n-homotopic to X.

Now we can introduce *strong surfaces*.

An object X is said to be *separating* if $\overline{X}$ has exactly two 6-connected components. A separating set X is said to be *strongly separating* if any point in X is 6-adjacent to both 6-connected components of $\overline{X}$. It was proved in (Malgouyres, 1996) that the class of strongly separating sets, together with some of its natural subclasses (see also (Bertrand and Malgouyres, 1996)), can not be *locally characterized*. Therefore we have to introduce a stronger (but reasonable) condition on strongly separating subsets for which such a phenomenon does not occur. The property on which is based the notion of a strong surface, for which a natural analogue can be stated and proved for surfaces in the continuous framework, is based on the notion of strong homotopy:

Definition 4. Let X be a strongly separating set, and let A and B be the two 6-connected components of $\overline{X}$. The set X is called a (*simple*) *strong n-surface* if A is strongly n-homotopic to $X \cup A$ and B is strongly n-homotopic to $X \cup B$.

It was proved in (Bertrand and Malgouyres, 1996) that a Morgenthaler's simple 26-surface (Morgenthaler and Rosenfeld, 1981) must be both a strong 18-surface and a strong 26-surface, while the surfaces of Malgouyres, to appear, which also generalize the Morgenthaler's simple 26-surface, are strong 18-surfaces but are not, in general, strong 26-surfaces.

The following proposition is proved in (Bertrand and Malgouyres, 1996).

Proposition 5 Let X be a strongly separating set, and let A and B be the two 6-connected components of $\overline{X}$. The set X is a strong n-surface if and only if for any $x \in X$ the four following conditions are satisfied:

1. $T_n(x, A^x) = 1$ and $T_n(x, B^x) = 1$,
2. $T_{\bar{n}}(x, A^x) = 1$ and $T_{\bar{n}}(x, B^x) = 1$,
3. $\forall y \in N_n(x) \cap X$, $T_n(x, A^x \cup \{y\}) = 1$ and $T_n(x, B^x \cup \{y\}) = 1$,
4. $\forall y \in N_{\bar{n}}(x) \cap X$, $T_{\bar{n}}(x, A^x \cup \{y\}) = 1$ and $T_{\bar{n}}(x, B^x \cup \{y\}) = 1$.

In Fig. 2, several examples of local configurations are represented. The set X is represented by filled circles, and points of the union of the elements of $C_6^x(N_{26}(x) \cap \overline{X})$ are marked with circles or squares depending on which 6-connected component of $N_{26}(x) \cap \overline{X}$ they belong to. Provided the X is strongly separating, all configurations of Fig. 2 satisfy all conditions of Proposition 5 except that

- the configuration (b) does not satisfy condition 3 for the point y represented and for $n = 26$;
- the configuration (c), in the case $n = 26$, does not satisfy condition 3 for one of the points y represented, depending on whether the point M belongs to A^x or B^x;
- the configuration (d), in the case $n = 26$, does not satisfy condition 4 for the point y represented and the connected component marked with circles (see also Lemma 9);
- finally, to know whether the configuration (e) satisfies condition 3 for $n = 18$ and for the point y represented, we need to know to which 6-connected component of $\overline{X}$ the point M belongs.

Observe that, in this characterization of strong n-surfaces within the class of strongly separating subsets, all conditions are local (i.e. deal only with properties of the 26-neighbourhoods of points of X), *except* that the definition of A^x and B^x is global ($A^x = A \cap N_{26}(x)$ and $B^x = B \cap N_{26}(x)$). However, since all the conditions of Proposition 5 are symmetric in A^x and B^x, to check these properties the only thing we need is an *assignment*:

Definition 6. Let X be a strongly separating object and let A and B be the two 6-connected components of $\overline{X}$. A *labeling* for X is, for all $x \in X$, of a map $f_x: N_{26}(x) \cap \overline{X} \to \{0,1\}$ which is constant on 6-connected components of $N_{26}(x) \cap \overline{X}$.

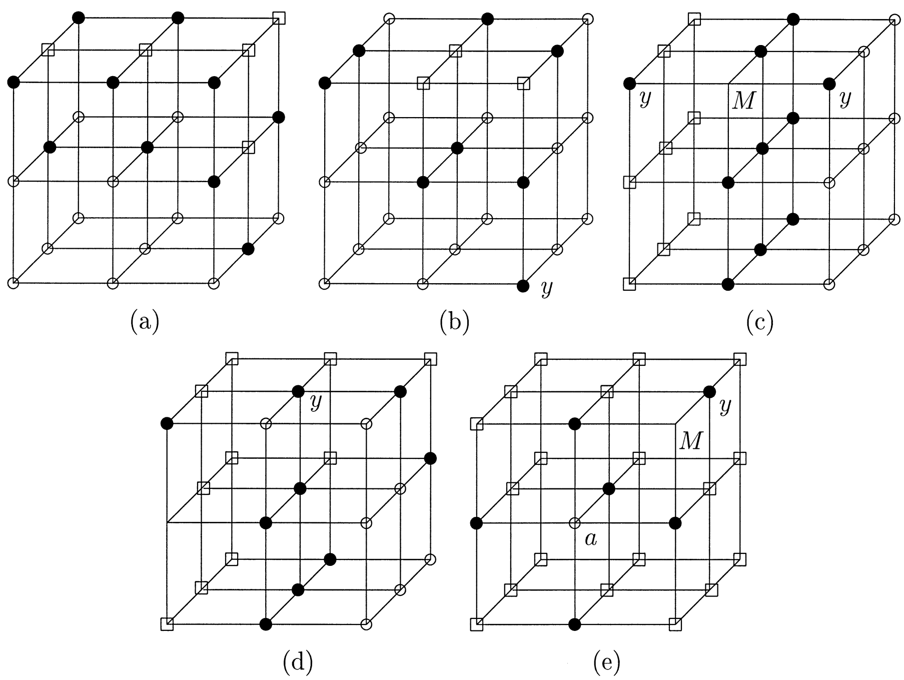


Fig. 2. Examples of local configurations.

Two labelings $\{f_x \mid x \in X\}$ and $\{g_x \mid x \in X\}$ for X are said to be the *same* if for any $x \in X$, either $g_x = f_x$, or $g_x = 1 - f_x$. Observe that if $\{f_x \mid x \in X\}$ is a labeling, then f_x is defined only on $N_{26}(x) \cap \overline{X}$.

Definition 7. An *assignment* for X is a labeling for X such that for $x \in X$ we have:

$$\left\{f_x^{-1}(0), f_x^{-1}(1)\right\} = \{A^x, B^x\}.$$

For example, in Fig. 2(e) with $n = 18$, if $\{f_x \mid x \in X\}$ is an assignment, then the point x satisfies condition 3 of Proposition 5 for the point y represented if and only if $f_x(M)$ is equal to the value of f_x at the point a (marked with a circle).

Notation. Let X be a strong surface, and A and B be the two 6-connected components of $\overline{X}$. From Condition 2 of Proposition 5 it follows that for $x \in X$, $card(C_6^x(A^x)) = 1$ and $card(C_6^x(B^x)) = 1$. We denote by A^{xx} (respectively B^{xx}), the unique element of $C_6^x(A^x)$ (respectively the unique element of $C_6^x(B^x)$) which is included in A (respectively in B).

4. A fundamental proposition

In this section, we prove two technical lemmas and a fundamental proposition about the structure of elements of $C_6(N_{26}(X) \cap \overline{X}) \setminus \{A^{xx}, B^{xx}\}$ where X is a strong n-surface and $x \in X$.

Lemma 8. Let $m \in \{6, 6^+, 18, 26\}$ and $d \in \mathbb{N}$. Let $D \subset \mathbb{Z}^3$ and $x \in \overline{D}$ such that $card(C_m(N_m^d(x,D))) = 1$, and $y \in N_m(x) \cap \overline{D}$. Then, $card(C_m(N_m^d(x, D \cup \{y\}))) = 1$ if and only if there exists an m-path π from y to x, with a length of at least 3 and at most $2d+1$, all the points of π belonging to D except x and y.

Proof. ''If'' Let $\pi = (y = P_1, \ldots, P_k)$, with P_k m-adjacent to x be an m-path with length $k \le 2d$, all points of π lying in $D \cup \{y\}$. It is sufficient to prove that $\pi \subset N_m^d(x, D \cup \{y\})$. Now, from the very definition of the geodesic neighbourhoods, $\{P_1, \ldots, P_d\} \subset N_m^d(x, D \cup \{y\})$, and $\{P_{d+1}, \ldots, P_k\} \subset N_m^d(x,D) \subset N_m^d(x, D \cup \{y\})$.

''Only if'' Hence we assume that $card(C_m(N_m^d(x, D \cup \{y\}))) = 1$. Since $card(C_m(N_m^d(x,D))) = 1$, the set $N_m^d(x,D)$ is nonempty. Since $card(C_m(N_m^d(x, D \cup \{y\}))) = 1$, there exists an m-path $\pi_1 = P_1, \ldots, P_k$ in $N_m^d(x, D \cup \{y\})$ from y to a point $P_k \in N_m^d(x,D)$. Moreover, we may assume that for $i = 2, \ldots, k-1$ the point P_i is contained in $N_m^d(x, D \cup \{y\}) \setminus N_m^d(x,D) \subset N_n^{d-1}(y, D \cup \{y\})$. Hence there is an m-path α from y to P_{k-1}, with a length of at most d, which is contained in $D \cup \{y\}$.

Now, the point P_k belongs to $N_m^d(x,D)$, hence there is an n-path β from P_k to x, with a length of at most $d+1$, all points of β except x lying in D. The concatenation $\alpha * \beta$ is an m-path with a length of at most $2d+1$ which satisfies the conditions required in Lemma 8. □

Notations. We set $l(6) = 5$, $l(6^+) = 7$, $l(26) = 3$ and $l(18) = 5$.

Lemma 9. Let X be a strong n-surface, and let A and B be the two 6-connected components of $\overline{X}$. Let $m \in \{n, \overline{n}\}$ and $x \in X$. Then, for any $y \in N_m(x) \cap X$, there exits an m-path π of length at least 3 and at most $l(m)$ from y to x, all points of π except x and y lying in A^x (similarly for B^x).

Proof. Lemma 9 follows directly from points 3 and 4 of Proposition 5 and Lemma 8. □

Definition 10. Let $X \subset \mathbb{Z}^3$ be strongly separating. We say that X is *labelable* if for any $x \in X$ we have:
1. $T_{\overline{n}}(x, \overline{X}) = 2$;
2. $\forall C \in C_6(N_{26}(x) \cap \overline{X}) \setminus C_6^x(N_{26}(x) \cap \overline{X})$,
   2.1. C has a single point: $C = \{M\}$ with $M = (a,b,c)$;
   2.2. M is not 18-adjacent to x;
   2.3. $Z = \{(a,j,k), (i,b,k), (i,j,c)\} \subset \overline{X}$.

In Fig. 2, only the configuration (c) is not labelable and it is condition 2(a) of the definition of labelable sets which is not satisfied in this case.

Observe that the definition of labelable sets is completely local. We shall see below that, for a labelable set X, we can construct a particular labeling which, in the case X is a strong surface, is an

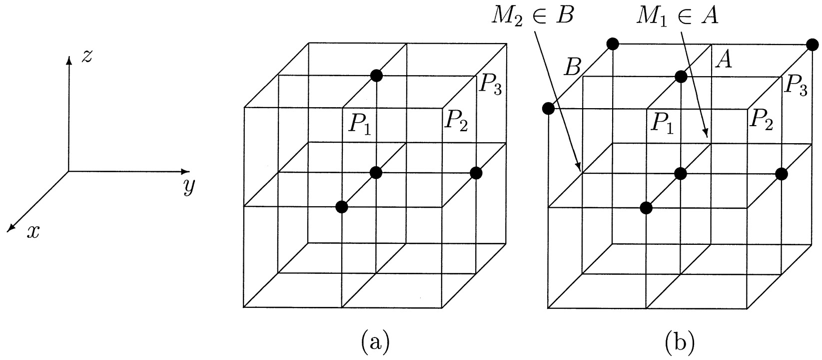


Fig. 3.

assignment. Now we can state and prove the fundamental proposition which characterizes the local structure of strong surfaces:

Proposition 11. Let X be a strong n-surface and let A and B be the two 6-connected components of $\overline{X}$. Then,

1. X is labelable,
2. let $x=(i,j,k)\in X$, $C=\{M\}=\{(a,b,c)\}\in C_6(N_{26}(x)\cap\overline{X})\setminus\{A^{xx},B^{xx}\}$ and let $Z=\{(a,j,k),(i,b,k),(i,j,c)\}$. If Z contains at least two points of A, then $M\in B^x$. Otherwise, $M\in A^x$.

Observe that point 2, in spite of its appearance, is symmetric in A and B.

Proof. Point 1 of the definition of a labelable set follows directly Proposition 5. We prove point 2(a) of the definition of labelable sets. The proofs of all other points, which could not appear here for the sake of brevity, use similar arguments and are rather easier.

So we consider $C\in C_6(N_{26}(x)\cap\overline{X})\setminus\{A^{xx},B^{xx}\}$ For convenience, we assume (up to a translation) that $i=j=k=0$.

First suppose that there are in C three points P_1, P_2, P_3 such that (P_1,P_2,P_3) is a 6-path and P_1 is 18-adjacent to P_3. Up to an isometry, we have $P_1=(1,0,1)$, $P_2=(1,1,1)$ and $P_3=(0,1,1)$. Besides, since P_1, P_2, and P_3 do not belong to $A^{xx}\cup B^{xx}$, the points $(0,0,1)$, $(1,0,0)$ and $(0,1,0)$ must be in X (see Fig. 3(a)).

Since $y=(0,0,1)\in N_6(x)\cap X$, from Lemma 9 it follows that the points $(-1,0,1)$ and $(0,-1,1)$ lie one in A^{xx} and one in B^{xx}, say $(-1,0,1)\in A^{xx}$ and $(0,-1,1)\in B^{xx}$. Since $(-1,0,1)$ and $(0,-1,1)$ must be respectively in A^{xx} and B^{xx}, they cannot be 6-connected to P_1 or P_3 in $N_{26}(x)\cap\overline{X}$. Therefore,

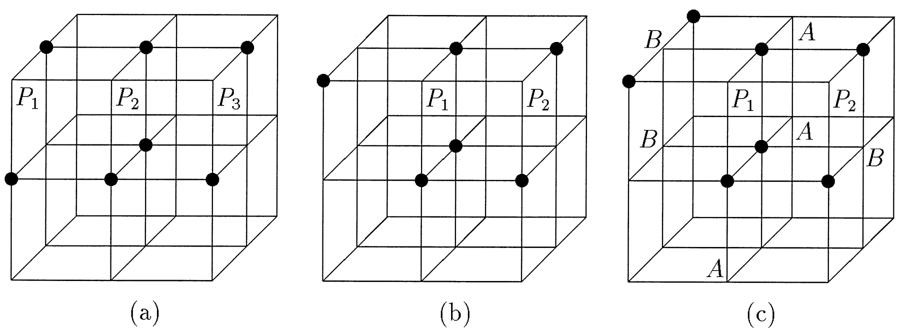


Fig. 4.

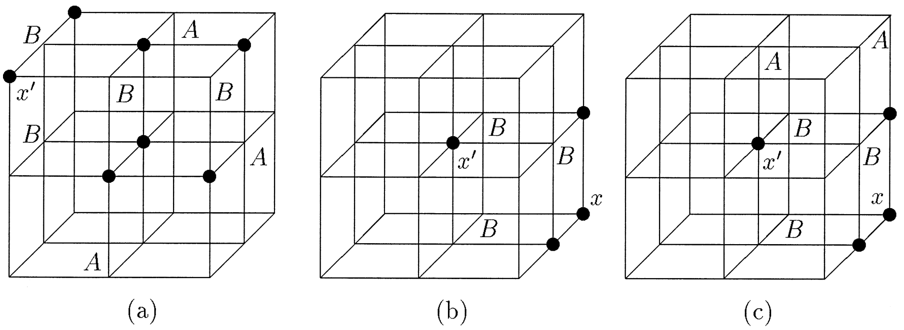


Fig. 5.

$(1,-1,1)\in X$ and $(-1,1,1)\in X$. Moreover, $(-1,-1,1)\in X$. Hence, since $(-1,0,1)$ and $(0,-1,1)$ are respectively in A^{xx} and B^{xx}, the points $M_1=(-1,0,0)$ and $M_2=(0,-1,0)$ must be respectively in A and B (see Fig. 3(b)).

Now, since $y=(0,1,0)\in N_6(x)\cap X$, it follows from Lemma 9 that there exists a 6-path $(0,1,0)=h_1,h_2,\dots,h_k=x$ with $h_i\in B$ for $2\le i\le k-1$, and $3\le k\le l(\overline{n})\le 7$. This implies that the point $M=(0,0,-1)\in B$. But similarly, $M\in A$, and we obtain a contradiction. Therefore, there cannot exist three points P_1, P_2, $P_3\in C$ such that (P_1,P_2,P_3) is a 6-path and P_1 is 18-adjacent to P_3. Therefore, C has at most three points and, up to an isometry, there are only four possible cases for C.

First case: If C has exactly three points P_1, P_2, and P_3. Up to an isometry, we may then assume that $P_1=(1,-1,1)$, $P_2=(1,0,1)$ and $P_3=(1,1,1)$ (see Fig. 4(a)).

Now, since $(0,0,1)\in N_6(x)\cap X$, it follows from Lemma 9 that $(0,0,1)$ is 6-adjacent to both A^{xx} and B^{xx} which is impossible since it is 6-adjacent to only one point of $\overline{C\cup X}$.

Second case: If C has exactly two points P_1 and P_2. Up to an isometry, we may assume that $P_1=(1,0,1)$ and $P_2=(1,1,1)$ (see Fig. 4(b)).

Since $(0,0,1)\in N_6(x)\cap X$, it follows from Lemma 9 that, say, $(-1,0,1)\in A^{xx}$ and $(0,-1,1)\in B^{xx}$. Then the point $(-1,-1,1)$ must be in X. Hence, since $(0,-1,1)\in B^{xx}$, the point $(0,-1,0)$ must be in B. Furthermore, since $(1,0,0)\in N_6(x)\cap X$ it follows from Lemma 9 that $(1,0,-1)\in A^{xx}$.

- If $(0,1,0)\in B$, since $(-1,0,1)\in A^{xx}$, the point $(-1,0,0)$ must lie in A. Now, since $(0,-1,0)\in B$ and $(0,1,0)\in B$, $t_6(x,B^x)\ge 2$ (see Fig. 4 (c)), and we obtain a contradiction.
- Hence $(0,1,0)\in A\cup X$. Moreover, $(0,1,0)$ can not be in X since it would not satisfy Lemma 9 with B. So we obtain: $(0,1,0)\in A$.

Since $(0,1,1)\in N_n(x)\cap X$, it follows from Lemma 9 that $(0,1,1)$ must be n-adjacent to a point of B. This point can be nowhere but in C, and therefore $C\subset B$ (see Fig. 5(a)).

Now we observe the 26-neighbourhood of the point $x'=(1,-1,1)$ (see Fig. 5(b)). We set $(p,q,r)=x'=(1,-1,1)$. In the case $n=26$, this configuration is impossible since $(p-1,q+1,r-1)\in N_{26}(x')\cap X$ and cannot be 26-adjacent to a point of $N_{26}(x')\cap A$, which contradicts Lemma 9. Hence $n=18$. Since $(p-1,q+1,r)\in N_{18}(x')\cap X$, it follows from Lemma 9 that $(p-1,q+1,r+1)\in A$ and $(p,q,r+1)\in A$ (see Fig. 5(c)). Since $(p,q+1,r-1)\in N_{18}(x')\cap X$, it follows from Lemma 9 that either $(p+1,q,r)\in A$, or $(p+1,q,r-1)\in A$. In both cases, B^{xx} cannot be connected and we obtain a contradiction. Therefore, we have proved that C is a singleton. □

5. Main results

In this section, we provide completely local conditions which characterize strong n-surfaces within strongly separating objects. We proceed as follows:

(1) In the previous section, we proved, assuming that an object X is a strong surface, some necessary conditions on elements of $C_6(N_{26}(x)\cap\overline{X})\setminus(A^{xx}\cup B^{xx})$, and in particular on the assignment defined by

$f_x(y)=0$ if $x\in A$ and $f_x(y)=1$ if $x\in B$. These necessary conditions enable us to define below a canonical labeling (Definition 6) for X;

(2) We show that for any strongly separating subset X, the labeling thus obtained is, in fact, an assignment for X so that the conditions of Proposition 5 can be checked with this assignment. Since the assignment is obtained by a *local* construction, the conditions thus obtained are completely local.

Given a labelable object X, we define the *labeling associated with* X as follows:

For $x=(i,j,k)\in X$,

(1) Since $T_{\bar{n}}(x,\overline{X})=2$ and X is strongly separating, it follows that $card(C_6^x(N_{26}(x)\cap\overline{X}))=2$. Let $C_6^x(N_{26}(x)\cap\overline{X})=\{\chi_1,\chi_2\}$. We define $f_x(M)=0$ if $M\in\chi_1$ and $f_x(M)=1$ if $M\in\chi_2$. Now the labeling has to be defined in elements of $C_6((N_{26}(x)\cap\overline{X}))\setminus(\chi_1\cup\chi_2)$.

(2) Let $C\in C_6(N_{26}(x)\cap\overline{X})\setminus\{\chi_1,\chi_2\}$. Since X is labelable, $C=\{M\}=\{(a,b,c)\}$ with $M\notin N_{18}(x)$. Let $Z=\{(a,j,k),(i,b,k),(i,j,c)\}\subset\overline{X}$. The labeling f_x is already defined on Z. We define:

$$\begin{cases} f_x(M)=1 & \text{if } \chi_1\cap Z \text{ contains at least two points},\\ f_x(M)=0 & \text{otherwise}.\end{cases}$$

The unique labeling $\{f_x \mid x\in X\}$ thus defined is called the *labeling associated with* X. Observe that the construction of the labeling associated with X is completely local.

Theorem 12. Let X be a strongly separating set. Then X is a strong n-surface if and only if X is labelable (let $\{f_x \mid x\in X\}$ be the labeling associated with X), and for any $x\in X$ the three following conditions are satisfied:
1. $T_n(x,f_x^{-1}(0))=1$ and $T_n(x,f_x^{-1}(1))=1$,
2. $\forall y\in N_n(x)\cap X$, $T_n(x,f_x^{-1}(0)\cup\{y\})=1$ and $T_n(x,f_x^{-1}(1)\cup\{y\})=1$,
3. $\forall y\in N_{\bar{n}}(x)\cap X$, $T_{\bar{n}}(x,f_x^{-1}(0)\cup\{y\})=1$ and $T_{\bar{n}}(x,f_x^{-1}(1)\cup\{y\})=1$.

For instance, the configuration of Fig. 2(c) does not satisfy the local condition of Theorem 12 since the set X containing this configuration cannot be labelable. Hence this configuration cannot appear in a strong surface (both for $n=18$ and $n=26$).

In opposite, in Fig. 2(e), the configuration can belong to a labelable set since $\{M\}$ is the unique element of $C_6(N_{26}(x)\cap\overline{X})\setminus C_6^x(N_{26}(x)\cap\overline{X})$, and it satisfies the condition imposed by Definition 10. Moreover, if we consider $\{f_x \mid x\in X\}$ the labelling associated with X, the point $f_x(M)$ is equal to $f_x(a)$, and the three conditions of Theorem 12 are satisfied for x (both for $n=18$ and $n=26$). Hence this configuration can appear in a strong surface.

Proof of Theorem 12. ''Only if'' If X is a strong n-surface, it follows from Proposition 11 that X is labelable and the labeling associated with X is in fact an assignment. Therefore, the conditions 1, 3 and 4 of Proposition 5 imply respectively the points 1, 2 and 3 of Theorem 12.

''If'' First we observe if $x\in X$ and two points M_1 and M_2 are points of two distinct elements of $C_6^x(N_{26}(x)\cap\overline{X})$, they must be in distinct 6-connected components of $\overline{X}$.

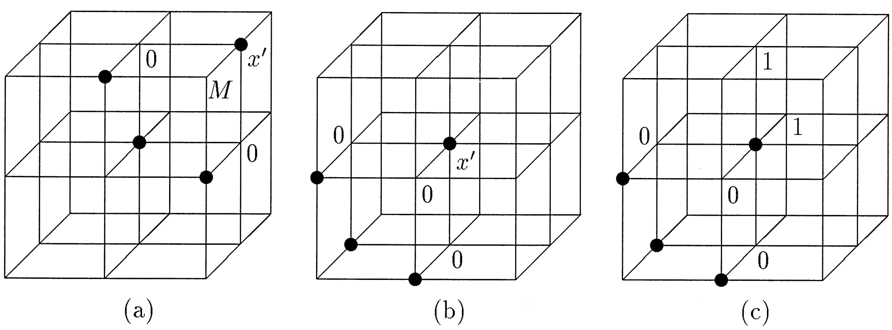


Fig. 6.

It suffices to prove that if X is labelable and satisfies points 1, 2 and 3 of Theorem 12, the labeling associated with X must be an assignment.

Let $x \in X$ and let $C = \{M\} = \{(a,b,c)\} \in C_6(N_{26}(x) \cap \overline{X})$. Up to a translation, we assume for convenience that $x = (0,0,0)$. We set $Z = \{(a,0,0),(0,b,0),(0,0,c)\}$. Up to an isometry, we assume for instance that $M = (1,1,1)$. Now assume that Z contains at least two points of $f_x^{-1}(0)$. Say for instance: $f_x((0,0,1)) = 0$ and $f_x((0,1,0)) = 0$ (see Fig. 6(a)).

Now we observe the configuration of $N_{26}(x')$ with $x' = (p,q,r) = (0,1,1)$. Since $T_n(x', f_{x'}^{-1}(0)) = 1$ and $T_{\overline{n}}(x', f_{x'}^{-1}(1)) = 1$, we have $card(C_6^{x'}(N_{26}(x') \cap \overline{X})) = 2$, and the two points $(0,0,1) = (p,q-1,r)$ and $(0,1,0) = (p,q,r-1)$, belonging to the same 6-connected component of $\overline{X}$, they must be in the same element of $C_6^{x'}(N_{26}(x') \cap \overline{X})$. Hence $f_{x'}((0,0,1)) = f_{x'}((0,1,0))$, say equals 0. By reductio ad absurdum, we assume that M is in the same 6-connected component of $\overline{X}$ as $(0,0,1)$ and $(0,1,0)$. Then, since $M \in N_6(x')$ we have $f_{x'}(M) = 0$ (see Fig. 6(b)).

If $n = 26$ and $(p+1,q-1,r-1) \in f_{x'}^{-1}(1)$, $T_{26}(x', f_{x'}^{-1}(1)) \geq 2$, which contradicts our hypothesis.

Hence, $n = 18$ or $(p+1,q-1,r-1) \in A \cup X$. In both cases, applying point 2 of our hypothesis, together with Lemma 8, to the points x' and $y = (p,q-1,r-1)$, we see that $(p-1,q,r) \in f_{x'}^{-1}(1)$. Similarly, applying point 2 of our hypothesis, together with Lemma 8, to the point $(p+1,q-1,r)$ we obtain: $(p,q,r+1) \in f_{x'}^{-1}(1)$ (see Fig. 6(c)). Now we see that $T_{\overline{n}}(x', f_{x'}^{-1}(0)) \geq 2$, which is impossible. Hence $f_{x'}(M) = 1 \neq f_{x'}((0,0,1))$ and M and $(0,0,1)$ must be in distinct 6-connected components of $\overline{X}$. □

6. Conclusion

We have given a local characterization of strong surfaces within strongly separating objects. However, the question of a complete local characterization of strong surfaces remains. In particular, the Jordan property should be derived from local conditions. We may wonder whether the Jordan property can be derived from the necessary local conditions presented in this paper. Finally we can say that generalizing the approach of strong surfaces to higher dimensions seems to be very difficult since the notions related to topology preservation are not as advanced as in dimension 3.

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